Globally optimal triangulations of minimum weight using Ant Colony Optimization metaheuristic

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ABSTRACT

Globally optimal triangulations are difficult to be found by deterministic methods as, for most type of criteria, no polynomial algorithm is known. In this work, we consider the Minimum Weight Triangulation (MWT) problem of a given set of n points in the plane. Our aim is to show how the Ant Colony Optimization (ACO) metaheuristic can be used to search for globally optimal triangulations of minimum weight. We present an experimental study for a set of instances for MWT problem. We create these instances since no reference to benchmarks for this problem were found in the literature. We assess through the experimental evaluation the applicability of the ACO metaheuristic for MWT problem.

Keywords: Triangulation, Minimum Weight Triangulation, Computational Geometry, ACO Metaheuristic.

I. INTRODUCTION

In Computational Geometry there are many problems that either are NP-hard or no polynomial algorithms are known. Therefore, we can find approximate solutions using metaheuristics. The optimization problems related to special geometric configurations are interesting to research due to their use in many fields of application. Triangulations are planar partitions, which received considerable attention mainly due to their applications, e.g., visibility, ray-shooting, kinetic collision detection, rigidity, guarding, etc. Minimizing the total length has been one of the main optimality criteria for triangulations. Indeed, the *Minimum Weight Triangulation* (MWT) minimize the sum of the edge lengths, providing a quality measure for determining how good is a structure. The complexity of computing a minimum weight triangulation has been one of the most longstanding open problems in Computational Geometry, introduced by Garey and Johnson [10] in their open problems list, and various approximation algorithms were proposed over time. Mulzer and Rote [23] recently showed that MWT problem is NP-hard.

Given the inherent difficulty of the above mentioned problem, the approximate algorithms arise as alternative candidates for solving MWT problem. These algorithms can obtain approximate solutions to the optimal solutions, and they can be specific for a particular problem or they can be part of a general applicable strategy in the resolution of different problems. The metaheuristic methods satisfy these properties.

A metaheuristic is an iterative generation process that guides the search of solutions intelligently combining different concepts of diverse fields as artificial intelligence [24], biological evolution [2], swarm intelligence [13], among others. These algorithms have a simple implementation and they can efficiently find good solutions for NP-hard optimization problems [22]. For the experimental study presented in this work we use the *Ant Colony Optimization* (ACO) metaheuristic.

According to the current state-of-the-art with respect to theoretical results about the problems considered in this investigation, we adopted to solve them a metaheuristic technique as the more appropriate approach to find nearly optimal solutions. Previous works about approximations on MWT problem using metaheuristic, were presented in [6] and [7], where we described the design of the ACO algorithms and gave the first steps in this research.

This paper is organized as follows. In the next Section, we present the theoretical aspects of triangulations. In Section III, we present the general overview of the ACO metaheuristic and in Section IV, we present the proposed ACO algorithm for solving the MWT problem. In Section V, we describe the MWT instances used and the details and results of the experimental study in which we analyze the sensitivity of some important parameters on the performance of the proposed ACO algorithm. Last section is reserved for the conclusions and future vision.

II. MINIMUM WEIGHT TRIANGULATION

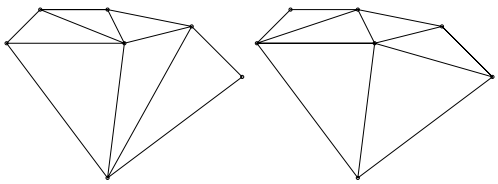
Let S be a set of points in the plane. A triangulation of S is a partition of the convex hull of S into triangles whose set of vertices is exactly S. The weight of a triangulation T is the sum of the Euclidean lengths of all the edges of T. The triangulation that minimizes this sum is named a *Minimum Weight Triangulation* of S and it is denoted by MWT(S).

Triangulation is one of the main topics in Computational Geometry and it is commonly used in a large set of applications, such as computer graphics, scientific visualization,

robotics, computer vision, and image synthesis, as well as in mathematical and natural science.

