DE GRUYTER OPEN Spec. Matrices 2015; 3:5–21

Rosário Fernandes*

The maximum multiplicity and the two largest multiplicities of eigenvalues in a Hermitian matrix whose graph is a tree

Abstract: The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree, M_1, was understood fully (from a combinatorial perspective) by C.R. Johnson, A. Leal-Duarte (Linear Algebra and Multilinear Algebra 46 (1999) 139-144). Among the possible multiplicity lists for the eigenvalues of Hermitian matrices whose graph is a tree, we focus upon $\overline{M_2}$, the maximum value of the sum of the two largest multiplicities when the largest multiplicity is M_1. Upper and lower bounds are given for $\overline{M_2}$. Using a combinatorial algorithm, cases of equality are computed for $\overline{M_2}$.

Keywords: Eigenvalue multiplicities; Symmetric matrices; Trees; Two largest multiplicities

MSC: 15A18, 05C38, 05C50

DOI 10.1515/spma-2015-0001
Received September 10, 2013; accepted December 4, 2014

1 Introduction

Let T be a tree on $n \geq 2$ vertices. We denote by $\mathcal{S}(T)$ the collection of all n-by-n complex Hermitian matrices whose graph is T. No restriction is placed upon the diagonal entries of matrices in $\mathcal{S}(T)$.

For convenience, when $A \in \mathcal{S}(T)$, we place in non-increasing order the multiplicities of the eigenvalues of A. We refer to such a list of multiplicities as the *unordered multiplicity list* and we denote it by $(m_1(A), m_2(A), \ldots, m_{k(A)}(A))$, where $k(A)$ is the number of distinct eigenvalues of A. So, $m_j(A)$ is the jth largest multiplicity of an eigenvalue in the multiplicity list of A.

Definition 1.1. Let $\mathcal{L}(T)$ be the set of all positive integer lists (unordered multiplicity lists) $(p_1, p_2, \ldots, p_s)$ satisfying:

(1) $p_1 \geq p_2 \geq \ldots \geq p_s \geq 1$;
(2) $\sum_{i=1}^{s} p_i = n$;
(3) There is an $A \in \mathcal{S}(T)$ with $(m_1(A), m_2(A), \ldots, m_{k(A)}(A)) = (p_1, p_2, \ldots, p_s)$.

For $j \geq 1$, we denote by

$$M_j(T) = \max_{(p_1,p_2,\ldots,p_s)\in\mathcal{L}(T)} (p_1 + \ldots + p_j).$$

It is well known that $M_1(T)$ is equal to the path cover number $P(T)$, the smallest number of nonintersecting induced paths of T that cover all the vertices of T; this is the same as $\max(p - q)$, where p is the number of paths remaining when q vertices have been removed from T in such a way as to leave only induced paths [3].

Remark 1.2. In [7] a combinatorial algorithm was given to compute $M_2(T)$. It is easy to see that if $(p_1, p_2, \ldots, p_s) \in \mathcal{L}(T)$ then

***Corresponding Author: Rosário Fernandes:** Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, 2829-516 Caparica, Portugal, E-mail: mrff@fct.unl.pt

(1) $p_1 \leq M_1(T)$.
(2) $p_1 + p_2 \leq M_2(T)$.
(3) $p_1 + p_2 \geq 2$, $p_2 \neq 0$ (because if T is a tree and $A \in \mathcal{S}(T)$ then the largest and the smallest eigenvalues of A have multiplicities one. So, each list in $\mathcal{L}(T)$ has at least two 1's, [4]).
(4) Using the definition of $M_1(T)$, there exists $(p_1, p_2, \ldots, p_s) \in \mathcal{L}(T)$ such that $p_1 = M_1(T)$.

Given $M_1(T)$ and $M_2(T)$, we cannot say there exists a list $(p_1, p_2, \ldots, p_s) \in \mathcal{L}(T)$ such that $p_1 = M_1(T)$ and $p_2 = M_2(T) - M_1(T)$. For example, [7], the double star $D_{3,3}$ has $M_1(D_{3,3}) = 4$, $M_2(D_{3,3}) = 6$ but $(4, 2, 1, 1) \notin \mathcal{L}(D_{3,3})$ (we can prove this using the Parter-Wiener theorem [5]). $M_1(D_{3,3}) = 4$ because $(4, 1, 1, 1, 1) \in \mathcal{L}(D_{3,3})$, for example, consider the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

$M_2(D_{3,3}) = 6$ because $(3, 3, 1, 1) \in \mathcal{L}(D_{3,3})$, for example, consider the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

So, it is important to know when given $M_2(T)$, we can say that there is a list $(p_1, p_2, \ldots, p_s) \in \mathcal{L}(T)$ such that $p_1 = M_1(T)$ and $p_2 = M_2(T) - M_1(T)$.

Let $\overline{M_2}(T)$ (or simply $\overline{M_2}$) denote the maximum value of the sum of the two largest integers among the lists $(p_1, p_2, \ldots, p_s) \in \mathcal{L}(T)$, when $p_1 = M_1(T)$, i.e.,

$$\overline{M_2}(T) = \max_{(M_1(T), p_2, \ldots, p_s) \in \mathcal{L}(T)} (M_1(T) + p_2).$$

Using the definition of $M_2(T)$, we have $\overline{M_2}(T) \leq M_2(T)$. In this paper we give upper and lower bounds for $\overline{M_2}$ and in some cases, a method for calculating $\overline{M_2}$.

2 Assignments

Let T be a tree on $n \geq 2$ vertices. If $A \in \mathcal{S}(T)$ and v is a vertex of T then $A(v)$ denotes the principal submatrix of A resulting from deleting row and column associated with v, and $m_A(\lambda)$ denotes the multiplicity of eigenvalue λ of matrix A. The Parter theorem, [8], indicates that if $A \in \mathcal{S}(T)$ and $m_A(\lambda) \geq 2$, then there is at least one vertex v of T, of degree at least 3, such that $m_{A(v)}(\lambda) = m_A(\lambda) + 1$. Moreover, v may be chosen so that λ is an eigenvalue of at least three principal submatrices of A associated with branches of T at v. So, we refer to any vertex v of degree greater or equal to 3 as a *high-degree vertex*, or HDV. The Parter theorem was refined by Wiener [9] and more fully in [5]. A vertex v of T is a *Parter vertex* for $A \in \mathcal{S}(T)$ and λ when $m_A(\lambda) \geq 1$ and $m_{A(v)}(\lambda) = m_A(\lambda) + 1$. The Parter theorem guarantees the existence of at least one Parter HDV for any multiple eigenvalue. If a principal submatrix of A associated with some branch at v again has λ as a multiple eigenvalue, then this theorem may again be applied to that branch. Parter vertices for λ may be removed in this fashion until (fully) fragmenting T into many subtrees when λ occurs as an eigenvalue in such a submatrix associated with the subtree at most once. Such a set of Parter vertices is called a *fully fragmented Parter* set for λ, and it is known that each successive Parter vertex is also a Parter vertex for A and λ in the original tree.

If X is a set or collection (or graph), then $|X|$ denotes the cardinality of (number of vertices in) X. If V is a set of vertices and X is a graph then $V \cap X$ denotes the set of vertices in both V and X. If X is a tree then $\mathcal{P}(X)$ denotes the collection of all subtrees of X, including X.

Definition 2.1. [7] (Assignment) *Let T be a tree on $n \geq 2$ vertices and let*

$$\left(p_1, p_2, \ldots, p_k, 1^{n-\sum_{i=1}^{k} p_i}\right)$$

be a non-increasing list of positive integers, with $\sum_{i=1}^{k} p_i \leq n$. The notation 1^l denotes that the last l entries of the list are 1. Note that some of the p_i's may be 1. An assignment *$\mathcal{A}$ of $\left(p_1, p_2, \ldots, p_k, 1^{n-\sum_{i=1}^{k} p_i}\right)$ to T is a collection $\mathcal{A} = ((\mathcal{A}_1, V_1), \ldots, (\mathcal{A}_k, V_k))$ of k collections $\mathcal{A}_i$ of subtrees of T and k collections V_i of vertices of T, with the following properties.*

(1) (Specification of Parter vertices) *For each integer i between 1 and k,*
   (1a) *Each subtree in $\mathcal{A}_i$ is a connected component of $T - V_i$.*
   (1b) $|\mathcal{A}_i| = p_i + |V_i|$.
   (1c) *For each vertex $v \in V_i$, there exists a vertex x adjacent to v such that x is in one of the subtrees in $\mathcal{A}_i$.*

(2) (No overloading) *We require that no subtree S of T is assigned more than $|S|$ integers; define*

$$c_i(S) = |\mathcal{A}_i \cap \mathcal{P}(S)| - |V_i \cap S|,$$

the difference between the number of subtrees contained in S and the number of Parter vertices in S for the ith integer. So, we require that

$$\sum_{i=1}^{k} \max(0, c_i(S)) \leq |S|, \text{ for each } S \in \mathcal{P}(T).$$

If this condition is violated at any subtree, then that subtree *is said to be* overloaded.

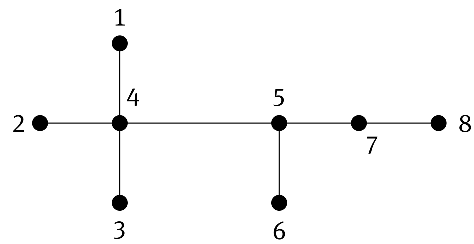
Definition 2.2. [7] *A collection $\mathcal{A} = ((\mathcal{A}_1, V_1), \ldots, (\mathcal{A}_k, V_k))$ of k collections $\mathcal{A}_i$ of subtrees of T and k collections V_i of vertices of T is:*

(1) *an* assignment candidate *of $\left(p_1, p_2, \ldots, p_k, 1^{n-\sum_{i=1}^{k} p_i}\right)$ to T when $\mathcal{A}$ satisfies condition* 1, *but not necessarily* 2 *of* Definition 2.1.
(2) *a* near-assignment *of $\left(p_1, p_2, \ldots, p_k, 1^{n-\sum_{i=1}^{k} p_i}\right)$ to T when $\mathcal{A}$ satisfies conditions $1a$, $1b$, 2, but not necessarily $1c$ of* Definition 2.1.
(3) *a* near-assignment candidate *of $\left(p_1, p_2, \ldots, p_k, 1^{n-\sum_{i=1}^{k} p_i}\right)$ to T when $\mathcal{A}$ satisfies conditions $1a$, $1b$, but not necessarily $1c$ or 2 of* Definition 2.1.

