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Impact of Black-Scholes Assumptions on Delta Hedging

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Abstract

In this work we are going to evaluate the different assumptions used in the Black-Scholes-Merton pricing model, namely log-normality of returns, continuous interest rates, inexistence of dividends and transaction costs, and the consequences of using them to hedge different options in real markets, where they often fail to verify. We are going to conduct a series of tests in simulated underlying price series, where alternatively each assumption will be violated and every option delta hedging profit and loss analysed. Ultimately we will monitor how the aggressiveness of an option payoff causes its hedging to be more vulnerable to profit and loss variations, caused by the referred assumptions.

Keywords

Black-Scholes-Merton Model, Assumptions, Delta Hedging, Hedging Quality, Profit and Loss, Option Payoffs, Market Discontinuities, Volatility, Interest Rate, Dividends, Transaction Costs

Contents

Acronyms

ATM At the Money

BS model Black-Scholes Model

CBOE Chicago Board Options Exchange

GBM Geometric Brownian Motion

ITM In the Money

OTM Out of the Money

OTC Over-the-Counter

PnL Profit and Loss

STD Standard Deviation

Chapter 1

Introduction

Options are nowadays a very useful financial instrument, both for speculative and hedging purposes. Call options began trading in a standardised manner right after the Chicago Board Options Exchange (CBOE) was established in 1973. Still in 1973, Fisher Black and Myron Scholes published a paper [1] where they extended a set of assumptions made by Robert Merton earlier that year, to describe a simple model, which would calculate the fair value of an option. This model revealed to be incredibly useful in financial markets, and is still considered to be the principal method to price options. Since then the trading volume of these have increased substantially, and in 2014 the number of traded option contracts in the CBOE was once again the highest ever, with an average of more than 5 million option contracts traded per day, equivalent to an annual dollar volume of 580 trillion.

Nevertheless the CBOE offers just a fraction of the options present in financial markets. Besides calls and put options, a vast number of more complicated options have appeared since 1973. These options evolved with markets' complexity, designed to suit a variety of different financial institutions' and investors' needs, the majority being traded over-the-counter (OTC), some becoming very complex, with payoffs dependent on a number of variables. According to the Bank for International Settlements the notional amount outstanding for options within the global OTC derivatives market in the first half of 2015 was 59 trillion US dollars. These have much less liquidity, in fact a considerable percentage of them are tailor made, traded one time and held until maturity.

Although almost every bank, hedge-fund, pension-fund, insurer as well as certain investors deal with OTC options, not everyone is able or willing to produce and hedge them. Because of these options' complexity, manufacturing them using market's securities to hedge their payoffs has gradually become a specialised field of finance. That is because the models used to hedge these securities, such as the Black-Scholes model, fail to have their assumptions verified in real market conditions, causing the final profit from selling these products to be uncertain. For that reason only a smaller portion of banks do produce these options. By scaling their derivatives portfolios to mitigate the risks of isolated options, these banks manage to produce more exotic payoffs with low risk and distribute them to clients or other financial institutions, which in turn may or may not trade them.

In this work we are going to explore this problem by focusing on equity options, more specifically on single underlying, single observation options, and on the problem of hedging them given the real market scenario. We are going to assess the option hedging quality produced by the Black-Scholes model, the most used model for this purpose, given that some of its assumptions fail to verify, and evaluate which of them produce the most severe hedging skew across the life of different option payoffs.

1.1 Related Work

Delta hedging in general and how to effectively do it has been subject of study for a number of years. The incredible number of variables affecting it and the random nature of markets, makes this topic very complex. Some years after the publication of the original Black and Scholes paper, Galai [2] assessed the market efficiency of the CBOE.

More recently an analysis of delta hedging using the Black and Scholes framework was done by Gillula [3], where he examines the option liquidity problem, caused when a dealer trades a large volume of options moving the market. Another interesting delta hedging study was done by Furst [4] which explored the effects of not knowing the true underlying volatility. Arthur Sepp from Bank of America Merrill Lynch conducted a similar delta hedging PnL study on the trade off between hedging frequency and transaction costs [5].

1.2 Overview

This thesis is divided into three main parts, that will allow us to understand the problem of hedging an option, the methodology used to evaluate its quality and the obtained results. In this introductory chapter we briefly presented how extensive the option market has become, as well as its structure. In the next chapter we will briefly go through the Black-Scholes model, how it works, its assumptions and how these fail to verify in real market conditions. Chapter 3 will contain the detailed methodology used to test the model and assess the hedging quality. In the fourth chapter, an explanation of how each of the models' assumptions were tested can be found, as well as the quantitative evaluation of their impact on hedging quality. As for the final chapter, we will conclude by analysing the results of chapter 4, as well as suggesting future work that can be developed as a continuation of this work project.

Chapter 2

The Black-Scholes Model

Options existed much before the BS model was developed, and in fact there were a number of models available to price them and calculate their expected payoff, the majority however relied on arbitrary parameters. For example Sprenkle [6] derived a formula to calculate the value of warrants in 1965, which was similar to the BS model's formula, only that it had two unknown input parameters that traders would have to estimate, such as the potential growth of the stock price until maturity and a discount factor according to the stock's risk. Nonetheless it also used a similar set of assumptions as the BS model, very much like various other previous researchers had done.