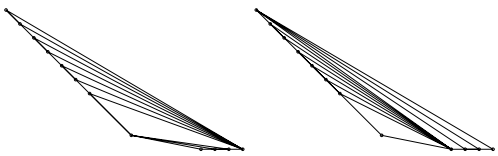
MWT problem has a long and rich history, dating back to the 1970s. As far as authors' knowledge, the MWT problem was first considered by Düppe and Gottschalk [8] who proposed a greedy algorithm which always adds the shortest edge to the triangulation. Later, Shamos and Hoey [29] suggested using the Delaunay triangulation as a minimum weight triangulation. Lloyd [20] provided examples which show that both proposed algorithms usually do not compute the MWT. Similarly, Gilbert [11] and Klincsek [16] independently showed how to compute a minimum weight triangulation of a simple polygon in $O(n^3)$ time by dynamic programming.

Approaching the problem from other direction, researchers were looking for triangulations that approximate the MWT. The Delaunay triangulation is not a good candidate, since it may be longer by a factor of $\Omega(n)$ (see Figure 1) [14] [21]. The greedy triangulation approximates the MWT by a factor of $\Omega(n)$ (see Figure 2) [21] [18] [19]. Plaisted and Hong [25] showed how to approximate the MWT up to a factor of $O(log\ n)$ in $O(n^2 log\ n)$ time. Levcopoulos and Krznaric [19] introduced quasi-greedy triangulations, which approximate the MWT within a constant factor. Remy and Steger [27] discovered an approximation scheme for MWT that runs in quasi-polynomial time: for every fixed ε, it finds a (1 + ε)-approximation in $n^{O(log^8 n)}$ time.



(a) Delaunay triangulation (b) Minimum weight triangulation

Fig. 1. Two examples of possible triangulations for the same set of points



(a) Greedy triangulation (b) Minimum weight triangulation

Fig. 2. Two examples of possible triangulations for the same set of points

From the point of view of metaheuristics, many papers present solutions to problems at the field of Graphical Computation. In 1992, Sen and Zheng [28] proposed an algorithm to approximate the minimum weight triangulation using Simulated Annealing but in many proofs they obtain solutions "near" to the ideal ones. The neighborhood is obtained with a flip in a random edge of the current triangulation. In 1993, Wu and Wainwright [32] approximated the minimum weight triangulation using a genetic algorithm where the recombination and mutation operators are the same, such that both of them make a flip to obtain the neighbors. Qin et al. [26] also use a genetic algorithm and they proposed new operators for recombination and mutation. Capp and Julstrom [3] present a new weight codification of the triangulations to use it in a genetic algorithm. In the previous mentioned works, the experimental evaluation is rather poor and they do not describe the quality of the obtained solutions. In 2001, Kolingerova and Ferko [17] presented a genetic optimization, which recombination operator is named DeWall and the mutation operator makes a flip in the selected individual. The principal weakness of this method is the time demand.

The complexity of the computation was one of the more interesting opened problems in Geometry Computacional until Mulzer and Rote demonstrated in 2006 that MWT's construction is a NP-hard problem [23].

III. Ant Colony Optimization Metaheuristic

The Ant Colony Optimization metaheuristic involves a family of algorithms in which a colony of artificial ants cooperate in finding good solutions to difficult discrete optimization problems [1]. Cooperation is a key design component of ACO algorithms: the choice is to allocate the computational resources to a set of relatively simple agents (artificial ants) that communicate indirectly by stigmergy. Thus, good quality solutions are an emergent property of the agents cooperative interaction.

An artificial ant in an ACO algorithm is a stochastic constructive procedure that incrementally builds a solution by adding opportunely defined solution components to a partial solution under construction. Therefore, the ACO metaheuristic can be applied to any combinatorial optimization problem for which a constructive graph can be defined. Each edge (i, j) in the graph represents a posible path and it has associated two information sources that guide the ant moves: pheromone trails and heuristic information. The pheromone trail, denoted by τ_{ij}, encodes a long-term memory about the entire ant search process, and is updated by the ants themselves. The heuristic information, denoted by η_{ij}, represents *a priori* information about the problem instance or run-time information provided by a source different from the ants. In many cases η is the cost, or an estimate of the cost, of adding the component or connection to the solution under construction.

