



Nova School of Business and Economics

Universidade Nova de Lisboa

Thesis, presented as part of the requirements for the Degree of Doctor of  
Philosophy in Economics

# Essays on Bounded Rationality: Individual Decision and Strategic Interaction

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June 30, 2015

A Thesis carried out on the PhD in Economics Program, under the supervision of  
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## Abstract

Economics is a social science which, therefore, focuses on people and on the decisions they make, be it in an individual context, or in group situations. It studies human choices, in face of needs to be fulfilled, and a limited amount of resources to fulfill them. For a long time, there was a convergence between the normative and positive views of human behavior, in that the ideal and predicted decisions of agents in economic models were entangled in one single concept. That is, it was assumed that the best that could be done in each situation was exactly the choice that would prevail. Or, at least, that the facts that economics needed to explain could be understood in the light of models in which individual agents act as if they are able to make ideal decisions. However, in the last decades, the complexity of the environment in which economic decisions are made and the limits on the ability of agents to deal with it have been recognized, and incorporated into models of decision making in what came to be known as the bounded rationality paradigm. This was triggered by the incapacity of the unboundedly rationality paradigm to explain observed phenomena and behavior. This thesis contributes to the literature in three different ways.

[Chapter 1](#) is a survey on bounded rationality, which gathers and organizes the contributions to the field since [Simon \(1955\)](#) first recognized the necessity to account for the limits on human rationality. The focus of the survey is on theoretical work rather than the experimental literature which presents evidence of actual behavior that differs from what classic rationality predicts. The general framework is as follows. Given a set of exogenous variables, the economic agent needs to choose an element from the choice set that is available to him, in order to optimize the expected value of an objective function (assuming his preferences are representable by such a function). If this problem is too complex for the agent to deal with, one or more of its elements is simplified. Each bounded rationality theory is categorized according to the most relevant element it simplifies.

[Chapter 2](#) proposes a novel theory of bounded rationality. Much in the same fashion as [Conlisk \(1980\)](#) and [Gabaix \(2014\)](#), we assume that thinking is costly in the sense that agents have to pay a cost for performing mental

operations. In our model, if they choose not to think, such cost is avoided, but they are left with a single alternative, labeled the default choice. We exemplify the idea with a very simple model of consumer choice and identify the concept of *isofin curves*, i.e., sets of default choices which generate the same utility net of thinking cost. Then, we apply the idea to a linear symmetric Cournot duopoly, in which the default choice can be interpreted as the most natural quantity to be produced in the market. We find that, as the thinking cost increases, the number of firms thinking in equilibrium decreases. More interestingly, for intermediate levels of thinking cost, an equilibrium in which one of the firms chooses the default quantity and the other best responds to it exists, generating asymmetric choices in a symmetric model. Our model is able to explain well-known regularities identified in the Cournot experimental literature, such as the adoption of different strategies by players (Huck *et al.*, 1999), the inter temporal rigidity of choices (Bosch-Domènech & Vriend, 2003) and the dispersion of quantities in the context of difficult decision making (Bosch-Domènech & Vriend, 2003).

Chapter 3 applies a model of bounded rationality in a game-theoretic setting to the well-known turnout paradox in large elections, pivotal probabilities vanish very quickly and no one should vote, in sharp contrast with the observed high levels of turnout. Inspired by the concept of rhizomatic thinking, introduced by Bravo-Furtado & Côte-Real (2009a), we assume that each person is self-delusional in the sense that, when making a decision, she believes that a fraction of the people who support the same party decides alike, even if no communication is established between them. This kind of belief simplifies the decision of the agent, as it reduces the number of players he believes to be playing against – it is thus a bounded rationality approach. Studying a two-party first-past-the-post election with a continuum of self-delusional agents, we show that the turnout rate is positive in all the possible equilibria, and that it can be as high as 100%. The game displays multiple equilibria, at least one of which entails a victory of the bigger party. The smaller one may also win, provided its relative size is not too small; more self-delusional voters in the minority party decreases this threshold size. Our model is able to explain some empirical facts, such as the possibility that a close election

leads to low turnout ([Geys, 2006](#)), a lower margin of victory when turnout is higher ([Geys, 2006](#)) and high turnout rates favoring the minority ([Bernhagen & Marsh, 1997](#)).

**Keywords:** Bounded Rationality, Cournot Oligopoly, Voter Turnout



# Acknowledgements

I am grateful to Fundação para a Ciência e Tecnologia for a 48 month scholarship, which provided an essential financial support for the completion of this thesis.

Also, I thank Nova School of Business and Economics, for allowing me to work as a teaching assistant. Not only for the financial support, but, most importantly, for giving me the opportunity to teach. All the math that is in the thesis would probably be replaced by conjectures and vague propositions, hoping no one would notice it, if I had not learned everything I did while teaching. Of course, I also have to mention all the assistance to the completion of this thesis directly provided by the school, either by supporting the attendance of conferences, organizing seminars to discuss work in progress, or introducing annual assessments, which always made me feel uncomfortable. And that is exactly what I needed.

Specifically, I have to recognize the contributions from all faculty members with whom I talked about the contents of this thesis. While some times the ideas I presented deserved to get an unpronounceable answer, they were nothing but kind and helped me find the right direction, which I hope to have done. João Furtado deserves a special mentioning here. Not only did he directly contributed to the third chapter of this thesis, by, being smart as usual, creating the concept of rhizomatic thinking, but he was also incredibly available to debate ideas, in discussions that lasted for hours and usually ended up as philosophical arguments. Thank you for all the interest you put in my ideas.

I also have to thank the faculty staff and students in the Université de Lausanne, for all their insights on my thesis during the semester I spent there. Luís Santos-Pinto, in special, was available at all times and was an amazing guide in the field of

Behavioral Economics. The first and second chapters of this thesis have an enormous contribution from him. And the football moves he taught me, which I thought were not anatomically possible, were also an inspiration in life.

I am very grateful to all my friends and students for eventually learning that a PhD thesis is like Voldemort. It should be known as the one that shall not be mentioned. I know I did not always reacted in the most pleasant way when you asked about the delivery date, and I am sorry for that, especially because you were only showing that you cared. The good news is that I now have an answer: it is done.

There are some friends which did more than simply ask about this thesis. And they are part of the reason why I now have an answer to all the others. I thought about mentioning them by name, but preferred not to. This way, I can say to anyone who asks: ‘Do you think I would forget you?’

I would like to thank my family members one by one, but, unfortunately, it is a rather big family. So, let me thank all of them for tolerating my ‘occasional’ bad mood. You were all a bigger help than you can imagine. But please do not ask me again to explain what the thesis is about. Just read it.

I think my two advisors, Susana Peralta and Guido Maretto, do not realize how much they contributed to this thesis. It simply would not exist if it were not for them. You are the people I am most strongly grateful to, but I find it difficult to express it in words. So, let me just say this: thank you for being such good friends.

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# Introduction

Economics can be thought of as the study of the decisions and actions of people who have to use the available resources to fulfill their needs. It is exactly this focus on people that makes Economics a social science. And its socialness has advantages and disadvantages. On the plus side, it is not a closed science, centered on itself, but can be used to observe, analyze and predict real phenomena, events and actions, and therefore to better understand the world we live in. However, its human centering prevents it from having hypotheses as testable and predictions as accurate as some natural sciences. In fact, human activity is ruled by human decisions and these are the result of each person's brain functioning. And here, scientists face two major problems. First, in spite of the advances in neurology and neuroeconomics, we are still unable to analyze the exact way decisions are made and, therefore, cannot predict with certainty what every individual will do in every situation. Second, even if we could do that for a specific person, we would still be far from being able to explain most social phenomena, as it seems that different people act differently in similar contexts.

Facing these difficulties, Economics had to find a way to go on. And it did so, by assuming a simple hypothesis: unbounded rationality. What does this mean? It means that, an individual, when faced with a problem, for which some information is available to him, chooses, from the different available options, the best for him. In other words, the economic scientist calculates the optimal action the individual should take and assumes he takes it. This technique is not always unrealistic. Some problems are so simple that it is reasonable to assume that most people will make the optimal choice. In other cases, although each person does not optimize, the

gathering of everyone's decisions cancels individuals mistakes and the aggregate result is somewhat close to the optimum. In these cases, the assumption of individual unbounded rationality does not harm the validity of the collective result. And in other situations, although people do not optimize the problem they face, they end up choosing the optimal action, either by randomizing, or by using another decision process which coincidentally results in selecting the optimum. However, it seems unreasonable to assume that unbounded rationality is used by all agents in all situations. There are some situations in which the optimal decision is so hard to find for the modeler (who can take the time and use the auxiliary resources needed to formalize the problem, find how to solve it, and execute calculations), that it seems implausible that an individual, when faced with the problem, having to make a decision on a short period of time, and relying on nothing but himself, is able to find the optimal solution by other than chance.

Even if one is happy with the results obtained from the use of unbounded rationality, the concern with this kind of issues seems important at least from a procedural point of view. When you first contact with Economics, you learn that models are a simplified version of reality, which allows you to study it in a tractable way and to get predictions which otherwise would be impossible to get. Then, one of the most important tasks faced by a modeler is to choose which aspects of reality should be simplified or ignored, and which ones should be kept unchanged. It is understandable that, in Economics, rationality is normally chosen as one aspect to be simplified, if nothing else, because we don't understand entirely how it works. However, being human decisions as important as they are in economic models, the level of simplification that unbounded rationality implies makes some authors believe that it should be replaced by some other concept closer to the way in which people actually think. Even if this does not improve the quality of the results obtained, the movement in this direction is at least an honest attempt to make economic models more realistic regarding one of its core features: decision processes.

It was in this spirit that bounded rationality came to existence. Its creation is probably due to Herbert A. Simon, who, in [Simon \(1955\)](#), affirms that "Broadly

stated, the task is to replace the global rationality of economic man with a kind of rational behavior that is compatible with the access to information and the computational capacities that are actually possessed by organisms, including man, in the kinds of environments in which such organisms exist.” Since then, many authors acknowledged the existence of this problem and have proposed some solutions. However, bounded rationality is very difficult to model in an unified way, because this would imply the knowledge of all determinants of human behavior, which we are still far from getting. Indeed, the most difficult object of investigation for man seems to be man himself. And this partly explains why bounded rationality has spread in so many different directions.

