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Coherent presentation for the hypoplactic monoid of rank n

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À minha família, amigos, aos Sopas e à Sofia

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Abstract

In this thesis, we construct a coherent presentation for the hypoplactic monoid of rank n and characterize the confluence diagrams associated with it, then we use the theory of quasi-Kashiwara operators and quasi-crystal graphs to prove that all confluence diagrams can be obtained from those diagrams whose vertices are highest-weight words. To do so, we first give a complete rewriting system for the hypoplactic monoid of rank n, then, using an extension of the Knuth–Bendix completion procedure called the homotopical completion procedure, we compute the previously mentioned coherent presentation, which, from a viewpoint of Monoidal Category Theory, gives us a family of generators of the relations amongst the relations. These coherent presentations are used for representations of monoids and are particularly useful to describe actions of monoids on categories. The theoretical background is given without proof, since the main purpose of this thesis is to present new results.

Keywords: Monoid, Presentation, Complete rewriting system, Homotopy relation, Finite derivation type, Homotopical completion procedure, Coherent presentation, Confluence diagram, Plactic monoid, Hypoplactic monoid

Resumo

Nesta tese, construímos uma apresentação coerente para o monóide hipopláctico de característica n e caracterizamos os diagramas de confluência associados, utilizando depois a Teoria dos operadores quasi-Kashiwara e dos grafos quasi-cristais para provar que todos os diagramas de confluência podem ser obtidos dos diagramas cujos vértices são palavras de maior peso. De forma a realizar esta tarefa, construímos primeiro um sistema de reescrita completo para o monóide hipopláctico de característica n e depois, utilizando o procedimento de completude homotópica, uma extensão do procedimento de completude de Knuth–Bendix, computamos a apresentação coerente atrás referida, que, dum ponto de vista de Teoria de Categorias Monoidais, nos dá uma família de geradores das relações entre as relações. Estas apresentações coerentes são usadas para representações de monóides e são particularmente úteis para descrever ações de monóides em categorias. A fundamentação teórica é dada sem demonstrações, dado que o principal objetivo desta tese é apresentar novos resultados.

Palavras-chave: Monóide, Apresentação, Sistema de reescrita completo, Relação de homotopia, Tipo de derivação finita, Procedimento de completude homotópica, Apresentação coerente, Diagrama de confluência, Monóide pláctico, Monóide hipopláctico

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Chapter 1

Introduction

In Semigroup and Monoid Theory, one of the most interesting and widely-studied problems is the word problem, first introduced in Group Theory by M. Dehn [7]. Given a presentation $\langle \mathcal{A} \mid \mathcal{R} \rangle$ for a monoid M, where $\mathcal{A}$ is an alphabet and $\mathcal{R}$ is a rewriting system, we can formulate the word problem in the following way: for $u, v \in \mathcal{A}^*$, decide if $u \leftrightarrow^*_{\mathcal{R}} v$, where $\leftrightarrow^*_{\mathcal{R}}$ is the Thue congruence generated by $\mathcal{R}$. If $\mathcal{R}$ is finite and complete, then the word problem is solved using the "normal form algorithm", that is, for $u, v \in \mathcal{A}^*$, we compute normal forms u_0 and v_0 for u and v, respectively, then we conclude that $u \leftrightarrow^*_{\mathcal{R}} v$ if and only if $u_0 = v_0$. The property of having solvable word problem is invariant for any finite presentation defining the same monoid, however the property of having a finite and complete rewriting system is not invariable under monoid presentations.

In [24], C. Squier, along with F. Otto and Y. Kobayashi, introduced the concept of *finite derivation type* (*FDT*), a combinatorial property of presentations of monoids, and showed that if a monoid is presented by a finite complete rewriting system, then it is *FDT*. This property is also an invariant property of finite monoid presentations. Squier's theory has recently been further developed by Guiraud, Malbos and Mimram [9], using the language of strict monoidal categories and higher-dimensional variations of them. In this paper, they introduce the concept of *coherent presentation* and give an extension of the Knuth–Bendix completion procedure, called the *homotopical completion procedure*, that allows one to obtain a coherent presentation from a noetherian rewriting system for the monoid being studied.

On the other hand, the plactic monoid, first studied by Schensted [21] and Knuth [14], and studied later in depth by Lascoux and Schützenberger [16], is an important tool in several aspects of representation theory and algebraic combinatorics, with applications in a wide range of areas. It can be defined using the Knuth relations, or Young tableaux and Schensted's algorithm. In [4], a finite complete rewriting was constructed for the plactic monoid of rank n plac_n and from it, in [10], a coherent presentation for plac_n was computed. The plactic monoid can also be defined using the theory of Kashiwara operators and the crystal graph [13]. A similar structure, the hypoplactic monoid, studied in depth by Novelli [19], initially defined using either the hypoplactic relations or quasi-ribbon tableaux and Krob–Thibon's algorithm, was also defined using the theory of quasi-Kashiwara operators and the quasi-crystal graph in [3].

The main purpose of this thesis is the construction of a coherent presentation for the hypoplactic monoid of rank n and the characterization of the confluence diagrams associated with it, using the theory of quasi-Kashiwara operators and quasi-crystal graphs to reduce the number of relevant diagrams. Given the significant extent of the theoretical background and the fact that this thesis presents new results, we have chosen to give the background without proof, otherwise the thesis would be exceedingly large compared to the new content presented. However, we give several definitions which, while not used directly in the obtained results, are fundamental to understand the concepts we deal with and the tools used.

In Chapter 2, we give the theoretical background, in Combinatorial Semigroup Theory, needed to reach the definitions of *FDT* and coherent presentations, and related results. In Section 2.1, we present fundamental Semigroup Theory concepts and results. In Section 2.2, we recall the concepts of presentations and rewriting systems and other important concepts and results of Combinatorial Semigroup Theory. In Section 2.3, we present basic definitions and theorems regarding graphs. In Section 2.4, we finally present the concepts of *FDT* and coherent presentation and give the homotopical completion procedure, which will be the main tool used to construct a coherent presentation for hypo_n.

In Chapter 3, we start by giving some background on the plactic monoid, including two possible definitions, one via the Knuth relations, the other via Young tableaux and Schensted's algorithm, and the Robinson–Schensted–Knuth correspondence. Afterwards, we introduce the Kashiwara operators and the crystal graph, restricted to the context of plac_n, and use them to give another definition of plac_n. We also give some important properties of the crystal graph and its interaction with the combinatorics of Young tableaux. Then, we give a finite complete rewriting system on the column alphabet, which gives us a presentation of plac_n from which we compute a coherent presentation for plac_n, using the homotopical completion procedure. We also characterize the related confluence diagrams.

Chapter 4 mirrors the first three Sections of Chapter 3, since we first give some background on the hypoplactic monoid, including two possible definitions, one via the hypoplactic relations, the other via quasi-ribbon tableaux and the Krob–Thibon algorithm, and an analogue of the Robinson–Schensted–Knuth correspondence. Then, we introduce the quasi-Kashiwara operators and the quasi-crystal graph, restricted to the context of hypo_n, and use them to give another definition of hypo_n. Afterwards, we present some important properties of the quasi-crystal graph and its interaction with the combinatorics of quasi-ribbon tableaux, which are used in the final results of this thesis.

Finally, in Chapter 5, we present new results and their respective proofs. We first give a complete rewriting system, on the alphabet $\mathcal{A}_n$, for hypo_n, then we introduce the concept of *uniform presentation* and prove that the associated presentation for hypo_n is indeed uniform with respect to the crystal structure. Afterwards, as mentioned before, we use the homotopical completion procedure to compute a coherent presentation for hypo_n. The main bulk of this chapter is the characterization of the confluence diagrams associated with the coherent presentation. In the final part of this chapter, first we extend the concept of uniform presentations to extended presentations, introducing the concept of *uniform extended presentations*. Then, we use the aforementioned properties of the quasi-crystal graph to prove that the coherent presentation for hypo_n that we computed before is uniform with respect to the crystal structure, in other words, that we only require those diagrams whose vertices are highest-weight words in order to construct all other confluence diagrams, since the quasi-Kashiwara operators preserve the structure of these diagrams.

CHAPTER 2

Introduction to Combinatorial Semigroup Theory

This chapter contains the basic concepts and theorems that will be used throughout this thesis. It mostly follows Chapter 1 of [18], except for Section 2.4, which follows [9] and [10]. In the first section, we present basic definitions and results on Semigroup Theory. We follow with a section on *presentations* and *string rewriting systems*. In the next section, we present basic definitions and results on *graphs*. Finally, we introduce the concept of *coherent presentation*, using the language of Combinatorial Semigroup Theory. This concept and related results expand on the theory developed by C. Squier in the late 1980's and early 1990's and were first introduced in [9], using the language of strict monoidal categories and higher-dimensional variations of them.

2.1 Basic concepts and results on Semigroup Theory

In this section, we will present concepts and results from fundamental Semigroup Theory, necessary for the understanding of this thesis. These and other fundamental results can be found in [12]. Some definitions regarding partial orders and admissible relations are taken from [2].

Let S be a non-empty set and let $\cdot$ be a binary operation on S, that is, a mapping from $S \times S$ into S. We will refer to $\cdot$ as multiplication and, for $x, y \in S$, we represent $x \cdot y$, the image of the pair (x, y) by $\cdot$, simply by xy.

The pair $(S, \cdot)$ is a *semigroup* if $\cdot$ is an associative binary operation on S. Instead of $(S, \cdot)$, we usually write just S. Let $x_1, \dots, x_n$ ($n \in \mathbb{N}$) be elements of S, then, we can write $x_1 \cdots x_n$ without any ambiguity, as a consequence of the associative property.

A semigroup S is said to have an *identity* element 1_S if, for any $x \in S$, $x1_S = x = 1_S x$. If it exists, then it is unique. If a semigroup has an identity element, it is called a *monoid*.

Given a semigroup $(S,\cdot)$, it is always possible to extend it to obtain a monoid $(S^1,*)$: If $(S,\cdot)$ is already a monoid, then $(S,\cdot) = (S^1,*)$; otherwise, we add an element $1 \notin S$, take $S^1 := S \cup \{1\}$, and define $*$ in the following way: For $x,y \in S$, $x * y = x \cdot y$, $x * 1 = x, 1 * x = x$ and $1 * 1 = 1$.

Let M be a monoid and let $x \in M$. We say x has an *inverse* if there exists an element x' in M such that $xx' = x'x = 1_M$. If every element of M has an inverse, we say M is a *group*.

Given semigroups S and T, we say T is a *subsemigroup* of S if $T \subseteq S$ and $t_1, t_2 \in T$ implies that $t_1 t_2 \in T$. If T is a subsemigroup of S, and is also a monoid, then T is called a *submonoid*; If T is also a group, then T is called a *subgroup*. Note that S need not necessarily be a monoid, and that, even if S is a monoid, then the identities of S and T need not coincide.

Let A be a non-empty subset of a semigroup S and let $\mathscr{A}$ be the collection of all subsemigroups of S that contain A. The intersection $\bigcap_{T\in\mathscr{A}}$ contains A and not only is a subsemigroup of S, it is also the least subsemigroup of S containing A. It is called the *subsemigroup of S generated by A*, and is denoted by $\langle A\rangle$. If $S = \langle A\rangle$, then we say that S is *generated by A* (or that *A generates S*), and the elements of A are called *generators* of S. If A is a finite set that generates S, we say that S is a *finitely generated semigroup*.

We can also define the *submonoid of a monoid M generated by A* in a similar manner: Let $\mathscr{A}$ be the collection of all submonoids of M that contain $A \cup \{1_M\}$. The intersection $\bigcap_{T\in\mathscr{A}}$ contains A and not only is a submonoid of M, it is also the least submonoid of M with identity 1_M containing A. It is called the *submonoid of S generated by A*, and is denoted by $\langle A\rangle$. Similarly, if $M = \langle A\rangle$, then we say that M is *generated by A as a monoid* (or that *A generates M as a monoid*), and the elements of A are called *generators* of M. If A is a finite set that generates M, we say that M is a *finitely generated monoid*.

Let $\rho \subseteq S \times S$ be a binary relation on S. We say that ρ is:

- *reflexive* if $x \,\rho\, x$, for all $x \in S$;
- *symmetric* if $x \,\rho\, y$ then $y \,\rho\, x$, for all $x,y \in S$;
- *anti-symmetric* if $x \,\rho\, y$ and $y \,\rho\, x$ then $x = y$, for all $x,y \in S$;
- *transitive* if $x \,\rho\, y$ and $y \,\rho\, z$ then $x \,\rho\, z$, for all $x,y,z \in S$.

If ρ is reflexive, symmetric and transitive, it is said to be an *equivalence relation*. An equivalence relation on S partitions the set S into *equivalence classes*, such that each class only contains elements ρ-related to one another.

If ρ is reflexive, anti-symmetric and transitive, it is said to be a *(strict) partial order*. The most common symbols used for partial orders are $\leq, \preceq$, and $\sqsubseteq$. We write $x < y$ to denote that $x \leq y$ and $x \neq y$. A *linear order* is a strict partial order such that either $x < y, x = y$ or $x > y$, for $x,y \in S$. We say that a partial order is *well-founded* if there is no infinite chain of the form $x_1 > x_2 > \dots$, for $x_i \in S, i \in \mathbb{N}$. A linear and well-founded order is called a *well-ordering*. We say that $<$ is *admissible* if, for all $x,y,u,v \in S$, whenever $x < y$ then $uxv < uyv$.

An equivalence relation ρ on S is said to be *right (left) compatible* if $a \,\rho\, b$ implies $ax \,\rho\, bx$ ($a \,\rho\, b \Rightarrow xa \,\rho\, xb$), for any $a,b,x \in S$. If ρ is both left and right compatible, it is called a *congruence*.

Let S,T be semigroups. A mapping $\phi : S \to T$ from S to T is called a *homomorphism* if, for any $x,y \in S$, we have $\phi(xy) = \phi(x)\phi(y)$. A homomorphism ϕ is called a *monomorphism* or *isomorphism* if it is, respectively, injective or bijective. If there exists an isomorphism $\phi : S \to T$, we say that S and T are *isomorphic* and write $S \cong T$.

Let $\phi : S \to T$ be a homomorphism between semigroups S and T. Then, ϕ induces a congruence on S, called the *kernel* of ϕ, denoted by $\ker\phi$ and given by

$$\ker\phi = \{(x,y) \in S \times S \mid \phi(x) = \phi(y)\}.$$

Let S be a semigroup and ρ a congruence on S. Consider the quotient set of S by ρ, denoted by S/ρ. For any $x \in S$, let $[x]_\rho$ be the ρ-class of x, that is, $[x]_\rho = \{y \in S \mid y \,\rho\, x\}$. We define a multiplication on S/ρ in the following way: For $x,y \in S$, $[x]_\rho[y]_\rho := [xy]_\rho$. With this multiplication, the quotient set S/ρ is a semigroup and is called the *quotient* of S by ρ. Furthermore, the natural mapping $\rho^\natural : S \to S/\rho$, given by $x \mapsto [x]_\rho$, for any $x \in S$, is an epimorphism.

The following well known result can be found in [11, Theorem 5.4].

Theorem 2.1.1. *Let $\phi : S \to T$ be a homomorphism between semigroups and let ρ be a congruence on S such that $\rho \subseteq \ker\phi$. Then, there exists a homomorphism $\psi : S/\rho \to T$ such that $\psi \circ \rho^\natural = \phi$. Moreover, ψ is injective if and only if $\rho = \ker\phi$.*

Let M be a monoid and X a non-empty set. A mapping $\chi : M \times X \to X$ is said to be a *left action* of M on X if it satisfies the equalities $\chi(m_1, \chi(m_2, x)) = \chi(m_1 m_2, x)$ and $\chi(1_M, x) = x$, for all $m_1, m_2 \in M, x \in X$. We also say that M *acts on X on the left* and we usually represent the element $\chi(m,x)$ by $m \cdot x$, which allows us to rewrite the previous equalities in the form $m_1 \cdot (m_2 \cdot x) = (m_1 m_2) \cdot x$ and $1_M \cdot x = x$. Similarly, we can define a *right action* of a monoid on a set. We say that a monoid M *acts on* a set X (or that there exists an *action* of M on X), if there exists simultaneously a left and a right action of M on X satisfying the following equality, called the compatible property: $(m_1 \cdot x) \cdot m_2 = m_1 \cdot (x \cdot m_2)$ for any $m_1, m_2 \in M, x \in X$. Note that every monoid acts on itself by multiplication, both on the left and on the right.

2.2 Alphabets, Presentations and Rewriting Systems

In this section, we recall the concepts of presentations and rewriting systems and their application on the study of semigroups, which gives rise to the field of Combinatorial Semigroup Theory. For further information on these subjects, see, for example, [12], [20] or [2].

Let $\mathcal{A}$ be a non-empty set, which we will refer to as an *alphabet*. The elements of $\mathcal{A}$ are called *letters* and finite sequences of letters are called *words* over the alphabet $\mathcal{A}$. The *length* of a word w is the number of letters that form w and is denoted by $|w|$. For $a \in \mathcal{A}$, the number of times the element a appears in a word w is denoted by $|w|_a$. The empty sequence is called the *empty word*, has length zero and is denoted by ε. For any two words u, v over $\mathcal{A}$, we write $u = v$ if they are equal as words.

Suppose $w = w_1 \dots w_k$ is a word over $\mathcal{A}$, with $w_1, \dots, w_k \in \mathcal{A}$. For $1 \leq i \leq j \leq k$, we say $w_i \cdots w_j$ is a *factor* of w. (Note that a factor must be made up of consecutive letters.) For $i_1, \dots, i_m \in \{1, \dots, k\}$ such that $i_1 < \cdots < i_m$, we say that $w_{i_1} \cdots w_{i_m}$ is a *subsequence* of w. (Note that a subsequence may not be necessarily made up of consecutive letters, unlike a factor.)

The set of all non-empty words over $\mathcal{A}$ is denoted by $\mathcal{A}^+$, and the set of all words over $\mathcal{A}$, including the empty word, is denoted by $\mathcal{A}^*$. When equipped with the binary operation of concatenation of words, $\mathcal{A}^+$ forms a semigroup, called the *free semigroup over* $\mathcal{A}$, and $\mathcal{A}^*$ forms a monoid, with the empty word as the identity element, and is called the *free monoid over* $\mathcal{A}$.

Throughout this text, we will consider $\mathcal{A}$ to be the set of natural numbers viewed as an infinite ordered alphabet: $\mathcal{A} = \{1 < 2 < 3 < \cdots\}$. Also, for $n \in \mathbb{N}$, we will denote by $\mathcal{A}_n$ the set of the first n natural numbers viewed as a finite ordered alphabet: $\mathcal{A}_n = \{1 < 2 < \cdots < n\}$.

A *weak composition* α is a finite sequence $(\alpha_1, \dots, \alpha_m)$ with terms in $\mathbb{N} \cup \{0\}$. The terms α_h up to the last non-zero terms of the sequence are the *parts* of α. The *length* of α, denoted by $l(\alpha)$, is the number of its parts. The *weight* of α, denoted by $|\alpha|$, is the sum of its parts, that is, $|\alpha| = \alpha_1 + \cdots + \alpha_m$. For example if $\alpha = (0,1,3,0,2,0)$ then $l(\alpha) = 5$ and $|\alpha| = 6$. We shall identify weak compositions whose parts are the same, that is, weak compositions which only differ in a tail of terms 0.

A *composition* is a weak composition whose parts are exclusively in $\mathbb{N}$. For a composition $\alpha = (\alpha_1, \dots, \alpha_{l(\alpha)})$, let us denote by $D(\alpha)$ the set $\{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \cdots + \alpha_{l(\alpha)-1}\}$.

We say that a non-increasing finite sequence $\lambda = (\lambda_1, \dots, \lambda_m)$ with terms in $\mathbb{N}$ is a *partition*. Note that a partition is a particular kind of weak composition, thus, we define and denote the *length* and *weight* of λ in the exact same way as before.

We now define the *weight function* (not to be confused with the weight of a weak composition), which informally is the function that counts the number of times each element appears in a word. More formally, it is defined by

$$\mathrm{wt} : \mathcal{A}^* \to (\mathbb{N} \cup \{0\})^{\mathcal{A}}, \quad w \mapsto (|w|_1, |w|_2, \dots).$$

Since words are finite sequences, then $\mathrm{wt}(\cdot)$ has an infinite tail of elements 0, thus we only consider its prefix up to the last non-zero term. Hence $\mathrm{wt}(\cdot)$ is a weak composition. We compare weights using the following order:

$$(\alpha_1, \alpha_2, \dots) \leq (\beta_1, \beta_2, \dots) \Leftrightarrow \sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{k} \beta_i,$$

for any $k \in \mathbb{N}$.

When $\mathrm{wt}(w_1) < \mathrm{wt}(w_2)$, for words $w_1, w_2 \in \mathcal{A}^*$, we say that w_1 has *lower weight* than w_2 (and that w_2 has *higher weight* than w_1).

We now relate alphabets with semigroups and introduce notions that allow us to apply combinatorial results to Semigroup Theory.