These values are used by the ants to make probabilistic decisions on how to move on the graph. The ants act concurrently and independently and although each ant is complex enough to find a solution to the problem, which is probably poor, good-quality solutions can only emerge as the result of the collective interaction among the ants. This is obtained via indirect communication mediated by the information that ants read or write in the variables storing pheromone trail values. It is a distributed learning process in which the single agents, the ants, are not adaptive

[1]ACO metaheuristic has also been successfully applied to continuous problems, however, in this paper we only consider and describe its application to discrete problems.

themselves but, on the contrary, adaptively modify the way the problem is represented and perceived by other ants [5]. There are two additional process for updating pheromone and the daemon actions. The pheromone updating is the process by which the pheromone trails are modified. The trail values can either increase, as ants deposit pheromone on the components or connections they use, or decrease, due to pheromone evaporation. The daemon procedure is used to implement centralized actions which cannot be performed by single ants. Examples of daemon actions are the activation of a local optimization procedure, or the collection of global information that can be used to decide whether it is useful or not to deposit additional pheromone to bias the search process from a nonlocal perspective. The daemon can observe the path found by each ant in the colony and select one or a few ants, like those that built the best solutions in the algorithm iteration that allowed to deposit additional pheromone on the connections they used.

A. *The general ACO algorithm*

In this section we present a general ACO algorithm (Algorithm 1) and a description of its main components. After that, the next two sections respectively describe in detail the specific component of the general ACO algorithm (function *BuildSolutionk*) that have to be adapted for each of the problem studied here, i.e., MWT.

Algorithm 1 General-ACO

```
Initialize
for c ∈ {1, ..., C} do
  for k ∈ {1, ..., K} do
    BuildSolutionk
    EvaluateSolution
  end for
  SaveBestSolutionSoFar
  UpdateTrails
end for
ReturnBestSolution
```

Main components of Algorithm 1:

- *Initialize*: this process initializes the parameters considered for the algorithm. The initial trail of pheromone is associated to each edge, τ_0; it is an small positive value, in general, the same for all edges. The quantity of ants of the colony, *K*. The weights that define the proportion in which they will affect the heuristic information and pheromone trails in the probabilistic transition rule, named respectively β y α. *C* is the maximum number of cycles.
- *BuildSolutionk*: this process begins with a partial empty solution which is extended at each step by adding a feasible solution component chosen from the current solution neighbors; i.e., to find a route on the construction graph guided by the mechanism that defines the set of feasible neighbors with regard to the partial solution. The choice of a feasible neighbor is done in a probabilistic way in every step of the construction, depending on the used ACO variant. In this work, the selection rule for the solutions construction is based on the following probabilistic model:

$$P_{ij} = \begin{cases} \dfrac{\tau_{ij}^{\alpha} \cdot \eta_{ij}^{\beta}}{\sum_{h \in F(i)} \tau_{ih}^{\alpha} \cdot \eta_{ih}^{\beta}}, & j \in F(i); \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

  - $F(i)$ is the set of feasible points for point i.
  - τ_{ij} is the pheromone value associated to edge (i, j).
  - η_{ij} is the heuristic value associated to edge (i, j).
  - α and β are positives parameters for determining the relative importance of the pheromone with respect to the heuristic information.
- *EvaluateSolution*: evaluates and saves the best solution found by ant k in the current cycle.
- *SaveBestSolutionSoFar*: saves the best solution found for all cycles so far.
- *UpdateTrails*: increases the pheromone level in the promising paths, and is decreased in other case. First, all the pheromone values are decreased by means of the process of evaporation. Then, the pheromone level is increased when good solutions appear. The following equation is used:

$$\tau_{ij} = (1 - \rho)\tau_{ij} + \Delta\tau_{ij} \quad (2)$$

  - $\rho \in (0, 1]$ is the factor of persistence of the trail.
  - $\Delta\tau_{ij} = \sum_{k=1}^{K} \Delta^k \tau_{ij}$ is the accumulation of trail, proportional to the quality of the solutions.
  - $\Delta^k \tau_{ij} = \begin{cases} Q/L_k, & \text{when ant } k \text{ used edge } (i, j); \\ 0, & \text{in other case.} \end{cases}$
  - Q is a constant depending of the problem; it usually set to 1.
  - L_k is the objective value of the solution *k*.