This diversity motivated [Chapter 1](#). It is a survey on bounded rationality, covering contributions to the field from Herbert Simon’s pioneering work in *satisficing* ([Simon, 1955](#)) to recent developments, such as the notion of *sparsity* ([Gabaix, 2014](#)). To select what literature we should analyze, we had to define a criterion. Especially because bounded rationality is part of a wider field, Behavioral Economics. This recent field in economic history gathers contributions from different sciences, such as psychology, sociology and neuroscience, to construct a view of human behavior that accounts for the factors that influence the human mind. However, Behavioral Economics does not necessarily assume that rationality is bounded. In fact, concepts like *reciprocity*, *fairness* or *loss aversion* do not imply that people find it difficult to solve any specific problem. Instead, their human nature influences their decisions, because they care about certain aspects that traditional Economics failed to acknowledge, or decided to ignore. The perspective we take here is that bounded rationality takes its place whenever the contrast between the complexity of the decisional environment and people’s abilities to solve problems affect the way in which such problems are viewed or dealt with. That is, we only focus on problems which are simplified or transformed in such a way that they become easier to analyze according to people’s views on them.

After restricting the literature, organization was an issue. But once again the definition we consider for bounded rationality came to the rescue. If a problem is

simplified, then we should be able to identify what is changed relatively to the classic perspective. But, if we are to do this, we need to have in mind what constitutes any economic problem. This led us to the recognition of six elements. If a choice has to be made, one or more alternatives must be available. This means a choice set, the first element, must be defined. Then, some kind of ranking of elements of the choice set must exist, because otherwise no decision was needed. Thus, preferences of the decision maker should be specified and, if possible, represented by an objective function, the second element. If, indeed such function exists, and its optimization is difficult, we need to specify how an agent tries to reach the best decision possible. That is, the third element, an optimizer operator, defines the procedure used to make a choice. And, technically, a decision maker needs to understand what, in the problem at hand, requires a decision. This generates the set of decision variables, the fifth element. Finally, the problem, specifically the objective function and the choice set, may be influenced by factors out of control of the decision maker. These external elements, parameters, are the fifth element we analyze. Finally, the decision maker may not possess all the information that affects the problem, but still need to make a decision. This is the basis for the sixth element, the uncertainty operator. Notice that models of strategic interaction between different players, may, in general, be accommodated in this framework, because each player is solving his own problem, and does not control other players' decisions. However, given the specificity of such models and the importance they have in economic modeling, we cover them separately. After the review of the literature in bounded rationality, we make two contributions to the field, one affecting individual decision and the other specifically designed for strategic interaction contexts. They are presented in [Chapter 2](#) and [Chapter 3](#), respectively.

Our modeling proposal in [Chapter 2](#) is extremely simple. Deciding is costly and, if an agent wants to be able to find the optimal solution of his problem, he needs to bear the cost of thinking. But that leaves that raises the question of what he does if he decides not to think. Our assumption is that all agents are equipped with what we call a default choice, that is, an option that the agent can choose without any mental effort. It can be interpreted as intuition ([Kahneman, 2003](#)), a

suggestion from other people (Choi *et al.* , 2003), the optimal choice associated with a standard, although not necessarily true, view of the world (Gabaix, 2014) or in other ways. What is important about it is that it provides the agent an alternative to thinking. A costly thinker then has to decide, besides an option from the original choice, whether to think on the problem, or to stick to his default choice. If he does decide to think, the cost of doing so is subtracted to the payoff he gets, and this frames thinking as any other scarce resource resource in economic problems. Its benefit and cost must be compared, in order before a decision about its use is made.

Before moving to the main application of the concept, we apply it to a very simple consumer problem, for illustrative reasons. Interpreting the default choice as the result of intuition, we conclude that consumers with a good intuition decide not to think, because the benefit in finding the optimal choices is outweighed by the cost it implies. And to have a good intuition, in this context, means to have a default bundle that the consumer highly values. On the other hand, consumers who find it more difficult to think rely more on their intuition.

Then, we drive our attention to the main application of the chapter: Cournot oligopoly. We study a duopoly model, with linear costs and demand, and a common default quantity and thinking cost. And find that the higher the thinking cost, the less firms are thinking in equilibrium. And all combinations of thinking decisions are possible. This means that it is possible to have symmetric firms making different thinking decisions which, in this model, implies that they also make different quantity decisions. The fact that quantities are strategic substitutes helps to understand why this happens. For example, if the default quantity is high, if one of the firms does not think and produce it, in an aggressive move. The other firm, by best responding, chooses a low quantity. And this makes the default quantity feel better for the non thinking firm, because it is closer to the best response of this firm than its rival's. Hence, there is room for a thinking cost, neither too high nor too low, that makes thinking attractive only for the firm that is actually thinking. This model can provide some intuition for some results in the Cournot experimental literature (Huck *et al.* , 1999, Rassenti *et al.* , 2000, Huck *et al.* , 2002, Bosch-Domènech &

Vriend, 2003). Strategy heterogeneity is obtained endogenously when the thinking decision is different between firms. Inter temporal rigidity, that is, repetition of choices across periods is a result from a dynamic extension of our model, which predicts that, if the thinking cost decreases over time at all but one specific rate, there are some periods in which quantities do not change and, moreover, the firms eventually learn to play the Nash equilibrium and do it forever without thinking. And an increase in quantity dispersion when decisions are harder to make is also a consequence of our model, if we allow for different default quantities and it is obtained in a more pronounced way if more than two firms are considered.

Chapter 3 was written in co-authorship with Susana Peralta. Contrary to the model in Chapter 2, in which the concept we introduced has a direct impact on individual decisions (although it also indirectly influences the outcome of game-theoretical models with players that are boundedly rational in the way we define), in this chapter, we directly approach strategic interaction between players. Based on the notion of *rhizomatic thinking* (Bravo-Furtado & Côte-Real, 2009a), we say that a player is *self-delusional* if, when deciding, believes that a fraction of like-minded players necessarily take the same action as he does. This means that self-deluded players believe their decisions have a higher impact on the game outcome than they really have, because the players they assume to act as they do, may, in fact behave differently. This type of belief simplifies the game, because the number of players whose strategies need to be forecast is reduced. A natural application for this concept is the problem of voter turnout. Indeed, in large elections, the small expected impact of each single vote combined with a positive cost of voting, even if very low, results in the theoretical prediction of no voting, which is clearly contradicted by reality. The empowerment that self-delusion provides to potential voters may then explain why it is that they actually vote.

We model a first-past-the-post election, with two parties, and a level of self-delusion, defined as the fraction of like-minded players that a player believes to act like him, distributed uniformly in each set of partisans. Like-minded agents are interpreted as the players that prefer the same party. We get that a positive

and possibly high turnout. In fact, if both parties have the same support size, an equilibrium in which all players vote is possible. The majority party may always win the election, whereas a victory of the minority is possible, if the parties' support sizes are not too different. When parties get closer in support, we can have higher or lower turnout rates. An increase in turnout rate is expected in these conditions, because the minority supporters feel they have a real chance of seeing their party win the election, whereas the majority supporters become less confident on a victory. On the other hand, players vote when they believe they can induce at least a tie with their vote, and perceive voting to be higher than it actually is. This means that, if the deluded part of perceived voting is more important than actual voting, the minimum level of self-delusion that is required for a majority supporter to vote increases when there are more potential voters in the minority party. But this takes players that are no longer self-deluded enough from the voting set, thus reducing turnout. However, we do get that equilibria in which the minority win have the highest turnout rates. We add an extension to our model that allows for different distributions of self-delusion between parties. This shows that this type of belief may actually benefit the people who have it. If the minority has no chance of winning the election because of lack of support, but its supporters become more self-deluded, winning can become a possibility. On the contrary, a minority win may become impossible because of an increase in self-delusion in the majority party supporters. This points out that parties, other than trying to gain new supporters, may concentrate efforts in raising the feelings of group identity and connectedness among the ones they already have.





# Chapter 1

## The Simplified Economic Problem: A Survey on Bounded Rationality

### 1.1 Introduction

Behavioral Economics is a relatively recent field. Its creation was a response to concerns about the assumptions on human behavior in economic models, as were made until then, and their impact on the quality of predictions. Contrary to the predominant practice at the time, this field had a clear focus on the connection between human reasoning and observed decisions. It proposed to reach to other sciences, such as psychology, sociology and human science, in order to better understand the way in which people actually think, and construct more realistic and successful models of human behavior. This movement made possible the appearing of novel theories such as *hyperbolic discounting* (Laibson, 1997), *prospect theory* (Kahneman & Tversky, 1979), *fairness and reciprocity* (Rabin, 1993) and *altruism* Levine (1998). These examples show that one possible way of altering the classic rationality paradigm is to assume that the problem at which it looks may be ill-specified. That is, differences between actual behavior and the predictions from traditional economic models may stem from the fact that they have different goals than is usually assumed. It is not that they are not able to solve a problem, but that they want to solve a different one. However, some authors in this field focus on

a different perspective. They consider that some problems are too hard to be solved by agents with bounds on their reasoning abilities. It is not just a matter of how limited an agent is, but, more generally, of what results from the contrast between his difficulties in thinking and the complexity of the decision he faces. And this idea is the core of *bounded rationality*.

The contributions to the literature on this theme have been the object of some surveys. [March \(1978\)](#) centers his analysis on the problem of choice and discusses how it is affected by bounded rationality and changes in the way preferences are modeled. [Camerer \(1998\)](#) mainly focuses on experiments that either motivate the creation of theoretical models, or test them. [Conlisk \(1996b\)](#) presents evidence of bounded rationality, shows the success of some papers in this area, discusses the objections against it and defends that it is a part of Economics on its own right, as it deals with the use of a scarce resource, reasoning. [Lipman \(1995\)](#) reviews papers which treat bounded rationality as an information processing problem.