Proposition 2.2.1. *Let M be a monoid. For any alphabet $\mathcal{A}$ and any mapping $\theta : \mathcal{A} \to S$, there is a unique extension of θ to a homomorphism from $\mathcal{A}^*$ into M, also denoted by θ, defined by $\theta(a_1 \cdots a_n) = (\theta a_1) \cdots (\theta a_n)$, for any $a_1, \ldots, a_n \in \mathcal{A}$. The image of this homomorphism is the submonoid of M generated by $\theta(\mathcal{A})$, and this submonoid is equal to M if and only if θ is surjective.*

A *monoid presentation* $\mathcal{P}$ is a pair $\langle \mathcal{A} \mid \mathcal{R} \rangle$ such that $\mathcal{R}$ is a binary relation in the free monoid over the alphabet $\mathcal{A}$. The set $\mathcal{R}$ is know as a *rewriting system* and its elements as *rewriting rules*. We say that $\mathcal{P}$ is finite if both $\mathcal{A}$ and $\mathcal{R}$ are finite.

Let $\mathcal{R}$ be a rewriting system over $\mathcal{A}^*$. We define a binary relation $\to_{\mathcal{R}}$ on $\mathcal{A}^*$, called a *single-step reduction*, in the following way: For any $u, v \in \mathcal{A}^*$,

$$u \to_{\mathcal{R}} v \Leftrightarrow (u = w_1 r_{+1} w_2) \wedge (v = w_1 r_{-1} w_2),$$

for some $(r_{+1}, r_{-1}) \in \mathcal{R}$ and $w_1, w_2 \in \mathcal{A}^*$. We denote the transitive and reflexive closure of $\to_{\mathcal{R}}$ by $\to_{\mathcal{R}}^*$, and the equivalence relation that $\to_{\mathcal{R}}$ induces by $\leftrightarrow_{\mathcal{R}}^*$. Note that this equivalence relation is in fact the smallest congruence on the free monoid $\mathcal{A}^*$ that contains $\mathcal{R}$, called the *Thue congruence* generated by $\mathcal{R}$.

We say $\mathcal{R}$ is:

- *noetherian* if there is no infinite descending chain $w_1 \to_{\mathcal{R}} w_2 \to_{\mathcal{R}} \cdots$, with $w_n \in \mathcal{A}^+, n \in \mathbb{N}$;
- *confluent* if, for $u, w_1, w_2 \in \mathcal{A}^*$, whenever $u \to_{\mathcal{R}}^* w_1$ and $u \to_{\mathcal{R}}^* w_2$ then there exists $v \in \mathcal{A}^*$ such that $w_1 \to_{\mathcal{R}}^* v$ and $w_2 \to_{\mathcal{R}}^* v$;
- *locally confluent* if, for $u, w_1, w_2 \in \mathcal{A}^*$, whenever $u \to_{\mathcal{R}} w_1$ and $u \to_{\mathcal{R}} w_2$ then there exists $v \in \mathcal{A}^*$ such that $w_1 \to_{\mathcal{R}}^* v$ and $w_2 \to_{\mathcal{R}}^* v$.

If $\mathcal{R}$ is both noetherian and confluent, it is called *complete*.

Let $u \in \mathcal{A}^*$. If there is no word $v \in \mathcal{A}^*$ such that $u \to_{\mathcal{R}} v$, we say that u is *irreducible*. If $u, v \in \mathcal{A}^*$ are such that $u \leftrightarrow_{\mathcal{R}}^* v$ and v is irreducible, we say v is a *normal form* for u.

The next results are consequences of Lemma 1.1.10, Corollary 1.1.8, and Theorem 1.1.12, respectively, in [2].

Proposition 2.2.2. *Let $\mathcal{R}$ be a noetherian rewriting system on an alphabet $\mathcal{A}$. Then, for every $u \in \mathcal{A}^*$, u has at least one normal form.*

Proposition 2.2.3. *Let $\mathcal{R}$ be a confluent rewriting system on an alphabet $\mathcal{A}$. Then, for every $u \in \mathcal{A}^*$, u has at most one normal form.*

Corollary 2.2.4. *Let $\mathcal{R}$ be a complete rewriting system on an alphabet $\mathcal{A}$. Then, for every $u \in \mathcal{A}^*$, u has a unique normal form.*

Let $u, v \in \mathcal{A}^*$. The words u and v are said to *overlap* if, up to symmetry, one of the two following cases occur:

(i) v is a factor of u, that is, there exist $a, c \in \mathcal{A}^*$ such that $u = avc$; or

(ii) u overlaps with v on the left, that is, there exist words a, b, c over the alphabet $\mathcal{A}$, with b non-empty, such that $u = ab$ and $v = bc$.

Furthermore, if both u and v are left sides of rewriting rules in $\mathcal{R}$, that is, there exist $u', v' \in \mathcal{A}^*$ such that $(u, u'), (v, v') \in \mathcal{R}$, then in case (i) and if whenever a and c are both empty, then $u' \neq v'$, then the pair of words $\{u', av'c\}$ is called a *critical pair*. In case (ii) we say that $uc = av$ is an *overlap ambiguity* of $\mathcal{R}$ and the pair $\{u'c, av'\}$ is also a *critical pair* of $\mathcal{R}$.

We say that a critical pair $\{u, v\}$ of $\mathcal{R}$ is *resolved* if there exists $w \in \mathcal{A}^*$ such that $u \rightarrow^*_{\mathcal{R}} w$ and $v \rightarrow^*_{\mathcal{R}} w$.

The following result follows from [1, Corollary 6.2.5] and [2, Theorem 1.1.13].

Proposition 2.2.5. *Let $\mathcal{R}$ be a noetherian rewriting system on an alphabet $\mathcal{A}$. The following conditions are equivalent:*

- *$\mathcal{R}$ is confluent;*
- *$\mathcal{R}$ is locally confluent;*
- *All critical pairs of $\mathcal{R}$ are resolved.*

Note that, by Proposition 1.5.10 in [12], given a rewriting system $\mathcal{R}$ on an alphabet $\mathcal{A}$ and words $u, v \in \mathcal{A}^*$, we have $u \leftrightarrow^*_{\mathcal{R}} v$ if and only if there is a finite sequence of words $w_0, \ldots, w_n \in \mathcal{A}^*, n \in \mathbb{N}$ such that $w_0 = u, w_n = v$ and either $w_i \rightarrow_{\mathcal{R}} w_{i+1}$ or $w_i \leftarrow_{\mathcal{R}} w_{i+1}$, for all $i = 1, \ldots, n-1$.

Proposition 2.2.6 ([2, Theorem 2.2.4])**.** *Let $\mathcal{R}$ be a rewriting system on an alphabet $\mathcal{A}$. Then, the following two statements are equivalent:*

- *$\mathcal{R}$ is noetherian;*
- *There exists an admissible well-founded partial order $<$ on $\mathcal{A}^*$ that is compatible with $\mathcal{R}$ (in the sense that $v < u$ for each rule $(u, v) \in \mathcal{R}$).*

Definition 2.2.7 (The length-plus-lexicographic order [2, Definition 2.2.2(d)])**.** We define the *length-plus-lexicographic order*, denoted by $<_{lenlex}$, induced by the natural order on $\mathcal{A}$ in the following way: Let $u = u_1 \cdots u_k, v = v_1 \cdots v_l$ be words in $\mathcal{A}^*$. Then,

$$u <_{lenlex} v \quad \Leftrightarrow \quad (k < l) \ \vee \ \Big(k = l \ \wedge \ (\exists i)\big(u_i < v_i \ \wedge \ (\forall j < i)(u_j = v_j)\big)\Big).$$

It is easy to see that the length-plus-lexicographic order is an admissible well-ordering, thus, it is an admissible well-founded partial order on $\mathcal{A}^*$. If a rewriting system $\mathcal{R}$ on $\mathcal{A}$ is compatible with the length-plus-lexicographic order, then it is noetherian.

The quotient of the free monoid $\mathcal{A}^*$ by the Thue congruence $\leftrightarrow^*_{\mathcal{R}}$ is called the *monoid defined by* the presentation $\mathcal{P} = \langle \mathcal{A} \mid \mathcal{R} \rangle$ and is denoted by $M(\mathcal{P})$. Consider the natural mapping $\rho : \mathcal{A} \to M(\mathcal{P}), a \mapsto [a]_{\leftrightarrow^*_{\mathcal{R}}}$. The homomorphism extension of ρ to $\mathcal{A}^+$ is an epimorphism from $\mathcal{A}^*$ onto $M(\mathcal{P})$, by Proposition 2.2.1, hence $\rho(\mathcal{A})$ generates $M(\mathcal{P})$. By this reason, the elements of $\mathcal{A}$ are called the *generating symbols*. If there is no ambiguity, it is usual to identify a word on $\mathcal{A}$ with its corresponding congruence class of $M(\mathcal{P})$, hence we identify the generating symbols with the generators of $M(\mathcal{P})$ and $\mathcal{A}$ with the generating set of $M(\mathcal{P})$.

Let $u, v \in \mathcal{A}^*$. If $u \leftrightarrow^*_{\mathcal{R}} v$, we say that u and v represent the same element of $M(\mathcal{P})$ and denote it by $u \equiv_{\mathcal{R}} v$. We also say that $M(\mathcal{P})$ *satisfies the relation* $u \equiv v$. Since, by definition, $M(\mathcal{P})$ satisfies all relations in $\mathcal{R}$, a rewriting rule (r_{+1}, r_{-1}) is also called a *defining relation* and written in the form $r_{+1} \equiv r_{-1}$.

Let M be a monoid and $\rho : \mathcal{A} \to M$ a mapping from $\mathcal{A}$ to M. If its homomorphism extension is an epimorphism from $\mathcal{A}^*$ onto M, we call the alphabet $\mathcal{A}$ a *generating set for* M. Also, if $\leftrightarrow^*_{\mathcal{R}} = \ker\rho$, for a rewriting system $\mathcal{R}$, we say that M is *defined by* $\mathcal{P} = \langle \mathcal{A} \mid \mathcal{R} \rangle$. In this case, due to Theorem 2.1.1, there exists an isomorphism $\psi : M(\mathcal{P}) \to M$ such that $\psi \circ \phi = \rho$, where $\phi : \mathcal{A}^* \to M(\mathcal{P})$ is the natural homomorphism. More generally, a monoid M is said to be *defined by* a monoid presentation $\mathcal{P} = \langle \mathcal{A} \mid \mathcal{R} \rangle$ if M and $M(\mathcal{P})$ are isomorphic. It is also possible to identify elements of $\mathcal{A}^*$ with elements of M, by extending the identification presented above, under the mapping ρ. If $\rho(u) \equiv \rho(v)$, we say that M *satisfies* the relation $u \equiv v$, for $u, v \in \mathcal{A}^*$.

To define the notions of *semigroup presentation* and of semigroup $S(\mathcal{P})$ *defined by* $\mathcal{P} = \langle \mathcal{A} \mid \mathcal{R} \rangle$, just replace $\mathcal{A}^*$ with $\mathcal{A}^+$, in the definitions given above. For most of the text, we will work mostly with monoid presentations, and we shall refer to them just as presentations, as long as there is no confusion with semigroup presentations.

2.3 Graphs

In this section we will present some basic definitions and theorems regarding graphs, according to Serre [22].

An *(oriented) graph* is a quintuple $\Gamma = (V, E, \iota, \tau, ^{-1})$, where $V = V(\Gamma)$ is the (non-empty) set of *vertices*, $E = E(\Gamma)$ is the set of *edges*, and $\iota : E \to V$ and $\tau : E \to V$ are mappings, respectively called the *initial* and *terminal mapping*. Given $e \in E$, the vertices ιe and τe are respectively know as the *start* and *end* of e, and are collectively known as the *extremities* of e. Orientation on the graph is given by the *inverse mapping* $^{-1} : E \to E$, a mapping that satisfies, for all $e \in E$, $e \neq e^{-1}$, $\iota(e^{-1}) = \tau(e)$, $\tau(e^{-1}) = \iota(e)$ and $(e^{-1})^{-1} = e$.

A non-empty *path* p on Γ is a finite sequence $(e_1, \ldots, e_n)$ of edges $e_i \in E$, with $n \in \mathbb{N}$, such that $\tau e_i = \iota e_{i+1}$, for $i = 1, \ldots, n-1$. It is usual to write p in the form $e_1 \cdots e_n$. Since

p has n elements, we say p has *length* n and write $l(p) = n$. We also extend the notions of start, end and extremities to paths, by defining $\iota p := \iota e_1$ and $\tau p := \tau e_n$. If, for vertices $u, v \in V$, $\iota p = u$ and $\tau p = v$ (or $\iota p = v$ and $\tau p = u$), we say p *joins* u and v. We say a path p is *closed* if $\iota p = \tau p$. We define the *inverse path* of p as the path $e_n^{-1} \cdots e_1^{-1}$ and denote it by p^{-1}. For each $v \in V$, we define an *empty path* 1_v with no edges, such that $\iota 1_v = \tau 1_v = v$ and $1_v^{-1} = 1_v$.

Definition 2.3.1. Given a presentation $\mathscr{P} = \langle \mathscr{A} \mid \mathscr{R} \rangle$, define a unique graph associated to $\mathscr{P}$, denoted by $\Gamma(\mathscr{P})$. Its set of vertices is the free monoid $\mathscr{A}^*$ (or the free semigroup $\mathscr{A}^+$), and the edges are quadruples of the form $e = (w_1, r_{+1} = r_{-1}, \epsilon, w_2)$, where $w_1, w_2 \in \mathscr{A}^*, (r_{+1}, r_{-1}) \in \mathscr{R}$ and $\epsilon = \pm 1$.

The initial and terminal vertices and the inverse mapping are defined, respectively, by $\iota e = w_1 r_\epsilon w_2, \tau e = w_1 r_{-\epsilon} w_2$ and $e^{-1} = (w_1, r_{+1} = r_{-1}, -\epsilon, w_2)$. We say that an edge is positive if $\epsilon = +1$ and negative otherwise. Also, for each word $w \in \mathscr{A}^*$, there is an empty path 1_w with no edges.

Note that, given any words $u, v \in \mathscr{A}^*$, we have $u \to_{\mathscr{R}} v$ if and only if there is a positive edge e of $\Gamma(\mathscr{P})$ such that $\iota e = u$ and $\tau e = v$. Thus, we have $u \leftrightarrow_{\mathscr{R}}^* v$ if and only if there is a path in $\Gamma(\mathscr{P})$ that joins u and v.

Let $\Gamma = (V, E, \iota, \tau, ^{-1})$ be a graph and let M be a monoid. We say that M *acts on the left of the graph* Γ if M acts on the left of the sets V and E, respectively, and, for any $m \in M, e \in E$, we have $\iota(m \cdot e) = m \cdot \iota e, \tau(m \cdot e) = m \cdot \tau e$ and $(m \cdot e)^{-1} = m \cdot e^{-1}$. We can extend this action to paths in the following way: given edges $e_1, \ldots, e_n \in E$ and $m \in M$, for $p = e_1 \ldots e_n$, we define $m \cdot p := (m \cdot e_1) \cdots (m \cdot e_n)$. We define a *right action of* M on Γ in a similar way. We say that M *acts* on Γ if M simultaneously acts on the left and on the right on Γ and if both actions on the set of vertices and on the set of edges are compatible.

Definition 2.3.2. The concatenation product in $\mathscr{A}^*$ induces natural left and right actions of $\mathscr{A}^*$ on $\Gamma(\mathscr{P})$, in the following way: For any $x, y \in \mathscr{A}^*$ and any vertex $w \in \mathscr{A}^*$, we define $x \cdot w = xw$ and $w \cdot y = wy$; and for any edge $e = (u, r, \epsilon, v)$, we define $x \cdot e = (xu, r, \epsilon, v)$ and $e \cdot y = (u, r, \epsilon, vy)$. Both actions are compatible, thus $\mathscr{A}^*$ acts on $\Gamma(\mathscr{P})$.

Let $\Gamma = (V, E, \iota, \tau, ^{-1})$ be a graph. Let V_0 be a subset of V and E_0 be a subset of E. The quintuple $\Gamma_0 = (V_0, E_0, \iota, \tau, ^{-1})$ is a *subgraph* of Γ if, for all $e \in E_0$, we have $e^{-1} \in E_0$ and $\iota e, \tau e \in V_0$. If E_0 is the set of all edges of Γ with both extremities in V_0, that is, $E_0 = \{e \in E \mid \iota e, \tau e \in V_0\}$, then the subgraph of Γ defined by V_0 and E_0 is known as the *full subgraph* defined by V_0 and denoted by Γ_{V_0}.

We say a graph $\Gamma = (V, E, \iota, \tau, ^{-1})$ is *connected* if any two vertices in it are joined by a path. It is easy to see that the binary relation on V, defined by u being related to v if and only if there is a path starting in u and ending in v, for $u, v \in V$, is in fact an equivalence relation. The full subgraphs whose vertex sets are the equivalence classes of this relation are known as the *connected components* of Γ.

Remark 2.3.3. Let $\mathscr{P} = \langle \mathscr{A} \mid \mathscr{R} \rangle$. Recalling Definition 2.3.1, each congruence class of $\leftrightarrow^*_{\mathscr{R}}$ is a connected component of the graph $\Gamma(\mathscr{P})$. Thus the set of elements of the monoid $M(\mathscr{P})$ (or the semigroup $S(\mathscr{P})$) is in bijection with the set of the connected components $\pi_0(\Gamma(\mathscr{P}))$ of $\Gamma(\mathscr{P})$.

Let Γ_1 and Γ_2 be graphs. A *mapping of graphs* ϕ from Γ_1 to Γ_2 is a pair of mappings $\phi_V : V(\Gamma_1) \to V(\Gamma_2)$ and $\phi_E : E(\Gamma_1) \to E(\Gamma_2)$, such that, for all $e \in E$, $\phi_E(e)$ is a path on Γ_2 starting at $\phi_V(\iota e)$ and ending at $\phi_V(\tau e)$, and $\phi_E(e^{-1}) = (\phi_E(e))^{-1}$. As long as there is no confusion, we shall write both ϕ_V and ϕ_E as ϕ. This map can be extended to paths by defining $\phi(1_v) := 1_{\phi(v)}$, for all $v \in V(\Gamma_1)$, and $\phi(p) = \phi(e_1)\cdots\phi(e_n)$, for a non-empty path $p = e_1 \dots e_n$, with $n \in \mathbb{N}$.

2.4 Homotopy relations, finite derivation type and coherent presentations

In this section, we will present three important concepts: the concept of *homotopy relations* and the concept of *finite derivation type (FDT)*, a finiteness property of semigroup presentations, first introduced by C. Squier in the 1990's and further studied by F. Otto and Y. Kobayashi (see [24]), and the concept of *coherent presentation*, which, as we have said before, were first introduced in [9], using the language of strict monoidal categories and higher-dimensional variations of them.

Let $\mathscr{P} = \langle \mathscr{A} \mid \mathscr{R} \rangle$ be a finite monoid presentation and let $\Gamma(\mathscr{P})$ be the graph associated with it. Consider the sets $P\big(\Gamma(\mathscr{P})\big)$ of all paths in $\Gamma(\mathscr{P})$ and $P^{(2)}\big(\Gamma(\mathscr{P})\big)$ of all ordered pairs of paths in $\Gamma(\mathscr{P})$ which have a common start and a common end. An equivalence relation $\sim$ on $P^{(2)}\big(\Gamma(\mathscr{P})\big)$ is called a *homotopy relation* if it satisfies the following conditions:

(H1) For any edges e_1 and e_2 of $\Gamma(\mathscr{P})$, we have

$$(e_1 \cdot \iota e_2)(\tau e_1 \cdot e_2) \sim (\iota e_1 \cdot e_2)(e_1 \cdot \tau e_2).$$

(H2) If $p \sim q$, then, for any $x, y \in \mathscr{A}^*$, we have $x \cdot p \cdot y \sim x \cdot q \cdot y$;

(H3) If $p, q_1, q_2, r \in P\big(\Gamma(\mathscr{P})\big)$ are such that $\tau p = \iota q_1 = \iota q_2$, $\tau q_1 = \tau q_2 = \iota r$ and $q_1 \sim q_2$, then $pq_1r \sim pq_2r$;

(H4) If $p \in P\big(\Gamma(\mathscr{P})\big)$, then $pp^{-1} \sim 1_{\iota p}$.

Notice that the collection of all homotopy relations on the set of paths in $\Gamma(\mathscr{P})$ is closed under arbitrary intersection, and that $P^{(2)}\big(\Gamma(\mathscr{P})\big)$ is itself a homotopy relation. Thus, for any subset $X \subseteq P^{(2)}\big(\Gamma(\mathscr{P})\big)$, there is a unique smallest homotopy relation $\sim_X$ on the set of paths in $\Gamma(\mathscr{P})$ that contains X, called the *homotopy relation generated by* X.