  Pheromone evaporation avoids a fast convergence of the algorithm. In addition, this way of forgetting allows the exploration of new areas of the search space. The update of the pheromone trail can be done according to one of the following criterions: *elitist* and *not elitist*. In the elitist case, the best found solution is used to give an additional reinforcement to the levels of pheromone. The not elitist one uses the solutions found by all the ants to give an additional reinforcement to the levels of pheromone.

IV. THE PROPOSED ACO ALGORITHM FOR MWT (ACO-MWT)

Considering Algorithm 1 described in the previous section, we present *BuildSolutionk* function for the MWT problem described in Algorithm 2. The other functions in the General-ACO (Algorithm 1) remain the same for ACO-MWT.

BuildSolutionk process showed below is an improvement of the algorithm described in [6]. The current version is better than the previous one in the sense that the intersection procedure is invoked fewer times. It must be noticed that in the initial version of the algorithm, we start with an empty solution, i.e., we just have the set of

points S. In this improvement, we take the β-$skeleton$ idea. Let p and q be two distinct points in S with $\beta > 1$. Following Kirkpatrick and Radke [15], the edge pq is included in the β-$skeleton$ β(S) of S if the two circles of diameter $\beta \cdot |pq|$ and passing through both p and q do not enclose any point in S. Interestingly, β(S) is a subgraph of each MWT(S) provided that β is large enough. The original bound $\beta \geq \sqrt{2}$ in Keil [12] was improved later in Cheng and Xu [4] to $\beta > 1.1768$. So, we start with a partial solution composed by the edges of the Relative Neighborhood Graph (RNG) which is called 2-$skeleton$ because is easier to compute than the 1.1768-skeleton.
BuildSolutionk works as follows, each ant builds a triangulation of a given set or instance S, starting from an initial random point. At each step, the algorithm adds a new edge (i, j) if there is no intersection between (i, j) and the edges of the (partial) solution S_k. In this case, i is a feasible point for j and vice versa. If the current point has no feasible points, it selects the next reference point according to one of the following criterions: i) random selection; ii) select the point with the largest quantity of feasible points; or, iii) select the point with the lowest quantity of feasible points.

Algorithm 2 BuildSolutionk

```
S_k ← edges of RNG
i ← SelectInitialPoint(S)
while S is not triangulated do
  F_i ← FeasiblePoints(i, S_k)
  if F_i = ∅ then
    i ← SelectPoint(S, S_k)
    F_i ← FeasiblePoints(i, S_k)
  end if
  j ← SelectPointProb(F_i)
  if not IntersectSolution(i, j, S_k) then
    S_k ← S_k ∪ (i, j)
    i ← j
  end if
  UpdateFeasiblePoints(i, j)
end while
```

In order to better understand the behavior of Algorithm 2, the respective main components are described in the following:

- *SelectInitialPoint*(S): returns a point $p \in S$, randomly selected.
- *FeasiblePoints*(i, S_k): returns a set of points $p \in S$, such that the edge (i, p) could not intersect with the edges of the solution S_k. Note that this function may return points that are no feasible for *p*.
- *SelectPoint*(S, S_k): returns a point $p \in S$, such that $FeasiblePoints(p, S_k)$ has at least a point. p is selected according yo one of the criteria mentioned previously.
- *SelectPointProb*(F_i): returns a point $j \in F_i$ chosen according to Equation 1, where η_{ij} is 1/d_{ij}, and d_{ij} is the Euclidean distance between i and j.
- *IntersectSolution*(i, j, S_k): returns true if the edge (i, j) intersects at least one edge of the solution S_k; returns false, in other case.
- *UpdateFeasiblePoints*(i, j): updates the feasible points of i and j, i.e., the points i and j are not more feasible with respect to each other.

