q-analogs of group divisible designs

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Abstract

A well known class of objects in combinatorial design theory are group divisible designs. Here, we introduce the q-analogs of group divisible designs. It turns out that there are interesting connections to scattered subspaces, q-Steiner systems, design packings and q^r-divisible projective sets.

We give necessary conditions for the existence of q-analogs of group divsible designs, construct an infinite series of examples, and provide further existence results with the help of a computer search.

One example is a $(6, 3, 2, 2)_2$ group divisible design over $\mathrm{GF}(2)$ which is a design packing consisting of 180 blocks that such every 2-dimensional subspace in $\mathrm{GF}(2)^6$ is covered at most twice.

1 Introduction

The classical theory of q-*analogs* of mathematical objects and functions has its beginnings as early as in the work of Euler [Eul53]. In 1957, Tits [Tit57] further suggested that combinatorics of sets could be regarded as the limiting case $q \to 1$ of combinatorics of vector spaces over the finite field $\mathrm{GF}(q)$. Recently, there has been an increased interest in studying q-analogs of combinatorial designs from an applications' view. These q-analog structures can be useful in network coding and distributed storage, see e.g. [GPe18].

It is therefore natural to ask which combinatorial structures can be generalized from sets to vector spaces over $\mathrm{GF}(q)$. For combinatorial designs, this question was first studied by Ray-Chaudhuri [BRC74], Cameron [Cam74a, Cam74b] and Delsarte [Del76] in the early 1970s.

Specifically, let $\mathrm{GF}(q)^v$ be a vector space of dimension v over the finite field $\mathrm{GF}(q)$. Then a t-$(v, k, \lambda)_q$ subspace design is defined as a collection of

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k-dimensional subspaces of $\mathrm{GF}(q)^v$, called blocks, such that each t-dimensional subspace of $\mathrm{GF}(q)^v$ is contained in exactly λ blocks. Such t-designs over $\mathrm{GF}(q)$ are the q-analogs of conventional designs. By analogy with the $q \to 1$ case, a t-$(v, k, 1)_q$ subspace design is said to be a q-Steiner system, and denoted $S(t, k, v)_q$.

Another well-known class of objects in combinatorial design theory are *group divisible designs* [MG07]. Considering the above, it therefore seems natural to ask for q-analogs of group divisible designs.

At first glance, this seems like a somewhat artificial task without much justification. But quite surprisingly, it turns out that q-analogs of group divisible designs have interesting connections to scattered subspaces which are central objects in finite geometry, as well as to coding theory via q^r-divisible projective sets. We will also discuss the connection to q-Steiner systems [BEÖ+16] and to design packings [EZ18].

Let K and G be sets of positive integers and let λ be a positive integer. A (v, K, λ, G)-*group divisible design* of index λ and order v is a triple $(V, \mathcal{G}, \mathcal{B})$, where V is a finite set of cardinality v, $\mathcal{G}$, where $\#\mathcal{G} > 1$, is a partition of V into parts (groups) whose sizes lie in G, and $\mathcal{B}$ is a family of subsets (blocks) of V (with $\#B \in K$ for $B \in \mathcal{B}$) such that every pair of distinct elements of V occurs in exactly λ blocks or one group, but not both. See—for example—[MG07, Han75] for details. We note that the "groups" in group divisible designs have nothing to do with group theory.

The q-analog of a combinatorial structure over sets is defined by replacing subsets by subspaces and cardinalities by dimensions. Thus, the q-analog of a group divisible design can be defined as follows.

Definition 1 *Let K and G be sets of positive integers and let λ be a positive integer. A q-*analog of a group divisible design *of index λ and order v — denoted as $(v, K, \lambda, G)_q$-*GDD *— is a triple $(V, \mathcal{G}, \mathcal{B})$, where*

- *V is a vector space over $\mathrm{GF}(q)$ of dimension v,*
- *$\mathcal{G}$ is a vector space partition[1] of V into subspaces (groups) whose dimensions lie in G, and*
- *$\mathcal{B}$ is a family of subspaces (blocks) of V,*

that satisfies

1. *$\#\mathcal{G} > 1$,*
2. *if $B \in \mathcal{B}$ then $\dim B \in K$,*
3. *every 2-dimensional subspace of V occurs in exactly λ blocks or one group, but not both.*

*A $(v, K, \lambda, \{g\})_q$-GDD is called g-*uniform*. Subsequently, if K or G are one-element sets, we denote it by small letters, e.g. $(v, k, \lambda, g)_q$-GDD for $K = \{k\}$ and $G = \{g\}$.*

[1] A set of subspaces of V such that every 1-dimensional subspace is covered exactly once is called vector space partition.

In the rest of the paper we study the case $K = \{k\}$ and $G = \{g\}$. The latter implies that the vector space partition $\mathcal{G}$ is a partition of the 1-dimensional subspaces of V in subspaces of dimension g. In finite geometry such a structure is known as $(g-1)$-*spread*. Additionally, we will only consider so called *simple* group divisible designs, i.e. designs without multiple appearances of blocks.

A possible generalization would be to require the last condition in Definition 1 for every t-dimensional subspace of V, where $t \geq 2$. For $t = 1$ such a definition would make no sense.

An equivalent formulation of the last condition in Definition 1 would be that every block in $\mathcal{B}$ intersects the spread elements in dimension of at most one. The q-analog of concept of a *transversal design* would be that every block in $\mathcal{B}$ intersects the spread elements exactly in dimension one. But for q-analogs this is only possible in the trivial case $g = 1$, $k = v$. However, a related concept was defined in [ES13].

Among all 2-subspaces of V, only a small fraction is covered by the elements of $\mathcal{G}$. Thus, a $(v, k, \lambda, g)_q$-GDD is "almost" a 2-$(v, k, \lambda)_q$ subspace design, in the sense that the vast majority of the 2-subspaces is covered by λ elements of $\mathcal{B}$. From a slightly different point of view, a $(v, k, \lambda, g)_q$-GDD is a 2-$(v, k, \lambda, g)_q$ *packing design* of fairly large size, which are designs where the condition "each t-subspace is covered by exactly λ blocks" is relaxed to "each t-subspace is covered by at most λ blocks" [BKW18a]. In Section 6 we give an example of a $(6, 3, 2, 2)_2$-GDD consisting of 180 blocks. This is the largest known 2-$(6, 3, 2)_2$ packing design.