The major breakthrough however came when Black and Scholes [1] used the capital asset pricing model to derive a formula to value calls and put options, while Merton [7] demonstrated that the return on a portfolio consisting of options and their underlyings would be the risk free rate, thus not existing arbitrage opportunities when holding it. The full set of assumptions for european options are:

a) The risk free interest rate is known, constant through time and the same for all maturities;
b) The stock price follows a random walk in continuous time, with normal distributed returns and constant volatility;
c) There are no dividends nor other distributions paid by the underlying stock;
d) Security trading is continuous and there are no transactions costs nor taxes;
e) All securities are perfectly divisible and it is possible to borrow at the risk free rate;
f) There are no penalties for short selling and full use of its proceeds is allowed.

In general some of these assumptions are not just used in the BS model but in several areas of finance as well. Despite most of them not being verifiable in financial markets, they can sometimes taken to be true as a workaround in order to simplify the mathematics involving financial formulae. Still some of these assumptions have been relaxed with the development of more extended models that considers a certain problem. For example the next model is a small extension of the original Black-Scholes-Merton formula to calculate the value of a call option, which can be derived through several ways that we will not cover here, but can be found in [1] (through variation of the option price as a function of its underlying, and through the capital asset pricing model) and [8] (from the binomial tree option pricing model):

$$C = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) \tag{2.1}$$

$$with \quad d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \quad , \quad d_2 = d_1 - \sigma\sqrt{T}$$

where S is the underlying spot price, K the option strike price, T the time to expiry, r the risk free rate, σ the underlying's volatility and δ its dividend yield. Here the dividend yield δ is added to the original BS model in order to relax assumption c), considering that δ is the stock dividend yield payout throughout the life of the option.

If we take δ to be zero we can rearrange the terms to better relate 2.1 to a call option payoff according to:

$$C = e^{-rT}(Se^{rT} N(d_1) - KN(d_2)) \quad \Longrightarrow \quad C = e^{-rT} \hat{E}[\max(0, S_T - K)]$$

to observe that the value of the call in a risk neutral world would be its expected payoff discounted at the risk free rate, as the term $Se^{rT}N(d_1)$ is the expected value of a variable that is equal to S_T or to 0 depending on whether the option was exercised or not, and $KN(d2)$ the probability that the strike price will have to be paid at maturity.

Other types of european options can be derived using this formula. The european put option BS model formula is the negative of (2.1) with d_1 and d_2 both negative, and the cash-or-nothing call option is similar to the absolute of the rightmost part of (2.1), following the analogy just made in the previous chapter, thus with K equal to the cash amount paid by the option. These basic three options, call, put and binary can be used and added up to calculate more complex options, as long as observing solely at maturity.

2.1 Delta Hedging

Following the premise of Black and Scholes, if options are correctly priced it should not be possible to arbitrage profits out of combining long and/or short positions in options and respective underlyings. In fact as first suggested by Merton, it is possible to hedge an option by holding a portfolio with a variable amount of stock over time, and guarantee its return would match the one of the option over its lifespan. This is called delta hedging and is practised by option sellers.

When an option is sold, the seller typically receives the premium and delivers an option contract to the buyer. If it is the case that the option was originally produced by the seller, then he is obliged to guarantee the option payoff to its counterpart. In the case of a stock option, its value is linked to the underlying until expiry according to (2.1), and will eventually reach maturity in or out of the money. For the most part, option sellers do not want to take the risk of betting that the option will mature worthless and prefer to hedge the option payoff as suggested by Merton, in order to guarantee the option premium they just received.

According to the BS model and from (2.1) we can see that, with $\delta = 0$ and all variables held constant, the option will change in value $N(d_1)$ times the change in price of the underlying at any given time, as described by:

$$\Delta = \frac{dC}{dS} = N(d_1) \tag{2.2}$$

This basically means that if at any given moment the seller holds Δ amount of the underlyings' stock, he will be delta hedged and indifferent to any movement the underlying might have at that particular moment. According to the Black and Scholes assumptions if the seller does this continuously, by at all times having Δ amount of stock and the remainder borrowed or invested at the risk free rate throughout the life of the option, than the return of the hedging portfolio will match the one of the option.

The delta, here stated as Δ, is usually expressed in percentage points. For example, a Δ of 15% on a 70\$ strike, 90\$ call option on 10 shares of an underlying currently trading at 35\$, means that according to the BS model, one should own at that time $15\% \times 10 \times 70\$ = 105\$$ of stock, or $105\$/35\$ = 3$ shares of the underlying, and borrow $105\$ - 90\$ = 15\$$ at the risk free rate for that purpose. We can easily demonstrate such time step demonstration of delta hedging using the binomial tree [9].

2.2 Assumptions

From the mechanics of delta hedging we rapidly understand why some of the Black and Scholes assumptions exist in the first place. Starting from the bottom, assumption f) is a real obstacle as it requires collateral posting and a counterpart willing to borrow the stock for the desired period which often is not the case. However in today's markets, for big financial institutions and particularly for liquid underlyings it is not a problem that would affect the quality of hedging. Assumption e) is also not much of a problem since option sellers often benefit from having vast portfolios with options on sizable notionals, and with common shared underlyings between them. So not being able to buy a fraction of a 90\$ share when the dollar delta is $10,000\$$, meaning that 10\$ or 10 basis points of δ are unhedged is not a major concern.

However assumptions a), b), c) and d) are a real concern to option sellers, since there is little that they can do, apart from considering different models, diversifying underlyings and payoffs, hedging with more options or ultimately raising option prices to an amount which they think portrays the risks of imperfect hedging. In this work we attempt to grasp the effect of these four assumptions on the delta hedging quality of different payoffs, and how is the quality of the hedge affected if these fail to verify, here is why:

Interest Rates Are never constant and in fact vary substantially according to the state of the economy and monetary police, as seen recently.