In [7] a simplification of assignments of the list $(p_1, p_2, 1^l)$ is considered.

Lemma 2.3. (*Overloading Lemma*) *If T is a tree and $\mathcal{A}$ is an assignment candidate (or a near-assignment candidate) of the list $(p_1, p_2, 1^l)$ to T, but $\mathcal{A}$ is not an assignment (or a near-assignment, respectively), then there must exist a single vertex in T that is overloaded by $\mathcal{A}$.*

Example 2.4. *Let T be the following tree*



and let $(3, 2, 1^3)$ *be a list.*

If we consider $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ *where*

$$\mathcal{A}_1 = T - \{4, 5\},\ \mathcal{A}_2 = T - \{5\},\ V_1 = \{4, 5\} \text{ and } V_2 = \{5\},$$

then $\mathcal{A}_1$ *has* 5 *connected components and* $\mathcal{A}_2$ *has* 3 *connected components. So,* $|\mathcal{A}_1| = 5$ *and* $|\mathcal{A}_2| = 3$.

$\mathcal{A}$ *is an* assignment candidate *of* $(3, 2, 1^3)$ *to* T *but not an assignment because the subtree* $\{6\}$ *of* T *satisfies*

$$\max(0, c_1(\{6\})) + \max(0, c_2(\{6\})) = 1 + 1 = 2 > 1 = |\{6\}|.$$

If we consider $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$, *where*

$$\mathcal{A}_1' = T - \{4\},\ \mathcal{A}_2' = T - \{5\}\ V_1' = \{4\} \text{ and } V_2' = \{5\}$$

then $\mathcal{A}_1'$ *has* 4 *connected components and* $\mathcal{A}_2'$ *has* 3 *connected components. So,* $|\mathcal{A}_1'| = 4$ *and* $|\mathcal{A}_2'| = 3$.

$\mathcal{A}'$ *satisfies condition* 1 *of* Definition 2.1.

If $S = \{1\}$ *or* $S = \{2\}$ *or* $S = \{3\}$, *then*

$$\max(0, c_1(S)) + \max(0, c_2(S)) = 1 + 0 = |S|.$$

If $S = \{4\}$ *or* $S = \{5\}$ *or* $S = \{7\}$ *or* $S = \{8\}$, *then*

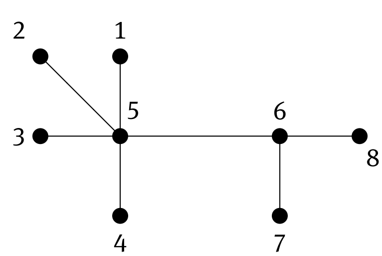
$$\max(0, c_1(S)) + \max(0, c_2(S)) = 0 + 0 < |S| = 1.$$

If $S = \{6\}$ *then*

$$\max(0, c_1(S)) + \max(0, c_2(S)) = 0 + 1 = |S|.$$

Using Lemma 2.3, $\mathcal{A}'$ *is an* assignment *of* $(3, 2, 1^3)$ *to* T.

Example 2.5. *Let* T *be the following tree*



and let $(2, 2, 1^4)$ *be a list.*

If we consider $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$, *where*

$$\mathcal{A}_1 = T - \{5, 6, 7, 8\},\ \mathcal{A}_2 = T - \{6\},\ V_1 = \{5, 6\} \text{ and } V_2 = \{6\}$$

then $\mathcal{A}_1$ *has* 4 *connected components and* $\mathcal{A}_2$ *has* 3 *connected components. So,* $|\mathcal{A}_1| = 4$ *and* $|\mathcal{A}_2| = 3$.

$\mathcal{A}$ *is a* near-assignment *of* $(2, 2, 1^4)$ *to* T (*to prove condition* 2 *of* Definition 2.1 *use* Lemma 2.3) *but not an assignment because* $6 \in V_1$ *and there is not a vertex of* T *adjacent to* 6 *in a subtree of* $\mathcal{A}_1$.

Using the Overloading Lemma (Lemma 2.3), another important result appears.

Lemma 2.6. *Let T be a tree. Then*

there exists a near-assignment of the list $(p_1, p_2, 1^l)$ to T if and only if there exists an assignment of the list $(p_1, p_2, 1^l)$ to T.

Proof Suppose there exists a near-assignment $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ of the list $(p_1, p_2, 1^l)$ to T. If $\mathcal{A}$ satisfies $1c$ of Definition 2.1, then $\mathcal{A}$ is an assignment of $(p_1, p_2, 1^l)$ to T.

Suppose that $\mathcal{A}$ does not satisfy $1c$. Then V_1 or V_2 does not satisfy $1c$. Suppose, without loss of generalization that V_1 does not satisfy $1c$. So, there exists a vertex $v_1 \in V_1$ such that there is not a vertex x adjacent to v_1 in a subtree of $\mathcal{A}_1$.

Since $|\mathcal{A}_1| = p_1 + |V_1|$, remove v_1 from V_1 and remove a subtree R_1 from $\mathcal{A}_1$. We obtain $\mathcal{A}_1' = \mathcal{A}_1 \setminus R_1$ and $V_1' = V_1 \setminus \{v_1\}$. Since $|\mathcal{A}_1'| = p_1 + |V_1'|$, we conclude that $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2, V_2))$ is a near-assignment candidate of the list $(p_1, p_2, 1^l)$ to T.

If $\mathcal{A}'$ is not a near-assignment, by Lemma 2.3, there must exist a single vertex y in T that is overloaded by $\mathcal{A}'$. Using the fact that $\mathcal{A}$ is a near-assignment, $y = v_1$. But v_1 does not belong to $\mathcal{A}_1'$. Consequently, $S = \{v_1\}$ satisfies condition 2 of Definition 2.1. Contradiction. Therefore, $\mathcal{A}'$ is a near-assignment.

If $\mathcal{A}'$ satisfies $1c$ of Definition 2.1, then $\mathcal{A}'$ is an assignment of $(p_1, p_2, 1^l)$ to T. If $\mathcal{A}'$ does not satisfy $1c$ of Definition 2.1, repeat the process.

Repeating this process we obtain an assignment because $p_1, p_2 \geq 1$ and in each process we have a collection of subtrees of T satisfying condition $1a$ of Definition 2.1.

Conversely, the proof is trivial. □

Definition 2.7. *If $A \in \mathcal{S}(T)$ and S is a subgraph of T then*

1. *$A[S]$ denotes the principal submatrix of A lying on rows and columns associated with the vertices of S.*
2. *$A(S)$ denotes the principal submatrix of A resulting from deleting rows and columns associated with the vertices of S.*

Using the interlacing theorem for Hermitian matrices [2], if x is a vertex of T (tree) and λ is an eigenvalue of $A \in \mathcal{S}(T)$, then there is a simple relation between $m_{A(x)}(\lambda)$ and $m_A(\lambda)$:

$$m_{A(x)}(\lambda) = m_A(\lambda) - 1 \quad \text{or} \quad m_{A(x)}(\lambda) = m_A(\lambda) \quad \text{or} \quad m_{A(x)}(\lambda) = m_A(\lambda) + 1.$$

Definition 2.8. [7] *Let T be a tree on $n \geq 2$ vertices. We call an assignment $\mathcal{A} = ((\mathcal{A}_1, V_1), \ldots, (\mathcal{A}_k, V_k))$ of $\left(p_1, p_2, \ldots, p_k, 1^{n-\sum_{i=1}^k p_i}\right)$ to T realizable if there exists a matrix $B \in \mathcal{S}(T)$ with unordered multiplicity list $\left(p_1, p_2, \ldots, p_k, 1^{n-\sum_{i=1}^k p_i}\right)$, such that, for each i between 1 and k, if s_i is the eigenvalue of B associated with p_i, i.e, $m_B(s_i) = p_i$, then:*

1. *For each subtree R of T in $\mathcal{A}_i$, $m_{B[R]}(s_i) = 1$.*
2. *For each connected component Q of $T - V_i$ that is not in $\mathcal{A}_i$, $m_{B[Q]}(s_i) = 0$.*
3. *For each $x \in V_i$, x is a Parter vertex for B and s_i.*

Remark 2.9. *Note that if $C \in \mathcal{S}(T)$ is a matrix that satisfies conditions 1 and 2 of* Definition 2.8, *then for each i between 1 and k, $m_C(s_i) = p_i \geq 1$.*

Using the interlacing theorem for Hermitian matrices, if $x \in V_i$, then $m_{C(x)}(s_i)$ is equal to

$$m_C(s_i) - 1 \quad \text{or} \quad m_C(s_i) \quad \text{or} \quad m_C(s_i) + 1.$$

By conditions 1 and 2 of Definition 2.8, *$m_{C(V_i)}(s_i) = |\mathcal{A}_i|$. But $\mathcal{A}$ is an assignment, so, $|\mathcal{A}_i| = p_i + |V_i|$. Thus,*

$$m_{C(x)}(s_i) = m_C(s_i) + 1.$$

Therefore, C satisfies Definition 2.8. □

Using the last remark, we can rewrite Definition 2.8.