We also review the literature on bounded rationality, but do it guided by a specific view of the concept and how it impacts economic modeling. In Economics, a problem arises when there is the need to allocate limited resources to achieve some goal. That is, whenever there is the need to make a choice between different options. Formally, we think of a general economic problem as:

$$\text{Opt}_x E(u(\alpha, x)) \text{ s. t. } x \in S(\alpha) \tag{1.1}$$

There are six elements we identify in (1.1). They are the following:

1. *Opt*: The optimizer operator. Represents the procedure that is followed to achieve the intended goal. May or may not result in the choice of the best available option.
2. *u*: The objective function. Is derived (if possible) from the preferences the agent has on the available options. Guides the choice process, by indicating the goal to be attained.

3.  $E$ : The uncertainty operator. An agent may not possess all the information he needs to solve the problem, but still need to make a decision. In this case, he must adopt a procedure to deal with the uncertainty he faces, represented by this operator.
4.  $S$ : The choice set. Includes the options the agent considers. May be endogenous, if the agent has the possibility to filter the options to choose from.
5.  $\alpha$ . The parameter set. Consists of all factors that influence the problem, over which the agent has no control. Includes other players' decisions in traditional game theory.
6.  $x$ : The set of decision variables. Expresses what the agent can control and needs to decide upon.

A bounded rationality model should then simplify an economic problem, in at least one of these elements. We classify what we consider to be the main papers on this theme according to the simplified element that is more relevant in their analysis. Game-theoretic models, given their specificity, are analyzed in their own section, although it would be possible to fit them in one of the six categories, as individual decisions need to be done even in contexts of strategic interaction. Our focus is on theoretical contributions, not on experimental evidence that motivates them. Of course, this is not an exhaustive analysis of the literature on this theme, but a presentation of the papers we feel to be more representative of the advances in this area.

The optimizer operator is the focus of [Section 1.2](#). The analysis of the objective function is divided into two parts. In [Section 1.2.2](#), the original problem is replaced by a new one, with which the agent can more easily cope. In [Section 1.2.3](#), the cost in making decisions is explicitly accounted for and directly affects the objective function. The remaining elements of the problem are the object of [Section 1.2.4](#) and game-theoretical models are gathered in [Section 1.2.4](#). [Section 1.4](#) concludes.

## 1.2 Individual Choice

In this section, we present the main theoretical papers which change the classic problem of individual decision. In [Section 1.2.1](#), we focus on the optimizer operator. Changes to the objective function are studied in [Section 1.2.2](#), if they are mere simplifications, and in [Section 1.2.3](#), if they consist on the adding of a thinking cost. The other elements of the economic problem, uncertainty operator, choice set, parameters and decision variables are analyzed in [Section 1.2.4](#).

### 1.2.1 How to choose

We start with models that assume that people, when confronted with a choice problem, do not try to select the best option available. In the papers we analyze, people search the choice set until finding a satisfactory option ([Simon, 1955](#), [Gigerenzer & Goldstein, 1996](#)), adjust their choices over time towards the ones that are revealed to be more successful ([Arthur, 1991, 1994](#)) and allocate their mental resources according to not necessarily optimal rules ([Radner & Rothschild, 1975](#)).

One possible method of making choices which are not necessarily optimal is to define what a good payoff would be and try to achieve it. That is, instead of taking the best option from a choice set, take the first one which implies a minimum utility level. This is the basis for the concept of *satisficing*, labeled this way by the formal launcher of the concept in [Simon \(1956\)](#). In a previous paper ([Simon, 1955](#)), he developed a framework that allows to adapt a general optimization problem to this concept. In a particular way, [Gigerenzer & Goldstein \(1996\)](#) also employs the satisficing idea. He assumes that, when comparing choices, people look at the characteristics of each of them. Sequentially, they try to rank these characteristics, and stop when they believe to have found conclusive evidence of what is the best option. This does not necessarily results in an optimal choice but, instead, search is stopped when the agent deciding is comfortable with his option.

[Simon \(1955\)](#) is generally regarded as the launcher of the bounded rationality concept. He recognizes that the environment in which decisions are made is hard

and that the human mind is limited in the way it deals with it, implying that optimization may not be possible in some situations and should be replaced with a different choice mechanism. In this paper, the notion of satisficing, although still not labeled this way, is presented for the first time. A satisficer first defines an *aspiration level*, that is, a minimum utility level which he is willing to accept. Then, a sequential search is performed in the choice set, and it stops when an alternative that guarantees at least the aspiration level is found, and this is the alternative selected.

More formally, the author proposes the following choice procedure. There is a general choice set,  $A$ , and a subset of it,  $\tilde{A}$ , is known by the agent.  $S$  is the set of consequences the selection of each element in  $A$  may have. The utility each consequence in  $S$  brings to the agent is defined by  $V$ . The agent may have a more limited information on the relationship between  $\tilde{A}$  and  $S$  and just know what are the possible consequences of each action he can take (if this is the case, each  $a$  in  $A$  is linked to a subset of  $S$ ), or he may be more informed, and be able to specify the probability that each possible consequence of a given action will indeed occur if the action is selected. Within this framework, classic rationality would result in the use, for example, of a maxmin rule or expected utility maximization. The behavioral alternative suggested is the simplification of  $V$ , which would map elements in  $S$  into a very simple set, such as  $\{0, 1\}$ , or  $\{-1, 0, 1\}$ . The elements of the former are interpreted as *unsatisfactory* and *satisfactory*, whereas the elements of the latter represent *lose*, *draw* and *win*. Focusing on the first set (the one which is more often mentioned by the author), in the satisficing perspective, the objective of the agent is to find an element in  $\tilde{A}$  that maps into a set of consequences that are considered satisfactory. In order to attain this, the agent should try to find an action in  $\tilde{A}$  which maps to a subset of  $S$  that contains only satisfactory alternatives. This idea is also applied to the more complex problem in which  $V$  is a vector function, be it because different features of the same object are being compared or because more than one agent is involved, and an obvious generalization of the procedure is advanced: search for an action which consequence is satisfactory in all aspects of the problem. More formally, if there are  $n$  aspects to considered,  $k_i$  is the aspiration level of aspect

$i \in \{1, \dots, n\}$ , and the agent's objective is to find an  $a \in \tilde{A}$  that maps into a subset of  $S$  containing only consequences  $s$  such that  $\forall i \in \{1, \dots, n\}, V_i(s) \geq k_i$ .

The model also allows for dynamic considerations. For instance, aspiration levels may vary with time, decreasing if satisfactory solutions are hard to find and increasing otherwise, and the choice in each period may be influenced by the results of the choices taken in the past. A final note to mention that the author refers the possibility of accounting for the cost of making complex decisions, which would generate a new, more general problem which could then be optimally solved. This route, however, is not followed, which is justified by the ignorance of the agent of such costs and its inability to compare the utility of each action and the cost of choosing it. Therefore, it is safe to assume that Simon, in his seminal contribution, was hinting at costly thinking, which was developed years later by [Conlisk \(1980\)](#), for example, as a promising research avenue.

[Gigerenzer & Goldstein \(1996\)](#), in a paper published in a psychology journal, present three heuristics to be used when making inductive inferences. They focus on the problem of making a comparative assessment of a set of alternatives, when such comparison is not easy to make directly. Instead, the agent resorts to some features of the alternative objects, labeled cues, which he thinks are good indicators of which choice he should make. Trying to define fast and efficient heuristics, the authors propose three different evaluation methods which make use of cues, but do not require the knowledge of all of them.

Formally, a choice is to me made in a set of  $n$  objects,  $A = \{a^1, \dots, a^n\}$ . Object  $k$  has a value  $t(a^k)$ , and the agent's goal is to select the highest value choice. There is the possibility that the agent has never heard of one or more of the objects. The recognition of object  $k$  by the agent is defined by the binary variable  $r^k \in \{0, 1\}$ , where 0 stands for ignorance and 1 for recognition. There is a set of  $m$  binary cues, which can take the value  $-1$  or  $1$ , where the former represents a negative and the latter a positive signal. Cue  $i$ , for object  $k$ , assumes the value  $c_i^k \in \{-1, 1\}$ . When investigating  $c_i^k$ , the agent can either believe it to be  $1$  or  $-1$  or do not assume anything about it. The agent's belief about  $c_i^k$  is  $b_i^k \in \{+, -, ?\}$ , where  $+$ ,

– and ? represent, respectively, the ideas that  $c_i^k$  is 1, –1 and unknown. If the agent does not recognize one object, he assumes all its cues to be unknown, i.e.,  $\forall (k \in \{1, \dots, n\} : r^k = 0), \forall i \in \{1, \dots, m\}, b_i^k = ?$ . Cue  $i$  has an ecological validity, represented by  $v_i$ , which measures its ability in correctly comparing the values of objects. Specifically, it is the fraction of times that an object has a higher value than another, whenever the cue is positive for the best object and negative for the other. That is:

$$v_i = \frac{\# \{(k, l) \in \{1, \dots, n\}^2 : c_i^k = 1 \wedge c_i^l = -1 \wedge t(a^k) > t(a^l)\}}{\# \{(k, l) \in \{1, \dots, n\}^2 : c_i^k = 1 \wedge c_i^l = -1\}}$$

The first heuristic proposed by the authors is the *Take the Best Algorithm*. The agent selects two objects,  $a^k$  and  $a^l$ , to compare. If he recognizes only one of them, that is the one he chooses. If he recognizes neither, he randomly selects one. If both objects are recognized, the agent ranks the cues by their ecological validity. He sequentially compares his beliefs about the cues, starting with the one with the highest economic validity. If, for cue  $i$ , his beliefs are  $b_i^k = +$  and  $b_i^l \neq +$ , object  $k$  is selected. Otherwise, the agent moves to the next cue. If this process leads to no objection selection, a random choice is made. If there are no random choices made, this heuristic is transitive, and, independently of the ordering of objects' pairing, the resulting choice is always the same. The *Take the Last Algorithm* differs from this in that cues are ordered not by their ecological validity, but rather by their order of use. That is, when comparing two objects, the first cue to be analyzed is the one that settled the last comparison of objects the agent made. In the *Minimalist Algorithm*, the use of cues is randomly ordered. Notice that, while the Take the Best Algorithm demands the knowledge of ecological validities and the Take the Last Algorithm the memory of discriminating cues, the Minimalist Algorithm needs no information about cues. And they all have in common the use of a subset of cues, which can be very small.

