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Licenciado em Matemática

A coalgebraic approach to fuzzy automata

Dissertação para obtenção do Grau de Mestre em

Matemática e Aplicações - Álgebra, Lógica e Computação

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FACULDADE DE CIÊNCIAS E TECNOLOGIA
UNIVERSIDADE NOVA DE LISBOA

Março, 2016

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Acknowledgements

I would like to thank my thesis advisor Prof. Luís Monteiro. His knowledge and patience were key to introduce me to the Coalgebra Theory. His support and guidance helped me throughout the research and writing of this thesis. I would like to express my sincere gratitude for being my supervisor in the research project *QAIS* (see below), where the research for this thesis has been done.

I would also like to thank my thesis co-advisor Prof. António Malheiro. His suggestions and encouragement helped me throughtout the process of structuring and writing this thesis. His successive reviews of the writing allowed me to submit a better thesis.

I would also like to thank Prof. João Nuno Gonçalves Faria Martins. His mentoring on Category Theory was very important in the research and the writing of this thesis. Whenever I needed help, Prof. João Martins was available either in his office or by email. Unfortunately, for bureaucratic reasons, Prof. João Martins could not be accounted as co-advisor, although he acted as one.

I would also like to thank the Departamento de Matemática da Faculdade de Ciências e Tecnologia da Universidade Nova de Lisboa. In particular, I would like to acknowledge all the professors who taught me making my last seven years in this department wonderfull.

Finally, I must thank my parents and my sister. Their unfailing support makes my life easier allowing me to focus on what is really important.

FCT
Fundação para a Ciência e a Tecnologia
MINISTÉRIO DA CIÊNCIA, TECNOLOGIA E ENSINO SUPERIOR

I, the author, was funded by FCT under the project *QAIS: Análise quantitativa de sistemas reactivos: fundamentos e algoritmos*, PTDC/EIA-CCO/122240/2010.

Abstract

In this thesis, we make a coalgebraic description of fuzzy automata allowing their integration in much general context. Thus, results obtained indivudually to fuzzy automata end up to be consequence of their coalgebraic description. In particular, a coalgebraic definition of the fuzzy language recognized by a fuzzy automaton is obtained. And, by defining a monad for fuzzy sets, a functor that describes a determinization process via a generalization of the powerset construction is obtained.

Keywords: Fuzzy automaton, Fuzzy language, Coalgebra, Determinization, Fuzzy set, Fuzzy relation

Resumo

Nesta tese, faz-se uma abordagem co-algébrica aos autómatos vagos, permitindo a sua integração num contexto mais geral. Deste modo, resultados obtidos especificamente para autómatos vagos são consequência da descrição co-algébrica elaborada. É obtida uma definição co-algébrica da linguagem vaga reconhecida por autómatos vagos e, com a definição de uma mónada para os conjuntos vagos, é obtido um functor que descreve um processo de determinização para os autómato vagos.

Palavras-chave: Autómato vago, Linguagem vaga, Co-álgebra, Determinização, Conjunto vago, Relação vaga

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Chapter 1

Introduction

In Computer Science, an adequate algebraic formalization for general discrete algebraic systems, in addition accommodating the a) classical notions of automata, b) formal languages and computatibily, c) computation models for programming languages semantics, has proven to be a difficult task. Recently this difficulty was overcome by using the theory of coalgebras.

From a categorical point of view the notion of a coalgebra is dual to that of an algebra. The axioms of coalgebras are exactly dual to the axioms of algebras of a functor in Category Theory. Thus, the Theory of Coalgebras is a subject in the Category Theory.

Fuzzy automata are a type of transition system, where sets and relations are fuzzy. Several transition systems are already studied from a coalgebraic point of view (e.g. [Rut00; Sil+13]), therefore our coalgebraic description of fuzzy automata integrates them in a much general context. This framework allowed us to obtain the following main results: a) a coalgebraic definition of the fuzzy language recognized by a fuzzy automaton, b) the definition of a functor that describes the determinization process of a fuzzy automata via a generalization of the powerset construction, c) a coalgebraic definition of bisimulation on fuzzy automata allowing the construction of a quotient fuzzy automaton. To achieve these results, a generalization of the powerset monad and its algebras were obtained for fuzzy sets. We obtained more results such as a coalgebraic description of trace equivalence, namely bisimilarity, behaviour and semantical state equivalence for fuzzy automata, but they were not included in this thesis for time reasons.

Chapter 2

Categories

In this chapter, we present some basic definitions in Category Theory. We assume that the foundations of the theory are known, such as the definitions of category and functor. See [Adá+90; ML98] for a full introduction.

In Section 2.1, we introduce most of the notation that will be used in the rest of the thesis, we also present some basic results that will be useful further. In Section 2.2, we construct a one-to-one correspondence between monads and Kleisli triples allowing us to use either of the concepts depending on the context. Finally, in Section 2.3, we present the category of the algebras of a monad and then show how morphism can be extend and functors can be lifted to this category.

2.1 Basics on categories

Let $\mathbf{C}$ be a category, we denote the classes of its objects and morphisms by $\mathrm{Ob}(\mathbf{C})$ and $\mathrm{Mor}(\mathbf{C})$, respectively. For simplicity, we write $X \in \mathbf{C}$ for $X \in \mathrm{Ob}(\mathbf{C})$ and f in $\mathbf{C}$ for $f \in \mathrm{Mor}(\mathbf{C})$. Unless specified beforehand, X, Y and Z are arbitrary objects in (a context category) $\mathbf{C}$. We denote by $\mathbf{C}(X,Y)$ the set of morphisms from X to Y, i.e. with domain X and codomain Y, in $\mathbf{C}$, and adopt the arrow notation $f : X \to Y$ or $X \xrightarrow{f} Y$ to denote $f \in \mathbf{C}(X,Y)$. Also, we denote the indentity morphism on X by id_X, and given morphisms $f : X \to Y$ and $g : Y \to Z$, we write $g \circ f$ for their composition (from X to Z). The notation $X \cong Y$ means that X is isomorphic to Y, i.e. there are morphisms $f : X \to Y$ and $f^{-1} : Y \to X$ such that $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$.

Given categories $\mathbf{D}$ and $\mathbf{E}$, to denote a functor F from $\mathbf{C}$ to $\mathbf{D}$ we also use the arrow notation $F : \mathbf{C} \to \mathbf{D}$ or $\mathbf{C} \xrightarrow{F} \mathbf{D}$. However, for functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{E}$, we write simply GF for their composition (from $\mathbf{C}$ to $\mathbf{E}$). For a functor $T : \mathbf{C} \to \mathbf{C}$ on $\mathbf{C}$, we define inductively

$$T^0 = \mathrm{Id}_{\mathbf{C}} \qquad \text{and} \qquad T^{n+1} = TT^n, \qquad n \in \mathbb{N},$$

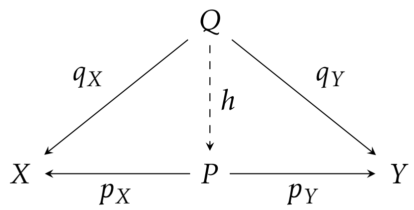
where $\mathrm{Id}_{\mathbf{C}}$ is the identity functor on $\mathbf{C}$ and $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of natural numbers.

We consider **Set** to be the category with objects all sets, morphisms all functions between them and the usual function composition. For sets X and Y, instead of $\mathbf{Set}(X, Y)$, we write Y^X to denote the set of all functions from X to Y. We also consider that a number $n \in \mathbb{N}$ may represent a set with exactly n elements, namely $1 = \{*\}$ and $2 = \{0, 1\}$. Note that $X \cong Y$ means that there is a bijection from X to Y.

Finally, when we consider a family $(a_i)_{i\in I}$, unless said otherwise, it is a set-indexed family, i.e. I is a set. A family $(f_i : X_i \to Y_i)_{i\in I}$ denotes a family of morphisms $(f_i)_{i\in I}$ such that $f_i : X_i \to Y_i$, for each $i \in I$.

2.1.1 Products

Definition 2.1. Let X and Y be objects in a category $\mathbf{C}$. A *product* of X and Y is a pair $(P, (p_X, p_Y))$ consisting of an object $P \in \mathbf{C}$ and morphisms $p_X : P \to X$ and $p_Y : P \to Y$ in $\mathbf{C}$ (called the *projections*), such that for any object $Q \in \mathbf{C}$ and morphisms $q_X : Q \to X$ and $q_Y : Q \to Y$ in $\mathbf{C}$, there exists a unique morphism $h : Q \to P$ which makes the diagram



commute, i.e. $q_X = p_X \circ h$ and $q_Y = p_Y \circ h$. Since a product (when exists) is unique up to isomorphism, we denote P by $X \times Y$ and h by $\langle q_X, q_Y \rangle$.

In general, a *product* of a family $(X_i)_{i\in I}$ of objects in $\mathbf{C}$, indexed by a set I, is a pair $(\prod_{i\in I} X_i, (p_i)_{i\in I})$, where $\prod_{i\in I} X_i \in \mathbf{C}$ and $(p_j : \prod_{i\in I} X_i \to X_j)_{j\in I}$, such that for any object $Q \in \mathbf{C}$ and family of morphisms $(q_i : Q \to X_i)_{i\in I}$ in $\mathbf{C}$, there exists a unique morphism $\langle q_i \rangle_{i\in I} : Q \to \prod_{i\in I} X_i$ satisfying $q_j = p_j \circ \langle q_i \rangle_{i\in I}$, for each $j \in I$.

Let X and Y be objects in a category $\mathbf{C}$ and assume that their product $(X \times Y, (p_X, p_Y))$ exists. If $h : Q \to X \times Y$ is a morphism in $\mathbf{C}$, then h is unambiguously determined by $q_X = p_X \circ h : Q \to X$ and $q_Y = p_Y \circ h : Q \to Y$, since $h = \langle q_X, q_Y \rangle$. Therefore, any morphism from an object $Q \in \mathbf{C}$ to a product $X \times Y$ will be usually described by its composition with the projections.

Given objects X, X', Y, Y', for which products $(X \times Y, (p_X, p_Y))$ and $(X' \times Y', (p_{X'}, p_{Y'}))$ exist, and morphisms $f : X \to X'$ and $g : Y \to Y'$ in a category $\mathbf{C}$, we define $f \times g : X \times Y \to X' \times Y'$ to be the only morphism that makes commutative the following diagram

$$\begin{array}{ccccc} X & \xleftarrow{p_X} & X \times Y & \xrightarrow{p_Y} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times g} & & \downarrow{\scriptstyle g} \\ X' & \xleftarrow[p_{X'}]{} & X' \times Y' & \xrightarrow[p_{Y'}]{} & Y', \end{array}$$

and so $f \times g = \langle f \circ p_X, g \circ p_Y \rangle$.

In **Set**, a product of any sets X and Y exists and it corresponds (up to bijection) to the cartesian product

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\},$$

with the respective projections

$$p_X(x, y) = x \qquad \text{and} \qquad p_Y(x, y) = y, \qquad (x, y) \in X \times Y.$$

Given functions $q_X : Q \to X$ and $q_Y : Q \to Y$, we have that

$$\langle q_X, q_Y \rangle(z) = (q_X(z), q_Y(z)), \qquad z \in Q.$$

And, for any functions $f : X \to X'$ and $g : Y \to Y'$, we have that

$$(f \times g)(x, y) = (f(x), g(y)), \qquad (x, y) \in X \times Y.$$

Note that, for sets X, Y and Z,

$$Z^{X \times Y} \cong (Z^Y)^X.$$

The process that maps a function $f : X \times Y \to Z$ to a function $f_c : X \to Z^Y$, such that $f_c(x)(y) = f(x, y)$ (with $x \in X$ and $y \in Y$), is called *currying*[1]. On the other hand, the inverse process which maps a function $f : X \to Z^Y$ to a function $f_u : X \times Y \to Z$, such that $f_u(x, y) = f(x)(y)$ (with $(x, y) \in X \times Y$), is called *uncurrying*. For the sake of simplicity, we drop the subscript letters (c and u) letting f denote both functions, where the context identifies which is being used.

Finally, the product of a family of sets $(X_i)_{i \in I}$, indexed by a set I, is again the (general) cartesian product

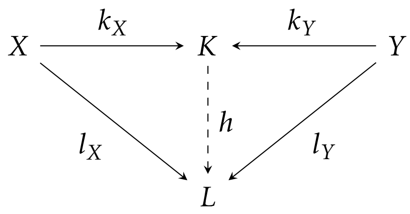
$$\prod_{i \in I} X_i = \{(x_i)_{i \in I} \mid x_i \in X_i \text{ for each } i \in I\}$$

together with the projections $(p_j : \prod_{i \in I} X_i \to X_j)_{j \in I}$ where $p_j(x_i)_{i \in I} = x_j$. For a family of functions $(q_i : Q \to X_i)_{i \in I}$, we have $\langle q_i \rangle_{i \in I}(z) = (q_i(z))_{i \in I}$, for all $z \in Q$.

[1] a common term in functional programming, named after H. B. Curry who introduced the method.

2.1.2 Coproducts

Definition 2.2. Let X and Y be objects in a category $\mathbf{C}$. A *coproduct* of X and Y is a pair $(K,(k_X,k_Y))$ consisting of an object $K \in \mathbf{C}$ and morphisms $k_X : X \to K$ and $k_Y : Y \to K$ in $\mathbf{C}$ (called the *coprojections*), such that for any object $L \in \mathbf{C}$ and morphisms $l_X : X \to L$ and $l_Y : Y \to L$, there exists a unique morphism $h : K \to L$ which makes the diagram



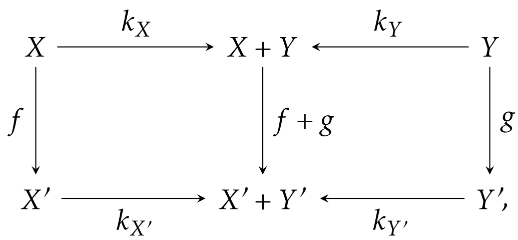
commute, i.e. $l_X = h \circ k_X$ and $l_Y = h \circ k_Y$. Since a coproduct (when exists) is unique up to isomorphism, we denote K by $X + Y$ and h by $[l_X, l_Y]$.

In general, a *coproduct* of a family $(X_i)_{i\in I}$ of objects in $\mathbf{C}$, indexed by a set I, is a pair $(\sum_{i\in I} X_i, (k_i)_{i\in I})$, where $\sum_{i\in I} X_i \in \mathbf{C}$ and $(k_j : X_j \to \sum_{i\in I} X_i)_{j\in I}$, such that for any object $L \in \mathbf{C}$ and family of morphisms $(l_i : X_i \to L)_{i\in I}$ in $\mathbf{C}$, there exists a unique morphism $[l_i]_{i\in I} : \sum_{i\in I} X_i \to L$ satisfying $l_j = [l_i]_{i\in I} \circ k_j$, for each $j \in I$.

We remark that coproducts could be defined as the dual of products. In that case, $(K,(k_X,k_Y))$ is a coproduct of X and Y in a category $\mathbf{C}$ if it is a product of X and Y in the opposite category $\mathbf{C}^{\mathrm{op}}$ [Adá+90, Definition 3.5]. Also, the following notes are dual to the ones done for products.

Let X and Y be objects in a category $\mathbf{C}$ and assume that their coproduct $(X+Y,(k_X,k_Y))$ exists. If $h : X + Y \to L$ is a morphism in $\mathbf{C}$, then h is unambiguously determined by $l_X = h \circ k_X : X \to L$ and $l_Y = h \circ k_Y : Y \to L$, since $h = [l_X, l_Y]$. Therefore, any morphism from a coproduct $X + Y$ to an object $L \in \mathbf{C}$ will be usually described by its composition with the coprojections.

Given objects X, X', Y, Y', for which coproducts $(X + Y, (k_X, k_Y))$ and $(X' + Y', (k_{X'}, k_{Y'}))$ exist, and morphisms $f : X \to X'$ and $g : Y \to Y'$ in a category $\mathbf{C}$, we define $f + g : X + Y \to X' + Y'$ to be the only morphism that makes commutative the following diagram



and so $f + g = [k_{X'} \circ f, k_{Y'} \circ g]$.

In **Set**, a coproduct of any sets X and Y exists and it corresponds (up to bijection) to the disjoint union

$$X + Y = (X \times \{1\}) \cup (Y \times \{2\}) = \{(x,1) \mid x \in X\} \cup \{(y,2) \mid y \in Y\},$$

with the respective coprojections (also known as *canonical injections*)

$$k_X(x) = (x, 1) \qquad \text{and} \qquad k_Y(y) = (y, 2), \qquad x \in X,\ y \in Y.$$

Given functions $l_X : X \to L$ and $l_Y : Y \to L$, we have that

$$[l_X, l_Y](z, i) = \begin{cases} l_X(z) & \text{if } z \in X, i = 1 \\ l_Y(z) & \text{if } z \in Y, i = 2, \end{cases} \qquad (z, i) \in X + Y.$$

And, for any functions $f : X \to X'$ and $g : Y \to Y'$, we have that

$$(f + g)(z, i) = \begin{cases} (f(z), i) & \text{if } z \in X, i = 1 \\ (g(z), i) & \text{if } z \in Y, i = 2, \end{cases} \qquad (z, i) \in X + Y.$$

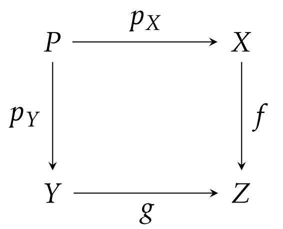
In general, the coproduct of a family of sets $(X_i)_{i \in I}$, indexed by a set I, is again the (general) disjoint union

$$\sum_{i \in I} X_i = \bigcup_{i \in I} (X_i \times \{i\}) = \{(x, i) \mid x \in X_i, i \in I\}$$

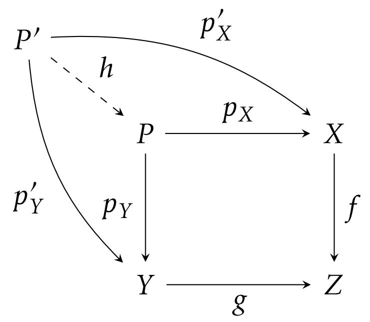
together with the coprojections $(k_j : X_j \to \sum_{i \in I} X_i)_{j \in I}$ where $k_j(x) = (x, j)$. For a family of function $(l_i : X_i \to L)_{i \in I}$, we have $[l_i]_{i \in I}(x, j) = l_j(x)$, for all $(x, j) \in \sum_{i \in I} X_i$.

2.1.3 Pullbacks

Definition 2.3. Let $f : X \to Z$ and $g : Y \to Z$ be morphisms with the same codomain in a category $\mathbf{C}$. A *pullback* of f and g is a pair $(P, (p_X, p_Y))$ consisting of an object $P \in \mathbf{C}$ and morphisms $p_X : P \to X$ and $p_Y : P \to Y$ in $\mathbf{C}$ such that the diagram



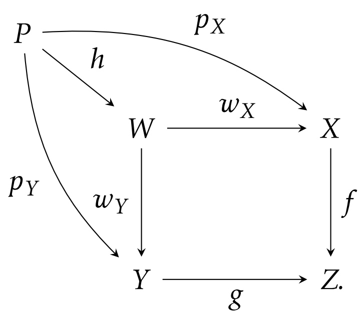
commutes, i.e. $f \circ p_X = g \circ p_Y$, and for any object $P' \in \mathbf{C}$ and morphisms $p'_X : P' \to X$ and $p'_Y : P' \to Y$ with $f \circ p'_X = g \circ p'_Y$, there exists a unique morphism $h : P' \to P$ which makes the diagram



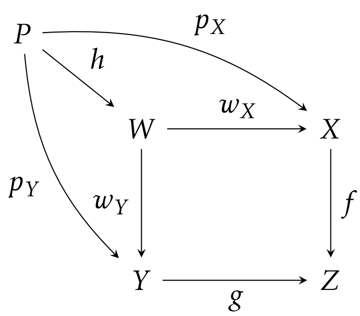
commute, i.e. $p'_X = p_X \circ h$ and $p'_Y = p_Y \circ h$. If we do not require h to be unique (for each object and morphisms), then $(P,(p_X,p_Y))$ is called a *weak pullback*.

Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in a category $\mathbf{C}$. We remark that if $(P,(p_X,p_Y))$ and $(P',(p'_X,p'_Y))$ are pullbacks of f and g, then there is an isomorphism $h : P' \to P$ with $p'_X = p_X \circ h$ and $p'_Y = p_Y \circ h$, and so a pullback (when exists) is unique up to isomorphism. Although this does not apply to weak pullbacks, we still have the following result.

Theorem 2.4. *Let $(P,(p_X,p_Y))$ be a weak pullback of morphisms $f : X \to Z$ and $g : Y \to Z$ in a category $\mathbf{C}$. A pair $(W,(w_X,w_Y))$, where $W \in \mathbf{C}$, $w_X : W \to X$ and $w_Y : W \to Y$, is a weak pullback of f and g if, and only if, $f \circ w_X = g \circ w_Y$ and there is $h : P \to W$ such that $p_X = w_X \circ h$ and $p_Y = w_Y \circ h$, i.e. the following diagram is commutative*

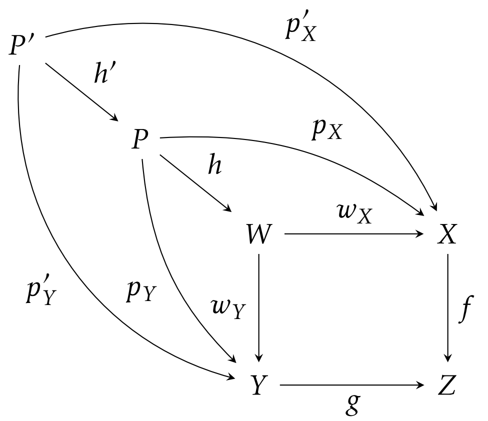


Proof. Assume that $(W,(w_X,w_Y))$ is a weak pullback of f and g. In particular, $f \circ w_X = g \circ w_Y$. Since $(P,(p_X,p_Y))$ is a weak pullback, $f \circ p_X = g \circ p_Y$. Thus, there is $h : P \to W$ which makes the diagram



commute, i.e. $p_X = w_X \circ h$ and $p_Y = w_Y \circ h$, due to $(W,(w_X,w_Y))$ be a weak pullback of f and g.

Conversely, suppose that $f \circ w_X = g \circ w_Y$, and let $h : P \to W$ be such that $p_X = w_X \circ h$ and $p_Y = w_Y \circ h$. If $P' \in \mathbf{C}$, $p'_X : P' \to X$ and $p'_Y : P' \to Y$ are such that $f \circ p'_X = g \circ p'_Y$, then there is $h' : P' \to P$ which makes the diagram



commute, since $(P,(p_X,p_Y))$ is a weak pullback. Therefore, $(W,(w_X,w_Y))$ is a weak pullback of f and g. □

In **Set**, a pullback of any functions $f : X \to Z$ and $g : Y \to Z$ exists. For instance, a pullback of $f : X \to Z$ and $g : Y \to Z$ is the set

$$P = \{(x,y) \in X \times Y \mid f(x) = g(y)\}$$

together with the projections (restricted to P) $p_X : P \to X$ and $p_Y : P \to Y$ defined by

$$p_X(x,y) = x \quad \text{and} \quad p_Y(x,y) = y, \qquad (x,y) \in P.$$

Since any other pullback of f and g is bijective to $(P,(p_X,p_Y))$, we will use it by default.

Example 2.5. In **Set**, let $f : X \to Y$ be a function. A pullback of f with itself is formed by

$$\ker(f) = \{(x,y) \in X \times X \mid f(x) = f(y)\},$$

the kernel of f, and the projections (restricted to $\ker(f)$) $p_1 : \ker(f) \to X$ and $p_2 : \ker(f) \to X$ where

$$p_1(x,y) = x \qquad \text{and} \qquad p_2(x,y) = y, \qquad (x,y) \in \ker(f).$$

Definition 2.6. Let **C** and **D** be categories, and let f and g be morphisms with common codomain in **C**. A functor $F : \mathbf{C} \to \mathbf{D}$ is said to *preserve (weak) pullbacks of f and g* provided that for every (weak) pullback $(P,(p_X,p_Y))$ of f and g in **C**, $(F(P),(F(p_X),F(p_Y)))$ is a (weak) pullback of $F(f)$ and $F(g)$ in **D**.

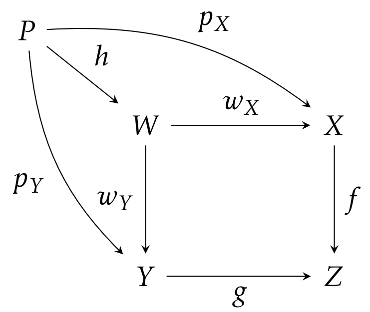
If $F : \mathbf{C} \to \mathbf{D}$ preserves (weak) pullbacks of any two morphisms with common codomain in **C**, then F is said to *preserve (weak) pullbacks*.

Note that if $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{E}$ are functors that preserve (weak) pullbacks, then their composition $GF : \mathbf{C} \to \mathbf{E}$ also preserves (weak) pullbacks.

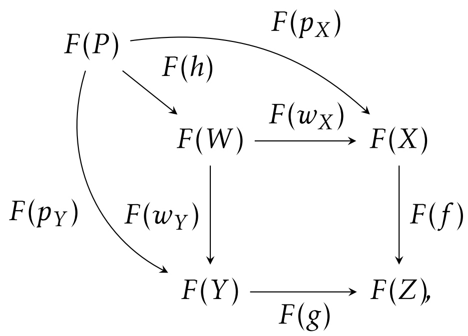
The following result will be very useful when proving that a specific functor preserves weak pullbacks.

Theorem 2.7. *Let $F : \mathbf{C} \to \mathbf{D}$ be a functor between categories* $\mathbf{C}$ *and* $\mathbf{D}$*, and let $f : X \to Z$ and $g : Y \to Z$ be morphisms in* $\mathbf{C}$*. If there is a weak pullback $(P, (p_X, p_Y))$ of f and g such that $(F(P), (F(p_X), F(p_Y)))$ is a weak pullback of $F(f)$ and $F(g)$ in* $\mathbf{D}$*, then F preserves weak pullbacks of f and g.*

Proof. Suppose $(P, (p_X, p_Y))$ is a weak pullback of f and g such that $(F(P), (F(p_X), F(p_Y)))$ is a weak pullback of $F(f)$ and $F(g)$. Let $(W, (w_X, w_Y))$ be any weak pullback of f and g, then there is $h : P \to W$ which makes the diagram

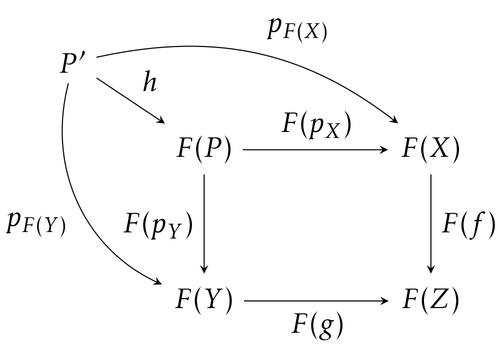


commute, by Theorem 2.4. Thus, the following diagram is also comutative



and so $(F(W), (F(w_X), F(w_Y)))$ is a weak pullback of $F(f)$ and $F(g)$, by Theorem 2.4. Therefore, F preserves weak pullbacks of f and g. □

As we have already seen, in **Set** there are pullbacks of any two functions with common codomain. By Theorems 2.4 and 2.7, to prove that a functor $F : \mathbf{Set} \to \mathbf{Set}$ preserves weak pullbacks, it is sufficient to show that for any functions $f : X \to Z$ and $g : Y \to Z$ there is $h : P' \to F(P)$ which makes the diagram



commute, where $(P,(p_X,p_Y))$ is a pullback of f and g, and $(P',(p_{F(X)},p_{F(Y)}))$ is a pullback of $F(f)$ and $F(g)$.

Example 2.8. (i) Let A be a set. Consider the functor $F = (-)^A : \mathbf{Set} \to \mathbf{Set}$ which maps a set X to the set X^A of all functions from A to X, and maps each function $f : Y \to Z$ to $f^A = f \circ (-) : Y^A \to Z^A$, where

$$f^A(h) = f \circ h, \qquad h \in Y^A.$$

Given functions $f : X \to Z$ and $g : Y \to Z$, let $(P,(p_X,p_Y))$ be their pullback, where

$$P = \{(x,y) \in X \times Y \mid f(x) = g(y)\},$$

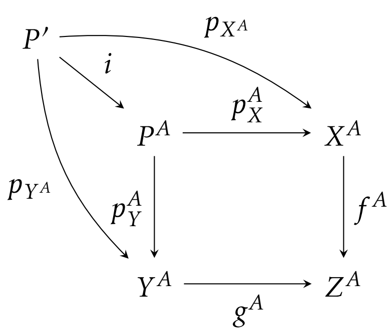
and p_X and p_Y are the projections (restricted to P). Also, let $(P',(p_{X^A},p_{Y^A}))$ be the pullback of f^A and g^A, where

$$P' = \{(h,k) \in X^A \times Y^A \mid f^A(h) = g^A(k)\} = \{(h,k) \in X^A \times Y^A \mid f \circ h = g \circ k\},$$

and p_{X^A} and p_{Y^A} are the projections (restricted to P'). Note that, if $(h,k) \in P'$, then $(h(a),k(a)) \in P$, for any $a \in A$. Define $i : P' \to P^A$ which maps each $(h,k) \in P'$ to $i(h,k) : A \to P$ defined by

$$i(h,k)(a) = (h(a),k(a)), \qquad a \in A.$$

Thus, the diagram



commutes, which implies that $(P^A,(p_X^A,p_Y^A))$ is a weak pullback, by Theorem 2.4. Consequently, F preserves weak pullbacks, by Theorem 2.7.

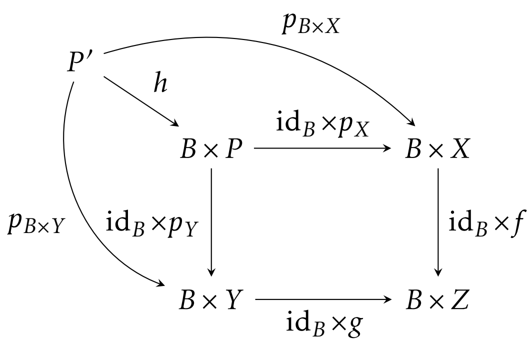
(ii) Let B be a set. Consider the functor $F = B \times (-) : \mathbf{Set} \to \mathbf{Set}$ which maps each set X to $B \times X$ and each function f to $\mathrm{id}_B \times f$. Given functions $f : X \to Z$ and $g : Y \to Z$, let $(P,(p_X,p_Y))$ be their pullback (as defined in (i)). Also, let $(P',(p_{B\times X},p_{B\times Y}))$ be the pullback of $\mathrm{id}_B \times f$ and $\mathrm{id}_B \times g$, where

$$\begin{aligned} P' &= \{((b,x),(b',y)) \in (B \times X) \times (B \times Y) \mid \mathrm{id}_B \times f(b,x) = \mathrm{id}_B \times g(b',y)\} \\ &= \{((b,x),(b',y)) \in (B \times X) \times (B \times Y) \mid b = b', f(x) = g(y)\}, \end{aligned}$$

and $p_{B\times X}$ and $p_{B\times Y}$ are the projections (restricted to P'). Note that $((b,x),(b',y)) \in P'$ if, and only if, $b = b'$ and $(x,y) \in P$. Define $h : P' \to B \times P$ by

$$h((b,x),(b',y)) = (b,(x,y)), \qquad ((b,x),(b',y)) \in P'.$$

Thus, the diagram

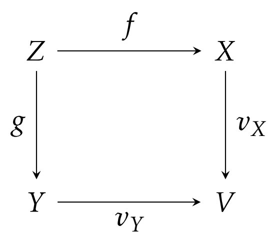


commutes. By Theorems 2.4 and 2.7, we have that F preserves weak pullbacks. Moreover, since h is a bijection, $(B \times P, (\mathrm{id}_B \times p_X, \mathrm{id}_B \times p_Y))$ is a pullback of $\mathrm{id}_B \times f$ and $\mathrm{id}_B \times g$, and therefore F preserves pullbacks.

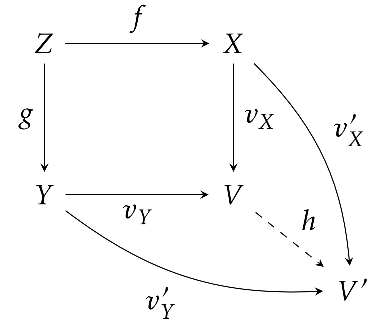
(iii) By composing the functors of the previous examples, we have that the functor $B \times (-)^A : \mathbf{Set} \to \mathbf{Set}$ also preserves weak pullbacks, for any sets A and B.

2.1.4 Pushouts

Definition 2.9. Let $f : Z \to X$ and $g : Z \to Y$ be morphisms with the same domain in a category $\mathbf{C}$. A *pushout* of f and g is a pair $(V, (v_X, v_Y))$ consisting of an object $V \in \mathbf{C}$ and morphisms $v_X : X \to V$ and $v_Y : Y \to V$ in $\mathbf{C}$, such that the diagram



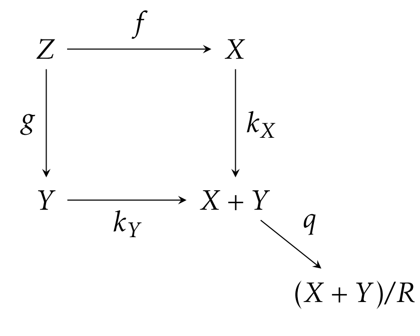
commutes, i.e. $v_X \circ f = v_Y \circ g$, and for any object $V' \in \mathbf{C}$ and morphisms $v'_X : X \to V'$ and $v'_Y : Y \to V'$ with $v'_X \circ f = v'_Y \circ g$, there exists a unique morphism $h : V \to V'$ which makes the diagram



commute, i.e. $v'_X = h \circ v_X$ and $v'_Y = h \circ v_Y$.

Observe that a pushout is the dual notion of a pullback. However, in **Set**, the construction of a pushout requires a little more work than the construction of pullbacks.

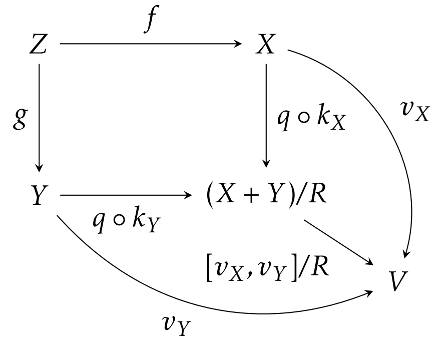
In **Set**, the pushout of functions $f : Z \to X$ and $g : Z \to Y$ exists and may be obtained as follows. Consider the coproduct $(X + Y, (k_X, k_Y))$ of X and Y, and let $R \subseteq (X+Y)\times(X+Y)$ be the smallest equivalence relation on $X+Y$ such that $(k_X(f(z)), k_Y(g(z))) \in R$, for all $z \in Z$, i.e. if S is an equivalence relation on $X + Y$ containing all pairs $(k_X(f(z)), k_Y(g(z)))$, $z \in Z$, then $R \subseteq S$. Since R is an equivalence relation on $X + Y$, denote the R-equivalence class of $a \in X + Y$ by $[a]_R$ and denote by $(X + Y)/R$ the quotient set of $X + Y$ by R, i.e. the set of all R-equivalence classes. Define $q : X + Y \to (X + Y)/R$ the quotient function which maps each $a \in X + Y$ to its R-equivalence class $[a]_R$. Thus, we have a diagram



where $q \circ k_X \circ f = q \circ k_Y \circ g$. Also, if V is a set, and $v_X : X \to V$ and $v_Y : Y \to V$ are functions for which $v_X \circ f = v_Y \circ g$, then $\ker[v_X, v_Y]$ is an equivalence relation on $X + Y$ that contains all pairs $(k_X(f(z)), k_Y(g(z)))$, $z \in Z$, which implies $R \subseteq \ker[v_X, v_Y]$, and so there is a function $[v_X, v_Y]/R : (X + Y)/R \to V$ defined by

$$([v_X, v_Y]/R)([a]_R) = [v_X, v_Y](a), \qquad a \in (X + Y)/R,$$

that makes the diagram



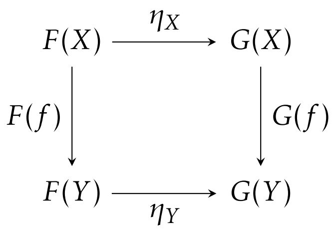
commute. Moreover, $[v_X, v_Y]/R$ is the unique function that makes such diagram commute, since if $h : (X+Y)/R \to V$ is such that $h \circ q \circ k_X = v_X$ and $h \circ q \circ k_Y = v_Y$, then

$$h \circ q = [h \circ q \circ k_X, h \circ q \circ k_Y] = [v_X, v_Y] = ([v_X, v_Y]/R) \circ q,$$

and so $h = [v_X, v_Y]/R$ because q is surjective. Therefore, $((X+Y)/R, (q \circ k_X, q \circ k_Y))$ is a pushout of f and g.

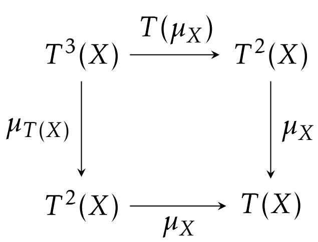
2.2 Monads and Kleisli triples

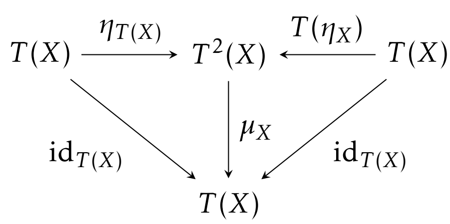
Let $F, G : \mathbf{C} \to \mathbf{D}$ be functors between two categories $\mathbf{C}$ and $\mathbf{D}$. A *natural transformation* η from F to G, denoted by $\eta : F \to G$ or $F \xrightarrow{\eta} G$, is a class of morphisms $(\eta_X : F(X) \to G(X))_{X \in \mathbf{C}}$ in $\mathbf{D}$, indexed by the objects of $\mathbf{C}$, such that for any objects $X, Y \in \mathbf{C}$ and morphism $f : X \to Y$ in $\mathbf{C}$ the following diagram



commutes, i.e. $\eta_Y \circ F(f) = G(f) \circ \eta_X$. Note that a natural transformation can also be regarded as a function $\eta : \mathrm{Ob}(\mathbf{C}) \to \mathrm{Mor}(\mathbf{D})$ with $X \mapsto (\eta_X : F(X) \to G(X))$ satisfying the previous property.