We note that a q-analog of a group divisible design can be also seen as a special graph decomposition over a finite field, a concept recently introduced in [BNW]. It is indeed equivalent to a decomposition of a complete m-partite graph into cliques where: the vertices are the points of a projective space $\mathrm{PG}(n, q)$; the parts are the members of a spread of $\mathrm{PG}(n, q)$ into subspaces of a suitable dimension; the vertex-set of each clique is a subspace of $\mathrm{PG}(n, q)$ of a suitable dimension.

2 Preliminaries

For $1 \leq m \leq v$ we denote the set of m-dimensional subspaces of V, also called *Grassmannian*, by $\begin{bmatrix} V \\ m \end{bmatrix}_q$. It is well known that its cardinality can be expressed by the *Gaussian coefficient*

$$\#\begin{bmatrix} V \\ m \end{bmatrix}_q = \begin{bmatrix} v \\ m \end{bmatrix}_q = \frac{(q^v - 1)(q^{v-1} - 1) \cdots (q^{v-m+1} - 1)}{(q^m - 1)(q^{m-1} - 1) \cdots (q - 1)}.$$

Definition 2 *Given a spread in dimension v, let $\begin{bmatrix} V \\ k \end{bmatrix}_q'$ be the set of all blocks that contain no 2-dimensional subspace which is already covered by the spread.*

The intersection between a k-dimensional subspace $B \in \begin{bmatrix} V \\ 2 \end{bmatrix}_q'$ and all elements of the spread is at most one-dimensional. In finite geometry such a subspace $B \in \begin{bmatrix} V \\ k \end{bmatrix}_q'$ is called *scattered subspace with respect to* $\mathcal{G}$ [BBL00, BL00].

In case $g = 1$, i.e. $\mathcal{G} = \begin{bmatrix} V \\ 1 \end{bmatrix}_q$, no 2-dimensional subspace is covered by this trivial spread. Then, $(V, \mathcal{B})$ is a 2-$(v, k, \lambda)_q$ subspace design. See [BKW18b, BKW18a] for surveys about subspace designs and computer methods for their construction.

Let $g \cdot s = v$ and $V = \mathrm{GF}(q)^v$. Then, the set of 1-dimensional subspaces of $\mathrm{GF}(q^g)^s$ regarded as g-dimensional subspaces in the q-linear vector space $\mathrm{GF}(q)^v$, i.e.

$$\mathcal{G} = \begin{bmatrix} \mathrm{GF}(q^g)^s \\ 1 \end{bmatrix}_{q^g},$$

is called *Desarguesian spread.*

A t-spread $\mathcal{G}$ is called *normal* or *geometric*, if $U, V \in \mathcal{G}$ then any element $W \in \mathcal{G}$ is either disjoint to the subspace $\langle U, V\rangle$ or contained in it, see e.g. [Lun99]. Since all normal spreads are isomorphic to the Desarguesian spread [Lun99], we will follow [Lav16] and denote normal spreads as Desarguesian spreads.

If $s \in \{1, 2\}$, then all spreads are normal and therefore Desarguesian. The automorphism group of a Desarguesian spread $\mathcal{G}$ is $\mathrm{P\Gamma L}(s, q^g)$.

"Trivial" q-analogs of group divisible designs. For subspace designs, the empty set as well as the the set of all k-dimensional subspaces in $\mathrm{GF}(q)^v$ always are a designs, called *trivial designs.* Here, it turns out that the question if trivial q-analogs of group divisible designs exist is rather non-trivial.

Of course, there exists always the trivial $(v, k, 0, g)_q$-GDD $(V, \mathcal{G}, \{\})$. But it is not clear if the set of all scattered k-dimensional subspaces, i.e. $(V, \mathcal{G}, {\begin{bmatrix} V \\ k \end{bmatrix}_q}')$, is always a q-GDD. This would require that every subspace $L \in \begin{bmatrix} V \\ 2 \end{bmatrix}_q$ that is not covered by the spread, is contained in the same number $\lambda_{\max}$ of blocks of ${\begin{bmatrix} V \\ k \end{bmatrix}_q}'$. If this is the case, we call $(V, {\begin{bmatrix} V \\ k \end{bmatrix}_q}', \mathcal{G})$ the *complete* $(v, k, \lambda_{\max}, g)_q$-GDD.

If the complete $(v, k, \lambda_{\max}, g)_q$-GDD exists, then for any $(v, k, \lambda, g)_q$-GDD $(V, \mathcal{G}, \mathcal{B})$ the triple $(V, \mathcal{G}, {\begin{bmatrix} V \\ 2 \end{bmatrix}_q}' \setminus \mathcal{B})$ is a $(v, k, \lambda_{\max} - \lambda, g)_q$-GDD, called the *supplementary* q-GDD.

For a few cases we can answer the question if the complete q-GDD exists, or in other words, if there is a $\lambda_{\max}$. In general, the answer depends on the choice of the spread. In the smallest case, $k = 3$, however, $\lambda_{\max}$ exists for all spreads.

Lemma 1 *Let $\mathcal{G}$ be a $(g-1)$-spread in V and let L be a 2-dimensional subspace which is not contained in any element of $\mathcal{G}$. Then, L is contained in*

$$\lambda_{\max} = \begin{bmatrix} v-2 \\ 3-2 \end{bmatrix}_q - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \begin{bmatrix} g-1 \\ 3-2 \end{bmatrix}_q$$

blocks of ${\begin{bmatrix} V \\ 3 \end{bmatrix}_q}'$.