V. Experimental Evaluation

In this work we present an ACO algorithm for the MWT problem. The proposed ACO algorithm is represented by an Ant System (AS), a particular instance of the class of ACO algorithms. We show the experimental phase for sets of instances of size 40, 80, 120, and 160 points. Through the experimental evaluation, we assess the applicability of the ACO metaheuristic for MWT problem.
To the best knowledge of the authors, there not exist collections of instances in the literature for MWT problem. According to that, we design an *instances generator*. Therefore, we have generated respectively a collection of 10 instances of size 40/80/120/160/200; i.e., a total of 50 instances for the problem. Each instance is called LDn-i where n denotes the size of the i-instance, with $1 \leq i \leq 10$. The instance generator uses different functions of CGAL Library [1]. The points are randomly generated, uniformly distributed and for each point (x, y), the coordinates $x, y \in [0, 1000]$. Especially for MWT problem, these instances must be pre-processed to guarantee non collinear points which a *sine qua non* property that has to be met to find suitable solutions for the problem. Particulary, in this work, we present the experimental research for four instances of size 40, 80, 120, and 160 respectively (LD40-1, LD40-2, LD40-3, LD40-4, LD80-1, LD80-2, LD80-3 LD80-4, LD120-1, LD120-2, LD120-3, LD120-4, LD160-1, LD160-2, LD160-3, and LD160-4). The ACO-MWT algorithm was implemented in C and run on BACO parallel cluster, composed by 60 PCs, with a 3.0 GHz Pentium-4 processor each one and 90 PCs with a 2.4 GHz Core 2 Quad processor each one, under CONDOR batch queuing system.
In this evaluation phase, we used the following parameter values: $\alpha = 1$; $\beta = 1$ and 5; and $\rho = 0.10$, 0.25, and 0.50. *elit* = 1 and 0, where 1 means that the trail is updated in a elitist way; in other case, the updating is done in a not elitist way. *criterion* = 1, 2, and 3, is used for selecting a point in the *SelectPoint*(S, S_k) procedure in ACO-MWT algorithm. For *criterion* = 1 the point is chosen randomly; for *criterion* = 2, the chosen point has the largest quantity of feasible points; and for *criterion* = 3, the chosen point has the lowest quantity of feasible points. The number of cycles, *C*, is 1000; the number of ants, *K*, is 50. For each parameter setting, given below, 30 runs were performed by using different random seeds. For each instance, the experiment were done with the twelve parameter setting, according to combinations of parameters before detailed. We obtain average, median, best, and standard deviation values, considering the objective function (*weight*). We show the results according to the four best parameter settings considering the smaller weights, i.e., the four *Best* values. We only show the results for the best four parameter settings since the results for the remaining ones were in general of lower quality with respect to the best found values.
Next, we show the experimental results, considering the above presented settings. Each parameter setting is denoted

TABLE I
MWT: RESULTS FOR FOUR INSTANCES OF 40 POINTS.

Par. Setting	Average	Median	Best	Std. Dev.
LD401-1-0.25-1	5497920	5499201	5493047	3093
LD401-1-0.50-1	5500288	5501497	5493047	4543
LD401-5-0.10-1	5501427	5502009	5493047	4890
LD401-5-0.25-1	5502441	5502009	5493047	4547
LD401-5-0.50-1	5500754	5501988	5493047	5518
LD402-1-0.25-1	4666083	4666261	4661242	2511
LD402-1-0.10-1	4665869	4665657	4660495	2830
LD402-5-0.50-1	4664708	4664817	4659553	2927
LD402-1-0.50-1	4666420	4666984	4659553	3191
LD402-5-0.25-1	4665475	4664817	4659553	3693
LD402-5-0.10-1	4665988	4665789	4659553	3812
LD403-5-0.25-1	5519150	5519777	5502567	6516
LD403-1-0.10-1	5520802	5521320	5503301	6966
LD403-5-0.10-1	5519544	5519625	5510241	5353
LD403-5-0.50-1	5517745	5519181	5510241	5657
LD404-1-0.25-1	5748259	5747745	5745772	2316
LD404-1-0.50-1	5748852	5748473	5745772	1946
LD404-5-0.50-1	5751695	5750729	5745772	4002
LD404-1-0.10-1	5749372	5748757	5747725	1950
LD404-5-0.10-1	5751877	5750729	5747725	3170
LD404-5-0.25-1	5751976	5750729	5747725	5157

TABLE II
MWT: RESULTS FOR FOUR INSTANCES OF 80 POINTS.