Definition 2.10. A *monad* (T, η, μ) on a category $\mathbf{C}$ consists of a functor $T : \mathbf{C} \to \mathbf{C}$ and two natural transformations $\eta : \mathrm{Id}_{\mathbf{C}} \to T$ (called the *unit*) and $\mu : T^2 \to T$ (called the *multiplication*), such that the diagrams





commute, i.e. $\mu_X \circ T(\mu_X) = \mu_X \circ \mu_{T(X)}$ (the *associative law*), $\mu_X \circ \eta_{T(X)} = \mathrm{id}_{T(X)}$ (the *left unit law*) and $\mu_X \circ T(\eta_X) = \mathrm{id}_{T(X)}$ (the *right unit law*), for each $X \in \mathbf{C}$.

For simplicity, we represent a monad by its functor, when its unit and its multiplication are clear from the context.

In the following example, we present a monad on **Set**, namely the powerset monad, that will be used through the rest of the text.

Example 2.11. Consider the powerset functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ that maps each set X to the set of all its subsets

$$\mathcal{P}(X) = \{S \mid S \subseteq X\},$$

and maps each function $f : Y \to Z$ to $\mathcal{P}(f) : \mathcal{P}(Y) \to \mathcal{P}(Z)$ defined by

$$\mathcal{P}(f)(S) = \{f(s) \mid s \in S\}, \qquad S \in \mathcal{P}(Y).$$

For each set X, define $\eta_X : X \to \mathcal{P}(X)$ by

$$\eta_X(x) = \{x\}, \qquad x \in X,$$

and define $\mu_X : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X)$ by

$$\mu_X(W) = \bigcup_{S \in W} S, \qquad W \in \mathcal{P}(\mathcal{P}(X)).$$

Let $\eta = (\eta_X)_{X \in \mathbf{Set}}$ and $\mu = (\mu_X)_{X \in \mathbf{Set}}$. Then, $(\mathcal{P}, \eta, \mu)$ is a monad on **Set** called the *powerset monad*.

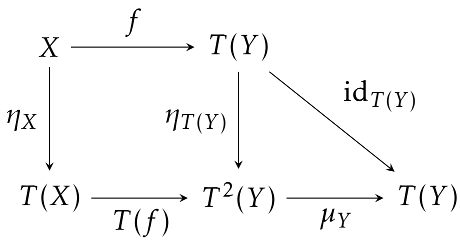
Let (T, η, μ) be a monad on a category $\mathbf{C}$. For a morphism $f : X \to T(Y)$ in $\mathbf{C}$, with $X, Y \in \mathbf{C}$, define

$$\overline{f} = \mu_Y \circ T(f) : T(X) \to T(Y).$$

Note that

$$\overline{\eta_X} = \mu_X \circ T(\eta_X) = \mathrm{id}_{T(X)},$$

by the right unit law, for each $X \in \mathbf{C}$. Also, for a morphism $f : X \to T(Y)$, the diagram



commutes, because η is a natural transformation from $\mathrm{Id}_{\mathbf{C}}$ to T and by the left unit law, and so

$$\overline{f} \circ \eta_X = \mu_Y \circ T(f) \circ \eta_X = f.$$

Finally, given morphisms $f : X \to T(Y)$ and $g : Y \to T(Z)$, the diagram

$$\begin{CD} T(X) @>T(f)>> T^2(Y) @>T^2(g)>> T^3(Z) @>T(\mu_Z)>> T^2(Z) \\ @. @V\mu_YVV @VV\mu_{T(Z)}V @VV\mu_ZV \\ @. T(Y) @>>T(g)> T^2(Z) @>>\mu_Z> T(Z) \end{CD}$$

commutes, because μ is a natural transformation from T^2 to T and by the associative law, and thus

$$\overline{\overline{g} \circ f} = \mu_Z \circ T(\mu_Z) \circ T^2(g) \circ T(f) = \mu_Z \circ T(g) \circ \mu_Y \circ T(f) = \overline{g} \circ \overline{f}.$$

From the previous construction, we have seen how to obtain a so-called Kleisli triple from a monad.

Definition 2.12. A *Kleisli triple* $(T, \eta, \overline{})$ on a category $\mathbf{C}$ consists of a function $T : \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{C})$, a family of morphisms $\eta = (\eta_X : X \to T(X))_{X \in \mathbf{C}}$ in $\mathbf{C}$, indexed by the objects of $\mathbf{C}$, and an extension operation $\overline{}$ that for each morphism $f : X \to T(Y)$ in $\mathbf{C}$ assigns a morphism $\overline{f} : T(X) \to T(Y)$ in $\mathbf{C}$, such that the following properties hold
(K1) $\overline{\eta_X} = \mathrm{id}_{T(X)}$,
(K2) $\overline{f} \circ \eta_X = f$, and
(K3) $\overline{\overline{g} \circ f} = \overline{g} \circ \overline{f}$,
for any morphisms $f : X \to T(Y)$ and $g : Y \to T(Z)$ and objects $X, Y, Z \in \mathbf{C}$.

Observe that properties (K2) and (K3) are equivalent to say that both diagrams

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & T(X) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \overline{f}} \\ & & T(Y) \end{array} \qquad\qquad \begin{array}{ccc} T(X) & \xrightarrow{\overline{f}} & T(Y) \\ & {\scriptstyle \overline{\overline{g} \circ f}}\searrow & \downarrow{\scriptstyle \overline{g}} \\ & & T(Z) \end{array}$$

commute, for any morphisms $f : X \to T(Y)$ and $g : Y \to T(Z)$.

Let $(T, \eta, \overline{})$ be a Kleisli triple on a category $\mathbf{C}$. For each morphism $f : X \to Y$ in $\mathbf{C}$, define

$$T(f) = \overline{\eta_Y \circ f} : T(X) \to T(Y).$$

For each $X \in \mathbf{C}$, we have that

$$T(\mathrm{id}_X) = \overline{\eta_X} = \mathrm{id}_{T(X)},$$

by (K1), and for each morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathbf{C}$, we have that

$$T(g) \circ T(f) = \overline{\eta_Z \circ g} \circ \overline{\eta_Y \circ f} \overset{\text{by (K3)}}{=} \overline{\overline{\eta_Z \circ g} \circ \eta_Y \circ f} \overset{\text{by (K2)}}{=} \overline{\eta_Z \circ g \circ f} = T(g \circ f).$$

Therefore, T is a functor on $\mathbf{C}$. By (K2), the diagram

$$\begin{CD} X @>{\eta_X}>> T(X) \\ @V{f}VV @VV{T(f) = \overline{\eta_Y \circ f}}V \\ Y @>>{\eta_Y}> T(Y) \end{CD}$$

commutes, for any morphism $f : X \to Y$. Hence, η is a natural transformation from $\mathrm{Id}_{\mathbf{C}}$ to T. For each $X \in \mathbf{C}$, define

$$\mu_X = \overline{\mathrm{id}_{T(X)}} : T^2(X) \to T(X).$$

Observe that

$$\begin{aligned} \mu_Y \circ T^2(f) &= \overline{\mathrm{id}_{T(Y)}} \circ \overline{\eta_{T(Y)} \circ \overline{\eta_Y \circ f}} \\ &= \overline{\overline{\mathrm{id}_{T(Y)}} \circ \eta_{T(Y)} \circ \overline{\eta_Y \circ f}} && \text{by (K3)} \\ &= \overline{\mathrm{id}_{T(Y)} \circ \overline{\eta_Y \circ f}} && \text{by (K2)} \\ &= \overline{\overline{\eta_Y \circ f} \circ \mathrm{id}_{T(X)}} \\ &= \overline{\eta_Y \circ f} \circ \overline{\mathrm{id}_{T(X)}} && \text{by (K3)} \\ &= T(f) \circ \mu_X, \end{aligned}$$

and so the diagram

$$\begin{CD} T^2(X) @>{\mu_X}>> T(X) \\ @V{T^2(f)}VV @VV{T(f)}V \\ T^2(Y) @>>{\mu_Y}> T(Y) \end{CD}$$

commutes, for any morphism $f : X \to Y$ in $\mathbf{C}$. Thus, $\mu = (\mu_X)_{X\in\mathbf{C}}$ is a natural transformation from T^2 to T. Finally, for each $X \in \mathbf{C}$, we have

$$\begin{aligned}
\mu_X \circ T(\mu_X) &= \overline{\mathrm{id}_{T(X)}} \circ \overline{\eta_{T(X)} \circ \overline{\mathrm{id}_{T(X)}}} \\
&= \overline{\overline{\mathrm{id}_{T(X)}} \circ \eta_{T(X)} \circ \overline{\mathrm{id}_{T(X)}}} && \text{by (K3)} \\
&= \overline{\mathrm{id}_{T(X)} \circ \overline{\mathrm{id}_{T(X)}}} && \text{by (K2)} \\
&= \overline{\overline{\mathrm{id}_{T(X)}} \circ \mathrm{id}_{T^2(X)}} \\
&= \overline{\mathrm{id}_{T(X)}} \circ \overline{\mathrm{id}_{T^2(X)}} && \text{by (K3)} \\
&= \mu_X \circ \mu_{T(X)}
\end{aligned}$$

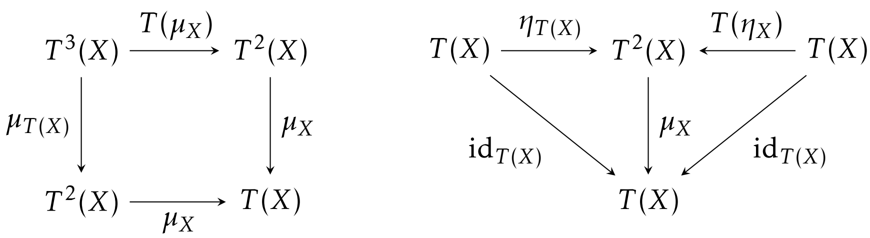
(the associative law),

$$\mu_X \circ \eta_{T(X)} = \overline{\mathrm{id}_{T(X)}} \circ \eta_{T(X)} \overset{\text{by (K2)}}{=} \mathrm{id}_{T(X)}$$

(the left unit law), and

$$\mu_X \circ T(\eta_X) = \overline{\mathrm{id}_{T(X)}} \circ \overline{\eta_{T(X)} \circ \eta_X} \overset{\text{by (K3)}}{=} \overline{\overline{\mathrm{id}_{T(X)}} \circ \eta_{T(X)} \circ \eta_X} \overset{\text{by (K2)}}{=} \overline{\eta_X} \overset{\text{by (K1)}}{=} \mathrm{id}_{T(X)}$$

(the right unit law), implying that the diagrams



commute. Therefore, (T,η,μ) is a monad on $\mathbf{C}$.

At first, we have seen how to construct a Kleisli triple from a monad and now we have seen how to construct a monad from a Kleisli triple (cf. [Man76]). It is easy to check that if we start with a monad, construct a Kleisli triple and obtain a monad from the Kleisli triple, then the monad we obtained is exactly the monad we began with. Also, if we start with a Kleisli triple, obtain a monad and construct a Kleisli triple from the monad, then the resulting Kleisli triple is the same that we began with. Thus, we have proven the following result.

Theorem 2.13. *Let $\mathbf{C}$ be a category.*

(i) Given a monad (T,η,μ) on $\mathbf{C}$, for each morphism $f : X \to T(Y)$ in $\mathbf{C}$ define $\overline{f} : T(X) \to T(Y)$ by $\overline{f} = \mu_Y \circ T(f)$. Then $(T,\eta,\overline{\ })$ is a Kleisli triple on $\mathbf{C}$.

(ii) Given a Kleisli triple $(T,\eta,\overline{\ })$ on $\mathbf{C}$, for each morphism $f : X \to Y$ in $\mathbf{C}$ define $T(f) : T(X) \to T(Y)$ by $T(f) = \overline{\eta_Y \circ f}$. Also, for each object $X \in \mathbf{C}$ define $\mu_X : T^2(X) \to T(X)$ by $\mu_X = \overline{\mathrm{id}_X}$, and let $\mu = (\mu_X)_{X\in\mathbf{C}}$. Then $T : \mathbf{C} \to \mathbf{C}$ is a functor and (T,η,μ) is a monad on $\mathbf{C}$.

Moreover, (i) and (ii) are inverse of each other.

To conclude this section, we describe the morphism extension for the powerset monad presented in Example 2.11.

Example 2.14. Consider the powerset monad $\mathcal{P}$. For a function $f : X \to \mathcal{P}(Y)$, we have that $\overline{f} : \mathcal{P}(X) \to \mathcal{P}(Y)$ is defined by

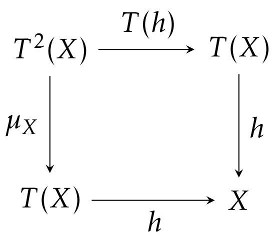
$$\overline{f}(S) = \bigcup_{x \in S} f(x),$$

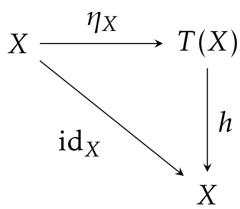
for each $S \in \mathcal{P}(X)$.

2.3 Algebras for a monad

Definition 2.15. Let (T, η, μ) be a monad on a category $\mathbf{C}$.

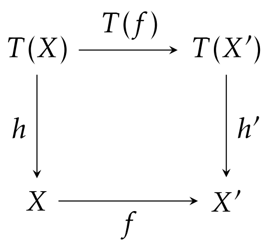
(i) An *algebra of* T, or simply a T*-algebra*, is a pair (X, h) consisting of an object $X \in \mathbf{C}$ and a morphism $h : T(X) \to X$ in $\mathbf{C}$, which makes both diagrams





commute, i.e. $h \circ T(h) = h \circ \mu_X$ and $h \circ \eta_X = \mathrm{id}_X$. Also, X is called the *carrier of the* T*-algebra* and α is called the T*-algebra structure*.

(ii) A *homomorphism of* T*-algebras* $f : (X, h) \to (X', h')$ is a morphism $f : X \to X'$ in $\mathbf{C}$ for which the diagram



commutes, i.e. $h' \circ T(f) = f \circ h$.

(iii) The class of all T-algebras together with their homomorphisms and composition as in $\mathbf{C}$ form a category known as the *Eilenberg-Moore category of* T, denoted by $\mathcal{EM}(T)$. The *forgetful functor* $\mathcal{U} : \mathcal{EM}(T) \to \mathbf{C}$ maps each T-algebra (X, h) to its carrier X and maps each homomorphism of T-algebras $f : (X, h) \to (X', h')$ to $f : X \to X'$ in $\mathbf{C}$.

In general, the algebras of a monad need not to be (isomorphic to) algebras in the usual sense of Universal Algebra [BS81]. Although every variety, regarded as a category, is isomorphic to the Eilenberg-Moore category of some monad [ML98, Theorem §VI.8.1].

The following example, which will be important throughout the text, describes the Eilenberg-Moore category of the powerset monad introduced in Example 2.11.

Example 2.16. Consider the powerset monad $\mathcal{P}$. For a set X and a function $h : \mathcal{P}(X) \to X$, we have that (X, h) is a $\mathcal{P}$-algebra if, and only if,

$$h(\bigcup_{S \in W} S) = h(\{h(S) \mid S \in W\}) \qquad \text{and} \qquad h(\{x\}) = x,$$

for any $W \in \mathcal{P}(\mathcal{P}(X))$ and $x \in X$. Let (X, h) be a $\mathcal{P}$-algebra. Since $\mathcal{P}(X)$ is nonempty and $h : \mathcal{P}(X) \to X$, then X is nonempty. Define a binary operation $\vee$ on X (called *join*) by

$$x \vee y = h(\{x, y\}), \qquad x, y \in X,$$

and let $\leq$ be a partial order relation on X defined by

$$x \leq y \iff x \vee y = y, \qquad x, y \in X.$$

Then, $(X, \vee)$ is a *join-semilattice* (cf. [Grä11]). Moreover, the supremum, or the least upper bound, of any subset of X exists and is given by

$$\bigvee S = h(S), \qquad S \subseteq X,$$

i.e. $(X, \vee)$ has arbitrary joins. And so $(X, \vee)$ is a *complete join-semilattice*.

Also, a homomorphism of $\mathcal{P}$-algebras $f : (X, h) \to (X', h')$ is a homomorphism of the resulting complete join-semilattices $f : (X, \vee) \to (X', \vee')$, since

$$f(\bigvee S) = f(h(S)) = h'(\mathcal{P}(f)(S)) = h'(\{f(x) \mid x \in S\}) = \bigvee_{x \in S}{}' f(x)$$

for any $S \subseteq X$.

Conversely, for a complete join-semilattice $(X, \vee)$, define $h : \mathcal{P}(X) \to X$ by

$$h(S) = \bigvee S, \qquad S \in \mathcal{P}(X).$$

Then, (X, h) is a $\mathcal{P}$-algebra. Also, a homomorphism of complete join-semilattices $f : (X, \vee) \to (X', \vee')$ is a homomorphism of the resulting $\mathcal{P}$-algebras $f : (X, h) \to (X', h')$, because

$$f(h(S)) = f(\bigvee S) = \bigvee_{x \in S}{}' f(x) = \bigvee{}' \mathcal{P}(f)(S) = h'(\mathcal{P}(f)(S))$$

for any $S \in \mathcal{P}(X)$.

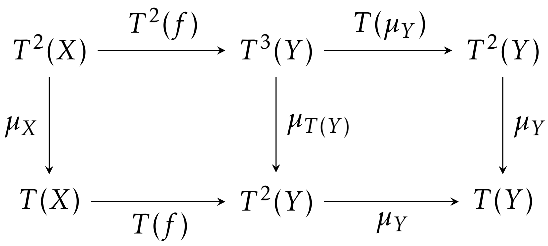
Therefore, $\mathcal{EM}(\mathcal{P})$ is isomorphic to the category of complete join-semilattices and their homomorphisms.

We remark that the complete join-semilattices do not form a variety, although they are still (isomorphic to) algebras of a monad, namely $\mathcal{P}$-algebras.

2.3.1 Generalizing morphism extension to the algebras of a monad

Let (T, η, μ) be a monad on a category $\mathbf{C}$. For any object $X \in \mathbf{C}$, by the associative law and left unit law in Definition 2.10, (X, μ_X) is a T-algebra called the *free T-algebra over X*. As in Universal Algebra [BS81, Proposition II.§10.10], the free T-algebra over an object $X \in \mathbf{C}$ also has an *universal property* [Man76, see 1.4.12], which we now present.

For instance, given a morphism $f : X \to T(Y)$ in $\mathbf{C}$, the diagram

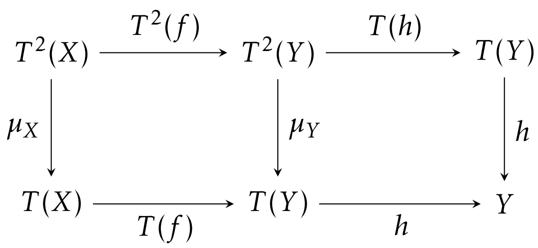


commutes, and so the extension $\overline{f} = \mu_Y \circ T(f) : T(X) \to T(Y)$ of f is a homomorphism of T-algebras from $(T(X), \mu_X)$ to $(T(Y), \mu_Y)$. The morphism extension $\overline{}$ can be generalized to T-algebras as follows.

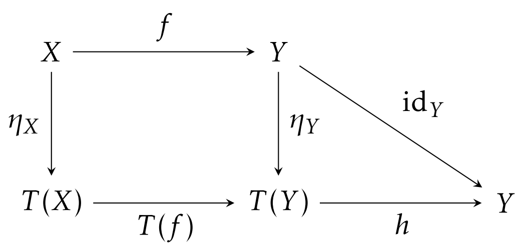
Let $f : X \to Y$ be a morphism in $\mathbf{C}$ and let (Y, h) be a T-algebra. Define

$$\overline{f}_h = h \circ T(f) : T(X) \to Y.$$

Since $\mu : T^2 \to T$ is a natural transformation and (Y, h) is a T-algebra, the diagram



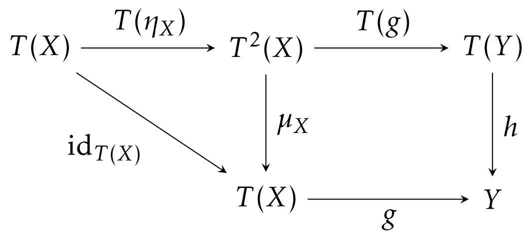
commutes, and then $h \circ T(\overline{f}_h) = \overline{f}_h \circ \mu_X$, which implies that $\overline{f}_h$ is a homomorphism of T-algebras from $(T(X), \mu_X)$ to (Y, h). Also, the diagram



commutes, because $\eta : \mathrm{Id}_{\mathbf{C}} \to T$ is a natural transformation and (Y, h) is a T-algebra, and so $f = \overline{f}_h \circ \eta_X$. In the following theorem, we show that $\overline{f}_h$ is the unique morphism in $\mathbf{C}$ that satisfies such conditions.

Theorem 2.17. *Let* (T, η, μ) *be a monad on a category* $\mathbf{C}$*. Given a* T*-algebra* (Y, h) *and a morphism* $f : X \to Y$ *in* $\mathbf{C}$*,* $\overline{f}_h = h \circ T(f) : T(X) \to Y$ *in* $\mathbf{C}$ *is the unique homomorphism of* T*-algebras from* $(T(X), \mu_X)$ *to* (Y, h) *such that* $f = \overline{f}_h \circ \eta_X$*.*

Proof. Let (Y, h) be a T-algebra and $f : X \to Y$ a morphism in $\mathbf{C}$. As we saw above, $\overline{f}_h = h \circ T(h)$ is a homomorphism of T-algebras from $(T(X), \mu_X)$ to (Y, h) and $f = \overline{f}_h \circ \eta_X$. If $g : (T(X), \mu_X) \to (Y, h)$ is a homomorphism of T-algebras with $f = g \circ \eta_X$, and since $\mu_X \circ T(\eta_X) = \mathrm{id}_{T(X)}$ by the right unit law in Definition 2.10, then the diagram



commutes, hence $g = h \circ T(g) \circ T(\eta_X) = h \circ T(g \circ \eta_X) = h \circ T(f) = \overline{f}_h$. □

When h is clear from the context, we write simply $\overline{f}$ instead of $\overline{f}_h$.

In the following example, we present the function extension for the algebras of the powerset monad $\mathcal{P}$, see Example 2.11.

Example 2.18. Given a set X, a $\mathcal{P}$-algebra (Y, h) and a function $f : X \to Y$, then $\overline{f} : \mathcal{P}(X) \to Y$ is defined by

$$\overline{f}(S) = h(\{f(x) \mid x \in S\}), \qquad S \in \mathcal{P}(X),$$

and thus $\overline{f}$ is the unique homomorphism of $\mathcal{P}$-algebras from (X, μ_X) to (Y, h) such that $\overline{f}(\{x\}) = f(x)$, for all $x \in X$.

In other words, by Example 2.16, consider $\mathcal{P}$-algebras as complete join-semilattices. Then, for a set X, the free $\mathcal{P}$-algebra over X is in correspondence with $(\mathcal{P}(X), \cup)$, the complete join-semilattice of all subsets of X together with set union. And for a complete join-semilattice $(Y, \vee)$ and a function $f : X \to Y$, $\overline{f} : \mathcal{P}(X) \to Y$ is defined by

$$\overline{f}(S) = \bigvee_{x \in S} f(x), \qquad S \in \mathcal{P}(X).$$

In particular, $\overline{f}$ is the unique homomorphism of complete join-semilattices from $(\mathcal{P}(X), \cup)$ to $(Y, \vee)$ such that $\overline{f}(\{x\}) = f(x)$, for all $x \in X$.

2.3.2 Functor liftings to the Eilenberg-Moore category of a monad

Now, we introduce the notion of a lifting of a functor to the Eilenberg-Moore category of a monad, and then describe a one-to-one correspondence between liftings and distributive laws. See also [Joh75].

Definition 2.19. Let (T,η,μ) be a monad on a category $\mathbf{C}$, and consider the forgetful functor $\mathcal{U} : \mathcal{EM}(T) \to \mathbf{C}$. Given a functor $F : \mathbf{C} \to \mathbf{C}$, a *lifting of F to $\mathcal{EM}(T)$* is a functor $\widehat{F} : \mathcal{EM}(T) \to \mathcal{EM}(T)$ such thar the diagram

$$\begin{array}{ccc} \mathcal{EM}(T) & \xrightarrow{\widehat{F}} & \mathcal{EM}(T) \\ {\scriptstyle \mathcal{U}}\downarrow & & \downarrow{\scriptstyle \mathcal{U}} \\ \mathbf{C} & \xrightarrow[F]{} & \mathbf{C} \end{array}$$

commutes, i.e. $\mathcal{U}\widehat{F} = F\mathcal{U}$.

Let (T,η,μ) be a monad on a category $\mathbf{C}$, and let $F : \mathbf{C} \to \mathbf{C}$ be a functor with a lifting $\widehat{F} : \mathcal{EM}(T) \to \mathcal{EM}(T)$ to $\mathcal{EM}(T)$. For a T-algebra (X,h), observe that $F(X)$ is the carrier of the T-algebra $\widehat{F}(X,h)$, and for a homomorphism of T-algebras $f : (X,h) \to (X',h')$, $F(f)$ is a homomorphism of T-algebras from $\widehat{F}(X,h) \to \widehat{F}(X',h')$. In particular, for each object $X \in \mathbf{C}$, there exists a morphism $h_X : TFT(X) \to FT(X)$ in $\mathbf{C}$ such that $(FT(X),h_X) = \widehat{F}(T(X),\mu_X)$, and thus $h_X \circ \eta_{FT(X)} = \mathrm{id}_{FT(X)}$ and $h_X \circ T(h_X) = h_X \circ \mu_X$. Define $\rho_X = h_X \circ TF(\eta_X) : TF(X) \to FT(X)$ in $\mathbf{C}$, for each $X \in \mathbf{C}$. Let $\rho = (\rho_X)_{X\in\mathbf{C}}$.

For any morphism $f : X \to Y$ in $\mathbf{C}$, since $T(f)$ is a homomorphism of T-algebras from $(T(X),\mu_X)$ to $(T(Y),\mu_Y)$, then $FT(f)$ is a homomorphism of T-algebras from $(FT(X),h_X)$ to $(FT(Y),h_Y)$, and since η is a natural transformation from $\mathrm{Id}_{\mathbf{C}}$ to T, we have that the following diagram

$$\begin{array}{ccccc} TF(X) & \xrightarrow{TF(\eta_X)} & TFT(X) & \xrightarrow{h_X} & FT(X) \\ {\scriptstyle TF(f)}\downarrow & & \downarrow{\scriptstyle TFT(f)} & & \downarrow{\scriptstyle FT(f)} \\ TF(Y) & \xrightarrow[TF(\eta_Y)]{} & TFT(Y) & \xrightarrow[h_Y]{} & FT(Y) \end{array}$$

commutes, which implies $FT(f) \circ \rho_X = \rho_Y \circ TF(f)$. Hence, ρ is a natural transformation from TF to FT.

Also, for any $X \in \mathbf{C}$, the diagram

$$\begin{array}{ccccc} F(X) & \xrightarrow{F(\eta_X)} & FT(X) & & \\ {\scriptstyle \eta_{F(X)}}\downarrow & & \downarrow{\scriptstyle \eta_{FT(X)}} & \searrow{\scriptstyle \mathrm{id}_{FT(X)}} & \\ TF(X) & \xrightarrow[TF(\eta_X)]{} & TFT(X) & \xrightarrow[h_X]{} & FT(X) \end{array}$$

commutes, because η is a natural transformation from $\mathrm{Id}_{\mathbf{C}}$ to T and $(FT(X),h_X)$ is a T-algebra. Thus, $\rho_X \circ \eta_{F(X)} = F(\eta_X)$ for every object $X \in \mathbf{C}$.

Finally, for any $X \in \mathbf{C}$, the diagram

$$\begin{array}{ccccc}
T^2F(X) & \xrightarrow{T^2F(\eta_X)} & T^2FT(X) & \xrightarrow{T(h_X)} & TFT(X) \\
{\scriptstyle \mu_{F(X)}}\downarrow & & \downarrow{\scriptstyle \mu_{FT(X)}} & & \downarrow{\scriptstyle h_X} \\
TF(X) & \xrightarrow[TF(\eta_X)]{} & TFT(X) & \xrightarrow[h_X]{} & FT(X)
\end{array}$$

commutes, because μ is a natural transformation from T^2 to T and $(FT(X), h_X)$ is a T-algebra, and so $h_X \circ T(\rho_X) = \rho_X \circ \mu_{F(X)}$. Also, since μ_X is a homomorphism of T-algebras from $(T^2(X), \mu_{T(X)})$ to $(T(X), \mu_X)$, by the associative law (Definition 2.10), then $F(\mu_X)$ is a homomorphism of T-algebras from $(FT^2(X), h_{T(X)})$ to $(FT(X), h_X)$, and since $\mu_X \circ \eta_{T(X)} = \mathrm{id}_{T(X)}$ by the left unit law (Definition 2.10), the diagram

$$\begin{array}{ccccc}
TFT(X) & \xrightarrow{TF(\eta_{T(X)})} & TFT^2(X) & \xrightarrow{h_{T(X)}} & FT^2(X) \\
 & {\scriptstyle \mathrm{id}_{TFT(X)} = TF(\mathrm{id}_{T(X)})}\searrow & \downarrow{\scriptstyle TF(\mu_X)} & & \downarrow{\scriptstyle F(\mu_X)} \\
 & & TFT(X) & \xrightarrow[h_X]{} & FT(X)
\end{array}$$

commutes, that is $F(\mu_X) \circ \rho_{T(X)} = h_X$. Therefore, the diagram

$$\begin{array}{ccccc}
T^2F(X) & \xrightarrow{T(\rho_X)} & TFT(X) & \xrightarrow{\rho_{T(X)}} & FT^2(X) \\
{\scriptstyle \mu_{F(X)}}\downarrow & & {\scriptstyle h_X}\searrow & & \downarrow{\scriptstyle F(\mu_X)} \\
TF(X) & & \xrightarrow[\rho_X]{} & & FT(X)
\end{array}$$

commutes, i.e. $F(\mu_X) \circ \rho_{T(X)} \circ T(\rho_X) = \rho_X \circ \mu_{F(X)}$, for every object $X \in \mathbf{C}$.

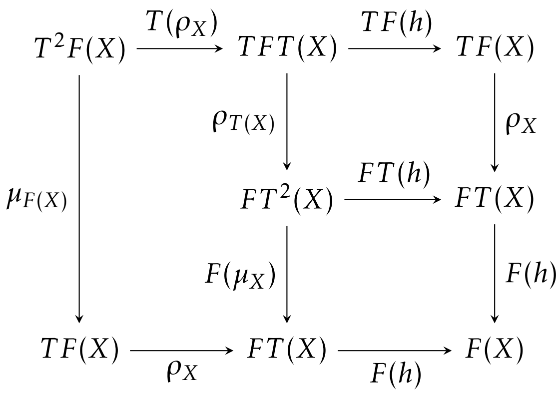
The previous results shows that ρ is a distributive law of T over F.

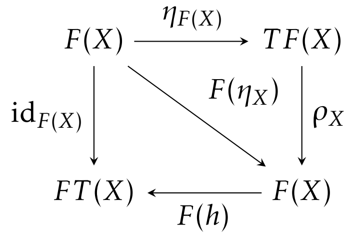
Definition 2.20. Let (T, η, μ) be a monad on a category $\mathbf{C}$, and let $F : \mathbf{C} \to \mathbf{C}$ be a functor on the same category. A *distributive law* of T over F is a natural transformation $\rho : TF \to FT$ such that the following diagrams

$$\begin{array}{ccc}
F(X) & \xrightarrow{\eta_{F(X)}} & TF(X) \\
 & {\scriptstyle F(\eta_X)}\searrow & \downarrow{\scriptstyle \rho_X} \\
 & & FT(X)
\end{array}
\qquad
\begin{array}{ccccc}
T^2F(X) & \xrightarrow{T(\rho_X)} & TFT(X) & \xrightarrow{\rho_{T(X)}} & FT^2(X) \\
{\scriptstyle \mu_{F(X)}}\downarrow & & & & \downarrow{\scriptstyle F(\mu_X)} \\
TF(X) & & \xrightarrow[\rho_X]{} & & FT(X)
\end{array}$$

commute, i.e. $\rho_X \circ \eta_{F(X)} = F(\eta_X)$ and $F(\mu_X) \circ \rho_{T(X)} \circ T(\rho_X) = \rho_X \circ \mu_{F(X)}$, for all $X \in \mathbf{C}$.

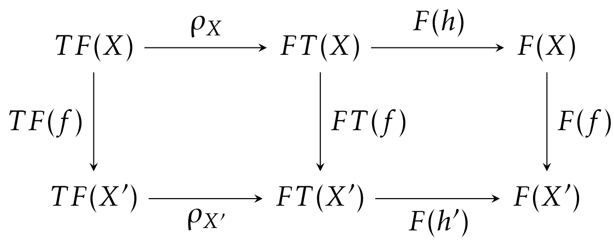
Let (T, η, μ) be a monad on a category $\mathbf{C}$ and $F : \mathbf{C} \to \mathbf{C}$ be a functor such that there is a distributive law $\rho : TF \to FT$. Given a T-algebra (X, h), observe that both diagrams





commute, and so $T(F(h) \circ \rho_X) \circ (F(h) \circ \rho_X) = \mu_{F(X)} \circ (F(h) \circ \rho_X)$ and $(F(h) \circ \rho_X) \circ \eta_{F(X)} = \mathrm{id}_{F(X)}$. Hence, $(F(X), F(h) \circ \rho_X)$ is a T-algebra.

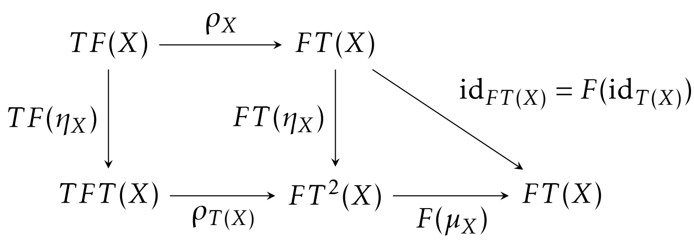
Also, for any homomorphism of T-algebras $f : (X, h) \to (X', h')$, the diagram



commutes, and thus $F(f)$ is a homomorphism of T-algebras from $(F(X), F(h) \circ \rho_X)$ to $(F(X'), F(h') \circ \rho_{X'})$.

Therefore, we can define $\widehat{F} : \mathcal{EM}(T) \to \mathcal{EM}(T)$, a lifting of F to $\mathcal{EM}(T)$, by $\widehat{F}(X, h) = (F(X), F(h) \circ \rho_X)$, for each T-algebra (X, h), and $\widehat{F}(f) = F(f)$, for each homomorphism of T-algebras.

Finally, for any object $X \in \mathbf{C}$, note that $\widehat{F}(T(X), \mu_X) = (FT(X), F(\mu_X) \circ \rho_{T(X)})$ and since the diagram



commutes, then $\rho_X = (F(\mu_X) \circ \rho_{T(X)}) \circ TF(\eta_X)$.

Now, we have proved the following theorem.

Theorem 2.21. *Let (T,η,μ) be a monad on a category* $\mathbf{C}$*, and let $F:\mathbf{C}\to\mathbf{C}$ be a functor on the same category.*

(i) Given a lifting $\widehat{F}:\mathcal{EM}(T)\to\mathcal{EM}(T)$ of F to $\mathcal{EM}(T)$, define $\rho_X = h_X\circ TF(\eta_X): TF(X)\to FT(X)$, where $h_X: TFT(X)\to FT(X)$ is such that $(FT(X),h_X)=\widehat{F}(T(X),\mu_X)$, for each $X\in\mathbf{C}$. Then, $\rho=(\rho_X)_{X\in\mathbf{C}}$ is a distributive law of T over F.

(ii) Given a distributive law $\rho: TF\to FT$ of T over F, define $\widehat{F}:\mathcal{EM}(T)\to\mathcal{EM}(T)$ by $\widehat{F}(X,h)=(F(X),F(h)\circ\rho_X)$, for each T-algebra (X,h), and $\widehat{F}(f)=F(f)$ for each homomorphism of T-algebras f. Then, $\widehat{F}$ is a lifting of F to $\mathcal{EM}(T)$.

Moreover, (i) and (ii) are inverse of each other.

At last, we present a lifting of the functor $2\times(-)^A$ to the Eilenberg-Moore category of the powerset monad.

Example 2.22. Consider the powerset monad $\mathcal{P}$, see Example 2.11, and the functor $D = 2\times(-)^A:\mathbf{Set}\to\mathbf{Set}$, see Example 2.8(iii), where $2=\{0,1\}$ and A is a nonempty set. Given a $\mathcal{P}$-algebra (X,h), define $h_1:\mathcal{P}(2\times X^A)\to 2$ by

$$h_1(S)=\max\{z\mid (z,f)\in S\},\qquad S\in\mathcal{P}(2\times X^A),$$

and define $h_2:\mathcal{P}(2\times X^A)\to X^A$ by

$$h_2(S)(a)=h(\{f(a)\mid (z,f)\in S\}),\qquad a\in A, S\in\mathcal{P}(2\times X^A),$$

and then we have a $\mathcal{P}$-algebra $(2\times X^A,\langle h_1,h_2\rangle)$. Moreover, we have a lifting $\widehat{D}:\mathcal{EM}(\mathcal{P})\to\mathcal{EM}(\mathcal{P})$ of D to $\mathcal{EM}(\mathcal{P})$ defined by $\widehat{D}(X,h)=(2\times X^A,\langle h_1,h_2\rangle)$, where h_1 and h_2 are defined as above for each $\mathcal{P}$-algebra (X,h), and $\widehat{D}(f)=\mathrm{id}_2\times f^A$ for each homomorphism of $\mathcal{P}$-algebras f.

In other words, if we consider $\mathcal{P}$-algebras as complete join-semilattices (Example 2.16), we have that $2=\{0,1\}$ is a complete join-semilattice where $0<1$. For a complete join-semilattice $(X,\vee)$, we obtain another complete join-semilattice on X^A where join is defined pointwise. And thus, $D(X)=2\times X^A$, being the product of two complete join-semilattices, is a complete join-semilattice with join defined pairwise. Also, given a homomorphism of complete join-semilattices $f:X\to X'$, then $\mathrm{id}_2\times f^A$ is a homomorphism of the resulting complete join-semilattices $2\times X^A$ and $2\times(X')^A$.

By Theorem 2.21, $\widehat{D}$ is in correspondence with a distributive law $\rho:\mathcal{P}D\to D\mathcal{P}$, where $\rho_X=\langle\rho_{1,X},\rho_{2,X}\rangle:\mathcal{P}(2\times X^A)\to 2\times(\mathcal{P}(X))^A$ is defined by

$$\rho_{1,X}(S)=\max\{z\mid (z,f)\in S\},\qquad S\in\mathcal{P}(2\times X^A),$$

and

$$\rho_{2,X}(S)(a)=\{f(a)\mid (z,f)\in S\},\qquad a\in A, S\in\mathcal{P}(2\times X^A)$$

for each set X.