Proof. Every 2-dimensional subspace L is contained in $\begin{bmatrix} v-2 \\ 3-2 \end{bmatrix}_q$ 3-dimensional subspaces of $\begin{bmatrix} V \\ 3 \end{bmatrix}_q$. If L is not contained in any spread element, this means that L intersects $\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q$ different spread elements and the intersections are 1-dimensional. Let S be one such spread element. Now, there are $\begin{bmatrix} g-1 \\ 1 \end{bmatrix}_q$ choices among the 3-dimensional subspaces in $\begin{bmatrix} V \\ 3 \end{bmatrix}_q$ which contain L to intersect S in dimension two. Therefore, L is contained in

$$\lambda_{\max} = \begin{bmatrix} v-2 \\ 3-2 \end{bmatrix}_q - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \begin{bmatrix} g-1 \\ 3-2 \end{bmatrix}_q$$

blocks of $\begin{bmatrix}V\\3\end{bmatrix}_q'$. $\square$

In general, the existence of $\lambda_{\max}$ may depend on the spread. This can be seen from the fact that the maximum dimension of a scattered subspace depends on the spread, see [BL00]. However, for a Desarguesian spread and $g = 2$, $k = 4$, we can determine $\lambda_{\max}$.

Lemma 2 *Let $\mathcal{G}$ be a Desarguesian $(g-1)$-spread in V and let L be a 2-dimensional subspace which is not contained in any element of $\mathcal{G}$. Then, L is contained in*

$$\lambda_{\max} = \begin{bmatrix}v-2\\4-2\end{bmatrix}_q - 1 - q\begin{bmatrix}2\\1\end{bmatrix}_q\begin{bmatrix}v-4\\1\end{bmatrix}_q - \begin{bmatrix}v\\1\end{bmatrix}_q \Big/ \begin{bmatrix}2\\1\end{bmatrix}_q + \begin{bmatrix}4\\1\end{bmatrix}_q \Big/ \begin{bmatrix}2\\1\end{bmatrix}_q$$

blocks of $\begin{bmatrix}V\\4\end{bmatrix}_q'$.

Proof. Every 2-dimensional subspace L is contained in $\begin{bmatrix}v-2\\4-2\end{bmatrix}_q$ 4-dimensional subspaces. If L is not covered by the spread this means that L intersects $\begin{bmatrix}2\\1\end{bmatrix}_q$ spread elements $S_1, \ldots, S_{q+1}$, which span a 4-dimensional space F. All other spread elements are disjoint to L. Since $L \leq F$, we have to subtract one possibility. For each $1 \leq i \leq q+1$, $\langle S_i, L\rangle$ is contained in $q\begin{bmatrix}v-4\\1\end{bmatrix}_q$ 4-dimensional subspaces with a 3-dimensional intersection with F. All other spread elements S' of F satisfy $\langle S', L\rangle = F$. If S'' is one of the $\begin{bmatrix}v\\1\end{bmatrix}_q/\begin{bmatrix}2\\1\end{bmatrix}_q - \begin{bmatrix}4\\1\end{bmatrix}_q/\begin{bmatrix}2\\1\end{bmatrix}_q$ spread elements disjoint from F, then $F'' := \langle S'', L\rangle$ intersects F in dimension 2. Moreover, F'' does not contain any further spread element, since otherwise F'' would be partitioned into q^2+1 spread elements, where $q+1$ of them have to intersect L. Thus, L is contained in exactly $\lambda_{\max}$ elements from $\begin{bmatrix}V\\4\end{bmatrix}_q'$. $\square$

3 Necessary conditions on $(v, k, \lambda, g)_q$

The necessary conditions for a (v, k, λ, g)-GDD over sets are $g \mid v$, $k \leq v/g$, $\lambda(\frac{v}{g} - 1)g \equiv 0 \pmod{k-1}$, and $\lambda\frac{v}{g}(\frac{v}{g} - 1)g^2 \equiv 0 \pmod{k(k-1)}$, see [Han75].

For q-analogs of GDDs it is well known that $(g-1)$-spreads exist if and only if g divides v. A $(g-1)$-spread consists of $\begin{bmatrix}v\\1\end{bmatrix}_q/\begin{bmatrix}g\\1\end{bmatrix}_q$ blocks and contains

$$\begin{bmatrix}g\\2\end{bmatrix}_q \cdot \begin{bmatrix}v\\1\end{bmatrix}_q \Big/ \begin{bmatrix}g\\1\end{bmatrix}_q$$

2-dimensional subspaces.

Based on the pigeonhole principle we can argue that if B is a block of a $(v, k, \lambda, g)_q$ q-GDD then there can not be more points in B than the number of spread elements, i.e. if $\begin{bmatrix}k\\1\end{bmatrix}_q \leq \begin{bmatrix}v\\1\end{bmatrix}_q/\begin{bmatrix}g\\1\end{bmatrix}_q$. It follows that (see [BL00, Theorem 3.1])

$$k \leq v - g\,. \tag{1}$$

This is the q-analog of the restriction $k \leq v/g$ for the set case.

If $\mathcal{G}$ is a Desarguesian spread, it follows from [BL00, Theorem 4.3] for the parameters $(v, k, \lambda, g)_q$ to be admissible that

$$k \leq v/2\,.$$

By looking at the numbers of 2-dimensional subspaces which are covered by spread elements we can conclude that the cardinality of $\mathcal{B}$ has to be

$$\#\mathcal{B} = \lambda \frac{\begin{bmatrix} v \\ 2 \end{bmatrix}_q - \begin{bmatrix} g \\ 2 \end{bmatrix}_q \cdot \begin{bmatrix} v \\ 1 \end{bmatrix}_q / \begin{bmatrix} g \\ 1 \end{bmatrix}_q}{\begin{bmatrix} k \\ 2 \end{bmatrix}_q}. \tag{2}$$

A necessary condition on the parameters of a g-uniform q-(k, λ) GDD is that the cardinality in (2) is an integer number.