Par. Setting	Average	Median	Best	Std. Dev.
LD801-5-0.50-1	6271586	6273781	6242505	14337
LD801-5-0.25-1	6271507	6275369	6249124	13911
LD801-1-0.25-1	6287660	6289344	6256190	14223
LD801-5-0.25-0	6312084	6313977	6257491	15609
LD802-5-0.25-1	7640159	7643473	7605383	13945
LD802-5-0.50-1	7637904	7638408	7607462	15751
LD802-5-0.10-1	7640725	7642497	7610007	16196
LD802-1-0.10-1	7648258	7645077	7611405	22608
LD803-5-0.10-1	5863919	5865538	5836037	13482
LD803-5-0.50-1	5867149	5866309	5843634	14250
LD803-1-0.50-1	5880349	5884690	5845840	15361
LD803-1-0.25-1	5879002	5882230	5848638	16061
LD804-5-0.50-1	6277069	6283664	6217040	23328
LD804-1-0.50-1	6273397	6271736	6221908	23681
LD804-1-0.10-1	6274697	6275648	6225424	23752
LD804-1-0.25-1	6268067	6270581	6228084	17899

by ($instance$-β-ρ-$elit$). α and $criterion$ are not shown because they are the same for all the cases ($\alpha = 1$ and $criterion = 1$). The decimal numbers are not showed because they are not significant.

We analyze the performance of the ACO-MWT algorithm over four instances of 40, 80, 120, and 160 points. In Tables I, II, III, and IV we show the results according to the four best parameter settings with respect to the smaller weights (Best values). The Table V is a summary of the previous tables and shows that the best weights are obtained using configurations with $\beta = 5$, $elit = 1$ and ρ between 0.1 and 0.5, i.e., we obtained better results giving more relevance to the heuristic information and updating the trails in a elitist way.

We compare ACO-MWT algorithm among the Delaunay Triangulation (DT). The Table VI shows the smaller weights found of each strategy. From the displayed results it can seen that the ACO-MWT algorithm found the smaller weights for all cases. ACO-MWT managed to reduce the weights between 1% and 5% with regard to the DT strategy, but for LD40-4 instance achieved a reduction of 8%.

We carried out the boxplot method to visualize the distri-

TABLE III
MWT: RESULTS FOR FOUR INSTANCES OF 120 POINTS.

Par. Setting	Average	Median	Best	Std. Dev.
LD1201-5-0.25-1	9361401	9361368	9325984	18424
LD1201-5-0.50-1	9364442	9361221	9331139	22122
LD1201-5-0.10-1	9366316	9361576	9333488	20569
LD1201-1-0.50-1	9393130	9398060	9345181	22710
LD1202-5-0.10-1	6019316	6020394	5962099	23256
LD1202-5-0.25-1	6022598	6027282	5979832	20284
LD1202-5-0.50-1	6026150	6030549	5995484	21177
LD1202-1-0.10-1	6052288	6059251	5996347	25249
LD1203-5-0.10-1	8661456	8661549	8632306	16552
LD1203-5-0.25-1	8658617	8659753	8632574	11813
LD1203-5-0.50-1	8663020	8668620	8633526	17104
LD1203-1-0.10-1	8704510	8706670	8658672	21258
LD1204-5-0.50-1	7802093	7804348	7762612	18435
LD1204-5-0.10-1	7797742	7798414	7766328	14877
LD1204-1-0.10-1	7831163	7832325	7774480	23019
LD1204-5-0.25-1	7798003	7794526	7776160	13279

TABLE IV
MWT: RESULTS FOR FOUR INSTANCES OF 160 POINTS.

