Wavelets bases defined over tetrahedra

Liliana B. Boscardín
Depto. de Matemática, Universidad Nacional del Sur
Bahía Blanca, B8000CPB, Argentina

Liliana R. Castro
Depto. de Matemática, Universidad Nacional del Sur
Bahía Blanca, B8000CPB, Argentina

Silvia M. Castro
Depto. de Cs. e Ing. de la Computación, Universidad Nacional del Sur
Bahía Blanca, B8000CPB, Argentina

Armando De Giusti
Facultad de Informática, Universidad Nacional de La Plata
La Plata, 1900, Argentina

ABSTRACT

In this paper we define two wavelets bases over tetrahedra which are generated by a regular subdivision method. One of them is a basis based on vertices while the other one is a Haar-like basis that form an unconditional basis for $L^p(T, \Sigma, \mu), 1 < p < \infty$, being μ the Lebesgue measure and Σ the $\sigma-$algebra of all tetrahedra generated from a tetrahedron T by the chosen subdivision method. In order to obtain more vanishing moments, the lifting scheme has been applied to both of them.
Keywords: Wavelets, multiresolution, lifting, volume modeling.

1. INTRODUCTION

Three dimensional scenes contain highly detailed geometric models that are rapidly emerging as the next frontier requirement for many applications such as those involving Internet 3D models for complex virtual environments, collaborative CAD, interactive visualization and multi-player video games. Therefore construction of better 3D surfaces and volume models is necessary in order to allow high quality approximations of big data sets with good storage space and better transmission time performance.

Wavelets have been making appearing in many pure and applied areas of science and engineering as a versatile tool for representing general functions or big data sets. Computer graphics, with its many and varied computational problems, is no exception to this rule since wavelets are a good approach for solving the situations described above.

An example of approximation of volumetric data in 3D using wavelets parameterized on a rectangular grid in $\mathbb{R}^3$ was presented in [7] for the case of applications in computer graphics. Although this is a simple way of constructing wavelets for surface or volume representation, this method has an important drawback: it cannot be used without introducing degeneracies for representing surfaces or volumes defined on general domains of arbitrary topological type, like spherical domains.

Lounsbery [6] and Stollnitz *et al.* [10] were the first who introduced wavelets from a different point of view, defining them on different bounded domains with arbitrary topology through scaling refinable functions. For this purpose, the multiresolution analysis (hereafter MRA) is extended to functions defined on surfaces by creating scaling refinable function. This approach was generalized *a posteriori* by Sweldens ([11], [12]) using the *lifting scheme*. Schröder and Sweldens [9] proved later that subdivision and lifting provide an efficient methodology for costum-design construction of wavelets and focused their work on wavelet representation of functions defined on a sphere. Nielson [8] also defined Haar wavelets over the sphere. Both constructions begin with a triangular net and, using subdivision as a construction tool, generate wavelets on arbitrary topological bidimensional domains.

Following similar ideas as those ones used for surfaces, we have examined the cases of representing volumes and functions defined on a tetrahedral grid. In this paper we present two different wavelets bases: a vertex basis for a tetrahedron and a Haar like wavelet basis defined over a tetrahedron. Vertex wavelets can be found, for example, in [5]. The main difference between ours and Lorentz and Oswald's [5] is that we give an explicit expression for the wavelet filters. The vertex and Haar bases are the first step for defining each such wavelet basis over an object which is represented by tetrahedra. In both cases we applied the lifting scheme to increase the number of vanishing moments of the

wavelets. Other techniques for representing volume data are given in [2] and [4].
This paper is organized as follows: in Section 1 we describe the vertex based wavelets, we show an example and present the lifted vertex wavelets. In Section 2 we define the Haar-like wavelets, we show an example representing a density function defined on a tetrahedron using these waveletes and also apply the lifting scheme for obtaining more vanishing moments. Finally, in Section 3 we draw the conclusions and outline the future work.

2. VERTEX WAVELETS

For the two dimensional case, Schröeder and Sweldens in [9] present four vertex bases: lazy, linear, butterfly and quadratic. Each of them depend on the neighbors used for a given vertex. In this section we build a vertex basis for a tetrahedrical net. It does not have an analytic expression but we give the analysis and synthesis steps for computing the scaling and wavelets coefficients. The chosen tetrahedron subdivision causes the new vertex to be on the midpoint of the tetrahedron edges. Then, when passing from a resolution level j to a resolution level $j+1$, six new vertices will be added on one tetrahedron. In the resolution level j we have an index set $\mathcal{K}_j$ to index the vertices of the tetrahedron. In level $j+1$, the set of vertices is $\mathcal{K}_{j+1} = \mathcal{K}_j \cup \mathcal{M}_j$, $\mathcal{M}_j$ being the set of indices corresponding to the inserted ones (see Figure 1). If $m \in \mathcal{M}_j$ is the index of a given vertex we may think of it as being on the midpoint of some parent edge. The neighboring vertices considered for defining the vertex basis are the endpoints of that parent edge and we denote them with indices u and v.