Definition 2.8 *Let T be a tree on $n \geq 2$ vertices. We call an assignment $\mathcal{A} = ((\mathcal{A}_1, V_1), \ldots, (\mathcal{A}_k, V_k))$ of $\left(p_1, p_2, \ldots, p_k, 1^{n-\sum_{i=1}^{k} p_i}\right)$ to T realizable if there exists a matrix $B \in \mathcal{S}(T)$ with unordered multiplicity list $\left(p_1, p_2, \ldots, p_k, 1^{n-\sum_{i=1}^{k} p_i}\right)$, such that, for each i between 1 and k, if s_i is the eigenvalue of B associated with p_i, i.e, $m_B(s_i) = p_i$, then:*

(1) *For each subtree R of T in $\mathcal{A}_i$, $m_{B[R]}(s_i) = 1$.*
(2) *For each connected component Q of $T - V_i$ that is not in $\mathcal{A}_i$, $m_{B[Q]}(s_i) = 0$.*

Definition 2.10. *If T is a tree on $n \geq 2$ vertices, $\mathcal{A}$ is a realizable assignment of $\left(p_1, p_2, \ldots, p_k, 1^{n-\sum_{i=1}^{k} p_i}\right)$ to T and $B \in \mathcal{S}(T)$ is a matrix that satisfies Definition 2.8, then we say that B realizes the assignment $\mathcal{A}$.*

There are assignments that are not realizable. For instance see Example 2.3 in [7]. However when we study the list $(p_1, p_2, 1^l)$ we have the following result.

Theorem 2.11. [7] *Given a tree T on $n = p_1 + p_2 + l$ vertices, a near-assignment of the list (p_1, p_2, l^l) to T, $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$, and any distinct real numbers α and β, then there exists $A \in \mathcal{S}(T)$ satisfying the following conditions:*

If R is a connected component of $T - V_1$, then

$$\alpha \text{ is an eigenvalue of } A[R] \text{ if and only if } R \in \mathcal{A}_1.$$

Similarly, if S is a connected component of $T - V_2$, then

$$\beta \text{ is an eigenvalue of } A[S] \text{ if and only if } S \in \mathcal{A}_2.$$

Using Lemma 2.6, Theorem 2.11 and the new version of Definition 2.8 we obtain the following result.

Theorem 2.12. *Given a tree T on $n = p_1 + p_2 + l$ vertices, a near-assignment $\mathcal{A}$ of the list (p_1, p_2, l^l) to T, and any distinct real numbers α and β, then*

(1) *there exists a realizable assignment $\mathcal{B}$ of $\left(p_1, p_2, 1^l\right)$ to T.*
(2) *there exists $A \in \mathcal{S}(T)$ that realizes the assignment $\mathcal{B}$ with $m_A(\alpha) = p_1$ and $m_A(\beta) = p_2$.*

Therefore, we immediately have as a consequence:

Corollary 2.13. *For any tree T, if there exists a near-assignment of the list $(M_1(T), p_2, 1^l)$ to T, then*

$$\overline{M_2}(T) \geq M_1(T) + p_2.$$

3 Upper and lower bounds for $\overline{M_2}$

In this section, using the reduction theorem for M_2, [7], we directly compute $\overline{M_2}$ for particular trees. For other kind of trees, we give bounds on $\overline{M_2}$.

In [7], the authors directly computed M_2 for generalized stars (for the notion of generalized star see [6]).

Definition 3.1. [6] *Let T be a tree and x_0 be a vertex of T. A* generalized star *T with central vertex x_0 is a tree such that $T - \{x_0\}$ is a union of paths (arms), each one of them is adjacent to x_0 by an endpoint.*

Proposition 3.2. [7] *Let T be a generalized star on $n \geq 2$ vertices, with f arms of length 1 and g arms of length at least 2. Then:*

(A) *If $g \geq 2$, then $M_2(T) = f + 2g - 2$.*

(B) *If $g \leq 1$ and T is not a path, then $M_2(T) = f + g$.*
(C) *If T is a path, then $M_2(T) = 2$.*

Definition 3.3. [7] (*Peripheral HDV, peripheral arm*) *Given a tree T and a high-degree vertex v, v is a* peripheral HDV *of T if and only if there is a branch of T at v that contains all the other high-degree vertices in T. A* peripheral arm *of a tree T is a branch of T at a peripheral HDV such that the branch does not itself contain any HDV.*

Definition 3.4. *Throughout this section, we will consider a* peripheral HDV *v in a tree T.*

The subtree of T consisting of v and its peripheral arms will be called S - however, if v is the only HDV in *T, we will let S be v and all but one of its peripheral arms (chosen arbitrarily). The point is that S should be a generalized star containing everything except a single branch of T at v.*

Let w be the one vertex adjacent to v that is not in S. We denote by $(T-S)+_w K_1$ the tree obtained from $T-S$ by putting a vertex adjacent to w.

Theorem 3.5. [7] (M_2 Reduction Theorem) *Let T be a tree and v a peripheral HDV, with S as defined earlier in this section. Suppose that S has f arms of length 1 and g arms of length at least 2. Then:*

(A) *If $g \geq 2$, then $M_2(T-S) = M_2(T) - f - 2g + 2$.*
(B) *If $g \leq 1$, then $M_2((T-S)+_w K_1) = M_2(T) - f - g + 1$.*

In [1] a class of trees was introduced that contains the generalized stars, the superstars.

Definition 3.6. [1] *Let T be a tree and x_0 be a vertex of T. A* superstar *T with central vertex x_0 is a tree such that $T - \{x_0\}$ is a union of paths.*

The focus of this section is to directly compute $\overline{M_2}$ for a subclass of superstars.

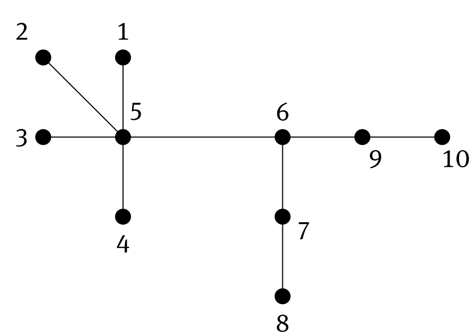
Definition 3.7. *Let T be a superstar with central vertex x_0. A* small pincer *of T is a path, P, of $T - \{x_0\}$ such that:*

(1) *P is adjacent to x_0 by a vertex u of degree two in P.*
(2) *At least one path of $P - u$ is a vertex.*

Definition 3.8. *Let T be a superstar with central vertex x_0. T is a small superstar if all paths of $T - \{x_0\}$ are small pincers or are adjacent to x_0 by an endpoint (arms).*

Example 3.9. *The superstar T of* Example 2.4 *is a small superstar with cental vertex 4. The superstar T of* Example 2.5 *is a small superstar with central vertex 5. All stars and generalized stars are small superstars.*

The following superstar is not a small superstar



Definition 3.10. *Let T be a tree and $\mathcal{A}$ an assignment of $(M_1(T), p_2, 1^l)$ to T.*

(1) *We refer to $\mathcal{A}$ as an $\overline{M_2}$ assignment to T.*
(2) *If $M_1(T) + p_2 = \overline{M_2}(T)$, we refer to $\mathcal{A}$ as an $\overline{M_2}$-maximal assignment to T.*

Remark 3.11. *Let T be a tree and $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ an $\overline{M_2}$ assignment of $(M_1(T), p_2, 1^l)$ to T. Because $M_1(T) = |\mathcal{A}_1| - |V_1|$,*

1. *All components of $T - V_1$ are in $\mathcal{A}_1$.*
2. *We can assume that if $v \in V_1$ then v is a HDV.*
3. *Since all components of $T - V_1$ are paths, if v if a peripheral HDV of degree greater or equal to 4 in T then $v \in V_1$.*
4. *If v is a peripheral HDV, $v \in V_1$ and all peripheral arms have length 1 then they are in $\mathcal{A}_1$ and no one is in $\mathcal{A}_2$ (see Lemma 2.3).*

Remark 3.12. *Let T be a tree and $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ an $\overline{M_2}$-maximal assignment of $(M_1(T), p_2, 1^l)$ to T. Because $\overline{M_2}(T) = |\mathcal{A}_1| - |V_1| + |\mathcal{A}_2| - |V_2|$,*

1. *All components of $T - V_2$ with more than one vertex are in $\mathcal{A}_2$.*
2. *We can assume that if $v \in V_2$ then v is a HDV.*
3. *All components of $T - V_2$ with one vertex that are not components of $T - V_1$ are in $\mathcal{A}_2$.*
4. *If v is a peripheral HDV, $v \in V_1$ and all peripheral arms have length 1 then using Remark 3.11, 4, we conclude that $v \notin V_2$.*
5. *If v if a peripheral HDV, $v \in V_1$ and all peripheral arms have length 1, except one, then there is an $\overline{M_2}$-maximal assignment of $(M_1(T), p_2, 1^l)$ to T such that $v \notin V_2$.*

Remark 3.13. *In some proofs we construct an $\overline{M_2}$-maximal (or simply an $\overline{M_2}$) assignment of $(M_1(T), p_2, 1^l)$ to T. for some integer p_2. In these cases, first we construct an $\overline{M_2}$ assignment of $(M_1(T), p_2, 1^l)$ to T, $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ by putting the elements in $\mathcal{A}_1$ and in V_1, next we put the elements in $\mathcal{A}_2$ and in V_2, using Remarks 3.11 and 3.12. This construction is in such a way that $M_1(T) = |\mathcal{A}_1| - |V_1|$ and $M_1(T) + p_2 = |\mathcal{A}_1| - |V_1| + |\mathcal{A}_2| - |V_2|$. After using Lemma 2.3 we conclude condition 2 of Definition 2.1 and by Corollary 2.13, we say that $\overline{M_2}(T) \geq M_1(T) + p_2$.*

Proposition 3.14. *Let T be a small superstar on $n \geq 2$ vertices, with f arms of length 1, g arms of length at least 2 and h small pincers, with $f + g \geq 2$ or $h \geq 2$. Then:*

(A) *If $g \geq 2$, then $\overline{M_2}(T) = 2h + f + 2g - 2$.*
(B) *If $g \leq 1$ and T is not a path, then $\overline{M_2}(T) = 2h + f + g$.*
(C) *If T is a path, then $\overline{M_2}(T) = 2$.*

Proof Let x be the central vertex of T. If S is a small pincer of T, by Theorem 3.5,

$$M_2((T - S) +_x K_1) = M_2(T) - 1.$$

Since T has h small pincers,

$$M_2(T') = M_2(T) - h,$$

where T' is obtained from T by removing all small pincers and by putting h vertices adjacent to x. Consequently, T' is a generalized star with $f + h$ arms of length 1 and g arms of length at least 2. Using Proposition 3.2

$$M_2(T') = \begin{cases} f + h + 2g - 2 & \text{if } g \geq 2 \\ f + h + g & \text{if } g \leq 1 \text{ and } T' \text{ is not a path} \\ 2 & \text{if } T' \text{ is a path.} \end{cases}$$

Therefore,

$$M_2(T) = \begin{cases} f + 2h + 2g - 2 & \text{if } g \geq 2 \\ f + 2h + g & \text{if } g \leq 1 \text{ and } T' \text{ is not a path} \\ 2 + h & \text{if } T' \text{ is a path.} \end{cases}$$

Note that if T' is a path with $h = 2$ and $f = g = 0$ then T is not a path and $M_2(T) = M_2(T') + h = 2 + 2 = 4 = f + 2h + g$. By hypothesis, if $h < 2$ then $f + g \geq 2$. In this case, if T' is a path then $h = 0$ and $f + g = 2$. Consequently, T is a path.