Normality of Returns Despite returns' distribution approximating a normal one over a long period of time, they tend to present fat tails and a slight skew related to abnormal events. The volatility is also not at all constant, and as one of the main variables affecting the value of an option, its miscalculation can have big impact on hedging.

Dividends Despite having relaxed assumption c) to include dividends, they are still paid at discrete times, also there are sometimes companies that alter the payout strategy. This can have considerable impacts on option prices with such underlyings, specially with dividend cuts, since it may or may not make an option instantly in or out of the money.

Transaction Costs Markets are not frictionless, and every transaction is sure to pay a fee and a bid-ask spread, which is counterintuitive to the idea of continuous rebalancing. They are known beforehand but it is still quite difficult to find the optimum trade off between trading frequency and hedging quality [5].

2.3 Geometric Brownian Motion

For more complex path dependent options, the regular BS model does not apply. However it is still possible to price them by simulating a high number of underlying price paths until maturity, according to the model's assumptions, calculating the payoff of each one, and discounting the average at the risk free. This methodology is known as Monte Carlo.

In assumption b) it is stated that prices follow a random walk, meaning that future prices are unaffected by the present or the past prices. The normality of returns and log-normality of stock prices can then be simulated through:

$$S_{t+\Delta t} = S_t \exp\left[\left(\mu_t - \delta_t - \frac{\sigma_t^2}{2}\right)\Delta t + \sigma_t W_t \sqrt{\Delta t}\right] \tag{2.3}$$

which relies on Black and Scholes assumptions to simulate a price of a given stock with current price S_t, volatility σ_t and dividend yield δ_t in Δt units of time. The growth rate μ_t is taken to be the risk free rate and W is a normal random variable called a wiener process, which guarantees that the geometric brownian motion (GBM) has a zero expected change and a variance rate of one per year, just like a regular stock, as in assumption b).

Chapter 3

Methodology

Throughout the previous chapter we got to understand how the BS model works, the reason behind its assumptions and the importance of each of them for option pricing. We also got to understand how does one hedge an option payoff, but also how to calculate the value of more complex options through GBM. In order to assess how the hedging quality can be prejudiced if the model's assumptions fail to verify, it is very important to choose the desired options in which we are going to test the model beforehand. That is what we will start by doing in this chapter, such as explaining the characteristics of each option, and why they were thought to be appropriate for analysing each of the BS model's assumptions. After that follows a more practical step by step explanation of the methodology used to do so.

3.1 Options Used

One of the goals of this work, besides quantifying the severeness of the BS model's assumptions under real market conditions, is to comprehend how different options behave if each of the assumptions fail to verify. Here we opted to test the hedging quality in four distinct options with four distinct payoffs, them being, a fixed strike asian call option, a vanilla call option, a binary call option and an up-and-in call option, explained next. We chose these options since they are broadly used and popular in the financial industry. They have distinct characteristics, which make their price and delta react differently for the same price movement on the underlying, which ultimately may or may not lead to an impact on the results. These were divided based on how they react to market changes:

Smooth Fixed Strike Asian Call
Regular Vanilla Call
Aggressive Binary Call, Up-and-In Call

Take note that the mentioned reactions also depend on the intrinsic value of the option, commonly referred to as moneyness, as well as the time to maturity along with a number of other factors. In order to guarantee a fair analysis across the mentioned options, we will impose before testing that all of them have the same maturity, start at-the-money (ATM) and have the same value or price at $t = 0$, to which the asian call option's will be used as a reference. To better understand these four options and why they were used, a brief explanation on each of them follows. For this analysis we are just going to consider single underlying, single observation European options, for reasons explained in the end of this section.

3.1.1 Fixed Strike Asian Call Option

Asian options are call or put options that take into consideration some type of average of the underlying price throughout a number of fixed predefined observations. There are two types of asian options, average price options which pay the spread, if any, between the average underlying price and a fixed strike, and average strike options which pay the spread, if any, between the maturity underlying price and the average underlying price. For simplicity and analogy to the other options used, we are going to consider average price asian call options, whose payoff formula is:

$$C_{AveragePrice}^{Asian} = \max(0, S_{avg} - K) \tag{3.1}$$

where K is the fixed strike and S_{avg} is the average underlying price at maturity. We will consider the average to be the arithmetic average of the daily underlying close prices during the life of the option and the strike the initial underlying price, as in:

$$S_{avg} = \frac{1}{T}\sum_{t=1}^{t=T} S_t \quad and \quad K = S_0 \tag{3.2}$$

where T is the total number of days that the option will last, S_t is the close price of day t and S_0 is the close price at $t = 0$.