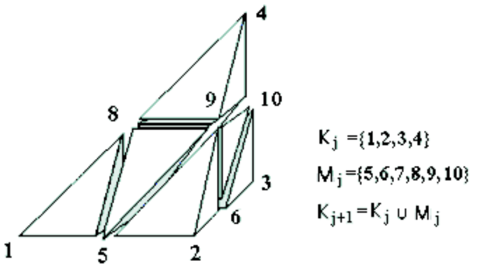

Figure 1: Tetrahedron vertices indexation

In what follows, we note the scaling and wavelet or detail coefficients corresponding to a given function at the resolution j, by $c_{j,k}$ and $d_{j,k}$ respectively. As always, the coefficients $c_{0,k}$ correspond to the coarsest approximation and the process begins with a given set of coefficients $c_{N,k}$ where N is the finest resolution level. As we shall consider an interpolating scheme for deriving a vertex basis, the unlifted scaling coefficients are just subsampled in the analysis and upsampled in the synthesis. Meanwhile for computing the wavelet coefficients it will be necessary to make some calculations.

Analysis

$$\begin{aligned} \forall k \in \mathcal{K}_j,\ c_{j,k} &\doteq c_{j+1,k}, \\ \forall m \in \mathcal{M}_j,\ d_{j,m} &\doteq c_{j+1,m} - \sum_{k \in \mathcal{K}_m} s_{j,k,m} c_{j,k}. \end{aligned} \quad (1)$$

Synthesis

$$\begin{aligned} \forall m \in \mathcal{K}_j,\ c_{j+1,k} &\doteq c_{j,k} \\ \forall m \in \mathcal{M}_j,\ c_{j+1,m} &\doteq d_{j,m} + \sum_{k \in \mathcal{K}_m} s_{j,k,m} c_{j,k}. \end{aligned} \quad (2)$$

Following the above scheme, we begin with a basic interpolatory form for analysis and synthesis

$$\begin{aligned} d_{j,m} &\doteq c_{j+1,m} - \frac{1}{2}\left(c_{j+1,u} + c_{j+1,v}\right) \\ c_{j+1,m} &\doteq d_{j,m} + \frac{1}{2}\left(c_{j,u} + c_{j,v}\right), \end{aligned}$$

that is $s_{j,u,m} = s_{j,v,m} = \frac{1}{2}$ in equations (1) and (2).

Multiresolution analysis

In order to define the sequence of nested spaces V_j needed for the MRA, we use a sequence of tetrahedra, beginning with a *basic net* T_0 that is a single tetrahedron T. Then, the spaces V_j are defined as:

$$V_j \doteq \{f : T_j \to \mathbb{R} : f \text{ is linear on each tetra-} \\ \text{hedron of } T_j\}.$$

These are nested spaces since any linear function on the tetrahedra of T_j is also linear on the tetrahedra of T_{j+1}. As the space V_j contains only piecewise linear functions, any member of V_j is uniquely determined by its values on the vertices of T_j. As usual, the V_j are spanned by the the scaling functions $\varphi_{i,j}$ and the wavelets $\psi_{i,j}$ are the basis functions of the complement spaces W_j. As always, any a function g defined on a tetrahedral net with vertices $c_{J,i} = (x_{J,i}, y_{J,i}, z_{J,i})$ is defined as

$$g(\mathbf{x}) = \sum_{i \in v(VT_J)} c_{J,i} \varphi_{J,i}(\mathbf{x}) \quad (3)$$

where $\mathbf{x} \in T_0$ and $v(T_J)$ is the set of indices that indexes the T_J vertices. Finally, we have the following wavelet decomposition of g:

$$\begin{aligned} g(\mathbf{x}) &= \sum_{i \in v(T_J)} c_{J,i} \varphi_{J,i}(\mathbf{x}) \quad (4) \\ &+ \sum_{j=0}^{J-1} \sum_{i \in (v(T_{j+1}) - v(T_j))} d_{j,i} \psi_{j,i}(\mathbf{x}), \end{aligned}$$

where the scaling functions $\varphi_{j,i}$ and the wavelets $\psi_{j,i}$ are the vertex ones while the inner product of two functions $f,\ g \in V_j(VT_0)$, $j < \infty$, the *inner product* of f and g is defined by

$$\langle f, g \rangle \doteq \sum_{\tau \in \Delta(T_j)} \frac{1}{Volume(\tau)} \int_\tau fg dV \quad (5)$$

being $\Delta(T_j)$ the set of all tetrahedra of T_j and τ a tetrahedron of $\Delta(T_j)$. So defined, these inner products do not depend on the geometric position of the vertices of T, so it is possible to pre-compute them.

Example
We choose to represent a density function on a tetrahedron using the scaling functions approximation. This density is mapped on color on each vertex of a tetrahedra. The idea is to treat each color component (red, green and blue) as a scalar function defined on the base mesh VT_0. Each color function can be converted to multiresolution form using the filter bank analysis and the wavelet coefficients. In Figure 2 and Figure 3 we show one step in the analysis and one step in the synthesis, respectively.