We say that $\mathscr{P}$ is of *finite derivation type (FDT)* if there is a finite subset $X \subseteq P^{(2)}\big(\Gamma(\mathscr{P})\big)$ which generates $P^{(2)}\big(\Gamma(\mathscr{P})\big)$ as a homotopy relation, that is, $P^{(2)}\big(\Gamma(\mathscr{P})\big)$ is the only homotopy relation on the set of paths in $\Gamma(\mathscr{P})$ that contains X.

Theorem 2.4.1 ([24, Theorem 4.3]). *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be finite monoid presentations defining the same monoid. Then, $\mathcal{P}_1$ is of FDT if and only if $\mathcal{P}_2$ is of FDT.*

Thus, having *FDT* is an invariant property of finitely presented monoids, hence it makes sense to refer to *FDT* monoids.

Recall the notion of critical pair of a rewriting system given in Section 2.2. Let e_1, e_2 be positive edges in $\Gamma(\mathcal{P})$, with $\iota e_1 = \iota e_2$, for a presentation $\mathcal{P} = \langle \mathcal{A} \mid \mathcal{R} \rangle$. We say the pair (e_1, e_2) is a *critical pair of edges* if the left-hand sides of the underlying rewriting rules overlap and lead to a critical pair. A *resolution* of a critical pair of edges (e_1, e_2) is a pair of paths (p_1, p_2) such that $\iota p_1 = \tau e_1, \iota p_2 = \tau e_2, \tau p_1 = \tau p_2$ and all edges of both p_1 and p_2 are positive. For any resolvable critical pair (e_1, e_2), fix a resolution (p_1, p_2). Denote by B the set

$$\Big\{(e_1 p_1, e_2 p_2) \mid (e_1, e_2) \textit{ is a critical pair of } \mathcal{R}, \textit{ and } \{p_1, p_2\} \textit{ is the corresponding resolution } \Big\}. \tag{2.4.1}$$

Theorem 2.4.2 ([24, Theorem 5.2]). *Let $\mathcal{P} = \langle \mathcal{A} \mid \mathcal{R} \rangle$ be a presentation, where $\mathcal{R}$ is a complete rewriting system, and let $\Gamma(\mathcal{P})$ be the graph associated with it. Let $B \subseteq P^{(2)}\big(\Gamma(\mathcal{P})\big)$ be defined as above. Then, B generates $P^{(2)}\big(\Gamma(\mathcal{P})\big)$ as a homotopy relation.*

Observe that if $\mathcal{R}$ if finite, then B is also finite, thus $\mathcal{P}$ is of *FDT*.

Theorem 2.4.3 ([24, Theorem 5.3]). *Let M be a finitely presented monoid. Let $\mathcal{P} = \langle \mathcal{A} \mid \mathcal{R} \rangle$ be a presentation, where $\mathcal{R}$ is a finite complete rewriting system. If M is presented by $\mathcal{P}$, then M is FDT.*

Now, we are able to introduce some definitions, first given by [9], but presented here using the language of Combinatorial Semigroup Theory.

An *extended presentation* of a monoid M is a pair $\langle \mathcal{P} \mid \mathcal{C} \rangle$, where $\mathcal{P} = \langle \mathcal{A} \mid \mathcal{R} \rangle$ is a presentation of M, $\mathcal{R}$ is a rewriting system and $\mathcal{C}$ is a subset of $P^{(2)}\big(\Gamma(\mathcal{P})\big)$ in which the pairs are oriented, that is, it is an analogue of a string-rewriting system for paths, with the restriction that the paths in each pair have the same start and end. We can also write an extended presentation as a triple $\langle \mathcal{A} \mid \mathcal{R} \mid \mathcal{C} \rangle$. An extended presentation is *finite* if both $\mathcal{P}$ and $\mathcal{C}$ are finite.

A *coherent presentation* is an extended presentation such that $\mathcal{C}$ generates $P^{(2)}\big(\Gamma(\mathcal{P})\big)$. Thus, if a monoid M admits a finite coherent presentation, it is *FDT*.

In the remaining of this section we provide tools to be able to construct coherent presentations. Let $\mathcal{P} = \langle \mathcal{A} \mid \mathcal{R} \rangle$ be a presentation. The following four types of transformations of $\langle \mathcal{A} \mid \mathcal{R} \rangle$ are called *elementary Tietze transformations*:

(T_1) - **Add a generator:** For $w \in \mathcal{A}^*$ and $a \notin \mathcal{A}$, add a to $\mathcal{A}$ and (w, a) to $\mathcal{R}$;

(T_2) - **Delete a generator:** For $a \in \mathcal{A}$ and $w \in (\mathcal{A} \setminus \{a\})^*$ such that $w \rightarrow_{\mathcal{R}} a$,

1. remove a from $\mathcal{A}$;

2. remove (w,a) from $\mathscr{R}$;
3. for any $(u,v) \in \mathscr{R}$, replace any factor a of u and v by w;

(T_3) - **Add a relation:** For $u,v \in \mathscr{A}^*$ such that $u \to^*_{\mathscr{R}} v$ but $(u,v) \notin \mathscr{R}$, add (u,v) to $\mathscr{R}$;

(T_4) - **Delete a relation:** For $u,v \in \mathscr{A}^*$ such that $u \to^*_{\mathscr{R}'} v$, where $\mathscr{R}' = \mathscr{R} \setminus \{(u,v)\}$, remove (u,v) from $\mathscr{R}$.

We say that a (finite) *Tietze transformation* is a (finite) sequence of elementary Tietze transformations.

In [8], a corresponding notion of Tietze transformations was introduced for extended presentations. Let $\langle \mathscr{A} \mid \mathscr{R} \mid \mathscr{C} \rangle$ be an extended presentation. The following six types of transformations of $\langle \mathscr{A} \mid \mathscr{R} \mid \mathscr{C} \rangle$ are called *elementary Tietze transformations*:

(T_1^*) - **Add a generator:** For $w \in \mathscr{A}^*$ and $a \notin \mathscr{A}$, add a to $\mathscr{A}$ and (w,a) to $\mathscr{R}$;

(T_2^*) - **Delete a generator:** For $a \in \mathscr{A}$ and $w \in (\mathscr{A} \setminus \{a\})^*$ such that $w \to_{\mathscr{R}} a$,

1. remove a from $\mathscr{A}$;
2. remove (w,a) from $\mathscr{R}$;
3. for any $(u,v) \in \mathscr{R}$, replace any factor a of u and v by w;
4. for any $(f,g) \in \mathscr{C}$, remove any occurrence of (w,a) in f and g;
5. for any $(f,g) \in \mathscr{C}$, replace any occurrence of a rule (u,v) in f and g by the rule (u',v'), where u or v have a factor a and (u',v') is obtained by replacing a in u and v by w;

(T_3^*) - **Add a relation:** For $u,v \in \mathscr{A}^*$ such that $u \to^*_{\mathscr{R}} v$ but $(u,v) \notin \mathscr{R}$,

1. add (u,v) to $\mathscr{R}$;
2. add (f,g) to $\mathscr{C}$, where $f = (u,v)$ and $g = (u,w)$, for $w \in \mathscr{A}^* \setminus \{v\}$ such that $u \to_{\mathscr{R}} w$ and $w \to^*_{\mathscr{R}} v$;

(T_4^*) - **Delete a relation:** For $u,v \in \mathscr{A}^*$ such that $u \to^*_{\mathscr{R}'} v$, where $\mathscr{R}' = \mathscr{R} \setminus \{(u,v)\}$,

1. remove (u,v) from $\mathscr{R}$;
2. for any $(f,g) \in \mathscr{C}$, remove any occurrence of (u,v) in f and g;

(T_5^*) - **Add a pair of paths:** For $f \sim_{\mathscr{C}} g$ but $(f,g) \notin \mathscr{C}$, add (f,g) to $\mathscr{C}$;

(T_6^*) - **Delete a pair of paths:** For $(f,g) \in \mathscr{C}$ such that $f \sim_{\mathscr{C}'} g$, where $\mathscr{C}' = \mathscr{C} \setminus \{(f,g)\}$, remove (f,g) from $\mathscr{C}$.

The notion of (finite) *Tietze transformation* is analogous to the previous case.

Theorem 2.4.4 ([8, Theorem 2.1.3]). *The monoids presented by two (finite) extended presentations are isomorphic if, and only if, there exists a (finite) Tietze transformation between them.*

Thus, if a monoid M is presented by $\mathscr{P} = \langle \mathscr{A} \mid \mathscr{R} \rangle$, where $\mathscr{R}$ is a noetherian rewriting system, we can build a coherent presentation for M: We start with the extended presentation $\langle \mathscr{P} \mid \mathscr{C} \rangle$, where $\mathscr{C}$ is the empty set. Then, for each critical pair of edges (e_1, e_2) of $\langle \mathscr{P} \mid \mathscr{C} \rangle$,

- if (e_1, e_2) admits a resolution, fix one resolution (p_1, p_2), then add $(e_1 p_1, e_2 p_2)$ to $\mathscr{C}$;
- otherwise, since $\mathscr{R}$ is noetherian, both τe_1 and τe_2 have normal forms. Let $u_1, u_2 \in \mathscr{A}^*$ be those normal forms, let p_1 be the path from τe_1 to u_1 and p_2 be the path from τe_2 to u_2. Let $<$ be the admissible well-founded partial order on $\mathscr{A}^*$ that is compatible with $\mathscr{R}$ (see 2.2.6).
  - If $v < u$, add (u_1, u_2) to $\mathscr{R}$. Let e_3 be the edge with start u_1 and end u_2. Add $(e_1 p_1 e_3, e_2 p_2)$ to $\mathscr{C}$.
  - Otherwise add (u_2, u_1) to $\mathscr{R}$. Let e_4 be the edge with start u_2 and end u_1. Add $(e_1 p_1, e_2 p_2 e_4)$ to $\mathscr{C}$.

This procedure is called the *homotopical completion procedure* and can be seen in much greater detail in [9]. The main feature of this homotopical completion procedure is that extends the Knuth–Bendix completion procedure (see for instance [2, Subsection 2.4]) into a tool for computing coherent presentations, by keeping track of homotopy generators created when adding new rules. Note that, in general, the procedure is not guaranteed to terminate.

In particular, if $\mathscr{R}$ is a complete rewriting system, then we can construct a coherent presentation for M in the following way: By Theorem 2.4.2, we consider the subset $\mathscr{C}$ of $P^{(2)}\big(\Gamma(\mathscr{P})\big)$ as defined by (2.4.1). Thus, the extended presentation $\langle \mathscr{P} \mid \mathscr{C} \rangle$ is a coherent presentation for M. Note that, since $\mathscr{R}$ is a complete rewriting system, to obtain $\langle \mathscr{P} \mid \mathscr{C} \rangle$ from $\mathscr{P} = \langle \mathscr{A} \mid \mathscr{R} \rangle$, we apply a Tietze transformation to $\mathscr{P}$ that consists only in elementary Tietze transformations of type (T_5^*).

CHAPTER 3

THE PLACTIC MONOID

In this chapter, we shall discuss three possible ways to define the plactic monoid: via *generators and relations*, *tableaux and insertion*, and *crystals*, and also the interaction of the crystal structure with the combinatorics of Young tableaux (following [3]). We shall also present a finite complete rewriting system for the plactic monoid of rank n, from which a convergent presentation for it can be computed (following [4] and [10]).

3.1 The plactic monoid, Young tableaux and insertion

Consider the ordered alphabet $\mathcal{A} = \{1 < 2 < \ldots\}$. The *plactic monoid*, denoted by plac, is presented by $\langle \mathcal{A} \mid \mathcal{R}_{\text{plac}} \rangle$, where $\mathcal{R}_{\text{plac}}$ is the set of relations of the form

$$\begin{aligned} &(cab, acb) \quad \text{with } a \le b < c\,; \\ &(bca, bac) \quad \text{with } a < b \le c\,, \end{aligned}$$

known as the *Knuth relations*.

Let $n \in \mathbb{N}$ and consider the finite ordered alphabet $\mathcal{A}_n = \{1 < 2 < \cdots < n\}$. The *plactic monoid* of rank n, denoted by plac_n, is presented by $\langle \mathcal{A}_n \mid \mathcal{R}_{\text{plac}} \rangle$, where in this case the set of defining relations $\mathcal{R}_{\text{plac}}$ is naturally restricted to $\mathcal{A}_n^* \times \mathcal{A}_n^*$.

We now proceed to introduce Young tableaux and related concepts, and then present an equivalent definition of the plactic monoid using these tools.

A *Young diagram* of shape λ, where λ is a partition, is a grid of cells, with left-justified rows such that the h-th row has λ_h cells, for $h = 1, \ldots, l(\lambda)$. In this text, Young diagrams will be top-left-aligned, that is, row length will be non-increasing top to bottom. If a Young diagram has shape $(1, 1, \ldots, 1)$, it is called a *column diagram* and is said to have *column shape*. For example, the Young diagram of shape $(4,3,2)$ is

$$\begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \\ \square & \square & & \end{array} \quad . \tag{3.1.1}$$

A *Young tableau* is a Young diagram filled with symbols from $\mathcal{A}$ such that entries in each row are non-decreasing from left to right, and entries in each column are (strictly) increasing from top to bottom. For example, a Young tableau of shape $(4,3,2)$ is

$$\begin{array}{cccc} 1 & 2 & 2 & 5 \\ 2 & 3 & 6 & \\ 4 & 5 & & \end{array} \quad . \tag{3.1.2}$$

A Young tableau of shape $(1,1,\ldots,1)$ is called a *column*.

A *standard Young tableau* of shape λ is a Young tableau with entries from $\{1,\ldots,|\lambda|\}$ such that each symbol appears exactly once, entries in each row are increasing from left to right, and entries in each column are increasing from top to bottom. For example, a standard Young tableau of shape $(4,3,2)$ is

$$\begin{array}{cccc} 1 & 3 & 5 & 6 \\ 2 & 7 & 8 & \\ 4 & 9 & & \end{array} \tag{3.1.3}$$

A *tabloid* is a grid of cells, filled with symbols from $\mathcal{A}$, obtained by concatenating columns, such that entries in each column are strictly increasing from top to bottom. Compared to a tableau, there is no restriction on the relative heights of columns, nor is there a condition on the order of entries in a row. Note that a tableau is a special case of a tabloid and that the shape of a tabloid cannot in general be expressed using a partition. An example of a tabloid is

$$\begin{array}{ccccc} 5 & 3 & 4 & 1 & 2 \\ & 4 & 5 & & 8 \\ & 6 & 7 & & \\ & & 9 & & \end{array} \quad . \tag{3.1.4}$$

Let $w = w_1 \cdots w_k$ be a word in $\mathcal{A}^*$, with $w_i \in \mathcal{A}$, for $i = 1,\ldots,k$. We say w is a *row word* if $w_i \leq w_{i+1}$ for all $i = 1,\ldots,k-1$. We say w is a *column word* if $w_i > w_{i+1}$ for all $i = 1,\ldots,k-1$.

The *column reading* $C(T)$ of a tabloid T is the word in $\mathcal{A}^*$ obtained by reading its columns from left to right, and reading each column from bottom to top. For example, the column reading of (3.1.2) is 421 532 62 5 and the column reading of (3.1.4) is 5 643 9754 1 82.

Let $w \in \mathcal{A}^*$. Note that every word over $\mathcal{A}^*$ has a factorization into maximal decreasing factors. Let $w^{(1)} \cdots w^{(k)}$ be such a factorization of w. Let $\mathrm{Toid}(w)$ be the tabloid whose h-th column has height $|w^{(h)}|$ and is filled with the symbols of $w^{(h)}$, for $h = 1,\ldots,k$. Then, $C(\mathrm{Toid}(w)) = w$. If w is the column reading of a Young tableau T, it is called a *tableau word*. By definition, it is immediate that w is a tableau word if and only if $\mathrm{Toid}(w)$ is a

Young tableau. Thus, we conclude that not every word in $\mathcal{A}^*$ is a tableau word. Also note that the column reading of a column matches the definition of a column word, and the column reading of a row matches the definition of a row word.

We will now see how the plactic monoid can be defined using Young tableaux, by introducing an insertion algorithm that computes a (unique) Young tableau $P(w)$ from a word $w \in \mathcal{A}^*$.

Algorithm 3.1.1 (Schensted's algorithm)**.**

Input: A Young tableau T and a symbol $a \in \mathcal{A}$.

Output: A Young tableau $T \leftarrow a$.

Method:

- If a is greater than or equal to every entry in the topmost row of T, add a as an entry at the rightmost end of the topmost row of T and output the resulting tableau.
- Otherwise, let z be the leftmost entry in the top row of T that is strictly greater than a. Replace z by a in the topmost row and recursively insert z into the tableau formed by the rows of T below the topmost (note that the recursion may end with an insertion into an 'empty row' below the existing rows of T).

Let $w = w_1 \cdots w_k$ be a word in $\mathcal{A}^*$. By applying the algorithm iteratively, we can compute a unique Young tableau $P(w)$: Starting with the empty word, we iteratively insert the symbols $w_1, \ldots, w_k \in \mathcal{A}$ in order. After inserting the last symbol, we obtain the tableau $P(w_1 \cdots w_k)$. This algorithm also allows us to compute a standard Young tableau $Q(w)$, in the following way:

Algorithm 3.1.2.

Input: A word $w = w_1 \cdots w_k$, where $w_i \in \mathcal{A}$, for $i = 1, \ldots, k$.

Output: A Young tableau $P(w)$ and a standard Young tableau $Q(w)$.

Method: Start with an empty Young tableau P_0 and an empty standard Young tableau Q_0. For each $i = 1, \ldots, k$, insert the symbol w_i into P_{i-1} as per Algorithm 3.1.1; let P_i be the resulting Young tableau. Add a cell filled with i to the standard tableaux Q_{i-1} in the same place as the unique cell that lies in P_i but not in P_{i-1}; let Q_i be the resulting standard Young tableau. Output P_k for $P(w)$ and Q_k for $Q(w)$.

The map $w \mapsto \big(P(w), Q(w)\big)$ is the well known *Robinson–Schensted–Knuth correspondence*, that is, a bijection between words in $\mathcal{A}^*$ and pairs consisting of a Young tableau over $\mathcal{A}$ and a standard Young tableau of the same shape (For more information on the subject, see [17, Subsection 5.3]). For example, the sequence of pairs (P_i, Q_i) produced during the application of Algorithm 3.1.2 to the word 3231 is:

$$(,), \quad \left(\boxed{3}, \boxed{1}\right), \quad \left(\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\right), \quad \left(\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right), \quad \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}\right).$$

Thus $P(3231) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$ and $Q(3231) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$.

The following result states the key combinatorial facts about tableaux:

Theorem 3.1.3 ([17, Theorem 5.1.1])**.** *Let $w \in \mathcal{A}^*$. The number of columns in $P(w)$ is equal to the length of the longest non-decreasing subsequence in w. The number of rows in $P(w)$ is equal to the length of the longest decreasing subsequence in w.*

Thus, we are now able to present an alternative definition of the plactic monoid in terms of tableaux. Define $\equiv_{\text{plac}}$ in the following way: For words $u, v \in \mathcal{A}^*$,

$$u \equiv_{\text{plac}} v \Leftrightarrow P(u) = P(v).$$

Using this definition, it follows that $\equiv_{\text{plac}}$ is in fact a congruence on $\mathcal{A}^*$ (see [14]). Thus, the plactic monoid is the factor monoid $\mathcal{A}^*/\equiv_{\text{plac}}$. The congruence $\equiv_{\text{plac}}$, known as the *plactic congruence*, naturally restricts to a congruence on $\mathcal{A}_n^*$, and hence the plactic monoid of rank n is the factor monoid $\mathcal{A}_n^*/\equiv_{\text{plac}}$.

Note that if w is a tableau word, then $w = C(P(w))$ and $\text{Toid}(w) = P(w)$. Hence the tableau words in $\mathcal{A}^*$ (respectively, $\mathcal{A}_n^*$) form a set of normal forms, called a cross-section, for plac (respectively, plac_n).

3.2 Kashiwara operators and the crystal graph

We will now introduce the concepts of crystal graphs and Kashiwara operators, in the context of plac_n. For a more general introduction to crystal bases, see [6].

The *Kashiwara operators* $\tilde{e}_i$ and $\tilde{f}_i$, with $i \in \{1, \ldots, n-1\}$, are partially defined operators on $\mathcal{A}_n^*$. They are described in a combinatorial way using the bracketing rule. The definitions of $\tilde{e}_i$ and $\tilde{f}_i$ start from the *crystal basis* for plac_n, which will form a connected component of the crystal graph:

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{3} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n .$$

Each operator $\tilde{f}_i$ is defined so that it replaces a symbol a with the end symbol of a directed edge labelled by i whenever such an edge starts at a, and each operator $\tilde{e}_i$ is defined so that it replaces a symbol a with the start symbol of a directed edge labelled by i whenever such an edge ends at a:

$$a \xrightarrow{i} \tilde{f}_i(a); \quad \tilde{e}_i(a) \xrightarrow{i} a.$$

Thus, by looking at the crystal basis given before, we have that:

- $\tilde{e}_i(i+1) = i$, $\tilde{e}_i(j)$ is undefined for $j \neq i+1$;
- $\tilde{f}_i(i) = i+1$, $\tilde{f}_i(j)$ is undefined for $j \neq i$.