There are two important aspects about this option. First, since the average smooths out the underlying's volatility and thus the volatility of the terminal payoff, asian option's delta and intrinsic value variations are more subtle than those of the options presented below. In fact these variations become smoother as the number of averaging observations increases. This is why we consider asian options to be very relevant for this study, as they set a benchmark on how precise can the delta hedging be under certain conditions, given how robust they are to market changes. The second aspect is that asian options are path dependent, meaning that their payoff depends not only on the maturity price but also on the course of the underlying price throughout the life of the option. Hence the Black-Scholes closed formula solution is not suitable to price these options. One should simulate instead a very high number of geometric brownian paths as described in section 2.3. After doing so we can value the asian option at that moment, considering both past and future simulated prices according to:

$$C_t^{AveragePrice} = \langle \max(0, S_{avg}^n - K) \rangle \quad S_{avg}^n = \frac{1}{T}\left(\sum_{i=1}^{i=t} S_i + \sum_{j=t+1}^{j=T} S_j^n\right) \quad n = 1, \dots, N \quad (3.3)$$

where t is the valuation day, S_i is the close price of past day i, S_j^n is the path's n simulated price for day j and N the total number of simulated paths. Bear in mind that when using Monte Carlo for daily price simulation one must take Δt in (2.3) to be one day and simulate S_i for each day consecutively from $t+1$ to T. An advantage of this method is that one can make use of varying (2.3) inputs, σ, δ and μ for different days, according to the market. However such daily price simulation is computationally expensive, as it requires dozens of thousands of every day simulations for an accurate price.

Asian options are very popular among investors and asset managers, since they are able to guarantee a good upside while being cheaper than a regular call option, precisely due to their lower volatility as mentioned above.

3.1.2 Vanilla Call Option

Vanilla options are the most standard options in finance, as suggested by their name. They observe the underlying price at maturity and pay the difference, if any, to the fixed strike depending on whether it is call or a put. Much like the asian option we are going to consider a vanilla call option, however in order to guarantee that the vanilla option has the same starting value as the asian, we shall add a parameter to the payoff, known to finance as leverage that will be used to adjust the initial price of the vanilla call option, as in:

$$C^{Vanilla} = leverage \times \max(0, S_T - K) \quad (3.4)$$

Generally, as one could tell through the BS model's formula if the underlying's volatility is higher the option price will be higher as well. Since the asian option smooths out the underlying volatility as previously mentioned, the call option will be more expensive *ceteris paribus*. For that reason $leverage$ should in this case be expected to be lower than 100%, as in Fig. 3.1(a) where $leverage = 70\%$.

3.1.3 Binary Call Option

Binary options are the easiest to understand. The call version of these only pays a given predefined amount should the underlying price reach maturity above the option's strike, they are also referred to as cash-or-nothing options for this reason. The payoff is:

$$C^{Binary} = Coupon \quad if \quad S_T > K \quad (3.5)$$

where $Coupon$ will be our variable to set the binary call option value equal to the asian's at $t = 0$. In Fig. 3.1(b) a binary call option payoff with $S = 100$ and $Coupon = 20$ is shown.

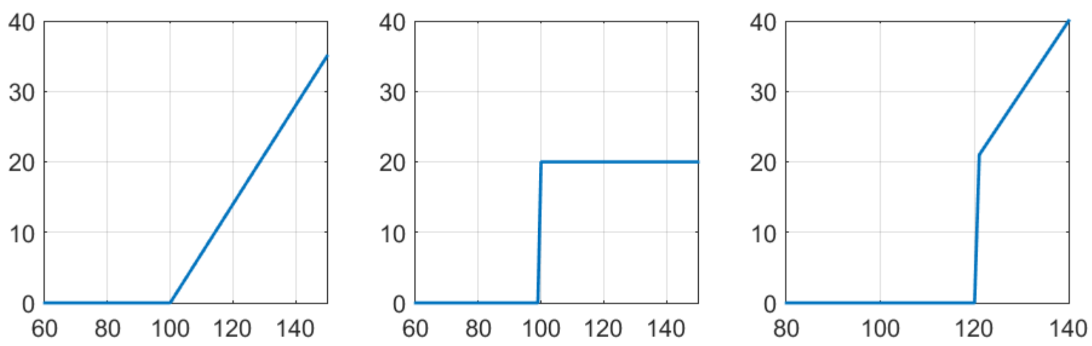
One of the motifs to feature binary and vanilla options, is because they are among the most traded both OTC or through the CBOE. Nevertheless, as seen above each was assigned to different reaction categories, aggressive and regular respectfully. As seen the in Fig. 3.1(a) and (b), the vanilla call option has a subtler payoff transition between being out-of-the-money (OTM) and in-the-money (ITM), than the binary call option, which either pays or not. This will cause underlying price moves around the strike to cause wider delta changes in the later, that may or not exacerbate hedging profit and loss (PnL) during that period.

3.1.4 Up-and-In Call Options

Besides the strike price, barrier call and put options can have two types of price barriers, either knock-in which if not reached cause the payoff to be zero, or knock-out which pay zero if the barrier is reached at any time during the life of the option. On top of that, if the barrier is above the strike it is called an up-barrier otherwise it is called a down-barrier. In order to check whether the barrier price was reached, these options can have the underlying price observed at maturity, continuously or at multiple observations throughout the life of the option, as stipulated in its contract. In this work we made use of an up-and-in call option observing at maturity, meaning that the option pays the spread, if any, between the strike and the underlying price, if and only if the underlying price is higher than the barrier price, as displayed in Fig. 3.1(c), following:

$$C^{UpAndIn} = \max(0, S_T - K) \quad iff \quad S_T > H \tag{3.6}$$

where H is the price barrier. In this work we take the barrier H to be greater than the



(a) Vanilla Call (b) Binary Call (c) Up-and-In Call

Figure 3.1: Option Payoffs at Maturity.
x axis - Underlying Price, y axis - Payoff w/ strike at 100, and barrier at 120 for c)

strike K, notice that if $H = 0$ this option would behave exactly like the vanilla call. As with the previous two options, we will make use of a parameter to set the initial price of the option identical to the asian, which in this case will be precisely the barrier H.