Any fixed 1-dimensional subspace P is contained in $\begin{bmatrix} v-1 \\ 1 \end{bmatrix}_q$ 2-dimensional subspaces. Further, P lies in exactly one block of the spread and this block covers $\begin{bmatrix} g-1 \\ 1 \end{bmatrix}_q$ 2-dimensional subspaces through P. Those 2-dimensional subspaces are not covered by blocks in $\mathcal{B}$. All other 2-dimensional subspaces containing P are covered by exactly λ k-dimensional blocks. Such a block contains P and there are $\begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q$ 2-dimensional subspaces through P in this block. It follows that P is contained in exactly

$$\lambda \frac{\begin{bmatrix} v-1 \\ 1 \end{bmatrix}_q - \begin{bmatrix} g-1 \\ 1 \end{bmatrix}_q}{\begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q} \tag{3}$$

k-dimensional blocks and this number must be an integer. The number (3) is the *replication number* of the point P in the q-GDD.

Up to now, the restrictions (1), (2), (3), as well as g divides v, on the parameters of a $(v, k, \lambda, g)_q$-GDD are the q-analogs of restrictions for the set case. But for q-GDDs there is a further necessary condition whose analog in the set case is trivial.

Given a multiset of subspaces of V, we obtain a corresponding multiset $\mathcal{P}$ of points by replacing each subspace by its set of points. A multiset $\mathcal{P} \subseteq \begin{bmatrix} V \\ 1 \end{bmatrix}_q$ of points in V can be expressed by its weight function $w_{\mathcal{P}}$: For each point $P \in V$ we denote its multiplicity in $\mathcal{P}$ by $w_{\mathcal{P}}(P)$. We write

$$\#\mathcal{P} = \sum_{P \in V} w_{\mathcal{P}}(P) \quad \text{and} \quad \#(\mathcal{P} \cap H) = \sum_{P \in H} w_{\mathcal{P}}(P)$$

where H is an arbitrary hyperplane in V.

Let $1 \le r < v$ be an integer. If $\#\mathcal{P} \equiv \#(\mathcal{P} \cap H) \pmod{q^r}$ for every hyperplane H, then $\mathcal{P}$ is called *q^r-divisible*.[2] In [KK17, Lemma 1] it is shown that the multiset $\mathcal{P}$ of points corresponding to a multiset of subspaces with dimension at least k is q^{k-1}-divisible.

Lemma 3 ([KK17, Lemma 1]) *For a non-empty multiset of subspaces of V with m_i subspaces of dimension i let $\mathcal{P}$ be the corresponding multiset of points. If $m_i = 0$ for all $0 \le i < k$, where $k \ge 2$, then*

$$\#\mathcal{P} \equiv \#(\mathcal{P} \cap H) \pmod{q^{k-1}}$$

for every hyperplane $H \le V$.

[2]Taking the elements of $\mathcal{P}$ as columns of a generator matrix gives a linear code of length $\#\mathcal{P}$ and dimension k whose codewords have weights being divisible by q^r.

PROOF. We have $\#\mathcal{P} = \sum_{i=0}^{v} m_i \begin{bmatrix} v \\ i \end{bmatrix}_q$. The intersection of an i-subspace $U \le V$ with an arbitrary hyperplane $H \le V$ has either dimension i or $i-1$. Therefore, for the set $\mathcal{P}'$ of points corresponding to U, we get that $\#\mathcal{P} = \begin{bmatrix} i \\ 1 \end{bmatrix}_q$ and that $\#(\mathcal{P}' \cap H)$ is equal to $\begin{bmatrix} i \\ 1 \end{bmatrix}_q$ or $\begin{bmatrix} i-1 \\ 1 \end{bmatrix}_q$. In either case, it follows from $\begin{bmatrix} i \\ 1 \end{bmatrix}_q \equiv \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_q \pmod{q^{i-1}}$ that

$$\#(\mathcal{P}' \cap H) \equiv \begin{bmatrix} i \\ 1 \end{bmatrix}_q \pmod{q^{i-1}}.$$

Summing up yields the proposed result. □

If there is a suitable integer λ such that $w_{\mathcal{P}}(P) \le \lambda$ for all $P \in V$, then we can define for $\mathcal{P}$ the complementary weight function

$$\bar{w}_\lambda(P) = \lambda - w(P)$$

which in turn gives rise to the *complementary* multiset of points $\bar{\mathcal{P}}$. In [KK17, Lemma 2] it is shown that a q^r-divisible multiset $\mathcal{P}$ leads to a multiset $\overline{\mathcal{P}}$ that is also q^r-divisible.

Lemma 4 ([KK17, Lemma 2]) *If a multiset $\mathcal{P}$ in V is q^r-divisible with $r < v$ and satisfies $w_{\mathcal{P}}(P) \le \lambda$ for all $P \in V$ then the complementary multiset $\bar{\mathcal{P}}$ is also q^r-divisible.*

PROOF. We have

$$\#\bar{\mathcal{P}} = \begin{bmatrix} v \\ 1 \end{bmatrix}_q \lambda - \#\mathcal{P} \quad \text{and} \quad \#(\bar{\mathcal{P}} \cap H) = \begin{bmatrix} v-1 \\ 1 \end{bmatrix}_q \lambda - \#(\mathcal{P} \cap H)$$

for every hyperplane $H \le V$. Thus, the result follows from $\begin{bmatrix} v \\ 1 \end{bmatrix}_q \equiv \begin{bmatrix} v-1 \\ 1 \end{bmatrix}_q \pmod{q^r}$ which holds for $r < v$. □

These easy but rather generally applicable facts about q^r-divisible multiset of points are enough to conclude:

Lemma 5 *Let $(V, \mathcal{G}, \mathcal{B})$ be a $(v, k, \lambda, g)_q$-GDD and $2 \le g \le k$, then q^{k-g} divides λ.*

PROOF. Let $P \in \begin{bmatrix} V \\ 1 \end{bmatrix}_q$ be an arbitrary point. Then there exists exactly one spread element $S \in \mathcal{G}$ that contains P. By $\mathcal{B}_P$ we denote the elements of $\mathcal{B}$ that contain P. Let S' and $\mathcal{B}'_P$ denote the corresponding subspaces in the factor space V/P.