This definition is extended to $\mathcal{A}_n^* \setminus \mathcal{A}_n$ by the recursion:

$$\tilde{e}_i(uv) = \begin{cases} \tilde{e}_i(u)v & \text{if } \tilde{\epsilon}_i(u) > \tilde{\phi}_i(u); \\ u\tilde{e}_i(v) & \text{if } \tilde{\epsilon}_i(u) \leq \tilde{\phi}_i(u), \end{cases}$$

$$\tilde{f}_i(uv) = \begin{cases} \tilde{f}_i(u)v & \text{if } \tilde{\epsilon}_i(u) \geq \tilde{\phi}_i(u); \\ u\tilde{f}_i(v) & \text{if } \tilde{\epsilon}_i(u) < \tilde{\phi}_i(u), \end{cases}$$

where $\tilde{\epsilon}_i$ and $\tilde{\phi}_i$ are auxiliary maps defined by

$$\tilde{\epsilon}_i(w) = \max\Big\{k \in \mathbb{N} \cup \{0\} \mid \underbrace{\tilde{e}_i \cdots \tilde{e}_i}_{k \text{ times}}(w) \text{ is defined}\Big\}$$

$$\tilde{\phi}_i(w) = \max\Big\{k \in \mathbb{N} \cup \{0\} \mid \underbrace{\tilde{f}_i \cdots \tilde{f}_i}_{k \text{ times}}(w) \text{ is defined}\Big\}.$$

Note that the definitions of $\tilde{e}_i$ and $\tilde{f}_i$ are not circular, since they depend, via $\tilde{\epsilon}_i$ and $\tilde{\phi}_i$, only on $\tilde{e}_i$ and $\tilde{f}_i$ applied to strictly shorter words. The recursion stops when $\tilde{e}_i$ and $\tilde{f}_i$ are applied to single letters, since we have already defined these applications by using the crystal basis. Also note that, although not immediate, it is possible to see that these operators are not only well-defined, but are also mutually inverse whenever they are defined, that is, if $\tilde{e}_i(w)$ is defined, then $w = \tilde{f}_i(\tilde{e}_i(w))$ (and if $\tilde{f}_i(w)$ is defined, then $w = \tilde{e}_i(\tilde{f}_i(w))$).

The *crystal graph* for plac_n, denoted by $\Gamma(\text{plac}_n)$, is the directed labelled graph with vertex set $\mathcal{A}_n^*$ and, for $u, v \in \mathcal{A}_n^*$, an edge from u to v labelled by i if and only if $u = \tilde{f}_i(v)$ (or, equivalently, $\tilde{e}_i(u) = v$). Note that the operators $\tilde{e}_i$ and $\tilde{f}_i$ preserve length. Therefore, since there are finitely many words in $\mathcal{A}_n^*$ of each length, each connected component in the crystal graph is finite. For any $w \in \mathcal{A}_n^*$, denote the connected component of $\Gamma(\text{plac}_n)$ that contains the vertex w by $\Gamma(\text{plac}_n, w)$.

A *crystal isomorphism* between two connected components is a weight-preserving labelled digraph isomorphism. In other words, if a map $\theta : \Gamma(\text{plac}_n, u) \to \Gamma(\text{plac}_n, v)$ verifies the following properties, then it is called a crystal isomorphism:

- θ is bijective;
- $\text{wt}(\theta(w)) = \text{wt}(w)$, for all $u \in \Gamma(\text{plac}_n, u)$;
- For all $w, w' \in \Gamma(\text{plac}_n, u)$, there is an edge $u \xrightarrow{i} v$ if and only if there is an edge $\theta(u) \xrightarrow{i} \theta(v)$.

The equivalent way of defining the plactic congruence $\equiv_{\text{plac}}$ using the crystal graph $\Gamma(\text{plac}_n)$ is as follows: For words $u, v \in \mathcal{A}_n^*$, $u \equiv_{\text{plac}} v$ if and only if there exists a crystal isomorphism $\theta : \Gamma(\text{plac}_n, u) \to \Gamma(\text{plac}_n, v)$ such that $\theta(u) = v$. In other words, u and v are related by the plactic congruence if and only if they appear in the same position in isomorphic connected components of the crystal graph.

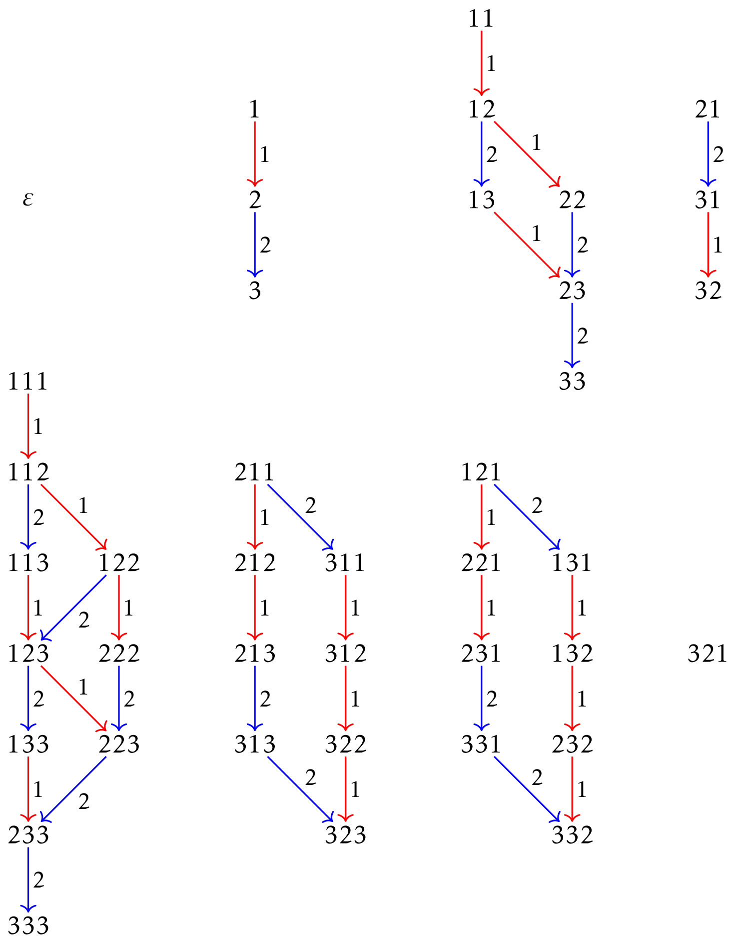


Figure 3.1: Part of the crystal graph for plac_3. Note that each connected component consists of words of the same length. In particular, the empty word ε is an isolated vertex, and the words of length 1 form a single connected component, which is the crystal basis for plac_3. The two connected components whose highest-weight words are 211 and 121 are isomorphic. However, the component consisting of the isolated vertices 321 and ε are not, since they have different weights. (This figure is taken from [3, Fig. 1].)

3.3 Properties of the crystal graph

It is easy to see from the definition that the length of the longest path with edges only labelled by i and ending (respectively, starting) in w, for a fixed $i \in \{1, \ldots, n-1\}$ and word $w \in \mathcal{A}_n^*$, is $\tilde{\varepsilon}_i(w)$ (respectively, $\tilde{\phi}_i(w)$).

An important property of the operators $\tilde{e}_i$ and $\tilde{f}_i$ is that they increase and decrease weight, respectively, whenever they are defined, that is, if $\tilde{e}_i$ (respectively $\tilde{f}_i$) is defined,

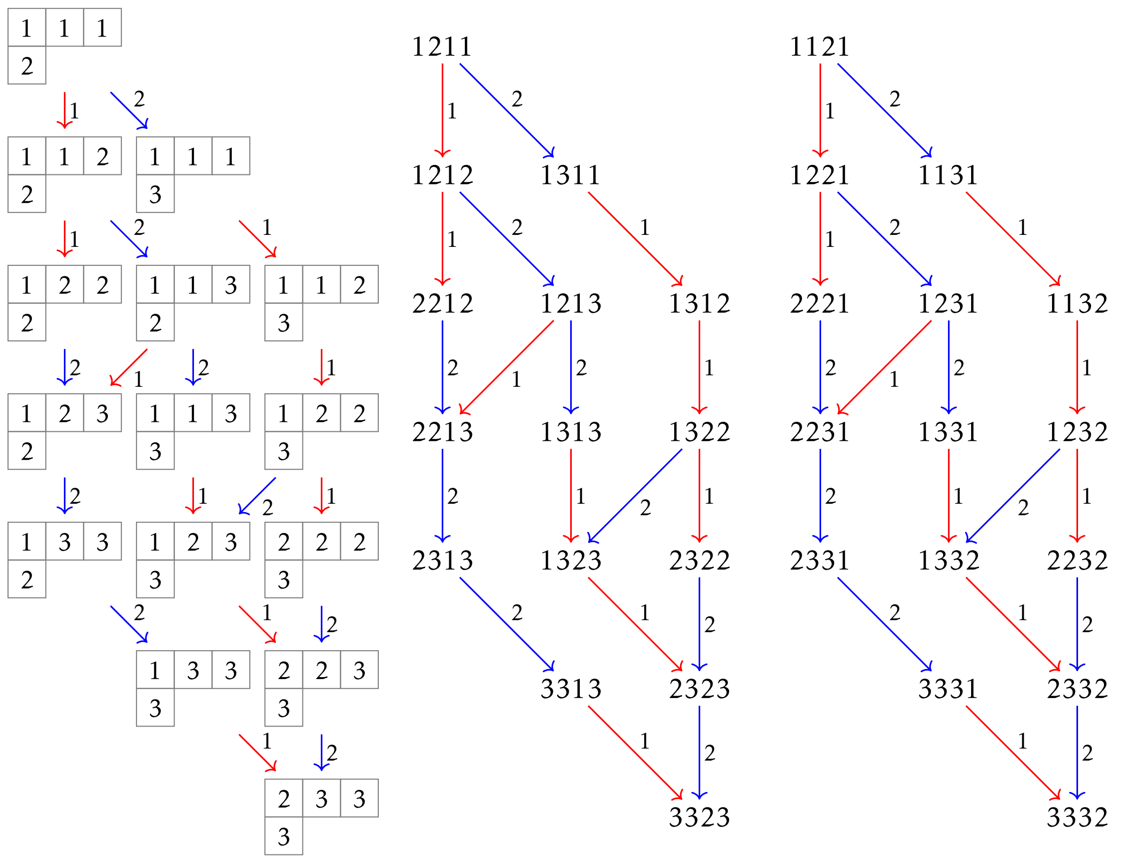


Figure 3.2: Three isomorphic components of the crystal graph for plac_3. In the component containing column readings of tableaux, the tableaux themselves are shown instead of words. (This figure is taken from [3, Fig. 2])

then $\text{wt}(\tilde{e}_i(w)) > \text{wt}(w)$ (respectively, $\text{wt}(\tilde{f}_i(w)) < \text{wt}(w)$). This happens because when we apply the operator $\tilde{e}_i$ to a word, it replaces a letter $i+1$ with the letter i, thus decreasing the $(i+1)$-th component and increasing the i-th component of the weight, which results in an increase with respect to the weight order defined in Subsection 2.2. Similarly, $\tilde{f}_i$ replaces a letter i with a letter $i+1$, whenever defined, thus it decreases weight. Because of this, these operators are also known as the Kashiwara *raising* and *lowering* operators, respectively.

Another important property of the operators $\tilde{e}_i$ and $\tilde{f}_i$ is that they preserve the property of being a tableau word and the shape of the corresponding tableau (see [13, Section 3]). Also, all tableau words corresponding to tableaux of a given shape, with entries in $\mathcal{A}_n$, are located in the same connected component.

Note that, since every connected component in $\Gamma(\text{plac}_n)$ is finite, there is at least a vertex in each component whose weight is higher than all other vertices in that component. In fact, this vertex is unique (see [23] for proofs and background) and is called the *highest-weight* vertex. Note that this means there is no operator $\tilde{e}_i$ defined on this vertex.

Each connected component in $\Gamma(\text{plac}_n)$ corresponds to exactly one standard tableau, in the sense that, if u, v are words in $\mathcal{A}_n^*$, then they are located in the same connected component if and only if their corresponding standard tableaux, $Q(u)$ and $Q(v)$, obtained via the Robinson–Schensted–Knuth correspondence, are equal. Thus, considering a word $w \in \mathcal{A}_n^*$, the Robinson–Schensted–Knuth correspondence $w \mapsto \big(P(w), Q(w)\big)$ allows us to first locate its connected component $\Gamma(\mathrm{plac}_n, w)$, via $Q(w)$, and then locate w in that component via $P(w)$.

An interesting characterization of highest-weight tableau words is the following: a tableau word is highest-weight if and only if its weight is equal to the shape of the corresponding tableau, that is, a tableau word whose corresponding tableau has shape λ is highest-weight if and only if, for each $i \in \mathcal{A}_n$, the number of symbols i it contains is λ_i. Thus, a tableau whose reading is a highest-weight word must contain only symbols i on its ith row, for all $i \in \{1, \ldots, l(\lambda)\}$.

3.4 Column presentation and complete rewriting system for the plactic monoid of rank *n*

In this section, we present the construction of a finite complete rewriting system for the plactic monoid of rank n, and the resulting column presentation, following [4].

Recall that plac_n is presented by $\langle \mathcal{A}_n \mid \mathcal{R}_{\mathrm{plac}} \rangle$, where

$$\mathcal{R}_{\mathrm{plac}} = \Big\{ (acb, cab) \mid a \leq b < c \Big\} \cup \Big\{ (bac, bca) \mid a < b \leq c \Big\}.$$

To construct a finite complete rewriting system presenting plac_n, we introduce a new set of generators. Let

$$\mathcal{C}_n = \Big\{ c_\alpha \mid \alpha \in \mathcal{A}_n^+ \text{ is a column } \Big\}.$$

The idea is that each symbol c_α represents the symbol α of plac_n, hence the symbols $c_1, c_2, \ldots, c_n$ represent the original generating set for plac_n and thus $\mathcal{C}_n$ also generates plac_n. We shall refer to this set as the *column alphabet*. Also notice that, since the set of columns is finite, the set $\mathcal{C}_n$ is finite.

Let α, β be columns such that $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_l$, with $u_1, \ldots, u_k, v_1, \ldots, v_l \in \mathcal{A}$, are their respective column readings. We write $\alpha \succeq \beta$ if and only if $k \geq l$ and $u_i \leq v_i$, for all $i = 1, \ldots, l$. Notice that $\alpha \succeq \beta$ if and only if α can appear immediately to the left of β in the planar representation of a tableau.

Define a set of rewriting rules $\mathcal{S}$ on $\mathcal{C}_n^*$ as follows:

$$\mathcal{S} = \Big\{ c_\alpha c_\beta \to c_\gamma \mid \alpha \not\succeq \beta \wedge P(\alpha\beta) \text{ consists of one column } \gamma \Big\} \cup \\ \cup \Big\{ c_\alpha c_\beta \to c_\gamma c_\delta \mid \alpha \not\succeq \beta \wedge P(\alpha\beta) \text{ consists of two columns, with left column } \gamma \text{ and right column } \delta \Big\}. \quad (3.4.1)$$

In [4], it is proven that $\langle \mathscr{C}_n \mid \mathcal{S} \rangle$ presents plac_n and that $(\mathscr{C}_n \mid \mathcal{S})$ is a finite complete rewriting system. The proof relies on three important tools: the length-plus-lexicographic order, the uniqueness of the tableau obtained from Schensted's algorithm and the following lemma:

Lemma 3.4.1 ([4, Lemma 3.1]). *Suppose α and β are columns with $\alpha \not\succeq \beta$. Then $P(\alpha\beta)$ has at most two columns. Furthermore, if $P(\alpha\beta)$ has exactly two columns, the left column has more symbols than α.*

3.5 Coherent presentation for the plactic monoid of rank *n*

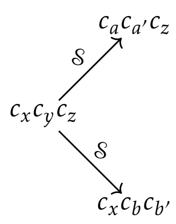
In [10], the homotopical completion procedure was applied to the presentation $\langle \mathscr{C}_n \mid \mathcal{S} \rangle$ in order to obtain a coherent presentation for plac_n. Since $(\mathscr{C}_n \mid \mathcal{S})$ is a finite complete rewriting system, the main contribution of this article was the explicit construction of the confluence diagrams, that is, the diagrams representing the critical pairs of edges and their resolutions.

Theorem 3.5.1 ([10, Theorem 3.2.2.]). *Consider the extended presentation $\langle \mathscr{C}_n \mid \mathcal{S} \mid \mathcal{X} \rangle$, where $\mathscr{C}_n$ is the column alphabet, $\mathcal{S}$ is as defined in (3.4.1) and $\mathcal{X}$ is as defined in (2.4.1), that is, if for any resolvable critical pair (e_1, e_2) of $\mathcal{S}$, we fix a resolution (p_1, p_2), then*

$$\mathcal{X} = \Big\{ (e_1 p_1, e_2 p_2) \mid (e_1, e_2) \text{ is a critical pair of } \mathcal{S}, \text{ and } (p_1, p_2) \text{ is the correspondent resolution} \Big\}.$$

Then, $\langle \mathscr{C}_n \mid \mathcal{S} \mid \mathcal{X} \rangle$ is a coherent presentation for plac_n.

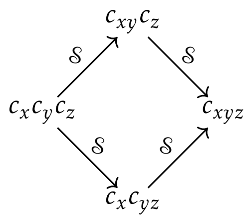
Left-hand side of rules from the presentation $\langle \mathscr{C}_n \mid \mathcal{S} \rangle$ can overlap creating an overlap ambiguity of the form $c_x c_y c_z$, for any columns x, y, z such that $x \not\succeq y$ and $y \not\succeq z$, which can be represented diagrammatically in the form



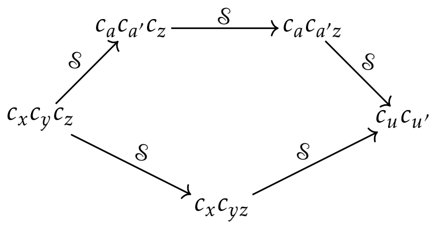
where a, a' denote the two columns of the tableau $P(xy)$ and b, b' denote the two columns of the tableau $P(yz)$. Note that some of these columns may be empty, thus their corresponding symbols in $\mathscr{C}_n$ will be the empty word.

Since, by Lemma 3.4.1, for columns α, β such that $\alpha \not\succeq \beta$, $P(\alpha\beta)$ has, at most, two columns, we have four types of critical pairs of edges. We will use a diagrammatic representation of vertices and edges to represent each of those cases, and obtain what are the so-called *confluence diagrams*:

- ([10, Lemma 3.2.3.]) If $P(xy)$ has only one column and $P(yz)$ also has only one column, then we have the following confluence diagram:

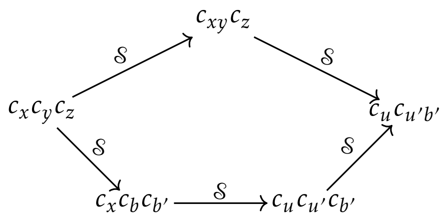


- ([10, Lemma 3.2.3.]) If $P(xy)$ has two columns and $P(yz)$ has only one column, then we have the following confluence diagram:



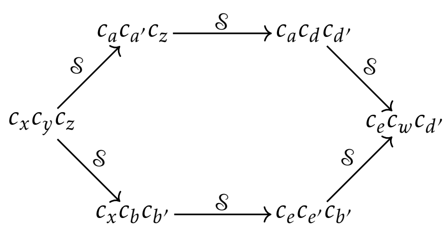
  where a, a' denote the two columns of the tableau $P(xy)$ and u, u' denote the two columns of the tableau $P(xyz)$;

- ([10, Lemma 3.2.3.]) If $P(xy)$ has only one column and $P(yz)$ has two columns, then we have the following confluence diagram:



  where b, b' denote the two columns of the tableau $P(yz)$ and u, u' denote the two columns of the tableau $P(xb)$;

- ([10, Lemma 3.2.3.]) If $P(xy)$ has two columns and $P(yz)$ also has two columns, then we have the following confluence diagram:



  where a, a' denote the two columns of the tableau $P(xy)$, b, b' denote the two columns of the tableau $P(yz)$, d, d' denote the two columns of the tableau $P(a'z)$, e, e' denote the two columns of the tableau $P(xb)$ and e, w, d' denote the three columns of the tableau $P(xyz)$. Note that, in this case, $P(xyz)$ always has three columns.

CHAPTER 4

The hypoplactic monoid

In this chapter, similarly to the previous one, we shall discuss three possible ways to define the *hypoplactic monoid*: via *generators and relations*, *quasi-ribbon tableaux and insertion*, and *quasi-crystals*, and the interaction of the quasi-crystal structure with the combinatorics of quasi-ribbon tableaux (following [3]).