A slight variation of this option, a short down-and-in put known as reverse convertible, is popular in OTC option markets. One version of the reverse convertible, has its payoff essentially equal to Fig. 3.1(c) only inverted through the x and y axis. Such option has only potential downside so an additional coupon is guaranteed to the investor, bound to receive it entirely should the underlying price be above the barrier at maturity.

Much like the binary option explained above up-and-in calls have an aggressive behaviour with varying market conditions for underlying prices close to the barrier.

The reason of only having call options, which ought to gain value should the underlying price go up, considered for the tests is really not relevant for this analysis, as we could as well use puts, long or short options. What is important is that all options value evolve identically according to the underlying price, meaning that for the same underlying price move, all options' price should move in similar direction, whether it matches the underlying's or not. For the same reason we considered all options to share a single underlying as well as only observing at maturity, thus European. This way we can construct a more robust analysis on Black-Scholes assumptions, when comparing different outcomes during the identical lifespan of the four options. We can also be more confident relating possible results to the aggressiveness of their payoffs, rather than underlying characteristics and observation timing, which may or may not interfere on the quality of delta hedging.

3.2 Underlying Price Series

For the purpose of evaluating the impact of the assumptions used in the BS model, it is pivotal that we are able to induce assumption violations on the underlying throughout the duration of the option, as well as to control the severity and timing of those violations. This hampers the use of actual equities traded in the stock market, since it is extremely difficult to find real historical prices in a given market condition, that contain the several violations that we want to assess. A good alternative is to simulate these price series using GBM, which allow us to control the characteristics of the underlying price series, such as volatility and interest rates, that would otherwise be impossible. Dividends and transaction costs although easier to control using real market data, can also be faithfully and easily reproduced using GBM, even to a higher precision. Remember that for the sake of this analysis, no importance is given to the underlying itself and no requirement exists regarding the use of real market data.

We will make use of the formula in (2.3) to simulate daily prices for the time period we want to consider, similarly as what we would do to value an asian option, but only taking the simulated prices as the ones we will use to value and hedge the four options. To reproduce stock prices in this manner, we will have to feed the GBM with inputs such as volatility, interest rates and dividend yield, which are used to model the intrinsic characteristics of the price motion. These inputs can be real ones found in the market or in any particular stock but also fabricated according to our testing requirements. In the next chapter we will always mention what kind of inputs were used in a particular results' table or figure.

In addition to the referred inputs we also have to fabricate a normal distributed wiener process W, with zero mean and unitary variance. This random variable will be responsible for the random nature of stock prices. In this work we will simulate 20,000 underlying price paths, figure commonly used in option pricing software and in the financial industry, which means we will have to create an array of 20,000 normally distributed numbers for each of the T days of the option's life. To do so we will take the inverse cumulative standardised normal distribution function of a sequence of equally spaced numbers between $1/(20,000+1)$ and $20,000/(20,000+1)$, which we will then randomly shuffle for each day, and use in the GBM paths. It is very important that these random numbers are held constant throughout the tests, so as to guarantee that the results differ only due to different inputs and payoffs. Take a look at Fig.(B.1) to see different simulated price paths for different inputs.

Creating and analysing 20,000 underlying price paths enable us to cover a wide range of price behaviours, including the most extreme ones encountered at the tails of the normal distribution. Altogether they should provide us a picture of how the quality of delta hedging across different options is jeopardised or not when inputs vary.

3.3 Impact on Delta Hedging Quality

In the beginning of this work we ought to measure how the Black and Scholes assumptions' impact the hedging of an option portfolio, when these assumptions are just to expensive to make. We will do it by evaluating how the hedging of the different options is affected, thus how faithful is the hedging portfolio to the sold option portfolio. Take note that the hedge portfolio should evolve exactly alike the options' portfolio, having similar day to day returns, so at maturity it has generated the capital to cover the expense of the payoff, if any. The day t return of the option portfolio is just:

$$R_t^{Option} = \frac{C_t - C_{t-1}}{C_{t-1}} \tag{3.7}$$

where C_t and C_{t-1} are how much the option portfolio is worth at t and $t-1$. Remember that for the sake of testing we might use an option portfolio of only one option with a single underlying, which is what we chose to do. If that is the case the hedging portfolio day t return is:

$$\begin{aligned} R_t^{Hedge} = & \left(\frac{S_t - S_{t-1}}{S_{t-1}} + \delta_t x\right)\Delta_t K + \\ & (C_0 - \Delta_t K)(1+r_t)^{\tau} - (C_0 - \Delta_t K) + \\ & b\|\Delta_t - \Delta_{t+1}\|K + f \end{aligned} \tag{3.8}$$

where S_t and S_{t-1} are the underlying price at t and $t-1$, δ_t is the dividend payout if any at time t, Δ_t is the delta of the option and K the strike price of the option, so that $\Delta_t K$ is the amount invested in underlying's stock at that moment. This first row is the equity return of the hedge portfolio. The following row respects the interest rate return as C_0 is the initial price of the option, equivalent to the premium received at $t = 0$, so $(C_0 - \Delta_t K)$ is the money we have to borrow or invest at the risk free rate, τ is the period between last and current valuation, in this case one day, and r_t is the risk free rate verified for that period. The last row are the transaction costs, where b is the bid ask spread and f the brokerage or fixed fees.