These algorithms are compared, in simulations, with a series of integration algorithms, which make use of all the information available, making them closer to the classic rationality paradigm. The objects are 83 German cities and their value was



their population. There were 9 cues used, such as the property of being a national capital or the existence of a local university. The percentage of cities recognized, as well as the percentage of cues known (they assumed that agents either knew the true value of the cue or assumed it to be unknown) varied through simulations. They conclude that, predictably, the heuristic algorithms are faster, in the sense that the number of cues they use is smaller and, to some surprise, that the Take the Best algorithm was the one that had a higher inference accuracy, on average. One curious phenomenon resulted from simulations. For the heuristic algorithms, accuracy was maximized when only some of the objects were recognized. The reason is that the simulations were setup with the property that, with an 80% probability, a recognized object had a higher value than an unrecognized one. Thus, an excessive number of recognized objects led to a small number of contests decided by the recognition variable, wasting a quite accurate criteria.

Another not necessarily optimal method of choice is sampling and adjustment, that is, finding the merits of each available option by trying it out. A bounded rational agent that is unable to instantly find an optimum may nonetheless be able to observe the results of his choices, choosing more often the options which perform better through time. This is the basis idea of two papers of the same author, [Arthur \(1991\)](#) and [Arthur \(1994\)](#). An important distinction between them is the object of the agents' observations. In [Arthur \(1991\)](#), the agent is undecided about what element from a choice set he should take, whereas in [Arthur \(1994\)](#) his indecision is relative to selection rules, that is, criteria he can use to make a choice. Besides, in [Arthur \(1991\)](#), there is a mechanism of self-reinforcement, as the selection of an action in one period increases the probability of its selection in the future, but in [Arthur \(1994\)](#), the probability that a selection rule is chosen is not influenced by the fact that it was used in the past, as the agent always looks at all the rules at all times, giving them all an equal opportunity.

[Arthur \(1991\)](#) constructs a very simple model of choice that reacts to the success of the different alternatives chosen through time. He assumes each possible action has an unknown payoff that is distributed according to a stationary distribution. An

agent has a prior belief about the payoff quality of each action and selects an action using a randomizing profile that takes this belief into account. After an action is randomly selected in this fashion, its realized payoff is registered and the belief is updated, leading to a new random choice.

Specifically, there are  $N$  actions, indexed by  $i$ . Action  $i$ 's payoff is positive and distributed according to the stationary distribution  $\Phi^i$ , with expected value  $\phi_i$ . For each period  $t \in \{0, 1, \dots\}$ ,  $S_t = (S_t^i)_{i \in \{1, \dots, N\}}$  is a vector of strengths associated with each action. The sum of actions' strengths in period  $t$  is  $C_t$  and, in this period, the probability that each action is chosen is its relative strength. That is, if  $p_t^i$  is the probability that action  $i$  is chosen in period  $t \geq 1$ ,  $p_t^i = \frac{S_t^i}{C_t}$ . After the action selection, the vector of strengths is updated. First, the payoff realized by the chosen action is added to its strength. Then, strengths are normalized so that their sum is  $C_{t+1} = C(t+1)^\nu$ . This implies that, if action  $j$  is chosen in period  $t$ , its realized payoff at that time is  $\pi_t^j$ , and  $e^j$  is the  $j$ th unit vector,  $S_{t+1} = \frac{C(t+1)^\nu}{C_t^\nu + \pi_t^j} (S_t + \pi_t^j e^j)$ .<sup>1</sup> The stochastic evolution of the probability of choosing each action is shown to be the following:

$$\forall t \in \{1, \dots\}, \forall i \in \{1, \dots, N\}, p_{t+1}^i = p_t^i \left( 1 + \frac{\phi^i - \sum_{j=1}^N (p_t^j \phi^j)}{C_t^\nu} + \frac{\xi_t}{C_t^\nu} \right) \quad (1.2)$$

In (1.2),  $\xi_t$  is a zero mean disturbance. These dynamics are important in understanding and confronting the concepts of *exploration* and *exploitation*. The former refers to the analysis of the search set, achieved by selecting different actions over time and allowing to find good alternatives, while the latter means the repeated selection of an alternative which has proven to be good enough. Suppose  $k$  is the action with the highest expected payoff and  $l$  also has a high payoff, but not as high. The factors that contribute to the exploitation of  $l$ , if it is chosen in early periods, or to the exploration of the choice set and subsequent finding of  $k$  are understood by observing (1.2). The expression  $\phi^i - \sum_{j=1}^N (p_t^j \phi^j)$  stands for the difference between

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<sup>1</sup>The agent is assumed to have an initial belief about the actions' strengths, which is represented by the strictly positive vector  $S_0$ . As no choice is made in period 0,  $S_1 = \frac{C}{C_0} S_0$

the expected payoff of action  $i$  and the expected payoff the agent gets, given the probabilities he defines in period  $t$ , and is positive for sure when  $i = k$  and also positive when  $i = l$ , if  $l$  has a high enough expected payoff. If  $k$  is significantly better than its alternatives, this expression attains a high positive value, and  $p_t^k$  tends to increase, even if  $k$  is not selected often in early periods, preventing the agent's choice from being locked in a different action. On the other hand, if either  $C$  or  $\nu$  are low, the rate of growth of  $p_t$ , if  $l$  is chosen with some regularity in early periods, is high, and this action may always be chosen from a certain point in time, even if it is not the optimal action. Restricting  $\nu$  to be in  $[0, 1]$ , the author states that only when  $\nu = 1$  the locking in of action  $k$  is guaranteed. The model is then used to calibrate the values of  $C$  and  $\nu$  against some experiments where subjects had to choose between two actions for 100 periods, and the calibrated model replicated quite well the results of the experiments.

[Arthur \(1994\)](#) argues that human reasoning is essentially inductive, in opposition to the deductive thinking classic rationality assumes. Intuition is defined as a set of beliefs, rules and selection methods, which depend on the context in which they are formed, and evolve over time, with the most successful ones being reinforced and the others discarded. Not knowing how to deal with a specific problem, an agent constructs some models of choice which he can cope with, and compares the results of their application to the problem at hand. Over time, the most successful ones become the most often used.

He exemplifies this idea with a congestion problem. There is a bar to which 100 people consider to go once a week. Going is enjoyable if and only if less than 60 people are present, and there is no communication between people. The only information available is the bar attendance in the last few weeks. With this information, an agent acts intuitively by predicting the attendance in each week in one of several different ways. He may say it is the same as it was in the previous week, a rounded average of the four previous weeks' attendance or the trend in the previous 8 weeks. These rules for deciding which option is the best, without any calculation to support their optimality, are usually referred to as *rules of thumb*. In each week, the agent

uses the method that has proven more accurate at the date, in a process that is constantly updated. The author performs a computer simulation to test this theory, defining that each of the 100 agents knows a subset of the whole set of acting rules. He concludes that, on average, 60% of the rules being used led to attendance, and states that it seems natural that this game is attracted to a Nash equilibrium, where each player goes to the bar with a probability of 60%.

Radner & Rothschild (1975) has in common with Arthur (1994) the fact that it uses rules of thumb to solve a complex enough problem to have an optimal solution that is hard to find. They deal with the problem of selecting how much effort to dedicate to different tasks and consider that the higher the fraction of the available effort that is dedicated to a task in a given period of time, the higher is the expected value of the change in the quality of its results in the following period.

Formally, the agent has to choose  $a_i(t)$ , the fraction of effort available to dedicate to activity  $i \in \{1, \dots, I\}$  in period  $t \in \{0, 1, \dots\}$ . Activity  $i$  in period  $t$  attains the performance level  $U_i(t)$ . The evolution of this level from period  $t$  to the next is  $Z_i(t+1) = U_i(t+1) - U_i(t)$  and is assumed to follow a random walk. More specifically,  $E(Z_i(t+1)) = a_i(t)\eta_i + (1 - a_i(t))(-\epsilon_i)$ , with  $\eta_i, \epsilon_i > 0$ .

Three rules of thumb are studied. The first one is time invariant and is named *constant proportions behavior*. It is based on the distribution of effort by activities in the same way in every period, that is  $\forall t \in \{0, 1, \dots\}, \forall i \in \{1, \dots, I\}, a_i(t) = a_i$ . The second rule focuses on controlling damages and is named *putting out fires*. It prescribes that, in each period, all effort is devoted to the worse-performing activity in the previous period. In case there is more than one, the first criteria of selection has a stay-put logic: choose the one that was selected before. If none of them fits this criteria, the one with the lowest index is chosen. Finally, the *staying with the winner* rule favors the best performing activities. It states that, in each period, the activity with the highest performance concentrates all effort, with the lowest index number serving again as a tie break rule. These rules are evaluated with the aid of two concepts: *survival*, which is verified when no activity performance is ever lower than a certain threshold, and the average growth of activity performance over time. They

find that, if activities do not require much effort to have improved performances, that is, if  $\eta = (\eta_i)_{i \in \{1, \dots, I\}}$  and  $\epsilon = (\epsilon_i)_{i \in \{1, \dots, I\}}$  have, respectively, high and low components, there is at least one behavior that generates survival with positive probability, and at least one of them is a constant proportions behavior. Given some conditions on  $Z$ , they also find that the use of the putting out fires rule implies that all activities' performances will almost surely have the same average growth rate. The repeated choosing of the staying with the winner rule will eventually select one single activity as the object of all effort, although the specific activity and period of time in which this selection begins cannot be known in advance. This means that the chosen activity will have a performance with a positive average growth, while this measure is negative for all the remainder activities.

Notice still that the staying with the winner rule is close in spirit to the adaptation method employed in [Arthur \(1991\)](#) and [Arthur \(1994\)](#), because it predicts that success attracts choice. On the contrary, the putting out fires rule goes in the opposite way.