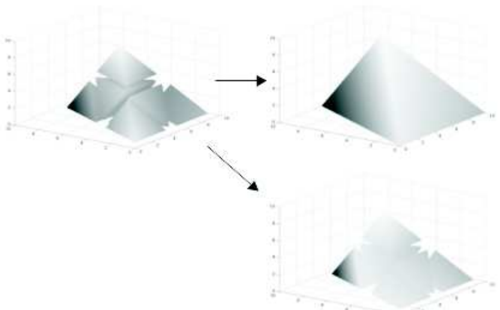

Figure 2: One step in analysis

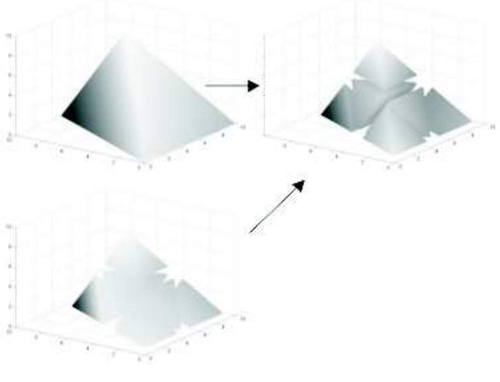

Figure 3: One step in synthesis

Lifting the vertex wavelets
An important property of a family of wavelets ψ_γ, $\gamma \in \mathcal{G}$, $\mathcal{G}$ a set of indices, is to have vanishing moments. We say that the wavelet ψ_γ has N vanishing moments if there exist N linearly independent polynomials $P_i, 0 \leq i < N$ such that:

$$\langle \psi_\gamma, P_i \rangle = 0 \tag{6}$$

We now proceed to use the lifting scheme in order to obtain wavelets with one vanishing moment. We start with the wavelets proposed by Schröder and Sweldens [9] given by

$$\psi_{j,m} = \varphi_{j+1,m} - s_{j,u,m}\varphi_{j,u} - s_{j,v,m}\varphi_{j,v}. \tag{7}$$

That is, the wavelet on a midpoint of an edge is defined as a linear combination of the scaling function on the midpoint $(j+1, m)$ and two scaling functions defined on a coarser resolution computed on the vertices (u, v). The weights $s_{j,*,m}$ are chosen in such a way that the resulting wavelet has null integral

$$\begin{aligned}\int_V \psi_{j,m} dV \quad &= \quad \int_V \varphi_{j+1,m} dV - s_{j,u,m} \int_V \varphi_{j,u} dV \\ &\quad - s_{j,v,m} \int_V \varphi_{j,v} dV. \end{aligned} \tag{8}$$

Then,

$$s_{j,*,m} = \frac{\int_V \varphi_{j+1,m} dv}{2 \int_V \varphi_{j,*}} = \frac{I_{j+1,m}}{2 I_{j+1,*}}. \tag{9}$$

The integral $\int_V \varphi dV$ can be approximated at the finest resolution level using a quadrature method and then recursively computed on the coarser levels using the refinement relationships. Then it is possible to express $\psi_{j,m}$ as follows

$$\psi_{j,m} = \varphi_{j+1,m} - \frac{I_{j+1,m}}{2 I_{j+1,*}} \varphi_{j,k}, \tag{10}$$

so the analysis and synthesis algorithms for computing the fast wavelet transform that give the lifted set of coefficients are the following:

Analysis
A1. Compute the detail coefficients

$$\forall m \in \mathcal{M}_j, d_{j,m} \doteq c_{j+1,m} - \frac{1}{2} \left(c_{j+1,u} + c_{j+1,v} \right). \tag{11}$$

A2. Calculate the coefficients $c_{j,k}$

$$\begin{array}{ll} \forall k \in \mathcal{K}_j, & c_{j,k} = c_{j+1,k} \\ \forall m \in \mathcal{M}_j, & \left\{ \begin{array}{l} c_{j,u} = c_{j,u} + s_{j,u,m} d_{j,m} \\ c_{j,v} = c_{j,v} + s_{j,v,m} d_{j,m}. \end{array} \right. \\ \forall u, v \in \mathcal{K}_j, & \end{array} \tag{12}$$

Synthesis
S1. Compute the $c_{j+1,k}$

$$\begin{array}{ll} \forall k \in \mathcal{K}_j, & c_{j+1,k} = c_{j,k} \\ \forall m \in \mathcal{M}_j, & \left\{ \begin{array}{l} c_{j,u} = c_{j,u} - s_{j,u,m} d_{j,m} \\ c_{j,v} = c_{j,v} - s_{j,v,m} d_{j,m}. \end{array} \right. \\ \forall u, v \in \mathcal{K}_j, & \end{array} \tag{13}$$