Chapter 3

Coalgebras

In this chapter, we introduce the notion of a coalgebra of a functor. We generalize some results of Universal Coalgebra (see [Rut00]) to coalgebras of a functor on an arbitrary category, instead of using functors on the category of sets. See also [Adá05; Jac12].

Throughout this chapter, we illustrate the main definition and theorems in examples using deterministic and nondeterministic automata. In theses examples, we obtain well-known results from a coalgebraic point of view. See [Eil74; How91] for a classical introduction to the Theory of Automata. Moreover, in Chapter 5, we will see these examples as concrete case of a general theory.

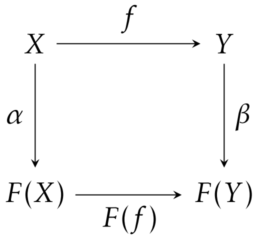
3.1 The category of coalgebras

In this section, we study some basic results in the category of coalgebras of a functor. The notion of a coalgebra of a functor comes from the duality with the notion of an algebra of a functor (see [JR11]).

Definition 3.1. Let $F : \mathbf{C} \to \mathbf{C}$ be a functor on a category $\mathbf{C}$.

(i) A *coalgebra of F*, or simply an *F-coalgebra*, is a pair (X, α) consisting of an object $X \in \mathbf{C}$ and a morphism $\alpha : X \to F(X)$ in $\mathbf{C}$. Also, F is called the *type* of the coalgebra, X is called the *carrier of the F-coalgebra*, or *state space*, and α is called the *F-coalgebra structure* or the *transition structure*.

(ii) A *homomorphism of F-coalgebra $f : (X, \alpha) \to (Y, \beta)$* is a morphism $f : X \to Y$ in $\mathbf{C}$ that makes the diagram

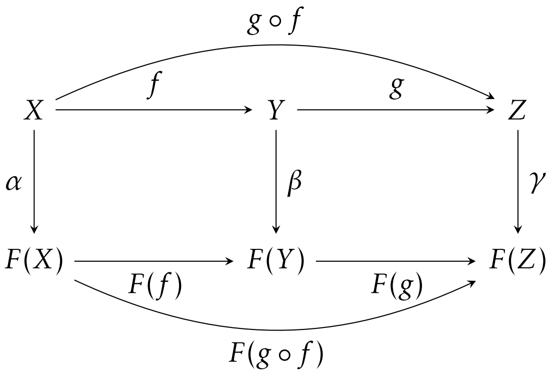


commute, i.e. $\beta \circ f = F(f) \circ \alpha$.

Let $F : \mathbf{C} \to \mathbf{C}$ be a functor on a category $\mathbf{C}$. Given an F-coalgebra (X, α), observe that

$$\alpha \circ \mathrm{id}_X = \alpha = \mathrm{id}_{F(X)} \circ \alpha = F(\mathrm{id}_X) \circ \alpha$$

and so id_X is a homomorphism of F-coalgebras from (X, α) to itself. Also, if $f : (X, \alpha) \to (Y, \beta)$ and $g : (Y, \beta) \to (Z, \gamma)$ are two homomorphisms of F-coalgebras, the diagram



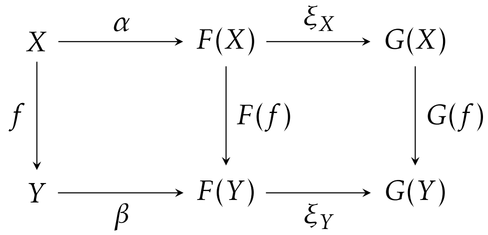
commutes, then $g \circ f : (X, \alpha) \to (Z, \gamma)$ is a homomorphism of F-coalgebras. Therefore, we can define a category of F-coalgebras as follows.

Definition 3.2. Let $F : \mathbf{C} \to \mathbf{C}$ be a functor on a category $\mathbf{C}$. We denote by $\mathrm{CoAlg}(F)$ the category of all F-coalgebras and their homomorphisms with composition defined as in $\mathbf{C}$.

Given two functors on the same category, we can define a functor between the categories of their coalgebras, whenever there is a natural transformation between them.

Theorem 3.3. *Let $F, G : \mathbf{C} \to \mathbf{C}$ be two functors on a category $\mathbf{C}$, and let $\xi : F \to G$ be a natural transformation. Given any homomorphism of F-coalgebras $f : (X, \alpha) \to (Y, \beta)$, then f is a homomorphism of G-coalgebras from $(X, \xi_X \circ \alpha)$ to $(Y, \xi_Y \circ \beta)$. Therefore, ξ induces a functor from $\mathrm{CoAlg}(F)$ to $\mathrm{CoAlg}(G)$ mapping each F-coalgebra (X, α) to $(X, \xi_X \circ \alpha)$ and mapping each homomorphism of F-coalgebras f to the homomorphism of G-coalgebras f.*

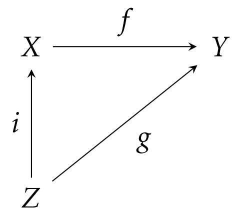
Proof. Let $f : (X, \alpha) \to (Y, \beta)$ be a homomorphism of F-coalgebras, that is a morphism $f : X \to Y$ in $\mathbf{C}$ such that $\beta \circ f = F(f) \circ \alpha$. Note that $\xi_X \circ \alpha : X \to G(X)$ and $\xi_Y \circ \beta : Y \to F(Y)$, and thus $(X, \xi_X \circ \alpha)$ and $(Y, \xi_Y \circ \beta)$ are both G-coalgebras. Since ξ is a natural transformation from F to G, the diagram

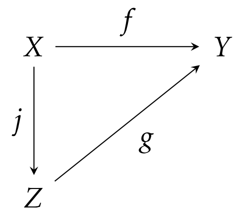


commutes, then $G(f) \circ \xi_X \circ \alpha = \xi_Y \circ \beta \circ f$. Hence, f is a homomorphism of G-coalgebras from $(X, \xi_X \circ \alpha)$ to $(Y, \xi_Y \circ \beta)$. Therefore, any homomorphism of F-coalgebras $f : (X, \alpha) \to (Y, \beta)$ is a homomorphism of G-coalgebras from $(X, \xi_X \circ \alpha)$ to $(Y, \xi_Y \circ \beta)$, and clearly the functor induced by ξ is well-defined. $\square$

In the following result we describe how to construct a coalgebra, which will be used later.

Theorem 3.4. *Let $F : \mathbf{C} \to \mathbf{C}$ be a functor on a category $\mathbf{C}$. Let $f : (X, \alpha) \to (Y, \beta)$ be a homomorphism between F-coalgebras (X, α) and (Y, β). Given $Z \in \mathbf{C}$ and $g : Z \to Y$ in $\mathbf{C}$, if there are morphisms $i : Z \to X$ and $j : X \to Z$ in $\mathbf{C}$ such that both diagrams*





commute, i.e. $g = f \circ i$ and $f = g \circ j$, then g is a homomorphism of F-coalgebras from $(Z, F(j) \circ \alpha \circ i)$ to (Y, β).

Proof. Let $Z \in \mathbf{C}$ and $g : Z \to Y$ in $\mathbf{C}$ be such that there are $i : Z \to X$ and $j : X \to Z$ in $\mathbf{C}$ with $g = f \circ i$ and $f = g \circ j$. Recall that $F(f) \circ \alpha = \beta \circ f$, because $f : (X, \alpha) \to (Y, \beta)$ is a homomorphism. Then

$$\begin{aligned} F(g) \circ F(j) \circ \alpha \circ i &= F(g \circ j) \circ \alpha \circ i \\ &= F(f) \circ \alpha \circ i \\ &= \beta \circ f \circ i \\ &= \beta \circ g, \end{aligned}$$

which implies that g is a homomorphism of F-coalgebras from $(Z, F(j) \circ \alpha \circ i)$ to (Y, β). $\square$

For any functor $F : \mathbf{C} \to \mathbf{C}$, the existence of coproducts (Definition 2.2) or pushouts (Definition 2.9) depends only whether they exist in $\mathbf{C}$, as shown in the following two results.

Theorem 3.5. *Let $F : \mathbf{C} \to \mathbf{C}$ be a functor on a category $\mathbf{C}$.*

(i) Given a family $(X_i, \alpha_i)_{i\in I}$ of F-coalgebras indexed by a set I. If there exists a coproduct $(\sum_{i\in I} X_i, (k_i)_{i\in I})$ of $(X_i)_{i\in I}$ in $\mathbf{C}$, then there exists $\zeta : \sum_{i\in I} X_i \to F(\sum_{i\in I} X_i)$ such that $((\sum_{i\in I} X_i, \zeta), (k_i)_{i\in I})$ is a coproduct of $(X_i, \alpha_i)_{i\in I}$ in $\mathrm{CoAlg}(F)$.

(ii) Given two homomorphisms of F-coalgebras, with common domain, $f : (Z, \gamma) \to (X, \alpha)$ and $g : (Z, \gamma) \to (Y, \beta)$. If there is a pushout $(V, (v_X, v_Y))$ of $f : Z \to X$ and $g : Z \to Y$ in $\mathbf{C}$, then there is $\xi : V \to F(V)$ such that $((V, \xi), (v_X, v_Y))$ is a pushout of f and g in $\mathrm{CoAlg}(F)$.

Proof. (i) Let $(X_i, \alpha_i)_{i\in I}$ be a family of F-coalgebras, and assume there is a coproduct $(\sum_{i\in I} X_i, (k_i)_{i\in I})$ of $(X_i)_{i\in I}$ in $\mathbf{C}$. Note that $F(k_j) \circ \alpha_j : X_j \to F(\sum_{i\in I} X_i)$ for each $j \in I$. Define $\zeta = [F(k_i) \circ \alpha_i]_{i\in I} : \sum_{i\in I} X_i \to F(\sum_{i\in I} X_i)$ to be the unique morphism that makes the diagram

$$\begin{array}{ccc} X_j & \xrightarrow{\;k_j\;} & \sum_{i\in I} X_i \\ {\scriptstyle \alpha_j}\downarrow & & \downarrow{\scriptstyle \zeta} \\ F(X_j) & \xrightarrow[F(k_j)]{} & F(\sum_{i\in I} X_i) \end{array}$$

commute, i.e. $\zeta \circ k_j = F(k_j) \circ \alpha_j$, for all $j \in I$. Thus, k_j is a homomorphism of F-coalgebras from (X_j, α_j) to $(\sum_{i\in I} X_i, \zeta)$, for each $j \in I$.

Also, let (Y, β) be an F-coalgebra and let $(f_i : (X_i, \alpha_i) \to (Y, \beta))_{i\in I}$ be a family of homomorphisms of F-coalgebras. In particular, $f_i : X_i \to Y$ in $\mathbf{C}$, for each $i \in I$, and so there is a unique morphism $[f_i]_{i\in I} : \sum_{i\in I} X_i \to Y$ in $\mathbf{C}$ such that $[f_i]_{i\in I} \circ k_j = f_j$, for every $j \in I$. Observe that

$$\beta \circ [f_i]_{i\in I} \circ k_j = \beta \circ f_j = F(f_j) \circ \alpha_j$$

and

$$F([f_i]_{i\in I}) \circ \zeta \circ k_j = F([f_i]_{i\in I}) \circ F(k_j) \circ \alpha_j = F(f_j) \circ \alpha_j,$$

which implies $\beta \circ [f_i]_{i\in I} \circ k_j = F([f_i]_{i\in I}) \circ \zeta \circ k_j$, for each $j \in I$. Then, we have

$$\beta \circ [f_i]_{i\in I} = [\beta \circ [f_i]_{i\in I} \circ k_j]_{j\in I} = [F([f_i]_{i\in I}) \circ \zeta \circ k_j]_{j\in I} = F([f_i]_{i\in I}) \circ \zeta$$

which means that the diagram

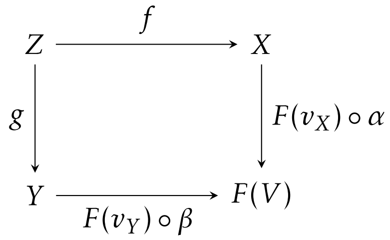
$$\begin{array}{ccc} \sum_{i\in I} X_i & \xrightarrow{\;[f_i]_{i\in I}\;} & Y \\ {\scriptstyle \zeta}\downarrow & & \downarrow{\scriptstyle \beta} \\ F(\sum_{i\in I} X_i) & \xrightarrow[F([f_i]_{i\in I})]{} & F(Y) \end{array}$$

commutes, and so $[f_i]_{i\in I}$ is a homomorphism of F-coalgebras from $(\sum_{i\in I} X_i, \zeta)$ to (Y,β). Since $[f_i]_{i\in I}$ is the unique morphism in **C** such that $[f_i]_{i\in I} \circ k_j = f_j$, for all $j \in I$, it follows that it is also the unique homomorphism of F-coalgebras from $(\sum_{i\in I} X_i, \zeta)$ to (Y,β) that verifies such condition. Hence, $((\sum_{i\in I} X_i, \zeta), (k_i)_{i\in I})$ is a coproduct of $(X_i, \alpha_i)_{i\in I}$ in CoAlg(F).

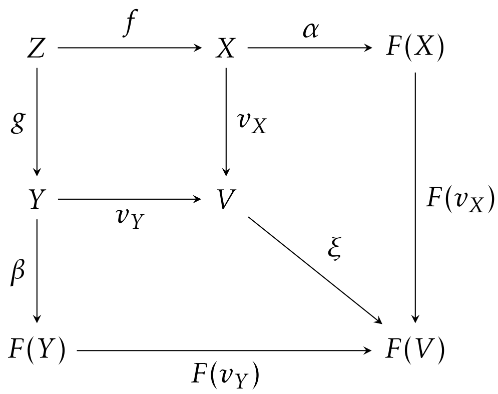
(ii) Consider two homomorphisms of F-coalgebras, with common domain, $f : (Z,\gamma) \to (X,\alpha)$ and $g : (Z,\gamma) \to (Y,\beta)$, and assume there is a pushout $(V,(v_X,v_Y))$ of $f : Z \to X$ and $g : Z \to Y$ in **C**. Note that

$$\begin{aligned} F(v_X)\circ\alpha\circ f &= F(v_X)\circ F(f)\circ\gamma \\ &= F(v_X\circ f)\circ\gamma \\ &= F(v_Y\circ g)\circ\gamma \\ &= F(v_Y)\circ F(g)\circ\gamma \\ &= F(v_Y)\circ\beta\circ g, \end{aligned}$$

so the diagram

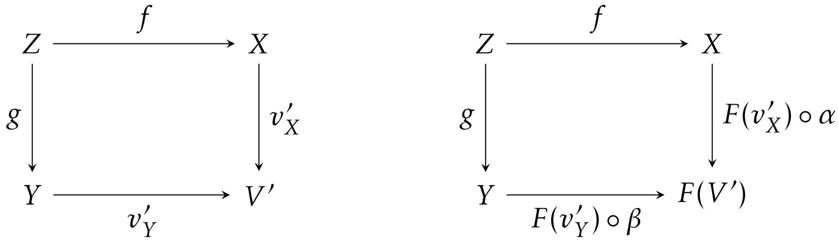


commutes. Then, since $(V,(v_X,v_Y))$ is a pushout of f and g in **C**, there is a unique morphism $\xi : V \to F(V)$ in **C** such that the following diagram



commutes. Thus, v_X is a homomorphism of F-coalgebras from (X,α) to (V,ξ) and v_Y is a homomorphism of F-coalgebras from (Y,β) to (V,ξ) with $v_X \circ f = v_Y \circ g$.

Now, let (V',ξ') be an F-coalgebra, and let $v'_X : (X,\alpha) \to (V',\xi')$ and $v'_Y : (Y,\beta) \to (V',\xi')$ be homomorphisms of F-coalgebras with $v'_X \circ f = v'_Y \circ g$. Similarly to what is done above, we have $F(v'_X)\circ\alpha\circ f = F(v'_Y)\circ\beta\circ g$. Then, both diagrams



commute. Since $(V,(v_X,v_Y))$ is a pushout of f and g in **C**, there are unique morphisms $h_1 : V \to V'$ and $h_2 : V \to F(V')$ in **C** such that $v'_X = h_1 \circ v_X$, $v'_Y = h_1 \circ v_Y$, $F(v'_X) \circ \alpha = h_2 \circ v_X$ and $F(v'_Y) \circ \beta = h_2 \circ v_Y$. Also, we have

$$\xi' \circ h_1 \circ v_X = \xi' \circ v'_X = F(v'_X) \circ \alpha$$

and

$$\xi' \circ h_1 \circ v_Y = \xi' \circ v'_Y = F(v'_Y) \circ \beta,$$

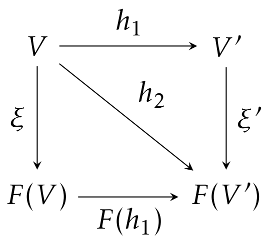
and so $\xi' \circ h_1 = h_2$ by uniqueness. On the other hand, we have

$$F(h_1) \circ \xi \circ v_X = F(h_1) \circ F(v_X) \circ \alpha = F(h_1 \circ v_X) \circ \alpha = F(v'_X) \circ \alpha$$

and

$$F(h_1) \circ \xi \circ v_Y = F(h_1) \circ F(v_Y) \circ \beta = F(h_1 \circ v_Y) \circ \beta = F(v'_Y) \circ \beta,$$

which implies $F(h_1) \circ \xi = h_2$ by uniqueness. Hence, the diagram



commutes, and thus h_1 is a homomorphism of F-coalgebras from (V,ξ) to (V',ξ'). Since h_1 is the unique morphism in **C** such that $v'_X = h_1 \circ v_X$ and $v'_Y = h_1 \circ v_Y$, it follows that it is the only homomorphism of F-coalgebras from (V,ξ) to (V',ξ') that verifies such conditions. Therefore, $((V,\xi),(v_X,v_Y))$ is a pushout of $f : (Z,\gamma) \to (X,\alpha)$ and $g : (Z,\gamma) \to (Y,\beta)$ in CoAlg(F). □

For instance, if $F : \mathbf{Set} \to \mathbf{Set}$ is any functor on the category **Set**, we have coproducts of any set-indexed family of F-coalgebras and pushouts of any two homomorphisms of F-coalgebras, with common domain, by the previous theorem.

In other cases, for a functor $F : \mathbf{C} \to \mathbf{C}$ on a category **C**, it is useful if F verifies certain properties. In the following result, we show how to construct an F-coalgebra from a weak pullback (Definition 2.3) in **C** when F preserves weak pullbacks (Definition 2.6).

Theorem 3.6. *Let $F : \mathbf{C} \to \mathbf{C}$ be a functor on a category $\mathbf{C}$ that preserves weak pullbacks. Given two homomorphisms of F-coalgebras, with common codomain, $f : (X,\alpha) \to (Z,\gamma)$ and $g : (Y,\beta) \to (Z,\gamma)$, if $(P,(p_X,p_Y))$ is a weak pullback of $f : X \to Z$ and $g : Y \to Z$ in $\mathbf{C}$, then there exists $\pi : P \to F(P)$ such that p_X is a homomorphism of F-coalgebras from (P,π) to (X,α) and p_Y is a homomorphism of F-coalgebras from (P,π) to (Y,β).*

Proof. Let $f : (X,\alpha) \to (Z,\gamma)$ and $g : (Y,\beta) \to (Z,\gamma)$ be homomorphisms of F-coalgebras, with common codomain, and assume there is a weak pullback $(P,(p_X,p_Y))$ of $f : X \to Z$ and $g : Y \to Z$ in $\mathbf{C}$. Observe that

$$F(f) \circ \alpha \circ p_X = \gamma \circ f \circ p_X = \gamma \circ g \circ p_Y = F(g) \circ \beta \circ p_Y.$$

This implies that the diagram

$$\begin{array}{ccc}
P & \xrightarrow{\alpha \circ p_X} & F(X) \\
{\scriptstyle \beta \circ p_Y}\downarrow & & \downarrow{\scriptstyle F(f)} \\
F(Y) & \xrightarrow[F(g)]{} & F(Z)
\end{array}$$

commutes. Since F preserves weak pullbacks, $(F(P),(F(p_X),F(p_Y)))$ is a weak pullback of $F(f) : F(X) \to F(Z)$ and $F(g) : F(Y) \to F(Z)$ in $\mathbf{C}$, and thus there exists $\pi : P \to F(P)$ such that the following diagram

$$\begin{array}{ccccc}
P & & \xrightarrow{\quad p_X \quad} & & X \\
\downarrow & \searrow^{\pi} & & & \downarrow{\scriptstyle \alpha} \\
{\scriptstyle p_Y}\downarrow & & F(P) & \xrightarrow{F(p_X)} & F(X) \\
\downarrow & & {\scriptstyle F(p_Y)}\downarrow & & \downarrow{\scriptstyle F(f)} \\
Y & \xrightarrow[\beta]{} & F(Y) & \xrightarrow[F(g)]{} & F(Z)
\end{array}$$

commutes. Therefore, p_X is a homomorphism of F-coalgebras from (P,π) to (X,α) and p_Y is a homomorphism of F-coalgebras from (P,π) to (Y,β). $\square$

Finally, we present two examples of how deterministic and nondeterministic automata can be modelled by coalgebras. These examples will be used throughout this chapter.

Example 3.7. In the following we describe automata (as coalgebras) without initial state or set of initial states which will be introduced in Examples 3.10 and 3.14. Also, let A be an *input alphabet*, i.e. a nonempty set whose elements are called *(input) letters*.

(i) *Deterministic automata*. Consider $D = 2 \times (-)^A : \mathbf{Set} \to \mathbf{Set}$, where $2 = \{0,1\}$, to be a functor on the category **Set** mapping each set X to $2 \times X^A$, and mapping each function $f : X \to Y$ to $\mathrm{id}_2 \times f^A : 2 \times X^A \to 2 \times Y^A$ defined by

$$(\mathrm{id}_2 \times f^A)(z,h) = (z, f \circ h), \qquad (z,h) \in 2 \times X^A.$$

A deterministic automaton is a D-coalgebra $(X, \langle o,d \rangle : X \to 2 \times X^A)$ where X is the *set of states*, $o : X \to 2$ determines whether a state $x \in X$ is *final* ($o(x) = 1$) or not ($o(x) = 0$), and $d : X \to X^A$ is the *transition function*. In a D-coalgebra $(X, \langle o,d \rangle)$, we say that the input letter $a \in A$ causes a transition from $x \in X$ to $y \in X$, denoted by $x \xrightarrow{a} y$, if $y = d(x)(a)$.

Given D-coalgebras $(X, \langle o,d \rangle$ and $(X', \langle o',d' \rangle)$, a function $f : X \to Y$ is a homomorphism of D-coalgebras from $(X, \langle o,d \rangle$ to $(X', \langle o',d' \rangle)$ if, and only if, $o(x) = o(f(x))$ and $d'(f(x))(a) = f(d(x)(a))$ for all $x \in X$ and $a \in A$. In other words, this last condition says that if $x \xrightarrow{a} y$ then $f(x) \xrightarrow{a} f(y)$, for every $x,y \in X$ and $a \in A$.

(ii) *Nondeterministic automata*. Let $N = D\mathcal{P} = 2 \times (\mathcal{P}(-))^A : \mathbf{Set} \to \mathbf{Set}$ be a functor on the category **Set** defined by the composition of the powerset functor (Example 2.11) with D (defined above). A nondeterministic automaton is an N-coalgebra $(X, \langle o,d \rangle : X \to 2 \times (\mathcal{P}(X))^A)$ where X is the *set of states*, $o : X \to 2$ determines whether a state $x \in X$ is *final* ($o(x) = 1$) or not ($o(x) = 0$), and $d : X \to (\mathcal{P}(X))^A$ maps each state $x \in X$ and letter $a \in A$ to the *set of next states*. In this case, we write $x \xrightarrow{a} y$ if $y \in d(x)(a)$, for $x \in X$ and $a \in A$.

Given N-coalgebras $(X, \langle o,d \rangle$ and $(X', \langle o',d' \rangle)$, a function $f : X \to Y$ is a homomorphism of N-coalgebras from $(X, \langle o,d \rangle$ to $(X', \langle o',d' \rangle)$ if, and only if,

$$o(x) = o(f(x)) \qquad \textit{and} \qquad d'(f(x))(a) = \mathcal{P}(f)(d(x)(a)) = \bigcup_{y \in d(x)(a)} f(y)$$

for all $x \in X$ and $a \in A$. From the second condition, we have if $x \xrightarrow{a} y$ then $f(x) \xrightarrow{a} f(y)$ and also if $f(x) \xrightarrow{a} x'$ then there is $z \in X$ such that $x' = f(z)$ and $x \xrightarrow{a} z$, for every $x,y \in X$, $x' \in X'$ and $a \in A$.

3.2 Final coalgebras

Definition 3.8. Let $F : \mathbf{C} \to \mathbf{C}$ be a functor on a category $\mathbf{C}$. An F-coalgebra (Ω, ω) is *final* if for any F-coalgebra (X, α) there exists exactly one homomorphism of F-coalgebras $\ell_{(X,\alpha)}$ from (X,α) to (Ω,ω), i.e. there exists a unique morphism $\ell_{(X,\alpha)} : X \to \Omega$ in $\mathbf{C}$ which makes the diagram

$$\begin{CD} X @>{\ell_{(X,\alpha)}}>> \Omega \\ @V{\alpha}VV @VV{\omega}V \\ F(X) @>>{F(\ell_{(X,\alpha)})}> F(\Omega) \end{CD}$$

commute. Thus a final F-coalgebra is a final object in the category $\mathrm{CoAlg}(F)$.

Consider a functor $F : \mathbf{C} \to \mathbf{C}$ on a category $\mathbf{C}$ for which there is a final F-coalgebra (Ω, ω). Given an F-coalgebra (X, α), we usually denote $\ell_{(X,\alpha)}$ simply by ℓ_X, when the F-coalgebra structure is understood, or even by ℓ, when the F-coalgebra is clear from the context. Since a final F-coalgebra is a final object in $\mathrm{CoAlg}(F)$, then id_Ω is the only homomorphism of F-coalgebras from (Ω, ω) to itself, i.e. $\ell_{(\Omega,\omega)} = \mathrm{id}_\Omega$.

Note that for certain functors there is no final coalgebra. Although when a final coalgebra exists, it has the following property.

Theorem 3.9. *Let $F : \mathbf{C} \to \mathbf{C}$ be a functor on a category $\mathbf{C}$, and let (Ω, ω) be a final F-coalgebra. Then $\omega : \Omega \to F(\Omega)$ is an isomorphism in $\mathbf{C}$.*

Proof. Note that $F(\omega) : F(\Omega) \to F(F(\Omega))$, and so $(F(\Omega), F(\omega))$ is an F-coalgebra. Since (Ω, ω) is a final F-coalgebra, there exists exactly one morphism $\ell : F(\Omega) \to \Omega$ such that $\omega \circ \ell = F(\ell) \circ F(\omega)$. Thus, the diagram

$$
\begin{array}{ccccc}
\Omega & \xrightarrow{\ \omega\ } & F(\Omega) & \xrightarrow{\ \ell\ } & \Omega \\
{\scriptstyle \omega}\downarrow & & \downarrow{\scriptstyle F(\omega)} & & \downarrow{\scriptstyle \omega} \\
F(\Omega) & \xrightarrow[F(\omega)]{} & F(F(\Omega)) & \xrightarrow[F(\ell)]{} & F(\Omega)
\end{array}
$$

commutes, which implies that $\ell \circ \omega$ is a homomorphism of F-coalgebras from (Ω, ω) to itself. Hence, $\ell \circ \omega = \mathrm{id}_\Omega$ because (Ω, ω) is final. Now, we have that

$$\omega \circ \ell = F(\ell) \circ F(\omega) = F(\ell \circ \omega) = F(\mathrm{id}_\Omega) = \mathrm{id}_{F(\Omega)}.$$

Therefore, ω is an isomorphism in $\mathbf{C}$. □

The functor $D = 2 \times (-)^A$ of deterministic automata, presented in Example 3.7(i), has a final coalgebra described in the following example. For further details see [Rut98].

Example 3.10. Let A be an input alphabet. Denote the set of all finite words over A by A^*, where the empty word is denoted by ε and the concatenation of two words $u, v \in A^*$ is denoted by uv. Thus, A^* is (the carrier of) the free monoid over A. A *language* over A is any subset of A^*.

Consider the functor $D = 2 \times (-)^A : \mathbf{Set} \to \mathbf{Set}$ of deterministic automata defined in Example 3.7(i). Let $\Omega = \{L \mid L \subseteq A^*\}$ be the set of all languages over A. The *coalgebra of languages* is a D-coalgebra $(\Omega, \langle e, t\rangle : \Omega \to 2 \times \Omega^A)$ where

$$e(L) = \begin{cases} 1 & \text{if } \varepsilon \in L \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad t(L)(a) = \{w \in A^* \mid aw \in L\},$$

for each $L \in \Omega$ and each $a \in A$.

Given a D-coalgebra $(X, \langle o, d\rangle : X \to 2 \times X^A)$, the transition function $d : X \to X^A$ can be extended up to $d^* : X \to X^{A^*}$ inductively defined by

$$d^*(x)(\varepsilon) = x$$

for any $x \in X$, and

$$d^*(x)(aw) = d^*(d(x)(a))(w)$$

for any $x \in X$, $a \in A$ and $w \in A^*$. Define a function $\ell : X \to \Omega$ mapping each state $x \in X$ to the language it accepts

$$\ell(x) = \{w \in A^* \mid o(d^*(x)(w)) = 1\}.$$

For any $x \in X$ and $a \in A$, note that

$$o(x) = 1 \iff \varepsilon \in \ell(x) \iff e(\ell(x)) = 1$$

and

$$\begin{aligned} \ell(d(x)(a)) &= \{w \in A^* \mid o(d^*(d(x)(a))(w)) = 1\} \\ &= \{w \in A^* \mid o(d^*(x)(aw)) = 1\} \\ &= \{w \in A^* \mid aw \in \ell(x)\} \\ &= t(\ell(x))(a). \end{aligned}$$

This implies that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad \ell \quad} & \Omega \\ {\scriptstyle \langle o,d\rangle}\downarrow & & \downarrow{\scriptstyle \langle e,t\rangle} \\ 2 \times X^A & \xrightarrow[\mathrm{id}_2 \times \ell^A]{} & 2 \times \Omega^A \end{array}$$

commutes, and so ℓ is a homomorphism of D-coalgebras from $(X, \langle o, d\rangle)$ to $(\Omega, \langle e, t\rangle)$. Moreover, ℓ is unique. Thus, $(\Omega, \langle e, t\rangle)$ is a final D-coalgebra.

Note that if we consider a deterministic automaton as a D-coalgebra $(X, \langle o, d\rangle)$ with an initial state $x_0 \in X$, then the language recognized by this automaton is $\ell(x_0)$.

On the other hand, the functor $N = D\mathcal{P} = 2 \times (\mathcal{P}(-))^A$ of nondeterministic automata, presented in Example 3.7(ii), does not have a final coalgebra. If (Ω, ω) were a final N-coalgebra, then $\omega : \Omega \to 2 \times (\mathcal{P}(\Omega))^A$ would be a bijection, by Theorem 3.9, which is a contradiction because the cardinality of $2 \times (\mathcal{P}(\Omega))^A$ is strictly greater than the cardinality of Ω (recall that A is a nonempty set).

3.3 Determinization

The powerset construction is a well-known determinization process that allows to obtain a deterministic automaton from a nondeterministic automaton, preserving many properties such as language recognizability. This process can be generalized to coalgebras of functors that satisfy certain conditions. See also [Jac+12; Sil+13].

Consider a monad (T, η, μ) on a category $\mathbf{C}$ and a functor $F : \mathbf{C} \to \mathbf{C}$ with a lifting $\widehat{F} : \mathcal{EM}(T) \to \mathcal{EM}(T)$ to the Eilenberg-Moore category of T (see Definitions 2.10, 2.15 and 2.19). By Definition 3.1, an $\widehat{F}$-coalgebra is a pair $((X,h),\alpha)$ consisting of a T-algebra (X,h) and a homomorphism of T-algebras $\alpha : (X,h) \to \widehat{F}(X,h)$. And a homomorphism of $\widehat{F}$-coalgebras $f : ((X,h),\alpha) \to ((X',h'),\alpha')$ is a morphism $f : X \to X'$ in $\mathbf{C}$ such that f is both a homomorphism of T-algebras from (X,h) to (X',h') and a homomorphism of F-coalgebras from (X,α) to (X',α').

By Theorem 2.21, there exists a distributive law $\rho : TF \to FT$ of T over F such that $\widehat{F}(X,h) = (F(X), F(h) \circ \rho_X)$, for each T-algebra (X,h). Given an FT-coalgebra (X,α), we have $\alpha : X \to FT(X)$ where $(FT(X), F(\mu_X) \circ \rho_{T(X)}) = \widehat{F}(T(X), \mu_X)$ is a T-algebra, and so the extension

$$\overline{\alpha} = F(\mu_X) \circ \rho_{T(X)} \circ T(\alpha) : T(X) \to FT(X)$$

is the unique homomorphism of T-algebras from (X,μ_X) to $\widehat{F}(X,\mu_X)$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & T(X) \\ & {\searrow}^{\alpha} & \downarrow{\scriptstyle \overline{\alpha}} \\ & & FT(X) \end{array}$$

commute, i.e. $\overline{\alpha} \circ \eta_X = \alpha$, by Theorem 2.17. Therefore, $((T(X),\mu_X),\overline{\alpha})$ is an $\widehat{F}$-coalgebra.

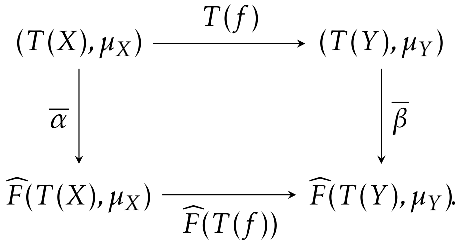
Let $f : (X,\alpha) \to (Y,\beta)$ be a homomorphism of FT-coalgebras, that is $\beta \circ f = FT(f) \circ \alpha$. Note that $T(f)$ is a homomorphism of T-algebras from $(T(X),\mu_X)$ to $(T(Y),\mu_Y)$, because μ is a natural transformation from T^2 to T, and so $FT(f) = \widehat{F}(T(f))$ is also a homomorphism of T-algebras from $\widehat{F}(T(X),\mu_X)$ to $\widehat{F}(T(Y),\mu_Y)$. Since ρ is a natural transformation from TF to FT, the diagram

$$\begin{array}{ccccccc} T(X) & \xrightarrow{T(\alpha)} & TFT(X) & \xrightarrow{\rho_{T(X)}} & FT^2(X) & \xrightarrow{F(\mu_X)} & FT(X) \\ \downarrow{\scriptstyle T(f)} & & \downarrow{\scriptstyle TFT(f)} & & \downarrow{\scriptstyle FT^2(f)} & & \downarrow{\scriptstyle FT(f)} \\ T(Y) & \xrightarrow[T(\beta)]{} & TFT(Y) & \xrightarrow[\rho_{T(Y)}]{} & FT^2(Y) & \xrightarrow[F(\mu_Y)]{} & FT(Y) \end{array}$$

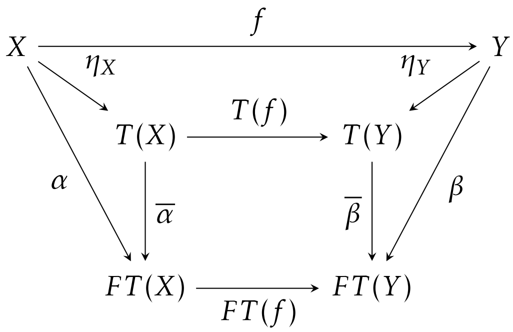
commutes, and thus

$$\overline{\beta} \circ T(f) = F(\mu_Y) \circ \rho_{T(Y)} \circ T(\beta) \circ T(f) = FT(f) \circ F(\mu_X) \circ \rho_{T(X)} \circ T(\alpha) = FT(f) \circ \overline{\alpha}$$

which implies that the following diagram is commutative in $\mathcal{EM}(T)$



Hence, $T(f)$ is a homomorphism of $\widehat{F}$-coalgebras from $((T(X),\mu_X),\overline{\alpha})$ to $((T(Y),\mu_Y),\overline{\beta})$. Also, observe that the diagram

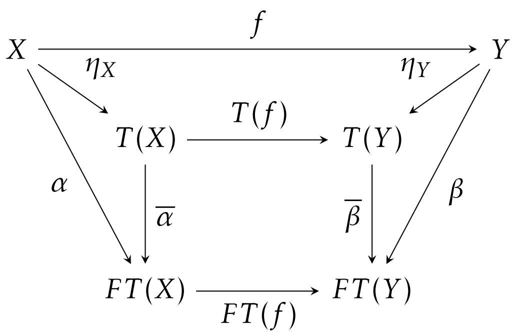


commutes, because η is a natural transformation from $\mathrm{Id}_{\mathbf{C}}$ to T.

We proved the following theorem that gives a coalgebraic interpretation of the determinization process [Jac+12].

Theorem 3.11. *Let (T,η,μ) be a monad on a category $\mathbf{C}$, and let $F:\mathbf{C}\to\mathbf{C}$ be a functor with a lifting $\widehat{F}:\mathcal{EM}(T)\to\mathcal{EM}(T)$ to the Eilenberg-Moore category of T. For each FT-coalgebra (X,α), define $H(X,\alpha)=((T(X),\mu_X),\overline{\alpha})$, where $\overline{\alpha}$ is the unique homomorphism of T-algebras from (X,μ_X) to $\widehat{F}(X,\mu_X)$ such that $\overline{\alpha}\circ\eta_X=\alpha$. For each homomorphism of FT-coalgebras f, define $H(f)=T(f)$. Then, H is a functor from* $\mathrm{CoAlg}(FT)$ *to* $\mathrm{CoAlg}(\widehat{F})$*.*

In particular, the following diagram



commutes, for any homomorphism of FT-coalgebras $f:(X,\alpha)\to(Y,\beta)$.

In the following example, we apply the previous theorem to the functor $N = D\mathcal{P} = 2 \times (\mathcal{P}(-))^A$ of nondeterministic automata. And basically, we get the determinization via the powerset construction as a particular case of the coalgebraic determinization.

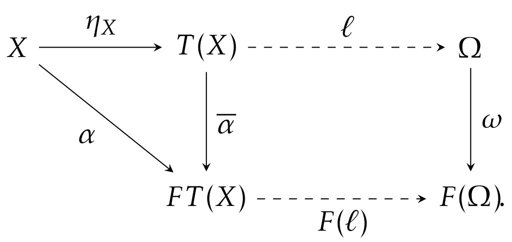
Example 3.12. Consider the functor $N = D\mathcal{P} = 2\times(\mathcal{P}(-))^A : \mathbf{Set} \to \mathbf{Set}$ of nondeterministic automata, introduced in Example 3.7(ii), where $(\mathcal{P}, \eta, \mu)$ is the powerset monad presented in Example 2.11. Let $\widehat{D} : \mathcal{EM}(\mathcal{P}) \to \mathcal{EM}(\mathcal{P})$ be the lifting of D to $\mathcal{EM}(\mathcal{P})$ defined in Example 2.22. Observe that $\widehat{D}$-coalgebras are basically deterministic automata in the category of complete join-semilattices, by Example 2.16.