We observe that every point of $\begin{bmatrix} S' \\ 1 \end{bmatrix}_q$ is disjoint to the elements of $\mathcal{B}'_P$ and that every point in $\begin{bmatrix} V/P \\ 1 \end{bmatrix}_q \setminus \begin{bmatrix} S' \\ 1 \end{bmatrix}_q$ is met by exactly λ elements of $\mathcal{B}'_P$ (all having dimension $k-1$). We note that $\mathcal{B}'_P$ gives rise to a q^{k-2}-divisible multiset $\mathcal{P}$ of points. So, its complement $\bar{\mathcal{P}}$, which is the λ-fold copy of S', also has to be q^{k-2}-divisible. For every hyperplane H not containing S', we have $\#(\bar{\mathcal{P}} \cap H) = \lambda \begin{bmatrix} g-2 \\ 1 \end{bmatrix}_q$ and $\#\bar{\mathcal{P}} = \lambda \begin{bmatrix} g-1 \\ 1 \end{bmatrix}_q$. Thus, $\lambda q^{g-2} = \#\bar{\mathcal{P}} - \#(\bar{\mathcal{P}} \cap H) \equiv 0 \pmod{q^{k-2}}$, so that q^{k-g} divides λ. □

We remark that the criterion in Lemma 5 is independent of the dimension v of the ambient space. Summarizing the above we arrive at the following restrictions.

Theorem 1 *Necessary conditions for a* $(v,k,\lambda,g)_q$*-GDD are*

1. *g divides v,*
2. *$k \le v-g$,*
3. *the cardinalities in (2), (3) are integer numbers,*
4. *if $2 \le g \le k$ then q^{k-g} divides λ.*

If these conditions are fulfilled, the parameters $(v,k,\lambda,g)_q$ *are called* admissible.

Table 1 contains the admissible parameters for $q=2$ up to dimension $v=14$. Column λ_Δ gives the minimum value of λ which fulfills the above necessary conditions. All admissible values of λ are integer multiples of λ_Δ. In column $\#\mathcal{B}$ the cardinality of $\mathcal{B}$ is given for $\lambda=\lambda_\Delta$. Those values of $\lambda_{\max}$ that are valid for the Desarguesian spread only are given in italics, where the values for $(v,g,k)=(8,4,4)$ and $(9,3,4)$ have been checked by a computer enumeration.

For the case $\lambda=1$, the online tables [HKKW16]

`http://subspacecodes.uni-bayreuth.de`

may give further restrictions, since $\mathcal{B}$ is a constant dimension subspace code of minimum distance $2(k-1)$ and therefore

$$\#\mathcal{B} \le A_q(v,2(k-1);k).$$

The currently best known upper bounds for $A_q(v,d;k)$ are given by [HHK$^+$17, Equation (2)] referring back to partial spreads and $A_2(6,4;3)=77$ [HKK15], $A_2(8,6;4)=257$ [HHK$^+$17] both obtained by exhaustive integer linear programming computations, see also [KK17].

4 q-GDDs and q-Steiner systems

In the set case the connection between Steiner systems 2-$(v,k,1)$ and group divisible designs is well understood.

Theorem 2 ([Han75, Lemma 2.12]) *A* 2-$(v+1,k,1)$ *design exists if and only if a* $(v,k,1,k-1)$*-GDD exists.*

There is a partial q-analog of Theorem 2:

Theorem 3 *If there exists a* 2-$(v+1,k,1)_q$ *subspace design, then a* $(v,k,q^2,k-1)_q$*-GDD exists.*

Proof. Let V' be a vector space of dimension $v+1$ over $\mathrm{GF}(q)$. We fix a point $P \in \begin{bmatrix} V' \\ 1 \end{bmatrix}_q$ and define the projection

$$\pi : \mathrm{PG}(V') \to \mathrm{PG}(V'/P), \quad U \mapsto (U+P)/P.$$

For any subspace $U \le V'$ we have

$$\dim(\pi(U)) = \begin{cases} \dim(U)-1 & \text{if } P \le U, \\ \dim(U) & \text{otherwise.} \end{cases}$$

Table 1: Admissible parameters for $(v, k, \lambda, g)_2$-GDDs with $v \le 14$.

v	g	k	λ_Δ	$\lambda_{\max}$	$\#\mathcal{B}$	$\#\mathcal{G}$
6	2	3	2	12	180	21
6	3	3	3	6	252	9
8	2	3	2	60	3060	85
8	2	4	4	*480*	1224	85
8	4	3	7	42	10200	17
8	4	4	7	*14*	2040	17
9	3	3	1	118	6132	73
9	3	4	10	*1680*	12264	73
10	2	3	14	252	347820	341
10	2	4	28	*10080*	139128	341
10	2	5	8		8976	341
10	5	3	21	210	507408	33
10	5	4	35		169136	33
10	5	5	15		16368	33
12	2	3	2	1020	797940	1365
12	2	4	28	*171360*	2234232	1365
12	2	5	40		720720	1365
12	2	6	16		68640	1365
12	3	3	3	1014	1195740	585
12	3	4	2		159432	585
12	3	5	1860		33480720	585
12	3	6	248		1062880	585
12	4	3	1	1002	397800	273
12	4	4	7		556920	273
12	4	5	62		1113840	273
12	4	6	124		530400	273
12	6	3	1	930	393120	65
12	6	4	1		78624	65
12	6	5	155		2751840	65
12	6	6	31		131040	65
14	2	3	2	4092	12778740	5461
14	2	4	4	*2782560*	5111496	5461
14	2	5	248		71560944	5461
14	2	6	496		34076640	5461
14	2	7	32		536640	5461
14	7	3	21	3906	133161024	129
14	7	4	35		44387008	129
14	7	5	465		133161024	129
14	7	6	651		44387008	129
14	7	7	63		1048512	129

Let $\mathcal{D} = (V', \mathcal{B}')$ be a 2-$(v+1,k,1)_q$ subspace design. The set

$$\mathcal{G} = \{\pi(B) \mid B \in \mathcal{B}', P \in B\}$$

is the derived design of $\mathcal{D}$ with respect to P [KL15], which has the parameters 1-$(v, k-1, 1)_q$. In other words, it is a $(k-2)$-spread in V'/P. Now define

$$\mathcal{B} = \{\pi(B) \mid B \in \mathcal{B}', P \notin B\} \text{ and } V = V'/P\,.$$

We claim that $(V, \mathcal{G}, \mathcal{B})$ is a $(v, k, q^2, k-1)_q$-GDD.