Par. Setting	Average	Median	Best	Std. Dev.
LD1601-5-0.10-1	7515805	7515852	7489134	16875
LD1601-5-0.50-1	7527522	7527589	7495482	17124
LD1601-5-0.25-1	7518991	7516890	7496947	15200
LD1601-1-0.50-1	7588850	7588794	7510242	24740
LD1602-5-0.50-1	7091344	7089629	7057185	18160
LD1602-5-0.25-0	7132276	7136565	7057845	18387
LD1602-5-0.25-1	7095758	7095337	7066699	17190
LD1602-5-0.10-1	7097824	7098349	7073680	13596
LD1603-5-0.50-1	8797618	8803769	8748156	19596
LD1603-5-0.10-1	8797039	8796674	8750828	21039
LD1603-5-0.25-1	8809061	8810073	8770606	17587
LD1603-1-0.10-1	8844513	8843003	8809559	24466
LD1604-5-0.50-1	6223382	6225358	6184695	19044
LD1604-5-0.10-1	6223522	6224625	6194364	15100
LD1604-5-0.25-1	6225932	6225323	6197952	16147
LD1604-1-0.25-1	6261122	6263084	6216992	20906

TABLE V
ACO-MWT: SUMMARY OF RESULTS FOR FOUR INSTANCES OF 40, 80, 120, AND 160 POINTS.

β	ρ	$elit$
1 (33.3%)	0.10 (33.3%)	0 (2.9%)
5 (66.7%)	0.25 (33.3%)	1 (97.1%)
	0.50 (33.3%)	

TABLE VI
ACO-MWT: COMPARING RESULTS BETWEEN ACO-MWT AND DT.

Instance	DT	ACO-MWT
LD40-1	5666348	5493047
LD40-2	4722381	4661242
LD40-3	5663032	5502567
LD40-4	6289829	5745772
LD80-1	6462038	6242505
LD80-2	8081573	7605383
LD80-3	6143637	5836037
LD80-4	6460311	6217040
LD120-1	9581142	9325984
LD120-2	6149825	5962099
LD120-3	8948084	8632306
LD120-4	8111182	7762612
LD160-1	7837804	7489134
LD160-2	7144975	7057185
LD160-3	8891459	8748156
LD160-4	6315497	6184695