S2. Use the $c_{j+1,k}$ already computed in order to compute the $c_{j+1,m}$

$$\forall m \in \mathcal{M}_j, \quad c_{j+1,m} \doteq d_{j,m} + \frac{1}{2} \left(c_{j+1,u} + c_{j+1,v} \right). \tag{14}$$

3. HAAR-LIKE WAVELETS

In this section we build a wavelets basis on the tetrahedron following the process given by Sweldens and Girardi [3]. Throughout this section, (T, Σ, μ) is the measure space in which T is a tetrahedron with volume V, Σ is the $\sigma-$algebra of all tetrahedra generated by Beys subdivision method [1] and μ is the Lebesgue measure. With this procedure we shall obtain Haar-like wavelets on the measure space (T, Σ, μ) that form an unconditional basis for $L^p(T, \Sigma, \mu)$, $1 < p < \infty$. The concepts of tree, forest, generation and root used in this section are defined in [3].

Let τ be the tree such that $\{T_\alpha : \alpha \in \tau\}$ is the set of all subtetrahedra obtained by the subdivision method we have considered. The collection $\{T_\alpha : \alpha \in \tau\}$ is a nested partition of the tetrahedron T and the tree τ has only one root element ρ for which $T_\rho = T$. As the initial tetrahedron has volume V, a subtetrahedron T_α of the *n-th* generation has volume $V/8^n$. The wavelet basic building blocks are the scaling functions $\{\varphi_\alpha : \alpha \in \tau\}$, defined by:

$$\varphi_\alpha = \left(\frac{V}{8^n}\right)^{-1/p} \chi(T_\alpha),$$

being $\chi(T_\alpha)$ the characteristic function of T_α.

The wavelets are indexed by a set G which consists of a set G^* along with the index $\rho \in \tau$ for which $T_\rho = T$, being G^* the set $G^* = \cup_{\alpha \in \tau} G(\alpha)$, where $G(\alpha)$ contains seven elements $\beta_i, i = 1, \ldots, 8, \beta_i < \beta_{i+1}$ and is constructed by building a logarithmic tree $\tilde{\tau}$ amongst the children of α.

Then $G(\alpha) = \{(\alpha, \varsigma) \in \{\alpha\} \times \tilde{\tau} : \#C(\varsigma) = 2\}$, where $C(\varsigma)$ are the children of ς. The elements $\varsigma \in \tilde{\tau}$ are :

$$\begin{array}{ll} \varsigma^1 = \alpha, & \varsigma^2 = \{\beta_1, \beta_2, \beta_3, \beta_4\}, \\ \varsigma^3 = \{\beta_5, \beta_6, \beta_7, \beta_8\}, & \varsigma^4 = \{\beta_1, \beta_2\}, \\ \varsigma^5 = \{\beta_3, \beta_4\} & \varsigma^6 = \{\beta_5, \beta_6\} \\ \varsigma^7 = \{\beta_7, \beta_8\}. & \end{array}$$

As we consider a logarithmic tree, $C(\varsigma) = 2, \forall \varsigma \in \tilde{\tau}$. The wavelets ψ_{γ^i} generated by the elements $\gamma^i = (\alpha, \varsigma^i) \in G(\alpha), i = 1, \ldots, 7$, where ς_1^i and ς_2^i are the children of ς^i, are defined as follows:

$$\psi_{\gamma^1} = 2^{-1/p} \left(\chi(P_{\gamma^1}) - \chi(N_{\gamma^1})\right) \frac{(4V)^{-1/p}}{(8^{n+1})^{-1/p}},$$

$$\begin{cases} P_{\gamma^1} = \cup_{j \in \zeta_1^1} T_{\beta_j} \\ N\gamma^1 = \cup_{j \in \zeta_2^1} T_{\beta_j} \end{cases}$$

$$\psi_{\gamma 2} = 2^{-1/p} \left(\chi(P_{\gamma^2}) - \chi(N_{\gamma^2})\right) \frac{(2V)^{-1/p}}{(8^{n+1})^{-1/p}}$$

$$\begin{cases} P_{\gamma 2} = \cup_{j \in \zeta_1^2} T_{\beta_j} \\ N\gamma^2 = \cup_{j \in \zeta_2^2} T_{\beta_j} \end{cases}$$

$$\psi_{\gamma^3} = 2^{-1/p} \left(\chi(P_{\gamma^3}) - \chi(N_{\gamma^3})\right) \frac{(2V)^{-1/p}}{(8^{n+1})^{-1/p}},$$

$$\begin{cases} P_{\gamma 3} = \cup_{j \in \zeta_1^3} T_{\beta_j} \\ N\gamma^3 = \cup_{j \in \zeta_2^3} T_{\beta_j} \end{cases}$$

$$\psi_{\gamma^4} = 2^{-1/p} \left(\chi(P_{\gamma^4}) - \chi(N_{\gamma^4}\right) \frac{V^{-1/p}}{(8^{n+1})^{-1/p}},$$