So, we conclude that

$$M_2(T) = \begin{cases} f + 2h + 2g - 2 & \text{if } g \geq 2 \\ f + 2h + g & \text{if } g \leq 1 \text{ and } T \text{ is not a path} \\ 2 & \text{if } T \text{ is a path.} \end{cases}$$

Since $\overline{M_2}(T) \leq M_2(T)$, we have

(A) If $g \geq 2$, then $\overline{M_2}(T) \leq 2h + f + 2g - 2$.
(B) If $g \leq 1$ and T is not a path, then $\overline{M_2}(T) \leq 2h + f + g$.
(C) If T is a path, then $\overline{M_2}(T) \leq 2$.

Conversely, since T is a tree,

$$M_1(T) = \begin{cases} f + h + g - 1 & \text{if } f + g \geq 2 \\ h + 1 & \text{if } f + g \leq 1. \end{cases}$$

We are going to construct an $\overline{M_2}$ assignment of $(M_1(T), p_2, 1^l)$, for some integer p_2, to T (see Remark 3.13).

Case 1 If $f+g \geq 2$, we put the central vertex of T in V_1 and we put the $f+h+g$ paths obtained by removing the central vertex of T in $\mathcal{A}_1$.

If $g \geq 2$, we put the central vertex of T in V_2 and we put the $h + g$ paths of length at least 2, obtained by removing the central vertex of T in $\mathcal{A}_2$. So, $|\mathcal{A}_1| - |V_1| = f + g + h - 1 = M_1(T)$. Using Remark 3.13, $\overline{M_2}(T) \geq f + h + g - 1 + h + g - 1 = f + 2h + 2g - 2$.

If $g \leq 1$, we put the central vertex of each small pincer of T in V_2, we put the $2h + 1$ subtrees obtained by removing the central vertex of all small pincers of T in $\mathcal{A}_2$. Since $|\mathcal{A}_1| - |V_1| = f + g + h - 1 = M_1(T)$, using Remark 3.13, $\overline{M_2}(T) \geq f + h + g - 1 + 2h + 1 - h = f + 2h + g$.

Note that if T is a path and $f+g \geq 2$ then $f+g = 2$ and $h = 0$. Thus, if $g = 2$, then $\overline{M_2}(T) \geq f+2h+2g-2 = 2$ and if $g \leq 1$, then $\overline{M_2}(T) \geq f + 2h + g = 2$.

Case 2 If $f + g \leq 1$ then $h \geq 2$ and T is not a path. We put the central vertex of each small pincer of T in V_1 and we put the $2h + 1$ subtrees obtained by removing the central vertex of all small pincers of T in $\mathcal{A}_1$. We put the central vertex of T in V_2 and we put the $f + h + g$ paths obtained by removing the central vertex of T in $\mathcal{A}_2$. Since $|\mathcal{A}_1| - |V_1| = h + 1 = M_1(T)$, by Remark 3.13, $\overline{M_2}(T) \geq h + 1 + f + g + h - 1 = f + g + 2h$.

Consequently,

(A) If $g \geq 2$, then $\overline{M_2}(T) \geq 2h + f + 2g - 2$.
(B) If $g \leq 1$ and T is not a path, then $\overline{M_2}(T) \geq 2h + f + g$.
(C) If T is a path, then $\overline{M_2}(T) \geq 2$.

Therefore,

(A) If $g \geq 2$, then $\overline{M_2}(T) = 2h + f + 2g - 2$.
(B) If $g \leq 1$ and T is not a path, then $\overline{M_2}(T) = 2h + f + g$.
(C) If T is a path, then $\overline{M_2}(T) = 2$. □

Proposition 3.15. *Let T be a tree and v a peripheral HDV, with S as defined earlier in this section. Suppose that S has 3 arms of length 1 and 0 arms of length at least 2 and $T \neq S$. Then*

$$\overline{M_2}(T - S) + 2 \leq \overline{M_2}(T) \leq \overline{M_2}(T - S) + 3.$$

Proof Let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an $\overline{M_2}$-maximal assignment to T. We are going to construct an $\overline{M_2}$ assignment to $T - S$, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ (see Remark 3.13). Note that $M_1(T - S) = M_1(T) - 2$. Because v has degree 4, by Remark 3.11, 3, $v \in V_1$, the peripheral arms of S are in $\mathcal{A}_1$ and no one is in $\mathcal{A}_2$. Using Remark 3.12, 4, $v \notin V_2$. So, let F be the component of $T - V_2$ containing S. By Remark 3.12, 1, F is in $\mathcal{A}_2$.

Let $\mathcal{A}_1' = \mathcal{A}_1 \setminus \{\text{the peripheral arms of } S\}$, $V_1' = V_1 \setminus \{v\}$, $V_2' = V_2$ and

$$\mathcal{A}_2' = \begin{cases} \mathcal{A}_2 \setminus \{F\} & \text{if } \mathcal{A}_2 \neq \{F\} \\ T - S & \text{if } \mathcal{A}_2 = \{F\} \end{cases}.$$

By Remark 3.13, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ is an $\overline{M_2}$ assignment to $T - S$ and $\overline{M_2}(T - S) \geq \overline{M_2}(T) - 2 - 1 = \overline{M_2}(T) - 3$.

Let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an $\overline{M_2}$-maximal assignment to $T - S$. We are going to construct an $\overline{M_2}$ assignment to T, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ (see Remark 3.13). Note that $M_1(T) = M_1(T - S) + 2$. Let w be the vertex of $T - S$ adjacent to v in T. If $w \notin V_2$ then let R be the component of $(T - S) - V_2$ containing w and let P be the component of $T - V_2$ containing S.

Let $\mathcal{A}_1' = \mathcal{A}_1 \cup \{\text{the peripheral arms of } S\}$, $V_1' = V_1 \cup \{v\}$, $V_2' = V_2$ and

$$\mathcal{A}_2' = \begin{cases} (\mathcal{A}_2 \setminus \{R\}) \cup \{P\} & \text{if } R \in \mathcal{A}_2 \text{ and } w \notin V_2 \\ \mathcal{A}_2 & \text{otherwise} \end{cases}.$$

By Remark 3.13, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ is an $\overline{M_2}$ assignment to T and $\overline{M_2}(T) \geq \overline{M_2}(T - S) + 2$. □

Proposition 3.16. *Let T be a tree and v a peripheral HDV, with S as defined earlier in this section. Suppose that S has* 1 *arm of length* 1 *and* 1 *arm of length at least* 2 (*or T has* 2 *arms of length* 1 *and* 0 *arms of length at least* 2) *and $T \neq S$. Then*

$$\overline{M_2}(T - S) + 1 \leq \overline{M_2}(T) \leq \overline{M_2}(T - S) + 2.$$

Proof Let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an $\overline{M_2}$-maximal assignment to T. We are going to construct an $\overline{M_2}$ assignment to $T - S$, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ (see Remark 3.13). Note that $M_1(T - S) = M_1(T) - 1$.

Using Remark 3.11, 1, if v is in V_1, then the peripheral arms of $S - v$ are in $\mathcal{A}_1$. Using Remark 3.12, 5, without loss of generality, we can assume that $v \notin V_2$. Let F be the component of $T - V_2$ containing S. By Remark 3.12, 1 and 3, F is in $\mathcal{A}_2$.

Let $\mathcal{A}_1' = \mathcal{A}_1 \setminus \{\text{the peripheral arms of } S\}$, $V_1' = V_1 \setminus \{v\}$, $V_2' = V_2$ and

$$\mathcal{A}_2' = \begin{cases} \mathcal{A}_2 \setminus \{F\} & \text{if } \mathcal{A}_2 \neq \{F\} \\ T - S & \text{if } \mathcal{A}_2 = \{F\} \end{cases}.$$

By Remark 3.13, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ is an $\overline{M_2}$ assignment to $T - S$ and $\overline{M_2}(T - S) \geq \overline{M_2}(T) - 1 - 1 = \overline{M_2}(T) - 2$.

If v is not in V_1, since v has degree 3 in T, then $w \in V_1$. By Remark 3.11, 1, S is in $\mathcal{A}_1$. By Remark 3.12, 3, we can assume, without loss of generality, that $v \in V_2$ and the peripheral arms of S are in $\mathcal{A}_2$.

Let $\mathcal{A}_1' = \mathcal{A}_1 \setminus \{S\}$, $V_1' = V_1$, $V_2' = V_2 \setminus \{v\}$, $\mathcal{A}_2' = \mathcal{A}_2 \setminus \{\text{the peripheral arms of } S\}$.

By Remark 3.13, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ is an $\overline{M_2}$ assignment to $T - S$ and $\overline{M_2}(T - S) \geq \overline{M_2}(T) - 1 - 1 = \overline{M_2}(T) - 2$.

