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Abstract

In the present paper we identify both blow–up and global existence behaviors for a simple but very rich kinetic equation describing of a swarm formation.

Keywords: Blow–up, Global existence, Kinetic equation

1. Introduction

In paper Parisot, Lachowicz (2015) a model of swarming behavior of an individual population was proposed and studied. The main aim was the macroscopic (*hydrodynamic*) limit. The mathematical structure that was proposed seems very reach and interesting from mathematical point of view. Let $f = f(t, x, v)$ be a probability density (p.d.) of individuals at time $t \geq 0$ and position $x \in \mathbb{R}^d$ with velocity $v \in \mathbb{V}$; $\mathbb{V} \subset \mathbb{R}^d$, the set of velocities of the individuals, is a bounded domain. The evolution of populations at the mesoscopic scale is defined by the nonlinear integro–differential Boltzmann–like equation, see Parisot, Lachowicz (2015),

$$\begin{aligned}&\partial_t f(t,x,v) + v\cdot\nabla_x f(t,x,v) = \tfrac{1}{\varepsilon}Q[f](t,x,v)\\&= \tfrac{1}{\varepsilon}\int_{\mathbb{V}}\Big(T[f(t,x,\,.\,)](w,v)f(t,x,w) - T[f(t,x,\,.\,)](v,w)f(t,x,v)\Big)\mathrm{d}w\end{aligned} \tag{1}$$

with the initial data $f(0,x,v) = f_0(x,v)$. The parameter ε corresponds to the Knudsen number and the macroscopic limit is defined by $\varepsilon \to 0$. The nonlinear operator Q describes interactions between individuals. The turning rate $T[f](v,w)$ measures the probability for an individual with velocity v to change velocity into w. Macroscopic limit for a simpler (two–velocities) kinetic equation was studied in Banasiak, Lachowicz (2013) (see also Banasiak, Lachowicz (2014)). In the context of modeling of preferential choice one should mention the paper Boissard, Degond, Motsch (2013) where a collision model was proposed describing ant trail formation.

In Ref. Parisot, Lachowicz (2015) the following general nonlinear case

$$T[f(t,x,\,.\,)](v,w) = \sigma\beta(v,w) f^{\gamma_{\rho,x}}(t,x,w)\,, \tag{2}$$

was considered, where the interaction rate β, the attractiveness coefficient γ, and σ characterize the interaction between the individual agents. Parisot, Lachowicz (2015) proposed results of global existence in the space homogeneous case for any set of collision parameters σ and γ except the so–called positive gregarious interaction, i.e. $\sigma = 1$ and $\gamma > 1$. The aim of the present paper is the analysis of simpler (but still rich) equation in this case. More general equation and some details of the present approach will be given in Ref. Lachowicz, Leszczyński, Parisot (2016).

2. Mathematical analysis of the space homogeneous case

We focus on the space homogeneous case, i.e. all functions and parameters are assumed to be independent of x. Moreover we assume that σ, γ, β are constants: $\sigma = \beta = 1$, and $\gamma > 1$.

Throughout the paper the L^p–norm in the velocity space $\mathbb{V}$ is denoted by $\|\phi\|_p = (\int_{\mathbb{V}} \phi^p \mathrm{d}v)^{\frac{1}{p}}$. It is easy to see that any solution preserves the nonnegativity of the initial datum and the L^1–norm of the nonnegative initial datum. Therefore Eq. (1) can be simplified to the following equation

$$\partial_t f = f^\gamma - \|f\|_\gamma^\gamma f\,, \quad \text{with} \quad f(0,v) = f_0(v)\,, \qquad t \geq 0\,,\ v \in \mathbb{V}\,. \tag{3}$$

Let $z(t) = \int_0^t \|f(s,\,.\,)\|_\gamma^\gamma \, \mathrm{d}s$. It fulfills

$$\mathrm{d}_t z(t) = e^{-\gamma z(t)} \int_{\mathbb{V}} \left(f_0^{1-\gamma} - (\gamma - 1) \int_0^t e^{-(\gamma-1)z(s)} \mathrm{d}s \right)^{-\frac{\gamma}{\gamma-1}} \mathrm{d}v\,. \tag{4}$$