$$\begin{cases} P_{\gamma 4} = \cup_{j \in \zeta_1^4} T_{\beta_j} \\ N\gamma^4 = \cup_{j \in \zeta_2^4} T_{\beta_j} \end{cases}$$

$$\psi_{\gamma^5} = 2^{-1/p} \left(\chi(P_{\gamma^5}) - \chi(N_{\gamma^5})\right) \frac{V^{-1/p}}{(8^{n+1})^{-1/p}},$$

$$\begin{cases} P_{\gamma 5} = \cup_{j \in \zeta_1^5} T_{\beta_j} \\ N\gamma^5 = \cup_{j \in \zeta_2^5} T_{\beta_j} \end{cases}$$

$$\psi_{\gamma^6} = 2^{-1/p} \left(\chi(P_{\gamma^6}) - \chi(N_{\gamma^6})\right) \frac{V^{-1/p}}{(8^{n+1})^{-1/p}},$$

$$\begin{cases} P_{\gamma^6} = \cup_{j \in \zeta_1^6} T_{\beta_j} \\ N\gamma^6 = \cup_{j \in \zeta_2^6} T_{\beta_j} \end{cases}$$

$$\psi_{\gamma^7} = 2^{-1/p} \left(\chi(P_{\gamma^7}) - \chi(N_{\gamma^7})\right) \frac{V^{-1/p}}{(8^{n+1})^{-1/p}},$$

$$\begin{cases} P_{\gamma^7} = \cup_{j \in \zeta_1^7} T_{\beta_j} \\ N\gamma^7 = \cup_{j \in \zeta_2^7} T_{\beta_j} \end{cases}$$

Graphically, the signs of the wavelets thus defined can be easily visualized in the following scheme. Each rectangle has eight squares and each square represents one of the sons of T_α:

+	+
+	+
-	-
-	-

ψ_{γ^1}

+	+
-	-
0	0
0	0

ψ_{γ^2}

0	0
0	0
+	+
-	-

ψ_{γ^3}

+	-
0	0
0	0
0	0

ψ_{γ^4}

0	0
+	-
0	0
0	0

ψ_{γ^5}

0	0
0	0
+	-
0	0

ψ_{γ^6}

0	0
0	0
0	0
+	-

ψ_{γ^7}

+	+
+	+
+	+
+	+

ψ_ρ

As the tree has a root element $\rho = T$, the wavelet ψ_ρ corresponding to that element is defined by:

$$\psi_\rho = (V)^{-1/p} \chi(T).$$

The generation function defined on G^* is $g((\alpha, \zeta)) = g(\alpha)$.

Finally we take:

$$\Psi = \{\psi_\gamma : \gamma \in G\}$$

and Ψ has the following properties:

1. Ψ is normalized, *i.e.* : $||\Psi||_p = 1, \forall \gamma \in G$.
2. $\int_T \psi_\gamma d\mu = 0, \forall \gamma \in G^*$.
3. If $\gamma, \gamma\,' \in G$, $\int_T \psi_\gamma \psi_{\gamma'} d\mu = 0$.
4. Ψ is an unconditional normalized basis for $L^p(T, \Sigma, \mu)$.

If $\alpha \in \tau$, then:

$$\text{span}\{\varphi_\beta : \beta \in C_l(\alpha)\} = \text{span}\{\varphi_\alpha, \psi_\gamma : \gamma \in G(\alpha)\},$$

and this extends over several generations since $\forall i \in \mathbb{N}$ is

$$\operatorname{span}\left\{\varphi_\beta : \beta \in C_l^i(\alpha)\right\} = \operatorname{span}\{\varphi_\alpha, \psi_\gamma : \gamma \in \cup_{j=0}^{i-1} G(C^j(\alpha))\}.$$

Multiresolution analysis
In this section we show how our wavelets, which are a special case of second generation wavelets, fit into the MRA concept. From now on we consider the case $p = 2$ for which the basis $\{\varphi_\beta : \beta \in C_l(\alpha)\}$ and $\{\psi_\gamma : \gamma \in G\}$ are self duals. So if $f \in L^2$,

$$f = \sum_{\gamma \in G} \langle f, \psi_\gamma \rangle \psi_\gamma, \tag{15}$$

being the convergence unconditional. The coefficients in the expansion (15) are given by the fast wavelet transform. We must write expressions that relate the basis functions, similar to the two scale relation for wavelets, in order to compute this transform. These are given by R_1 and R_2 defined below:
(R1) If $\alpha \in \tau$ and $\gamma \in G^*$, then:

$$\varphi_\alpha = \sum_{\beta \in C_l(\alpha)} \frac{1}{\sqrt{8}} \varphi_\beta, \psi_\gamma = \sum_{\beta \in S(\gamma)} g_{\gamma,\beta} \varphi_\beta,$$

where $S(\gamma) = C_l(\alpha)$, if $\gamma \in G(\alpha)$ and the $g_{\gamma,\beta}$ are the following:

$$\begin{cases} g_{\gamma^1,\beta_1} = g_{\gamma^1,\beta_2} = g_{\gamma^1,\beta_3} = g_{\gamma^1,\beta_4} = \frac{1}{\sqrt{8}}, \\ g_{\gamma^1,\beta_5} = g_{\gamma^1,\beta_6} = g_{\gamma^1,\beta_7} = g_{\gamma^1,\beta_8} = -\frac{1}{\sqrt{8}}. \end{cases}$$

$$\begin{cases} g_{\gamma^2,\beta_1} = g_{\gamma^2,\beta_2} = \frac{1}{2}, \\ g_{\gamma^2,\beta_3} = g_{\gamma^2,\beta_4} = -\frac{1}{2}, \\ g_{\gamma^2,\beta_5} = g_{\gamma^2,\beta_6} = g_{\gamma^2,\beta_7} = g_{\gamma^2,\beta_8} = 0. \end{cases}$$

$$\begin{cases} g_{\gamma^3,\beta_1} = g_{\gamma^3,\beta_2} = g_{\gamma^3,\beta_3} = g_{\gamma^3,\beta_4} = 0, \\ g_{\gamma^3,\beta_5} = g_{\gamma^3,\beta_6} = \frac{1}{2}, \\ g_{\gamma^3,\beta_7} = g_{\gamma^3,\beta_8} = -\frac{1}{2}. \end{cases}$$

$$\begin{cases} g_{\gamma^4,\beta_1} = \frac{1}{\sqrt{2}}, \\ g_{\gamma^4,\beta_2} = -\frac{1}{\sqrt{2}}, \\ g_{\gamma^4,\beta_3} = g_{\gamma^4,\beta_4} = g_{\gamma^4,\beta_5} = g_{\gamma^4,\beta_6} = g_{\gamma^4,\beta_7} = \\ = g_{\gamma^4,\beta_8} = 0. \end{cases}$$

$$\begin{cases} g_{\gamma 5,\beta_1} = g_{\gamma^5,\beta_2} = g_{\gamma^5,\beta_5} = g_{\gamma^5,\beta_6} = g_{\gamma^5,\beta_7} = \\ = g_{\gamma^5,\beta_8} = 0, \\ g_{\gamma^5,\beta_3} = \frac{1}{\sqrt{2}}, \\ g_{\gamma^5,\beta_4} = -\frac{1}{\sqrt{2}}. \end{cases}$$

$$\begin{cases} g_{\gamma^6,\beta_1} = g_{\gamma^6,\beta_2} = g_{\gamma^6,\beta_3} = g_{\gamma^6,\beta_4} = g_{\gamma^6,\beta_7} = \\ = g_{\gamma^6,\beta_8} = 0, \\ g_{\gamma^6,\beta_5} = \frac{1}{\sqrt{2}}, \\ g_{\gamma^6,\beta_6} = -\frac{1}{\sqrt{2}}. \end{cases}$$

$$\begin{cases} g_{\gamma^7,\beta_1} = g_{\gamma^7,\beta_2} = g_{\gamma^7,\beta_3} = g_{\gamma^7,\beta_4} = g_{\gamma^7,\beta_5} = \\ = g_{\gamma^7,\beta_6} = 0, \\ g_{\gamma^7,\beta_7} = \frac{1}{\sqrt{2}}, \\ g_{\gamma^7,\beta_8} = -\frac{1}{\sqrt{2}}. \end{cases}$$

(R2) For

$$\beta \neq \rho, \varphi_\beta = \frac{1}{\sqrt{8}} \varphi_{p(\beta)} + \sum_{\gamma \in S^*(\beta)} g_{\gamma,\beta} \psi_\gamma,$$

where: $S^*(\beta) = G(\alpha)$ if $\beta \in C_l(\alpha)$.
For $k \in g(\tau)$, let $G_k = \{\gamma \in G^* : g(\gamma) = k\}$ and consider the subspaces V_k and W_k of L^2, where:

$$V_k = \text{clos span}\{\varphi_\beta : \beta \in \mathcal{F}_k^*\}$$

$$W_k = \text{clos span}\{\psi_\gamma : \gamma \in G_k\}$$

By viewing $\mathcal{F}_{k-1}^* \cup \mathcal{F}_k^*$ as a two generation tree, it follows that:

$$V_k = V_{k-1} + W_{k-1}$$

and then V_k has another basis:

$$\left\{\varphi_\alpha : \alpha \in \mathcal{F}_{k-1}^*\right\} \cup \{\psi_\gamma : \gamma \in G_{k-1}\}$$

So a function $f \in V_k$ has two representations as:

$$f = \sum_{\beta \in \mathcal{F}_k^*} a_\beta \varphi_\beta$$

as well as:

$$f = \sum_{\alpha \in \mathcal{F}_{k-1}^*} a_\alpha \varphi_\alpha + \sum_{\gamma \in G_{k-1}} c_\gamma \psi_\gamma,$$

where:

$$a_\alpha = \sum_{\beta \in C_l(\alpha)} \frac{1}{\sqrt{8}} a_\beta \quad c_\gamma = \textstyle\sum_{\beta \in S(\gamma)} g_{\gamma,\beta} a_\beta,$$

$$a_\beta = \frac{1}{\sqrt{8}} a_{p(\beta)} + \sum_{\gamma \in S^*(\beta)} g_{\gamma,\beta} c_\gamma,$$

being $g(\beta) = k$ and $g(\alpha) = g(\gamma) = k - 1$.