Given a $D\mathcal{P}$-coalgebra $(X, \langle o, d\rangle : X \to 2 \times (\mathcal{P}(X))^A)$, by Theorem 3.11, we have a $\widehat{D}$-coalgebra $((\mathcal{P}(X), \mu_X), \langle \overline{o}, \overline{d}\rangle)$ where $\overline{o} : \mathcal{P}(X) \to 2$ and $\overline{d} : \mathcal{P}(X) \to (\mathcal{P}(X))^A$ are defined by

$$\overline{o}(S) = \max\{o(x) \mid x \in S\} \qquad \text{and} \qquad \overline{d}(S)(a) = \bigcup_{x\in S} d(x)(a),$$

for every $S \in \mathcal{P}(X)$ and $a \in A$. In particular, $(\mathcal{P}(X), \langle \overline{o}, \overline{d}\rangle)$ is a deterministic automaton as it is a D-coalgebra. Also, a set $S \in \mathcal{P}(X)$ is final in $(\mathcal{P}(X), \langle \overline{o}, \overline{d}\rangle)$ if, and only if, there is a state $x \in S$ that is final in $(X, \langle o, d\rangle)$. And for $S, S' \in \mathcal{P}(X)$ and $a \in A$, $S \xrightarrow{a} S'$ in $(\mathcal{P}(X), \langle \overline{o}, \overline{d}\rangle)$ if, and only if, for every state $x' \in S'$ there exists a state $x \in S$ such that $x \xrightarrow{a} x'$ in $(X, \langle o, d\rangle)$. Note that $(\mathcal{P}(X), \langle \overline{o}, \overline{d}\rangle)$ is basically the determinization of $(X, \langle o, d\rangle)$ via the powerset construction.

In the terms of Theorem 3.11, assume that F has a final coalgebra (Ω, ω) (see Definition 3.8). Given an FT-coalgebra (X, α), by Theorem 3.11, we have an $\widehat{F}$-coalgebra $((T(X), \mu_X), \overline{\alpha})$, and thus an F-coalgebra $(T(X), \overline{\alpha})$. Since (Ω, ω) is a final F-coalgebra, there exists a unique homomorphism of F-coalgebras $\ell : (T(X), \overline{\alpha}) \to (\Omega, \omega)$. Then, we have that the following diagram



commutes. Moreover, we can obtain a final $\widehat{F}$-coalgebra whenever F has a final coalgebra.

Theorem 3.13. *Let (T, η, μ) be a monad on a category $\mathbf{C}$, and let $F : \mathbf{C} \to \mathbf{C}$ be a functor with a lifting $\widehat{F} : \mathcal{EM}(T) \to \mathcal{EM}(T)$ to the Eilenberg-Moore category of T. By Theorem 2.21, define a distributive law $\rho : TF \to FT$ of T over F such that $\widehat{F}(X, h) = (F(X), F(h) \circ \rho_X)$, for every T-algebra (X, h). If there exists a final F-coalgebra (Ω, ω), then $\widehat{F}$ has a final coalgebra $((\Omega, \kappa), \omega)$ where $\kappa : T(\Omega) \to \Omega$ is the unique morphism in $\mathbf{C}$ making the diagram*

$$\begin{array}{ccc} T(\Omega) & \overset{\kappa}{\dashrightarrow} & \Omega \\ {\scriptstyle \rho_\Omega \circ T(\omega)}\Big\downarrow & & \Big\downarrow{\scriptstyle \omega} \\ FT(\Omega) & \underset{F(\kappa)}{\dashrightarrow} & F(\Omega) \end{array}$$

commute, i.e. $\omega \circ \kappa = F(\kappa) \circ \rho_\Omega \circ T(\omega)$.

Proof. Suppose that F has a final coalgebra (Ω, ω). We have that $(T(\Omega), \rho_X \circ T(\omega))$ is an F-coalgebra, then there exists exactly one morphism $\kappa : T(\Omega) \to \Omega$ in $\mathbf{C}$ such that the diagram

$$\begin{array}{ccc} T(\Omega) & \overset{\kappa}{\dashrightarrow} & \Omega \\ {\scriptstyle \rho_\Omega \circ T(\omega)}\Big\downarrow & & \Big\downarrow{\scriptstyle \omega} \\ FT(\Omega) & \underset{F(\kappa)}{\dashrightarrow} & F(\Omega) \end{array}$$

commutes. Since η is a natural transformation from $\mathrm{Id}_{\mathbf{C}}$ to T and ρ is a distributive law of T over F (Definition 2.20), the diagram

$$\begin{array}{ccccc} \Omega & \xrightarrow{\omega} & F(\Omega) & & \\ {\scriptstyle \eta_\Omega}\Big\downarrow & & {\scriptstyle \eta_{F(\Omega)}}\Big\downarrow & \overset{F(\eta_\Omega)}{\searrow} & \\ T(\Omega) & \underset{T(\omega)}{\longrightarrow} & TF(\Omega) & \underset{\rho_\Omega}{\longrightarrow} & FT(\Omega) \end{array}$$

commutes, then η_Ω is a homomorphism of F-coalgebras from (Ω, ω) to $(T(\Omega), \rho_\Omega \circ T(\omega))$. Thus, $\kappa \circ \eta_\Omega$ is a homomorphism of F-coalgebras from (Ω, ω) to itself, which implies that $\kappa \circ \eta_\Omega = \mathrm{id}_\Omega$, by finality of (Ω, ω). Also, both diagrams

$$\begin{array}{ccccc} T^2(\Omega) & \xrightarrow{T(\rho_\Omega \circ T(\omega))} & TFT(\Omega) & \xrightarrow{\rho_{T(\Omega)}} & FT^2(\Omega) \\ {\scriptstyle T(\kappa)}\Big\downarrow & & \Big\downarrow{\scriptstyle TF(\kappa)} & & \Big\downarrow{\scriptstyle FT(\kappa)} \\ T(\Omega) & \underset{T(\omega)}{\longrightarrow} & TF(\Omega) & \underset{\rho_\Omega}{\longrightarrow} & FT(\Omega) \end{array}$$

and

$$
\begin{array}{ccccccc}
T^2(\Omega) & \xrightarrow{T^2(\omega)} & T^2F(\Omega) & \xrightarrow{T(\rho_\Omega)} & TFT(\Omega) & \xrightarrow{\rho_{T(\Omega)}} & FT^2(\Omega) \\
{\scriptstyle \mu_\Omega}\downarrow & & \downarrow{\scriptstyle \mu_{F(\Omega)}} & & & & \downarrow{\scriptstyle F(\mu_\Omega)} \\
T(\Omega) & \xrightarrow[T(\omega)]{} & TF(\Omega) & & \xrightarrow[\rho_\Omega]{} & & FT(\Omega)
\end{array}
$$

commute, because ρ is a distributive law of T over F and μ is a natural transformation from T^2 to T, and hence both $T(\kappa)$ and μ_Ω are homomorphisms of F-coalgebras from $(T^2(\Omega), \rho_{T(\Omega)} \circ T(\rho_\Omega) \circ T^2(\omega))$ to $(T(\Omega), \rho_\Omega \circ T(\omega))$. Therefore, both $\kappa \circ T(\kappa)$ and $\kappa \circ \mu_\Omega$ are homomorphisms of F-coalgebras from $(T^2(\Omega), \rho_{T(\Omega)} \circ T(\rho_\Omega) \circ T^2(\omega))$ to (Ω, ω), which implies that $\kappa \circ T(\kappa) = \kappa \circ \mu_\Omega$, by finality of (Ω, ω). Hence, (Ω, κ) is a T-algebra.

By the definition of κ, the diagram

$$
\begin{array}{ccc}
T(\Omega) & \xrightarrow{T(\omega)} & TF(\Omega) \\
{\scriptstyle \kappa}\downarrow & & \downarrow{\scriptstyle F(\kappa) \circ \rho_\Omega} \\
\Omega & \xrightarrow[\omega]{} & F(\Omega)
\end{array}
$$

commutes, then ω is a homomorphism of T-algebras from (Ω, κ) to $\widehat{F}(\Omega, \kappa) = (F(\Omega), F(\kappa) \circ \rho_\Omega)$. Thus, $((\Omega, \kappa), \omega)$ is an $\widehat{F}$-coalgebra.

Let $((X, h), \alpha)$ be an $\widehat{F}$-coalgebra. Note that (X, α) is an F-coalgebra, and so there exists a unique homomorphism of F-coalgebras $\ell : (X, \alpha) \to (\Omega, \omega)$, because (Ω, ω) is a final F-coalgebra. We have that $\omega \circ \kappa = F(\kappa) \circ \rho_\Omega \circ T(\omega)$ and $\omega \circ \ell = F(\ell) \circ \alpha$, and since α is a homomorphism of T-algebras from (X, h) to $\widehat{F}(X, h) = (F(X), F(h) \circ \rho_X)$ and ρ is (in particular) a natural transformation from TF to FT, then both diagrams

$$
\begin{array}{ccccc}
T(X) & \xrightarrow{h} & X & \xrightarrow{\ell} & \Omega \\
{\scriptstyle \rho_X \circ T(\alpha)}\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \omega} \\
FT(X) & \xrightarrow[F(h)]{} & F(X) & \xrightarrow[F(\ell)]{} & F(\Omega)
\end{array}
$$

and

$$
\begin{array}{ccccc}
T(X) & \xrightarrow{T(\ell)} & T(\Omega) & \xrightarrow{\kappa} & \Omega \\
{\scriptstyle T(\alpha)}\downarrow & & \downarrow{\scriptstyle T(\omega)} & & \\
TF(X) & \xrightarrow{TF(\ell)} & TF(\Omega) & & \downarrow{\scriptstyle \omega} \\
{\scriptstyle \rho_X}\downarrow & & \downarrow{\scriptstyle \rho_\Omega} & & \\
FT(X) & \xrightarrow[FT(\ell)]{} & FT(\Omega) & \xrightarrow[F(\kappa)]{} & F(\Omega)
\end{array}
$$

commute, which implies that both $\ell \circ h$ and $\kappa \circ T(\ell)$ are homomorphisms of F-coalgebras from $(T(X), \rho_X \circ T(\alpha))$ to (Ω, ω). Hence, $\ell \circ h = \kappa \circ T(\ell)$ because (Ω, ω) is a final F-coalgebra, and so ℓ is a homomorphism of T-algebras from (X, h) to (Ω, κ). Therefore, ℓ is a homomorphism of $\widehat{F}$-coalgebras from $((X, h), \alpha)$ to $((\Omega, \kappa), \omega)$.

Finally, if $\ell' : ((X, h), \alpha) \to ((\Omega, \kappa), \omega)$ is a homomorphism of $\widehat{F}$-coalgebras, then ℓ' is also a homomorphism of F-coalgebras from (X, α) to (Ω, ω) which implies $\ell' = \ell$ due to (Ω, ω) be a final F-coalgebra. Thus, ℓ is the unique homomorphism of $\widehat{F}$-coalgebras from $((X, h), \alpha)$ to $((\Omega, \kappa), \omega)$.

We have that $((\Omega, \kappa), \omega)$ is a final $\widehat{F}$-coalgebra. □

In Example 3.10 we saw that language recognition arises from a final coalgebra for deterministic automata. By the previous theorem, we get a similar notion, from a coalgebraic point of view, for nondeterministic automata.

Example 3.14. Consider the functor $N = D\mathcal{P} = 2 \times (\mathcal{P}(-))^A : \mathbf{Set} \to \mathbf{Set}$ of nondeterministic automata, described in Example 3.7(ii), where $(\mathcal{P}, \eta, \mu)$ is the powerset monad presented in Example 2.11. Let $\widehat{D} : \mathcal{EM}(\mathcal{P}) \to \mathcal{EM}(\mathcal{P})$ and $\rho : \mathcal{P}D \to D\mathcal{P}$ be the lifting of D to $\mathcal{EM}(\mathcal{P})$ and the distributive law of $\mathcal{P}$ over D, respectively, defined in Example 2.22. And let $(\Omega, \langle e, t \rangle)$, where $\Omega = \{L \mid L \subseteq A^*\}$, be the coalgebra of languages described in Example 3.10, which is a final D-coalgebra. Define $\kappa : \mathcal{P}(\Omega) \to \Omega$ by

$$\kappa(X) = \bigcup_{L \in X} L, \qquad X \in \mathcal{P}(\Omega),$$

and define $\langle e', t' \rangle = \rho_\Omega \circ \mathcal{P}(\langle e, t \rangle) : \mathcal{P}(\Omega) \to D\mathcal{P}(\Omega)$ which is given by

$$e'(X) = \max\{e(L) \mid L \in X\} \qquad \text{and} \qquad t'(X)(a) = \{t(L)(a) \mid L \in X\}$$

for each $X \in \mathcal{P}(\Omega)$ and $a \in A$. Observe that the diagram

$$
\begin{array}{ccc}
\mathcal{P}(\Omega) & \xrightarrow{\kappa} & \Omega \\
{\scriptstyle \langle e', t' \rangle = \rho_\Omega \circ \mathcal{P}(\langle e, t \rangle)}\downarrow & & \downarrow{\scriptstyle \langle e, t \rangle} \\
D\mathcal{P}(\Omega) & \xrightarrow[\mathrm{id}_2 \times \kappa^A]{} & D(\Omega)
\end{array}
$$

commutes. Then, by Theorem 3.13, $((\Omega, \kappa), \langle e, t\rangle)$ is a final $\widehat{D}$-coalgebra.

Given a $D\mathcal{P}$-coalgebra $(X, \langle o, d\rangle)$, we obtain a $\widehat{D}$-coalgebra $((\mathcal{P}(X), \mu_X), \langle \overline{o}, \overline{d}\rangle)$ as described in Example 3.12, and then there exists a unique homomorphism of $\widehat{D}$-coalgebras $\ell : ((\mathcal{P}(X), \mu_X), \langle \overline{o}, \overline{d}\rangle) \to ((\Omega, \kappa), \langle e, t\rangle)$ defined by

$$\ell(S) = \{w \in A^* \mid \overline{o}(\overline{d}^*(S)(w)) = 1\}, \qquad S \in \mathcal{P}(X).$$

For any $S \in \mathcal{P}(X)$ and $a_1, a_2, \ldots, a_n \in A^*$, where $n \in \mathbb{N}$, note that $a_1 a_2 \ldots a_n \in \ell(S)$ if and only if there exists $x_0 \in S$ such that $x_0 \xrightarrow{a_1} x_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} x_n$ in $(X, \langle o, d\rangle)$ and $o(x_n) = 1$, for some $x_{i+1} \in d(x_i)(a_{i+1})$ with $0 \leq i < n$. Then, ℓ maps each set $S \in \mathcal{P}(X)$ to the language recognized by the nondeterministic automaton $(X, \langle o, d\rangle)$ with S as set of initial states.

Chapter 4

Fuzzy sets

In this chapter, we study fuzzy sets with membership degrees in a complete residuated lattice and define a monad for them. Fuzzy sets were first introduced in [Zad65] to model collections of objects where the question of whether an object belongs to a collection is answered by a value in a certain structure. See also [Gog67; Win07].

This chapter is divided in three parts. First, in Section 4.1, we introduce complete residuated lattices that we will use as structure of membership degrees for fuzzy sets, and state some results that will be useful to work with fuzzy sets. We also define the notion of a module over a complete residuated lattice in Section 4.1.1.

Second, in Section 4.2, we give a brief introduction to fuzzy sets and fuzzy relations. For a comprehensive introduction we refer to [Běl02; BV05]. Then, in Section 4.2.2, we relate fuzzy sets and modules over a complete residuated lattice.

Finally, in Section 4.3, we make a categorical approach to fuzzy sets by defining a monad for them. And then, in Section 4.3.1, we present a isomorphism between algebras of the fuzzy-set monad and the category of modules over a complete residuated lattice.

4.1 Residuated lattices

In this section we introduce an algebraic structure that will be used as structure of truth values for the underlying fuzzy logic. Since fuzzy logic admits many truth values (between the traditional ones: absolutely true and absolutely false), this structure gains extreme importance. In concrete applications, the structure of truth values is the basis of our resoning and judgment. So we choose to work on complete residuated lattices which are general enough to be used in many different problems and still a rich structure where basic logical notions can be "soundly" modelled. Thus, along the section we also do some notes about the logical motivation and interpretation of some properties.

Definition 4.1. A *residuated lattice* is an algebra $\mathcal{K} = (K, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ with four binary and two nullary operations that satifies:

(RL1) $(K, \wedge, \vee, 0, 1)$ is a lattice with the least element 0 and the greatest element 1 for the partial order $\leq$ defined by

$$x \leq y \iff x \vee y = y, \qquad x, y \in K;$$

(RL2) $(K, \otimes, 1)$ is a commutative monoid with the unit 1;

(RL3) $\otimes$ and $\rightarrow$ form an adjoint pair, i.e. the following property holds

$$x \leq y \rightarrow z \iff x \otimes y \leq z, \qquad x, y, z \in K.$$

The operations $\otimes$ and $\rightarrow$ are called *multiplication* and *residuum*, respectively. Also, if $(K, \wedge, \vee, 0, 1)$ is a complete lattice, then $\mathcal{K}$ is called a *complete residuated lattice*.

In this thesis, the structures of truth values of fuzzy logic will be complete residuated lattices. Let $\mathcal{K} = (K, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be such a structure. Condition (RL1) guarantees that the set of truth values K is partially ordered (where infima and suprema of every two truth values exist) and contains the least truth value 0 (representing "absolutely false") and the greatest truth value 1 (representing "absolutely true"). The (general) infimum $\bigwedge$ and (general) supremum $\bigvee$ are intended for modelling of the general and existential quantifier, respectively.

For logical connectives, the multiplication $\otimes$ and residuum $\rightarrow$ are intended for modelling of the conjunction and implication, respectively. Condition (RL2) provides that the multiplication satisfies some properties that we want for the conjunction. Condition (RL3) reflects a generalization of the inference rule *modus ponens* (from classical bivalent logic) and ensures that the residuum gives the greatest truth value for which it holds.

Although residuated lattices are derived from relatively simple logical assumptions, we will see that they originate a rich structure of truth values. Since we intended to generalize classical bivalent logic, the two-element Boolean algebra (i.e. the structure of truth of classical bivalent logic) arises as a residuated lattice on $\{0, 1\}$.

Example 4.2. Let $K = 2 = \{0, 1\}$. Define $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$, for $x, y \in K$. Then $(K, \wedge, \vee, 0, 1)$ is a (complete) lattice. Also, define $x \otimes y = x \wedge y$ and $x \rightarrow y = \max(1 - x, y)$, for $x, y \in K$. Thus, $(K, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is a (complete) residuated lattice, which is basically the two-element Boolean algebra (to match the standard definition [BS81, Definition IV§1.3], consider the complementation operation $'$ given by $x' = x \rightarrow 0$, $x \in K$). Moreover, this is the unique residuated lattice on $2 = \{0, 1\}$ (with $0 < 1$), which we denote by $\mathbf{2}$.

Furthermore, most of the structures of truth values used for fuzzy logic are (complete) residuated lattices as shown in the following example.

Example 4.3. Let $K = [0, 1]$ be the interval of all real numbers between 0 and 1. Note that $(K, \min, \max, 0, 1)$ is a complete lattice.

(i) Define

$$x \otimes y = \max(x + y - 1, 0)$$
$$x \to y = \min(1 - x + y, 1),$$

for $x, y \in [0,1]$. Then, $(K, \min, \max, \otimes, \to, 0, 1)$ is a complete residuated lattice that corresponds to the *standard Lukasiewicz algebra*.

(ii) Define

$$x \otimes y = \min(x, y)$$
$$x \to y = \begin{cases} 1 & \text{if } x \le y \\ y & \text{if } y < x \end{cases}$$

for $x, y \in [0,1]$. Then, $(K, \min, \max, \otimes, \to, 0, 1)$ is a complete residuated lattice that corresponds to the *standard Gödel algebra*.

(iii) Define

$$x \otimes y = x \cdot y$$
$$x \to y = \begin{cases} 1 & \text{if } x \le y \\ y/x & \text{if } y < x \end{cases}$$

(where $\cdot$ and $/$ represent the usual multiplication and division of real numbers), for $x, y \in [0,1]$. Then, $(K, \min, \max, \otimes, \to, 0, 1)$ is a complete residuated lattice that corresponds to the *standard product algebra*.

The following theorem presents some basic properties of residuated lattices. Each of them has an interpretation from the logical point of view that complements the previous discussion. For instance, Definition 4.1 does not (directly) state the result of $x \otimes 0$, although one would expect it to be 0.

Theorem 4.4. *Let $\mathcal{K} = (K, \wedge, \vee, \otimes, \to, 0, 1)$ be a residuated lattice. The following hold for any $x, y \in K$:*

(i) $x \otimes 0 = 0$;

(ii) $1 \to x = x$;

(iii) $x \otimes (x \to y) \le y$;

(iv) $x \le y$ *if, and only if,* $x \to y = 1$;

(v) $x \otimes y \le x \wedge y$;

(vi) the set $\{z \in K \mid x \otimes z \le y\}$ *is nonempty and* $x \to y$ *is its supremum.*

Proof. Let $x, y \in K$. (i) Since 0 is the least element, we have $0 \le x \to 0$, then by adjunction $x \otimes 0 = 0 \otimes x \le 0$ which implies $x \otimes 0 = 0$.

(ii) By adjunction, $x \otimes 1 = x \le x$ implies $x \le 1 \to x$ and $1 \to x \le 1 \to x$ implies $1 \to x = (1 \to x) \otimes 1 \le x$. Thus, $1 \to x = x$.

(iii) From $x \to y \le x \to y$, it follows that $x \otimes (x \to y) = (x \to y) \otimes x \le y$.

(iv) By adjunction, we have that

$$x \le y \iff 1 \otimes x \le y \iff 1 \le x \to y \iff x \to y = 1.$$

(v) Applying the previous result, $y \to y = 1$ (since $y \le y$) and then $x \le y \to y$ implies that $x \otimes y \le y$. Since $\otimes$ is commutative, we also have $x \otimes y = y \otimes x \le x$. Thus, $x \otimes y \le x \wedge y$.

(vi) Let $S = \{z \in K \mid x \otimes z \le y\}$. By (iii), $x \to y \in S$. If z is such that $x \otimes z \le y$ (or $z \otimes x \le y$, by commutativity of $\otimes$), then $z \le x \to y$. Therefore, $x \to y = \bigvee S$. □

From these last properties it is easy to check that $\otimes$ and $\to$ behave like the classical conjunction and implication on $\{0,1\}$, respectively. It supports our claim in Example 4.2 that the only two-element residuated lattice is indeed the two-element Boolean algebra.

Theorem 4.5. *Let $\mathcal{K} = (K, \wedge, \vee, \otimes, \to, 0, 1)$ be a residuated lattice. The multiplication $\otimes$ is order-preserving in both arguments and the residuum $\to$ is order-preserving in the second and order-reversing in the first argument, i.e.*

$$\begin{array}{cc} x \otimes y_1 \le x \otimes y_2 & y_1 \otimes x \le y_2 \otimes x \\ x \to y_1 \le x \to y_2 & y_2 \to x \le y_1 \to x \end{array}$$

for any $x, y_1, y_2 \in K$, where $y_1 \le y_2$.

Proof. Let $x, y_1, y_2 \in K$ with $y_1 \le y_2$. From $y_2 \otimes x \le y_2 \otimes x$, we get $y_2 \le x \to (y_2 \otimes x)$ (by adjunction) and thus $y_1 \le x \to (y_2 \otimes x)$ which implies $y_1 \otimes x \le y_2 \otimes x$. Therefore, $\otimes$ is order-preserving in the first argument and, since $\otimes$ is commutative, it follows for the second argument.

From Theorem 4.4(iii), we have $(x \to y_1) \otimes x \le y_1$ and thus $(x \to y_1) \otimes x \le y_2$ which implies $x \to y_1 \le x \to y_2$. Therefore, $\to$ is order-preserving in the second argument.

Since $(y_2 \to x) \otimes y_2 \le x$ (Theorem 4.4(iii)) and $\otimes$ is order-preserving (in the second argument), we have $(y_2 \to x) \otimes y_1 \le (y_2 \to x) \otimes y_2$. Thus, $(y_2 \to x) \otimes y_1 \le x$ that implies $y_2 \to x \le y_1 \to x$. Therefore, $\to$ is order-reversing in the first argument. □

Note that the order preservation (or reversion) has a special relevance from the logical point of view, because it tells how the truth degree of a conjunction or implication depends on the truth degree of the propositions involved. Moreover, the following result can be seen as a generalization of Theorem 4.5.

Theorem 4.6. *Let $\mathcal{K} = (K, \wedge, \vee, \otimes, \to, 0, 1)$ be a complete residuated lattice. The following (distributive) rules hold for any $x, y_i \in K$, $i \in I$, where I is an index set:*

(i) $x \otimes (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \otimes y_i)$;

(ii) $x \to (\bigwedge_{i \in I} y_i) = \bigwedge_{i \in I} (x \to y_i)$;

(iii) $(\bigvee_{i \in I} y_i) \to x = \bigwedge_{i \in I} (y_i \to x)$.

Proof. Let $x, y_i \in K$, $i \in I$ (where I is an index set). (i) For each $j \in I$, we have $y_j \leq \bigvee_{i\in I} y_i$ and, since $\otimes$ is order-preserving (Theorem 4.5), $x \otimes y_j \leq x \otimes (\bigvee_{i\in I} y_i)$. If z is such that $x \otimes y_j \leq z$ (or, equivalently, $y_j \leq x \to z$), for each $j \in I$, then $\bigvee_{i\in I} y_i \leq x \to z$ which implies $x \otimes (\bigvee_{i\in I} y_i) \leq z$. Therefore, $x \otimes (\bigvee_{i\in I} y_i) = \bigvee_{i\in I}(x \otimes y_i)$.

(ii) For each $j \in I$, we have $\bigwedge_{i\in I} y_i \leq y_j$ and, since $\to$ is order-preserving in the second argument (Theorem 4.5), $x \to (\bigwedge_{i\in I} y_i) \leq x \to y_j$. If z is such that $z \leq x \to y_j$ (or, equivalently, $z \otimes x \leq y_j$), for each $j \in I$, then $z \otimes x \leq \bigwedge_{i\in I} y_i$ which implies $z \leq x \to (\bigwedge_{i\in I} y_i)$. Therefore, $x \to (\bigwedge_{i\in I} y_i) = \bigwedge_{i\in I}(x \to y_i)$.

(iii) For each $j \in I$, $y_j \leq \bigvee_{i\in I} y_i$ and, since $\to$ is order-reversing in the first argument (Theorem 4.5), $(\bigvee_{i\in I} y_i) \to x \leq y_j \to x$. If z is such that $z \leq y_j \to x$ (or, equivalently, $z \otimes y_j \leq x$), for each $j \in I$, then $z \otimes (\bigvee_{i\in I} y_i) = \bigvee_{i\in I}(z \otimes y_i) \leq x$ (from (i)) which implies $z \leq (\bigvee_{i\in I} y_i) \to x$. Therefore, $(\bigvee_{i\in I} y_i) \to x = \bigwedge_{i\in I}(y_i \to x)$. □

In Theorem 4.6, we could assume $\bigwedge_{i\in I} y_i$ and $\bigvee_{i\in I} y_i$ to exist instead of $\mathcal{K}$ to be complete. From this fact, we have that

$$x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$$
$$x \to (y \wedge z) = (x \to y) \wedge (x \to z)$$
$$(x \vee y) \to z = (x \to z) \wedge (y \to z)$$

in any residuated lattice.

From the properties already described, it is easy to check that any residuated lattice satisfies the following identities

$$\begin{aligned} x \to (x \vee y) &= 1 \\ (x \otimes (x \to y)) \vee y &= y \\ (x \otimes y) \to z &= x \to (y \to z). \end{aligned}$$

On the other hand, if an algebra $\mathcal{K} = (K, \wedge, \vee, \otimes, \to, 0, 1)$ satisfies (RL1) and (RL2) from Definition 4.1 and these three identities, then $\mathcal{K}$ is a residuated lattice. Thus, the class of all residuated lattices is a variety of algebras [Běl02, Theorem 2.18].

The following example generalizes Example 4.3 (note that the standard Lukasiewicz algebra is an MV-algebra).

Example 4.7. The following are alternative descriptions of well-known types of algebras which mostly match the usual definitions so that in [Běl02, Definition 2.15] they are given as the main definitions.

(i) A *Heyting algebra* (or *Brouwerian lattice*) is a residuated lattice $(K, \wedge, \vee, \otimes, \to, 0, 1)$ satisfying

$$x \otimes y = x \wedge y, \qquad x, y \in K.$$

(ii) A *BL-algebra* is a residuated lattice $(K,\wedge,\vee,\otimes,\rightarrow,0,1)$ satisfying

$$x \wedge y = x \otimes (x \rightarrow y),$$
$$(x \rightarrow y) \vee (y \rightarrow x) = 1, \qquad x,y \in K.$$

(iii) An *MV-algebra* is a residuated lattice $(K,\wedge,\vee,\otimes,\rightarrow,0,1)$ satisfying

$$x \vee y = (x \rightarrow y) \rightarrow y, \qquad x,y \in K.$$

(iv) A $\prod$-*algebra* (*product algebra*) is a residuated lattice $(K,\wedge,\vee,\otimes,\rightarrow,0,1)$ which is a BL-algebra and satisfies

$$x \wedge (x \rightarrow 0) = 0,$$
$$(z \rightarrow 0) \rightarrow 0 \leq ((x \otimes z) \rightarrow (y \otimes z)) \rightarrow (x \rightarrow y), \qquad x,y,z \in K.$$

(v) A *G-algebra* (*Gödel algebra*) is a residuated lattice $(K,\wedge,\vee,\otimes,\rightarrow,0,1)$ which is a BL-algebra and satisfies

$$x \otimes x = x, \qquad x \in K.$$

(vi) A *Boolean algebra* is a residuated lattice which is both a Heyting algebra and an MV-algebra.

On the subsequent sections, $\mathcal{K} = (K,\wedge,\vee,\otimes,\rightarrow,0,1)$ is an arbitrary complete residuated lattice, with at least two elements (i.e. $0 \neq 1$), that will be used as structure of truth values for the underlying fuzzy logic.

4.1.1 $\mathcal{K}$-modules

Complete join-semilattices (Example 2.16) are also called sup-lattices (cf. [JT84]), because a complete join-semilattice $(X,\vee)$ has arbitrary meets defined by

$$\bigwedge S = \bigvee \{x \in X \mid x \leq s \text{ for all } s \in S\}, \qquad S \subseteq X,$$

and so $(X,\wedge,\vee)$ is a complete lattice. However, a homomorphism of complete join-semilattices may not preserve meets and thus it is not a homomorphism of complete lattices.

Definition 4.8. (i) A $\mathcal{K}$-*module* $\mathcal{M} = (M,\vee,*)$ is a complete join-semilattice $(M,\vee)$ together with an action $*: K \times M \rightarrow M$, $(k,m) \mapsto k * m$, that satisfies the following properties

$$1 * m = m, \qquad a * (b * m) = (a \otimes b) * m,$$
$$(\bigvee_{i\in I} a_i) * m = \bigvee_{i\in I}(a_i * m), \qquad a * (\bigvee_{i\in I} m_i) = \bigvee_{i\in I}(a * m_i),$$

for any $a,b \in K$, $m \in M$, and any families $(a_i)_{i\in I}$ and $(m_i)_{i\in I}$ of elements of K and M, respectively.

(ii) A *homomorphism of $\mathcal{K}$-modules* $f : \mathcal{M} \to \mathcal{M}'$ is a function $f : M \to M'$ such that

$$f(\bigvee_{i\in I} m_i) = \bigvee_{i\in I}{}' f(m_i) \qquad \text{and} \qquad f(a * m) = a *' f(m),$$

for any $a \in K$, $m \in M$ and family $(m_i)_{i\in I}$ of elements of M. Homomorphisms of $\mathcal{K}$-modules are also called *linear maps*.

(iii) The category with objects all $\mathcal{K}$-modules and morphisms all linear maps between them is denoted by $\mathcal{K}$-**Mod**.

This terminology is due to the resemblances with the definition of modules over a ring. It is also defined modules over a quantale (see [Res00; Ros90] for a general reference about quantales and quantale modules), and since $\mathcal{K}$ is a special type of quantale whose multiplication is commutative and 1 is its unit, $\mathcal{K}$-modules are a special case of modules over a quantale.

Given $\mathcal{K}$-modules $\mathcal{M}$ and $\mathcal{M}'$. Note that a linear map $f : \mathcal{M} \to \mathcal{M}'$ is a homomorphism of complete join-semilattices from $(M, \vee)$ to $(M', \vee')$ which commutes with the action of each element $a \in K$ (i.e. $f(a * m) = a *' f(m)$, for all $m \in M$).

Example 4.9. $(K, \vee, \otimes)$ is a $\mathcal{K}$-module (by Theorem 4.6(i)).

We can also obtain $\mathcal{K}$-modules from other $\mathcal{K}$-modules as described in the following two theorems.

Theorem 4.10. *(i) Given a $\mathcal{K}$-module $\mathcal{M} = (M, \vee, *)$ and a set X. We have a $\mathcal{K}$-module $\mathcal{M}^X = (M^X, \vee^X, *^X)$, where $\vee^X$ and $*^X$ are defined pointwise, i.e.*

$$(\bigvee_{i\in I}{}^X f_i)(x) = \bigvee_{i\in I} f_i(x) \qquad \text{and} \qquad (a *^X f)(x) = a * f(x), \qquad x \in X,$$

for any $f \in M^X$, $a \in K$, and any family of functions $(f_i : X \to M)_{i\in I}$.

(ii) Given a linear map $f : \mathcal{M} \to \mathcal{M}'$, then $f^X : M^X \to (M')^X$ defined by

$$f^X(h) = f \circ h, \qquad h \in M^X,$$

is a linear map from $\mathcal{M}^X$ to $(\mathcal{M}')^X$

Proof. (i) Since M is nonempty, because it is the carrier of a complete join-semilattice, then M^X is nonempty. For any functions $h_1, h_2, h_3 \in M^X$, we have that

$$(h_1 \vee^X h_2)(x) = h_1(x) \vee h_2(x) = h_2(x) \vee h_1(x) = (h_2 \vee^X h_1)(x),$$
$$(h_1 \vee^X h_1)(x) = h_1(x) \vee h_1(x) = h_1(x),$$

and

$$\begin{aligned}((h_1 \vee^X h_2) \vee^X h_3)(x) &= (h_1(x) \vee h_2(x)) \vee h_3(x) \\ &= h_1(x) \vee (h_2(x) \vee h_3(x)) = (h_1 \vee^X (h_2 \vee^X h_3))(x)\end{aligned}$$

for all $x \in X$. This implies that $(M^X, \vee^X)$ is a join-semilattice, where the partial order is given by

$$h_1 \leq^X h_2 \iff \forall x \in X.\, h_1(x) \leq h_2(x), \qquad h_1, h_2 \in M^X.$$

Let $(h_i)_{i \in I}$ be a family of functions from X to M. Define $s : X \to M$ by

$$s(x) = \bigvee_{i \in I} h_i(x), \qquad x \in X.$$

Then, by definition of $\leq^X$, we have that $h_i \leq^X s$ for all $i \in I$. If $u : X \to M$ is such that $h_i \leq^X u$ for all $i \in I$, that is $h_i(x) \leq u(x)$ for all $x \in X$ and $i \in I$, then

$$s(x) = \bigvee_{i \in I} h_i(x) \leq u(x),$$

for all $x \in X$, which implies $s \leq^X u$. Hence, s is the supremum of $(h_i)_{i \in I}$. Consequently, $(M^X, \vee^X)$ is a complete join-semilattice.

For any $h \in M^X$ and $a, b \in K$, we have that

$$\begin{aligned}(1 *^X h)(x) &= 1 * h(x) = h(x)\\ (a *^X (b *^X h))(x) &= a * (b * h(x)) = (a \otimes b) * h(x) = ((a \otimes b) *^X h)(x),\end{aligned}$$

for all $x \in X$. And for any families $(a_i)_{i \in I}$ of elements of K and $(h_i)_{i \in I}$ of functions from X to M, we have that

$$\begin{aligned}((\bigvee_{i \in I} a_i) *^X h)(x) &= (\bigvee_{i \in I} a_i) * h(x) = \bigvee_{i \in I} a_i * h(x) = \bigvee_{i \in I} (a_i *^X h)(x) = (\bigvee_{i \in I}^X a_i *^X h)(x)\\ (a *^X (\bigvee_{i \in I}^X h_i))(x) &= a * (\bigvee_{i \in I}^X h_i)(x) = a * (\bigvee_{i \in I} h_i(x)) = \bigvee_{i \in I} a * h_i(x) = (\bigvee_{i \in I}^X a *^X h_i)(x),\end{aligned}$$

for every $x \in X$. Therefore, $(M^X, \vee^X, *^X)$ is a $\mathcal{K}$-module.

(ii) Let $f : \mathcal{M} \to \mathcal{M}'$ be a linear map. Then, for a family $(h_i)_{i \in I}$ of functions from X to M, we have that

$$f^X(\bigvee_{i \in I}^X h_i)(x) = f(\bigvee_{i \in I} h_i(x)) = \bigvee_{i \in I}{}' f(h_i(x)) = (\bigvee_{i \in I}{}'^X f^X(h_i))(x)$$

for all $x \in X$. And for any $a \in K$ and $h \in M^X$, we have that

$$f^X(a *^X h)(x) = f(a * h(x)) = a *' f(h(x)) = (a *'^X f^X(h))(x),$$

for all $x \in X$. Therefore, f^X is a linear map from $\mathcal{M}^X$ to $(\mathcal{M}')^X$. □

Theorem 4.11. *(i) Given two $\mathcal{K}$-modules $\mathcal{M}$ and $\mathcal{M}'$. Then, we have a $\mathcal{K}$-module $\mathcal{M} \times \mathcal{M}' = (M \times M', \vee^\times, *^\times)$, where $\vee^\times$ and $*^\times$ are defined componentwise, i.e.*

$$\bigvee_{i \in I}^\times (m_i, m_i') = (\bigvee_{i \in I} m_i, \bigvee_{i \in I}{}' m_i') \qquad \text{and} \qquad a *^\times (m, m') = (a * m, a *' m')$$

for any $(m, m') \in M \times M'$, $a \in K$, and any family $(m_i, m_i')_{i \in I}$ of elements of $M \times M'$).

(ii) Given two linear maps $f : \mathcal{M}_1 \to \mathcal{M}_2$ and $f' : \mathcal{M}_1' \to \mathcal{M}_2'$, then $f \times f'$ is a linear map from $\mathcal{M}_1 \times \mathcal{M}_1'$ to $\mathcal{M}_2 \times \mathcal{M}_2'$.