Equation (4) determines global existence or blow–up for Eq. (3). Let $u(t) = \int\limits_0^t e^{-(\gamma-1)z(s)}\mathrm{d}s$. The function $u = u(t)$ is increasing and (as we will see) concave. A blow–up occurs for $T > 0$ such that

$$(\gamma - 1)\,\|f_0\|_\infty^{\gamma-1}\,u(T) = 1\,. \tag{5}$$

The ODE for u reads $\mathrm{d}_t u(t) = e^{-(\gamma-1)z(t)}$ and we have

$$\begin{aligned}&\mathrm{d}_t^2 u = -(\gamma-1)\,e^{-(\gamma-1)z}\mathrm{d}_t z = \\ &-(\gamma-1)\,\mathrm{d}_t u\,(\mathrm{d}_t u)^{\frac{\gamma}{\gamma-1}} \int\limits_{\mathbb{V}} \left(f_0^{1-\gamma} - (\gamma-1)\,u\right)^{-\frac{\gamma}{\gamma-1}}\mathrm{d}v\,.\end{aligned}$$

By integration we obtain

$$\mathrm{d}_t u = \Big(\int\limits_{\mathbb{V}} f_0(v)\Big(1 - (\gamma-1)\,f_0^{\gamma-1}u\Big)^{\frac{1}{1-\gamma}}\mathrm{d}v\Big)^{1-\gamma}\,. \tag{6}$$

We consider first the case when $\mathbb{W} = \{v \in \mathbb{V} \,:\, f_0(v) = \|f_0\|_\infty\}$ is such that $|\mathbb{W}| > 0$, where $|\,.\,|$ denotes the Lebesgue measure. We have

Theorem 1. *Let the probability density f_0 be in $L^\infty(\mathbb{V})$ and $|\mathbb{W}| > 0$. Then, for any $T > 0$, there exists a unique solution of Eq. (3) in $C^1\left([0,T[\,; L^\infty(\mathbb{V})\right)$. Moreover, the solution is nonnegative.*

Proof. We are going to find an inequality of the type $\mathrm{d}_t u\,(\ldots) \leq 1$. We have

$$\begin{aligned}&\mathrm{d}_t u\,\Bigg(|\mathbb{W}|\,\|f_0\|_\infty\,(1 - (\gamma-1)\,\|f_0\|_\infty^{\gamma-1}u)^{\frac{1}{1-\gamma}} \\ &+ \int\limits_{W'} f_0(v)\left(1 - (\gamma-1)\,f_0^{\gamma-1}u\right)^{\frac{1}{1-\gamma}}\mathrm{d}v\Bigg)^{\gamma-1} = 1\end{aligned}$$

where $\mathbb{W}' = \mathbb{V} \setminus \mathbb{W}$. Keeping in mind that the derivative $\mathrm{d}_t u$ is nonnegative and $(\gamma-1)\,\|f_0\|_\infty^{\gamma-1}\,u(t) \leq 1$, cf. Eq. (5), we obtain the desired inequality

$$\mathrm{d}_t u\left(|\mathbb{W}|\,\|f_0\|_\infty\left(1 - (\gamma-1)\,\|f_0\|_\infty^{\gamma-1}u\right)^{\frac{1}{1-\gamma}}\right)^{\gamma-1} \leq 1\,.$$

We may consider the IVP for the *comparison equation*

$$\mathrm{d}_t U = (|\mathbb{W}|\,\|f_0\|_\infty)^{1-\gamma}\left(1 - (\gamma-1)\,\|f_0\|_\infty^{\gamma-1}U\right)\,, \qquad U(0) = 0\,.$$

It implies $u(t) \le U(t)$, $t \ge 0$, where U is given by

$$U(t) = (\gamma-1)^{-1} \|f_0\|_\infty^{1-\gamma} \left(1 - e^{-t\,(\gamma-1)|\mathbb{W}|^{1-\gamma}}\right). \tag{7}$$

Therefore the global existence follows.

□

Case $|\mathbb{W}| = 0$, as we will see, is more complex. We denote the RHS of Eq. (6) by Φ. The function Φ is defined on $[0, u_0[$, where $u_0 = (\gamma-1)^{-1}\|f_0\|_\infty^{1-\gamma}$. Let $\Phi(u_0) = \lim\limits_{u \uparrow u_0} \Phi(u)$.