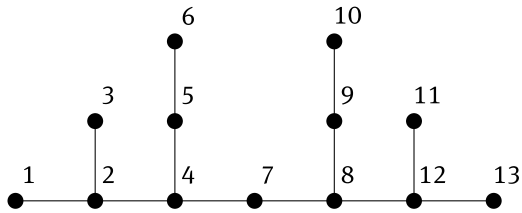
Let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an $\overline{M_2}$-maximal assignment to $T - S$. We are going to construct an $\overline{M_2}$ assignment to T, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ (see Remark 3.13). Note that $M_1(T) = M_1(T - S) + 1$. Let w be the vertex of $T - S$ adjacent to v in T. If $w \notin V_2$ then let F be the component of $(T - S) - V_2$ containing w and let P be the component of $T - V_2$ containing S.

Let $\mathcal{A}_1' = \mathcal{A}_1 \cup \{\text{the peripheral arms of } S\}$, $V_1' = V_1 \cup \{v\}$, $V_2' = V_2$ and

$$\mathcal{A}_2' = \begin{cases} (\mathcal{A}_2 \setminus \{F\}) \cup \{P\} & \text{if } F \in \mathcal{A}_2 \text{ and } w \notin V_2 \\ \mathcal{A}_2 & \text{otherwise} \end{cases}.$$

By Remark 3.13, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ is an $\overline{M_2}$ assignment to T and $\overline{M_2}(T) \geq \overline{M_2}(T - S) + 1$. □

Example 3.17. *Let T be the following tree*



Let H be the subtree obtained from T by removing vertices 11, 12, 13. *By* Proposition 3.16,

$$\overline{M_2}(H) + 1 \leq \overline{M_2}(T) \leq \overline{M_2}(H) + 2.$$

Since H is a small superstar with central vertex 4, *by* Proposition 3.14, $\overline{M_2}(H) = 4 + 0 + 4 - 2 = 2$. *So,*

$$5 \leq \overline{M_2}(T) \leq 6.$$

4 An algorithm for $\overline{M_2}$

The purpose of this section is to find simple reductions of the initial tree in such a way that we know the effect of each reduction on $\overline{M_2}$. The process may be continued until a small superstar, for which $\overline{M_2}$ is known (Proposition 3.14), or until a subtree for which $\overline{M_2}$ has bounds (Section 3).

Definition 4.1. *(Peripheral SHDV, peripheral super path) Let T be a tree that is not a small superstar. A* peripheral superstar high degree vertex (*SHDV*) *v of T is an HDV vertex of T such that*

[(1) *there is a unique subtree of $T - v$, R, that contains high-degree vertices;*
[(2) *$T - R$ is a small superstar;*
[(3) *if $w \in R$ and w is adjacent to v, then w does not satisfy* 1, 2.

A peripheral super path *of T at v (v is a SHDV) is a path of $(T - R) - v$. There are two kinds of peripheral super paths of T at v (SHDV): peripheral arms and small pincers.*

Example 4.2. *Consider the tree T of* Example 3.17.

The vertices 4 *and* 8 *are peripheral superstar high degree vertices.*

The vertex 2 *is not a peripheral superstar high degree vertex because it is adjacent to vertex* 4 *and this vertex satisfies conditions* 1 *and* 2 *of* Definition 4.1.

The subtree of T generated by vertices 1, 2, 3 *is a peripheral super path of T at* 4, *but it is not a peripheral arm of T at* 4 *(it is a small pincer).*

Definition 4.3. *Throughout this section, we will consider a* peripheral SHDV *v in a tree T that is not a small superstar. The* subtree *of T* consisting of *v* and its peripheral super paths *will be called Q. Let w be the one vertex adjacent to v that is not in Q.*

Remark 4.4. *Let T be a tree and $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ an $\overline{M_2}$ assignment of $(M_1(T), p_2, 1^l)$ to T. Because $M_1(T) = |\mathcal{A}_1| - |V_1|$,*

(1) *All components of $T - V_1$ are in $\mathcal{A}_1$.*
(2) *We can assume that if $v \in V_1$ then v has degree greater than two in T.*
(3) *Since all components of $T - V_1$ are paths, if v is a peripheral SHDV of degree greater or equal to* 4 *in T then $v \in V_1$ or there is at most one peripheral arm adjacent to v and the central vertex of each small pincer adjacent to v is in V_1.*
(4) *If v is a peripheral SHDV, $v \in V_1$ and all peripheral super paths adjacent to v have length* 1 *then they are in $\mathcal{A}_1$ and no one is in $\mathcal{A}_2$ (see Lemma 2.3).*

Remark 4.5. *Let T be a tree and $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ an $\overline{M_2}$-maximal assignment of $(M_1(T), p_2, 1^l)$ to T. Because $\overline{M_2}(T) = |\mathcal{A}_1| - |V_1| + |\mathcal{A}_2| - |V_2|$,*

1. *All components of $T - V_2$ with more than one vertex are in $\mathcal{A}_2$.*
2. *We can assume that if $v \in V_2$ then v has degree greater than two in T.*
3. *All components of $T - V_2$ with one vertex that are not components of $T - V_1$ are in $\mathcal{A}_2$.*
4. *If v is a peripheral SHDV, $v \in V_1$ and all peripheral super paths adjacent to v have length 1, then using Remark 4.4, 4, we conclude that $v \notin V_2$.*
5. *If v is a peripheral SHDV, $v \in V_1$ and all peripheral super paths adjacent to v have length 1, except one, then there is an $\overline{M_2}$-maximal assignment of $(M_1(T), p_2, 1^l)$ to T such that $v \notin V_2$.*

Remark 4.6. *In some proofs we construct an $\overline{M_2}$-maximal (or simply an $\overline{M_2}$) assignment of $(M_1(T), p_2, 1^l)$ to T. for some integer p_2. In these cases, first we construct an $\overline{M_2}$ assignment of $(M_1(T), p_2, 1^l)$ to T, $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ by putting the elements in $\mathcal{A}_1$ and in V_1, next we put the elements in $\mathcal{A}_2$ and in V_2, using Remarks 4.4 and 4.5. This construction is in such a way that $M_1(T) = |\mathcal{A}_1| - |V_1|$ and $M_1(T) + p_2 = |\mathcal{A}_1| - |V_1| + |\mathcal{A}_2| - |V_2|$. After using Lemma 2.3 we conclude condition 2 of Definition 2.1 and by Corollary 2.13, we say that $\overline{M_2}(T) \geq M_1(T) + p_2$.*

Proposition 4.7. *Let T be a tree that is not a small superstar and v a peripheral SHDV, with Q as defined earlier in this section. Suppose that Q has $h \geq 1$ small pincers and the degree of v in T is greater than 4. Let H be the graph obtained from T by removing one small pincer of Q. Then*

$$\overline{M_2}(H) = \overline{M_2}(T) - 2.$$

Proof By Proposition 3.16, $\overline{M_2}(H) \geq \overline{M_2}(T) - 2$.

Let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an $\overline{M_2}$-maximal assignment to H. We are going to construct an $\overline{M_2}$ assignment to T, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ (see Remark 4.6). Note that $M_1(T) = M_1(H) + 1$. Since the degree of v in T is greater than 4, we conclude that the degree of v in H is greater than 3. By Remark 4.4, 3 and Remark 4.5, 1, 3, we have $v \in V_1 \cup V_2$.

Suppose that $v \in V_1 \cap V_2$. Let P be the small pincer $T - H$.

Let $\mathcal{A}_1' = \mathcal{A}_1 \cup \{P\}$, $V_1' = V_1$, $V_2' = V_2$ and $\mathcal{A}_2' = \mathcal{A}_2 \cup \{P\}$.

By Remark 4.6, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ is an $\overline{M_2}$ assignment to T and $\overline{M_2}(T) \geq \overline{M_2}(H) + 2$.

Suppose that $v \in V_1 \setminus V_2$. Let x be the central vertex of the small pincer, P, of $T - H$.

Let $\mathcal{A}_1' = \mathcal{A}_1 \cup \{P\}$, $V_1' = V_1$, $\mathcal{A}_2' = \mathcal{A}_2 \cup \{$the peripheral arms of P at $x\}$ and $V_2' = V_2 \cup \{x\}$.

By Remark 4.6, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ is an $\overline{M_2}$ assignment to T and $\overline{M_2}(T) \geq \overline{M_2}(H) + 2$.

Suppose that $v \in V_2 \setminus V_1$. Let x be the central vertex of the small pincer, P, of $T - H$.

Let $V_1' = V_1 \cup \{x\}$, $\mathcal{A}_1' = \mathcal{A}_1 \cup \{$the peripheral arms of P at $x\}$., $\mathcal{A}_2' = \mathcal{A}_2 \cup \{P\}$ and $V_2' = V_2$.

By Remark 4.6, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ is an $\overline{M_2}$ assignment to T and $\overline{M_2}(T) \geq \overline{M_2}(H) + 2$.

Consequently, $\overline{M_2}(T) = \overline{M_2}(H) + 2$. □

Lemma 4.8. *Let T be a tree that is not a small superstar. Suppose that v is a peripheral SHDV in T with Q, w as defined earlier in this section. Then, there exists an $\overline{M_2}$-maximal assignment to T, $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$, in which $v \in V_1 \cup V_2$.*

Moreover,

1. *If v has at least two peripheral arms of length at least 2, then there exists an $\overline{M_2}$-maximal assignment, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ in which $v \in V_1' \cap V_2'$.*
2. *If v has at most one peripheral arm of length at least 2 and w has degree two in T, then there exists an $\overline{M_2}$-maximal assignment, $\mathcal{A}'' = ((\mathcal{A}_1'', V_1''), (\mathcal{A}_2'', V_2''))$ such that v is in exactly one V_1'' or V_2''.*
3. *If Q has f peripheral arms of length 1 and $g \leq 1$ peripheral arms of length at least 2, $f + g > 2$ and $\mathcal{A}''' = ((\mathcal{A}_1''', V_1'''), (\mathcal{A}_2''', V_2'''))$ is an $\overline{M_2}$-maximal assignment to T, then $v \in V_1'''$.*

Proof Let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an $\overline{M_2}$-maximal assignment to T in which $v \notin V_1 \cup V_2$. Suppose that Q has f peripheral arms of length 1 and g peripheral arms of length at least 2. We are going to construct an $\overline{M_2}$-maximal assignment to T, $\mathcal{B} = ((\mathcal{B}_1, U_1), (\mathcal{B}_2, U_2))$ (see Remark 4.6).