In order to prove this, let $L \in \begin{bmatrix} V \\ 2 \end{bmatrix}_q$ be a line not covered by any element in $\mathcal{G}$. Then $L = E/P$, where $E \in \begin{bmatrix} V' \\ 3 \end{bmatrix}_q$, $P \leq E$ and E is not contained in a block of the design $\mathcal{D}$. The blocks of $\mathcal{B}$ covering L have the form $\pi(B)$ with $B \in \mathcal{B}'$ such that $B \cap E$ is a line in E not passing through P. There are q^2 such lines and each line is contained in a unique block in $\mathcal{B}'$. Since these q^2 blocks B have to be pairwise distinct and do not contain the point P, we get that there are q^2 blocks $\pi(B) \in \mathcal{B}$ containing L. □

Since there are 2-$(13,3,1)_2$ subspace designs [BEÖ+16], by Theorem 3 there are also $(12,3,4,2)_2$-GDDs.

The smallest admissible case of a 2-$(v,3,1)_q$ subspace design is $v = 7$, which is known as a *q-analog of the Fano plane*. Its existence is a notorious open question for any value of q. By Theorem 3, the existence would imply the existence of a $(6,3,q^2,2)_q$-GDD, which has been shown to be true in [EH17] for any value of q, in the terminology of a "residual construction for the q-Fano plane". In Theorem 4, we will give a general construction of q-GDDs covering these parameters. The crucial question is if a $(6,3,q^2,2)_q$-GDD can be "lifted" to a 2-$(7,3,1)_q$ subspace design. While the GDDs with these parameters constructed in Theorem 4 have a large automorphism group, for the binary case $q = 2$ we know from [BKN16, KKW18] that the order of the automorphism group of a putative 2-$(7,3,1)_2$ subspace design is at most two. So if the lifting construction is at all possible for the binary $(6,3,4,2)_2$-GDD from Theorem 4, necessarily many automorphisms have to "get destroyed".

In Table 2 we can see that there exists a $(8,3,4,2)_2$-GDD. This might lead in the same way to a 2-$(9,3,1)_2$ subspace design, which is not known to exist.

5 A general construction

A very successful approach to construct t-(v,k,λ) designs over sets is to prescribe an automorphism group which acts transitively on the subsets of cardinality t. However for q-analogs of designs with $t \geq 2$ this approach yields only trivial designs, since in [CK79, Prop. 8.4] it is shown that if a group $G \leq \mathrm{P\Gamma L}(v,q)$ acts transitively on the t-dimensional subspaces of V, $2 \leq t \leq v-2$, then G acts transitively also on the k-dimensional subspaces of V for all $1 \leq k \leq v-1$.

The following lemma provides the counterpart of the construction idea for q-analogs of group divisible designs. Unlike the situation of q-analogs of designs, in this slightly different setting there are indeed suitable groups admitting the general construction of non-trivial q-GDDs, which will be described in the sequel. Itoh's construction of infinite families of subspace designs is based on a similar idea [Ito98].

Lemma 6 *Let $\mathcal{G}$ be a $(g-1)$-spread in* $\mathrm{PG}(V)$ *and let G be a subgroup of the stabilizer* $\mathrm{P\Gamma L}(v,q)_{\mathcal{G}}$ *of $\mathcal{G}$ in* $\mathrm{P\Gamma L}(v,q)$*. If the action of G on $\begin{bmatrix} V \\ 2 \end{bmatrix}_q \setminus \bigcup_{S\in\mathcal{G}} \begin{bmatrix} S \\ 2 \end{bmatrix}_q$ is transitive, then any union $\mathcal{B}$ of G-orbits on the set of k-subspaces which are scattered with respect to $\mathcal{G}$ yields a $(v,k,\lambda,g)_q$-GDD $(V,\mathcal{G},\mathcal{B})$ for a suitable value λ.*

Proof. By transitivity, the number λ of blocks in $\mathcal{B}$ passing through a line $L \in \begin{bmatrix} V \\ 2 \end{bmatrix}_q \setminus \bigcup_{S\in\mathcal{G}} \begin{bmatrix} S \\ 2 \end{bmatrix}_q$ does not depend on the choice of L. □

In the following, let $V = \mathrm{GF}(q^g)^s$, which is a vector space over $\mathrm{GF}(q)$ of dimension $v = gs$. Furthermore, let $\mathcal{G} = \begin{bmatrix} V \\ 1 \end{bmatrix}_{q^g}$ be the Desarguesian $(g-1)$-spread in $\mathrm{PG}(V)$. For every $\mathrm{GF}(q)$-subspace $U \leq V$ we have that

$$\dim_{\mathrm{GF}(q^g)}\big(\langle U\rangle_{\mathrm{GF}(q^g)}\big) \leq \dim_{\mathrm{GF}(q)}(U).$$

In the case of equality, U will be called *fat*. Equivalently, U is fat if and only if one (and then any) $\mathrm{GF}(q)$-basis of U is $\mathrm{GF}(q^g)$-linearly independent. The set of fat k-subspaces of V will be denoted by $\mathcal{F}_k$.

We remark that for a fat subspace U, the set of points $\{\langle x\rangle_{\mathrm{GF}(q^g)} : x \in U\}$ is a Baer subspace of V as a $\mathrm{GF}(q^g)$-vector space.

Lemma 7

$$\#\mathcal{F}_k = q^{(g-1)\binom{k}{2}} \prod_{i=0}^{k-1} \frac{q^{g(s-i)}-1}{q^{k-i}-1}.$$

Proof. A sequence of k vectors in V is the $\mathrm{GF}(q)$-basis of a fat k-subspace if and only if it is linearly independent over $\mathrm{GF}(q^g)$. Counting the set of those sequences in two ways yields

$$\#\mathcal{F}_k \cdot \prod_{i=0}^{k-1}(q^k - q^i) = \prod_{i=0}^{k-1}((q^g)^s - (q^g)^i),$$

which leads to the stated formula. □

We will identify the unit group $\mathrm{GF}(q)^*$ with the corresponding group of $s\times s$ scalar matrices over $\mathrm{GF}(q^g)$.