Theorem 2. *Let the probability density f_0 be in $L^\infty(\mathbb{V})$ and $|\mathbb{W}| = 0$.*

1. *If $\Phi(u_0) > 0$, then there is a blow–up in a finite time $T > 0$;*
2. *If $\Phi(u_0) = 0$, then*
   (a) *if $\int\limits_0^{u_0} (\Phi(u))^{-1}\,\mathrm{d}u < \infty$ then there is a blow–up in a finite time $T > 0$;*
   (b) *if $\int\limits_0^{u_0} (\Phi(u))^{-1}\,\mathrm{d}u = \infty$ then for each $T > 0$ there exists a unique solution on $[0, T]$.*

Proof. Assume first that $\Phi(u_0) > 0$. We have

$$\Phi'(u) = (1-\gamma)\left(\int\limits_{\mathbb{V}} f_0(v)\left(1-(\gamma-1)\,f_0^{\gamma-1}u\right)^{\frac{1}{1-\gamma}}\mathrm{d}v\right)^{-\gamma} \times$$
$$\int\limits_{\mathbb{V}} f_0^{\gamma}(v)\left(1-(\gamma-1)\,f_0^{\gamma-1}(v)\,u\right)^{\frac{1}{1-\gamma}}\mathrm{d}v \;<\; 0\,.$$

Therefore u is increasing and concave, in fact $\mathrm{d}_t^2 u = \Phi'(u)\mathrm{d}_t u$. Because the tangent of the straight line passing through the points $(0,0)$ and (T, u_0) is bigger than the tangent of the tangential to the curve defined by $u = u(t)$ in the point (T, u_0) we obtain that $\frac{u_0}{T} \ge \Phi(u_0)$ and $T \le \frac{u_0}{\Phi(u_0)}$. Analogously $T \ge \frac{u_0}{\Phi(0)}$. Thus the blow–up time T satisfies

$$\frac{u_0}{\Phi(0)} \le T \le \frac{u_0}{\Phi(u_0)}\,.$$

Assume next that $\Phi(u_0) = 0$. We may use the standard theory of ODE (see Walter (1998)). We have blow–up in a finite time provided that u_0 is a non–uniqueness point for the ODE $\mathrm{d}_t u = \Phi(u)$, i.e. $\int\limits_0^{u_0} (\Phi(u))^{-1}\,\mathrm{d}u < \infty$. The reason of this blow–up is that f has arbitrarily large values.

On the other hand we have global existence provided that u_0 is a uniqueness point for the ODE, i.e. $\int\limits_0^{u_0} (\Phi(u))^{-1}\,\mathrm{d}u = \infty$. Then $u(t)$ tends to u_0, but never reaches the limit value because the solution u_0 is unique for $\mathrm{d}_t u = \Phi(u)$. □

Note that two conditions $\Phi(u_0) > 0$ and $\Phi(u_0) = 0$ are equivalent to $\widetilde{f}_0(1-\widetilde{f}_0^{\gamma-1})^{\frac{1}{1-\gamma}} \in L^1(\mathbb{V})$ and $\widetilde{f}_0(1-\widetilde{f}_0^{\gamma-1})^{\frac{1}{1-\gamma}} \notin L^1(\mathbb{V})$, respectively, where we denote $\widetilde{f}_0\,(v) = f_0(v)\|f_0\|_\infty^{-1}$. We can now rephrase Item 2 of Theorem 2 as follows

Corollary 2.1. *Let p.d. f_0 be in $L^\infty(\mathbb{V})$, $|\mathbb{W}| = 0$, and $\widetilde{f}_0(1-\widetilde{f}_0^{\gamma-1})^{\frac{1}{1-\gamma}} \notin L^1(\mathbb{V})$.*

1. *If* $\int\limits_0^1 \Big(\int\limits_{\mathbb{V}} \widetilde{f}_0(v)(1-\widetilde{f}_0^{\gamma-1}(v)y)^{\frac{1}{1-\gamma}}\,\mathrm{d}v\Big)^{\gamma-1} \mathrm{d}y < \infty$, *then then there is a blow–up in a finite time.*
2. *If* $\int\limits_0^1 \Big(\int\limits_{\mathbb{V}} \widetilde{f}_0(v)(1-\widetilde{f}_0^{\gamma-1}(v)y)^{\frac{1}{1-\gamma}}\,\mathrm{d}v\Big)^{\gamma-1} \mathrm{d}y = \infty$, *then, for any time $T > 0$, there exists a unique solution of (3) in $C^1([0,T]; L^\infty(\mathbb{V}))$.*