A positive or negative difference between R_t^{Hedge} and R_t^{Option} is the profit or loss made at day t, which we will design as delta hedging PnL. If an options' portfolio is properly hedged its seller should expect not to make any gains or losses at any given day, as he will make the same amount of money in both portfolios. So a good quality hedging portfolio would have:

$$\min \sum_{t=0}^{t=T} \|R_t^{Hedge} - R_t^{Option}\| \quad , \quad \langle R_t^{Hedge} - R_t^{Option} \rangle = 0 \quad , \quad T \to \infty \tag{3.9}$$

Meaning that it is not only desired that the hedging is able to match the payoff at the end, as the average difference between returns during the option's life is zero. In the Appendix A we will also consider the standard deviation, skewness and kurtosis of the PnL to assess the hedging quality and overall impact of a given assumption under certain conditions.

3.4 Valuing Options and Deltas

Both for (3.7) and (3.8) above, it is required to know the value of C_t, so the option portfolio has to be calculated at every period. As mentioned in the previous section 3.1.1 we are going to use a path dependent asian option throughout the various tests, which requires the use of the Monte Carlo method to calculate the options' value at a given instant. However since we are taking as our testing sample 20,000 random price series, evaluating each of those using Monte Carlo would require another 20,000 simulations for the remaining t days of the option, meaning that it would be necessary to simulate at least 400,000,000 paths every day, just for the asian option. This would take hours on any average computer, so it is necessary to find a workaround to value the asian option. The other three options, the vanilla, binary and up-and-in call all have fairly simple closed formulas, but there is no closed formula solution to calculate arithmetic average price asian options. However if instead of using the arithmetic price average we consider the average to be the geometric one it is possible to derive a closed formula to value such asian option. The formulas are not of great interest for the understanding of the results, nevertheless it should be mentioned that all of them derive directly from the BS model and its assumptions, and are all presented in Appendix A.

For the valuation formula we are essentially using the same type of inputs used to create the simulated price series, however lagged one day behind. This implies that the market characteristics that drove the price for that one day are only observable at the end of the trading day, so as to be used for the new option pricing. Since we are just interested in studying BS model's assumptions, which are mostly about continuous volatility, interest rate and dividends, it is important that the series are similar so we can better interpret discontinuities between events in the market and variables fed to the BS model's formulas.

Much like the value of the option it is equally important to calculate their deltas. As we have seen in section 2.1, they are derivable from the option's formula as the first derivative with respect to the underlying price S. To the contrary of option pricing formulas, the vanilla call is about the only one who has a straightforward delta formula, which is $N(d_1)$, the other ones are considerably more complex or non-existant. However it is easy to calculate a fairly precise delta by varying of the underlying prices around the spot price, as in:

$$\Delta_t = \frac{C(S_t \times (1+h)) - C(S_t \times (1-h))}{2h} \quad , \quad h \to 0 \tag{3.10}$$

where $C(S)$ is the value of the option for underlying price S. Notice that (3.10) is simply the discrete formula for the first derivative. So in essence by varying the price of the underlying around its current price and calculating the change in the option value we are essentially measuring how much the option varies for changes in the underlying, thus the same delta as in (2.2).

Throughout this third chapter we went through all the practical aspects of the testing we will undertake next, such as the payoffs, price series and valuation methods used. It is important to comprehend how all results were derived, in order to interpret, question and ultimately modify them.

Chapter 4

Experimental Results

Before starting to test the assumptions we will first demonstrate some fundamentals regarding the model, presenting some graphs and figures that might be interesting before proceeding. Remember that for all tests, 20,000 price simulations were taken with equal market data inputs. A generic 100 strike and initial price were considered, allowing for an easier analogy to percentage returns. The four options were priced initially so as to guarantee they have identical prices to the asian option, by solving for the parameters mentioned in section 3.1.

First we shall compare the difference between the closed formula option pricing method and the Monte Carlo method. We used both methods on the same set of simulations. Looking at the results displayed in Table C.1, one can see that the differences between methods for the Vanilla, Binary and UpAndIn options are negligible. As we would expect the major difference occurs for the Asian option which results from a pricing approximation in the closed formula method mentioned in previous section 3.4, which causes the average pricing difference, standard deviation (STD), maximum and minimum difference to be much higher than the rest. The reason for the maximum difference to be much more negative than positive is due to the closed formula considering the geometric mean for S_{avg}. However similar results will not occur between the difference in the asian delta position calculated with both methods, since they are essentially calculated using the same underlying price variation scheme described as well in section 3.4.

4.1 When Assumptions are Met

Next we should set a basis scenario where no assumption is violated, to understand how each of the four options behave on an intentionally controlled market, with continuous volatility and interest rates, no dividends and transaction costs but only the randomness of the normally distributed returns to destabilise the hedging PnL. Take a look at Fig.(B.2) and Table (C.2). Since the annual risk free rate $r = -0.5\%$ the average price moves in a straight line from 100 to 99, over the course of two years, 520 days. In the same graph we can also see that the hedging PnL for the different options is fairly stable around the mean but destabilises considerably near the option's maturity. This is also comprehensible due to the options' gamma, the rate at which the delta varies with the change in price, which intensifies at maturity since any movement in the stock price will vehemently affect the final payoff and price of the option. Fig.(B.3) and Table (C.3) present the same data but just for a one year option, with more underlying volatility and 1% risk free rate. It is noticeable an increase in PnL volatility, although skewness and kurtosis having diminished, probably due to the reduced sample. The payoff aggressiveness also becomes more evident with a wider range of values across the four options.