Proof. (i) Since $\mathcal{M}$ and $\mathcal{M}'$ are $\mathcal{K}$-modules, the sets M and M' are both nonempty, and so $M \times M'$ is nonempty. For any $(m_1, m_1'), (m_2, m_2'), (m_3, m_3') \in M \times M'$, we have that

$$\begin{aligned}
(m_1, m_1') \vee^\times (m_1, m_1') &= (m_1 \vee m_1, m_1' \vee' m_1') = (m_1, m_1'),\\
(m_1, m_1') \vee^\times (m_2, m_2') &= (m_1 \vee m_2, m_1' \vee' m_2') = (m_2 \vee m_1, m_2' \vee' m_1')\\
&= (m_2, m_2') \vee^\times (m_1, m_1')
\end{aligned}$$

and

$$\begin{aligned}
(((m_1, m_1') \vee^\times (m_2, m_2')) \vee^\times (m_3, m_3')) &= ((m_1 \vee m_2) \vee m_3, (m_1' \vee' m_2') \vee' m_3')\\
&= (m_1 \vee (m_2 \vee m_3), m_1' \vee' (m_2' \vee' m_3'))\\
&= ((m_1, m_1') \vee^\times ((m_2, m_2') \vee^\times (m_3, m_3'))).
\end{aligned}$$

Then, $(M \times M', \vee^\times)$ is a join-semilattice where the partial order $\leq^\times$ is given by

$$(m_1, m_1') \leq^\times (m_2, m_2') \iff m_1 \leq m_2 \text{ and } m_1' \leq' m_2',$$

$(m_1, m_1'), (m_2, m_2') \in M \times M'$. Let $(m_i, m_i')_{i \in I}$ be a family of elements of $M \times M'$. Define

$$(s, s') = (\bigvee_{i \in I} m_i, \bigvee_{i \in I}{}' m_i').$$

Then $(m_i, m_i') \leq^\times (s, s')$ for all $i \in I$. If $(u, u') \in M \times M'$ are such that $(m_i, m_i') \leq^\times (u, u')$ for all $i \in I$, that is $m_i \leq u$ and $m_i' \leq' u'$ for all $i \in I$, then $s \leq u$ and $s' \leq' u'$ which implies that $(s, s') \leq^\times (u, u')$. Hence, (s, s') is the supremum of $(m_i, m_i')_{i \in I}$. Consequently, $(M \times M', \vee^\times)$ is a complete join-semilattice.

For any $(m, m') \in M \times M'$ and $a, b \in K$, we have that

$$\begin{aligned}
1 *^\times (m, m') &= (1 * m, 1 *' m') = (m, m')\\
a *^\times (b \vee^\times (m, m')) &= (a * (b * m), a *' (b *' m')) = ((a \otimes b) * m, (a \otimes b) *' m')\\
&= (a \otimes b) *^\times (m, m').
\end{aligned}$$

And for any families $(a_i)_{i \in I}$ of elements of K and $(m_i, m_i')_{i \in I}$ of elements of $M \times M'$, we have that

$$\begin{aligned}
(\bigvee_{i \in I} a_i) *^\times (m, m') &= ((\bigvee_{i \in I} a_i) * m, (\bigvee_{i \in I} a_i) *' m')\\
&= (\bigvee_{i \in I} a_i * m, \bigvee_{i \in I} a_i *' m') = \bigvee_{i \in I}{}^\times a_i *^\times (m, m')
\end{aligned}$$

and

$$\begin{aligned}
a *^\times (\bigvee_{i \in I}{}^\times (m_i, m_i')) &= (a * (\bigvee_{i \in I} m_i), a *' (\bigvee_{i \in I}{}' m_i'))\\
&= (\bigvee_{i \in I} a * m_i, \bigvee_{i \in I}{}' a *' m_i) = \bigvee_{i \in I}{}^\times a *^\times (m_i, m_i').
\end{aligned}$$

Therefore, $(M \times M', \vee^{\times}, *^{\times})$ is a $\mathcal{K}$-module.

(ii) Let $f : \mathcal{M}_1 \to \mathcal{M}_2$ and $f' : \mathcal{M}'_1 \to \mathcal{M}'_2$ be linear maps. For any family $(m_i, m'_i)_{i\in I}$ of elements of $M_1 \times M'_1$, we have that

$$(f \times f')(\bigvee_{i\in I}^{1\times}(m_i, m'_i)) = (f(\bigvee_{i\in I}^{1} m_i), f'(\bigvee_{i\in I}^{1'} m'_i))$$
$$= (\bigvee_{i\in I}^{2} f(m_i), \bigvee_{i\in I}^{2'} f'(m'_i)) = \bigvee_{i\in I}^{2\times}(f \times f')(m_i, m'_i).$$

And for any $a \in K$ and $(m, m') \in M_1 \times M'_1$, we have that

$$(f \times f')(a *^{1\times} (m, m')) = (f(a *^{1} m), f'(a *^{1'} m'))$$
$$= (a *^{2} f(m), a *^{2'} f'(m')) = a *^{2\times} (f \times f')(m, m').$$

Therefore, $f \times f'$ is a linear map from $\mathcal{M}_1 \times \mathcal{M}'_1$ to $\mathcal{M}_2 \times \mathcal{M}'_2$. □

Theorem 4.12. *Let $\mathcal{M}$ be a $\mathcal{K}$-module. The following hold for any $a, b \in K$, $m, n \in M$:*

(i) if $a \le b$, then $a * m \le b * m$;

(ii) if $m \le n$, then $a * m \le a * n$;

(iii) $0 * m = 0_M$, where 0_M is the least element of $\mathcal{M}$.

Proof. Let $a, b \in K$ and $m, n \in M$. (i) If $a \le b$, then

$$(a * m) \vee (b * m) = (a \vee b) * m = b * m$$

and so $a * m \le b * m$.

(ii) If $m \le n$, then

$$(a * m) \vee (a * n) = a * (m \vee n) = a * n$$

and thus $a * m \le a * n$.

(iii) Note that $0 = \bigvee \emptyset$ in $\mathcal{K}$, and $0_M = \bigvee \emptyset$ in M. Let $(a_i)_{i\in\emptyset}$ denote the empty family, then

$$0 * m = (\bigvee_{i\in\emptyset} a_i) * m = \bigvee_{i\in\emptyset} a_i * m = \bigvee \emptyset = 0_M.$$

by Definition 4.8(i). □

Observe that Definition 4.8(i) and Theorem 4.12(iii) determines how 0 and 1 act on any $\mathcal{K}$-module.

Example 4.13. Let 2 be the two-element (complete) residuated lattice in Example 4.2. For a complete join-semilattice $(M, \vee)$, we have that $(M, \vee, *)$ is a 2-module if, and only if, $* : 2 \times M \to M$ is given by

$$0 * m = 0_M \qquad \text{and} \qquad 1 * m = m, \qquad m \in M,$$

Also, a homomorphism of complete join-semilattices is a linear map of 2-modules. Thus, the category of 2-modules (Definition 4.8(iii)) is isomorphic to the category of complete join-semilattices (Example 2.16).

4.2 Fuzzy sets

In this section we present the notions of fuzzy sets and fuzzy relations (with truth values for membership degrees in a complete residuated lattice $\mathcal{K}$). We also define some basic operations (such as union of fuzzy sets and composition of fuzzy relations) that structure these notions.

Definition 4.14. Let X be a set.

(i) A *fuzzy subset* of X (over $\mathcal{K}$) is a function $\varphi : X \to K$. Given $x \in X$, $\varphi(x)$ is called the *membership degree of x in φ*.

(ii) In addition to K^X, the set of all fuzzy subsets of X is also denoted by $\mathcal{Z}(X)$.

The notation $\mathcal{Z}(X)$ will be mostly used in Section 4.3 to define a functor on **Set** that maps each set X to the set of all its fuzzy subsets $\mathcal{Z}(X)$. In general, we write $\varphi \in K^X$ or $\varphi : X \to K$, rather than $\varphi \in \mathcal{Z}(X)$, to be more explicit.

Let $\varphi \in K^X$ be a fuzzy subset of a set X. For each $x \in X$, $\varphi(x)$ can be interpreted as the truth value of "x is an element of φ" (or "x is in φ"). Thus, if $\varphi(x) \in \{0,1\}$ ($\subseteq K$), for all $x \in X$, then φ defines an ordinary subset $S = \{x \in X \mid \varphi(x) = 1\}$ of X, where φ is the characteristic function of S (i.e. $x \in S$, if $\varphi(x) = 1$, and $x \notin S$, if $\varphi(x) = 0$). Moreover, if there is a unique $x_0 \in X$ such that $\varphi(x_0) = 1$, and $\varphi(x) = 0$, for every $x \in X \setminus \{x_0\}$, then φ determines an element of X.

Definition 4.15. Let $\varphi \in K^X$ be a fuzzy subset of a set X.

(i) φ is called *crisp* if either $\varphi(x) = 0$ or $\varphi(x) = 1$, for all $x \in X$.

(ii) φ is called *crisp-deterministic* if there exists $x_0 \in X$ such that $\varphi(x_0) = 1$, and $\varphi(x) = 0$, for every $x \in X \setminus \{x_0\}$.

On the other hand, an ordinary subset S of a set X defines a crisp fuzzy subset φ of X where $\varphi(x) = 1$, if $x \in S$, and $\varphi(x) = 0$, if $x \in X \setminus S$. And an element $x_0 \in X$ defines a crisp-deterministic fuzzy subset φ of X where $\varphi(x_0) = 1$, and $\varphi(x) = 0$, for all $x \in X \setminus \{x_0\}$.

An application of fuzzy sets is for modelling of collections of objects where the question of whether an object is in a collection is answered by a truth value in a structure (namely a complete residuated lattice).

Example 4.16. (i) Consider $\mathcal{K}$ to be the standard Lukasiewicz algebra (Example 4.3(i)). Let $X = \{\text{black}, \text{brown}, \text{orange}, \text{red}\}$ and define a fuzzy subset φ of X by

$$\varphi(\text{black}) = 0, \quad \varphi(\text{brown}) = 0.4, \quad \varphi(\text{orange}) = 0.6, \quad \varphi(\text{red}) = 1.$$

The collection of "colors like red" (in X) may be represented by φ.

(ii) Consider $\mathcal{K}$ to be the standard Gödel algebra (Example 4.3(ii)). Let φ be a fuzzy subset of the natural numbers $\mathbb{N}$ defined by

$$\varphi(n) = \begin{cases} n/100 & \text{if } n \leq 100 \\ 1 & \text{otherwise,} \end{cases} \qquad n \in \mathbb{N}.$$

The collection of “large natural numbers” may be represented by φ.

(iii) Consider $\mathcal{K}$ to be the standard Gödel algebra (Example 4.3(ii)). Let φ be a fuzzy subset of the real numbers $\mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} 1+x & \text{if } -1 \leq x \leq 0 \\ 1-x & \text{if } 0 < x \leq 1 \\ 0 & \text{otherwise,} \end{cases} \qquad x \in \mathbb{R}.$$

The collection of “real numbers near 0” may be represented by φ.

Note that the structure of truth values plays a very important role when defining a fuzzy set. It is from where the membership degrees are choosen, which is itself a very important matter whether the choice represents our collection. Moreover, it also induces operation on the fuzzy sets.

Definition 4.17. Let X be a set. Define binary operations $\cap$, $\cup$, $\otimes^X$ and $\to^X$, and nullary operations $\emptyset$ and χ on K^X by

$$\begin{aligned} (\varphi \cap \psi)(x) &= \varphi(x) \wedge \psi(x), & (\varphi \cup \psi)(x) &= \varphi(x) \vee \psi(x), \\ (\varphi \otimes^X \psi)(x) &= \varphi(x) \otimes \psi(x), & (\varphi \to^X \psi)(x) &= \varphi(x) \to \psi(x), \\ \emptyset(x) &= 0, & \chi(x) &= 1, \end{aligned}$$

for all $x \in X$ and all $\varphi, \psi \in K^X$. Also, for a family $(\varphi_i)_{i \in I}$ of fuzzy subsets of X, indexed by a set I, define $(\bigcap_{i \in I} \varphi_i) : X \to K$ and $(\bigcup_{i \in I} \varphi_i) : X \to K$ by

$$(\bigcap_{i \in I} \varphi_i)(x) = \bigwedge_{i \in I} \varphi_i(x) \qquad \text{and} \qquad (\bigcup_{i \in I} \varphi_i)(x) = \bigvee_{i \in I} \varphi_i(x), \qquad x \in X.$$

Note that if X is the empty set, then K^X has exactly one element and thus the previous operations are trivially defined on K^X.

Let X be a set. The algebra $(K^X, \cap, \cup, \otimes^X, \to^X, \emptyset, \chi)$ is a complete residuated lattice, the (lattice) partial order $\subseteq$ is given by

$$\varphi \subseteq \psi \iff \forall x \in X.\, \varphi(x) \leq \psi(x), \qquad \varphi, \psi \in K^X,$$

where arbitrary meets and arbitrary joins correspond to $\bigcap$ and $\bigcup$, respectively. We usually omit the superscript X in $\otimes^X$ and $\to^X$, since it is always clear from the context when we are considering these operations on X or on K^X.

Example 4.18. Let $\mathbf{2}$ be the two-element (complete) residuated lattice in Example 4.2. For each set X, consider a function λ_X from the set of fuzzy subsets of X to the set of ordinary subsets of X, i.e. $\lambda_X : 2^X \to \mathcal{P}(X)$, defined by

$$\lambda_X(\varphi) = \{x \in X \mid \varphi(x) = 1\}, \qquad \varphi \in 2^X.$$

Let X be a set. λ_X is a bijection, which allows us to check that the operations on 2^X (from Definition 4.17) coincide with the usual ones on $\mathcal{P}(X)$. For $\varphi, \psi \in 2^X$, we have

$$(\varphi \cap \psi)(x) = 1 \iff x \in \lambda_X(\varphi) \text{ and } x \in \lambda_X(\psi),$$
$$(\varphi \cup \psi)(x) = 1 \iff x \in \lambda_X(\varphi) \text{ or } x \in \lambda_X(\psi), \qquad x \in X.$$

Thus, $\cap$ and $\cup$ on 2^X coincide with the intersection and union on $\mathcal{P}(X)$, respectively. Also

$$\varphi \subseteq \psi \iff \lambda_X(\varphi) \text{ is a subset of } \lambda_X(\psi), \qquad \varphi, \psi \in 2^X.$$

Similarly, $\bigcap$ and $\bigcup$ on 2^X correspond to arbitrary intersection and arbitrary union on $\mathcal{P}(X)$, respectively. Note that $\otimes$ coincides with $\cap$, since K has just two elements. Finally, for $\varphi, \psi \in 2^X$, we have

$$(\varphi \to \psi)(x) = 1 \iff x \in \lambda_X(\psi) \text{ or } x \in X \setminus \lambda_X(\varphi), \qquad x \in X.$$

4.2.1 Fuzzy relations

Recall that a relation between (ordinary) sets is a subset of their cartesian product. Thus, the following definition becomes very natural.

Definition 4.19. Let $X_1, X_2, \ldots, X_n$ be sets, $n \geq 2$. A *fuzzy relation* between $X_1, X_2, \ldots, X_n$ is a fuzzy subset of the cartesian product $X_1 \times X_2 \times \cdots \times X_n$. An *n-ary fuzzy relation* on a set X is a fuzzy subset of X^n.

Although fuzzy relations are fuzzy sets, some characterizations mean different properties. For instance, the terms introduced in Definition 4.15 for fuzzy sets have the following definition for fuzzy relations.

Definition 4.20. Let $\mu : X_1 \times X_2 \times \cdots \times X_n \to K$ be a fuzzy relation between sets $X_1, X_2, \ldots, X_n$, $n \geq 2$.

(i) μ is called *crisp* if either $\mu(x_1, x_2, \ldots, x_n) = 0$ or $\mu(x_1, x_2, \ldots, x_n) = 1$, for every $x_1 \in X_1, x_2 \in X_2, \ldots, x_n \in X_n$.

(ii) μ is called *crisp-deterministic* if for every $x_1 \in X_1, x_2 \in X_2, \ldots, x_{n-1} \in X_{n-1}$ there exists $x_n \in X_n$ such that $\mu(x_1, x_2, \ldots, x_{n-1}, x_n) = 1$, and $\mu(x_1, x_2, \ldots, x_{n-1}, y) = 0$, for all $y \in X_n \setminus \{x_n\}$.

Note that a crisp fuzzy relation is a crisp fuzzy subset of a cartesian product. However, this does not apply to crisp-deterministic fuzzy relations, where we have the following.

Let $\mu : X_1 \times X_2 \times \cdots \times X_n \to K$ be a fuzzy relation between sets $X_1, X_2, \ldots, X_n$, $n \geq 2$. For each $x_1 \in X_1, x_2 \in X_2, \ldots, x_{n-1} \in X_{n-1}$, consider a fuzzy subset $\mu_{x_1, x_2, \ldots, x_{n-1}}$ of X_n defined by

$$\mu_{x_1, x_2, \ldots, x_{n-1}}(x_n) = \mu(x_1, x_2, \ldots, x_n), \qquad x_n \in X_n.$$

Therefore, μ is a crisp-deterministic fuzzy relation between $X_1, X_2, \ldots, X_n$ if, and only if, $\mu_{x_1, x_2, \ldots, x_{n-1}}$ is a crisp-determistic fuzzy subset of X_n, for all $x_1 \in X_1, x_2 \in X_2, \ldots, x_{n-1} \in X_{n-1}$.

For each $x_1 \in X_1, x_2 \in X_2, \dots, x_n \in X_n$, $\mu(x_1, x_2, \dots, x_n)$ can be interpreted as the truth value of "$x_1, x_2, \dots, x_n$ are related by μ" (or "$x_1, x_2, \dots, x_n$ are μ-related"). Therefore, as fuzzy sets models collections of objects, fuzzy relations may be used for modelling of relationships between objects where the membertship degree (or *relationship degree*) is intended to answer the question of whether some objects are related.

Example 4.21 ([Běl02, Example 3.5])**.** Consider $\mathcal{K}$ to be any complete residuated lattice on $[0,1]$. Define a binary fuzzy relation μ on the set of real numbers $\mathbb{R}$ by

$$\mu(x,y) = \max(1 - |x-y|, 0) \qquad x, y \in \mathbb{R}$$

(where $|\cdot|$ represents the absolute value). Then μ may represent the relationship "being close real numbers".

Since fuzzy relations are fuzzy sets, the operations in Definition 4.17 apply to them.

Definition 4.22. Let X, Y and Z be sets.

(i) The *composition* of fuzzy relations $\mu \in K^{X\times Y}$ and $\nu \in K^{Y\times Z}$ is a fuzzy relation $\mu \diamond \nu \in K^{X\times Z}$ defined by

$$(\mu \diamond \nu)(x,z) = \bigvee_{y\in Y} (\mu(x,y) \otimes \nu(y,z)), \qquad (x,z) \in X \times Z.$$

(ii) The *composition* of a fuzzy set $\varphi \in K^X$ and a fuzzy relation $\mu \in K^{X\times Y}$ is a fuzzy set $\varphi \diamond \mu \in K^Y$ defined by

$$(\varphi \diamond \mu)(y) = \bigvee_{x\in X} (\varphi(x) \otimes \mu(x,y)), \qquad y \in Y.$$

(iii) The *composition* of a fuzzy relation $\mu \in K^{X\times Y}$ and a fuzzy set $\psi \in K^Y$ is a fuzzy set $\mu \diamond \psi \in K^X$ defined by

$$(\mu \diamond \psi)(x) = \bigvee_{y\in Y} (\mu(x,y) \otimes \psi(y)), \qquad x \in X.$$

(iv) The *composition* of fuzzy sets $\varphi, \psi \in K^X$ is an element $\varphi \diamond \psi \in K$ defined by

$$\varphi \diamond \psi = \bigvee_{x\in X} (\varphi(x) \otimes \psi(x)).$$

From Theorem 4.6(i), it is easy to verify that the composition of fuzzy structures (sets or relations) is associative, i.e., given sets X, Y and Z,

$$(\mu \diamond \nu) \diamond \rho = \mu \diamond (\nu \diamond \rho), \qquad (\varphi \diamond \mu) \diamond \nu = \varphi \diamond (\mu \diamond \nu),$$
$$(\mu \diamond \nu) \diamond \xi = \mu \diamond (\nu \diamond \xi), \qquad (\varphi \diamond \mu) \diamond \psi = \varphi \diamond (\mu \diamond \psi),$$

for any fuzzy relations $\mu \in K^{X\times Y}$, $\nu \in K^{Y\times Z}$ and $\rho \in K^{Z\times W}$, and any fuzzy sets $\varphi \in K^X$, $\psi \in K^Y$ and $\xi \in K^Z$. Therefore, we will omit parentheses when composing fuzzy structures.

Example 4.23. Let 2 be the two-element (complete) residuated lattice in Example 4.2, and let X, Y and Z be sets. Consider the functions $\lambda_X : 2^X \to \mathcal{P}(X)$, $\lambda_Y : 2^Y \to \mathcal{P}(Y)$, $\lambda_{X\times Y} : 2^{X\times Y} \to \mathcal{P}(X\times Y)$, and $\lambda_{Y\times Z} : 2^{Y\times Z} \to \mathcal{P}(Y\times Z)$ defined in Example 4.18. Let $\mu \in 2^{X\times Y}$. We have, for $\nu \in 2^{Y\times Z}$,

$$(\mu \diamond \nu)(x,z) = 1 \iff \exists y \in Y.\, (x,y) \in \lambda_{X\times Y}(\mu) \text{ and } (y,z) \in \lambda_{Y\times Z}(\nu), \qquad (x,z) \in X \times Z,$$

thus the composition of fuzzy relations corresponds to the usual composition of ordinary relations. For $\varphi \in 2^X$, we have

$$(\varphi \diamond \mu)(y) = 1 \iff \exists x \in \lambda_X(\varphi).\, (x,y) \in \lambda_{X\times Y}(\mu), \qquad y \in Y,$$

and so $\lambda_Y(\varphi\diamond\mu)$ is the subset of Y whose elements are $\lambda_{X\times Y}(\mu)$-related with some element of $\lambda_X(\varphi)$. Similarly, for $\psi \in 2^Y$,

$$(\mu \diamond \psi)(x) = 1 \iff \exists y \in \lambda_Y(\psi).\, (x,y) \in \lambda_{X\times Y}(\mu), \qquad x \in X.$$

Finally, for $\varphi, \varphi' \in 2^X$, we have

$$\varphi \diamond \varphi' = 1 \iff \exists x \in X.\, x \in \lambda_X(\varphi) \text{ and } x \in \lambda_X(\varphi'),$$

therefore the composition of fuzzy sets may be seen as a test for the resulting ordinary sets having common elements.

4.2.2 The $\mathcal{K}$-module of fuzzy sets

Given a set X, recall that $(K^X, \cap, \cup, \otimes^X, \to^X, \emptyset, \chi)$ is a complete residuated lattice (Definition 4.17). In particular, $(K^X, \cup)$ is a complete join-semilattice which becomes a $\mathcal{K}$-module (see Definition 4.8, Example 4.9 and Theorem 4.10) as follows.

Definition 4.24. Let X be a set. For every $a \in K$ and $\varphi \in K^X$, define $a \circledast \varphi : X \to K$ by

$$(a \circledast \varphi)(x) = a \otimes \varphi(x), \qquad x \in X.$$

$(K^X, \cup, \circledast)$ is called the *$\mathcal{K}$-module of fuzzy subsets of X*.

The $\mathcal{K}$-module of fuzzy subsets of a set X has a very good structure since the crisp-deterministic fuzzy subsets of X (Definition 4.15(ii)) works as a basis in the following sense.

Definition 4.25. For each set X, let $\eta_X : X \to K^X$ be defined by

$$\eta_X(x)(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise,} \end{cases} \qquad x, y \in X.$$

Theorem 4.26. *Let X be a set. Then for every $\varphi \in K^X$ there exist unique elements $a_x \in K$, for every $x \in X$, such that*

$$\varphi = \bigcup_{x\in X} a_x \circledast \eta_X(x).$$

Consequently, for any $\mathcal{K}$-module $\mathcal{M}$ and function $f : X \to M$, there exists a unique linear map $\overline{f} : (K^X, \cup, \circledast) \to \mathcal{M}$ such that $\overline{f}(\eta_X(x)) = f(x)$, for all $x \in X$.

Proof. Let $\varphi \in K^X$ and choose $a_x = \varphi(x)$, for each $x \in X$. Then

$$(\bigcup_{x\in X} a_x \circledast \eta_X(x))(y) = \bigcup_{x\in X}(a_x \circledast \eta_X(x))(y) = \bigcup_{x\in X} a_x \otimes \eta_X(x)(y) = a_y = \varphi(y)$$

(since $a_x \otimes \eta_X(x)(y) = a_x \otimes 0 = 0$, if $x \neq y$, by Theorem 4.4(i), and $a_y \otimes \eta_X(y)(y) = a_y \otimes 1 = a_y$), for all $y \in X$. Hence a_x, $x \in X$, are the unique elements such that $\varphi = \bigcup_{x\in X} a_x \circledast \eta_X(x)$.

Let $\mathcal{M}$ be a $\mathcal{K}$-module and $f : X \to M$ a function. Define $\overline{f} : K^X \to M$ by

$$\overline{f}(\varphi) = \bigvee_{x\in X} \varphi(x) * f(x), \qquad \varphi \in K^X.$$

Then $\overline{f}$ is indeed a linear map (Definition 4.8(ii)), since

$$\overline{f}(\bigcup_{i\in I}\varphi_i) = \bigvee_{x\in X}(\bigcup_{i\in I}\varphi_i)(x) * f(x) = \bigvee_{x\in X}(\bigvee_{i\in I}\varphi_i(x)) * f(x) = \bigvee_{x\in X}\bigvee_{i\in I}\varphi_i(x) * f(x) = \bigvee_{i\in I}\overline{f}(\varphi_i),$$

for any family $(\varphi_i)_{i\in I}$ of fuzzy subsets of X, and

$$\overline{f}(a \circledast \varphi) = \bigvee_{x\in X}(a \circledast \varphi)(x) * f(x) = \bigvee_{x\in X}(a \otimes \varphi(x)) * f(x) = \bigvee_{x\in X} a * (\varphi(x) * f(x)) = a * \overline{f}(\varphi),$$

for any $a \in K$ and $\varphi \in K^X$. Also,

$$\overline{f}(\eta_X(x)) = \bigvee_{y\in X} \eta_X(x)(y) * f(y) = f(x)$$

(since $\eta_X(x)(y) * f(y) = 0 * f(y) = 0_M$, whenever $y \neq x$, by Theorem 4.12(iii), and $\eta_X(x)(x) * f(x) = 1 * f(x) = f(x)$), for all $x \in X$. Finally, if $g : (K^X, \cup, \circledast) \to \mathcal{M}$ is a linear map such that $g(\eta_X(x)) = f(x)$, for each $x \in X$, then

$$\begin{aligned}
g(\varphi) &= g(\bigcup_{x\in X}\varphi(x) \circledast \eta_X(x)) \\
&= \bigvee_{x\in X} g(\varphi(x) \circledast \eta_X(x)) \\
&= \bigvee_{x\in X} \varphi(x) * g(\eta_X(x)) \\
&= \bigvee_{x\in X} \varphi(x) * f(x) = \overline{f}(\varphi),
\end{aligned}$$

for all $\varphi \in K^X$, and therefore $g = \overline{f}$. □

This last result motivates the following definition.

Definition 4.27. Given a set X, a $\mathcal{K}$-module $\mathcal{M}$ and a function $f : X \to M$, let $\overline{f} : K^X \to M$ be defined by

$$\overline{f}(\varphi) = \bigvee_{x \in X} \varphi(x) * f(x), \qquad \varphi \in K^X.$$

$\overline{f}$ is called the *linear extension of* f.

In other words, by Theorem 4.26, a linear extension of a function $f : X \to M$, where $\mathcal{M}$ is a $\mathcal{K}$-module, is the unique linear map $\overline{f} : (K^X, \cup, \circledast) \to \mathcal{M}$ such that $\overline{f}(\eta_X(x)) = f(x)$, for all $x \in X$.

Example 4.28. Let $\mathbf{2}$ be the two-element (complete) residuated lattice in Example 4.2. Let X be a set and $\mathcal{M}$ a $\mathbf{2}$-module (Example 4.13). For a function $f : X \to M$, we have $\overline{f} : 2^X \to M$ is such that

$$\overline{f}(\varphi) = \bigvee_{x \in X} \varphi(x) * f(x) = \bigvee_{\substack{x \in X \\ \varphi(x)=1}} f(x) = \bigvee_{x \in \lambda_X(\varphi)} f(x)$$

(where $\lambda_X : 2^X \to \mathcal{P}(X)$ is defined in Example 4.18), for all $\varphi \in 2^X$.

Observe the similarity between the linear extension in the previous example and the $\mathcal{P}$-algebra extension in Example 2.18, when $\mathcal{P}$-algebras are considered complete join-semilattices. This follows from the fact that the powerset monad is a particular case of the fuzzy-set monad presented in the next section.

4.3 A monad for fuzzy sets

In Definition 4.14(ii), we defined a function $\mathcal{Z}$ that maps a set X to the set of all its fuzzy subsets $\mathcal{Z}(X) = K^X$. For each set X, we also have a function $\eta_X : X \to \mathcal{Z}(X)$, by Definition 4.25. Let $\eta = (\eta_X)_{X \in \mathbf{Set}}$. Finally, given a function $f : X \to \mathcal{Z}(Y)$, we have that $(\mathcal{Z}(Y), \cup, \circledast)$ is a $\mathcal{K}$-module (Definition 4.24) and thus $\overline{f} : \mathcal{Z}(X) \to \mathcal{Z}(Y)$, the linear extension of f (Definition 4.27), is defined by

$$\overline{f}(\varphi)(y) = (\bigcup_{x \in X} \varphi(x) \circledast f(x))(y) = \bigvee_{x \in X} (\varphi(x) \circledast f(x))(y) = \bigvee_{x \in X} \varphi(x) \otimes f(x)(y),$$

for each $y \in Y$ and each $\varphi \in \mathcal{Z}(X)$. These definitions lead to the following result which uses the notion of Kleisli triple in Definition 2.12.

Theorem 4.29. *$(\mathcal{Z}, \eta, \overline{})$, as defined above, is a Kleisli triple.*

Proof. Let X, Y and Z be sets, and let $f : X \to \mathcal{Z}(Y)$ and $g : Y \to \mathcal{Z}(Z)$ be functions. First, we have $\overline{\eta_X} : \mathcal{Z}(X) \to \mathcal{Z}(X)$ is such that

$$\overline{\eta_X}(\varphi) = \bigcup_{x \in X} \varphi(x) \circledast \eta_X(x) = \varphi,$$

for all $\varphi \in \mathcal{Z}(X)$, by Theorem 4.26. Hence, $\overline{\eta_X} = \mathrm{id}_{\mathcal{Z}(X)}$.

Second, by Theorem 4.26, $\overline{f} : \mathcal{Z}(X) \to \mathcal{Z}(Y)$ is such that $\overline{f}(\eta_X(x)) = f(x)$, for each $x \in X$. Therefore, $\overline{f} \circ \eta_X = f$.

Finally, we have $\overline{\overline{g} \circ f} : \mathcal{Z}(X) \to \mathcal{Z}(Z)$ is such that

$$\begin{aligned}(\overline{\overline{g} \circ f})(\varphi)(z) &= \bigvee_{x \in X} \varphi(x) \otimes (\overline{g} \circ f)(x)(z) \\ &= \bigvee_{x \in X} \varphi(x) \otimes (\bigvee_{y \in Y} f(x)(y) \otimes g(y)(z)) \\ &= \bigvee_{y \in Y} (\bigvee_{x \in X} \varphi(x) \otimes f(x)(y)) \otimes g(y)(z) \\ &= \bigvee_{y \in Y} \overline{f}(\varphi)(y) \otimes g(y)(z) \\ &= \overline{g}(\overline{f}(\varphi))(z) = (\overline{g} \circ \overline{f})(\varphi)(z)\end{aligned}$$

(recall that $\otimes$ is distributive over $\bigvee$, by Theorem 4.6(i)) for any $z \in Z$ and $\varphi \in \mathcal{Z}(X)$. Hence, $\overline{\overline{g} \circ f} = \overline{g} \circ \overline{f}$. □

By Theorem 2.13, we have that a monad (Definition 2.10) can be obtained from a Kleisli triple. For each function $f : X \to Y$, define $\mathcal{Z}(f) = \overline{\eta_Y \circ f} : \mathcal{Z}(X) \to \mathcal{Z}(Y)$, that is

$$\mathcal{Z}(f)(\varphi) = \bigcup_{x \in X} \varphi(x) \circledast \eta_Y(f(x)), \qquad \varphi \in \mathcal{Z}(X).$$

Then $\mathcal{Z}$ becomes a functor on **Set**. And, for each set X, define $\mu_X = \overline{\mathrm{id}_{\mathcal{Z}(X)}} : \mathcal{Z}^2(X) \to \mathcal{Z}(X)$, where $\mathcal{Z}^2(X) = \mathcal{Z}(\mathcal{Z}(X)) = K^{(K^X)}$, that is

$$\mu_X(\Phi) = \bigcup_{\varphi \in \mathcal{Z}(X)} \Phi(\varphi) \circledast \varphi, \qquad \Phi \in \mathcal{Z}^2(X).$$

Let $\mu = (\mu_X)_{X \in \mathbf{Set}}$. Thus, we have the following result.

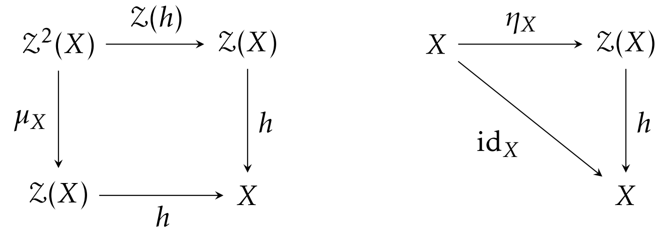
Corollary 4.30. *$(\mathcal{Z}, \eta, \mu)$, as defined above, is a monad on* **Set**.

The monad $(\mathcal{Z}, \eta, \mu)$ is called the *fuzzy-set monad*.

Example 4.31. Consider the two-element (complete) residuated lattice **2** in Example 4.2. For each set X, let $\lambda_X : \mathcal{Z}(X) \to \mathcal{P}(X)$ defined in Example 4.18, and let $\lambda = (\lambda_X)_{X \in \mathbf{Set}}$. Then, λ is a natural isomorphism between $\mathcal{Z}$ and $\mathcal{P}$, i.e. it is a natural transformation from $\mathcal{Z}$ to $\mathcal{P}$ and λ_X is bijective foe each set X.

4.3.1 $\mathcal{Z}$-algebras are $\mathcal{K}$-modules

Recall that a set X and a function $h : \mathcal{Z}(X) \to X$ form a $\mathcal{Z}$-algebra (Definition 2.15) if both diagrams



commute, i.e. $h \circ \mu_X = h \circ \mathcal{Z}(h)$ and $h \circ \eta_X = \mathrm{id}_X$. Thus, for a $\mathcal{Z}$-algebra (X, h), we have

$$h\Big(\bigcup_{\varphi \in \mathcal{Z}(X)} \Phi(\varphi) \circledast \varphi\Big) = h\Big(\bigcup_{\varphi \in \mathcal{Z}(X)} \Phi(\varphi) \circledast \eta_X(h(\varphi))\Big),$$

for all $\Phi \in \mathcal{Z}^2(X)$, and $h(\eta_X(x)) = x$, for each $x \in X$. Moreover, we have the following properties.

Theorem 4.32. *Let (X, h) be a $\mathcal{Z}$-algebra.*

(i) For any family $(\varphi_i)_{i \in I}$ of fuzzy subsets of X,

$$h\Big(\bigcup_{i \in I} \varphi_i\Big) = h\Big(\bigcup_{i \in I} \eta_X(h(\varphi_i))\Big).$$

(ii) For any $a \in K$ and $\varphi \in \mathcal{Z}(X)$,

$$h(a \circledast \varphi) = h(a \circledast \eta_X(h(\varphi))).$$

Moreover, for any $b \in K$,

$$h((a \otimes b) \circledast \varphi) = h(a \circledast \eta_X(h(b \circledast \varphi))).$$

Proof. (i) Let $(\varphi_i)_{i \in I}$ be a family of fuzzy subsets of X. Define $\Phi : \mathcal{Z}(X) \to K$ by

$$\Phi(\psi) = \begin{cases} 1 & \text{if } \psi \in \{\varphi_i \mid i \in I\} \\ 0 & \text{otherwise,} \end{cases} \qquad \psi \in \mathcal{Z}(X).$$

Note that $\Phi \in \mathcal{Z}^2(X)$ and $\Phi = \bigcup_{i \in I} \eta_{\mathcal{Z}(X)}(\varphi_i)$, by Theorem 4.26. Also,

$$\bigcup_{i \in I} \varphi_i = \bigcup_{\psi \in \mathcal{Z}(X)} \Phi(\psi) \circledast \psi = \mu_X(\Phi).$$

Thus, we have

$$h\Big(\bigcup_{i \in I} \varphi_i\Big) = h(\mu_X(\Phi)) = h(\mathcal{Z}(h)(\Phi)) = h\Big(\bigcup_{\psi \in \mathcal{Z}(X)} \Phi(\psi) \circledast \eta_X(h(\psi))\Big) = h\Big(\bigcup_{i \in I} \eta_X(h(\varphi_i))\Big).$$

(ii) Let $a \in K$ and $\varphi \in \mathcal{Z}(X)$. Define $\Psi : \mathcal{Z}(X) \to K$ by

$$\Psi(\psi) = \begin{cases} a & \text{if } \psi = \varphi \\ 0 & \text{otherwise,} \end{cases} \qquad \psi \in \mathcal{Z}(X).$$

Note that

$$a \circledast \varphi = \bigcup_{\psi \in \mathcal{Z}(X)} \Psi(\psi) \circledast \psi = \mu_X(\Psi)$$

and therefore

$$h(a \circledast \varphi) = h(\mu_X(\Psi)) = h(\mathcal{Z}(h)(\Psi)) = h(\bigcup_{\psi \in \mathcal{Z}(X)} \Psi(\psi) \circledast \eta_X(h(\psi))) = h(a \circledast \eta_X(h(\varphi))).$$

Moreover,

$$h((a \otimes b) \circledast \varphi) = h(a \circledast (b \circledast \varphi)) = h(a \circledast \eta_X(h(b \circledast \varphi))),$$

for any $b \in K$. □

Similarly to Example 2.16, now we are able to construct an isomorphism between the category of $\mathcal{Z}$-algebras $\mathcal{EM}(\mathcal{Z})$ and the category of $\mathcal{K}$-modules $\mathcal{K}$-**Mod**.