4.1 The hypoplactic monoid, quasi-ribbon tableaux and insertion

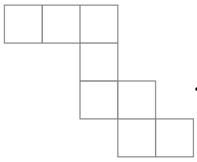
Consider the ordered alphabet $\mathcal{A} = \{1 < 2 < \dots\}$. The *hypoplactic monoid*, denoted by hypo, is presented by $\langle \mathcal{A} \mid \mathcal{R}_{\text{plac}} \cup \mathcal{R}_{\text{hypo}} \rangle$, where $\mathcal{R}_{\text{plac}}$ is the set of the Knuth relations given in Section 3.1 and $\mathcal{R}_{\text{hypo}}$ is the set of relations of the form

$$\begin{aligned} &(cadb, acbd) \quad \text{with } a \le b < c \le d\,; \\ &(dbca, badc) \quad \text{with } a < b \le c < d\,. \end{aligned} \tag{4.1.1}$$

Let $n \in \mathbb{N}$ and consider the finite ordered alphabet $\mathcal{A}_n = \{1 < 2 < \dots < n\}$. The *hypoplactic monoid* of rank n, denoted by hypo_n, is the monoid presented by $\langle \mathcal{A}_n \mid \mathcal{R}_{\text{plac}} \cup \mathcal{R}_{\text{hypo}} \rangle$, where in this case the sets of defining relations $\mathcal{R}_{\text{plac}}$ and $\mathcal{R}_{\text{hypo}}$ are naturally restricted to $\mathcal{A}_n^* \times \mathcal{A}_n^*$.

We now proceed to introduce quasi-ribbon tableaux and related concepts, and then present an alternative definition of the hypoplactic monoid using these tools. For further information, see [15] and [19].

Let α be a composition. A *ribbon diagram* of shape α is an array of cells, with α_h cells in the h-th row, for $h = 1, \dots, l(\alpha)$, and counting rows from top to bottom, aligned so that the leftmost cell in each row is below the rightmost cell of the previous row. For example, the ribbon tableau of shape $(3,1,2,2)$ is:



Notice that a ribbon diagram cannot contain a 2×2 subarray, that is, of the form . Also, in a ribbon diagram of shape α, the number of rows is $l(\alpha)$ and the number of cells is $|\alpha|$.

A *quasi-ribbon tableau* of shape α is a ribbon diagram of shape α filled with symbols from $\mathcal{A}$ such that entries in each row are non-decreasing left to right and entries in each column are strictly increasing from top to bottom. An example of a quasi-ribbon tableau of shape $(3,1,2,2)$ is:

$$\begin{array}{cccc} 1 & 2 & 4 & \\ & & 5 & \\ & & 6 & 6 \\ & & & 7 & 8 \end{array}. \qquad (4.1.2)$$

Note that:

- For each $a \in \mathcal{A}$, the symbols a in a quasi-ribbon tableau all appear in the same row, which must be the j-th for some $j \leq a$;
- The h row of a quasi-ribbon tableau cannot contain symbols from $\{1,\ldots,h-1\}$.

A *quasi-ribbon tabloid* is a ribbon diagram of shape α filled with symbols from $\mathcal{A}$ such that entries in each column are strictly increasing from top to bottom, without any restriction on rows. An example of a quasi-ribbon tabloid of shape $(3,1,2,2)$ is:

$$\begin{array}{cccc} 1 & 4 & 2 & \\ & & 5 & \\ & & 6 & 3 \\ & & & 7 & 8 \end{array} \qquad (4.1.3)$$

Note that a quasi-ribbon tableau is a special kind of quasi-ribbon tabloid.

A *recording ribbon* of shape α is a ribbon diagram of shape α filled with symbols from $\{1,\ldots,|\alpha|\}$, with each symbol appearing exactly once, such that entries in each row are increasing from left to right (the same as in the quasi-ribbon tableau) and entries in each column are decreasing from top to bottom (the opposite of the rule in a quasi-ribbon tableau). An example of a recording ribbon of shape $(3,1,2,2)$ is:

$$\begin{array}{cccc} 1 & 2 & 6 & \\ & & 5 & \\ & & 4 & 7 \\ & & & 3 & 8 \end{array}. \qquad (4.1.4)$$

The *column reading* $C(T)$ of a quasi-ribbon tabloid T is the word in $\mathcal{A}^*$ obtained by reading its columns from left to right, and reading each column from bottom to top. For example, the column reading of (4.1.2) is 1 2 654 76 8 and the column reading of (4.1.3) is 1 4 652 73 8.

Let $w \in \mathcal{A}^*$, and let $w^{(1)} \cdots w^{(k)}$ be its factorization into maximal decreasing factors. Let QRoid(w) be the quasi-ribbon tabloid whose h-th column has height $|w^{(h)}|$ and is filled with the symbols of $w^{(h)}$, for $h = 1, \ldots, k$. Then, $C(\text{QRoid}(w)) = w$. Note that each maximal decreasing factor of w corresponds to a column of QRoid(w). If w is the column reading of a quasi-ribbon tableau T, it is called a *quasi-ribbon word*. By definition, it is immediate that w is a quasi-ribbon word if and only if QRoid(w) is a quasi-ribbon tableau. Also, note that w is a quasi-ribbon word if and only if, for all $i = 1, \ldots, k-1$, the smallest symbol of $w^{(i+1)}$ is greater than or equal to the greatest symbol of $w^{(i)}$.

The following algorithm is an analogue of Schensted's algorithm. It allows us to compute a unique quasi-ribbon tableau $QR(w)$ from a word $w \in \mathcal{A}^*$.

Algorithm 4.1.1 ([15, §7.2])**.**

Input: A quasi-ribbon tableau T and a symbol $a \in \mathcal{A}$.

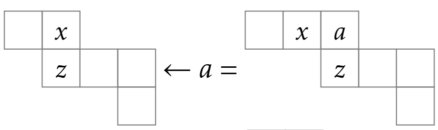
Output: A quasi-ribbon tableau $T \leftarrow a$.

Method:

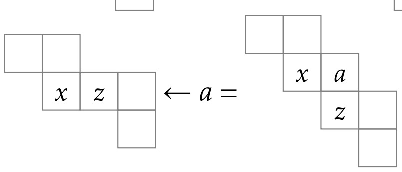
- If there is no entry in T that is less than or equal to a, output the quasi-ribbon tableau obtained by creating a new entry a and attaching (by its top-left-most entry) the quasi-ribbon tableau T to the bottom of a.

- If there is no entry in T that is greater than a, output the word obtained by creating a new entry a and attaching (by its bottom-right-most entry) the quasi-ribbon tableau T to the left of a.

- Otherwise, let x and z be the adjacent entries of the quasi-ribbon tableau T such that $x \leq a < z$. (Equivalently, let x be the right-most and bottom-most entry of T that is less than or equal to a, and let z be the left-most and top-most entry that is greater than a. Note that x and z could be either horizontally or vertically adjacent.) Take the part of T from the top left down to and including x, put a new entry a to the right of x and attach the remaining part of T (from z onwards to the bottom right) to the bottom of the new entry a, as illustrated here:



, where x and z are vertically adjacent;



, where x and z are horizontally adjacent.

Output the resulting quasi-ribbon tableau.

Let $w = w_1 \cdots w_k$ be a word in $\mathcal{A}^*$. By applying the algorithm iteratively, we can compute a unique quasi-ribbon tableau $P(w)$: Starting with the empty word, we iteratively insert the symbols from $\mathcal{A}$, $w_1, \ldots, w_k$ in order. After inserting the last symbol, we obtain the quasi-ribbon tableau $QR(w_1 \cdots w_k)$. This algorithm also allows us to compute a recording ribbon $RR(w)$, in the following way:

Algorithm 4.1.2 ([15, §7.2])**.**

Input: A word $w = w_1 \cdots w_k$, where $w_i \in \mathcal{A}$, for $i = 1, \ldots, k$.

Output: A quasi-ribbon tableau $QR(w)$ and a recording ribbon $RR(w)$.

Method: Start with an empty quasi-ribbon tableau Q_0 and an empty recording ribbon R_0. For each $i = 1, \ldots, k$, insert the symbol w_i into Q_{i-1} as per Algorithm 4.1.2; let Q_i be the resulting quasi-ribbon tableau. Build the recording ribbon R_i, which has the same shape as Q_i, by adding an entry i into R_{i-1} at the same place as w_i was inserted into Q_{i-1}. Output Q_k for $QR(w)$ and R_k for $RR(w)$.

For example, the sequence of pairs (Q_i, R_i) produced during the application of Algorithm 4.1.2 to the word 5231 is:

$$(,), \quad (\boxed{5}, \boxed{1}), \quad \left(\begin{array}{|c|}\hline 2\\\hline 5\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\right), \quad \left(\begin{array}{cc} \boxed{2} & \boxed{3} \\ & \boxed{5}\end{array}, \begin{array}{cc} \boxed{2} & \boxed{3} \\ & \boxed{1}\end{array}\right), \quad \left(\begin{array}{cc} \boxed{1} & \\ \boxed{2} & \boxed{3} \\ & \boxed{5}\end{array}, \begin{array}{cc} \boxed{4} & \\ \boxed{2} & \boxed{3} \\ & \boxed{1}\end{array}\right).$$

Thus $QR(5231) = \begin{array}{cc} \boxed{1} & \\ \boxed{2} & \boxed{3} \\ & \boxed{5}\end{array}$ and $RR(5231) = \begin{array}{cc} \boxed{4} & \\ \boxed{2} & \boxed{3} \\ & \boxed{1}\end{array}$.

Similarly to the plactic case, it is easy to see that the map $w \mapsto \big(QR(w), RR(w)\big)$ is a bijection between words in $\mathcal{A}^*$ and pairs consisting of a quasi-ribbon tableau over $\mathcal{A}$ and a recording ribbon of the same shape; this is an analogue of the Robinson–Schensted–Knuth correspondence. For example, if $QR(u) = \begin{array}{cc} \boxed{2} & \\ \boxed{3} & \boxed{3} \\ & \boxed{4}\end{array}$ and $RR(u) = \begin{array}{cc} \boxed{3} & \\ \boxed{2} & \boxed{4} \\ & \boxed{1}\end{array}$ then $u = 4323$.

Thus, we are now able to present an alternative definition of the hypoplactic monoid in terms of quasi-ribbon tableaux. First, we define the relation $\equiv_{\text{hypo}}$, called the *hypoplactic congruence* on $\mathcal{A}^*$, in the following way: For words $u, v \in \mathcal{A}^*$,

$$u \equiv_{\text{hypo}} v \Leftrightarrow QR(u) = QR(v).$$

This relation is also a congruence on $\mathcal{A}^*$ and it is the smallest congruence containing $\mathcal{R}_{\text{plac}}$ and $\mathcal{R}_{\text{hypo}}$ (see [19, §4]). Thus, the *hypoplactic monoid* is the factor monoid $\mathcal{A}^*/\equiv_{\text{hypo}}$. The congruence $\equiv_{\text{hypo}}$ naturally restricts to a congruence on $\mathcal{A}_n^*$, and so the *hypoplactic monoid* of rank n is the factor monoid $\mathcal{A}_n^*/\equiv_{\text{hypo}}$.

Note that if w is a quasi-ribbon word, then $w = C(QR(w))$ and $\text{QRoid}(w) = QR(w)$. Hence the quasi-ribbon words in $\mathcal{A}^*$ (respectively, $\mathcal{A}_n^*$) form a cross-section for hypo (respectively, hypo_n).

Theorem 4.1.3 ([19, Theorem 5.12]). *The smallest word with respect to the lexicographic order of a non-empty hypoplactic class is its quasi-ribbon word.*

4.2 Quasi-Kashiwara operators and the quasi-crystal graph

In this section, following [3], we will define the quasi-Kashiwara operators and the quasi-crystal graph and present some important results, one of which is that isomorphisms between components of this graph give rise to the hypoplactic monoid.

Let $n \in \mathbb{N}$ and $i \in \{1, \ldots, n-1\}$. For any given word $w \in \mathcal{A}_n^*$, we say w has an *i-inversion* if it contains a symbol $i+1$ to the left of a symbol i. Equivalently, w has an i-inversion if it contains a subsequence $(i+1)i$. If the word w does not have an i-inversion, we say it is *i-inversion-free*.

For each $i \in \{1, \ldots, n-1\}$, define the quasi-Kashiwara operators $\ddot{e}_i$ and $\ddot{f}_i$ on $\mathcal{A}_n^*$ as follows: Let $w \in \mathcal{A}_n^*$.

- If w has an i-inversion, both $\ddot{e}_i(w)$ and $\ddot{f}_i(w)$ are undefined;
- If w is i-inversion-free, but w contains at least one symbol $i+1$, then $\ddot{e}_i(w)$ is the word obtained from w by replacing the left-most symbol $i+1$ by i; if w contains no symbol $i+1$, then $\ddot{e}_i(w)$ is undefined;
- If w is i-inversion-free, but w contains at least one symbol i, then $\ddot{f}_i(w)$ is the word obtained from w by replacing the right-most symbol i by $i+1$; if w contains no symbol i, then $\ddot{f}_i(w)$ is undefined.

Paralleling the plactic case we define

$$\ddot{\varepsilon}_i(w) = \max\Big\{k \in \mathbb{N} \cup \{0\} \mid \underbrace{\ddot{e}_i \cdots \ddot{e}_i}_{k \text{ times}}(w) \text{ is defined}\Big\}$$

and

$$\ddot{\varphi}_i(w) = \max\Big\{k \in \mathbb{N} \cup \{0\} \mid \underbrace{\ddot{f}_i \cdots \ddot{f}_i}_{k \text{ times}}(w) \text{ is defined}\Big\},$$

for any $i \in \{1, \ldots, n-1\}$ and $w \in \mathcal{A}_n^*$. In this case, notice that if w has an i-inversion, then $\ddot{\varepsilon}_i(w) = \ddot{\varphi}_i(w) = 0$, and if w is i-inversion-free, then every symbol i is located to the left of every symbol $i+1$ in w, thus $\ddot{\varepsilon}_i(w) = |w|_{i+1}$ and $\ddot{\varphi}_i(w) = |w|_i$.

It is interesting to note that, if $\ddot{e}_i(w)$ (or $\ddot{f}_i(w)$) is defined, then $\tilde{e}_i(w)$ (respectively, $\tilde{f}_i(w)$) is also defined and $\ddot{e}_i(w) = \tilde{e}_i(w)$ (respectively, $\ddot{f}_i(w) = \tilde{f}_i(w)$) [3, Remark 1].

Lemma 4.2.1 ([3, Lemma 1]). *For all $i \in \{1, \ldots, n-1\}$, the operators $\ddot{e}_i$ and $\ddot{f}_i$ are mutually inverse, that is, for any $w \in \mathcal{A}_n^*$, if $\ddot{e}_i(w)$ is defined, then $w = \ddot{f}_i(\ddot{e}_i(w))$ (and if $\ddot{f}_i(w)$ is defined, then $w = \ddot{e}_i(\ddot{f}_i(w))$).*

The quasi-crystal graph for hypo_n, denoted by $\Gamma(\text{hypo}_n)$ is the labelled directed graph with vertex set $\mathcal{A}_n^*$ and, for all $u, v \in \mathcal{A}_n^*$ and $i \in \{1, \ldots, n-1\}$, an edge from u to v labelled by i if and only if $\ddot{f}_i(u) = v$ (or, equivalently by the previous Lemma, $\ddot{e}_i(v) = u$).

Note that the operators $\ddot{e}_i$ and $\ddot{f}_i$ preserve length. Therefore, since there are finitely many words in $\mathcal{A}_n^*$ of each length, each connected component in the quasi-crystal graph is finite. For any $w \in \mathcal{A}_n^*$, denote the connected component of $\Gamma(\text{hypo}_n)$ that contains the vertex w by $\Gamma(\text{hypo}_n, w)$. A *quasi-crystal isomorphism* between two connected components is a weight-preserving labelled digraph isomorphism.

Define a relation $\sim$ on the free monoid $\mathcal{A}_n^*$ as follows: For words $u, v \in \mathcal{A}_n^*$, $u \sim v$ if and only if there exists a quasi-crystal isomorphism $\theta : \Gamma(\text{hypo}_n, u) \to \Gamma(\text{hypo}_n, v)$ such that $\theta(u) = v$. That is, $u \sim v$ if and only if they appear in the same position in isomorphic connected components of the quasi-crystal graph. In fact, not only is this relation a congruence, it is equal to the hypoplactic congruence $\equiv_{\text{hypo}}$, thus the factor monoid $\mathcal{A}_n^*/\sim$ is actually the *hypoplactic monoid* of rank n (see the full proof in [3]).

4.3 Properties of the quasi-crystal graph

Similarly to the Kashiwara operators, the operators $\ddot{e}_i$ and $\ddot{f}_i$ increase and decrease weight, respectively, whenever they are defined, that is, if $\ddot{e}_i$ (or $\ddot{f}_i$) is defined, then $\text{wt}(\ddot{e}_i(w)) > \text{wt}(w)$ (respectively, $\text{wt}(\ddot{f}_i(w)) < \text{wt}(w)$). Because of this, these operators are also known as the quasi-Kashiwara *raising* and *lowering* operators, respectively.

Note that every vertex of $\Gamma(\text{hypo}_n)$ has at most one incoming and at most one outgoing edge with a given label.

Now we present some results which are relevant in the following chapter:

Proposition 4.3.1 ([3, Proposition 6]). *Let α be a composition.*

- *The set of quasi-ribbon words corresponding to quasi-ribbon tableaux of shape α forms a single connected component of* $\Gamma(\text{hypo}_n)$*;*
- *In this connected component, there is a unique highest-weight word w, which corresponds to the quasi-ribbon tableau of shape α whose jth row consists entirely of symbols j, for $j = 1, \ldots, l(\alpha)$. Furthermore,* $\text{wt}(w) = \alpha$.

Thus, the quasi-Kashiwara preserve shapes of quasi-ribbon tableaux. More generally, we have the following results, the first one a consequence of [3, Proposition 14]:

Proposition 4.3.2. *Let $i \in \{1, \ldots, n-1\}$. Let $w \in \mathcal{A}_n^*$.*

- *If the quasi-Kashiwara operator $\ddot{e}_i$ is defined on w, then $QRoid(\ddot{e}_i(w))$ and $QRoid(w)$ have the same shape;*
- *If the quasi-Kashiwara operator $\ddot{f}_i$ is defined on w, then $QRoid(\ddot{f}_i(w))$ and $QRoid(w)$ have the same shape.*

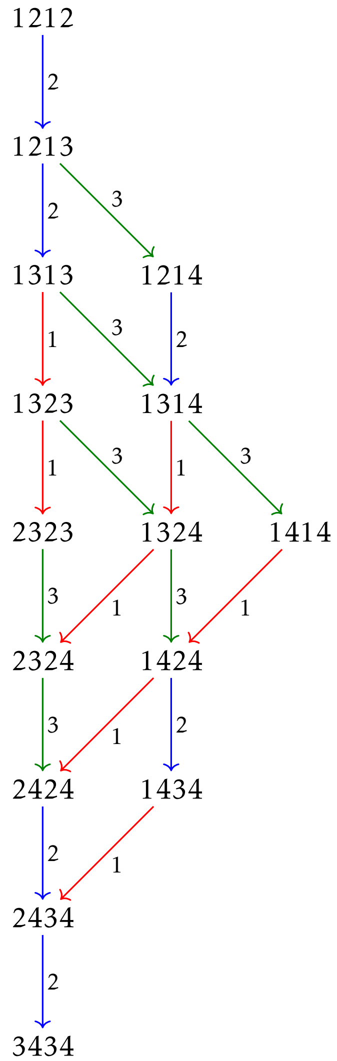


Figure 4.1: The isomorphic components $\Gamma_4(1212)$ and $\Gamma_4(2121)$ of the quasi-crystal graph Γ_4. (This figure is taken from [3, Fig. 3].)