In both cases we can tell that the Up-and-In option is by far the most difficult to hedge due to its abrupt payoff, while the binary option appears to be well behaved potentially due to the fact of having a fairly stable delta, since on average it passes from ATM to ITM or OTM, depending on the growth rate, right at the beginning due to the constant growth rate. Once ITM, the delta displayed in Fig.(B.4), will not change since the option is paying the coupon regardless of any subtle movements in the stock. From the mentioned the figures until now, it is also possible to see that the asian option has an average decreasing delta and hedge PnL, which tend to zero as time goes by. This is due to the different characteristics of such option, which takes an average value of S for the payoff, resulting in every realised closed price observation to have less of a contribution to the final average price payoff.

From Fig.(B.4) we can also spot a resemblance between delta behaviour to the average delta PnL in the preceding two graphs. This is only the case since there is a constant growth rate, thus the amount of delta each option has strongly influences the hedging PnL, notice how the option deltas and PnL amounts position in relation to each other.

All in all, with no market changes the model appears to work very well, it has an expected PnL close to zero, with very small volatility, as seen in the tables referred in the mentioned graphs. Now let us move on to analyse what happens when Black and Scholes assumptions are not met.

4.2 When Assumptions are Not Met

When realised volatility, interest rates or dividends, are different from the historical, implicit or estimated ones used in the BS model to price an option, assumptions are violated. Since markets evolve continuously it happens more often than not and option sellers are used to it. Fortunately more often than not these differences are subtle, and their effect on hedging PnL is expected to cancel out eventually, much like any random normal process. However in financial markets it happens that sometimes a small number of these differences are enough to affect the hedging portfolio immensely. These are created by major market discontinuities, difficult to predict and very difficult to account for, such as market crashes, profit warnings, volatility spikes and dividend cuts. We explore some of these next.

4.2.1 No Dividends

In Chapter 2 we referred that it was possible to account with the dividend yield when pricing an option. Which is very important since option sellers often use underlyings paying high dividend yields in order to lower their forward, making call options cheaper. Recall that essentially the dividend yield acts like a negative growth rate for the underlying, take a look at (2.1). But according to the BS model this dividend yield however has to be continuous, which excluding stock indexes, rarely occurs in the market. Instead what option sellers do when hedging, is to use the dividend yield of the forecasted dividends for every day pricing. It can be easily computed through:

$$\delta_t = ln\left(\frac{S(1+r_t)^{(T-t)} - \sum_i^n div_i}{S(1+r_t)^{(T-t)}}\right) \tag{4.1}$$

where div_i is the i^{th} dividend among a total of n forecasted for the remaining period $T-t$. Fig.(B.5) portrays a situation where an underlying pays quarterly dividends, with $\delta_t = 4\%$.

We can observe the average underlying price drop by the dividend amount, approximately 1%, and how the different options react. Notice that option prices at beginning cost around 3.8%, which is only possible at the expense of having a high dividend yield decreasing the return potential of the option, which is net of dividend. The asian option is the most affected in the first dividend payout, but the least in the last, the opposite happens to the Up-and-In, which is related to the option deltas at the time of the dividend payout.

One might ask, why are the mean option payoffs around 4% while the average underlying price has dropped bellow the strike of 100 in Fig.(B.5). Remember that the considered options never expire with a negative value, so when averaging the ones who paid and the ones who did not, the average is always going to be above zero. Notice that the final option payoffs are not far away from their initial valuations, since for the entire one year period the Black-Scholes assumptions were still valid, as the underlying did pay a 4% yield as expected.

In Table C.4, the average delta hedging PnL when quarterly dividends are received is compared to a similar market situation but with the underlying company cancelling the last two dividends, as of the second dividend payout. Evidently the later occurrence would prejudice the hedger, as he would have priced any call option cheaper than it would turned out to be. With no way to recoup the money the hedging portfolio would endure a considerable loss. Remember that the average values in the tables are averaged out across the number of days, so for example the Up-and-In option hedging portfolio would loose an average total of 45.60%! Nowadays it is possible however to hedge dividend cuts, through dividend futures.

4.2.2 Constant Interest Rates

In Fig.(B.6) we can see the actual annual rate that would have been used in a 2 year option, starting at the beginning of 2014 and expiring at the beginning of 2016. It was calculated from the Euribor rates and interpolated according to the time to maturity of the option.

Although in a very small scale the hedging PnL appears to react to the more violent changes in interest rates during 2014. Table C.5 shows that even for the dramatic decreasing in interest rates during the last two years, the PnL behaving is not too much different from the base case Table C.2. Same conclusion is drawn from Fig.(B.8) and Table C.6, where we inverted the trend to create an interest rate hike.

Overall changes in interest rates are subtle, and even a sustained changing rate trend during the entire life of the option appears to pose no barriers to a good delta hedging in any of the options.

4.2.3 No Transaction Costs

To assess how the PnL would be affected with transaction costs we shall assume no fixed brokerage fee, since every option is most likely sure to have their underlying's hedging position rebalanced every period. Maintaining the scenario of the inverted real rates, a bid-ask of one basis point would be negligible. If we augment that spread to 1% we can see the effects on Fig.(B.9). The biggest loss occurs at $t = 0$ when buying equity to cover the initial hedging position. The loss is proportional to the initial delta of the option. As time goes forward, options with low gamma will not present significant losses, while the contrary occurs to options with higher gamma, i.e. change in delta per change in underlying price. The later ones have a tendency for increasing transaction costs as the options reach maturity, since they require a greater equity re-balancing torwards the end.