If $f + g \geq 2$, then by Remark 4.4, 1, the component, R, of $T - V_1$ containing v is in $\mathcal{A}_1$. Note that the peripheral arms of Q might be in R.

Let $\mathcal{B}_1 = (\mathcal{A}_1 \setminus \{R\}) \cup \{\text{two peripheral arms of } Q\}$, $U_1 = V_1 \cup \{v\}$, $\mathcal{B}_2 = \mathcal{A}_2$ and $U_2 = V_2$.

By Remark 4.6 and the cardinality of $\mathcal{B}$, $\mathcal{B} = ((\mathcal{B}_1, U_1), (\mathcal{A}_2, V_2))$ is an $\overline{M_2}$-maximal assignment to T in which $v \in U_1$.

If $f + g \leq 1$, by Remark 4.4, 3 and Remark 4.5, 1 and 3, the central vertex of each small pincer of Q is in $V_1 \setminus V_2$. By Remark 4.5, 1, the component, R, of $T - V_2$ containing v, is in $\mathcal{A}_2$.

Let $\mathcal{B}_1 = \mathcal{A}_1$, $U_1 = V_1$, $\mathcal{B}_2 = (\mathcal{A}_1 \setminus \{R\}) \cup \{\text{two peripheral super paths of } Q\}$ and $U_2 = V_2 \cup \{v\}$.

By Remark 4.6 and the cardinality of $\mathcal{B}$, $\mathcal{B} = ((\mathcal{B}_1, U_1), (\mathcal{B}_2, U_2))$ is an $\overline{M_2}$-maximal assignment to T in which $v \in U_2$. So, there exists an $\overline{M_2}$-maximal assignment to T $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ in which $v \in V_1 \cup V_2$.

(1) By what we just proved, there exists an $\overline{M_2}$-maximal assignment to T,

$$\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2)),$$

in which $v \in V_1 \cup V_2$. Suppose without loss of generality that $v \in V_1 \setminus V_2$. We are going to construct an $\overline{M_2}$-maximal assignment to T, $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$, in which $v \in V'_1 \cap V'_2$. (see Remark 4.6). By Remark 4.5, 1 and 3, the component, R, of $T - V_2$ containing v, is in $\mathcal{A}_2$. Note that the peripheral arms of Q might be in R.
Let $\mathcal{A}'_1 = \mathcal{A}_1$,n $V'_1 = V_1$, $V'_2 = V_2 \cup \{v\}$ and

$$\mathcal{A}'_2 = (\mathcal{A}_2 \setminus \{R\}) \cup \{\text{two peripheral arms of length at least two of } Q\}.$$

Since $|\mathcal{A}_2| - |V_2| = |\mathcal{A}'_2| - |V'_2|$, by Remark 4.6 and the cardinality of $\mathcal{A}'$, $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$ is an $\overline{M_2}$-maximal assignment to T , in which $v \in V'_1 \cap V'_2$.

(2) By what we just proved, there exists an $\overline{M_2}$-maximal assignment,

$$\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2)),$$

in which $v \in V_1 \cup V_2$. Suppose $v \in V_1 \cap V_2$. We are going to construct an $\overline{M_2}$-maximal assignment to T, $\mathcal{A}'' = ((\mathcal{A}''_1, V''_1), (\mathcal{A}''_2, V''_2))$, in which $v \in V''_1 \setminus V''_2$. (see Remark 4.6) Using Remark 4.4, 1, each peripheral super path of Q is in $\mathcal{A}_1$. By Remark 4.5, 1, the longer arm of Q and the small pincers of Q are in $\mathcal{A}_2$ and there is not a peripheral arm of length 1 of Q in $\mathcal{A}_2$. By Remark 4.5, 2, $w \notin V_2$. Let R be the component of $T - V_2$ containing w and let F be the component of $T - ((V_2 \setminus \{v\}) \cup \{\text{the central vertex of each small pincer of } Q\})$ containing v and w. By Remark 4.5, 1, $F \in \mathcal{A}_2$.
Let $\mathcal{A}''_1 = \mathcal{A}_1$, $V''_1 = V_1$,

$$\begin{aligned}\mathcal{A}''_2 = \ & (\mathcal{A}_2 \setminus \{\text{the peripheral super paths of length at least two of } Q, R\}) \cup \\ & \cup \{\text{the peripheral arms of each small pincer of } Q, F\}\end{aligned}$$

and $V''_2 = (V_2 \setminus \{v\}) \cup \{\text{the central vertex of each small pincer of } Q\}$.
If Q does not have a longer arm or $R \notin \mathcal{A}_2$ then $|\mathcal{A}''_2| - |V''_2| > |\mathcal{A}_2| - |V_2|$. This is impossible because $\mathcal{A}$ is an $\overline{M_2}$-maximal assignment to T. So, $v \notin V_1 \cap V_2$.
If Q has a longer arm and $R \in \mathcal{A}_2$ then $|\mathcal{A}_2| - |V_2| = |\mathcal{A}''_2| - |V''_2|$. By Remark 4.6 and using the cardinality of $\mathcal{A}''$, $\mathcal{A}'' = ((\mathcal{A}''_1, V''_1), (\mathcal{A}''_2, V''_2))$ is an $\overline{M_2}$-maximal assignment to T, in which v in $V''_1 \setminus V''_2$.

(3) Let $\mathcal{A}''' = ((\mathcal{A}'''_1, V'''_1), (\mathcal{A}'''_2, V'''_2))$ be an $\overline{M_2}$-maximal assignment to T. By Remark 4.4, 1, each peripheral super path of Q belongs to $\mathcal{A}'''_1$ and $v \in V'''_1$. □

Lemma 4.9. *Let T be a tree that is not a small superstar. Suppose that v is a peripheral SHDV in T with Q, w as defined earlier in this section. Suppose that Q has f peripheral arms of length* 1 *and $g \leq 1$ peripheral arms of length at least* 2 *and the degree of w in T is* 2. *Then, there exists an $\overline{M_2}$-maximal assignment to T, $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$, in which:*

(1) *If $f + g \geq 1$, then $v \in V_1$ and the central vertex of each small pincer of Q belongs to V_2.*
(2) *If $f + g = 0$, then $v \in V_2$ and the central vertex of each small pincer of Q belongs to V_1.*

Proof

(1) By 2 of Lemma 4.8, let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an $\overline{M_2}$-maximal assignment to T such that v is exactly one V_1 or V_2.
If $f + g > 1$, since v is a peripheral SHDV and $w \notin V_1$ (the degree of w in T is 2), by Remark 4.4, 1, each peripheral super path of Q belongs to $\mathcal{A}_1$ and $v \in V_1$. In this case, because $v \notin V_2$ and $\mathcal{A}$ is an $\overline{M_2}$-maximal assignment to T, we conclude that the central vertex of each small pincer of Q is in V_2 and the peripheral arms of each small pincer of Q are in $\mathcal{A}_2$..
Suppose that $f + g = 1$ and $v \in V_2$. then by Remark 4.4, 1, the central vertex of each small pincer of Q is in V_1 and the peripheral arms of each small pincer of Q are in $\mathcal{A}_1$. By Remark 4.5, 1 and 3, the peripheral super paths of Q are in $\mathcal{A}_2$. Since w has degree two in T, we can assume that $w \notin V_1 \cup V_2$. We are going to construct an $\overline{M_2}$-maximal assignment to T, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$, in which $v \in V_1$ and the central vertex of each small pincer of Q is in V_2 (see Remark 4.6). Let R be the component of $T - V_1$ containing v, w. By Remark 4.4, 1, $R \in \mathcal{A}_1$. Let P be the component of $T - V_2$, containing w. Since $P \neq R$, by Remark 4.5, 1 and 3, $P \in \mathcal{A}_2$. Let B be the component of $T - ((V_2 \setminus \{v\}) \cup \{\text{the central vertex of each small pincer of } Q\})$, containing v and w. Let C be the component of $T - (V_1 \cup \{v\})$, containing w. Note that $B \neq C$.
Let

$$\begin{aligned} \mathcal{A}_1' = \; & (\mathcal{A}_1 \setminus \{\text{the peripheral arms of each small pincer of } Q, R\}) \cup \\ & \cup \{C, \; \text{the peripheral super paths of } Q\}, \end{aligned}$$

$V_1' = (V_1 \setminus \{\text{the central vertex of each small pincer of } Q\}) \cup \{v\}$,

$$\begin{aligned} \mathcal{A}_2' = \; & (\mathcal{A}_2 \setminus \{\text{the peripheral super paths of } Q, P\}) \cup \\ & \cup \{\text{the peripheral arms of each small pincer of } Q, B\} \end{aligned}$$

and $V_2' = (V_2 \setminus \{v\}) \cup \{\text{the central vertex of each small pincer of } Q\}$.
Since $|\mathcal{A}_1'| - |V_1'| = |\mathcal{A}_1| - |V_1|$ and $|\mathcal{A}_2'| - |V_2'| = |\mathcal{A}_2| - |V_2|$ and by Remark 4.6, we get an $\overline{M_2}$-maximal assignment to T, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$, where $v \in V_1'$ and the central vertex of each small pincer of Q belongs to V_2'.