Proposition 4.3.3 ([3, Proposition 9]). *In every connected component in* $\Gamma(\mathrm{hypo}_n)$*, there is a unique highest-weight word.*

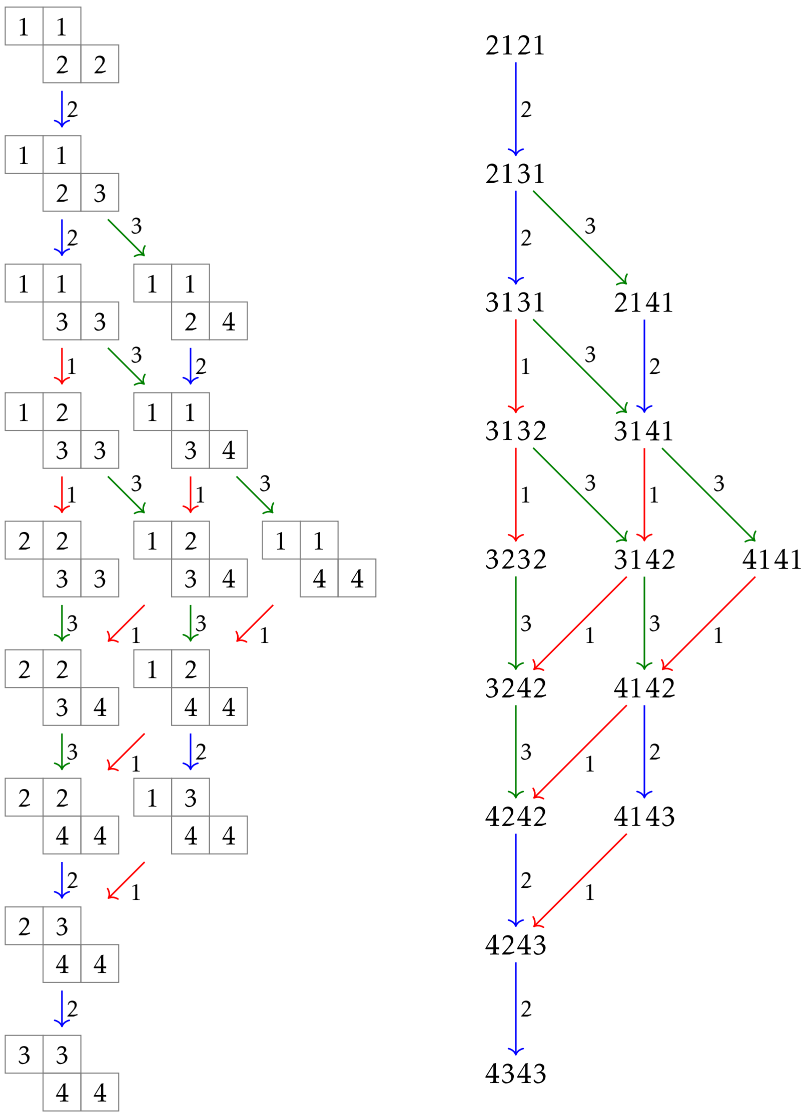


Figure 4.2: The isomorphic components $\Gamma_4(1212)$ (left) and $\Gamma_4(2121)$ (right) of the quasi-crystal graph Γ_4, with symbols of $\Gamma_4(1212)$ drawn as quasi-ribbon tableau instead of written as words. The component $\Gamma_4(1212)$ consists of all quasi-ribbon words whose quasi-ribbon tableaux have shape $(2,2)$. None of the words in $\Gamma_4(2121)$ is a quasi-ribbon word. (This figure is taken from [3, Fig. 4].)

CHAPTER 5

Coherent presentation for the hypoplactic monoid of rank n and characterization of the confluence diagrams

In this section, we present new results and their respective proofs. We first give a finite complete rewriting system $\mathcal{T}'$ for the hypoplactic monoid of rank n, then we introduce the concept of *uniform presentation* and prove that the presentation $\langle \mathcal{A}_n \mid \mathcal{T}' \rangle$ for hypo_n is indeed uniform with respect to the quasi-crystal structure. Then, proceeding as in Section 3.5, we use the homotopical completion procedure to compute a coherent presentation for hypo_n from $\langle \mathcal{A}_n \mid \mathcal{T}' \rangle$, and then we characterize the confluence diagrams. Afterwards, we extend the concept of uniform presentations to extended presentations, introducing the concept of *uniform extended presentations*. Finally, we prove that the coherent presentation for hypo_n that we computed before is uniform with respect to the quasi-crystal structure.

5.1 Column presentation and complete rewriting system for the hypoplactic monoid of rank n

Consider the two following rewriting systems on $\mathcal{A}_n^*$:

$$\mathcal{T} = \Big\{ w \to C\big(QR(w)\big) \mid w \in \mathcal{A}_n^* \wedge w \neq C\big(QR(w)\big) \wedge |w| \leq \max\{2n, 4\} \Big\};$$

$$\mathcal{T}' = \Big\{ w^{(1)}w^{(2)} \to C\big(QR(w^{(1)}w^{(2)})\big) \mid w^{(1)}, w^{(2)} \text{ are columns in } \mathcal{A}_n^* \text{ and } w^{(1)}w^{(2)} \text{ is not a quasi-ribbon word} \Big\}. \quad (5.1.1)$$

Recall that, by definition, hypo_n is presented by $\langle \mathcal{A}_n \mid \mathcal{R}_{\mathrm{plac}} \cup \mathcal{R}_{\mathrm{hypo}} \rangle$. In [5], it was proven not only that $\langle \mathcal{A}_n \mid \mathcal{T} \rangle$ presents hypo_n, but also that $\mathcal{T}$ is a finite complete rewriting system. Unfortunately, the definition of $\mathcal{T}$ is not suited to our needs, so we take inspiration from it and build the rewriting system $\mathcal{T}'$. This new system will serve as our starting point to obtain a coherent presentation for the hypoplactic monoid of rank n.

Proposition 5.1.1. *$(\mathcal{A}_n \mid \mathcal{T}')$ is a finite complete rewriting system presenting* hypo_n.

Proof. First, note that every rule in $\mathcal{T}'$ holds in hypo_n, since the quasi-ribbon words in $\mathcal{A}_n^*$ form a cross-section for hypo_n, therefore, for all $w \in \mathcal{A}_n^*$, $w \equiv_{\mathrm{hypo}_n} C\big(QR(w)\big)$. Thus, every rule in $\mathcal{T}'$ is a consequence of the relations in $\mathcal{R}_{\mathrm{plac}} \cup \mathcal{R}_{\mathrm{hypo}}$.

On the other hand, every relation in $\mathcal{R}_{\mathrm{plac}} \cup \mathcal{R}_{\mathrm{hypo}}$ is a consequence of the rules in $\mathcal{T}'$: Let $a, b, c, d \in \mathcal{A}_n$. Consider the following cases:

- For $a \le b < c$, the words ca and b are columns.

$$QR(cab) = QR(acb) = \begin{array}{|c|c|} \hline a & b \\ \hline \multicolumn{1}{c|}{} & c \\ \cline{2-2} \end{array},$$

hence $C\big(QR(cab)\big) = C\big(QR(acb)\big) = acb$ and $(cab, acb) \in \mathcal{T}'$.

- For $a < b \le c$, the words ca and b are columns.

$$QR(bca) = QR(bac) = \begin{array}{|c|c} \cline{1-1} a & \\ \hline b & \multicolumn{1}{c|}{c} \\ \hline \end{array},$$

hence $C\big(QR(bca)\big) = C\big(QR(bac)\big) = bac$ and $(bca, bac) \in \mathcal{T}'$.

Thus, every rule in $\mathcal{R}_{\mathrm{plac}}$ is a consequence of the relations in $\mathcal{T}'$.

- For $a \le b < c \le d$, the words ca and db are columns.

$$QR(cadb) = QR(acbd) = \begin{array}{|c|c|c} \cline{1-2} a & b & \\ \hline \multicolumn{1}{c|}{} & c & \multicolumn{1}{c|}{d} \\ \cline{2-3} \end{array},$$

hence $C\big(QR(cadb)\big) = C\big(QR(acdb)\big) = acbd$ and $(cadb, acdb) \in \mathcal{T}'$.

- For $a < b \le c < d$, the words ca and db are columns.

$$QR(dbca) = QR(badc) = \begin{array}{|c|c|} \cline{1-1} a & \multicolumn{1}{c}{} \\ \hline b & c \\ \hline \multicolumn{1}{c|}{} & d \\ \cline{2-2} \end{array},$$

hence $C\big(QR(dbca)\big) = C\big(QR(badc)\big) = badc$ and $(dbca, badc) \in \mathcal{T}'$.

Thus, every rule in $\mathcal{R}_{\mathrm{hypo}}$ is a consequence of the relations in $\mathcal{T}'$.

Therefore, since hypo_n is presented by $\langle \mathcal{A}_n \mid \mathcal{R}_{\mathrm{plac}} \cup \mathcal{R}_{\mathrm{hypo}} \rangle$, we conclude that hypo_n is also presented by $\langle \mathcal{A}_n \mid \mathcal{T}' \rangle$.

Note that there are only finitely many rules in $\mathcal{T}'$, since there are finitely many columns in $\mathcal{A}_n^*$ and $C\big(QR(w)\big)$ is uniquely determined.

Let $u, v \in \mathcal{A}_n^*$ and suppose that $(u,v) \in \mathcal{T}'$. Clearly, $u \neq v$ and $|u| = |v|$. By Theorem 4.1.3 we have $v <_{lex} u$ since v is a quasi-ribbon word. Considering the length-plus-lexicographic order as presented in Definition 2.2.7, we deduce that $v \leq_{lenlex} u$. Thus, since the length-plus-lexicographic order is an admissible well-ordering on $\mathcal{A}_n^*$ compatible with $\mathcal{T}'$, we conclude that $(\mathcal{A}_n \mid \mathcal{T}')$ is noetherian by Proposition 2.2.6.

Let $v \in \mathcal{A}_n^*$ be such that v is irreducible. We aim to show that v is a quasi-ribbon word. In order to obtain a contradiction, suppose that $v \neq C\big(QR(v)\big)$. Let $v^{(1)} \cdots v^{(k)}$ be the decomposition of v into maximal decreasing factors. Since $v \neq C\big(QR(v)\big)$, that is, v is not a quasi-ribbon word, there exists $i \in \{1, \ldots, k-1\}$ such that the smallest symbol in $v^{(i+1)}$ is less than the greatest symbol in $v^{(i)}$. Hence, also $v^{(i)}v^{(i+1)}$ is not a quasi ribbon word, that is,

$$v^{(i)}v^{(i+1)} \neq C\Big(QR(v^{(i)}v^{(i+1)})\Big).$$

But $v^{(i)}, v^{(i+1)}$ are columns, therefore $v^{(i)}v^{(i+1)} \rightarrow_{\mathcal{T}'} C\Big(QR(v^{(i)}v^{(i+1)})\Big)$, which implies that

$$v = v^{(1)} \cdots v^{(i)}v^{(i+1)} \cdots v^{(k)} \rightarrow_{\mathcal{T}'} v^{(1)} \cdots C\Big(QR(v^{(i)}v^{(i+1)})\Big) \cdots v^{(k)},$$

which is absurd, since v is irreducible. We have reached a contradiction.

Therefore, the irreducible words for $(\mathcal{A}_n \mid \mathcal{T}')$ are the quasi-ribbon words (note that a quasi-ribbon word is an irreducible word for $(\mathcal{A}_n \mid \mathcal{T}')$). Since the quasi-ribbon words in $\mathcal{A}_n^*$ form a cross-section for hypo_n, we conclude that $(\mathcal{A}_n \mid \mathcal{T}')$ is confluent.

Hence, $(\mathcal{A}_n \mid \mathcal{T}')$ is a finite complete rewriting system presenting hypo_n. □

We say that a presentation $\langle \mathcal{A}_n \mid \mathcal{R} \rangle$ for hypo_n, where $\mathcal{R}$ is a rewriting system on $\mathcal{A}$, is *uniform* with respect to the quasi-crystal structure if, for all defining relations (u,v) in $\mathcal{R}$, we have that:

- If $\ddot{e}_i(u)$ and $\ddot{e}_i(v)$ are both defined, then $\big(\ddot{e}_i(u), \ddot{e}_i(v)\big)$ is a defining relation in $\mathcal{R}$;
- If $\ddot{f}_i(u)$ and $\ddot{f}_i(v)$ are both defined, then $\big(\ddot{f}_i(u), \ddot{f}_i(v)\big)$ is a defining relation in $\mathcal{R}$.

Proposition 5.1.2. *The presentation $\langle \mathcal{A}_n \mid \mathcal{T}' \rangle$ for* hypo_n *is uniform with respect to the quasi-crystal structure.*

Proof. Let g be a quasi-Kashiwara operator. Note that, by Proposition 4.3.1, the quasi-Kashiwara operators preserve the property of being (or not) a quasi-ribbon word. Also note that, by Proposition 4.3.2, the quasi-Kashiwara operators preserve the shapes of quasi-ribbon tabloids, therefore, for $w \in \mathcal{A}_n^*$, if $QRoid(w)$ is made up of two columns, then, if g is defined on w, $QRoid(g(w))$ is also made up of two columns.

Suppose $w \in \mathcal{A}_n^*$ is such that $w = w^{(1)}w^{(2)}$, where $w^{(1)}, w^{(2)}$ are columns in $\mathcal{A}_n^*$ and w is not a quasi-ribbon word. Then, as consequence of the previous statements, if g is defined on w, $g(w)$ is also not a quasi-ribbon word and there are columns $u^{(1)}, u^{(2)}$ in $\mathcal{A}_n^*$ such that $g(w) = u^{(1)}u^{(2)}$. Thus, since w is the left-hand side of a rewriting rule in $\mathcal{T}'$, if g is defined on w, then $g(w)$ is also the left-hand side of a rewriting rule in $\mathcal{T}'$.

Recall that, for any $u, v \in \mathcal{A}_n^*$, we have that $u \equiv_{\text{hypo}} v$ if and only if they appear in the same position in isomorphic connected components of the quasi-crystal graph. Thus, for any word $u \in \mathcal{A}_n^*$, u and $C\big(QR(u)\big)$ appear in the same position in isomorphic connected components of the quasi-crystal graph, therefore, if g is defined on w, then g is defined on $C\big(QR(w)\big)$, hence $g\Big(C\big(QR(w)\big)\Big) = C\Big(QR\big(g(w)\big)\Big)$. In conclusion, if g is defined on w, then $\bigg(g(w), g\Big(C\big(QR(w)\big)\Big)\bigg)$ is a defining relation in $\mathcal{T}'$.

Thus, the presentation $\langle \mathcal{A}_n \mid \mathcal{T}' \rangle$ for hypo_n is *uniform* with respect to the quasi-crystal structure. □

Once again, recall that, for any $u, v \in \mathcal{A}_n^*$, we have that $u \equiv_{\text{hypo}} v$ if and only if they appear in the same position in isomorphic connected components of the quasi-crystal graph. Thus, for a quasi-Kashiwara operator g, g is defined on u if and only if it is defined on v.

Therefore, if a presentation $\mathcal{P} = \langle \mathcal{A}_n \mid \mathcal{R} \rangle$ for hypo_n is uniform with respect to the quasi-crystal structure, then, for any path p on $\Gamma(\mathcal{P})$ such that $p = p_1 \cdots p_k$, where $p_1, \ldots, p_k$ are edges on $\Gamma(\mathcal{P})$, if g is defined on an extremity of p_j, for any $j = 1, \ldots, k$, then g is defined on both extremities of p_j, for all $j = 1, \ldots, k$. Furthermore, the path $p' = p_1' \cdots p_k'$, where p_j' is the edge $\big(g(\iota p_j), g(\tau p_j)\big)$, for all $j = 1, \ldots, k$, is also a path in $\Gamma(\mathcal{P})$;

5.2 Coherent presentation for the hypoplactic monoid of rank n and characterization of the confluence diagrams

The following Theorem is an immediate consequence of Proposition 5.1.1 and the results presented in Section 2.4.

Theorem 5.2.1. *Consider the extended presentation $\langle \mathcal{A}_n \mid \mathcal{T}' \mid \mathcal{X} \rangle$, where $\mathcal{T}'$ is as defined in (5.1.1) and $\mathcal{X}$ is as defined in (2.4.1), that is, if, for any resolvable critical pair (e_1, e_2) of $\mathcal{T}'$, we fix a resolution (p_1, p_2), then*

$$\mathcal{X} = \Big\{(e_1p_1, e_2p_2) \mid (e_1, e_2) \text{ is a critical pair of } \mathcal{T}', \text{ and } (p_1, p_2) \text{ is the corresponding resolution}\Big\}.$$

Then, $\langle \mathcal{A}_n \mid \mathcal{T}' \mid \mathcal{X} \rangle$ is a coherent presentation for hypo_n.

By the definition of the rules in $\mathcal{T}'$, the presentation $\langle \mathcal{A}_n \mid \mathcal{T}' \rangle$ has exactly one kind of critical pair of edges, which can be represented diagrammatically in the form:

$$\begin{array}{ccc} & w^{(1)}w^{(2)_1}w^{(2)_2}w^{(2)_3}w^{(3)} & \\ \swarrow{\scriptstyle \mathcal{T}'} & & \searrow{\scriptstyle \mathcal{T}'} \\ C\Big(QR(w^{(1)}w^{(2)_1}w^{(2)_2})\Big)w^{(2)_3}w^{(3)} & & w^{(1)}w^{(2)_1}C\Big(QR(w^{(2)_2}w^{(2)_3}w^{(3)})\Big) \end{array}$$

for any columns $w^{(1)}, w^{(2)}, w^{(3)}$ in $\mathcal{A}_n^*$ such that $w^{(i)}w^{(i+1)}$ is not a quasi-ribbon word, for $i = 1, 2$, and such that $w^{(2)} = w^{(2)_1}w^{(2)_2}w^{(2)_3}$, with $w^{(2)_1}, w^{(2)_2}, w^{(2)_3} \in \mathcal{A}_n^*$ and $w^{(2)_1}, w^{(2)_3}$ possibly empty.

Since $\langle \mathcal{A}_n \mid \mathcal{T}' \rangle$ is complete, such a critical pair of edges is resolved. Thus, all confluence diagrams will have the following form:

$$\begin{array}{ccc} & w^{(1)}w^{(2)_1}w^{(2)_2}w^{(2)_3}w^{(3)} & \\ \swarrow{\scriptstyle \mathcal{T}'} & & \searrow{\scriptstyle \mathcal{T}'} \\ C\Big(QR(w^{(1)}w^{(2)_1}w^{(2)_2})\Big)w^{(2)_3}w^{(3)} & & w^{(1)}w^{(2)_1}C\Big(QR(w^{(2)_2}w^{(2)_3}w^{(3)})\Big) \\ \downarrow{\scriptstyle \mathcal{T}'} & & \downarrow{\scriptstyle \mathcal{T}'} \\ \vdots & & \vdots \\ \downarrow{\scriptstyle \mathcal{T}'} & & \downarrow{\scriptstyle \mathcal{T}'} \\ C\Big(QR(w^{(1)}w^{(2)})\Big)w^{(3)} & & w^{(1)}C\Big(QR(w^{(2)}w^{(3)})\Big) \\ \downarrow{\scriptstyle \mathcal{T}'} & & \downarrow{\scriptstyle \mathcal{T}'} \\ \vdots & & \vdots \\ \searrow{\scriptstyle \mathcal{T}'} & & \swarrow{\scriptstyle \mathcal{T}'} \\ & C\Big(QR(w^{(1)}w^{(2)}w^{(3)})\Big) & \end{array}$$

We shall prove that:

- For $i = 1, 2$, $QR(w^{(i)}w^{(i+1)})$ will have, at most, $n+1$ columns;
- $QR(w^{(1)}w^{(2)}w^{(3)})$ will have, at most, $2n+1$ columns;
- There exists a path from $C\Big(QR(w^{(1)}w^{(2)_1}w^{(2)_2})\Big)w^{(2)_3}w^{(3)}$ to $C\Big(QR(w^{(1)}w^{(2)})\Big)w^{(3)}$ that has at most $n+1$ edges;
- There exists a path from $w^{(1)}w^{(2)_1}C\Big(QR(w^{(2)_2}w^{(2)_3}w^{(3)})\Big)$ to $w^{(1)}C\Big(QR(w^{(2)}w^{(3)})\Big)$ that has at most $n+1$ edges;
- There exists a path from $C\Big(QR(w^{(1)}w^{(2)})\Big)w^{(3)}$ to $C\Big(QR(w^{(1)}w^{(2)}w^{(3)})\Big)$ that has at most n edges;
- There exists a path from $w^{(1)}C\Big(QR(w^{(2)}w^{(3)})\Big)$ to $C\Big(QR(w^{(1)}w^{(2)}w^{(3)})\Big)$ that has at most n edges.

Note that the length of the path from $C\Big(QR(w^{(1)}w^{(2)})\Big)w^{(3)}$ to $C\Big(QR(w^{(1)}w^{(2)}w^{(3)})\Big)$ may be different from the length of the path from $w^{(1)}C\Big(QR(w^{(2)}w^{(3)})\Big)$ to $C\Big(QR(w^{(1)}w^{(2)}w^{(3)})\Big)$:

For example, if we consider $w^{(1)} = 65432$, $w^{(2)} = 54321$ and $w^{(3)} = 4$, we have

$$\underline{65432}\ \underline{54321}\ 4 \to_{\mathcal{T}'} 21\ 32\ 43\ \underline{54}\ \underline{654} \to_{\mathcal{T}'} 21\ 32\ 43\ 4\ 54\ 65$$

and

$$65432\ \underline{54321}\ \underline{4} \to_{\mathcal{T}'} \underline{65432}\ \underline{4321}\ 54 \to_{\mathcal{T}'} 21\ 32\ 43\ \underline{654}\ \underline{54} \to_{\mathcal{T}'} 21\ 32\ 43\ 4\ 54\ 65.$$

Lemma 5.2.2. *Let $w^{(1)}, w^{(2)}$ be columns in $\mathcal{A}_n^*$ such that $w^{(1)}w^{(2)}$ is not a quasi-ribbon word. Then, $QR(w^{(1)}w^{(2)})$ will have, at most, $n+1$ columns.*

Proof. Consider the application of Algorithm 4.1.2 to compute $QR(w^{(1)}w^{(2)})$. Since $w^{(1)}$ is a column, then $QR(w^{(1)})$ is a quasi-ribbon tableau with a single column. Now, the insertion of symbols from $w^{(2)}$ into $QR(w^{(1)})$ can increase the number of columns by at most one for each inserted symbol (see Algorithm 4.1.1). Since $w^{(2)}$ is a column in $\mathcal{A}_n^*$, it has at most n symbols, and therefore $QR(w^{(1)}w^{(2)})$ has, at most, $n+1$ columns. □

Lemma 5.2.3. *Let $w^{(1)}, w^{(2)}, w^{(3)}$ be columns in $\mathcal{A}_n^*$ such that $w^{(1)}w^{(2)}$ and $w^{(2)}w^{(3)}$ are not quasi-ribbon words. Then, $QR(w^{(1)}w^{(2)}w^{(3)})$ will have at most $2n+1$ columns.*

Proof. The proof follows the reasoning of the proof of the previous lemma. In this case, each of $w^{(2)}$ and $w^{(3)}$ has at most n symbols, and therefore the insertion of the word $w^{(2)}w^{(3)}$ into $QR(w^{(1)})$ will increase the number of columns by, at most $2n$ columns. □

We now present a technical lemma, which will be necessary in order to prove further results.