Theorem 4.33. *(i) Let (X,h) be a $\mathcal{Z}$-algebra. Define a binary operation $\vee$ on X by*

$$x \vee y = h(\eta_X(x) \cup \eta_X(y)), \qquad x,y \in X,$$

and an action $*: K \times X \to X$ by

$$a * x = h(a \circledast \eta_X(x)), \qquad a \in K, x \in X.$$

Then $(X,\vee,*)$ is a $\mathcal{K}$-module.

(ii) If $f:(X,h) \to (X',h')$ is a homorphism of $\mathcal{Z}$-algebras, then f is a linear map between the resulting $\mathcal{K}$-modules by (i).

Proof. (i) First, we check that $(X,\vee)$ is a complete join-semilattice. Since $\mathcal{Z}(X)$ is nonempty and $h:\mathcal{Z}(X) \to X$, then X is nonempty. Let $x,y,z \in X$, observe that

$$x \vee x = h(\eta_X(x) \cup \eta_X(x)) = h(\eta_X(x)) = (h \circ \eta_X)(x) = \mathrm{id}_X(x) = x$$

(since (X,h) is a $\mathcal{Z}$-algebra),

$$x \vee y = h(\eta_X(x) \cup \eta_X(y)) = h(\eta_X(y) \cup \eta_X(x)) = y \vee x,$$

and

$$\begin{aligned}
(x \vee y) \vee z &= h(\eta_X(h(\eta_X(x) \cup \eta_X(y))) \cup \eta_X(z)) \\
&= h(\eta_X(h(\eta_X(x) \cup \eta_X(y))) \cup \eta_X(h(\eta_X(z)))) \\
&= h((\eta_X(x) \cup \eta_X(y)) \cup \eta_X(z)) \\
&= h(\eta_X(x) \cup (\eta_X(y) \cup \eta_X(z))) \\
&= h(\eta_X(h(\eta_X(x))) \cup \eta_X(h(\eta_X(y) \cup \eta_X(z)))) \\
&= h(\eta_X(x) \cup \eta_X(h(\eta_X(y) \cup \eta_X(z)))) = x \vee (y \vee z),
\end{aligned}$$

by Theorem 4.32(i). Thus, $\vee$ is idempotent, commutative and associative. And so $(X,\vee)$ is a join-semilattice, where the partial order $\leq$ is defined by

$$x \leq y \iff x \vee y = y, \qquad x,y \in X.$$

Let Y be a subset of X and consider $s = h(\bigcup_{y\in Y} \eta_X(y))$. For any $y \in Y$, we have

$$\begin{aligned} y \vee s &= h(\eta_X(y) \cup \eta_X(s)) \\ &= h(\eta_X(h(\eta_X(y))) \cup \eta_X(h(\bigcup_{z\in Y} \eta_X(z)))) \\ &= h(\eta_X(y) \cup (\bigcup_{z\in Y} \eta_X(z))) \\ &= h(\bigcup_{z\in Y} \eta_X(z)) = s, \end{aligned}$$

by Theorem 4.32(i), which implies $y \leq s$. If $u \in X$ is such that $y \leq u$, for all $y \in Y$, then

$$\begin{aligned} s \vee u &= h(\eta_X(s) \cup \eta_X(u)) \\ &= h(\eta_X(h(\bigcup_{y\in Y} \eta_X(y))) \cup \eta_X(h(\eta_X(u)))) \\ &= h((\bigcup_{y\in Y} \eta_X(y)) \cup \eta_X(u)) \\ &= h(\bigcup_{y\in Y} (\eta_X(y) \cup \eta_X(u))) \\ &= h(\bigcup_{y\in Y} \eta_X(h(\eta_X(y) \cup \eta_X(u)))) \\ &= h(\bigcup_{y\in Y} \eta_X(y \vee u)) \\ &= h(\bigcup_{y\in Y} \eta_X(u)) \\ &= h(\eta_X(u)) = u \end{aligned}$$

and so $s \leq u$. Hence, s is the supremum of Y. Thus,$(X,\vee)$ is a complete join-semilattice, where $\bigvee Y = h(\bigcup_{y\in Y} \eta_X(y))$ for any $Y \subseteq X$.

Second, we check that $(X,\vee,*)$ is a $\mathcal{K}$-module (Definition 4.8(i)). Let $a,b \in K$, $x \in X$, then

$$1 * x = h(1 \circledast \eta_X(x)) = h(\eta_X(x)) = x$$

and, by Theorem 4.32(ii),

$$a * (b * x) = h(a \circledast \eta_X(h(b \circledast \eta_X(x)))) = h((a \otimes b) \circledast \eta_X(x)) = (a \otimes b) * x.$$

Also, for a family $(a_i)_{i\in I}$ of elements of K,

$$\begin{aligned}(\bigvee_{i\in I} a_i) * x &= h((\bigvee_{i\in I} a_i) \circledast \eta_X(x))\\ &= h(\bigcup_{i\in I} a_i \circledast \eta_X(x))\\ &= h(\bigcup_{i\in I} \eta_X(h(a_i \circledast \eta_X(x))))\\ &= h(\bigcup_{i\in I} \eta_X(a_i * x)) = \bigvee_{i\in I} a_i * x\end{aligned}$$

(by Theorem 4.32(i)) and, for a family $(x_i)_{i\in I}$ of elements of X,

$$\begin{aligned}a * (\bigvee_{i\in I} x_i) &= h(a \circledast \eta_X(h(\bigcup_{i\in I} \eta_X(x_i))))\\ &= h(a \circledast (\bigcup_{i\in I} \eta_X(x_i)))\\ &= h(\bigcup_{i\in I} a \circledast \eta_X(x_i))\\ &= h(\bigcup_{i\in I} \eta_X(h(a \circledast \eta_X(x_i))))\\ &= h(\bigcup_{i\in I} \eta_X(a * x_i)) = \bigvee_{i\in I} a * x_i\end{aligned}$$

(by Theorem 4.32). Therefore, $(X, \vee, *)$ is indeed a $\mathcal{K}$-module.

(ii) Let $f : (X,h) \to (X',h')$ be a homomorphism of $\mathcal{Z}$-algebras (Definition 2.15(ii)), and let $(X,\vee,*)$ and $(X',\vee',*')$ be the $\mathcal{K}$-modules obtained from (X,h) and (X',h') by (i), respectively. Then

$$\begin{aligned}f(\bigvee_{i\in I} x_i) &= f(h(\bigcup_{i\in I} \eta_X(x_i)))\\ &= h'(\mathcal{Z}(f)(\bigcup_{i\in I} \eta_X(x_i)))\\ &= h'(\bigcup_{x\in X}(\bigcup_{i\in I} \eta_X(x_i))(x) \circledast \eta_{X'}(f(x)))\\ &= h'(\bigcup_{i\in I}\bigcup_{x\in X} \eta_X(x_i)(x) \circledast \eta_{X'}(f(x)))\\ &= h'(\bigcup_{i\in I} \eta_{X'}(f(x_i))) = \bigvee_{i\in I}{}' f(x_i),\end{aligned}$$

for any family $(x_i)_{i\in I}$ of elements of X, and

$$\begin{aligned}f(a * x) &= f(h(a \circledast \eta_X(x)))\\ &= h'(\mathcal{Z}(f)(a \circledast \eta_X(x)))\\ &= h'(\bigcup_{y\in X}(a \circledast \eta_X(x))(y) \circledast \eta_{X'}(f(y)))\\ &= h'(a \circledast \bigcup_{y\in X} \eta_X(x)(y) \circledast \eta_{X'}(f(y)))\\ &= h'(a \circledast \eta_{X'}(f(x))) = a *' f(x),\end{aligned}$$

for any $a \in K$ and $x \in X$. Hence, f is a linear map from $(X, \vee, *)$ to $(X', \vee', *')$. $\square$

From the previous construction, for a $\mathcal{Z}$-algebra (X, h), we have

$$h(\bigcup_{i\in I} \varphi_i) = h(\bigcup_{i\in I} \eta_X(h(\varphi_i))) = \bigvee_{i\in I} h(\varphi_i),$$

for any family $(\varphi_i)_{i\in I}$ of fuzzy subsets of X (by Theorem 4.32(i)), and

$$h(a \circledast \varphi) = h(a \circledast \eta_X(h(\varphi)) = a * h(\varphi),$$

for any $a \in K$ and $\varphi \in \mathcal{Z}(X)$ (by Theorem 4.32(ii)). Thus, h is a linear map from $(\mathcal{Z}(X), \cup, \circledast)$ to $(X, \vee, *)$. Moreover, $h(\eta_X(x)) = x$, for all $x \in X$, and so $h = \overline{\mathrm{id}_X}$ by Theorem 4.26, i.e. h is the linear extension of the identity function on X (Definition 4.27).

Theorem 4.34. *(i) Let $\mathcal{M}$ be a $\mathcal{K}$-module, and let $h = \overline{\mathrm{id}_M} : \mathcal{Z}(M) \to M$ be the linear extension of the identity function on M, i.e.*

$$h(\varphi) = \bigvee_{m\in M} \varphi(m) * m, \qquad \varphi \in \mathcal{Z}(M).$$

Then (M, h) is a $\mathcal{Z}$-algebra.

(ii) If $f : \mathcal{M} \to \mathcal{M}'$ is a linear map of $\mathcal{K}$-modules, then f is a homomorphism of the resulting $\mathcal{Z}$-algebras by (i).

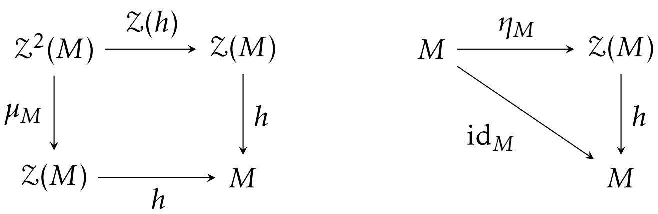
Proof. (i) Note that h is a linear map from $(\mathcal{Z}(X), \cup, \circledast)$ to $\mathcal{M}$ and $h(\eta_M(m)) = m$, for all $m \in M$. Thus, $h \circ \eta_M = \mathrm{id}_M$. Also, for any $\Phi \in \mathcal{Z}^2(M)$, we have

$$h(\mu_M(\Phi)) = h(\bigcup_{\varphi\in\mathcal{Z}(M)} \Phi(\varphi) \circledast \varphi) = \bigvee_{\varphi\in\mathcal{Z}(M)} h(\Phi(\varphi) \circledast \varphi) = \bigvee_{\varphi\in\mathcal{Z}(M)} \Phi(\varphi) * h(\varphi)$$

and

$$h(\mathcal{Z}(h)(\Phi)) = h(\bigcup_{\varphi\in\mathcal{Z}(M)} \Phi(\varphi) \circledast \eta_M(h(\varphi))) = \bigvee_{\varphi\in\mathcal{Z}(M)} \Phi(\varphi) * h(\eta_M(h(\varphi))) = \bigvee_{\varphi\in\mathcal{Z}(M)} \Phi(\varphi) * h(\varphi).$$

Hence, $h \circ \mu_M = h \circ \mathcal{Z}(h)$. Therefore, both diagrams



commute, and so (M, h) is a $\mathcal{Z}$-algebra.

(ii) Let $f : \mathcal{M} \to \mathcal{M}'$ be a linear map of $\mathcal{K}$-modules, and let (M,h) and (M',h') be the $\mathcal{Z}$-algebras obtained from $\mathcal{M}$ and $\mathcal{M}'$ by (i), respectively. For any $\varphi \in \mathcal{Z}(M)$, we have

$$\begin{aligned} h'(\mathcal{Z}(f)(\varphi)) &= h'(\bigcup_{m\in M} \varphi(m) \circledast \eta_{M'}(f(m))) \\ &= \bigvee_{m\in M}{}' \varphi(m) *' h'(\eta_{M'}(f(m))) \\ &= \bigvee_{m\in M}{}' \varphi(m) *' f(m) \\ &= f(\bigvee_{m\in M} \varphi(m) * m) = f(h(\varphi)), \end{aligned}$$

since $h = \overline{\mathrm{id}_M}$ and $h' = \overline{\mathrm{id}_{M'}}$. Hence, $h' \circ \mathcal{Z}(f) = f \circ h$, and so the diagram

$$\begin{CD} \mathcal{Z}(M) @>{\mathcal{Z}(f)}>> \mathcal{Z}(M') \\ @V{h}VV @VV{h'}V \\ M @>>{f}> M \end{CD}$$

commutes. Therefore, f is a homomorphism of $\mathcal{Z}$-algebras from (M,h) to (M',h'). $\square$

From the previous construction, for a $\mathcal{K}$-module $\mathcal{M}$, we have

$$h(\eta_M(x) \cup \eta_M(y)) = h(\eta_M(x)) \vee h(\eta_M(y)) = x \vee y$$

and

$$h(a \circledast \eta_M(x)) = a * h(\eta_X(x)) = a * x,$$

for any $x, y \in M$ and $a \in K$ (recall that $h = \overline{\mathrm{id}_M}$).

Observe that, if we start with a $\mathcal{Z}$-algebra, construct a $\mathcal{K}$-module by Theorem 4.33(i), and then obtain a $\mathcal{Z}$-algebra by Theorem 4.34(i), we end up with the same $\mathcal{Z}$-algebra that we started with. On the other hand, if we start with a $\mathcal{K}$-module, obtain a $\mathcal{Z}$-algebra by Theorem 4.34(i), and then construct a $\mathcal{K}$-module by Theorem 4.33(i), we end up with the same $\mathcal{K}$-module that we started with. Clearly, we have a similar interaction between Theorem 4.33(ii) and Theorem 4.34(ii) for homomorphisms. Therefore, Theorem 4.33 induces a functor from $\mathcal{EM}(\mathcal{Z})$ to $\mathcal{K}$-**Mod** (see Definitions 2.15(iii) and 4.8(iii)), and Theorem 4.34 induces its inverse.

Corollary 4.35. *The Eilenberg-Moore category of $\mathcal{Z}$ is isomorphic to the category of $\mathcal{K}$-modules and linear maps, i.e. $\mathcal{EM}(\mathcal{Z}) \cong \mathcal{K}$-**Mod**.*

Given a set X, recall that we defined $\mu_X = \overline{\mathrm{id}_{\mathcal{Z}(X)}}$ (see Corollary 4.30). Then $(\mathcal{Z}(X), \mu_X)$, the free $\mathcal{Z}$-algebra over X, is obtained from $(\mathcal{Z}(X), \cup, \circledast)$, the $\mathcal{K}$-modules of fuzzy subsets of X (Definition 4.24), by Theorem 4.33(i).

Finally, let (Y,h) be a $\mathcal{Z}$-algebra and $f : X \to Y$ a function. Consider the $\mathcal{K}$-module $(Y,\vee,*)$ constructed from (Y,h) by Theorem 4.33(i), and let $\overline{f} : X \to Y$ be the linear extension of f (Definition 4.24), which is a linear map from $(\mathcal{Z}(X),\cup,\circledast)$ to $(Y,\vee,*)$. Note that the $\mathcal{Z}$-algebras $(\mathcal{Z}(X),\mu_X)$ and (Y,h) are obtained from $(\mathcal{Z}(X),\cup,\circledast)$ and $(Y,\vee,*)$, respectively, by Theorem 4.34(i). Thus, $\overline{f}$ is a homomorphism of $\mathcal{Z}$-algebras from $(\mathcal{Z}(X),\mu_X)$ to (Y,h), by Theorem 4.34(ii). Since $f = \overline{f} \circ \eta_X$, then $\overline{f}$ is also the function extension for $\mathcal{Z}$-algebras as described in Section 2.3.1 (denoted by $\overline{f}_h$ there), by Theorem 2.17. Therefore, given the isomorphisms induced by Theorems 4.33 and 4.34, we have that the function extension for $\mathcal{Z}$-algebras corresponds to the linear extension for $\mathcal{K}$-modules.

Chapter 5

Fuzzy automata

In this chapter, we present a coalgebraic approach to fuzzy automata. In Section 5.1, we show how the classical notions of fuzzy automata and crisp-deterministic fuzzy automata can be modelled by coalgebras. Although we introduce the classical notions, we refer to [Ćir+12; Ign+08; Ign+10; JĆ14; Jan+14; Wec78] for more details. In Section 5.2, we introduce the notions of fuzzy languages and fuzzy languages recognized by fuzzy automata, see also [BLB06; BLB08]. Then, in Section 5.3, we show a coalgebraic description of the fuzzy language recognized by a crisp-deterministic fuzzy automata, by proving that the functor of crisp-deterministic automata has a final coalgebra. In Section 5.4, we show that the functor of fuzzy automata satisfies the properties for determinization, studied in Section 3.3, leading to a generalization of the powerset construction. Finally, in Section 5.4.1, we give a coalgebraic description of the fuzzy language recognized by a fuzzy automaton via determinization process. Finally, in Section 5.5, we describe bisimultion for the functor of fuzzy automata leading to the definition of a quotient coalgebras for this functor.

5.1 Coalgebras for fuzzy automata

In this section we introduce the notion of a fuzzy automaton, based on [Jan+14]. Then we show how this concept can be modelled from a coalgebraic point of view, leading to a framework where some well-known results on fuzzy automata arise from a much general theory.

Recall that $\mathcal{K} = (K, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is a complete residuated lattice (Definition 4.1), with at least two elements, and $(\mathcal{Z}, \eta, \mu)$ is the fuzzy-set monad (Corollary 4.30). Also, from now on let A be an arbitrary nonempty set, the *input alphabet*, whose elements are called *(input) letters*.

Definition 5.1. A *fuzzy automaton* (with input alphabet A and membership values over $\mathcal{K}$) is a quadruple $\mathcal{A} = (X, \sigma, \tau, \delta)$ consisting of a set X (called the *set of states*), two fuzzy subsets $\sigma : X \to K$ (called the *fuzzy set of initial states*) and $\tau : X \to K$ (called the *fuzzy set of final states*) of X, and a fuzzy relation $\delta : X \times A \times X \to K$ (called the *fuzzy transition relation*) between X, A and X.

Let $\mathcal{A} = (X, \sigma, \tau, \delta)$ be a fuzzy automaton. For a state $x \in X$, $\sigma(x)$ and $\tau(x)$ can be seen as the truth degree of "x is an initial state" (or input state) and "x is a final state" (or terminal state), respectively. For states $x, y \in X$ and a letter $a \in A$, $\delta(x, a, y)$ can be interpreted as the truth degree of "the input a causes a transition from x to y".

There are bijective correspondences

$$K^{X \times A \times X} \cong ((K^X)^A)^X \cong ((K^X)^X)^A \cong (K^{X \times X})^A$$

via currying and uncurrying (see Section 2.1.1). Thus, the fuzzy transition relation $\delta : X \times A \times X \to K$ may assume different forms. For the sake of simplicity, we still use δ to denote these other forms, despite of being formally different functions, letting the context decide which one it represents. Considering $\delta : X \to (K^X)^A$, called the *fuzzy transition function* (usually, this name is used for $\delta : X \times A \to K^X$, but in the coalgebraic approach our terminology will be more appropriate), δ assigns to each state $x \in X$ its fuzzy transitions function $\delta(x) : A \to K^X$ which maps an input letter $a \in A$ to the fuzzy subset of next states $\delta(x)(a)$. When $\delta : A \to (K^X)^X$, it assigns the *fuzzy state transition function* for each input letter $a \in A$, denoted by $\delta_a : X \to K^X$. Similarly, $\delta : A \to K^{X \times X}$ associates to each input letter $a \in A$ its induced *fuzzy state transition relation* $\delta_a : X \times X \to K$. In the end, regardless of the form we began with, we have

$$\delta(x, a, y) = \delta(x)(a)(y) = \delta_a(x)(y) = \delta_a(x, y).$$

Let A^* denote (the carrier of) the free monoid over A, where $\varepsilon \in A^*$ is the empty word and, for two words $u, v \in A^*$, $uv \in A^*$ is their concatenation. The fuzzy transition relation $\delta : X \times A \times X \to K$ is inductively extended up to $\delta^* : X \times A^* \times X \to K$ as follows: for any $x, y \in X$,

$$\delta^*(x, \varepsilon, y) = \eta_X(x)(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

and, for any $x, y \in X$, $w \in A^*$ and $a \in A$,

$$\delta^*(x, aw, y) = \bigvee_{z \in X} \delta(x, a, z) \otimes \delta^*(z, w, y).$$

Similarly, we can extend any other form of δ, and so we have

$$\delta^*(x, w, y) = \delta^*(x)(w)(y) = \delta^*_w(x)(y) = \delta^*_w(x, y),$$

for every $x, y \in X$ and $w \in A^*$. Considering $\delta^* : X \times A^* \times X \to K$, since $\otimes$ is distributive over $\bigvee$, by Theorem 4.6(i), we have that

$$\delta^*(x, a_1 a_2 \dots a_n, y) = \bigvee_{\substack{z_i \in X \\ 1 \le i < n}} \delta(x, a_1, z_1) \otimes \delta(z_1, a_2, z_2) \otimes \cdots \otimes \delta(z_{n-1}, a_n, y),$$

and considering $\delta^* : A^* \to K^{X \times X}$, we have that

$$\delta^*_{a_1 a_2 \dots a_n} = \delta_{a_1} \diamond \delta_{a_2} \diamond \cdots \diamond \delta_{a_n}$$

(Definition 4.22). for any $a_1, a_2, \dots, a_n \in A$ where $n \ge 2$.

For states $x, y \in X$ and a word $w \in A^*$, $\delta^*(x, w, y)$ can be interpreted as the truth degree of "the input word w causes a transition from x to y" or "the word labels a path from x to y".

Observe that, apart from the fuzzy set of initial states, a fuzzy automaton is described by a pair $(X, \langle \tau, \delta \rangle)$ consisting of a set X and a function $\langle \tau, \delta \rangle : X \to K \times (K^X)^A$.

Definition 5.2. The *fuzzy automata functor* $\mathcal{F} = K \times (\mathcal{Z}(-))^A$ is a functor $\mathcal{F} : \mathbf{Set} \to \mathbf{Set}$ on the category **Set** mapping each set X to

$$\mathcal{F}(X) = K \times (\mathcal{Z}(X))^A = K \times (K^X)^A,$$

and mapping each function $f : X \to Y$ to

$$\mathcal{F}(f) = \mathrm{id}_K \times (\mathcal{Z}(f))^A : K \times (K^X)^A \to K \times (K^Y)^A$$

defined by $\mathcal{F}(f)(k, \varphi) = (k, \mathcal{Z}(f) \circ \varphi)$ for each $(k, \varphi) \in K \times (K^X)^A$.

The functor $\mathcal{F}$ will be used as the type for a coalgebraic approach to fuzzy automata, such that $\mathcal{F}$-coalgebras shall bring a new perspective to their classical theory. For instance, we have the notion of homomorphism. By Definition 3.1(ii), a homomorphism of $\mathcal{F}$-coalgebras $f : (X, \langle \tau, \delta \rangle) \to (X', \langle \tau', \delta' \rangle)$ is a function $f : X \to X'$ which makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & X' \\ {\scriptstyle \langle \tau, \delta \rangle}\big\downarrow & & \big\downarrow{\scriptstyle \langle \tau', \delta' \rangle} \\ K \times (K^X)^A & \xrightarrow[\mathrm{id}_K \times (\mathcal{Z}(f))^A]{} & K \times (K^{X'})^A \end{array}$$

commute, that is $\langle \tau' \circ f, \delta' \circ f \rangle = \langle \tau, (\mathcal{Z}(f))^A \circ \delta \rangle$. Thus, a homomorphism $f : (X, \langle \tau, \delta \rangle) \to (X', \langle \tau', \delta' \rangle)$ is such that

$$\tau'(f(x)) = \tau(x)$$

(from $\tau' \circ f = \tau$), for all $x \in X$, and

$$\delta'(f(x))(a) = \mathcal{Z}(f)(\delta(x)(a)) = \bigcup_{y \in X} \delta(x)(a)(y) \circledast \eta_{X'}(f(y))$$

(from $\delta' \circ f = (\mathcal{Z}(f))^A \circ \delta$), for every $x \in X$ and $a \in A$, which implies that

$$\delta'(f(x))(a)(y') = \bigvee_{\substack{y \in X \\ f(y)=y'}} \delta(x)(a)(y) = \bigvee_{y \in f^{-1}(y')} \delta(x)(a)(y)$$

(where $f^{-1}(y') = \{y \in X \mid f(y) = y'\}$), for every $x \in X$, $a \in A$ and $y' \in X'$.

Example 5.3. Consider $\mathcal{K}$ to be the two-element (complete) residuated lattice **2** described in Example 4.2. Let $(X, \langle \tau, \delta \rangle : 2 \times (2^X)^A)$ be an $\mathcal{F}$-coalgebra, and let $\lambda_X : \mathcal{Z}(X) \to \mathcal{P}(X)$ defined in Example 4.18. Define

$$\langle o, d \rangle = (\mathrm{id}_2 \times (\lambda_X)^A) \circ \langle \tau, \delta \rangle = \langle \tau, (\lambda_X)^A \circ \delta \rangle : X \to 2 \times (\mathcal{P}(X))^A,$$

and so $(X, \langle o, d \rangle)$ is a nondeterministic automaton as it is an N-coalgebra (Example 3.7(ii)). For $x, y \in X$ and $a \in A$, we have that $x \xrightarrow{a} y$ in $(X, \langle o, d \rangle)$ if, and only if, $\delta(x)(a)(y) = 1$.

Moreover, $(\mathrm{id}_2 \times \lambda_X)_{X \in \mathbf{Set}}$ is a natural bijection between $\mathcal{F}$ and N, and therefore fuzzy automata with membership values in **2** are basically nondeterministic automata.

5.1.1 Crisp-deterministic fuzzy automata

Definition 5.4. A *crisp-deterministic fuzzy automaton* is a quadruple $\mathcal{A} = (X, x_0, \tau, d)$ consisting of a set X (called the *set of states*), an element $x_0 \in X$ (called the *initial state*), a fuzzy subset $\tau : X \to K$ of X (called the *fuzzy set of final states*), and a function $d : X \to X^A$ (called the *transition function*).

In the literature [Ćir+12; JĆ14], usually the transition function of a crisp-deterministic fuzzy automaton $\mathcal{A} = (X, x_0, \tau, d)$ is considered $d : X \times A \to X$, which makes no conflict with our definition, since there is a bijection between $X^{X \times A}$ and $(X^A)^X$ via currying.

Given a crisp-deterministic fuzzy automaton $\mathcal{A} = (X, x_0, \tau, d)$, define $\delta : (\eta_X)^A \circ d : X \to (K^X)^A$, that is

$$\delta(x)(a)(y) = \begin{cases} 1 & \text{if } y = d(x)(a) \\ 0 & \text{otherwise} \end{cases}$$

for every $x, y \in X$ and $a \in A$. Then, $(X, \eta_X(x_0), \tau, \delta)$ is a fuzzy automaton, where $\eta_X(x_0)$ is a crisp-deterministic subset of X (Definition 4.15(ii)), and considering $\delta : X \times A \times X \to K$ (via uncurrying), δ is a crisp-deterministic relation (Definition 4.20(ii)).

On the other hand, for a fuzzy automaton $\mathcal{A} = (X, \sigma, \tau, \delta)$ where σ and δ are both crisp-deterministic (in [Jan+14] this is given as the definition of crisp-deterministic fuzzy automaton), let $x_0 \in X$ be such that $\sigma(x_0) = 1$, and let $d : X \to X^A$ map each $x \in X$ and each $a \in A$ to the unique element $d(x)(a) \in X$ such that $\delta(x, a, d(x)(a))$. Then, (X, x_0, τ, d) is a crisp-deterministic fuzzy automaton.

In this case, $\eta_X(x_0)$ is called a *crisp-deterministic fuzzy set* and δ is called a *crisp-deterministic fuzzy transition relation* (or *crisp-deterministic fuzzy transition function*, when considering $\delta : X \to (K^X)^A$). Equivalently, we can obtain a crisp-deterministic fuzzy automaton from a fuzzy automaton where the fuzzy set of initial states and the fuzzy transition relation are both crisp-deterministic [JĆ14],

Let $\mathcal{A} = (X, x_0, \tau, d)$ be a crisp-deterministic automaton, the transition function $d : X \to X^A$ can be extended up to $d^* : X \to X^{A^*}$ inductively defined by

$$d^*(x)(\varepsilon) = x$$

for any $x \in X$, and

$$d^*(x)(aw) = d^*(d(x)(a))(w)$$

for any $x \in X$, $w \in A^*$ and $a \in A$.

Note that a crisp-deterministic fuzzy automaton is a type of Moore automaton, where the output set is the underlying set of a complete residuated lattice. Thus, apart from the initial state, crisp-deterministic fuzzy automata are coalgebras of the following functor.

Definition 5.5. The *crisp-deterministic fuzzy automata functor* $\mathcal{D}_K = K \times (-)^A$ is a functor $\mathcal{D}_K : \mathbf{Set} \to \mathbf{Set}$ on the category **Set** mapping each set X to

$$\mathcal{D}_K(X) = K \times X^A,$$

and mapping each function $f : X \to Y$ to

$$\mathcal{D}_K(f) = \mathrm{id}_K \times f^A : K \times X^A \to K \times Y^A$$

defined by $\mathcal{D}_K(f)(k,h) = (k, f \circ h)$ for each $(k,h) \in K \times X^A$.

We remark that, for the two-element complete residuated lattice $\mathbf{2}$ (Example 4.2), the crisp-deterministic fuzzy automata functor $\mathcal{D}_2$ is exactly the functor for deterministic automata defined in Example 3.7(ii).

For each set X, define $\xi_X = \mathcal{D}_K(\eta_X) : \mathcal{D}_K(X) \to \mathcal{D}_K(\mathcal{Z}(X))$. Let $\xi = (\xi_X)_{X \in \mathbf{Set}}$. Recall that η is the unit of the fuzzy-set monad, in particular η is a natural transformation from $\mathrm{Id}_{\mathbf{Set}}$ to $\mathcal{Z}$, and so, for any function $f : X \to Y$, the diagram

$$\begin{array}{ccc}
\mathcal{D}_K(X) & \xrightarrow{\mathcal{D}_K(f)} & \mathcal{D}_K(Y) \\
{\scriptstyle \xi_X = \mathcal{D}_K(\eta_X)}\big\downarrow & & \big\downarrow{\scriptstyle \xi = \mathcal{D}_K(\eta_Y)} \\
\mathcal{D}_K(\mathcal{Z}(X)) & \xrightarrow[\mathcal{D}_K(\mathcal{Z}(f))]{} & \mathcal{D}_K(\mathcal{Z}(Y))
\end{array}$$

commutes. Hence, ξ is a natural transformation from $\mathcal{D}_K$ to $\mathcal{D}_K\mathcal{Z}$. Note that $\mathcal{F} = \mathcal{D}_K\mathcal{Z}$. Therefore, by Theorem 3.3, ξ induced a functor from $\mathrm{CoAlg}(\mathcal{D}_K)$ to $\mathrm{CoAlg}(\mathcal{F})$, which maps a $\mathcal{D}_K$-coalgebra $(X, \langle \tau, d \rangle)$ to the $\mathcal{F}$-coalgebra $(X, \langle \tau, (\eta_X)^A \circ d \rangle)$ and maps a homomorphism to itself. This functor reflects the fact that a crisp-deterministic fuzzy automaton induces a special kind of fuzzy automaton, where the transition relation is crisp-deterministic, as discussed above.

5.2 Fuzzy languages

In this section, we introduce the notion of fuzzy language and define the (left) derivative operation. We also define the classical notion of the fuzzy language recognized by a (crisp-deterministic) fuzzy automaton. This notion will be obtained from a coalgebraic point of view in the next two sections.

Definition 5.6. A *fuzzy language* over an alphabet A (and with membership values over $\mathcal{K}$), or simply a *fuzzy language*, is a fuzzy subset of A^*, that is a function $\lambda : A^* \to K$.

The set of all fuzzy languages over A is $\mathcal{Z}(A^*) = K^{A^*}$ and so all operations on fuzzy sets in Definition 4.17 are appliable on fuzzy languages. Also, we have that A^* acts on fuzzy languages by the following operation.

Definition 5.7. Let $\lambda : A^* \to K$ be a fuzzy language and $u \in A^*$ be a word. The *(left) derivative of λ with respect to u* is a fuzzy language $u^{-1}\lambda : A^* \to K$ defined by

$$(u^{-1}\lambda)(v) = \lambda(uv), \qquad v \in A^*.$$

Given a fuzzy language $\lambda : A^* \to K$ and two words $u, v \in A^*$, note that

$$(v^{-1}(u^{-1}\lambda))(w) = (u^{-1}\lambda)(vw) = \lambda(uvw) = ((uv)^{-1}\lambda)(w)$$

for any $w \in A^*$. Hence, $v^{-1}(u^{-1}\lambda) = (uv)^{-1}\lambda$.

Fuzzy languages relates with fuzzy automata as follows.

Definition 5.8. (i) Given a fuzzy automaton $\mathcal{A} = (X, \sigma, \tau, \delta)$, the *fuzzy language recognized by $\mathcal{A}$*, denoted by $\mathcal{L}(\mathcal{A}) : A^* \to K$, is defined by

$$\mathcal{L}(\mathcal{A})(w) = \bigvee_{x,y \in X} \sigma(x) \otimes \delta^*(x, w, y) \otimes \tau(y), \qquad w \in A^*.$$

(ii) Given a crisp-deterministic fuzzy automaton $\mathcal{A} = (X, x_0, \tau, d)$, the *fuzzy language recognized by $\mathcal{A}$*, denoted by $\mathcal{L}(\mathcal{A}) : A^* \to K$, is defined by

$$\mathcal{L}(\mathcal{A})(w) = \tau(d^*(x_0)(w)), \qquad w \in A^*,$$

that is $\mathcal{L}(\mathcal{A}) = \tau \circ d(x_0)$.

(iii) A (crisp-deterministic) fuzzy automaton $\mathcal{A}$ is *(language) equivalent* to a (crisp-deterministic) fuzzy automaton $\mathcal{A}'$ if $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.

For any fuzzy automaton $\mathcal{A} = (X, \sigma, \tau, \delta)$, letting $\delta^* : A^* \to K^{X \times X}$, we have that

$$\mathcal{L}(\mathcal{A})(w) = \sigma \diamond \delta^*_w \diamond \tau, \qquad w \in A^*.$$

Given a (crisp-deterministic) fuzzy automaton $\mathcal{A}$ and a word $w \in A^*$, $\mathcal{L}(\mathcal{A})(w)$ can be seen as the truth degree of "the word w causes a transition from an initial state to a final state in $\mathcal{A}$". Thus, $\mathcal{L}(\mathcal{A})(w)$ is called the *degree of recognition of w by $\mathcal{A}$*.

5.3 A final coalgebra for crisp-deterministic fuzzy automata

In this section we show that the coalgebra of fuzzy languages, described below, is a final coalgebra (Definition 3.8) for the crisp-deterministic fuzzy automata functor $\mathcal{D}_K$ (Definition 5.5).

Definition 5.9. The *coalgebra of fuzzy languages* is a $\mathcal{D}_K$-coalgebra $(K^{A^*}, \langle \epsilon, t\rangle)$ where $\epsilon : K^{A^*} \to K$ is defined by

$$\epsilon(\lambda) = \lambda(\varepsilon), \qquad \lambda \in K^{A^*},$$

and $t : K^{A^*} \to (K^{A^*})^A$ is defined by

$$t(\lambda)(a) = a^{-1}\lambda, \qquad a \in A, \lambda \in K^{A^*}.$$

In the coalgebra of fuzzy languages, we show by induction, on the length of words $w \in A^*$, that $t^* : K^{A^*} \to (K^{A^*})^{A^*}$ is given by

$$t^*(\lambda)(w) = w^{-1}\lambda,$$

for every fuzzy language $\lambda \in K^{A^*}$. First, we have that $t^*(\lambda)(\varepsilon) = \varepsilon^{-1}\lambda = \lambda$ for any $\lambda \in K^{A^*}$. Assume $t^*(\lambda)(w) = w^{-1}\lambda$, for all $\lambda \in K^{A^*}$, as induction hypothesis (IH). Thus,

$$t^*(\lambda)(aw) = t^*(t(\lambda)(a))(w) = t^*(a^{-1}\lambda)(w) \overset{\text{(IH)}}{=} w^{-1}(a^{-1}\lambda) = (aw)^{-1}\lambda$$

for any $\lambda \in K^{A^*}$.

Observe that, given a fuzzy language $\lambda \in K^{A^*}$, the fuzzy language $\epsilon \circ t^*(\lambda) : A^* \to K$ is such that

$$(\epsilon \circ t^*(\lambda))(w) = \epsilon(t^*(\lambda)(w)) = \epsilon(w^{-1}\lambda) = (w^{-1}\lambda)(\varepsilon) = \lambda(w\varepsilon) = \lambda(w),$$

for every word $w \in A^*$. Hence, $\lambda = \epsilon \circ t^*(\lambda)$ is the fuzzy language recognized by the crisp-deterministic fuzzy automaton $\mathcal{A} = (K^{A^*}, \lambda, \epsilon, t)$. This is a particular case of the following result.

Theorem 5.10. *The coalgebra of fuzzy languages $(K^{A^*}, \langle \epsilon, t\rangle)$ is a final $\mathcal{D}_K$-coalgebra. Moreover, given a $\mathcal{D}_K$-coalgebra $(X, \langle \tau, d\rangle : X \to K \times X^A)$, the function $\ell : X \to K^{A^*}$ defined by*

$$\ell(x) = \tau \circ d^*(x) : A^* \to K, \qquad x \in X,$$

is the unique homomorphism of $\mathcal{D}_K$-coalgebras from $(X, \langle \tau, d\rangle)$ to $(K^{A^*}, \langle \epsilon, t\rangle)$, that is the unique function which makes the following diagram commute

$$\begin{array}{ccc} X & \overset{\ell}{\dashrightarrow} & K^{A^*} \\ {\scriptstyle\langle \tau, d\rangle}\big\downarrow & & \big\downarrow{\scriptstyle\langle \epsilon, t\rangle} \\ K \times X^A & \underset{\mathrm{id}_K \times \ell^A}{\dashrightarrow} & K \times (K^{A^*})^A. \end{array}$$

In particular, for any crisp-deterministic fuzzy automaton $\mathcal{A} = (X, x_0, \tau, d)$*, the unique homomorphism of* $\mathcal{D}_K$*-coalgebras* $\ell : (X, \langle \tau, d\rangle) \to (K^{A^*}, \langle \epsilon, t\rangle)$ *is such that* $\mathcal{L}(\mathcal{A}) = \ell(x_0)$*.*

Proof. Let $(X, \langle \tau, d\rangle : X \to K \times X^A)$ be any $\mathcal{D}_K$-coalgebra. Define $\ell : X \to K^{A^*}$ by $\ell(x) = \tau \circ d^*(x)$, for each $x \in X$. We have that

$$(\epsilon \circ \ell)(x) = \epsilon(\ell(x)) = \ell(x)(\varepsilon) = \tau(d^*(x)(\varepsilon)) = \tau(x),$$

for any $x \in X$, and so $\epsilon \circ \ell = \mathrm{id}_K \circ \tau$. Since

$$d^*(x)(aw) = d^*(d(x)(a))(w)$$

by definition of d^*, then

$$\begin{aligned}
(t \circ \ell)(x)(a)(w) &= t(\ell(x))(a)(w) \\
&= (a^{-1}(\ell(x)))(w) \\
&= \ell(x)(aw) && \text{by Definition 5.7} \\
&= \tau(d^*(x)(aw)) \\
&= \tau(d^*(d(x)(a))(w)) \\
&= \ell(d(x)(a))(w) \\
&= (\ell \circ d(x))(a)(w) \\
&= (\ell^A \circ d)(x)(a)(w),
\end{aligned}$$

for any $x \in X$, $a \in A$ and $w \in A^*$, and hence $t \circ \ell = \ell^A \circ d$. Therefore, the diagram

$$\begin{array}{ccc}
X & \xrightarrow{\ \ell\ } & K^{A^*} \\
{\scriptstyle \langle \tau, d\rangle}\downarrow & & \downarrow{\scriptstyle \langle \epsilon, t\rangle} \\
K \times X^A & \xrightarrow[\mathrm{id}_K \times \ell^A]{} & K \times (K^{A^*})^A
\end{array}$$

commutes, which implies that ℓ is a homomorphism of $\mathcal{D}_K$-coalgebras from $(X, \langle \tau, d\rangle)$ to $(K^{A^*}, \langle \epsilon, t\rangle)$.