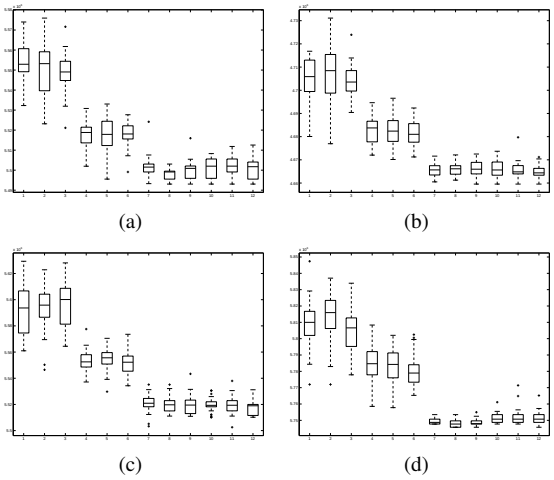


Fig. 3. MWT: Boxplots for (a) LD40-1, (b) LD40-2, (c) LD40-3, and (d) LD40-4

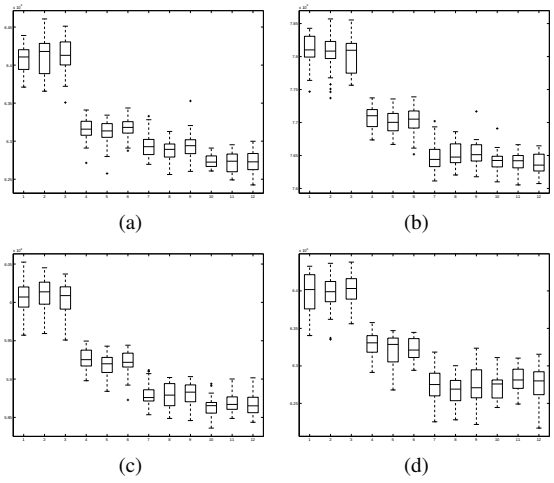


Fig. 4. MWT: Boxplots for (a) LD80-1, (b) LD80-2, (c) LD80-3, and (d) LD80-4

bution of the weights for each environment. The Figures 3, 4, 5, and 6 show the boxplots of the weights obtained for the 30 seeds for four instances for 40, 80, 120, and 160 points for the twelve configurations. The medians are similar for ρ between 0.10, 0.25 and 0.50. The algorithm is more robust when $elit = 1$ because the 50% of the values (values between the first and third quartile) are very closed around the median value. We obtained better results with $\beta = 5$ and $elit = 1$ (see parameter settings 7 to 12).

VI. Conclusion

In this work, we present the design of approximation algorithms for solving the Minimum Weight Triangulation problem for sets of points in the plane. The proposed ACO algorithm for the MWT is represented by an Ant System (AS), a particular instance of the class of ACO algorithms. We also detailed the generation of instances for the experimental evaluation, being this another contribution of this paper, since there are not available instances with special properties for building triangulations. Currently, we are continuing the experimental evaluation over a large collection of instances, where the obtained results could be used as benchmarks.

From this initial experimental phase we obtained preliminary results that will guide future experimentation. Currently, we are in the phase of applying a more methodological approach (e.g., SPOT toolbox[30]) for the experimental design following the directions of Bartz-Beielstein [31] and Fang et al.[9]. This methodological approach for parameter optimization will let us assess more accurately the behavior of our proposed algorithms under different parameter setting.

Future work will address the use different parameter setting for the ACO algorithms and the experimentation with the whole collection of instances generated. In addition, we aim at comparing the proposed algorithms against other strategies like greedy algorithms or alternative metaheuristics (e.g., evolutionary algorithm, particle swarm optimization algorithms).

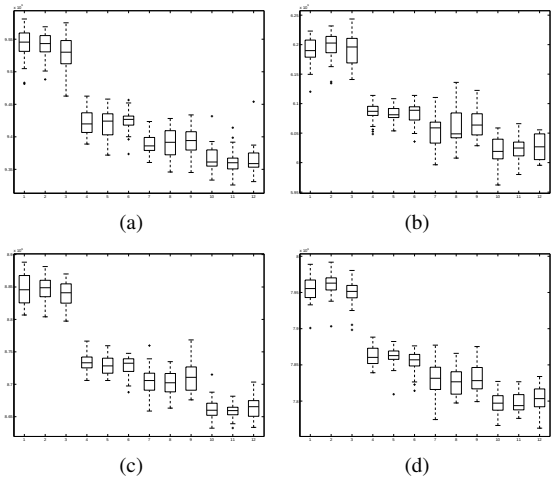


Fig. 5. MWT: Boxplots for (a) LD120-1, (b) LD120-2, (c) LD120-3, and (d) LD120-4

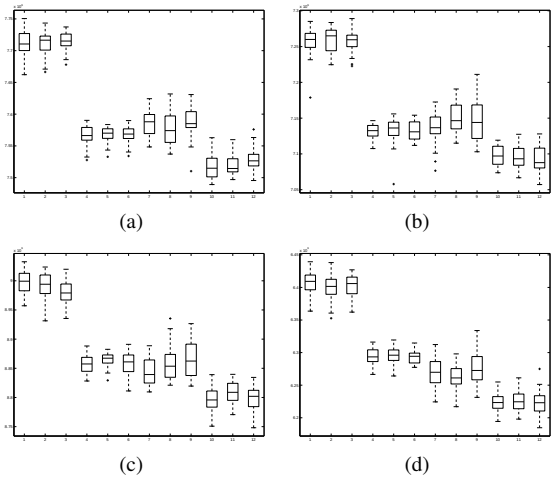


Fig. 6. MWT: Boxplots for (a) LD160-1, (b) LD160-2, (c) LD160-3, and (d) LD160-4

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