Therefore, one must be cautious when doing options on illiquid equities, such as some depositary receipts, specially more exotic options. Frequently, besides having a wide bid-ask spread, these securities also have low volume, which might causing big trades to move the market considerably.

4.2.4 Log-Normality of Prices

Prices being log-normal is a Black-Scholes assumption with multiple clauses, since it is essentially the same of assuming that returns follow a normal distribution which require the fulfilment of several conditions, such as constant volatility, no leptokurticity caused by fat tails and no skewness mostly caused by price gaps or market crashes. All of these are however very characteristic of the general markets behaviour. Next we are briefly going to analyse each of them.

Constant Volatility

Every market security has a varying volatility, due to market correlation, current events, earnings, news etc. After the price, volatility is the characteristic that most influence an option price, since it measures how wide of a price variation is the underlying likely to have during a period of time, hence the probability of the option's underlying surpassing or not a given strike or barrier. Fig.(B.10) containing a real security's 20 day historic rolling volatility series and Fig.(B.11) with the option's deltas and underlying price series, picture just that. Notice how different the deltas are to the base case deltas in Fig.(B.4), difference that can be almost entirely explained due to the underlying's volatility.

Like the abrupt changes in delta, the changes in option prices also react strongly to changes in volatility, as we can see from the PnL peaks in Fig.(B.10), however even for the same type of options, volatility changes can have symmetrical impacts. For most options evaluated at $t = 0$, a higher volatility will yield an higher option price, as it translates the expected range of price moves of the underlying. The wider the range the better, since the ranges for which the option does not pay are most often limited, like in a vanilla call option, where ranges below the strike are not exercised. But once the option starts to get ITM, than volatility also increases the chances of it going back OTM. This can be observed up close in Fig.(B.12), with the ITM binary option always reacting contrary to the remaining options as volatility varies, since it can only pay less than in its current state.

Unlike interest rates, changes in volatility are abrupt and consequent, as depicted in Table C.8. Across all options a changing volatility will have a significant impact on delta hedging PnL's volatility, specially for more aggressive payoffs such as the Binary and Up-and-In Options, the later close to having a 9% PnL standard deviation. Volatility swings throughout the option's life are very concerning, and it is impossible to hedge them through delta. Like a dividend cut, a volatility increase would instantaneously make any call option more expensive than its original valuation, precisely due to the increased probability for more extreme payoffs. Option sellers usually protect against such moves by performing another hedging technique called vega hedging. Vega is the rate of option price change for a change in volatility, and can be hedged by going long or short in an additional put or call options and delta hedging them as well.

Overall volatility is very important in option pricing, perhaps even more than prices themselves, since unlike prices which provide no probabilistic information, volatility gives a solid indication of how likely is the price to move. Which volatility to use in the BS model is indeed difficult to tell, and opinions diverge across financial markets. Since volatility is not palpable, every option seller can use whatever one they feel comfortable with, including volatilities calculated through other mathematical models.

Leptokurticity

In financial markets, large positive and negative returns tend to occur much more frequently than predicted by a normal distribution curve. This generates fat tailed distributions which can be approximately modelled by a t-distribution pictured in Fig.(B.13). We can use such distribution when creating the wiener series, just like in section 3.2. The results of underlying returns having such a characteristic will dramatically increase the PnL kurtosis as seen in Table C.9, in fact when comparing to the base case Table C.2, we can spot a relation between underlying returns' and delta hedging PnL kurtosis, the later being much bigger.

No Skewness

Earnings' announcements, profit warnings, dividend cuts or market crashes often cause a skew of returns, due to the great negative impact caused by these. In Table C.10 we can see the impact caused by a -4% price shock at $t = 160$, to be similar to the one provoked by a real volatility series in Table C.8, only that the skewness increased, in particular to the more aggressive payoffs. The mean PnL also dropped substantially for all options.

In the end, underlying return characteristics seem to have a clear effect in hedging PnL, since changes in volatility, as well as fat tails and skewness or price gaps, originate higher volatility in the PnL, higher kurtosis and higher skewness or radical profits or losses.

In the results we were able to see how different options react, if BS model's assumptions fail to verify. On the whole, the major threats for option hedging are both discontinuous volatility and dividend cuts, which can originate considerable hedging deficiencies.

Chapter 5

Conclusions

In the results we could see that when Black and Scholes assumptions are not violated, the difference between options is not as evident as when they are violated, specially in terms of delta hedging PnL volatility. This means that in a sense, more complex and exotic payoffs pose a bigger risk to option sellers, and much like in any market portfolio, diversification is thus very important, whether across payoffs, underlyings or maturities.

We also demonstrated that dividend cuts and volatility discontinuities have a much bigger impact than interest rate discontinuities or transaction costs. So perhaps big notionals on a single underlying might justify dividend hedging. Volatility can also be hedged by doing back to back options with a different counterpart, or vega hedging which was not covered in this work.

5.1 Future Work

This work only accounted for options with single underlying, single observation options, however with low interest rates, basket options and options with several discrete observations are becoming the norm. It would be interesting to see how such options would react to similar market discontinuities, and how it is possible for option sellers to protect themselves against these.

Since we have demonstrated that there is a relationship between option payoffs and each of the violated Black-Scholes assumptions, and since the effects are comprehensible, maybe it is possible to incorporate these assumptions' irregularity in option pricing so as to account for such market frictions.

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