Let ℓ' be a homomorphism of $\mathcal{D}_K$-coalgebras from $(X, \langle \tau, d\rangle)$ to $(K^{A^*}, \langle \epsilon, t\rangle)$. We prove by induction, on the length of words $w \in A^*$, that $\ell'(x)(w) = \ell(x)(w)$ for every $x \in X$. Since

$$\epsilon \circ \ell' = \mathrm{id}_K \circ \tau = \epsilon \circ \ell,$$

then

$$\ell'(x)(\varepsilon) = \epsilon(\ell'(x)) = \epsilon(\ell(x)) = \ell(x)(\varepsilon)$$

for any $x \in X$. Assume $\ell'(x)(w) = \ell(x)(w)$, for all $x \in X$, as induction hypothesis (IH). Thus,

$$\begin{aligned}
\ell'(x)(aw) &= t(\ell'(x))(a)(w) \\
&= \ell'(d(x)(a))(w) && \text{by } t \circ \ell' = (\ell')^A \circ d \\
&= \ell(d(x)(a))(w) && \text{by (IH)} \\
&= t(\ell(x))(a)(w) && \text{by } t \circ \ell = \ell^A \circ d \\
&= \ell(x)(aw).
\end{aligned}$$

Hence, $\ell' = \ell$.

Therefore, ℓ is the unique homomorphism of $\mathcal{D}_K$-coalgebras from $(X, \langle \tau, d \rangle)$ to $(K^{A^*}, \langle \epsilon, t \rangle)$. Consequently, the coalgebra of fuzzy languages $(K^{A^*}, \langle \epsilon, t \rangle)$ is a final $\mathcal{D}_K$-coalgebra.

Finally, given a crisp-deterministic fuzzy automaton $\mathcal{A} = (X, x_0, \tau, d)$, there exists a unique homomorphism of $\mathcal{D}_K$-coalgebras $\ell : (X, \langle \tau, d \rangle) \to (K^{A^*}, \langle \epsilon, t \rangle)$ as described above. Thus,

$$\mathcal{L}(\mathcal{A}) = \tau \circ d^*(x_0) = \ell(x_0)$$

by Definition 5.8(ii). □

By the previous theorem, we have a coalgebraic description of the fuzzy language recognized by a crisp-deterministic fuzzy automaton. Furthermore, having a final $\mathcal{D}_K$-coalgebra, we can do proofs by coinduction (see [Rut00]) on crisp-deterministic fuzzy automata as presented in the following example.

Example 5.11. Let $\lambda \in K^{A^*}$ be a fuzzy language. Let $X = \{w^{-1}\lambda \mid w \in A^*\}$, and define $\tau : X \to K$ and $d : X \to X^A$ by

$$\tau(\varphi) = \varphi(\varepsilon) \qquad \text{and} \qquad d(\varphi)(a) = a^{-1}\varphi,$$

for all $\varphi \in X$ and $a \in A$. Note that d is well-defined because $a^{-1}(w^{-1}\lambda) = (wa)^{-1}\lambda$, for any $a \in A$ and $w \in A^*$. Clearly, the inclusion function $i : X \to K^{A^*}$, given by $i(\varphi) = \varphi$ for all $\varphi \in X$, makes the diagram

$$\begin{array}{ccc}
X & \xrightarrow{\quad i \quad} & K^{A^*} \\
{\scriptstyle \langle \tau, d \rangle}\downarrow & & \downarrow{\scriptstyle \langle \epsilon, t \rangle} \\
K \times X^A & \xrightarrow[\mathrm{id}_K \times i^A]{} & K \times (K^{A^*})^A
\end{array}$$

commute. Thus, by Theorem 5.10, $\lambda = i(\varepsilon^{-1}\lambda)$ is the fuzzy language recognized by $\mathcal{A} = (X, \lambda, \tau, d)$. We have that $\mathcal{A}$ is called the *(left) derivative automaton of the fuzzy language* λ and we proved part of [Ign+10, Theorem 4.1].

By Theorem 5.10, which gives a final $\mathcal{D}_K$-coalgebra, we can also do definitions by coinduction (see [Rut00]) as follows.

Example 5.12. Define a function $r : K^{A^*} \to (K^{A^*})^A$ by

$$r(\lambda)(a)(w) = \lambda(wa), \qquad \lambda \in K^{A^*}, a \in A, w \in A^*.$$

By Theorem 5.10, there exists a unique function $\mathrm{rev} : K^{A^*} \to K^{A^*}$ which makes the diagram

$$\begin{array}{ccc} K^{A^*} & \xrightarrow{\ \mathrm{rev}\ } & K^{A^*} \\ {\scriptstyle \langle \epsilon, r\rangle}\big\downarrow & & \big\downarrow{\scriptstyle \langle \epsilon, t\rangle} \\ K \times (K^{A^*})^A & \xrightarrow[\mathrm{id}_K \times \mathrm{rev}^A]{} & K \times (K^{A^*})^A \end{array}$$

commute. Given a fuzzy language $\lambda \in A^*$, observe that

$$\begin{aligned} \mathrm{rev}(\lambda)(ab) &= \mathrm{rev}(r(\lambda)(a))(b) = \mathrm{rev}(r(r(\lambda)(a))(b))(\varepsilon) = \epsilon(r(r(\lambda)(a))(b)) \\ &= r(r(\lambda)(a))(b)(\varepsilon) = r(\lambda)(a)(\varepsilon b) = \lambda(\varepsilon ba) = \lambda(ba), \end{aligned}$$

for any letters $a, b \in A$. In general, we have that

$$\mathrm{rev}(\lambda)(\varepsilon) = \epsilon(\mathrm{rev}(\lambda)) = \epsilon(\lambda) = \lambda(\varepsilon)$$

and

$$\mathrm{rev}(\lambda)(a_1 a_2 \dots a_n) = \lambda(a_n \dots a_2 a_1)$$

for any letters $a_1, a_2, \dots, a_n \in A$, where $n \geq 1$. Thus, $\mathrm{rev}(\lambda)$ is the fuzzy language known as the *reverse fuzzy language of* λ.

5.4 Determinization of fuzzy automata

In this section we apply the coalgebraic determinization process, studied in Section 3.3, to the fuzzy automata functor $\mathcal{F}$ (Definition 5.2) leading to a generalization of the powerset construction. Also, we present a coalgebraic definition of the fuzzy language recognized by a fuzzy automaton.

Let $\mathcal{M} = (M, \vee, *)$ be a $\mathcal{K}$-module (Definition 4.8). By Theorem 4.10(i), we can obtain a $\mathcal{K}$-module $\mathcal{M}^A = (M^A, \vee^A, *^A)$. Also $(K, \vee, \otimes)$ is a $\mathcal{K}$-module (Example 4.9). Then, by Theorem 4.11(i), we have a $\mathcal{K}$-module $(\mathcal{D}_K(M), \vee^{\mathcal{D}_K}, *^{\mathcal{D}_K})$ where

$$\bigvee_{i\in I}^{\mathcal{D}_K}(k_i, h_i) = (\bigvee_{i\in I} k_i, \bigvee_{i\in I}^{A} h_i) \qquad \text{and} \qquad z *^{\mathcal{D}_K} (k, h) = (z \otimes k, z *^A h)$$

for any $z \in K$, $(k, h) \in K \times M^A$, and any family $(k_i, h_i)_{i\in I}$ of elements of $K \times M^A$.

Given a linear map $f : \mathcal{M} \times \mathcal{M}'$. By Theorem 4.10(ii), f^A is a linear map from $\mathcal{M}^A$ to $(\mathcal{M}')^A$. Since id_K is a linear map from $(K, \vee, \otimes)$ to itself, by Theorem 4.11, we have that $\mathcal{D}_K(f)$ is a linear map from $(\mathcal{D}_K(M), \vee^{\mathcal{D}_K}, *^{\mathcal{D}_K})$ to $(\mathcal{D}_K(M'), \vee^{\mathcal{D}_K}, *^{\mathcal{D}_K})$. Therefore, we proved the following result that will be useful throughout this section.

Theorem 5.13. *(i) Let $\mathcal{M}$ be a $\mathcal{K}$-module. Then, $(\mathcal{D}_K(M), \vee^{\mathcal{D}_K}, *^{\mathcal{D}_K})$, as defined above, is a $\mathcal{K}$-module.*

(ii) Let $f : \mathcal{M} \to \mathcal{M}'$ be a linear map between $\mathcal{K}$-modules $\mathcal{M}$ and $\mathcal{M}'$. Then, $\mathcal{D}_K(f)$ is a linear map between the resulting $\mathcal{K}$-modules by (i).

As usual, we omit the superscript $\mathcal{D}_K$ in $\vee^{\mathcal{D}_K}$ and $*^{\mathcal{D}_K}$, whenever the operations are clear from the context.

In particular, by Theorem 5.13, the $\mathcal{K}$-module $(\mathcal{Z}(X), \cup, \circledast)$ of fuzzy subsets of X (Definition 4.24) gives rise to a $\mathcal{K}$-module $(\mathcal{D}_K\mathcal{Z}(X), \vee, *)$, where

$$\bigvee_{i\in I}(k_i, \alpha_i) = (\bigvee_{i\in I} k_i, \bigcup_{i\in I}^{A} \alpha_i) \qquad \text{and} \qquad z * (k, \alpha) = (z \otimes k, z \circledast^A \alpha),$$

for any $(k, \alpha) \in K \times (\mathcal{Z}(X))^A$, $z \in K$ and any family $(k_i, \alpha_i)_{i\in I}$ of elements of $K \times (\mathcal{Z}(X))^A$.

The fuzzy automata functor $\mathcal{F}$ is the composition of $\mathcal{D}_K$ with $\mathcal{Z}$, where $(\mathcal{Z}, \eta, \mu)$ is the fuzzy-set monad (Corollary 4.30), that is $\mathcal{F} = \mathcal{D}_K\mathcal{Z}$. For each set X, we have that $\mathcal{D}_K(\eta_X) : \mathcal{D}_K(X) \to \mathcal{D}_K\mathcal{Z}(X)$, where $(\mathcal{D}_K\mathcal{Z}(X), \vee, *)$ is a $\mathcal{K}$-module, and so we can consider the linear extension (Definition 4.27) $\overline{\mathcal{D}_K(\eta_X)} : \mathcal{Z}\mathcal{D}_K(X) \to \mathcal{D}_K\mathcal{Z}(X)$ of $\mathcal{D}_K(\eta_X)$. This leads to the definition of a distributive law (Definition 2.20) as the following result shows.

Theorem 5.14. *For each set X, consider the $\mathcal{K}$-module $(\mathcal{D}_K\mathcal{Z}(X), \vee, *)$ obtained from the $\mathcal{K}$-module $(\mathcal{Z}(X), \cup, \circledast)$ of fuzzy subsets of X by Theorem 5.13(i), and let*

$$\rho_X = \overline{\mathcal{D}_K(\eta_X)} : K^{K\times X^A} \to K \times (K^X)^A$$

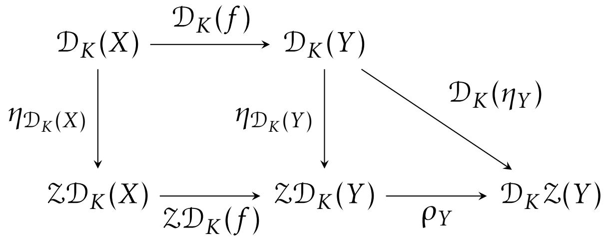
be the linear extension of $\mathcal{D}_K(\eta_X) : \mathcal{D}_K(X) \to \mathcal{D}_K\mathcal{Z}(X)$, i.e.

$$\begin{aligned}\rho_X(\varphi) &= \bigvee_{(k,h)\in K\times X^A} \varphi(k,h) * (k, \eta_X \circ h)\\ &= (\bigvee_{(k,h)\in K\times X^A} \varphi(k,h) \otimes k, \bigcup_{(k,h)\in K\times X^A}^{A} \varphi(k,h) \circledast^A (\eta_X \circ h))\end{aligned}$$

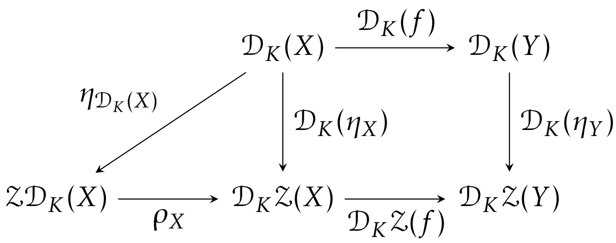
for all $\varphi \in K^{K\times X^A}$. Let $\rho = (\rho)_{X\in\mathbf{Set}}$. Then, ρ is a distributive law of $\mathcal{Z}$ over $\mathcal{D}_K$.

Proof. Recall that, for each set X, $\rho_X = \overline{\mathcal{D}_K(\eta_X)}$ is the unique linear map from the $\mathcal{K}$-module $(\mathcal{Z}\mathcal{D}_K(X), \cup, \circledast)$ of fuzzy subsets of $\mathcal{D}_K(X)$ to the $\mathcal{K}$-module $(\mathcal{D}_K\mathcal{Z}(X), \vee, *)$, obtained from $(\mathcal{Z}(X), \cup, \circledast)$ by Theorem 5.13(i), such that $\rho_X \circ \eta_{\mathcal{D}_K(X)} = \mathcal{D}_K(\eta_X)$, by Theorem 4.26 and Definition 4.27.

Let $f : X \to Y$ be any function between sets X and Y, and let $(\mathcal{D}_K\mathcal{Z}(X), \vee, *)$ and $(\mathcal{D}_K\mathcal{Z}(Y), \vee, *)$ be the $\mathcal{K}$-modules obtained from $(\mathcal{Z}(X), \cup, \circledast)$ and $(\mathcal{Z}(Y), \cup, \circledast)$, respectively, by Theorem 5.13(i). By definition of $\mathcal{Z}$ (Corollary 4.30), we have that $\mathcal{Z}\mathcal{D}_K(f) = \overline{\eta_{\mathcal{D}_K(Y)} \circ \mathcal{D}_K(f)}$ is the unique linear map from $(\mathcal{Z}\mathcal{D}_K(X), \cup, \circledast)$ to $(\mathcal{Z}\mathcal{D}_K(Y), \cup, \circledast)$ such that $\mathcal{Z}\mathcal{D}_K(f) \circ \eta_{\mathcal{D}_K(X)} = \eta_{\mathcal{D}_K(Y)} \circ \mathcal{D}_K(f)$. Thus, the diagram



commutes, and so $\rho_Y \circ \mathcal{Z}\mathcal{D}_K(f) \circ \eta_{\mathcal{D}_K(X)} = \mathcal{D}_K(\eta_Y) \circ \mathcal{D}_K(f)$. Also, $\mathcal{Z}(f) = \overline{\eta_Y \circ f}$ is the unique linear map from $(\mathcal{Z}(X), \cup, \circledast)$ to $(\mathcal{Z}(Y), \cup, \circledast)$ such that $\mathcal{Z}(f) \circ \eta_X = \eta_Y \circ f$. This implies that $\mathcal{D}_K\mathcal{Z}(f)$ is a linear map from $(\mathcal{D}_K\mathcal{Z}(X), \vee, *)$ to $(\mathcal{D}_K\mathcal{Z}(Y), \vee, *)$, by Theorem 5.13(ii), and that the diagram

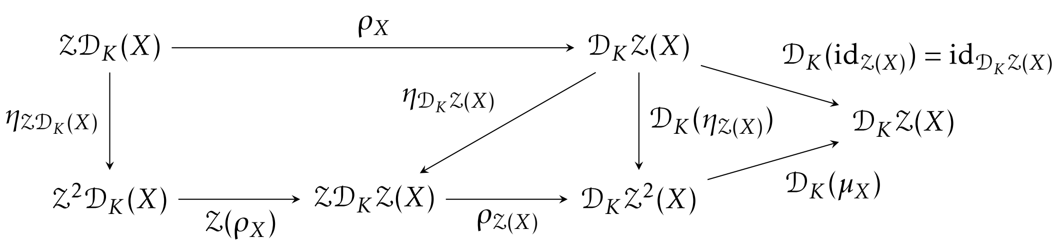


commutes, and so $\mathcal{D}_K\mathcal{Z}(f) \circ \rho_X \circ \eta_{\mathcal{D}_K(X)} = \mathcal{D}_K(\eta_Y) \circ \mathcal{D}_K(f)$. Therefore, both functions $\mathcal{D}_K\mathcal{Z}(f) \circ \rho_X$ and $\rho_Y \circ \mathcal{Z}\mathcal{D}_K(f)$ are linear maps from $(\mathcal{Z}\mathcal{D}_K(X), \cup, \circledast)$ to $(\mathcal{D}_K\mathcal{Z}(Y), \vee, *)$ and

$$\mathcal{D}_K\mathcal{Z}(f) \circ \rho_X \circ \eta_{\mathcal{D}_K(X)} = \mathcal{D}_K(\eta_Y) \circ \mathcal{D}_K(f) = \rho_Y \circ \mathcal{Z}\mathcal{D}_K(f) \circ \eta_{\mathcal{D}_K(X)},$$

which implies that $\mathcal{D}_K\mathcal{Z}(f) \circ \rho_X = \rho_Y \circ \mathcal{Z}\mathcal{D}_K(f)$, by Theorem 4.26. Consequently, ρ is a natural transformation from $\mathcal{Z}\mathcal{D}_K$ to $\mathcal{D}_K\mathcal{Z}$.

For each set X, by definition of μ_X (Corollary 4.30), we have that $\mu_X = \overline{\mathrm{id}_{\mathcal{Z}(X)}}$ is the unique linear map from $(\mathcal{Z}(\mathcal{Z}(X)), \cup, \circledast)$ to $(\mathcal{Z}(X), \cup, \circledast)$ such that $\mu_X \circ \eta_{\mathcal{Z}(X)} = \mathrm{id}_{\mathcal{Z}(X)}$. Given a set X, let $(\mathcal{D}_K\mathcal{Z}^2(X), \vee, *)$ and $(\mathcal{D}_K\mathcal{Z}(X), \vee, *)$ be the $\mathcal{K}$-modules obtained from $(\mathcal{Z}(\mathcal{Z}(X)), \cup, \circledast)$ and $(\mathcal{Z}(X), \cup, \circledast)$, respectively, by Theorem 5.13(i). Thus, $\mathcal{D}_K(\mu_X)$ is a linear map from $(\mathcal{D}_K\mathcal{Z}^2(X), \vee, *)$ to $(\mathcal{D}_K\mathcal{Z}(X), \vee, *)$ by Theorem 5.13(ii). Also, $\mathcal{Z}(\rho_X) = \overline{\eta_{\mathcal{D}_K\mathcal{Z}(X)} \circ \rho_X}$ is the unique linear map from $(\mathcal{Z}(\mathcal{Z}\mathcal{D}_K(X)), \cup, \circledast)$ to $(\mathcal{Z}(\mathcal{D}_K\mathcal{Z}(X)), \cup, \circledast)$ such that $\mathcal{Z}(\rho_X) \circ \eta_{\mathcal{Z}\mathcal{D}_K(X)} = \eta_{\mathcal{D}_K\mathcal{Z}(X)} \circ \rho_X$. And $\rho_{\mathcal{Z}(X)} = \overline{\mathcal{D}_K(\eta_{\mathcal{Z}(X)})}$ is the unique linear map from $(\mathcal{Z}\mathcal{D}_K\mathcal{Z}(X), \cup, \circledast)$ to $(\mathcal{D}_K\mathcal{Z}^2(X), \vee, *)$ such that $\rho_{\mathcal{Z}(X)} \circ \eta_{\mathcal{D}_K\mathcal{Z}(X)} = \mathcal{D}_K(\eta_{\mathcal{Z}(X)})$. Then, the diagram



commutes, that is $\mathcal{D}_K(\mu_X) \circ \rho_{\mathcal{Z}(X)} \circ \mathcal{Z}(\rho_X) \circ \eta_{\mathcal{Z}\mathcal{D}_K(X)} = \rho_X$. Hence

$$\mathcal{D}_K(\mu_X) \circ \rho_{\mathcal{Z}(X)} \circ \mathcal{Z}(\rho_X) \circ \eta_{\mathcal{Z}\mathcal{D}_K(X)} = \rho_X = \rho_X \circ \mathrm{id}_{\mathcal{Z}\mathcal{D}_K(X)} = \rho_X \circ \mu_{\mathcal{D}_K(X)} \circ \eta_{\mathcal{Z}\mathcal{D}_K(X)},$$

where both $\mathcal{D}_K(\mu_X) \circ \rho_{\mathcal{Z}(X)} \circ \mathcal{Z}(\rho_X)$ and $\rho_X \circ \mu_{\mathcal{D}_K(X)}$ are linear maps from $(\mathcal{Z}(\mathcal{Z}\mathcal{D}_K(X)), \cup, \circledast)$ to $(\mathcal{D}_K\mathcal{Z}(X), \vee, *)$, and therefore $\mathcal{D}_K(\mu_X) \circ \rho_{\mathcal{Z}(X)} \circ \mathcal{Z}(\rho_X) = \rho_X \circ \mu_{\mathcal{D}_K(X)}$ by Theorem 4.26. Consequently, for any set X, both diagrams

$$\begin{array}{ccc} \mathcal{D}_K(X) & \xrightarrow{\eta_{\mathcal{D}_K(X)}} & \mathcal{Z}\mathcal{D}_K(X) \\ & \searrow^{\mathcal{D}_K(\eta_X)} & \downarrow{\rho_X} \\ & & \mathcal{D}_K\mathcal{Z}(X) \end{array} \qquad \begin{array}{ccc} \mathcal{Z}^2\mathcal{D}_K(X) & \xrightarrow{\rho_{\mathcal{Z}(X)} \circ \mathcal{Z}(\rho_X)} & \mathcal{D}_K\mathcal{Z}^2(X) \\ \downarrow{\mu_{\mathcal{D}_K(X)}} & & \downarrow{\mathcal{D}_K(\mu_X)} \\ \mathcal{Z}\mathcal{D}_K(X) & \xrightarrow[\rho_X]{} & \mathcal{D}_K\mathcal{Z}(X) \end{array}$$

commute, and hence ρ is a distributive law of $\mathcal{Z}$ over $\mathcal{D}_K$. □

By Theorem 2.21, from the fact that ρ is a distributive law of $\mathcal{Z}$ over $\mathcal{D}_K$, we have the following.

Corollary 5.15. *Let ρ be the distributive law of $\mathcal{Z}$ ovar $\mathcal{D}_K$ described in Theorem 5.14. Define a functor $\widehat{\mathcal{D}_K} : \mathcal{EM}(\mathcal{Z}) \to \mathcal{EM}(\mathcal{Z})$ mapping each $\mathcal{Z}$-algebra (X, h) to*

$$\widehat{\mathcal{D}_K}(X, h) = (\mathcal{D}_K(X), \mathcal{D}_K(h) \circ \rho_X),$$

and mapping each homomorphism of $\mathcal{Z}$-algebras f to $\widehat{\mathcal{D}_K}(f) = \mathcal{D}_K(f)$. Then, $\widehat{\mathcal{D}_K}$ is a lifting of $\mathcal{D}_K$ to $\mathcal{EM}(\mathcal{Z})$.

We remark that the previous corollary reflects Theorem 5.13 on $\mathcal{EM}(\mathcal{Z})$, given the isomorphism between $\mathcal{K}$-**Mod** and $\mathcal{EM}(\mathcal{Z})$ induced by Theorems 4.33 and 4.34. In other words, denote by $G : \mathcal{K}\text{-}\mathbf{Mod} \to \mathcal{EM}(\mathcal{Z})$ the isomorphism induced by Theorem 4.34, with inverse $G^{-1} : \mathcal{EM}(\mathcal{Z}) \to \mathcal{K}\text{-}\mathbf{Mod}$ induced by Theorem 4.33, then

$$(\mathcal{D}_K(M), \vee^{\mathcal{D}_K}, *^{\mathcal{D}_K}) = G^{-1}\widehat{\mathcal{D}_K}G(M, \vee, *)$$

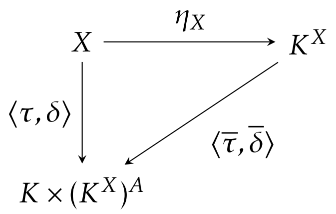
for any $\mathcal{K}$-module $(M, \vee, *)$. Thus, given a function $f : X \to \mathcal{D}_K\mathcal{Z}(X)$, the linear extension $\overline{f} : \mathcal{Z}(X) \to \mathcal{D}_K\mathcal{Z}(X)$ of f, which is the unique linear map from $(\mathcal{Z}(X), \cup, \circledast)$ to $(\mathcal{D}_K\mathcal{Z}(X), \vee, *)$ satisfying $\overline{f} \circ \eta_X = f$, is also the unique homomorphism of $\mathcal{Z}$-algebras from $(\mathcal{Z}(X), \mu_X)$ to $\widehat{\mathcal{D}_K}(\mathcal{Z}(X), \mu_X)$ satisfying the equality.

By Corollary 5.15, the fuzzy automata functor $\mathcal{F}$ is the composition of $\mathcal{D}_K$ with $\mathcal{Z}$, where $(\mathcal{Z}, \eta, \mu)$ is the fuzzy-set monad, and $\mathcal{D}_K$ has a lifting $\widehat{\mathcal{D}_K}$ to $\mathcal{EM}(\mathcal{Z})$. Thus, the functor $\mathcal{F} = \mathcal{D}_K\mathcal{Z}$ satifies the assumptions of Theorem 3.11 leading to a (coalgebraic) determinization method of fuzzy automata.

Theorem 5.16. *Consider the lifting $\widehat{\mathcal{D}_K}$ of $\mathcal{D}_K$ to $\mathcal{EM}(\mathcal{Z})$, by Corollary 5.15. For each $\mathcal{F}$-coalgebra $(X, \langle \tau, \delta \rangle : X \to K \times (K^X)^A)$, define $H(X, \langle \tau, \delta \rangle) = ((K^X, \mu_X), \langle \overline{\tau}, \overline{\delta} \rangle)$, where $\overline{\tau} : K^X \to K$ and $\overline{\delta} : K^X \to (K^X)^A$ are defined by*

$$\overline{\tau}(\varphi) = \bigvee_{x \in X} \varphi(x) \otimes \tau(x) \qquad \textit{and} \qquad \overline{\delta}(\varphi) = \bigcup_{x \in X}^{A} \varphi(x) \circledast^A \delta(x)$$

for all $\varphi \in K^X$, and then the diagram



commutes. For each homomorphism of $\mathcal{F}$-coalgebras, define $H(f)=\mathcal{Z}(f)$. Then, H is a functor from $\mathrm{CoAlg}(\mathcal{F})$ *to* $\mathrm{CoAlg}(\widehat{\mathcal{D}_K})$*.*

In particular, a fuzzy automaton (X,σ,τ,δ) gives rise to a crisp-deterministic fuzzy automaton $(K^X,\sigma,\overline{\tau},\overline{\delta})$.

Proof. This statement follows from Theorem 3.11, whenever we prove that $\langle\overline{\tau},\overline{\delta}\rangle$ is a homomorphism of $\mathcal{Z}$-algebras from $(\mathcal{Z}(X),\mu_X)$ to $\widehat{\mathcal{D}_K}(\mathcal{Z}(X),\mu_X)$ and $\langle\overline{\tau},\overline{\delta}\rangle\circ\eta_X=\langle\tau,\delta\rangle$, for any $\mathcal{F}$-coalgebra $(X,\langle\sigma,\delta\rangle)$.

Let $(X,\langle\tau,\delta\rangle)$ be an $\mathcal{F}$-coalgebra. By Theorem 4.26 and Definition 4.27, we have that $\overline{\tau}$ is the unique linear map from $(\mathcal{Z}(X),\cup,\circledast)$ to $(K,\vee,\otimes)$ such that $\overline{\tau}\circ\eta_X=\tau$, and $\overline{\delta}$ is the unique linear map from $(\mathcal{Z}(X),\cup,\circledast)$ to $((\mathcal{Z}(X))^A,\cup^A,\circledast^A)$ such that $\overline{\delta}\circ\eta_X=\delta$. Let $(\mathcal{D}_K\mathcal{Z}(X),\vee,*)$ be the $\mathcal{K}$-module obtained from $(\mathcal{Z}(X),\cup,\circledast)$ by Theorem 5.13(i). For any family $(\varphi_i)_{i\in I}$ of fuzzy subsets of X aand any $\varphi\in K^X,k\in K$, we have that

$$\langle\overline{\tau},\overline{\delta}\rangle(\bigcup_{i\in I}\varphi_i)=(\overline{\tau}(\bigcup_{i\in I}\varphi_i),\overline{\delta}(\bigcup_{i\in I}\varphi_i))=(\bigvee_{i\in I}\overline{\tau}(\varphi_i),\bigcup_{i\in I}^{A}\overline{\delta}(\varphi_i))=\bigvee_{i\in I}\langle\overline{\tau},\overline{\delta}\rangle(\varphi_i)$$
$$\langle\overline{\tau},\overline{\delta}\rangle(k\circledast\varphi)=(\overline{\tau}(k\circledast\varphi),\overline{\delta}(k\circledast\varphi))=(k\otimes\overline{\tau}(\varphi),k\circledast^A\overline{\delta}(\varphi))=k*\langle\overline{\tau},\overline{\delta}(\varphi\rangle,$$

and so $\langle\overline{\tau},\overline{\delta}\rangle$ is a linear map from $(\mathcal{Z}(X),\cup,\circledast)$ to $(\mathcal{D}_K\mathcal{Z}(X),\vee,*)$. Also, we have that

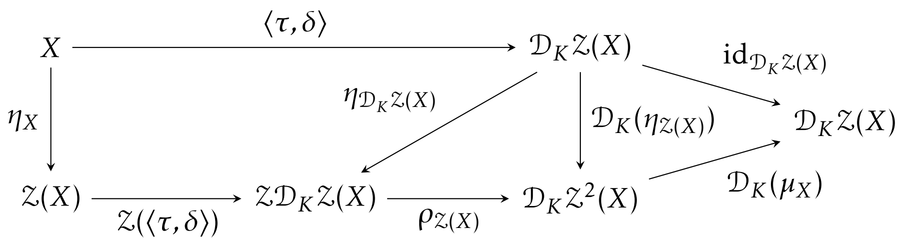
$$\langle\overline{\tau},\overline{\delta}\rangle(\eta_X(x))=(\overline{\tau}(\eta_X(x)),\overline{\delta}(\eta_X(x)))=(\tau(x),\delta(x))=\langle\tau,\delta\rangle(x),$$

for all $x\in X$, and hence $\langle\overline{\tau},\overline{\delta}\rangle\circ\eta_X=\langle\tau,\delta\rangle$.

Let ρ be the distributive law described in Theorem 5.14. As showed in the proof of Theorem 5.14, we have linear maps

$$(\mathcal{Z}(X),\cup,\circledast)\xrightarrow{\mathcal{Z}(\langle\tau,\delta\rangle)}(\mathcal{Z}\mathcal{D}_K\mathcal{Z}(X),\cup,\circledast)\xrightarrow{\rho_{\mathcal{Z}(X)}}(\mathcal{D}_K\mathcal{Z}^2(X),\vee,*)\xrightarrow{\mathcal{D}_K(\mu_X)}(\mathcal{D}_K\mathcal{Z}(X),\vee,*)$$

such that $\mathcal{Z}(\langle\tau,\delta\rangle)\circ\eta_X=\eta_{\mathcal{D}_K\mathcal{Z}(X)}\circ\langle\tau,\delta\rangle$, $\rho_{\mathcal{Z}(X)}\circ\eta_{\mathcal{D}_K\mathcal{Z}(X)}=\mathcal{D}_K(\eta_{\mathcal{Z}(X)})$ and $\mathcal{D}_K(\mu_X)\circ\mathcal{D}_K(\eta_{\mathcal{Z}(X)})=\mathrm{id}_{\mathcal{D}_K\mathcal{Z}(X)}$, which implies that the diagram



commutes. And so, both function $\langle\overline{\tau},\overline{\delta}\rangle$ and $\mathcal{D}_K(\mu_X)\circ\rho_{\mathcal{Z}(X)}\circ\mathcal{Z}(\langle\tau,\delta\rangle)$ are linear maps from $(\mathcal{Z}(X),\cup,\circledast)$ to $(\mathcal{D}_K\mathcal{Z}(X),\vee,*)$ and

$$\langle\overline{\tau},\overline{\delta}\rangle\circ\eta_X=\langle\tau,\delta\rangle=\mathcal{D}_K(\mu_X)\circ\rho_{\mathcal{Z}(X)}\circ\mathcal{Z}(\langle\tau,\delta\rangle)\circ\eta_X,$$

which implies that $\langle\overline{\tau},\overline{\delta}\rangle=\mathcal{D}_K(\mu_X)\circ\rho_{\mathcal{Z}(X)}\circ\mathcal{Z}(\langle\tau,\delta\rangle)$ by Theorem 4.26. Therefore, since $(\mathcal{D}_K\mathcal{Z}(X),\mathcal{D}_K(\mu_X)\circ\rho_{\mathcal{Z}(X)})=\widehat{\mathcal{D}_K}(\mathcal{Z}(X),\mu_X)$ by Corollary 5.15, we have that $\langle\overline{\tau},\overline{\delta}\rangle$ is the unique homomorphism of $\mathcal{Z}$-algebras from $(\mathcal{Z}(X),\mu_X)$ to $\widehat{\mathcal{D}_K}(\mathcal{Z}(X),\mu_X)$ satisfying $\langle\overline{\tau},\overline{\delta}\rangle\circ\eta_X=\langle\tau,\delta\rangle$, by Theorem 2.17. □

By the previous theorem, given a fuzzy automaton (X,σ,τ,δ), we can construct a crisp-deterministic fuzzy automaton $(K^X,\sigma,\overline{\tau},\overline{\delta})$. We remark that K^X may not be finite, even when X is finite. Also, since $\mathcal{EM}(\mathcal{Z})\cong\mathcal{K}$-**Mod** by Corollary 4.35, note that a $\widehat{\mathcal{D}_K}$-coalgebra induces, by adding an initial state, a crisp-deterministic fuzzy automaton on the category of $\mathcal{K}$-modules and linear maps.

5.4.1 Recognizing fuzzy languages

Recall that $\mathcal{K}$ is a complete residuated lattice with at least two elements. Thus, for any set X, the cardinality of K^X is strictly greatar than the cardinality of X.

Theorem 5.17. *The fuzzy automata functor $\mathcal{F}$ does not have a final coalgebra.*

Proof. Suppose that (Ω,ω) is a final $\mathcal{F}$-coalgebra. Then, by Theorem 3.9, $\omega:\Omega\to K\times(K^\Omega)^A$ is a bijection, which is a contradiction because the cardinality of $K\times(K^\Omega)^A$ is strictly greater than the cardinality of Ω, since K has at least two elemnts and A is nonempty. Consequently, a final $\mathcal{F}$-coalgebra does not exist. □

On the other hand, the coalgebra of fuzzy languages $(K^{A^*},\langle\epsilon,t\rangle)$, described in Definition 5.9, is a final $\mathcal{D}_K$-coalgebra by Theorem 5.10. Then, by Theorem 3.13, there exists a final $\widehat{\mathcal{D}_K}$-coalgebra as follows.

Theorem 5.18. *Let $(K^{A^*},\langle\epsilon,t\rangle)$ be the coalgebra of fuzzy languages, and let ρ be the distributive law of $\mathcal{Z}$ over $\mathcal{D}_K$ defined in Theorem 5.14. Then, the following diagram*

$$\begin{array}{ccc}
\mathcal{Z}^2(A^*) & \xrightarrow{\ \mu_{A^*}\ } & \mathcal{Z}(A^*) \\
{\scriptstyle \rho_{\mathcal{Z}(A^*)}\circ\mathcal{Z}(\langle\epsilon,t\rangle)}\downarrow & & \downarrow{\scriptstyle\langle\epsilon,t\rangle} \\
\mathcal{D}_K\mathcal{Z}^2(A^*) & \xrightarrow[\ \mathcal{D}_K(\mu_{A^*})\ ]{} & \mathcal{D}_K\mathcal{Z}(A^*)
\end{array}$$

commutes, i.e. $\langle\epsilon,t\rangle\circ\mu_{A^*}=\mathcal{D}_K(\mu_{A^*})\circ\rho_{\mathcal{Z}(A^*)}\circ\mathcal{Z}(\langle\epsilon,t\rangle)$. And therefore, $((K^{A^*},\mu_{A^*}),\langle\epsilon,t\rangle)$ is a final $\widehat{\mathcal{D}_K}$-coalgebra, for the lifting $\widehat{\mathcal{D}_K}$ of $\mathcal{D}_K$ to $\mathcal{EM}(\mathcal{Z})$ described in Corollary 5.15.