(2) By 2 of Lemma 4.8, let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an $\overline{M_2}$-maximal assignment to T such that v is exactly one V_1 or V_2. Since $f + g = 0$, v is a peripheral SHDV and $w \notin V_1$, if $v \in V_1$ then by Remark 4.4, 1, the peripheral super paths of Q are in $\mathcal{A}_1$. Let F be the component of $T - V_1$ containing w. By Remark 4.4, 1, $F \in \mathcal{A}_1$. Let H be the component of $T - (V_1 \setminus \{v\}) \cup \{\text{the central vertex of each small pincer of } Q\})$ containing w and v. Let

$$\begin{aligned} \mathcal{A}_1' = \; & (\mathcal{A}_1 \setminus \{\text{the peripheral super paths of } Q, F\}) \cup \\ & \cup \{\text{the peripheral arms of each small pincer of } Q, H\}, \end{aligned}$$

$V_1' = (V_1 \setminus \{v\}) \cup \{\text{the central vertex of each small pincer of } Q\}$. Since $|\mathcal{A}_1'| - |V_1'| = |\mathcal{A}_1| - |V_1| + 1|$ we conclude that $\mathcal{A}$ is not an $\overline{M_2}$-maximal assignment to T. Impossible. Consequently, $v \notin V_1$ and $v \in V_2$. Therefore, the central vertex of each small pincer of Q belongs to V_1. □

Theorem 4.10. ($\overline{M_2}$ Reduction Theorem) *Let T be a tree that is not a small superstar and v a peripheral SHDV, with Q, w as defined earlier in this section. Suppose that Q has f peripheral arms of length 1, g peripheral arms of length at least 2 and h small pincers. Then:*

(A) *If $g \geq 2$, then $\overline{M_2}(T - Q) = \overline{M_2}(T) - f - 2g - 2h + 2$.*
(B) *If $g \leq 1$ and the degree of w in T is 2, then $\overline{M_2}((T - Q) +_w K_1) = \overline{M_2}(T) - f - g - 2h + 1$, where $(+_w K_1)$ means that we put a vertex adjacent to w.*
(C) *If $g \leq 1$, the degree of w in T is greater than 2 and $f + g > 2$ then*

$$\overline{M_2}((T - Q) +_w S_4) = \overline{M_2}(T) - f - g - 2h + 3,$$

where S_4 is the star with 3 arms of length 1 and $(+_w S_4)$ means that S_4 is adjacent to w by the central vertex.

Proof Part A: Let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an $\overline{M_2}$-maximal assignment to $T - Q$. We are going to construct an $\overline{M_2}$ assignment to T, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ (see Remark 4.6).

Let $\mathcal{A}_1' = \mathcal{A}_1 \cup \{$the peripheral super paths of $Q\}$, $V_1' = V_1 \cup \{v\}$, $\mathcal{A}_2' = \mathcal{A}_2 \cup \{$ the peripheral super paths of lenght at least two of $Q\}$ and $V_2' = V_2 \cup \{v\}$.

Since $M_1(T) = M_1(T - Q) + f + g + h - 1$, by Remark 4.6, this creates an $\overline{M_2}$ assignment to T, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ and $\overline{M_2}(T) \geq \overline{M_2}(T - Q) + f + g + h - 1 + g + h - 1 = \overline{M_2}(T - Q) + f + 2g + 2h - 2$.

Conversely, by Lemma 4.8, 1, there exists an $\overline{M_2}$-maximal assignment to T, $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$, in which v is in $V_1 \cap V_2$. We are going to construct an $\overline{M_2}$ assignment to $T - Q$, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$. By Remarks 4.4, 1 and 4.5,1 , each of the $f + g + h$ peripheral super paths of Q might be in $\mathcal{A}_1$ and each of the $g + h$ peripheral super paths of length at least 2 of Q might be in $\mathcal{A}_2$.

Let $\mathcal{A}_1' = \mathcal{A}_1 \setminus \{$the peripheral super paths of $Q\}$, $V_1' = V_1 \setminus \{v\}$, $\mathcal{A}_2' = \mathcal{A}_2 \setminus \{$ the peripheral super paths of lenght at least two of $Q\}$ and $V_2' = V_2 \setminus \{v\}$.

Since $M_1(T - Q) = M_1(T) - f - g - h + 1$, by Remark 4.6, this creates an $\overline{M_2}$ assignment to $T - Q$, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ and $\overline{M_2}(T - Q) \geq \overline{M_2}(T) - f - g - h + 1 - g - h + 1 = \overline{M_2}(T) - f - 2g - 2h + 2$. So, we have $\overline{M_2}(T - Q) = \overline{M_2}(T) - f - 2g - 2h + 2$.

Part B: Let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an $\overline{M_2}$-maximal assignment to $(T - Q) +_w K_1$. We are going to construct an $\overline{M_2}$ assignment to T, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ (see Remark 4.6). Let R be the component of $((T - Q) +_w K_1) - V_1$ containing K_1 and let U be the component of $((T - Q) +_w K_1) - V_2$ containing K_1. Since degree of w in T is 2, without loss of generality, by Remarks 4.4, 2, and 4.5, 2, we can assume that $w \in R \cap U$. Consequently, $R \neq K_1$ and $U \neq K_1$. By Remarks 4.4, 1 and 4.5, 1, R is in $\mathcal{A}_1$ and U is in $\mathcal{A}_2$.

Suppose that $f + g \geq 1$. Let P be the component of $T - (V_1 \cup \{v\})$ containing w and let H be the component of $T - (V_2 \cup \{$the central vertex of each small pincer of $Q\})$ containing w and v.

Let $\mathcal{A}_1' = (\mathcal{A}_1 \setminus \{R\}) \cup \{$ the peripheral super paths of Q, $P\}$, $V_1' = V_1 \cup \{v\}$, $\mathcal{A}_2' = (\mathcal{A}_2 \setminus \{U\}) \cup \{$the peripheral arms of each small pincer of Q, $H\}$ and $V_2' = V_2 \cup \{$ the central vertex of each small pincer of $Q\}$.

Since $M_1(T) = M_1((T - Q) +_w K_1) + f + g + h - 1$, by Remark 4.6, this creates an $\overline{M_2}$ assignment to T, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ and $\overline{M_2}(T) \geq \overline{M_2}((T-Q)+_w K_1)+f+g+h-1+2h-h = \overline{M_2}((T-Q)+_w K_1)+f+g+2h-1$.

Suppose that $f + g = 0$. Let B be the component of $T - (V_2 \cup \{v\})$ containing w and let C be the component of $T - (V_1 \cup \{$the central vertex of each small pincer of $Q\}$ containing w and v.

Let $\mathcal{A}_1' = (\mathcal{A}_1 \setminus \{R\}) \cup \{$the peripheral arms of each small pincer of Q, $C\}$, $V_1' = V_1 \cup \{$ the central vertex of each small pincer of $Q\}$, $V_2' = V_2 \cup \{v\}$ and $\mathcal{A}_2' = (\mathcal{A}_2 \setminus \{U\}) \cup \{$the peripheral super paths of Q, $B\}$.

Since $M_1(T) = M_1((T - Q) +_w K_1) + h$, by Remark 4.6, this creates an $\overline{M_2}$ assignment to T, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ and $\overline{M_2}(T) \geq \overline{M_2}((T - Q) +_w K_1) + 2h - h + h - 1 = \overline{M_2}((T - Q) +_w K_1) + f + g + 2h - 1$.

Conversely, suppose that $f + g \geq 1$. By Lemma 4.9, 1, there exists an $\overline{M_2}$-maximal assignment to T, $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$, in which v is in V_1 and the central vertex of each small pincer of Q is in V_2. We are going to construct an $\overline{M_2}$ assignment to $(T - Q) +_w K_1$, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ (see Remark 4.6). By Remarks 4.4, 1 and 2, and 4.5, 1 and 3, each of the $f + g + h$ peripheral super paths of Q might be in $\mathcal{A}_1$, the peripheral arms of each small pincer of Q might be in $\mathcal{A}_2$ and $w \notin V_1 \cup V_2$. Let R be the component of $T - V_1$ containing w and let P be the component of $T - V_2$ containing v and w. By Remarks 4.4, 1 and 4.5, 1, $R \in \mathcal{A}_1$ and $P \in \mathcal{A}_2$. Let R' be the component of $((T - Q) +_w K_1) - (V_1 \setminus \{v\})$ containing w and K_1, and let P' be the component of $(T - Q) +_w K_1) - (V_2 \setminus \{$the central vertex of each small pincer of $Q\})$ containing w and K_1.

Let $\mathcal{A}_1' = (\mathcal{A}_1 \setminus \{R,$ the peripheral super paths of $Q\}) \cup \{R'\}$, $V_1' = V_1 \setminus \{v\}$, $\mathcal{A}_2' = (\mathcal{A}_2 \setminus \{P,$ the peripheral arms of each small pincer of $Q\}) \cup \{P'\}$ and $V_2' = V_2 \setminus \{$the central vertex of each small pincer of $Q\}$.

Since $M_1((T-Q)+_w K_1) = M_1(T)-f-g-h+1$, by Remark 4.6, this creates an $\overline{M_2}$ assignment to $(T-Q)+_w K_1$, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ and $\overline{M_2}((T - Q) +_w K_1) \geq \overline{M_2}(T) - f - g - h + 1 - 2h + h = \overline{M_2}(T) - f - g - 2h + 1$.

Suppose that $f + g = 0$. By Lemma 4.9, 2, there exists an $\overline{M_2}$-maximal assignment to T, $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$, in which v is in V_2, the central vertex of each small pincer of Q is in V_1 and $w \notin V_1 \cup V_2$. We are going to construct an $\overline{M_2}$ assignment to $(T - Q) +_w K_1$, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$. By Remarks 4.4, 1 and

4.5, 1 and 3, each of the h small pincers of Q might be in $\mathcal{A}_2$ and the peripheral arms of each small pincer of Q might be in $\mathcal{A}_1$. Let R be the component of $T - V_1$ containing v, w and let P be the component of $T - V_2$ containing w. By Remarks 4.4, 1 and 4.5, 1 and 3, $R \in \mathcal{A}_1$ and $P \in \mathcal{A}_2$. Let P' be the component of $((T - Q) +_w K_1) - (V_2 \setminus \{v\})$ containing $w$and K_1, and let R' be the component of $((T - Q) +_w K_1) - (V_1 \setminus$ {the central vertex of each small pincer of $Q\})$ containing w and K_1.