Example
To construct the wavelets it is necessary to consider the set of children of T_α, named $T_{\beta_i}, i = 1, \ldots, 8$ and order them (see Figure 4) as follows: the bottom ones are numbered in counterclockwise sense followed at last by the top one, and the interior ones are also numbered in counterclockwise sense beginning with the tetrahedron next to T_{β_1}.

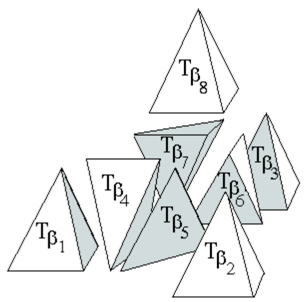

Figure 4: Numbered subtetrahedra

In order to show how the defined wavelets are used, we have first chosen the scaling functions approximation to represent a density function on a tetrahedon. We suppose that the data of this density

function correspond to the space V_3 and then we mapped this density on a grey colourmap whose range is between zero and one: the lowest density corresponds to the white and the highest to the black.
In Figure 5, we show the density function over a tetrahedron using the scaling functions and the wavelets at different resolutions. As in the previous example, the grey subtetrahedra represent an approximation of the density using the scaling functions while the coloured ones represent the value of the different coefficients in the wavelet decomposition. We can distinguish seven colours, each of them representing one of the details coefficients. When a coefficient is zero, the corresponding subtetrahedron is white. If two subtetrahedra have the same colour, the darker one indicates highest absolute value for its coefficient. Figure 5 shows, from top to bottom and from left to right, can see:
a) the given density function defined on V_3,
b) the density function on V_2 and the details on W_2,
c) the density function on V_1 and the details on W_2 y W_1,
d) the density function on V_0 and the details on W_0, W_1 and W_2.

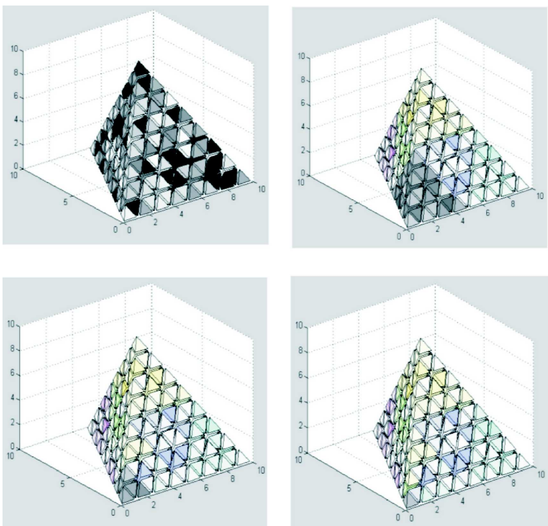

Figure 5: Three consecutive steps in the multiresolution decomposition of the density function

Lifting the Haar like wavelets
The basis functions constructed so far have only one vanishing moment but the MRA they generate is adequate for applying the lifting scheme [12]. In this section we build a new MRA in which the wavelets will have more vanishing moments. According to the lifting scheme, the new wavelets are:

$$\psi^{lif}_{\gamma^i} = \psi_{\gamma^i} - \sum_{\alpha \in \mathcal{F}^*_{n-1}} s_\alpha \varphi_\alpha, \tag{16}$$

where γ^i belongs to the $(n-1)-th$ generation and α belongs to the $(n-1)th$-generation. The coefficients s_α can be chosen so that the $\psi^{lif}_{\gamma^i}$ have more than one vanishing moment. Considering the tetrahedron with vertices in $(1,0,0), (0,1,0), (0,0,1)$ named T_0, we construct wavelets that kill the following polynomials:

$$x, \quad x+y+5, \quad x+z+5, \quad y, \quad z, \quad y+z, \quad xy, \quad xyz, \tag{17}$$

leading to the equations:

$$\langle \psi_{\gamma^i}, P \rangle = \sum_{\alpha \in \mathcal{F}^*_{n-1}} s_\alpha \langle \varphi_\alpha, P \rangle, \tag{18}$$

being P each one of the eight polynomials we have chosen. As three of them: x, $x+y+5$, $x+z+5$ are linearly independent on the tetrahedron T_0, the new wavelets will have three vanishing moments. We then obtain an 8×8 singular (but solvable) matrix problem for each $i = 1, \ldots, 8$ in the unknowns s_α. The entries of the linear system are integrals of the chosen polynomials over a tetrahedron specified by its vertices. These integrals were evaluated using barycentric coordinates. Beginning with the wavelet ψ_{γ^3} and doing one step in the subdivision, we found the following values of s_α:

-0.3103	0.0052	0.0883	0.0849
-0.1476	0.0079	0.0469	-0.0411
-0.0742	-0.0120	0.0153	-0.0019
-0.2369	-0.0147	0.0567	0.0357
0.4080	-0.0436	-0.1211	-0.0674
0.1233	0.1356	0.0167	0.0185
0.3018	0.0558	-0.0654	-0.0466
-0.0640	-0.1343	-0.0375	0.0180

Table 1: Coefficients s_α of the lifted wavelets

-0.1658	-0.0198	-0.0187
-0.0678	-0.0280	-0.0277
0.0203	-0.0149	-0.0140
-0.1661	-0.0066	-0.0050
0.2524	0.0325	0.0191
0.0252	0.0284	0.0411
0.1333	0.0299	0.0397
-0.0314	-0.0215	-0.0345

Table 2: Coefficients s_α of the lifted wavelets

Each column of Tables 1 and 2 corresponds to the coefficients s_α of one of the wavelets, being the first column the coefficients that corresponds to ψ_{γ^1} and so on.

4. CONCLUSIONS AND FUTURE WORK

Representing objects through their closure is a powerful mathematical modeling tool but this scheme has little similarity with real objects. On the other hand, modeling of solids has been made possible by constructive solid geometry (CSG). Though nowadays, both representations play an important role for modeling objects, these methods aren't good enough for modeling many real ones. Typically, the volumetric object is represented by a regular 3D grid of volumetric elements (voxels), but the data that represent a given object are sometimes sparse or irregularly designed. These cases call for alternative representations; for example, we could represent the object on a tetrahedrical domain on which its attributes (color or density) are defined. For representing such attributes it is necessary to define bases over tetrahedrical grids. As wavelets have been proved to be a powerful tool for representing general functions and large data sets accurately, we look for such bases in the field of wavelets. In this work we have taken the first step in defining wavelets over geometrical objects which can be partitioned by a net of tetrahedra. This was done by defining a wavelet vertex basis and a Haar-like wavelet basis over a tetrahedron. In both constructions, a generalization of MRA is used and the lifting scheme is applied to obtain more vanishing moments. In principle, it is the explicit nature of the construction which may have practical value, as we have shown in the examples. We believe that many other applications would benefit from them. Now, based on this contribution, we are working on new alternatives for compression of volume models as well as in the extension to more general 3D objects.

5. REFERENCES

[1] J. Bey, "Tetrahedral Grid Refinement", Computing, Vol. 55, No. 4, pp. 355-378, 1995.

[2] F. Dong and J. Shi, "Multiresolution Data Modeling for Irregular Data Fields based on Wavelets", IEEE IV, 1997.

[3] M. Girardi and W. Sweldens, "A New Class of Unbalanced Haar Wavelets that form an Unconditional Basis for L_p on General Measure Spaces, Journal of Fourier Analysis Appl., Vol. 3, pp. 457-474, 1997.

[4] D. J. Holliday and G. M. Nielson, "Progressive volume Models for Rectilinear Data using Tetrahedral Coons Volumes", http://prism.asu.esu/research/data/publications/paper00_pvmnitv.pdf, 2000.

[5] R. Lorentz and P. Oswald, "Multilevel Finite Riesz Basis for Sobolev Spaces", Proc. of the 9th Int. Conference on Domain Decomposition Methods in Sc. and Eng., Domain Decomposition Press, pp. 178-187, 1998.

[6] J. M. Lounsbery, Multiresolution Analysis for Surfaces of Arbitrary Topological Type, PhD. Thesis, University of Washington, Washington, Seattle, 1994.

[7] S. Muraki, "Approximation and Rendering of Volume Data using Wavelet Transforms", Proc. of Visualization'92, pp. 21-28, 1992.

[8] G. M. Nielson, I. H. Jung and J. Sung, "Haar Wavelets over Triangular Domains with Applications to Multiresolution Models for Flow over a Sphere", IEEE Visualization '97, 1997.

[9] P. Schröder and W. Sweldens, "Spherical Wavelets: Efficiently Representing Functions on the Sphere", ACM Proceedings of SIGGRAPH'95, pp. 161-172, 1995.

[10] E. Stollnitz, T. DeRose and D. Salesin, Wavelets for Computer Graphics: Theory and Applications, Morgan & Kaufmann Publishers Inc., 1996.

[11] W, Sweldens, "The Lifting Scheme: A New Philosophy in Biorthogonal Wavelet Constructions", Proc. of the SPIE, vol. 2569, pp.68-79, 1995.

[12] W, Sweldens, "The Lifting Scheme:Custom-Design Construction of Biorthogonal Wavelets", Appl. and Computational Harmonic Analysis, Vol. 3, pp. 186-200, 1996.

Received: Nov 2005. Accepted: Feb. 2006.