Lemma 8 *Consider the action of* $\mathrm{SL}(s,q^g)/\mathrm{GF}(q)^*$ *on the set of the fat k-subspaces of V. For $k < s$, the action is transitive. For $k = s$, $\mathcal{F}_k$ splits into $\frac{q^g-1}{q-1}$ orbits of equal length.*

Proof. Let U be a fat k-subspace of V and let B be an ordered $\mathrm{GF}(q)$-basis of U. Then B is an ordered $\mathrm{GF}(q^g)$-basis of $\langle U\rangle_{\mathrm{GF}(q^g)}$.

For $k < s$, B can be extended to an ordered $\mathrm{GF}(q^g)$-basis B' of V. Let A be the $(s\times s)$-matrix over $\mathrm{GF}(q^g)$ whose columns are given by B'. By scaling one of the vectors in $B' \setminus B$, we may assume $\det(A) = 1$. Now the mapping $V \to V, x \mapsto Ax$ is in $\mathrm{SL}(s,q^g)$ and maps the fat k-subspace $\langle e_1,\ldots,e_k\rangle$ to U (e_i denoting the i-th standard vector of V). Thus, the action of $\mathrm{SL}(s,q^g)/\mathrm{GF}(q)^*$ is transitive on $\mathcal{F}_k$.

It remains to consider the case $k = s$. Let A be the $(s\times s)$-matrix over $\mathrm{GF}(q^g)$ whose columns are given by B. As any two $\mathrm{GF}(q)$-bases of U can be mapped to each other by a $\mathrm{GF}(q)$-linear map, we see that up to a factor in $\mathrm{GF}(q)^*$, $\det(A)$ does not depend on the choice of B. Thus,

$$\det(U) := \det(A) \cdot \mathrm{GF}(q)^* \in \mathrm{GF}(q^g)^* / \mathrm{GF}(q)^*$$

is invariant under the action of $\mathrm{SL}(s, q^g)$ on $\mathcal{F}_k$. It is readily checked that every value in $\mathrm{GF}(q^g)^* / \mathrm{GF}(q)^*$ appears as the invariant $\det(U)$ for some fat s-subspace U, and that two fat s-subspaces having the same invariant can be mapped to each other within $\mathrm{SL}(s, q^g)$. Thus, the number of orbits of the action of $\mathrm{SL}(s, q^g)$ on $\mathcal{F}_s$ is given by the number $\#(\mathrm{GF}(q^g)^* / \mathrm{GF}(q)^*) = \frac{q^g-1}{q-1}$ of invariants. As $\mathrm{SL}(s, q^g)$ is normal in $\mathrm{GL}(s, q^g)$ which acts transitively on $\mathcal{F}_s$, all orbits have the same size. Modding out the kernel $\mathrm{GF}(q)^*$ of the action yields the statement in the lemma. □

Theorem 4 *Let V be a vector space over $\mathrm{GF}(q)$ of dimension gs with $g \geq 2$ and $s \geq 3$. Let $\mathcal{G}$ be a Desarguesian $(g-1)$-spread in $\mathrm{PG}(V)$. For $k \in \{3, \ldots, s-1\}$, $(V, \mathcal{G}, \mathcal{F}_k)$ is a $(gs, k, \lambda, g)_q$-GDD with*

$$\lambda = q^{(g-1)(\binom{k}{2}-1)} \prod_{i=2}^{k-1} \frac{q^{g(s-i)} - 1}{q^{k-i} - 1}.$$

Moreover, for each $\alpha \in \{1, \ldots, \frac{q^g-1}{q-1}\}$, the union $\mathcal{B}$ of any α orbits of the action of $\mathrm{SL}(s, q^g) / \mathrm{GF}(q)^*$ on $\mathcal{F}_s$ gives a $(gs, s, \lambda, g)_q$-GDD $(V, \mathcal{G}, \mathcal{B})$ with

$$\lambda = \alpha q^{(g-1)(\binom{s}{2}-1)} \prod_{i=2}^{s-2} \frac{q^{gi} - 1}{q^i - 1}.$$

Proof. We may assume $V = \mathrm{GF}(q^g)^s$ and $\mathcal{G} = \begin{bmatrix} V \\ 1 \end{bmatrix}_{q^g}$. The lines covered by the elements of $\mathcal{G}$ are exactly the non-fat $\mathrm{GF}(q)$-subspaces of V of dimension 2. By Lemma 6 and Lemma 8, $(V, \mathcal{G}, \mathcal{F}_k)$ is a GDD. Double counting yields $\#\mathcal{F}_2 \cdot \lambda = \#\mathcal{F}_k \cdot \begin{bmatrix} k \\ 2 \end{bmatrix}_q$. Using Lemma 7, this equation transforms into the given formula for λ.

In the case $k = s$, by Lemma 8, each union $\mathcal{B}$ of $\alpha \in \{1, \ldots, \frac{q^g-1}{q-1}\}$ orbits under the action of $\mathrm{SL}(s, q) / \mathrm{GF}(q)^*$ on $\mathcal{F}_s$ yields a GDD with

$$\lambda = \alpha q^{(g-1)(\binom{s}{2}-1)} \frac{q-1}{q^g-1} \prod_{i=2}^{s-1} \frac{q^{g(s-i)} - 1}{q^{s-i} - 1} = \alpha q^{(g-1)(\binom{s}{2}-1)} \prod_{i=2}^{s-2} \frac{q^{gi} - 1}{q^i - 1}.$$

□

Remark 1 *In the special case $g = 2$, $s = 3$ and $\alpha = 1$ the second case of Theorem 4 yields $(6, 3, q^2, 2)_q$-GDDs. These parameters match the "residual construction for the q-Fano plane" in [EH17].*