## 1.2.2 Optimize, but what?

In this section, maximization is restored as a way to solve problems. However, the function to be maximized is not the same as in classic rationality. If people are faced with a complex problem, they may be tempted to replace it with a different one, where optimization is easier. This is the case of the models in [de Palma \*et al.\* \(1994\)](#), [Gabaix \*et al.\* \(2006\)](#) and [Kőszegi & Szeidl \(2013\)](#). The first two are intimately connected by the notion of myopia. They propose that an agent splits the original problem in different smaller and easier to solve problems, but fail to take into account the effects the decisions they make in each of them have in what follows. However, their settings are very different: [de Palma \*et al.\* \(1994\)](#) analyzes consumers who prefer to buy one good at a time, instead of buying an entire bundle, and [Gabaix \*et al.\* \(2006\)](#) investigate the sequential costly analysis of choices with uncertain payoffs. [Kőszegi & Szeidl \(2013\)](#) also has agents that maximize an objective function with which they can easily cope, but, contrary to

the previously cited papers, there is no partition of the original problem. Instead, an objective function that puts more weight in the characteristics of an object that distinguish it from the other objects is the center of analysis.

de Palma *et al.* (1994) model a consumer who differs from the classic rationality paradigm in three ways. First, instead of deciding which consumption bundle to acquire, he splits his income through different periods and, in each of them, decides on which good to use it. Second, he cares not about the level of each good he has, but about the average rate of consumption of the available stock of each good. Finally, he is unable to see the true benefit from choosing any good, having a distorted perception of it. The bounded rationality of the agent is manifested in the way he simplifies the hard problem of choosing all goods' quantities at once, making more but easier decisions, and in the perception errors he makes as a result of his difficulty in processing information.

There is a set of  $n$  goods from which to choose in a game of duration  $T$ . The consumer has an income  $Y \in \mathbb{N}$ , of which he spends 1 unit in each period. The average income spent per unit of time is then  $y = \frac{Y}{T}$  and the length of each period is  $l = \frac{1}{y}$ . There are  $Y$  periods, which correspond to the intervals  $[(k-1), k]l$ , with  $k \in \{1, \dots, Y\}$ . The stock of goods the agent possesses in the beginning of period  $k$  is  $S^k = (S_i^k)_{i \in \{1, \dots, n\}}$ . The initial endowment of the agent is  $S^1$ . In period  $k$ , the agent consumes uniformly the stock of each good, but does not necessarily exhausts it. The intensity of consumption of good  $i$  is given by  $c_i \in [0, \frac{1}{l}]$ , and the consumption of good  $i$  in period  $k$  after an amount  $\tau \in [0, l]$  of time has elapsed since the beginning of the period is  $C_i^k(\tau) = \tau c_i S_i^k$ . The rate of consumption of good  $i$  in period  $k$  is then:

$$q_i^k = \frac{C_i^k(l)}{l} = c_i S_i^k$$

In the beginning of each period, the agent has to decide in which good to spend 1 unit of his income. Let  $x_i^k \in \{0, 1\}$  define if, in period  $k$ , the agent acquires good  $i$ , where 1 stands for yes and 0 for no. The price of good  $i$  is  $p_i$ , hence, if  $x_i^k = 1$ , the agent acquires  $\frac{1}{p_i}$  units of good  $i$ , which he adds to his stock. This means that

the available stock of each good in the beginning of each period has the following dynamics:  $\forall k \in \{2, \dots, Y\}, S_i^k = (1 - lc_i) S_i^{k-1} + \frac{x_i^k}{p_i}$ . The agent derives utility not from the possession of each good, but from its rate of consumption. That is, the objective function he seeks to maximize in each period,  $v$ , depends on  $(q_i^k)_{k \in \{1, \dots, n\}}$ , and is positively affected by each  $q_i^k$ . To solve his problem, the agent engages in a process of *melioration*. That is, instead of solving the harder global problem, he myopically chooses, in each period, the good which seems more attractive to him, ignoring the consequences of this choice in the subsequent periods. If he knew perfectly the consequences of choosing good  $i$  in period  $k$ , his objective, in this period, would be to maximize  $\Delta_i^k v$ , the increment in utility obtained from choosing good  $i$ . However, the agent is unable to know this function and maximizes, instead,  $\Delta_i^k u$ . It is assumed that  $\Delta_i^k u = \Delta_i^k v + \varepsilon_i^k$ . The error made in assessing  $\Delta_i^k v$  is  $\varepsilon_i^k$ , a random variable which variance represents the ability of the agent to choose. The lower the variance, the higher the ability. The case in which  $\forall i \in \{1, \dots, n\}, \forall k \in \{1, \dots, Y\}, \varepsilon_i^k = \frac{\varepsilon}{\mu}$  is studied, where the ability to choose is positively related to  $\mu$ . They get the very intuitive result that all good tend to be chosen with the same probability in each period, when  $\mu$  approaches 0, and that, when  $\mu$  approaches  $+\infty$ , the probability of choosing the good that maximizes  $\Delta v_i^k$  in period  $k$  tends to 1.

The stationary behavior of the model is then analyzed. The stationary value of the true utility function is  $\bar{v}$  and  $\sigma^2$  is defined as the variance of the ratio between the true marginal utility of good  $i$ ,  $\frac{\partial \bar{v}}{\partial q_i}$ , and its marginal cost,  $p_i$ . It is concluded that  $\frac{\partial \bar{v}}{\partial \mu} = y\sigma^2$ , which has two implications. First, the true utility obtained increases with the ability to choose, which means there is a kind of satisficing behavior, as in [Simon \(1955\)](#), in the sense that the agent is happy with a below optimal utility level. Second, the impact of the ability to choose in the stationary utility level depends on the average expense per unit of time and on the dispersion of the ratio of marginal utilities and costs. In fact, if either more money is available to spend, or the quality of alternatives is more dispersed, the consequences of the choices made become more important and an increase in the ability to choose is more valued.

The authors add some policy insights to the model. They claim that, if peo-

ple have the possibility of increasing their ability to choose by investing in their education, but fail to recognize the benefit of doing it, a coercive mechanism like mandatory schooling can benefit society. Also, if imperfect ability to choose is indeed a reality, manipulative advertising should be monitored. Finally, the social optimal level of product differentiation in the Hotelling model is increasing in the agents' ability to choose, and the encouragement of product diversity should take this into account.

[Gabaix et al. \(2006\)](#) studies a model of sequential investigation of objects with unknown payoffs. An agent is confronted with a choice, but does not know the value of each of the available alternatives. To resolve this uncertainty, he can investigate the objects' payoffs, one at a time. However, investigation is costly, which means the number and order of investigations is not irrelevant for the level of satisfaction the agent gets when he finally selects one object. If the agent solved this problem in an optimal way, this would be a costly thinking or an uncertainty handling model.<sup>2</sup> However, the authors' central idea is the maximization of an objective function different from the one a classic rational agent would maximize. They propose what they call the *Directed Cognition Algorithm* as a way to address this issue. It consists of a myopic view of the problem, treating the decision taken in each period as if it were the last one, and ignoring the impact that a decision made in one period has on the following ones. On contrary, the classic rationality approach, which predicts a global view of the problem, takes into account, in each period, the fact that the investigation can continue or not, depending on the results obtained to date. The Directed Cognition Algorithm is applied to two different problems.

The first problem is the simplest one. There are  $n$  objects,  $X_i$  being one of them. The payoff of object  $i$ ,  $U_i$ , is unknown, but can be uncovered upon investigation. Object  $i$  can either be a *loser*, in which case  $U_i = 0$ , or a *winner*, and  $U_i = V_i > 0$ . The probability attached by the agent to the fact that object  $i$  is a winner is  $p_i$ . In each period, the agent may either select or investigate the payoff of one of the objects, which then becomes known. When one object is finally chosen, the selection

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<sup>2</sup>In this case, the model would be analyzed in [Section 1.2.3](#) or [Section 1.2.4](#).



procedure stops. A classic rational agent associates each object with a *sure value* that, if available, would make him indifferent between investigating the object or accepting the value. The sure value for object  $i$  is  $Z_i \in [0, V_i]$ . If  $W_i = 0$ , the agent, confronted with this choice, would prefer to receive  $Z_i$ . But if  $W_i = V_i$ , the agent's choice would rely on  $X_i$ . This means that the expected benefit of investigation of object  $i$ , when the sure value  $Z_i$  is available, is  $p_i (V_i - Z_i)$ . The cost of investigating an object whose payoff is unknown is  $c_i$ . This means that the sure value of object  $i$  is:

$$Z_i = \begin{cases} 0 & , \quad p_i = 0 \\ V_i - \frac{c_i}{p_i} & , \quad 0 < p_i < 1 \\ V_i & , \quad p_i = 1 \end{cases}$$

Optimality is achieved if the object with the highest sure value is investigated in each period and selected in the next if it turns out to be a winner. Otherwise, a new investigation over the object with the next higher sure value is performed. If all objects are losers, the agent selects one of them randomly after investigating all of them. The Directed Cognition Algorithm proposes a simpler procedure. In period  $t$ ,  $S^t$  is the payoff of the best known winner at the time. If no winner is known at  $t$ ,  $S^t = 0$ . The investigation of object  $i$  at period  $t$  brings a benefit of  $V_i - S_t$ , if  $V_i > S_t$  and  $i$  reveals to be a winner, and 0 otherwise. The agent pays a cost of  $c_i$  to know this, except if  $i$ 's payoff is known in advance. The expected net benefit of investigating object  $i$  at period  $t$  is then:

$$G_i^t = \begin{cases} 0 & , \quad p_i = 0 \\ p_i \max \{0, V_i - S^t\} - c_i & , \quad 0 < p_i < 1 \\ \max \{0, V_i - S^t\} & , \quad p_i = 1 \end{cases}$$

In each period, the agent investigates the object with the highest expected net benefit of investigation. Uncovered winners are chosen after being revealed, and the finding of a loser implies a new investigation following the same rule. Again, if all objects are losers, one is chosen at random after all the investigations are performed. Notice that this algorithm, even though inducing the agent to act optimally if there

were no periods left for investigating, because the investigation with the highest expected net benefit is conducted, ignores the fact that the disclosure of losers results in another investigation cost. That is, it does not account for the fact that a myopic investigation in each period may lead to an excessive number of investigations over time. This algorithm was put against the classic rationality solution in a series of experiments, and it showed a higher fit to the data than its contestant.