Proof. We have

$$\int\limits_0^{u_0} (\Phi(u))^{-1}\,\mathrm{d}u = \int\limits_0^{(\gamma-1)^{-1}\|f_0\|_\infty^{1-\gamma}} \Big(\int\limits_{\mathbb{V}} f_0(v)\Big(1-(\gamma-1)f_0^{\gamma-1}(v)\,u\Big)^{\frac{1}{1-\gamma}}\,\mathrm{d}v\Big)^{\gamma-1} \mathrm{d}u\,,$$

and changing the variable $y := (\gamma-1)\,\|f_0\|_\infty^{\gamma-1}\,u$ yields

$$\int\limits_0^{u_0} (\Phi(u))^{-1}\,\mathrm{d}u = (\gamma-1)^{-1}\int\limits_0^1 \Big(\int\limits_{\mathbb{V}} \widetilde{f}_0(v)(1-\widetilde{f}_0^{\gamma-1}(v)y)^{\frac{1}{1-\gamma}}\,\mathrm{d}v\Big)^{\gamma-1} \mathrm{d}y\,.$$

Therefore by Theorem 2 the statement follows. □

Corollary 2.2. *Let p.d.* $f_0 \in L^\infty(\mathbb{V})$, $|\mathbb{W}| = 0$, $\widetilde{f}_0(1-\widetilde{f}_0^{\gamma-1})^{\frac{1}{1-\gamma}} \notin L^1(\mathbb{V})$. *Set* $g(v) = \widetilde{f}_0^{2-\gamma}(v)(1-\widetilde{f}_0^{\gamma-1}(v))^{\frac{\gamma-2}{\gamma-1}}$ *and* $h(v) = -\log\left(1-\widetilde{f}_0^{\gamma-1}(v)\right)$.

1. *If* $1 < \gamma < 2$ *and* $g \in L^1(\mathbb{V})$ *then the solution blows–up in a finite time;*
2. *If* $1 < \gamma \leq 2$ *and* $h \notin L^1(\mathbb{V})$ *then the solution is global;*
3. *If* $\gamma \geq 2$ *and* $h \in L^1(\mathbb{V})$ *then the solution blows–up in a finite time.*

Proof. The statement follows by Hölder's inequality and Corollary 2.1. □

Remark 1. *If* $\gamma = 2$ *then Items 3 and 4 of Corollary 2.2 give a sufficient and necessary condition for blow–up in terms of h. Let* $\alpha > 0$ *and* $\widetilde{f}_0(v) = (1-\exp(-|v|^{-\alpha}))$ *on* $\mathbb{V} =]-1,1[$. *Then, we have* $h \in L^1(\mathbb{V})$ *iff* $\alpha < 1$. *We may deliver examples of initial data such that the solution blows up in a finite time, e.g.* $\widetilde{f}_0(v) = 1-\exp\left(-|v|^{-\frac{1}{2}}\right)$, *and the solution is global, e.g.* $\widetilde{f}_0(v) = 1-\exp(-|v|^{-2})$.

References

J. Banasiak, M. Lachowicz, On a macroscopic limit of a kinetic model of alignment, Math. Models Methods Appl. Sci., 23 (14), 2013

J. Banasiak, M. Lachowicz, Methods of small parameter in mathematical biology, Birkhäuser, Boston 2014.

E. Boissard, P. Degond, S. Motsch, Trail formation based on directed pheromone deposition, J. Math. Biol. 66, 1267-1301, 2013.

M. Lachowicz, H. Leszczyński, M. Parisot, Blow–up and global existence of a kinetic equation of swarm formation, to appear.

H. Ninomiya, M. Fila, Reaction versus diffusion: blow–up induced and inhibited by diffusivity, Russian Math. Surveys 60, 1217–1235, 2005.

M. Parisot, M. Lachowicz, A kinetic model for the formation of swarms with nonlinear interactions, Kinetic Related Models, 9, 1, 131–164, 2016.

P. Quittner, P. Souplet, Superlinear Parabolic Problems. Blow–up, Global Existence and Steady States. Birkhäuser Advanced Texts, Basel, 2007.

W. Walter, Ordinary Differential Equations, Springer, New York 1998.