Remark 2 *A fat k-subspace ($k \in \{3, \ldots, s\}$) is always scattered with respect to the Desarguesian spread $\begin{bmatrix} V \\ 1 \end{bmatrix}_{q^g}$. The converse is only true for $g = 2$. Thus, Theorem 4 implies that the set of all scattered k-subspaces with respect to the Desarguesian line spread of $\mathrm{GF}(q)^{2s}$ is a $(2s, k, \lambda_{\max}, 2)_q$-GDD.*

Table 2: Existence results for $(v, k, \lambda, g)_q$-GDD for $q = 2$.

v	g	k	λ_Δ	λ_{max}	λ	comments
6	2	3	2	12	4	[EH17]
					2, 4, ..., 12	$\langle\sigma^7\rangle$
					4α, $\alpha = 1, 2, 3$	Thm. 4
6	3	3	3	6	3, 6	$\langle\sigma^{21}\rangle$
8	2	3	2	60	2, 58	$\langle\sigma, \phi^4\rangle$
					4, 6, ..., 54, 56, 60	$\langle\sigma, \phi\rangle$
8	2	4	4	480	20, 40, ..., 480	$\langle\sigma, \phi\rangle$
					160α, $\alpha = 1, 2, 3$	Thm. 4
8	4	3	7	42	7, 21, 35	$\langle\sigma\rangle$
					14, 28, 42	$\langle\sigma, \phi\rangle$
8	4	4	7	14	7, 14	$\langle\sigma\rangle$
9	3	3	1	118	2, 3, ..., 115, 116, 118	$\langle\sigma, \phi\rangle$
					16α, $\alpha = 1, \dots, 16$	Thm. 4
9	3	4	10	1680	30, 60, ..., 1680	$\langle\sigma, \phi\rangle$
10	2	3	14	252	14, 28, ..., 252	$\langle\sigma, \phi\rangle$
10	2	5	8		23040α, $\alpha = 1, \dots, 3$	Thm. 4
10	5	3	21	210	105, 210	$\langle\sigma, \phi^2\rangle$
12	2	3	2	1020	4	[BEÖ+16]
12	2	6	16		12533760α, $\alpha = 1, \dots, 3$	Thm. 4
12	3	4	2		21504α, $\alpha = 1, \dots, 7$	Thm. 4
12	4	3	1	1002	64α, $\alpha = 1, \dots, 15$	Thm. 4

6 Computer constructions

An element $\pi \in \mathrm{P\Gamma L}(v, q)$ is an automorphism of a $(v, k, \lambda, g)_q$-GDD if $\pi(\mathcal{G}) = \mathcal{G}$ and $\pi(\mathcal{B}) = \mathcal{B}$.

Taking the Desarguesian $(g-1)$-spread and applying the Kramer-Mesner method [KM76] with the tools described in [BKL05, BKW18b, BKW18a] to the remaining blocks, we have found $(v, k, \lambda, g)_q$-GDDs for the parameters listed in Tables 2, 3. In all cases, the prescribed automorphism groups are subgroups of the *normalizer* $\langle\sigma, \phi\rangle$ *of a Singer cycle group* generated by an element σ of order $q^v - 1$ and by the Frobenius automorphism ϕ, see [BKW18a]. Note that the presented necessary conditions for λ_Δ turn out to be tight in several cases.

Example. We take the primitive polynomial $1 + x + x^3 + x^4 + x^6$, together with the canonical Singer cycle group generated by

$$\sigma = \begin{pmatrix} 010000 \\ 001000 \\ 000100 \\ 000010 \\ 000001 \\ 110110 \end{pmatrix}$$

For a compact representation we will write all $\alpha \times \beta$ matrices X over $\mathrm{GF}(q)$

Table 3: Existence results for $(v, k, \lambda, g)_q$-GDD for $q = 3$.

v	g	k	λ_Δ	λ_{max}	λ	comments
6	2	3	3	36	9	[EH17]
					9α, $\alpha = 1, \ldots, 4$	Thm. 4
					12, 18	$\langle\sigma^{13}, \phi\rangle$
6	3	3	4	24	12, 24	$\langle\sigma^{14}, \phi\rangle$
8	2	4	9	9720	2430α, $\alpha = 1, \ldots, 4$	Thm. 4
8	4	3	13	312	52, 104, 156, 208, 260, 312	$\langle\sigma, \phi\rangle$
9	3	3	1	1077	81α, $\alpha = 1, \ldots, 13$	Thm. 4
10	2	5	27	22044960	5511240α, $\alpha = 1, \ldots, 4$	Thm. 4
12	2	6	81	439267872960	109816968240α, $\alpha = 1, \ldots, 4$	Thm. 4
12	3	4	3		5373459α, $\alpha = 1, \ldots, 13$	Thm. 4
12	4	3	1	29472	729α, $\alpha = 1, \ldots, 40$	Thm. 4

with entries $x_{i,j}$, whose indices are numbered from 0, as vectors of integers

$$[\sum_j x_{0,j}q^j, \ldots, \sum_j x_{\alpha-1,j}q^j],$$

i.e. $\sigma = [2, 4, 8, 16, 32, 27]$.

The block representatives of a $(6, 3, 2, 2)_2$-GDD can be constructed by prescribing the subgroup $G = \langle\sigma^7\rangle$ of the Singer cycle group. The order of G is 9, a generator is $[54, 55, 53, 49, 57, 41]$. The spread is generated by $[1, 14]$, under the action of G the 21 spread elements are partitioned into 7 orbits. The blocks of the GDD consist of the G-orbits of the following 20 generators.

$$[3, 16, 32], [15, 16, 32], [4, 8, 32], [5, 8, 32], [19, 24, 32], [7, 24, 32], [10, 4, 32],$$
$$[18, 28, 32], [17, 20, 32], [1, 28, 32], [17, 10, 32], [25, 2, 32], [13, 6, 32], [29, 30, 32],$$
$$[33, 12, 16], [38, 40, 16], [2, 36, 16], [1, 36, 16], [11, 12, 16], [19, 20, 8]$$

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