The same idea is applied to the more complicated problem of selection among objects with different features, each with its value, where each object's payoff is the sum of all its features' values. The value of one specific feature of all objects is known in advance, while the remaining are not before being investigated, and are distributed according to a zero mean distribution. In each period, the agent selects one object and a subset of its unknown features to investigate. He is familiar with the variance of the features' values and is assumed to always choose the ones with the highest variance, because they supply more information. When one action is investigated, more of its features' values are known, and the expected value of the object payoff is updated. In period  $t$ , if the agent investigates  $\Gamma^t$  features of object  $i$ , he adds the values uncovered to the expected payoff of object  $i$ , while the other objects' expected payoffs remain unchanged. The expected benefit of doing so,  $w_i^t$ , is the increase in the highest expected payoff among all objects, and is assumed to depend negatively on the distance between the expected payoffs of the best object and object  $i$  before investigation, and positively on the total variance of the features' values investigated. There is a cost of  $\kappa$  per feature investigated. As in the previous, simpler example, an agent following the Directed Cognition model acts myopically and selects, in each period, the object and features that maximize the ratio between the expected benefit and cost implied. If time is limited, the agent does so until time is exhausted, and selects the object with the highest expected payoff in the end. If there is an amount of time to be endogenously split by different versions of this game, the investigating operations in each version continue until the ratio benefit / cost falls below the marginal value of time of the agent. Notice that, even though [Gigerenzer & Goldstein \(1996\)](#) also deal with the problem of deciding which is the best choice by observing their features, the features they refer are not something

from which the agents extract utility, but simply indicators of which alternative is better. The comparison between these two models also helps to contrast the logic of the present section and the previous one. In both models, it is possible that agents only observe some of the features of each choice. However, if in [Gigerenzer & Goldstein \(1996\)](#) agents stop searching because they are happy with possibly sub-optimal decision they will make, in this model they do it because the cost of further search exceeds its benefits.

This model is confronted with experimental data and three satisficing models: one has the same investigation structure of the Directed Cognition model and the others fully explore one object or feature, before moving to another. They all have in common the fact that, when the stopping time is endogenous, the process stops when the object with the highest expected payoff is better than some aspiration level. Finally, the model of Elimination by Aspects, developed in [Tversky \(1972\)](#), in which objects are compared feature by feature, and are eliminated when their features are below some aspiration level, until only one remains, is also put to the test against the experimental data. In almost all criteria used, it is the standard Directed Cognition Model that takes the lead.

[Kőszegi & Szeidl \(2013\)](#) assume bounded rationality on agents by stating that they are incapable of correctly comparing the alternatives they have. Faced with objects that have a number of features, from which they derive utility, they are not able to calculate the true total benefit of the features of each choice. Instead, they integrate these benefits in a way that puts more weight in the features which have greater discrepancies in benefits between choices. In contrast to [Gigerenzer & Goldstein \(1996\)](#) and [de Palma \*et al.\* \(1994\)](#), the problem is not knowing the features of each object and the utility they create. Agents know all of these, but cannot integrate the information they have without being influenced by the salience of each feature.

An agent has to choose an object  $c$  in the choice set  $C \subseteq \mathbb{R}^K$ . Each of the  $k$  coordinates of  $c$  represents one of its features. The way the agent values feature  $k \in \{1, \dots, K\}$  is represented by  $u_k$ . Total utility is the sum of partial utilities

of features, which means an unboundedly rational agent would choose the object that maximizes  $U = \sum_{k=1}^K (u_k)$ . However, the agent focuses more on features in which there are higher utility differences. The weight attributed to feature  $k$  is  $w_k = g(\max_{c \in C} (u_k(c_k)) - \min_{c \in C} (u_k(c_k)))$ , where  $g$  is a strictly increasing function. The function maximized by the agent is a weighted sum of the utility of each feature,  $\tilde{U}$ , defined by the following expression:

$$\tilde{U}(c) = \sum_{k=1}^K (g_k u_k(c_k))$$

This model has four main implications. *Bias toward concentration* means that if object  $c$  is much better than the others in some features, and, in the remaining features, there are not very high discrepancies, chances are that  $c$  will be chosen. *Increasing concentration* implies that gathering the advantages one object has over the others in less features enhances the probability of its selection. *Rationality in balanced trade-offs* and *More rationality in more balanced choices* refer to the fact that agents tend to select the utility maximizing object when the number of features in which it is best than its alternatives is close to the number of features in which it is worse. All of these help us understand that agents with this type of bounded rationality are attracted to choices which are not necessarily the best, but that capture their attention for being particularly good in a few domains, although this bias is less important when the number of advantages and disadvantages of alternatives is close.

The model can help explain the importance of *framing effects*, which alter the choices people make simply by restating the problem they are faced with in different terms. In this context, the restating of the problem effectively changes it, as it can change the focus of people's attention. Also, it can be used to study intertemporal choice, if we consider that each choice profile is an object and the choice in each period is a feature. This interpretation sheds some light on why are people time-inconsistent, in the sense that the actions that they plan to make are not made when it comes the time, and present-biased if, in each period, they overvalue present benefits or costs relative to the future, not taking the action that would be best for

them in the long term. Contrary to hyperbolic discounting, which explains these phenomena with the way time is discounted, this model says that the reason for their existence lies on the fact that the salience of future costs or present benefits can induce a present-oriented choice, and that the plannings of actions extended in time and of the decision to be taken in each period imply different sets of alternatives, with different focus attractions.

### 1.2.3 Costly thinking

Decisions are, most times, not instantaneous nor easy to take. Besides requiring the gathering of information, they also imply the exertion of some mental effort and possibly some time to be made. Thus, for a boundedly rational agent, decision-making is not free but, instead, entails a cost. In this section, we study models which incorporate this cost explicitly in the problem in which it arises, thus changing the original objective function. The models we study either feature costly decisions (Conlisk, 1980, Reis, 2006b,a), costly uncertainty handling (Evans & Ramey, 1992, Conlisk, 1996a) or a solution to a conceptual problem arising from the introduction of costly thinking (Lipman, 1991).

Conlisk (1980), Reis (2006b) and Reis (2006a) are three models in which agents have to decide whether to optimize or not. They are able to find the optimal solution for all problems they have to solve, but the existence of a thinking cost may lead them to choose not to. However, the non-optimizing behavior of agents in the first paper is very different from the one in the other two. While in Conlisk (1980), a person has to decide, as a child, whether to always optimize or to always imitate what she sees, in Reis (2006b) and Reis (2006a), an agent, when deciding, has to design a plan of action for some periods in the future, during which he will not pay attention to news relevant to his decisions. Also, he needs to specify when he will decide again. Hence, imitation in the first paper is replaced by a fixed plan of actions which, in general, will not be optimal, due to external shocks.

Conlisk (1980) models a society where some people take optimal actions, paying a cost for doing so, while the rest do some kind of imitation, and are exempted of

such cost. In the model's dynamic setting, people's roles as *optimizers* or *imitators* is defined when they are children, and this decision is influenced by the success and prevalence of optimizer behavior in the past. The fraction of optimizers in the society evolves over time, and, intuitively, converges to 1 if optimizing is cheap enough, and to a lower value, if optimizing is expensive enough.

Agents are indexed by  $i \in \{1, \dots, M\}$  and time is indexed by  $t \in \{1, \dots\}$ . In period  $t$ , agent  $i$  has to choose a consumption basket  $x_i^t \in \mathbb{R}^n$ . His preferred choice in this period is  $w_i^t = Z^t + u_i^t$ , where  $Z^t$  is common to all society in that period and evolves over time with the adding of a zero mean random variable, and  $u_i^t$  is a zero mean random variable private to him. Utility is defined quadratically in terms of losses. The perception imitators have of  $Z^t$  does not necessarily correspond to its true value, and is defined by  $T^t$ . Its evolution through time is given by  $T^t = \lambda X^{t-1} + (1 - \lambda) T^{t-1}$ , where  $X^t$  is the average choice of all agents in period  $t - 1$ . Thus,  $T^t$  depends on the previous period's perception of  $Z^t$  and average choices, with higher values of  $\lambda$  making it more responsive to what society has been choosing. The utility function to be maximized,  $v$ , has the following expression:

$$v(x, w) = (x - w)^T Q (x - w)$$

An optimizer agent  $i$  chooses the optimal action,  $w_i^t$ , but pays a cost of  $C$  for doing so. An imitator avoids such cost, but is unable to select  $w_i^t$ . Instead, he chooses  $T^t + u_i^t + v_i^t$ . That is, he replaces  $Z^t$  with the impression he has of it, possibly with error, represented by  $v_i^t$ , a zero mean random variable. Hence, the average choice of optimizers and imitators is  $Z^t$  and  $T^t$ , respectively. The average society choice in period  $t$  is  $X^t = A^t Z^t + (1 - A^t) T^t$ . To close the model, one needs to define the fraction of optimizers in each period. For that, a performance indicator of optimizer relative to imitator behavior in period  $t$ ,  $D^t$ , is introduced. It is assumed to evolve according to the following rule:

$$D^t = \mu \left( \frac{E(\text{Imitator Loss}^t)}{\text{Optimizer Loss}} - 1 \right) + (1 - \mu) D^{t-1} \quad (1.3)$$

The higher the value of  $\mu$  in (1.3), the more the performance indicator reacts to the present quality of optimizers' decisions, relative to imitators. A person is a child for one period, and an adult for the rest of her life, with is finite and has the same duration for everyone. Population is stable, as the number of births and deaths is the same in each period. A role of imitator or optimizer is attributed to each child in each period, and she keeps her role for the rest of her life. The probability that a child born in period  $t$  becomes an optimizer is  $a^t = f(A^{t-1}, D^{t-1})$ , with  $f$  being increasing in both  $A^{t-1}$  and  $D^{t-1}$ . That is, the higher the fraction and performance level of imitators in the previous period, the higher the probability a child becomes an optimizer. The evolution of  $A^t$  is then studied, and two main conclusions are presented. First, if the average loss of imitators is always higher than  $C$ , the loss of optimizers,  $A^t$  converges to 1. That is, if imitators, who are exempted from an optimizing cost, can never outperform optimizers, they tend to disappear over time. Second, if the optimization cost is low enough,  $A^t$  does not converge to 1. In this case, an approximate value of the limit of  $A^t$  is obtained, which intuitively depends positively on the variance of imitator's optimal choices (as it increases their losses) and negatively on  $C$ , the optimizing cost, and  $\lambda$ , the speed of adjustment of the imitators' perception of  $Z^t$ . Although the model is not very micro-founded, in the sense that individual behavior is assumed and not derived, it displays an ingenious way to account for the fact that thinking is costly, and shows that optimizing and non-optimizing behavior may co-exist.