Let $\mathcal{A}_1' = (\mathcal{A}_1 \setminus \{R,$ the peripheral arms of each small pincer of $Q\}) \cup \{R'\}$, $V_1' = V_1 \setminus \{$ the central vertex of each small pincer of $Q\}$, $V_2' = V_2 \setminus \{v\}$ and $\mathcal{A}_2' = (\mathcal{A}_2 \setminus \{P,$ the peripheral super paths of $Q\}) \cup \{P'\}$.

Since $M_1((T - Q) +_w K_1) = M_1(T) - h$, by Remark 4.6, this creates an $\overline{M_2}$ assignment to $(T - Q) +_w K_1$, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ and $\overline{M_2}((T - Q) +_w K_1) \geq \overline{M_2}(T) - 2h + h - h + 1 = \overline{M_2}(T) - f - g - 2h + 1$.

So, we have $\overline{M_2}((T - Q) +_w K_1) = \overline{M_2}(T) - f - g - 2h + 1$.

Part C: Let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an $\overline{M_2}$-maximal assignment to $(T - Q) +_w S_4$. We are going to construct an $\overline{M_2}$ assignment to T, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$. Let x be the central vertex of S_4.

By Lemma 4.8, 3 and by Remark 4.4, 1, $x \in V_1$ and the peripheral arms of S_4 are in $\mathcal{A}_1$. By Remark 4.5, 4, $x \notin V_2$. Let R be the component of $((T - Q) +_w S_4) - V_2$ containing S_4. By Remark 4.5, 1, R is in $\mathcal{A}_2$. Let R' be the component of $T - (V_2 \cup \{$the central vertex of each small pincer of $Q\})$ containing v.

Let $\mathcal{A}_1' = (\mathcal{A}_1 \setminus \{$the peripheral arms of $S_4\}) \cup \{$the peripheral super paths of $Q\}$, $V_1' = (V_1 \setminus \{x\}) \cup \{v\}$, $V_2' = V_2 \cup \{$ the central vertex of each small pincer of $Q\}$ and $\mathcal{A}_2' = (\mathcal{A}_2 \setminus \{R\}) \cup \{$ the peripheral arms of each small pincer of Q, $R'\}$.

Since $M_1(T) = M_1((T - Q) +_w S_4) + f + g + h - 3$, by Remark 4.6, this creates an $\overline{M_2}$ assignment to T, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ and $\overline{M_2}(T) \geq \overline{M_2}((T-Q)+_w S_4)+f+g+h-3+2h-h = \overline{M_2}(T-Q+_w S_4)+f+g+2h-3$.

Conversely, let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an $\overline{M_2}$-maximal assignment to T. By Lemma 4.8 3, v is in V_1.

If $v \in V_2$ then by Remark 4.5, 1, the longer arm and the small pincers of Q are in $\mathcal{A}_2$. By Remark 4.4, 1 , each of the $f + g + h$ peripheral super paths of Q might be in $\mathcal{A}_1$. If $w \notin V_2$, then let F be the component of $T - V_2$ containing w. Let H be the component of $T - ((V_2 \setminus \{v\}) \cup \{$the central vertex of each small pincer of $Q\})$ containing v.

Let $\mathcal{B}_1 = \mathcal{A}_1$, $U_1 = V_1$, $\mathcal{B}_2 = (\mathcal{A}_2 \setminus \{F,$ the longer arm and the small pincers of $Q\} \cup \{$ the peripheral arms of each small pincer of Q, $H\}$ and $U_2 = (V_2 \setminus \{v\}) \cup \{$the central vertex of each small pincer of $Q\}$.

By Remark 4.6, this creates an $\overline{M_2}$ assignment to T, $\mathcal{B} = ((\mathcal{B}_1, U_1), (\mathcal{B}_2, U_2))$. Using the cardinality of $\mathcal{B}$ we conclude that $g = 1$, $w \notin V_2$ and $F \in \mathcal{A}_2$.

We are going to construct, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$, an $\overline{M_2}$ assignment to $(T-Q)+_w S_4$. Let x be the central vertex of S_4. Let R' be the component of $((T - Q) +_w S_4) - (V_2 \setminus \{v\})$ containing x.

Let $\mathcal{A}_1' = (\mathcal{A}_1 \setminus \{$the peripheral super paths of $Q\}) \cup \{$the peripheral arms of $S_4\}$, $V_1' = (V_1 \setminus \{v\}) \cup \{x\}$, $\mathcal{A}_2' = (\mathcal{A}_2 \setminus \{F,$ the longer arm and the small pincers of $Q\} \cup \{R'\}$ and $V_2' = V_2 \setminus \{v\}$.

Since $M_1((T-Q)+_w S_4) = M_1(T)-f-g-h+3$, by Remark 4.6, this creates an $\overline{M_2}$ assignment to $(T-Q)+_w S_4$, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ and $\overline{M_2}((T - Q) +_w S_4) \geq \overline{M_2}(T)-f-g-h+3-1-1-h+1+1 = \overline{M_2}(T)-f-g-2h+3$.

If $v \notin V_2$, using the maximality of $|\mathcal{A}_2| - |V_2|$, then the central vertex of each small pincer of Q is in V_2. We are going to construct an $\overline{M_2}$ assignment to $(T - Q) +_w S_4$, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$. By Remarks 4.4, 1 and 4.5, 1 and 3, each of the $f + g + h$ peripheral super paths of Q might be in $\mathcal{A}_1$ and the peripheral arms of each small pincer of Q might be in $\mathcal{A}_2$. Let R be the component of $T - V_2$ containing v. Let R' be the component of $((T - Q) +_w S_4) - V_2$ containing x (x is the central vertex of S_4).

Let $\mathcal{A}_1' = (\mathcal{A}_1 \setminus \{$the peripheral super paths of $Q\}) \cup \{$the peripheral arms of $S_4\}$, $V_1' = (V_1 \setminus \{v\}) \cup \{x\}$, $\mathcal{A}_2' = (\mathcal{A}_2 \setminus \{R,$ the peripheral arms of each small pincer of $Q \cup \{R'\}$ and $V_2' = V_2 \setminus \{$ the central vertex of each small pincer of $Q\}$.

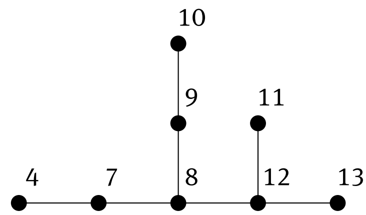
Since $M_1((T-Q)+_w S_4) = M_1(T)-f-g-h+3$, by Remark 4.6, this creates an $\overline{M_2}$ assignment to $(T-Q)+_w S_4$, $\mathcal{A}' = ((\mathcal{A}_1', V_1'), (\mathcal{A}_2', V_2'))$ and $\overline{M_2}((T - Q) +_w S_4) \geq \overline{M_2}(T) - f - g - h + 3 - 2h + h = \overline{M_2}(T) - f - g - 2h + 3$.

Consequently, we have $\overline{M_2}((T - Q) +_w S_4) = \overline{M_2}(T) - f - g - 2h + 3$. □

Example 4.11. *Let T be the tree of* Example 3.17. *Let Q be the subtree of T generated by vertices* 1, 2, 3, 4, 5, 6. *Since Q is a small superstar* (T *is not a small superstar*) *with* 1 *arm of length* 1, 1 *small pincer, and* 7 *is a vertex* *of T with degree* 2*, by* Theorem 4.10*,*

$$\overline{M_2}(T) = \overline{M_2}((T - Q) +_w K_1) + 2,$$

where w is the vertex 7*. So,* $(T - Q) +_w K_1$ *(that is a small superstar with central vertex* 8*) is the tree*



By Proposition 3.14,

$$\overline{M_2}(((T - Q) +_w K_1) - J) = 2 + 4 - 2 = 2.$$

Therefore,

$$\overline{M_2}(T) = 6.$$

□

Acknowledgement: This work was partially supported by *Fundação para a Ciência e Tecnologia* and was done within the activities of the *Centro de Estruturas Lineares e Combinatórias*.

References

[1] R. Fernandes. On the inverse eigenvalue problems: the case of superstars. *Electronic Journal of Linear Algebra* **18** (2009), 442-461.

[2] R. Horn and C.R. Johnson., Matrix Analysis, *Cambridge University Press* **New York** (1985).

[3] C.R. Johnson and A.Leal Duarte., The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree, *Linear and Multilinear Algebra* **46** (1999), 139-144.

[4] C.R. Johnson and A.Leal Duarte., On the possible multiplicities of the eigenvalues of a Hermitian matrix whose graph is a tree, *Linear Algebra and Applications* **248** (2002), 7-21.

[5] C.R. Johnson, A.Leal Duarte and C.M. Saiago., The Parter-Wiener theorem: refinement and generalization, *SIAM Journal on Matrix Analysis and Applications* **25 (2)** (2003), 352-361.

[6] C.R. Johnson, A.Leal Duarte and C.M. Saiago., Inverse eigenvalue problems and lists of multiplicities of eigenvalues for matrices whose graph is a tree: The case of generalized stars and double generalized stars, *Linear Algebra and its Applications* **373** (2003), 311-330.

[7] C.R. Johnson, C. Jordan-Squire and D.A. Sher., Eigenvalue assignments and the two largest multiplicities in a Hermitian matrix whose graph is a tree, *Discrete Applied Mathematics* **158** (2010), 681-691.

[8] S. Parter., On the eigenvalues and eigenvectors of a class of matrices, *Journal of the Society for Industrial and Applied Mathematics* **8** (1960), 376-388.

[9] G. Wiener., Spectral multiplicity and splitting results for a class of qualitative matrices, *Linear Algebra and its Applications* **61** (1984), 15-18.