Lemma 5.2.4. *Let α, β and γ be columns in $\mathcal{A}_n^*$ such that $\beta\gamma$ is a quasi-ribbon word. Consider the factorization of $C\big(QR(\alpha\beta)\big)$ into maximal decreasing factors $\eta^{(1)}, \dots, \eta^{(k)}$. Then $\eta^{(1)} \cdots \eta^{(k-1)} C\big(QR(\eta^{(k)}\gamma)\big)$ is a quasi-ribbon word.*

Proof. Let $\alpha_p, \dots, \alpha_1, \beta_q, \dots, \beta_1 \in \mathcal{A}_n^*$ be such that $\alpha = \alpha_p \cdots \alpha_1$ and $\beta = \beta_q \cdots \beta_1$.

If $\beta_q < \alpha_p$, then $QR(\alpha\beta_q)$ has right-most column with column reading $\alpha_p \cdots \alpha_s \beta_q$, for some $s \leq p$. Attending to Algorithm 4.1.2, since $\beta_i < \beta_q$, for any $1 \leq i < q$, the right-most column of $QR(\alpha\beta)$ will have the form $\alpha_p \cdots \alpha_s \beta_q \cdots \beta_t$, for some $1 \leq t \leq q$, and so $\eta^{(k)} = \alpha_p \cdots \alpha_s \beta_q \cdots \beta_t$.

Now suppose that $\beta_q \geq \alpha_p$. In this case β_q is inserted into $QR(\alpha)$ by attaching β_q by its bottom-most entry. thus $QR(\alpha\beta_q)$ has right-most column β_q. As in the other case, the remaining symbols of β will be inserted either in the right most column above β_q (if they are greater or equal that α_p) or in a column further left. Thus, the right-most column of $QR(\alpha\beta)$ has the form $\beta_q \cdots \beta_t$, for some $1 \leq t \leq q$, and so $\eta^{(k)} = \beta_q \cdots \beta_t$.

Since $\beta\gamma$ is a quasi-ribbon word, every symbol in γ is greater than or equal to β_q. Again by Algorithm 4.1.2, the tableau $QR(\eta^{(k)}\gamma) = QR(\eta^{(k)}) \leftarrow \gamma$ has the symbol β_t as its top-left most symbol. Therefore, $\eta^{(1)} \cdots \eta^{(k-1)} C\big(QR(\eta^{(k)}\gamma)\big)$ is a quasi-ribbon word. □

A symmetrical lemma can be stated, which is proven using the symmetrical version of the insertion algorithm, given in [5, Subsection 4.1].

Lemma 5.2.5. *Let α, β and γ be columns in $\mathcal{A}_n^*$ such that $\alpha\beta$ is a quasi-ribbon word. Consider the factorization of $C\Big(QR(\beta\gamma)\Big)$ into maximal decreasing factors $\eta^{(1)},\ldots,\eta^{(k)}$. Then $C\Big(QR(\alpha\eta^{(1)})\Big)\eta^{(2)}\cdots\eta^{(k)}$ is a quasi-ribbon word.*

Proposition 5.2.6. *Let w be a column and β be a quasi-ribbon word in $\mathcal{A}_n^*$ such that $w\beta$ is not a quasi-ribbon word. Suppose $QR(\beta)$ has r columns. There is a path, of length at most r, in $\Gamma\Big(\langle\mathcal{A}_n \mid \mathcal{T}'\rangle\Big)$ from $w\beta$ to $C\Big(QR(w\beta)\Big)$, where for each edge of these paths the rewriting rule has as left-hand side two of the maximal decreasing factors of the initial vertex.*

Proof. Consider the factorization of β into maximal decreasing factors $\beta^{(1)},\ldots,\beta^{(r)}$ (or equivalently, the column readings of the columns of $QR(\beta)$ from left to right).

Note that the rules in $\mathcal{T}'$ are applied to pairs of columns that do not constitute a quasi-ribbon word. Hence, if a rewriting rule is applied to $w\beta$, it must be applied to (some factor of) the columns w and $\beta^{(1)}$.

Consider the reading $\eta^{(1)}$ of the right-most column of $QR(w\beta^{(1)})$. In this way, we have $C\Big(QR(w\beta^{(1)})\Big) = \eta_1\eta^{(1)}$, for some word $\eta_1 \in \mathcal{A}_n^*$. Thus, if $w\beta^{(1)}$ is not a quasi-ribbon word, we have $(w\beta^{(1)}, \eta_1\eta^{(1)}) \in \mathcal{T}'$ and a rewriting rule can be applied to $w\beta = w\beta^{(1)}\cdots\beta^{(r)}$ and we get

$$w\beta^{(1)}\cdots\beta^{(r)} \to_{\mathcal{T}'} \eta_1\eta^{(1)}\beta^{(2)}\cdots\beta^{(r)}.$$

If $r = 1$ or $\eta^{(1)}\beta^{(2)}$ is a quasi-ribbon word, then also $\eta_1\eta^{(1)}\beta^{(2)}\cdots\beta^{(r)}$ is a quasi-ribbon word, and the result holds.

Otherwise, $\eta^{(1)}\beta^{(2)}$ is not a quasi-ribbon word, and a rewriting rule can be applied to $\eta_1\eta^{(1)}\beta^{(2)}\cdots\beta^{(r)}$. Let $\eta^{(2)}$ be the reading of the right-most column of $QR(\eta^{(1)}\beta^{(2)})$ and η_2 be such that $C\Big(QR(\eta^{(1)}\beta^{(2)})\Big) = \eta_2\eta^{(2)}$. We obtain the single-step reduction

$$\eta_1\eta^{(1)}\beta^{(2)}\cdots\beta^{(r)} \to_{\mathcal{T}'} \eta_1\eta_2\eta^{(2)}\beta^{(3)}\cdots\beta^{(r)}.$$

By Lemma 5.2.4, $\eta_1\eta_2\eta^{(2)}$ is a quasi-ribbon word. If $r = 2$ or $\eta^{(2)}\beta^{(3)}$ is a quasi-ribbon word, then also $\eta_1\eta_2\eta^{(2)}\beta^{(3)}\cdots\beta^{(r)}$ is a quasi-ribbon word, and the result holds.

Suppose that $\eta^{(2)}\beta^{(3)}$ is not a quasi-ribbon word. A reasoning similar to the one presented in the previous paragraph can be applied: We have $C\Big(QR(\eta^{(2)}\beta^{(3)})\Big) = \eta_3\eta^{(3)}$, with $\eta^{(3)}$ a column, and

$$\eta_1\eta_2\eta^{(2)}\beta^{(3)}\cdots\beta^{(r)} \to_{\mathcal{T}'} \eta_1\eta_2\eta_3\eta^{(3)}\beta^{(4)}\cdots\beta^{(r)}.$$

By Lemma 5.2.4, $\eta_2\eta_3\eta^{(3)}$ is a quasi-ribbon word. Note that since $\eta_1\eta_2\eta^{(2)}$ is a quasi-ribbon word, then also $\eta_1\eta_2\eta_3\eta^{(3)}$ is a quasi-ribbon word.

Proceeding in this way, we will obtain a sequence of reductions as follows:

$$\begin{array}{rcl} w^{(1)}C\big(QR(w^{(2)}w^{(3)})\big) & = & w^{(1)}\beta^{(1)}\cdots\beta^{(r)} \\ & \to_{\mathcal{T}'} & \eta_1\eta^{(1)}\beta^{(2)}\cdots\beta^{(r)} \\ & \to_{\mathcal{T}'} & \eta_1\eta_2\eta^{(2)}\beta^{(3)}\cdots\beta^{(r)} \\ & \to_{\mathcal{T}'} & \eta_1\eta_2\eta_3\eta^{(3)}\beta^{(4)}\cdots\beta^{(r)} \\ & \vdots & \vdots \\ & \to_{\mathcal{T}'} & \eta_1\cdots\eta_i\eta^{(i)}\beta^{(i+1)}\cdots\beta^{(r)} \\ & \vdots & \vdots \\ & \to_{\mathcal{T}'} & C\big(QR(w^{(1)}w^{(2)}w^{(3)})\big). \end{array}$$

This process will stop if i reaches r or if $\eta^{(i)}\beta^{(i+1)}$ is a quasi-ribbon word. As a consequence of Lemma 5.2.4 we deduce that $\eta_{k-1}\eta_k\eta^{(k)}$ is a quasi-ribbon word, for $k \leq i$, and so that $\eta_1\cdots\eta_i\eta^{(i)}$ is a quasi-ribbon word. Once the process stops we have quasi-ribbon words $\eta_1\cdots\eta_i\eta^{(i)}$, $\eta^{(i)}\beta^{(i+1)}$ and $\beta^{(i+1)}\cdots\beta^{(r)}$. Thus $\eta_1\cdots\eta_i\eta^{(i)}\beta^{(i+1)}\cdots\beta^{(r)}$ is a quasi-ribbon word, which must be equal to $C\big(QR(w\beta)\big)$. Thus, the length of the path from $w\beta$ to $C\big(QR(w\beta)\big)$ is i, which is at most r, since $i \leq r$.

□

A symmetrical proposition can be stated, which is proven using Lemma 5.2.5.

Proposition 5.2.7. *Let w be a column and β be a quasi-ribbon word in $\mathcal{A}_n^*$ such that βw is not a quasi-ribbon word. Suppose $QR(\beta)$ has r columns. There is a path, of length at most r, in $\Gamma\big(\langle \mathcal{A}_n \mid \mathcal{T}'\rangle\big)$ from βw to $C\big(QR(\beta w)\big)$, where for each edge of these paths the rewriting rule has as left-hand side two of the maximal decreasing factors of the initial vertex.*

The following corollary is immediate from Propositions 5.2.6 and 5.2.7

Corollary 5.2.8. *Let $w^{(1)}, w^{(2)}, w^{(3)}$ be columns in $\mathcal{A}_n^*$ such that $w^{(1)}w^{(2)}$ and $w^{(2)}w^{(3)}$ are not quasi-ribbon words and such that $w^{(2)} = w^{(2)_1}w^{(2)_2}w^{(2)_3}$, with $w^{(2)_1}, w^{(2)_2}, w^{(2)_3} \in \mathcal{A}_n^*$ and $w^{(2)_1}, w^{(2)_3}$ possibly empty. There is a path, of length at most $n+1$, in $\Gamma\big(\langle \mathcal{A}_n \mid \mathcal{T}'\rangle\big)$*

1. *from $C\big(QR(w^{(1)}w^{(2)_1}w^{(2)_2})\big)w^{(2)_3}w^{(3)}$ to $C\big(QR(w^{(1)}w^{(2)})\big)w^{(3)}$;*
2. *from $w^{(1)}w^{(2)_1}C\big(QR(w^{(2)_2}w^{(2)_3}w^{(3)})\big)$ to $w^{(1)}C\big(QR(w^{(2)}w^{(3)})\big)$,*

where for each edge of these paths the rewriting rule has as left-hand side two of the maximal decreasing factors of the initial vertex.

The following proposition gives us an improvement over the boundaries established in Propositions 5.2.6 and 5.2.7.

Proposition 5.2.9. *Let $w^{(1)}, w^{(2)}, w^{(3)}$ be columns in $\mathcal{A}_n^*$ such that $w^{(1)}w^{(2)}$ and $w^{(2)}w^{(3)}$ are not quasi-ribbon words. There is a path, of length at most n, in $\Gamma\big(\langle \mathcal{A}_n \mid \mathcal{T}'\rangle\big)$*

1. *from $w^{(1)}C\big(QR(w^{(2)}w^{(3)})\big)$ to $C\big(QR(w^{(1)}w^{(2)}w^{(3)})\big)$;*

2. *from* $C\Big(QR(w^{(1)}w^{(2)})\Big)w^{(3)}$ *to* $C\Big(QR(w^{(1)}w^{(2)}w^{(3)})\Big)$,

where for each edge of these paths the rewriting rule has as left-hand side two of the maximal decreasing factors of the initial vertex.

Proof. We shall only present the proof of the first case, since the proof of the second case is analogous due to Proposition 5.2.7, the symmetrical version of Proposition 5.2.6.

Let us consider the factorization of $C\Big(QR(w^{(2)}w^{(3)})\Big)$ into maximal decreasing factors $\beta^{(1)},\ldots,\beta^{(r)}$ (or equivalently, the column readings of the columns of $QR(C\Big(QR(w^{(2)}w^{(3)})\Big))$ from left to right). Note that, by Lemma 5.2.2, we have $r \leq n+1$. Thus, by the proof of Proposition 5.2.6, there is a path in $\Gamma\Big(\langle \mathcal{A}_n \mid \mathcal{T}' \rangle\Big)$, of length i, for $i \leq r \leq n+1$, from $w^{(1)}C\Big(QR(w^{(2)}w^{(3)})\Big)$ to $C\Big(QR(w^{(1)}w^{(2)}w^{(3)})\Big)$, of the form

$$\begin{array}{rcc} w^{(1)}C\Big(QR(w^{(2)}w^{(3)})\Big) & = & w^{(1)}\beta^{(1)}\cdots\beta^{(r)} \\ & \to_{\mathcal{T}'} & \eta_1\eta^{(1)}\beta^{(2)}\cdots\beta^{(r)} \\ & \to_{\mathcal{T}'} & \eta_1\eta_2\eta^{(2)}\beta^{(3)}\cdots\beta^{(r)} \\ & \to_{\mathcal{T}'} & \eta_1\eta_2\eta_3\eta^{(3)}\beta^{(4)}\cdots\beta^{(r)} \\ & \vdots & \vdots \\ & \to_{\mathcal{T}'} & \eta_1\cdots\eta_i\eta^{(i)}\beta^{(i+1)}\cdots\beta^{(r)} \\ & \vdots & \vdots \\ & \to_{\mathcal{T}'} & \eta_1\cdots\eta_{r-1}\eta^{(r-1)}\beta^{(r)} \\ & \to_{\mathcal{T}'} & C\Big(QR(w^{(1)}w^{(2)}w^{(3)})\Big). \end{array}$$

In order to obtain a contradiction, suppose that $i = n+1$, that is,

$$\begin{array}{rcc} w^{(1)}C\Big(QR(w^{(2)}w^{(3)})\Big) & = & w^{(1)}\beta^{(1)}\cdots\beta^{(r)} \\ & \to_{\mathcal{T}'}^{n+1} & C\Big(QR(w^{(1)}w^{(2)}w^{(3)})\Big). \end{array}$$

Therefore, we have $r = n+1$. Hence there is at least one symbol n in $\beta^{(1)}\cdots\beta^{(r)}$, otherwise we would have $w^{(2)}, w^{(3)} \in \mathcal{A}_{n-1}^*$, which implies, by Lemma 5.2.2, that $C\Big(QR(w^{(2)}w^{(3)})\Big)$ would have at most n columns, thus $r \leq n$. Let $\beta_q,\ldots,\beta_1 \in \mathcal{A}_n^*$ be such that $\beta^{(n+1)} = \beta_q\cdots\beta_1$. Thus, since n is the greatest symbol of $\mathcal{A}_n^*$, we have $\beta_q = n$.

Once again, recall that the rules in $\mathcal{T}'$ are applied to pairs of columns that do not constitute a quasi-ribbon word. Notice that, for any quasi-ribbon word $u \in \mathcal{A}_n^*$, the word $u\,n$ is still a quasi-ribbon word. Therefore, if $\beta^{(n+1)} = \beta_q\cdots\beta_1$ is to be the right-hand side of a rule in $\mathcal{T}'$, q must be greater than 1.

Then, since all symbols n must appear in the same row of a quasi-ribbon tableau, and $\beta^{(n+1)}$ has at least two symbols, with $\beta_q = n$, we conclude that $\beta^{(1)}\cdots\beta^{(n+1)}$ has one and only one symbol n, which occurs in $\beta^{(n+1)}$.

Let $\alpha^{(n+1)} = \beta_{q-1}\cdots\beta_1$. Notice that, since $\beta = \beta^{(1)}\cdots\beta^{(n+1)}$ is a quasi-ribbon word, $\beta^{(1)}\cdots\beta^{(n)}\alpha^{(n+1)}$ is also a quasi-ribbon word. Also notice that, by definition of $\alpha^{(n+1)}$, n does not occur in $\beta^{(1)}\cdots\beta^{(n)}\alpha^{(n+1)}$, thus it is a quasi-ribbon word in $\mathcal{A}_{n-1}^*$. Hence, by

Lemma 5.2.2, it has at most n columns. But $\beta^{(1)} \cdots \beta^{(n)} \alpha^{(n+1)}$ has the same number of columns as β, which has $n+1$ columns. Thus, we have reached a contradiction.

Hence, we conclude that $i \leq n$, hence the length of the path from $w^{(1)} C\Big(QR(w^{(2)} w^{(3)})\Big)$ to $C\Big(QR(w^{(1)} w^{(2)} w^{(3)})\Big)$ is at most n. □

Now we extend the definition of uniform presentations to extended presentations. Consider an extended presentation $\langle \mathscr{P} \mid \mathscr{C} \rangle$ for hypo$_n$, where $\mathscr{P} = \langle \mathscr{A}_n \mid \mathscr{R} \rangle$ is a uniform presentation for hypo$_n$. We say that $\langle \mathscr{P} \mid \mathscr{C} \rangle$ is a *uniform extended presentation* with respect to the quasi-crystal structure if the following conditions are verified: Let (p, q) be a pair of paths in $\mathscr{C}$ such that $p = p_1 \cdots p_r$ and $q = q_1 \cdots q_s$, where $p_1, \ldots, p_r, q_1, \ldots, q_s$ are edges in $\Gamma(\mathscr{P})$. Then,

- If $\ddot{e}_i$ is defined on an extremity of p_j or q_l, for any $j = 1, \ldots, r$ or $l = 1, \ldots, s$, then (p', q') is a pair of paths in $\mathscr{C}$, where $p' = p'_1 \cdots p'_r$ is such that p'_j is the edge $\Big(\ddot{e}_i(\iota p_j), \ddot{e}_i(\tau p_j)\Big)$, for all $j = 1, \ldots, r$, and $q' = q'_1 \cdots q'_s$ is such that q'_l is the edge $\Big(\ddot{e}_i(\iota q_l), \ddot{e}_i(\tau q_l)\Big)$, for all $l = 1, \ldots, s$;
- If $\ddot{f}_i$ is defined on an extremity of p_j or q_l, for any $j = 1, \ldots, r$ or $l = 1, \ldots, s$, then (p', q') is a pair of paths in $\mathscr{C}$, where $p' = p'_1 \cdots p'_r$ is such that p'_j is the edge $\Big(\ddot{f}_i(\iota p_j), \ddot{f}_i(\tau p_j)\Big)$, for all $j = 1, \ldots, r$, and $q' = q'_1 \cdots q'_s$ is such that q'_l is the edge $\Big(\ddot{f}_i(\iota q_l), \ddot{f}_i(\tau q_l)\Big)$, for all $l = 1, \ldots, s$.

Lemma 5.2.10. *Let α, β be columns in $\mathscr{A}_n^*$ such that*

$$\alpha = \alpha_k^1 \cdots \alpha_1^1 \alpha_k^2 \cdots \alpha_1^2 \text{ and } \beta = \beta_r^1 \cdots \beta_1^1 \beta_r^2 \cdots \beta_1^2,$$

where $\alpha_k^1, \ldots, \alpha_1^1, \alpha_k^2, \ldots \alpha_1^2, \beta_r^1, \ldots, \beta_1^1, \beta_r^2, \ldots \beta_1^2 \in \mathscr{A}_n^*$ are such that $r \neq 0$ in β_r^2, $k \neq 0$ in α_k^1 and $\beta_1^1 < \alpha_k^2$. Then, the following words are not quasi-ribbon words:

- $\alpha_k^1 \cdots \alpha_1^1 C\Big(QR(\alpha_k^2 \cdots \alpha_1^2 \beta)\Big)$;
- $C\Big(QR(\alpha \beta_r^1 \cdots \beta_1^1)\Big) \beta_r^2 \cdots \beta_1^2$;
- $\alpha_k^1 \cdots \alpha_1^1 C\Big(QR(\alpha_k^2 \cdots \alpha_1^2 \beta_r^1 \cdots \beta_1^1)\Big) \beta_r^2 \cdots \beta_1^2$.