Reis (2006b) and Reis (2006a) are two papers that apply the same concept, *optimal inattention*, to the producer and consumer problem, respectively. Optimal inattention applies to dynamic models, affected by shocks, which induces classic rational agents to make a choice every period, according to the news that affect their decisions. However, boundedly rational agents have a cost of making decisions, because it implies the collection of information and the execution of mental operations. Hence, instead of making new decisions every period, inattentive agents, when deciding, make plans for when to decide again and what to do in the meanwhile. In this sense, this idea is close in spirit to Evans & Ramey (1992). That paper focus on expectations about inflation, but the logic of reacting to what is happening in the

economy only in some periods and the fact that it is costly to do so are also there.

The basic framework is the same for both models. Time is continuous and indexed by  $t$ .  $D_i$  is the time at which the  $i^{\text{th}}$  plan is made, with  $i \in \mathbb{N}_0$ . The agent plans in the beginning, hence  $D_0 = 0$ . There is a shock in each period, represented by the stochastic vector  $x^t$ . In the case of producers, the shock affects demand and costs and, in the case of consumers, income. A plan made at time  $t$  implies a cost  $F(x^t)$ , which is an incentive for long periods of inattentiveness. Such plan has two parts: the value of  $D_{i+1}$ , that is, the time at which the next planning will take place, and  $z_i = [z^{D_i}, z^{D_{i+1}}[$ , the choices to be made until then. These choices refer to one of two variables, in each model. In the producer case, the firm has to decide between fixing prices or quantities and, in the consumer case, the decision is between fixing consumption or savings levels. Importantly, whatever is decided in a plan, has to depend only on what the agent knows at the time. All plan times and choices profiles are defined to maximize time-discounted total utility net of planning costs over time.

In the producer model, there is a monopolist, which faces a negatively sloped demand  $Q(x, P)$  and has a cost function  $C(x, Y)$ . At each planning period, he chooses whether to fix prices or quantities, according to the maximum profit he can get in both cases. Importantly, if  $t \in [D_i, D_{i+1}[$ , the maximum profit the monopolist expects to get at time  $t$  can only depend on  $x^{D_i}$ , because that was the information he obtained when he last updated. The sum of time-discounted expected profits net of planning costs then determines the set of planning times. It is shown that the length of inattentiveness depends positively on the planning cost, because firms avoid such cost by planning less often. On the contrary, it is decreasing in the expected loss in profits that results from following a predetermined plan, instead of optimizing. The more impact inattentiveness has on profits, the more often the monopolist decides to plan.

In the consumer model, income in period  $t$  is  $y(x^t)$ . The agent has to decide how much to consume,  $c^t$ . What is left,  $s^t = y(x^t) - c^t$ , is savings. The stock the agent possesses at time  $t$ ,  $a^t$ , generates returns according to the interest rate  $r$ .



Thus, from period  $t$  to the next,  $ra^t$  and  $s^t$  are added to the assets, and the planning cost is subtracted, if planning occurs. Borrowing is allowed, but Ponzi schemes are ruled out. The consumer derives utility  $u(c^t)$  from consumption  $c^t$ . The set of planning dates and paths of consumption or savings while the consumer is inattentive is then chosen to maximize time-discounted time utility. Note that, contrary to what happens in the producer model, where the planning cost is part of the objective functions, here its impact is felt in the assets' dynamics. A higher planning cost means there are less assets at planning times, which reduces consumption and, therefore, utility. When several consumers are considered, some interesting conclusions arise. As one would predict, consumption reacts slowly and with delay to news, as only some consumers pay attention to them instantaneously. Under some assumptions about functional forms, the length of inattentiveness is shown to depend on some parameters. If shocks are more volatile, or consumers are more risk averse, planning is more frequent, because consumers dislike the risk associated with not reacting in each period. If planning costs are high, the same effect occurs. And, if the interest rate increases, planning becomes more frequent. The reason is that the absence of instantaneous optimization leads consumers to generate sub-optimal savings. The higher the interest rate, the higher the impact of such sub-optimality.

[Evans & Ramey \(1992\)](#) and [Conlisk \(1996a\)](#) focus on costly uncertainty reduction. But, if in [Evans & Ramey \(1992\)](#), the uncertainty is about future events which cannot be known in advance, in [Conlisk \(1996a\)](#) the uncertainty is a consequence of the bounded rationality of agents, who are incapable of knowing what the optimal decision is. In this way, the thinking cost is a consequence of the need to form expectations in the former model, and to find the contemporaneous optimal solution in the latter.

[Evans & Ramey \(1992\)](#) present a model where the update of expectations is costly. Inflation evolves over time depending on the expectations the agents have about it. However, as changing previous expectations comes with a cost, there is the possibility that, in some periods and for some firms, expectations are not rational in the classic sense (although, given the cost of their formation, they are chosen in

an optimal way). Monetary authorities, being aware of such cost, can induce firms to update expectations or not, depending on the objectives they have. A Phillips curve relating output and inflation is obtained.

There is a continuum of firms in the interval  $[0, 1]$ . The logarithm of aggregate output and its price, henceforward named output and price, in period  $t$  is, respectively,  $y^t$  and  $p^t$ . The price firm  $\omega$  expects in period  $t$  is  $p_e^t(\omega)$ . This implies that the inflation rate firm  $\omega$  expects at period  $t$  is  $\beta^t(\omega) = p_e^t(\omega) - p^{t-1}$ . The assumed relationship between output and price, aggregate demand and money supply generate the following real inflation rate in period  $t$ :  $\Delta p^t = \frac{\gamma}{1+\gamma} \int_0^1 (\beta^t(\omega) d\omega) + \frac{1}{1+\gamma} (g + v^t)$ , where  $g$  is a drift parameter controlled by the monetary authority and  $v^t$  is white noise. The loss generated by an error in expectations,  $h$ , is assumed to be quadratic, and has the following expression:  $h(\Delta p, \beta) = k(\Delta p - \beta)^2$ , with  $k > 0$ . Notice that this expression implies that firms are myopic, since they only care about how much their expectations in each period are different from real inflation, neglecting the impact of their expectations on the long-run. In this sense, this model has some similarities with [de Palma \*et al.\* \(1994\)](#) and [Gabaix \*et al.\* \(2006\)](#). Firms pay a cost of  $c$  if they want to update their expectations from one period to the next. If a firm  $\omega$  chooses not to in period  $t$ , it keeps its expectation unchanged, that is,  $\beta_n^t(\omega) = \beta^{t-1}$ , where  $n$  stands for not update. Otherwise, it forms rational expectations and calculates expected inflation in period  $t$  in the right way, but using the firms' expectations about inflation from period  $t-1$ :  $\beta_u^t(\omega) = \frac{\gamma}{1+\gamma} \int_0^1 (\beta^{t-1}(\omega) d\omega) + \frac{1}{1+\gamma} g$ , where  $u$  stands for update.

In the rational expectations equilibrium, in which expectation updating is costless,  $\Delta p = g + \frac{1}{1+\gamma} v^t$ , hence the closer inflation expectations are from  $g$ , the more rational they are, and the lower the loss they imply. Intuitively, equilibria in one period in which no firm updates its expectation are possible if the previous period's expectations were close enough to  $g$ . In the same logic, equilibria in which all firms update are possible whenever the previous period's expectations were far from  $g$ . Assuming all firms begin with the same expectations, two equilibrium dynamics are possible. Either  $g$  is such that, in the first period, no firm wants to update,

which means expectations remain unchanged forever, or all firms want to update in this period. As the update implies the choosing of a weighted average between the period's expectations and  $g$ , its application results in an approximation of expectations to  $g$ . This means that, in some period, expectations are close enough to  $g$  so that all firms prefer not to update from then on. The monetary authority, knowing this, chooses  $g$  to force one of these two dynamics. The different values of  $g$  result in different long-run equilibria. It is possible to get a higher output if inflation is higher, but there are boundaries for output and price, as a result of the need to induce one specific type of equilibrium. The important message to take from this model is that the explanation for the fact that monetary policies can have a real effect in the economy may lie on the cost of expectation updating.

[Conlisk \(1996a\)](#) applies the notion of costly thinking to firms' decisions. With the goal of understanding what is the effect of costly thinking on the dispersion of market variables, he states that firms, unable to costlessly choose the optimal quantity in each period, are equipped with a deliberation technology that leads them to choose a weighted average between an approximation to optimal output and a default quantity. Bounded rationality is manifested in the fact that the higher the effort put in finding a good approximation to the ideal output, the higher the thinking cost. The influence of this cost in firms' choices is natural: a higher unit cost means less effort put on finding the ideal quantity and a chosen quantity closer to the default one. On the other hand, it can increase or attenuate market fluctuations, because its interaction with the rest of the model produces effects that go in opposite ways.