Proof. Consider $(\mathcal{D}_K\mathcal{Z}(A^*), \vee, *)$ to be the $\mathcal{K}$-module obtained from $(\mathcal{Z}(A^*), \cup, \circledast)$ by Theorem 5.13(i). Then, for any family $(\lambda_i)_{i\in I}$ of fuzzy languages and any $\lambda \in K^{A^*}, k \in K$, we have that

$$\epsilon(\bigcup_{i\in I}\lambda_i) = (\bigcup_{i\in I}\lambda_i)(\varepsilon) = \bigvee_{i\in I}\lambda_i(\varepsilon) = \bigvee_{i\in I}\epsilon(\lambda_i)$$
$$\epsilon(k\circledast\lambda) = (k\circledast\lambda)(\varepsilon) = k\otimes\lambda(\varepsilon) = k\otimes\epsilon(\lambda)$$

and

$$t(\bigcup_{i\in I}\lambda_i)(a)(w) = (\bigcup_{i\in I}\lambda_i)(aw) = \bigvee_{i\in I}\lambda_i(aw) = \bigvee_{i\in I}t(\lambda_i)(a)(w) = (\bigcup_{i\in I}^{A}t(\lambda_i))(a)(w)$$
$$t(k\circledast\lambda)(a)(w) = (k\circledast\lambda)(aw) = k\otimes\lambda(aw) = k\otimes t(\lambda)(a)(w) = (k\circledast^{A} t(\lambda))(a)(w),$$

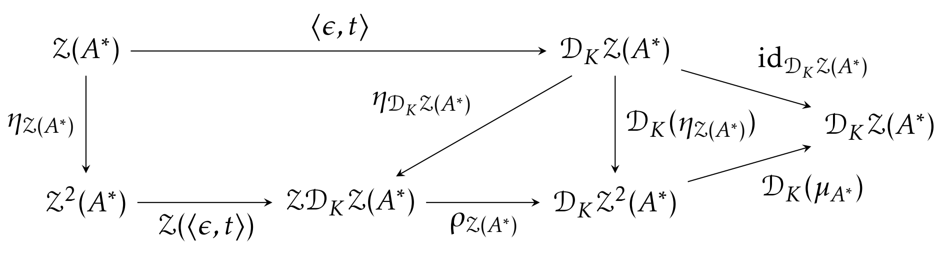
for all $a \in A$ and $w \in A^*$, and hence

$$\langle\epsilon,t\rangle(\bigcup_{i\in I}\lambda_i) = (\epsilon(\bigcup_{i\in I}\lambda_i), t(\bigcup_{i\in I}\lambda_i)) = (\bigvee_{i\in I}\epsilon(\lambda_i), \bigcup_{i\in I}^{A}t(\lambda_i)) = \bigvee_{i\in I}\langle\epsilon,t\rangle(\lambda_i)$$
$$\langle\epsilon,t\rangle(k\circledast\lambda) = (\epsilon(k\circledast\lambda), t(k\circledast\lambda)) = (k\otimes\epsilon(\lambda), k\circledast^{A} t(\lambda)) = k * \langle\epsilon,t\rangle(\lambda).$$

Thus, $\langle\epsilon,t\rangle$ is a linear map from $(\mathcal{Z}(X), \cup, \circledast)$ to $(\mathcal{D}_K\mathcal{Z}(X), \vee, *)$. By definition of μ_{A^*} (Corollary 4.30), we have that $\mu_{A^*} = \overline{\mathrm{id}_{\mathcal{Z}(A^*)}}$ is the unique linear map from $(\mathcal{Z}^2(A^*), \cup, \circledast)$ to $(\mathcal{Z}(A^*), \cup, \circledast)$ such that $\mu_{A^*}\circ\eta_{\mathcal{Z}(A^*)} = \mathrm{id}_{\mathcal{Z}(A^*)}$. As showed in the proof of Theorem 5.14, we have linear maps

$$(\mathcal{Z}^2(A^*), \cup, \circledast) \xrightarrow{\mathcal{Z}(\langle\epsilon,t\rangle)} (\mathcal{Z}\mathcal{D}_K\mathcal{Z}(A^*), \cup, \circledast) \xrightarrow{\rho_{\mathcal{Z}(A^*)}} (\mathcal{D}_K\mathcal{Z}^2(A^*), \vee, *) \xrightarrow{\mathcal{D}_K(\mu_{A^*})} (\mathcal{D}_K\mathcal{Z}(A^*), \vee, *)$$

such that $\mathcal{Z}(\langle\epsilon,t\rangle)\circ\eta_{\mathcal{Z}(A^*)} = \eta_{\mathcal{D}_K\mathcal{Z}(A^*)}\circ\langle\epsilon,t\rangle$, $\rho_{\mathcal{Z}(A^*)}\circ\eta_{\mathcal{D}_K\mathcal{Z}(A^*)} = \mathcal{D}_K(\eta_{\mathcal{Z}(A^*)})$ and $\mathcal{D}_K(\mu_{A^*})\circ\mathcal{D}_K(\eta_{\mathcal{Z}(A^*)}) = \mathrm{id}_{\mathcal{D}_K\mathcal{Z}(A^*)}$, which implies that the diagram



commutes. Hence, both functions $\langle\epsilon,t\rangle\circ\mu_{A^*}$ and $\mathcal{D}_K(\mu_{A^*})\circ\rho_{\mathcal{Z}(A^*)}\circ\mathcal{Z}(\langle\epsilon,t\rangle)$ are linear maps from $(\mathcal{Z}^2(A^*), \cup, \circledast)$ to $(\mathcal{D}_K\mathcal{Z}(A^*), \vee, *)$ and

$$\langle\epsilon,t\rangle\circ\mu_{A^*}\circ\eta_{\mathcal{Z}(A^*)} = \langle\epsilon,t\rangle = \mathcal{D}_K(\mu_{A^*})\circ\rho_{\mathcal{Z}(A^*)}\circ\mathcal{Z}(\langle\epsilon,t\rangle)\circ\eta_{\mathcal{Z}(A^*)},$$

which implies that $\langle\epsilon,t\rangle\circ\mu_{A^*} = \mathcal{D}_K(\mu_{A^*})\circ\rho_{\mathcal{Z}(A^*)}\circ\mathcal{Z}(\langle\epsilon,t\rangle)$ by Theorem 4.26, that is the diagram

$$\begin{array}{ccc} \mathcal{Z}^2(A^*) & \xrightarrow{\mu_{A^*}} & \mathcal{Z}(A^*) \\ {\scriptstyle \rho_{\mathcal{Z}(A^*)} \circ \mathcal{Z}(\langle \epsilon, t\rangle)} \downarrow & & \downarrow {\scriptstyle \langle \epsilon, t\rangle} \\ \mathcal{D}_K\mathcal{Z}^2(A^*) & \xrightarrow[\mathcal{D}_K(\mu_{A^*})]{} & \mathcal{D}_K\mathcal{Z}(A^*) \end{array}$$

commutes. Therefore, by Theorem 3.13, $((K^{A^*}, \mu_{A^*}), \langle \epsilon, t\rangle)$ is a final $\widehat{\mathcal{D}_K}$-coalgebra. $\square$

Given an $\mathcal{F}$-coalgebra $(X, \langle \tau, \delta\rangle)$, by Theorem 5.16, we can construct a $\widehat{\mathcal{D}_K}$-coalgebra $((K^X, \mu_X), \langle \overline{\tau}, \overline{\delta}\rangle)$, and then there exists (a unique) function $\ell : K^X \to K^{A^*}$ such that the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\eta_X} & K^X & \overset{\ell}{\dashrightarrow} & K^{A^*} \\ {\scriptstyle \langle \tau, \delta\rangle} \downarrow & \swarrow {\scriptstyle \langle \overline{\tau}, \overline{\delta}\rangle} & & & \downarrow {\scriptstyle \langle \epsilon, t\rangle} \\ K \times (K^X)^A & & \overset{}{\underset{\mathrm{id}_K \times \ell^A}{\dashrightarrow}} & & K \times (K^{A^*})^A \end{array}$$

commutes, by Theorem 5.18. The function ℓ allows us to obtain the fuzzy language recognized by a fuzzy automaton.

Theorem 5.19. *Let $\widehat{\mathcal{D}_K}$ be the lifting of $\mathcal{D}_K$ to $\mathcal{EM}(\mathcal{Z})$ in Corollary 5.15, and let $(K^{A^*}, \langle \epsilon, t\rangle)$ be the coalgebra of fuzzy languages. Given an $\mathcal{F}$-coalgebra $(X, \langle \tau, \delta\rangle)$, consider the $\widehat{\mathcal{D}_K}$-coalgebra $((K^X, \mu_X), \langle \overline{\tau}, \overline{\delta}\rangle)$ as described in Theorem 5.16. Define $\ell : K^X \to K^{A^*}$ by*

$$\ell(\sigma)(w) = \bigvee_{x,y \in X} \sigma(x) \otimes \delta^*(x)(w)(y) \otimes \tau(y), \qquad w \in A^*, \sigma \in K^X.$$

Then, the function ℓ is the unique homomorphism of $\widehat{\mathcal{D}_K}$-coalgebras from $((K^X, \mu_X), \langle \overline{\tau}, \overline{\delta}\rangle)$ to $((K^{A^*}, \mu_{A^*}), \langle \epsilon, t\rangle)$.

In particular, a fuzzy automaton $(X, \sigma, \tau, \delta)$ is (language) equivalent to its uduced crisp-deterministic fuzzy automaton $(K^X, \sigma, \overline{\tau}, \overline{\delta})$ by Theorem 5.16.

We provide two distinct proofs of this theorem. A proof by coinduction where we just have to show that ℓ is a homomorphism of $\mathcal{D}_K$-coalgebras. And a proof by induction, on the length of words in A^*, where we show that ℓ coincides with the unique homomorphism of $\mathcal{D}_K$-coalgebras from $(K^X, \langle \overline{\tau}, \overline{\delta}\rangle)$ to $(K^{A^*}, \langle \epsilon, t\rangle)$.

Coinductive proof of Theorem 5.19. Let $(X, \langle \tau, \delta\rangle)$ be an $\mathcal{F}$-coalgebra, and construct the $\widehat{\mathcal{D}_K}$-coalgebra $((K^X, \mu_X), \langle \overline{\tau}, \overline{\delta}\rangle)$ as described in Theorem 5.16. Define $\ell : K^X \to K^{A^*}$ by

$$\ell(\sigma)(w) = \bigvee_{x,y \in X} \sigma(x) \otimes \delta^*(x)(w)(y) \otimes \tau(y), \qquad w \in A^*, \sigma \in K^X.$$

By definition of δ^*, for any $x, y \in X$, we have that $\delta^*(x)(\varepsilon)(x) = 1$ and $\delta^*(x)(\varepsilon)(y) = 0$, whenever $x \neq y$. Thus,

$$\epsilon(\ell(\sigma)) = \ell(\sigma)(\varepsilon) = \bigvee_{x,y\in X} \sigma(x) \otimes \delta^*(x)(\varepsilon)(y) \otimes \tau(y) = \bigvee_{x\in X} \sigma(x) \otimes \tau(x) = \overline{\tau}(\sigma),$$

by Theorem 4.4(i) and the definition of $\overline{\tau}$, for any $\sigma \in K^X$. Hence, $\epsilon \circ \ell = \overline{\tau}$.

By definition of $\overline{\delta}$, we have that

$$\overline{\delta}(\sigma)(a)(z) = (\bigcup_{x\in X}^{A} \sigma(x) \circledast^{A} \delta(x))(a)(z) = \bigvee_{x\in X} \sigma(x) \otimes \delta(x)(a)(z)$$

for all $\sigma \in K^X$, $a \in A$ and $z \in X$. Since $\otimes$ is distributive over $\bigvee$, by Theorem 4.6(i), then

$$\begin{aligned}
\ell(\overline{\delta}(\sigma)(a))(w) &= \bigvee_{z,y\in X} \overline{\delta}(\sigma)(a)(z) \otimes \delta^*(z)(w)(y) \otimes \tau(y) \\
&= \bigvee_{z,y\in X} (\bigvee_{x\in X} \sigma(x) \otimes \delta(x)(a)(z)) \otimes \delta^*(z)(w)(y) \otimes \tau(y) \\
&= \bigvee_{x,z,y\in X} \sigma(x) \otimes \delta(x)(a)(z) \otimes \delta^*(z)(w)(y) \otimes \tau(y) \\
&= \bigvee_{x,y\in X} \sigma(x) \otimes (\bigvee_{z\in X} \delta(x)(a)(z) \otimes \delta^*(z)(w)(y)) \otimes \tau(y) \\
&= \bigvee_{x,y\in X} \sigma(x) \otimes \delta^*(x)(aw)(y) \otimes \tau(y) \\
&= \ell(\sigma)(aw) \\
&= t(\ell(\sigma))(a)(w),
\end{aligned}$$

for all $\sigma \in K^X$, $a \in A$ and $w \in A^*$. Hence, $t \circ \ell = \ell^A \circ \overline{\delta}$.

Therefore, the diagram

$$\begin{array}{ccc}
K^X & \xrightarrow{\quad \ell \quad} & K^{A^*} \\
{\scriptstyle \langle \overline{\tau}, \overline{\delta} \rangle} \downarrow & & \downarrow {\scriptstyle \langle \epsilon, t \rangle} \\
K \times (K^X)^A & \xrightarrow[\mathrm{id}_K \times \ell^A]{} & K \times (K^{A^*})^A
\end{array}$$

commutes, which implies that ℓ is a homomorphism of $\mathcal{D}_K$-coalgebras from $(K^X, \langle \overline{\tau}, \overline{\delta} \rangle)$ to $(K^{A^*}, \langle \epsilon, t \rangle)$. Moreover, ℓ is unique because $(K^{A^*}, \langle \epsilon, t \rangle)$ is a final $\mathcal{D}_K$-coalgebra, by Theorem 5.10. And since $((K^{A^*}, \mu_{A^*}), \langle \epsilon, t \rangle)$ is a final $\widehat{\mathcal{D}_K}$-coalgebra by Theorem 5.18, the function ℓ is the unique homomorphism of $\widehat{\mathcal{D}_K}$-coalgebras from $((K^X, \mu_X), \langle \overline{\tau}, \overline{\delta} \rangle)$ to $((K^{A^*}, \mu_{A^*}), \langle \epsilon, t \rangle)$, as showed in the proof of Theorem 3.13.

Finally, given a fuzzy automaton $\mathcal{A} = (X, \sigma, \tau, \delta)$ and its determinization, by Theorem 5.16, $\mathcal{A}' = (K^X, \sigma, \overline{\tau}, \overline{\delta})$. Then $\mathcal{L}(\mathcal{A}) = \ell(\sigma)$ as proved above, and $\mathcal{L}(\mathcal{A}') = \ell(\sigma)$ by Theorem 5.10. Consequently, $\mathcal{A}$ and $\mathcal{A}'$ are (language) equivalent. □

Inductive proof of Theorem 5.19. Let $(X, \langle \tau, \delta \rangle)$ be an $\mathcal{F}$-coalgebra, and consider the $\widehat{\mathcal{D}_K}$-coalgebra $((K^X, \mu_X), \langle \overline{\tau}, \overline{\delta} \rangle)$ as described in Theorem 5.16. Define $\ell : K^X \to K^{A^*}$ as above. Since $((K^{A^*}, \mu_{A^*}), \langle \epsilon, t \rangle)$ is a final $\widehat{\mathcal{D}_K}$-coalgebra by Theorem 5.18, there is a unique homomorphism of $\widehat{\mathcal{D}_K}$-coalgebras

$$\ell' : ((K^X, \mu_X), \langle \overline{\tau}, \overline{\delta} \rangle) \to ((K^{A^*}, \mu_{A^*}), \langle \epsilon, t \rangle).$$

In particular, the diagram

$$\begin{array}{ccc} K^X & \xrightarrow{\quad \ell' \quad} & K^{A^*} \\ {\scriptstyle \langle \overline{\tau}, \overline{\delta} \rangle} \downarrow & & \downarrow {\scriptstyle \langle \epsilon, t \rangle} \\ K \times (K^X)^A & \xrightarrow[\mathrm{id}_K \times (\ell')^A]{} & K \times (K^{A^*})^A \end{array}$$

commutes, and so we have that

$$\epsilon \circ \ell' = \overline{\tau} \qquad \text{and} \qquad t \circ \ell' = (\ell')^A \circ \overline{\delta}.$$

We show by induction, on the length of words $w \in A^*$, that $\ell'(\sigma) = \ell(\sigma)$ for any $\sigma \in K^X$. As we showed in the coinductive proof,

$$\ell'(\sigma)(\varepsilon) = \epsilon(\ell'(\sigma)) = \overline{\tau}(\sigma) = \bigvee_{x \in X} \sigma(x) \otimes \tau(x) = \ell(\sigma)(\varepsilon),$$

for any $\sigma \in K^X$. Assume $\ell'(\sigma)(w) = \ell(\sigma)(w)$, for all $\sigma \in K^X$, as induction hypothesis (IH). Then, for $a \in A$,

$$\ell'(\sigma)(aw) = t(\ell'(\sigma))(a)(w) = \ell'(\overline{\delta}(\sigma)(a))(w) = \ell(\overline{\delta}(\sigma)(a))(w),$$

by (IH), and

$$\begin{aligned} \ell(\overline{\delta}(\sigma)(a))(w) &= \bigvee_{z,y \in X} \overline{\delta}(\sigma)(a)(z) \otimes \delta^*(z)(w)(y) \otimes \tau(y) \\ &= \bigvee_{x,y \in X} \sigma(x) \otimes \delta^*(x)(aw)(y)\tau(y) = \ell(\sigma)(aw), \end{aligned}$$

as showed in the coinductive proof. Hence, $\ell' = \ell$. And therefore, ℓ is the unique homomorphism of $\widehat{\mathcal{D}_K}$-coalgebras from $((K^X, \mu_X), \langle \overline{\tau}, \overline{\delta} \rangle)$ to $((K^{A^*}, \mu_{A^*}), \langle \epsilon, t \rangle)$. The rest of the proof is analogous to the coinductive proof. □

By the previous theorem, we have a coalgebraic description of the fuzzy language recognized by a fuzzy automaton. Furthermore, the determinization process described in Theorem 5.16 generates a crisp-deterministic fuzzy automaton from a fuzzy automaton, which recognizes the same fuzzy language.

5.5 Bisimulations for fuzzy automata

In this section, we describe bisimulations for the coalgebras of the fuzzy automata functor $\mathcal{F}$ (Definition 5.1). We refer to [Rut00] for a proper introduction on bisimulations for coalgebras.

Let $(X, \langle\tau,\delta\rangle : X \to K \times (K^X)^A)$ and $(X', \langle\tau',\delta'\rangle : X' \to K \times (K^{X'})^A)$ be $\mathcal{F}$-coalgebras. A *bisimulation* between $(X, \langle\tau,\delta\rangle)$ and $(X', \langle\tau',\delta'\rangle)$ is a relation $R \subseteq X \times X'$ such that there exists an $\mathcal{F}$-transition structure $\langle\psi,\theta\rangle : R \to K \times (K^R)^A$ which makes the diagram

$$\begin{array}{ccccc}
X & \xleftarrow{\quad p \quad} & R & \xrightarrow{\quad p' \quad} & X' \\
\Big\downarrow{\scriptstyle \langle\tau,\delta\rangle} & & \Big\downarrow{\scriptstyle \langle\psi,\theta\rangle} & & \Big\downarrow{\scriptstyle \langle\tau',\delta'\rangle} \\
K \times (K^X)^A & \xleftarrow[\mathrm{id}_K \times (\mathcal{Z}(p))^A]{} & K \times (K^R)^A & \xrightarrow[\mathrm{id}_K \times (\mathcal{Z}(p'))^A]{} & K \times (K^{X'})^A
\end{array}$$

commute, where p and p' are the projections from R to X and X', respectively.

Consider a bisimulation R between $(X, \langle\tau,\delta\rangle)$ and $(X', \langle\tau',\delta'\rangle)$ with transition structure $\langle\psi,\theta\rangle : R \to K \times (K^R)^A$. First, we have

$$\tau \circ p = \psi \qquad \text{and} \qquad \tau' \circ p' = \psi,$$

which implies that, for all $(x,x') \in R$,

$$\tau(x) = \tau'(x').$$

Second, we have

$$\delta \circ p = (\mathcal{Z}(p))^A \circ \theta \qquad \text{and} \qquad \delta' = (\mathcal{Z}(p'))^A \circ \theta,$$

which implies that, for all $(x,x') \in R$ and $a \in A$,

$$\begin{aligned}
\delta(x)(a) &= \mathcal{Z}(p)(\theta(x,x')(a)) \\
&= \bigcup_{(y,y') \in R} \theta(x,x')(a)(y,y') \circledast \eta_X(p(y,y')) \\
&= \bigcup_{(y,y') \in R} \theta(x,x')(a)(y,y') \circledast \eta_X(y)
\end{aligned}$$

and similarly

$$\delta'(x')(a) = \mathcal{Z}(p')(\theta(x,x')(a)) = \bigcup_{(y,y') \in R} \theta(x,x')(a)(y,y') \circledast \eta_{X'}(y').$$

Note that, for $(x,x') \in R$, $a \in A$ and $y \in X$,

$$\delta(x)(a)(y) = \bigvee_{\substack{y' \in X' \\ (y,y') \in R}} \theta(x,x')(a)(y,y'),$$

where the supremum ranges over $y' \in X'$ such that $(y, y') \in R$, and so $\delta(x)(a)(y)$ can be seen as the truth degree of "there is $y' \in X'$ such that $(y, y') \in R$ and the input a causes a transition from (x, x') to (y, y')". Similarly, for $(x, x') \in R$, $a \in A$ and $y' \in X'$,

$$\delta'(x')(a)(y') = \bigvee_{\substack{y \in X \\ (y,y') \in R}} \theta(x, x')(a)(y, y'),$$

and so $\delta'(x')(a)(y')$ can be seen as the truth degree of "there is $y \in X$ such that $(y, y') \in R$ and the input a causes a transition from (x, x') to (y, y')".

Theorem 5.20. *Let R be a bisimulation between $\mathcal{F}$-coalgebras $(X, \langle \tau, \delta \rangle)$ and $(X', \langle \tau', \delta' \rangle)$, with $\mathcal{F}$-transition structure $\langle \psi, \theta \rangle : R \to K \times (K^R)^A$. Then, for all $a \in A$,*

(i) $\bigvee_{\substack{y' \in X' \\ (y,y') \in R}} \theta(x, x_1')(a)(y, y') = \bigvee_{\substack{y' \in X' \\ (y,y') \in R}} \theta(x, x_2')(a)(y, y')$, *for* $(x, x_1'), (x, x_2') \in R, y \in X$;

(ii) $\bigvee_{\substack{y \in X \\ (y,y') \in R}} \theta(x_1, x')(a)(y, y') = \bigvee_{\substack{y \in X \\ (y,y') \in R}} \theta(x_2, x')(a)(y, y')$, *for* $(x_1, x'), (x_2, x') \in R, y' \in X'$;

(iii) $\theta(x, x')(a)(y, y') \leq \delta(x)(a)(y) \wedge \delta'(x')(a)(y')$, *for any* $(x, x'), (y, y') \in R$;

(iv) $\delta(x)(a)(y) \leq \bigvee_{\substack{y' \in X' \\ (y,y') \in R}} \delta'(x')(a)(y')$, *for any* $(x, x') \in R$ *and* $y \in X$;

(v) $\delta'(x')(a)(y') \leq \bigvee_{\substack{y \in X \\ (y,y') \in R}} \delta(x)(a)(y)$, *for any* $(x, x') \in R$ *and* $y' \in X'$;

(vi) $\bigvee_{y \in X} \delta(x)(a)(y) = \bigvee_{y' \in X'} \delta'(x')(a)(y')$, *for any* $(x, x') \in R$.

Proof. We only prove (i), (iii), (iv) and (vi), since (ii) and (v) are proven as (i) and (iv), respectively. Let $a \in A$. (i) If $(x, x_1'), (x, x_2') \in R$ and $y \in X$, then

$$\bigvee_{\substack{y' \in X' \\ (y,y') \in R}} \theta(x, x_1')(a)(y, y') = \delta(x)(a)(y) = \bigvee_{\substack{y' \in X' \\ (y,y') \in R}} \theta(x, x_2')(a)(y, y'),$$

since $(R, \langle \psi, \theta \rangle)$ is a bisimulation.

(iii) For $(x, x'), (y, y') \in R$, we have

$$\delta(x)(a)(y) = \bigvee_{\substack{z' \in X' \\ (y,z') \in R}} \theta(x, x')(a)(y, z') \qquad \text{and} \qquad \delta'(x')(a)(y') = \bigvee_{\substack{z \in X \\ (z,y') \in R}} \theta(x, x')(a)(z, y'),$$

which implies that $\theta(x, x')(a)(y, z') \leq \delta(x)(a)(y)$ and $\theta(x, x')(a)(z, y') \leq \delta'(x')(a)(y')$, for any $(y, z'), (z, y') \in R$. In particular, $\theta(x, x')(a)(y, y') \leq \delta(x)(a)(y)$ and $\theta(x, x')(a)(y, y') \leq \delta'(x')(a)(y')$, and thus

$$\theta(x, x')(a)(y, y') \leq \delta(x)(a)(y) \wedge \delta'(x')(a)(y').$$

(iv) For $(x,x') \in R$ and $y \in X$, we have

$$\delta(x)(a)(y) = \bigvee_{\substack{y' \in X' \\ (y,y') \in R}} \theta(x,x')(a)(y,y') \le \bigvee_{\substack{y' \in X' \\ (y,y') \in R}} \delta'(x')(a)(y')$$

by (iii).

(vi) Let $(x,x') \in R$. By (iv), we have

$$\bigvee_{y \in X} \delta(x)(a)(y) \le \bigvee_{y \in X} \bigvee_{\substack{y' \in X' \\ (y,y') \in R}} \delta'(x')(a)(y') \le \bigvee_{y' \in X'} \delta'(x')(a)(y').$$

By (v), we have

$$\bigvee_{y' \in X'} \delta'(x')(a)(y') \le \bigvee_{y' \in X'} \bigvee_{\substack{y \in X \\ (y,y') \in R}} \delta(x)(a)(y) \le \bigvee_{y \in X} \delta(x)(a)(y).$$

And thus $\bigvee_{y \in X} \delta(x)(a)(y) = \bigvee_{y' \in X'} \delta'(x')(a)(y')$. □

5.5.1 Quotient fuzzy automata

Let R be a *bisimulation equivalence* on an $\mathcal{F}$-coalgebra $(X, \langle \tau, \delta \rangle)$, that is, a bisimulation between $(X, \langle \tau, \delta \rangle)$ and itself which is also an equivalence relation on X. Given $y \in X$, recall that the set $\{y' \in X \mid (y,y') \in R\}$ is the R-equivalence class of y, denoted by $[y]$. Thus, many results in Theorem 5.20 can be written in terms of equivalence classes. Moreover, the following theorem states two equalities which will allow us to describe the *quotient coalgebra* $(X/R, \langle \tau, \delta \rangle_{/R})$ of $(X, \langle \tau, \delta \rangle)$ by R.

Theorem 5.21. *Let R be a bisimulation equivalence on an $\mathcal{F}$-coalgebra $(X, \langle \tau, \delta \rangle)$. Then, for all $x \in X$,*

(i) $\tau(x_1) = \tau(x_2)$, *for any $x_1, x_2 \in [x]$;*

(ii) $\bigvee_{y' \in [y]} \delta(x_1)(a)(y') = \bigvee_{y' \in [y]} \delta(x_2)(a)(y')$, *for any $x_1, x_2 \in [x]$, $a \in A$ and $y \in X$;*

where $[x]$ denotes the R-equivalence class of x.

Proof. Let $x \in X$. (i) If $x_1, x_2 \in [x]$, then $(x_1, x_2) \in R$ and thus

$$\tau(x_1) = \tau(x_2),$$

since R is a bisimulation on $(X, \langle \tau, \delta \rangle)$.

(ii) Let $x_1, x_2 \in [x]$, $a \in A$, and $y \in X$. Since $(x_1, x_2) \in R$, for any $z \in [y]$,

$$\delta(x_1)(a)(z) \le \bigvee_{y' \in [z]} \delta(x_2)(a)(y') = \bigvee_{y' \in [y]} \delta(x_2)(a)(y'),$$

by Theorem 5.20(iv). Thus,

$$\bigvee_{y' \in [y]} \delta(x_1)(a)(y') \leq \bigvee_{y' \in [y]} \delta(x_2)(a)(y').$$

Similarly,

$$\bigvee_{y' \in [y]} \delta(x_2)(a)(y') \leq \bigvee_{y' \in [y]} \delta(x_1)(a)(y'),$$

due to $(x_2, x_1) \in R$. Therefore, $\bigvee_{y' \in [y]} \delta(x_1)(a)(y') = \bigvee_{y' \in [y]} \delta(x_2)(a)(y')$. □

Now, we describe the $\mathcal{F}$-transition structure for the quotient of an $\mathcal{F}$-coalgebra by a bisimuletion equivalence on it.

Theorem 5.22. *Let R be a bisimulation equivalence on an $\mathcal{F}$-coalgebra $(X, \langle \tau, \delta \rangle)$. Define $\tau_{/R} = \tau/R : X/R \to K$ by*

$$\tau_{/R}([x]) = \tau(x), \qquad [x] \in X/R$$

and $\delta_{/R} : X/R \to (K^{X/R})^A$ by

$$\delta_{/R}([x])(a)([y]) = \bigvee_{y' \in [y]} \delta(x)(a)(y'), \qquad [x], [y] \in X/R, a \in A.$$

Then, $\langle \tau_{/R}, \delta_{/R} \rangle$ is the unique function from $X/R \to K \times (K^{X/R})^A$ that makes the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad q \quad} & X/R \\ {\scriptstyle \langle \tau, \delta \rangle}\downarrow & & \downarrow{\scriptstyle \langle \tau_{/R}, \delta_{/R} \rangle} \\ K \times (K^X)^A & \xrightarrow[\mathrm{id}_K \times (\mathcal{Z}(q))^A]{} & K \times (K^{X/R})^A \end{array}$$

commute, i.e. $\langle \tau_{/R} \circ q, \delta_{/R} \circ q \rangle = \langle \tau, (\mathcal{Z}(q))^A \circ \delta \rangle$, where q denotes the quotient function. Therefore, $(X/R, \langle \tau_{/R}, \delta_{/R} \rangle)$ is the quotient coalgebra of $(X, \langle \tau, \delta \rangle)$ by R.

Proof. First, if $[x_1], [x_2] \in X/R$ with $[x_1] = [x_2]$, that is $(x_1, x_2) \in R$, then

$$\tau_{/R}([x_1]) = \tau(x_1) = \tau(x_2) = \tau_{/R}([x_2]),$$

by Theorem 5.21(i), and

$$\delta_{/R}([x_1])(a)([y]) = \bigvee_{y' \in [y]} \delta(x_1)(a)(y') = \bigvee_{y' \in [y]} \delta(x_2)(a)(y') = \delta_{/R}([x_2])(a)([y]),$$

for all $a \in A$ and $[y] \in X/R$, by Theorem 5.21(ii). Hence, $\tau_{/R}$ and $\delta_{/R}$ are both well-defined functions.

Finally, for any $x \in X$, $a \in A$ and $[y] \in X/R$, we have

$$\begin{aligned}((\mathcal{Z}(q))^A \circ \delta)(x)(a)([y]) &= \bigvee_{y' \in X} \delta(x)(a)(y') \otimes \eta_{X/R}(q(y'))([y]) \\ &= \bigvee_{y' \in X} \delta(x)(a)(y') \otimes \eta_{X/R}([y'])([y]) \\ &= \bigvee_{y' \in [y]} \delta(x)(a)(y') = \delta_{/R}([x])(a)([y])\end{aligned}$$

(note that $[y'] = [y]$ if, and only if, $y' \in [y]$), which implies that $(\mathcal{Z}(q))^A \circ \delta = \delta_{/R} \circ q$. Thus, $\langle \tau_{/R} \circ q, \delta_{/R} \circ q \rangle = \langle \tau, (\mathcal{Z}(q))^A \circ \delta \rangle$. And so, by [Rut00, Proposition 5.8], $\langle \tau_{/R}, \delta_{/R} \rangle$ is the unique function from X/R to $K \times (K^{X/R})^A$ such that

$$\langle \tau_{/R}, \delta_{/R} \rangle \circ q = (\mathrm{id}_K \times (\mathcal{Z}(q))^A) \circ \langle \tau, \delta \rangle.$$

□

In the conditions of the previous theorem, we remark that $\langle \tau_{/R}, \delta_{/R} \rangle$ is the unique function from X/R to $K \times (K^{X/R})^A$ which makes q a homomorphism of $\mathcal{F}$-coalgebras from $(X, \langle \tau, \delta \rangle)$ to $(X/R, \langle \tau_{/R}, \delta_{/R} \rangle)$.

Although the cardinality of X/R is lesser than or equal to the cardinality of X, we have that the $\mathcal{F}$-coalgebras $(X, \langle \tau, \delta \rangle)$ and $(X/R, \langle \tau_{/R}, \delta_{/R} \rangle)$ can recognize exactly the same fuzzy languages in the following sense.

Theorem 5.23. *Let R be a bisimulation equivalence on an $\mathcal{F}$-coalgebra $(X, \langle \tau, \delta \rangle)$. And let $(X/R, \langle \tau_{/R}, \delta_{/R} \rangle)$ be the quotient $\mathcal{F}$-coalgebra as defined in Theorem 5.22. Also, by Theorem 5.16, consider the $\widehat{\mathcal{D}_K}$-coalgebras $((K^X, \mu_X), \langle \overline{\tau}, \overline{\delta} \rangle)$ and $((K^{X/R}, \mu_{X/R}), \langle \overline{\tau_{/R}}, \overline{\delta_{/R}} \rangle)$ obtained from $(X, \langle \tau, \delta \rangle)$ and $(X/R, \langle \tau_{/R}, \delta_{/R} \rangle)$, respectively. Given the homomorphism of $\widehat{\mathcal{D}_K}$-coalgebras*

$$\ell_X : ((K^X, \mu_X), \langle \overline{\tau}, \overline{\delta} \rangle) \to ((K^{A^*}, \mu_{A^*}), \langle \epsilon, t \rangle)$$

and

$$\ell_{X/R} : ((K^{X/R}, \mu_{X/R}), \langle \overline{\tau_{/R}}, \overline{\delta_{/R}} \rangle) \to ((K^{A^*}, \mu_{A^*}), \langle \epsilon, t \rangle),$$

by Theorem 5.19, then

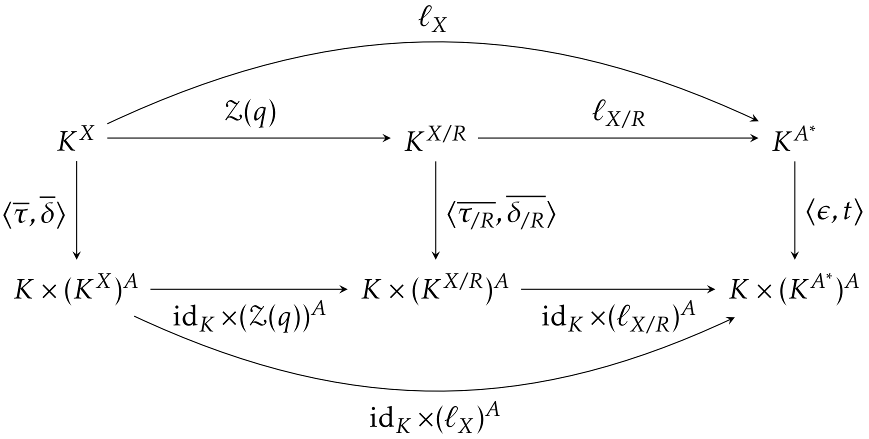
$$\{\ell_X(\sigma) \mid \sigma \in K^X\} = \{\ell_{X/R}(\sigma') \mid \sigma' \in K^{X/R}\}.$$

Furthermore, $\ell_X(\sigma) = \ell_{X/R}(\mathcal{Z}(q)(\sigma))$ and $\ell_{X/R}(\sigma') = \ell_X(\sigma' \circ q)$, for every $\sigma \in K^X$ and $\sigma' \in K^{X/R}$.

Proof. By Theorem 5.22, we have that the quotient function q is a homomorphism of $\mathcal{F}$-coalgebras from $(X, \langle \tau, \delta \rangle)$ to $(X/R, \langle \tau_{/R}, \delta_{/R} \rangle)$. Then, by Theorem 5.16, $\mathcal{Z}(q)$ is a homomorphism of $\widehat{\mathcal{D}_K}$-coalgebras from $((K^X, \mu_X), \langle \overline{\tau}, \overline{\delta} \rangle)$ to $((K^{X/R}, \mu_{X/R}), \langle \overline{\tau_{/R}}, \overline{\delta_{/R}} \rangle)$, and so we have a homomorphism of $\widehat{\mathcal{D}_K}$-coalgebras

$$\ell_{X/R} \circ \mathcal{Z}(q) : ((K^X, \mu_X), \langle \overline{\tau}, \overline{\delta} \rangle) \to ((K^{A^*}, \mu_{A^*}), \langle \epsilon, t \rangle).$$

Since ℓ_X is unique by Theorem 5.19, then $\ell_X = \ell_{X/R} \circ \mathcal{Z}(q)$, that is, the diagram



commutes. In particular, $\ell_X(\sigma) = \ell_{X/R}(\mathcal{Z}(q)(\sigma))$, for all $\sigma \in K^X$, which implies that

$$\{\ell_X(\sigma) \mid \sigma \in K^X\} \subseteq \{\ell_{X/R}(\sigma') \mid \sigma' \in K^{X/R}\}.$$

On the other hand, given $\sigma' \in K^{X/R}$, we have that

$$\mathcal{Z}(q)(\sigma' \circ q) = \bigvee_{x\in X} \sigma'(q(x)) \circledast \eta_{X/R}(q(x)) = \bigvee_{x\in X} \sigma'([x]) \circledast \eta_{X/R}([x]),$$

and thus,

$$\mathcal{Z}(q)(\sigma' \circ q)([x']) = \bigvee_{x\in X} \sigma'([x]) \otimes \eta_{X/R}([x])([x']) = \bigvee_{x\in[x']} \sigma'([x]) = \sigma'([x'])$$

(note that $[x] = [x']$ for all $x \in [x']$), for any $[x'] \in X/R$. Hence, $\mathcal{Z}(q)(\sigma' \circ q) = \sigma'$. Therefore, for any $\sigma' \in K^{X/R}$, we have

$$\ell_X(\sigma' \circ q) = \ell_{X/R}(\mathcal{Z}(q)(\sigma' \circ q)) = \ell_{X/R}(\sigma'),$$

which implies that

$$\{\ell_{X/R}(\sigma') \mid \sigma' \in K^{X/R}\} \subseteq \{\ell(\sigma) \mid \sigma \in K^X\}.$$

Consequently, $\{\ell_X(\sigma) \mid \sigma \in K^X\} = \{\ell_{X/R}(\sigma') \mid \sigma' \in K^{X/R}\}$. □

From the previous theorem, given a fuzzy automaton $\mathcal{A} = (X, \sigma, \tau, \delta)$ and a bisimulation equivalence R on the $\mathcal{F}$-coalgebra $(X, \langle\tau, \delta\rangle)$, we can construct a fuzzy automaton $\mathcal{A}/R = (X/R, \mathcal{Z}(q)(\sigma), \tau_{/R}, \delta_{/R})$ which is (language) equivalent to $\mathcal{A}$ and may have fewer states.

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