Proof. We will only prove that $\alpha_k^1 \cdots \alpha_1^1 C\Big(QR(\alpha_k^2 \cdots \alpha_1^2 \beta)\Big)$ is not a quasi-ribbon word. The proof of the other cases is analogous.

Suppose the bottom-most element of the left-most column of $QR(\alpha_k^2 \cdots \alpha_1^2 \beta)$ is greater than or equal to α_1^1. Therefore, we have that $\alpha_k^1 \cdots \alpha_1^1$ and the left-most column of $QR(\alpha_k^2 \cdots \alpha_1^2 \beta)$ do not form a column. In this case, since $\beta_1^1 < \alpha_k^2$, by the insertion algorithm, the top-most element of the left-most column of $QR(\alpha_k^2 \cdots \alpha_1^2 \beta)$ is less than α_k^1, thus $\alpha_k^1 \cdots \alpha_1^1 C\Big(QR(\alpha_k^2 \cdots \alpha_1^2 \beta)\Big)$ is not a quasi-ribbon word.

Suppose the bottom-most element of the left-most column of $QR(\alpha_k^2 \cdots \alpha_1^2 \beta)$ is less than α_1^1. Then, $\alpha_k^1 \cdots \alpha_1^1$ and the left-most column of $QR(\alpha_k^2 \cdots \alpha_1^2 \beta)$ form a column. Again, since $\beta_1^1 < \alpha_k^2$, by the insertion algorithm, the top-most element of the second column of $QR(\alpha_k^2 \cdots \alpha_1^2 \beta)$ is less than α_k^1, thus $\alpha_k^1 \cdots \alpha_1^1 C\Big(QR(\alpha_k^2 \cdots \alpha_1^2 \beta)\Big)$ is not a quasi-ribbon word. □

Proposition 5.2.11. *The coherent presentation for* hypo$_n$*,* $\langle \mathcal{A}_n \mid \mathcal{T}' \mid \mathcal{X} \rangle$*, given in Theorem 5.2.1 where for* $\mathcal{X}$ *the resolution paths are as described in Proposition 5.2.9, is a uniform extended presentation with respect to the quasi-crystal structure.*

Proof. Note that the underlying monoid presentation of $\langle \mathcal{A}_n \mid \mathcal{T}' \mid \mathcal{X} \rangle$ is the presentation $\langle \mathcal{A}_n \mid \mathcal{T}' \rangle$ for hypo$_n$, which we have proven to be a uniform presentation with respect to the quasi-crystal structure in Proposition 5.1.2.

For all critical pairs (e_1, e_2) of $\mathcal{T}'$, fix a resolution (p_1, p_2) as described in Proposition 5.2.9. Recall that

$$\mathcal{X} = \Big\{(e_1 p_1, e_2 p_2) \mid (e_1, e_2) \textit{ is a critical pair of } \mathcal{T}', \textit{ and } (p_1, p_2) \textit{ is the correspondent resolution} \Big\}.$$

Recall that the critical pairs of $\mathcal{T}'$ are of the form

$$\bigg(\Big(w^{(1)} w^{(2)_1} w^{(2)_2} w^{(2)_3} w^{(3)}\ ,\ C\Big(QR(w^{(1)} w^{(2)_1} w^{(2)_2})\Big) w^{(2)_3} w^{(3)}\Big),$$
$$\Big(w^{(1)} w^{(2)_1} w^{(2)_2} w^{(2)_3} w^{(3)}\ ,\ w^{(1)} w^{(2)_1} C\Big(QR(w^{(2)_2} w^{(2)_3} w^{(3)})\Big)\Big)\bigg),$$

where $w^{(1)}, w^{(2)}, w^{(3)}$ are columns in $\mathcal{A}_n^*$ such that $w^{(1)} w^{(2)}$ and $w^{(2)} w^{(3)}$ are not quasi-ribbon words and such that $w^{(2)} = w^{(2)_1} w^{(2)_2} w^{(2)_3}$, with $w^{(2)_1}, w^{(2)_2}, w^{(2)_3} \in \mathcal{A}_n^*$, and $w^{(2)_1}, w^{(2)_3}$ possibly empty. The underlying rewriting rules of the critical pairs are

$$\Big(w^{(1)} w^{(2)_1} w^{(2)_2}\ ,\ C\Big(QR(w^{(1)} w^{(2)_1} w^{(2)_2})\Big)\Big) \text{ and } \Big(w^{(2)_2} w^{(2)_3} w^{(3)}\ ,\ C\Big(QR(w^{(2)_2} w^{(2)_3} w^{(3)})\Big)\Big).$$

Let g be a a quasi-Kashiwara operator. First, we need to prove that, if g is defined on $w^{(1)} w^{(2)} w^{(3)}$, then

$$\bigg(\Big(g\Big(w^{(1)} w^{(2)_1} w^{(2)_2} w^{(2)_3} w^{(3)}\Big), g\Big(C\Big(QR(w^{(1)} w^{(2)_1} w^{(2)_2})\Big) w^{(2)_3} w^{(3)}\Big)\Big),$$
$$\Big(g\Big(w^{(1)} w^{(2)_1} w^{(2)_2} w^{(2)_3} w^{(3)}\Big), g\Big(w^{(1)} w^{(2)_1} C\Big(QR(w^{(2)_2} w^{(2)_3} w^{(3)})\Big)\Big)\Big)\bigg),$$

is also a critical pair of $\mathcal{T}'$.

We need to further consider three possible cases:

- $g(w^{(1)} w^{(2)_1} w^{(2)_2} w^{(2)_3} w^{(3)}) = g(w^{(1)} w^{(2)_1}) w^{(2)_2} w^{(2)_3} w^{(3)}$;
- $g(w^{(1)} w^{(2)_1} w^{(2)_2} w^{(2)_3} w^{(3)}) = w^{(1)} w^{(2)_1} g(w^{(2)_2}) w^{(2)_3} w^{(3)}$;
- $g(w^{(1)} w^{(2)_1} w^{(2)_2} w^{(2)_3} w^{(3)}) = w^{(1)} w^{(2)_1} w^{(2)_2} g(w^{(2)_3} w^{(3)})$.

In the first case, we have, by the definition of the quasi-Kashiwara operators, that

$$g(w^{(1)}w^{(2)_1}w^{(2)_2}w^{(2)_3}w^{(3)}) = g(w^{(1)}w^{(2)_1}w^{(2)_2})w^{(2)_3}w^{(3)} = g(w^{(1)}w^{(2)_1})w^{(2)_2}w^{(2)_3}w^{(3)},$$

thus $g(w^{(1)}w^{(2)_1}w^{(2)_2})$ is defined. Then, since $\langle \mathcal{A}_n \mid \mathcal{T}' \rangle$ is a uniform presentation for hypo$_n$ with respect to the quasi-crystal structure and $\Big(w^{(1)}w^{(2)_1}w^{(2)_2}\,,\, C\Big(QR(w^{(1)}w^{(2)_1}w^{(2)_2})\Big)\Big)$ is a defining relation in $\mathcal{T}'$, $\Big(g\Big(w^{(1)}w^{(2)_1}w^{(2)_2}\Big)\,,\, g\Big(C\Big(QR(w^{(1)}w^{(2)_1}w^{(2)_2})\Big)\Big)\Big)$ is not only defined, but is also a defining relation in $\mathcal{T}'$. Thus,

$$\Big(g\Big(w^{(1)}w^{(2)_1}w^{(2)_2}w^{(2)_3}w^{(3)}\Big)\,,\, g\Big(C\Big(QR(w^{(1)}w^{(2)_1}w^{(2)_2})\Big)w^{(2)_3}w^{(3)}\Big)\Big) =$$
$$= \Big(g\Big(w^{(1)}w^{(2)_1}\Big)w^{(2)_2}w^{(2)_3}w^{(3)}\,,\, g\Big(C\Big(QR(w^{(1)}w^{(2)_1}w^{(2)_2})\Big)\Big)w^{(2)_3}w^{(3)}\Big)$$

and

$$\Big(g\Big(w^{(1)}w^{(2)_1}w^{(2)_2}w^{(2)_3}w^{(3)}\Big)\,,\, g\Big(w^{(1)}w^{(2)_1}C\Big(QR(w^{(2)_2}w^{(2)_3}w^{(3)})\Big)\Big)\Big) =$$
$$= \Big(g\Big(w^{(1)}w^{(2)_1}\Big)w^{(2)_2}w^{(2)_3}w^{(3)}\,,\, g\Big(w^{(1)}w^{(2)_1}\Big)C\Big(QR(w^{(2)_2}w^{(2)_3}w^{(3)})\Big)\Big).$$

Note that $g(w^{(1)}w^{(2)_1}) = g(w^{(1)})w^{(2)_1}$ or $g(w^{(1)}w^{(2)_1}) = w^{(1)}g(w^{(2)_1})$. Since the quasi-Kashiwara operators maintain the shape of columns (see Proposition 4.3.2), we have that $g(w^{(1)})$ $\Big($or $g(w^{(2)_1})$, whichever is defined $\Big)$ is still a column. Note that, if we factorize $C\Big(QR(w^{(2)_2}w^{(2)_3}w^{(3)})\Big)$ into column words of maximal length, the last one must be different from $w^{(3)}$, otherwise, $w^{(2)_2}w^{(2)_3}w^{(3)}$ would be a quasi-ribbon word (Notice that $w^{(2)_2}w^{(2)_3}$ is also a column). Thus,

$$g\Big(C\Big(QR(w^{(1)}w^{(2)_1}w^{(2)_2})\Big)\Big)w^{(2)_3}w^{(3)} \neq g\Big(w^{(1)}w^{(2)_1}\Big)C\Big(QR(w^{(2)_2}w^{(2)_3}w^{(3)})\Big),$$

therefore,

$$\Big(\Big(g\Big(w^{(1)}w^{(2)_1}\Big)w^{(2)_2}w^{(2)_3}w^{(3)}\,,\, g\Big(C\Big(QR(w^{(1)}w^{(2)_1}w^{(2)_2})\Big)\Big)w^{(2)_3}w^{(3)}\Big),$$
$$\Big(g\Big(w^{(1)}w^{(2)_1}\Big)w^{(2)_2}w^{(2)_3}w^{(3)}\,,\, g\Big(w^{(1)}w^{(2)_1}\Big)C\Big(QR(w^{(2)_2}w^{(2)_3}w^{(3)})\Big)\Big)\Big).$$

is a critical pair of $\mathcal{T}'$.

The third case is analogous to the first, so we will now look at the second case. We have , by the definition of the quasi-Kashiwara operators, that

$$g(w^{(1)}w^{(2)_1}w^{(2)_2}w^{(2)_3}w^{(3)}) = g(w^{(1)}w^{(2)_1}w^{(2)_2})w^{(2)_3}w^{(3)} =$$
$$= w^{(1)}w^{(2)_1}g(w^{(2)_2}w^{(2)_3}w^{(3)}) = w^{(1)}w^{(2)_1}g(w^{(2)}_2)w^{(2)_3}w^{(3)},$$

thus $g(w^{(1)}w^{(2)_1}w^{(2)_2})$ and $g(w^{(2)_2}w^{(2)_3}w^{(3)})$ are defined. Then, since $\langle \mathcal{A}_n \mid \mathcal{T}' \rangle$ is a uniform presentation for hypo_n with respect to the quasi-crystal structure and both

$$\Big(w^{(1)}w^{(2)_1}w^{(2)_2}\,,\; C\Big(QR(w^{(1)}w^{(2)_1}w^{(2)_2})\Big)\Big) \text{ and } \Big(w^{(2)_2}w^{(2)_3}w^{(3)}\,,\; C\Big(QR(w^{(2)_2}w^{(2)_3}w^{(3)})\Big)\Big)$$

are defining relations in $\mathcal{T}'$,

$$\Big(g\Big(w^{(1)}w^{(2)_1}w^{(2)_2}\Big),\, g\Big(C\Big(QR(w^{(1)}w^{(2)_1}w^{(2)_2})\Big)\Big)\Big) \text{ and}$$
$$\Big(g\Big(w^{(2)_2}w^{(2)_3}w^{(3)}\Big),\, g\Big(C\Big(QR(w^{(2)_2}w^{(2)_3}w^{(3)})\Big)\Big)\Big)$$

are not only defined, but are also defining relations in $\mathcal{T}'$. Thus,

$$\Big(g\Big(w^{(1)}w^{(2)_1}w^{(2)_2}w^{(2)_3}w^{(3)}\Big),\, g\Big(C\Big(QR(w^{(1)}w^{(2)_1}w^{(2)_2})\Big)w^{(2)_3}w^{(3)}\Big)\Big) =$$
$$= \Big(w^{(1)}w^{(2)_1}g\Big(w^{(2)_2}\Big)w^{(2)_3}w^{(3)}\,,\; g\Big(C\Big(QR(w^{(1)}w^{(2)_1}w^{(2)_2})\Big)\Big)w^{(2)_3}w^{(3)}\Big)$$

and

$$\Big(g\Big(w^{(1)}w^{(2)_1}w^{(2)_2}w^{(2)_3}w^{(3)}\Big),\, g\Big(w^{(1)}w^{(2)_1}C\Big(QR(w^{(2)_2}w^{(2)_3}w^{(3)})\Big)\Big)\Big) =$$
$$= \Big(w^{(1)}w^{(2)_1}g\Big(w^{(2)_2}\Big)w^{(2)_3}w^{(3)}\,,\; w^{(1)}w^{(2)_1}g\Big(C\Big(QR(w^{(2)_2}w^{(2)_3}w^{(3)})\Big)\Big)\Big).$$

By the same reasoning as before, we have that $g(w^{(2)_2})$ is a column. Note that, if we factorize $C\Big(QR(w^{(2)_2}w^{(2)_3}w^{(3)})\Big)$ into column words of maximal length, the last one must be different from $w^{(3)}$ and its length must be different from $|w^{(3)}|$. Thus, since the quasi-Kashiwara operators preserve the shape of quasi-ribbon tableaux, we have that the length of the last column of $g\Big(C\Big(QR(w^{(2)_2}w^{(2)_3}w^{(3)})\Big)\Big)$ is different from the lenght of $w^{(3)}$, hence they are different, thus

$$g\Big(C\Big(QR(w^{(1)}w^{(2)_1}w^{(2)_2})\Big)\Big)w^{(2)_3}w^{(3)} \neq w^{(1)}w^{(2)_1}g\Big(C\Big(QR(w^{(2)_2}w^{(2)_3}w^{(3)})\Big)\Big),$$

thus

$$\Big(\Big(w^{(1)}w^{(2)_1}g\Big(w^{(2)_2}\Big)w^{(2)_3}w^{(3)}\,,\; g\Big(C\Big(QR(w^{(1)}w^{(2)_1}w^{(2)_2})\Big)\Big)w^{(2)_3}w^{(3)}\Big),$$
$$\Big(w^{(1)}w^{(2)_1}g\Big(w^{(2)_2}\Big)w^{(2)_3}w^{(3)}\,,\; w^{(1)}w^{(2)_1}g\Big(C\Big(QR(w^{(2)_2}w^{(2)_3}w^{(3)})\Big)\Big)\Big)\Big).$$

is a critical pair of $\mathcal{T}'$.

It remains to show that for each pair of paths as described in Proposition 5.2.9, whenever a quasi-crystal operator can be applied to the initial vertex (and hence to all vertices on the paths), the pair of paths that results of applying the operator to all the vertices and to the edges, is still a pair of paths in $\mathcal{X}$. For any edge e on a path as described in Proposition 5.2.9, the rewriting rule has as left-hand side two of the maximal decreasing factors of the initial vertex. Since $\langle \mathcal{A}_n \mid \mathcal{T}' \rangle$ is uniform, for any such edge, the edge e' resulting of applying a quasi-crystal operator (if possible) is also in $\Gamma(\langle \mathcal{A}_n \mid \mathcal{T}' \rangle)$.

Now, we want to prove that in the edge e' the underlying rewriting rule also has as left-hand side two of the maximal decreasing factors of the initial vertex. One problem that might arise is that, by coincidence, the right-hand side of the rule is the same whether the rule has this property or not. We will prove, by contradiction, that this situation does not occur.

Note that the quasi-crystal operators preserve the property of being a quasi-ribbon word. Let α, β be columns in $\mathcal{A}_n^*$ such that $\alpha\beta$ is the left-hand side of the underlying rewriting rule in e. Consequently, $C\Big(QR(\alpha\beta)\Big)$ is the right-hand side of the underlying rewriting rule in e. Suppose $\alpha = \alpha_k^1 \cdots \alpha_1^1 \alpha_k^2 \cdots \alpha_1^2$ and $\beta = \beta_r^1 \cdots \beta_1^1 \beta_r^2 \cdots \beta_1^2$, where $\alpha_k^1, \ldots, \alpha_1^1, \alpha_k^2, \ldots \alpha_1^2, \beta_r^1, \ldots, \beta_1^1, \beta_r^2, \ldots \beta_1^2 \in \mathcal{A}_n^*$ are such that $r \neq 0$ in β_r^2, $k \neq 0$ in α_k^1 and $\beta_1^1 < \alpha_k^2$. Notice that $g\Big(C\Big(QR(\alpha\beta)\Big)\Big)$ is also a quasi-ribbon word and that $g\Big(C\Big(QR(\alpha\beta)\Big)\Big) = C\Big(QR\Big(g(\alpha\beta)\Big)\Big)$.

Note that, by definition of the quasi-Kashiwara operators, if g is defined on a word $u = u_1 \cdots u_m$, then $g(u) = u_1 \cdots u_{i-1} g(u_i) u_{i+1} \cdots u_m$, for a certain $i \in \{1, \ldots, m\}$. Suppose that, in the edge e', the underlying rewriting rule does not have as left-hand side two of the maximal decreasing factors of the initial vertex. Then, $C\Big(QR\Big(g(\alpha\beta)\Big)\Big)$ has one of the following forms, whichever is defined:

- $g(\alpha_k^1 \cdots \alpha_1^1) C\Big(QR(\alpha_k^2 \cdots \alpha_1^2 \beta)\Big)$;
- $\alpha_k^1 \cdots \alpha_1^1 C\Big(QR\Big(g(\alpha_k^2 \cdots \alpha_1^2 \beta)\Big)\Big)$;
- $C\Big(QR\Big(g(\alpha \beta_r^1 \cdots \beta_1^1)\Big)\Big) \beta_r^2 \cdots \beta_1^2$;
- $C\Big(QR(\alpha \beta_r^1 \cdots \beta_1^1)\Big) g(\beta_r^2 \cdots \beta_1^2)$;
- $g(\alpha_k^1 \cdots \alpha_1^1) C\Big(QR(\alpha_k^2 \cdots \alpha_1^2 \beta_r^1 \cdots \beta_1^1)\Big) \beta_r^2 \cdots \beta_1^2$;
- $\alpha_k^1 \cdots \alpha_1^1 C\Big(QR\Big(g(\alpha_k^2 \cdots \alpha_1^2 \beta_r^1 \cdots \beta_1^1)\Big)\Big) \beta_r^2 \cdots \beta_1^2$;
- $\alpha_k^1 \cdots \alpha_1^1 C\Big(QR(\alpha_k^2 \cdots \alpha_1^2 \beta_r^1 \cdots \beta_1^1)\Big) g(\beta_r^2 \cdots \beta_1^2)$.

By Lemma 5.2.10, none of these words are quasi-ribbon words. Thus, we have reached a contradiction, since $C\Big(QR\Big(g(\alpha\beta)\Big)\Big)$ is a quasi-ribbon word.

Hence, we deduce that in the edge e' the underlying rewriting rule also has as left-hand side two of the maximal decreasing factors of the initial vertex. Therefore, the pair of paths resulting of applying a quasi-crystal operator is also a pair of paths in $\mathcal{X}$.

Thus, $\langle \mathcal{A}_n \mid \mathcal{T}' \mid \mathcal{X} \rangle$ is a uniform extended presentation with respect to the quasi-crystal structure. □

As a final consideration, note that the previous proposition allows us to construct, from a confluence diagram G of $\langle \mathcal{A}_n \mid \mathcal{T}' \mid \mathcal{X} \rangle$ (where for $\mathcal{X}$ the resolution paths are as described in Proposition 5.2.9), all confluence diagrams with initial vertices in the same quasi-crystal component in $\Gamma(\mathrm{hypo}_n)$ as the initial vertex of G. Thus, by Proposition 4.3.3, since every quasi-crystal component in $\Gamma(\mathrm{hypo}_n)$ has a unique highest-weight word, we only need to consider those confluence diagrams of $\langle \mathcal{A}_n \mid \mathcal{T}' \mid \mathcal{X} \rangle$ whose vertices are highest-weight words, in the sense that $\mathcal{X}$ is the set of all pairs of paths associated with the highest-weight confluence diagrams and all other diagrams obtained from them by applying the quasi-Kashiwara operators.

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