There are  $n^t$  firms producing a homogeneous good in period  $t \in \{1, \dots\}$ . In this period, firm  $i$  has to decide the quantity to produce,  $q_i^t$ . Demand is given by  $Q(p^t) = S(a - bp^t) + \sqrt{S}z^t$ , where  $S$  represents market size and  $z^t$  is a random variable, with zero mean and variance  $\sigma_z^2$ . Firm  $i$ 's production cost in period  $t$  is given by  $C_i^t(q_i) = W + \frac{q_i^2}{2w_i^t}$ , where  $w_i^t$  is a random variable with mean  $\mu$  and variance  $\sigma_w^2$ . The quantity that maximizes profits in period  $t$  is  $w_i^t p^t$ . As firms have to choose quantities before price is known, the quantity a rational firm  $i$  would choose in period

$t$  is  $q_i^{tr} = w_i^t E(p^t)$ . The quantity a boundedly rational firm  $i$  chooses to produce is  $q_i^{t*}$ . This firm has a default quantity, corresponding to the long-run average quantity produced by firms:  $q_i^d = \frac{E(\sum_{j=1}^n(q_j^*))}{n}$ . The firm does not know  $q_i^{tr}$ , but is able to find  $f_i^t(q_i^{tr}) = q_i^{tr} + \frac{u_i^t}{\sqrt{h_i^t}}$ , where  $u_i^t$  is a zero mean random variable with variance  $\sigma_u$ , and  $h_i^t \geq 0$  is the level of deliberation of firm  $i$  in period  $t$ . However, the firm pays a cost of  $c$  for each unit of  $h_i^t$  chosen. Bounded rationality is also expressed in the fact that the quantity chosen by firms is not what they consider to be the ideal quantity, but a weighted average of the perceived ideal and default quantities:  $q_i^{t*} = \alpha_i^t f_i^t(q_i^{tr}) + (1 - \alpha_i^t) q_i^d$ , where  $\alpha_i^t \in [0, 1]$  is to be chosen. The problem of firm  $i$  in period  $t$  is then:

$$\max_{h_i^t, \alpha_i^t} (E(v_i^t(h_i^t, \alpha_i^t)) = E(p^t q_i^{t*}(h_i^t, \alpha_i^t) - C_i^t(q_i^{t*}(h_i^t, \alpha_i^t)) - h_i^t c))$$

As firms are symmetric, so is the solution to his problem. Besides, the equilibrium is time invariant, which means that  $(h_i^{t*}, \alpha_i^{t*}) = (h^*, \alpha^*)$ . Free entry is assumed, so  $n^t = n$  is such that  $E(v_i^t(h^*, \alpha^*)) = 0$ .

Closed form solutions are obtained for the limit case of perfect competition, in which  $S \rightarrow +\infty$ . The first intuitive result presented is that, given some conditions, if  $c$  is large enough, no thinking occurs, and firms totally rely on their defaults, that is,  $h^* = \alpha^* = 0$ . If deliberation is too costly to be initiated, firms decide to abandon it completely, which makes the approximation to the ideal output totally uninformative, leading firms to simply choose their default quantity. When the parameters are such that some deliberation occurs and a positive weight is given to the approximation to the ideal output, the variation of deliberation cost provides intuitive comparative statics. When  $c$  is approaching 0,  $h^*$  is growing unboundedly and  $\alpha^*$  is getting closer to 1: thinking is getting almost free, hence firms tend to behave rationally in the classic sense. When  $c$  is increasing, both  $h^*$  and  $\alpha^*$  are near 0. In general, it is possible to say that price is increasing and the number of firms and individual quantities are decreasing in  $c$ . Deliberation cost is something firms want to avoid, just as production cost, so it makes sense that an increase in  $c$  contracts the market. As to the main question of the paper, that is, what

is the impact of the introduction of a deliberation cost in market fluctuations, no general rule is obtained. In what refers to firms, an increase in  $c$  or  $\sigma_u$  (a higher  $\sigma_u$  means a lower ability of firms to get a good approximation of  $q_i^{tr}$ ) implies a lower  $h^*$  and, therefore, a higher variability of firms' decisions, but also a higher  $\alpha^*$ , which means firms rely more on their default quantity, and the variance of firms' choices is reduced. The total effect is ambiguous. As to markets, an increase in  $c$  or  $\sigma_u$  raises the expected price and lowers the number of firms per consumer. An increase in the expected price generates a higher variance of the ideal quantity (remember that  $y_i^{tr} = w_i^t E(p^t)$ ), increasing market fluctuations, but this effect is counterbalanced by the reduction in the number of firms per consumer. Once again, no general conclusion can be presented.

All the models cited in this section suffer from a conceptual problem, labeled *infinite regress*. This problem arises when we assume that thinking is costly, and people are able to decide whether to think or not and by how much in an optimal way. If they are not able to choose the optimal action without incurring in a mental cost, why would they be able to costlessly solve the thinking problem? This could be solved by the stating that people pay a cost for solving this problem, but then how to solve the new problem created would also be an issue, and this reasoning continues indefinitely. [Conlisk \(1996a\)](#) acknowledges the existence of such problem, but, nonetheless, defends models which stop the reasoning process in the second level. That is, models in which thinking is costly, but thinking in thinking is costless. He provides two arguments that sustain this defense. The first states that, although logically imperfect, costly thinking models are an upgrade relatively to the unbounded rationality paradigm, as they account for the fact that some problems are hard to solve, even if they do it in an imperfect way. The second distinguishes the familiarity level of agents with any given problem and the problem on thinking about the first one. He claims that people are not very often confronted with each problem they have to solve in their daily lives, but are constantly solving thinking problems. Thus, costless optimization in the former problems seems implausible, but is natural in the latter. [Lipman \(1991\)](#), however, puts the infinite regress problem into a perspective that allows him to conclude that it is solvable. That is, given

some conditions, there is a level of thinking beyond which people's reasoning stop, because there is no advantage in deepening it.

To better understand the model [Lipman \(1991\)](#) develops, let us illustrate it with the example the author provides. There is a choice set  $A$ . At the simplest thinking level, the agent has to choose an element in  $A$ , so his decision set at this level is  $D^0 = A$ . Suppose an agent, who values money linearly, is offered the alternative of accepting  $x\$$ , or  $f(1)\$$ , where  $f$  is a non-trivial function. Then,  $D^0$  can be defined as  $\{0, 1\}$ , where 0 stands for accepting  $x\$$ . The optimal action would be to choose the  $x\$$  if and only if  $f(1) \leq x$ , but a boundedly rational agent is unable to find, with certainty, the value of  $f(1)$ . This uncertainty about each action's payoff is represented by  $S_0$ , which contains the payoffs the agent considers possible. In the example, the agent, who knows the consequences of accepting  $x\$$ , may believe that  $f(1) \in \{y_1^0, y_2^0\}$ , with  $y_1^0 < x < y_2^0$ .<sup>3</sup> Hence,  $S^0 = \{s_1^0, s_2^0\}$ , and, in the agent's mind,  $f(1) = y_1^0$ , if  $s_1^0$  occurs, and  $f(1) = y_2^0$  otherwise. If thinking is done in this setting, than the agent works with the utility function  $u^0 : D^0 \times S^0 \rightarrow \mathbb{R}$ . In the example,  $u^0$  has the following expression:

$$u^0(d^0, s^0) = \begin{cases} x & , \quad d^0 = 0 \\ y_k^0 & , \quad (d^0, s^0) = (1, s_k^0) \end{cases}$$

However, the agent may wish to refine his beliefs regarding  $S^0$ , and choose between the several methods of investigation he knows for doing so. The set of possible calculations to investigate  $S^0$  is  $C(S^0)$ . Supposing the agent in the example only considers using a computer for finding  $f(1)$ , we have that  $C(S^0) = \{c\}$ , where  $c$  represents the using of a computer. If thinking is at this level, the agent's decision set is  $D^1 = D^0 \cup C(S^0)$ . In the example,  $D^1 = \{0, 1, c\}$ . However, the consequences of investigation at this level may also carry some uncertainty, represented by  $S^1$ . The agent in the example may believe that using the computer allows him to find  $f(1)$ , but may imply a cost of  $y_1^1\$$  or  $y_2^1\$$ , in which case  $S^1 = \{s_1^1, s_2^1\}$ . At this level,

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<sup>3</sup>It is possible to more generally assume that  $y_1^0, y_2^0 \in \mathbb{R}$ , but the specification we make makes the problem non-trivial, avoiding that the agent is sure that  $f(1) < x$  or  $f(1) > x$ , in which case the decision would be easy and there would be no need to reduce uncertainty.

the utility function considered is  $u^1 : D^1 \times S^0 \times S^1 \rightarrow \mathbb{R}$ , which expression, in the example, is:

$$u^1(d^1, s^0, s^1) = \begin{cases} x & , \quad d^0 = 0 \\ y_k^0 & , \quad d^0 = 1 \wedge s^0 = s_k^0 \\ y_k^0 - y_l^1 & , \quad (d^0, s^0, s^1) = (c, s_k^0, s_l^1) \end{cases}$$

Once again, the agent can stop at this level and focus on  $u^1$ , or investigate the consequences of refining his beliefs about  $S^1$ . In the example, in order to know the cost of using a computer, the agent may, for example, read a book or conduct an Internet search about the topic. And the process goes on until eventually the decision set stabilizes at some level. That is, if there is a thinking level  $n \in \mathbb{N}$  such that  $D^{n+1} = D^n$  then  $S^{n+1} = \{s^{n+1}\}$  is a singleton, and thinking at levels above  $n$  does not add anything to the problem, as the uncertainty about  $D^n$ , represented by  $S^n$ , is solved by  $D^n$  itself. The author states that, if uncertainty at each level, or the complexity of the investigations to solve it, are limited, or if we allow long enough sequences of thinking levels, the process eventually stops. This is good news for the costly thinking models we study here. If  $n = 1$ , there is no need in thinking about thinking about thinking, and the problem of deciding whether and how to think may be solved optimally.

#### 1.2.4 An uncertain or different problem

In this section, we present models which focus not on how to solve a problem, nor on the objective function to maximize, but rather on one of the remaining defining elements of an economic problem. We first present two examples of methods for calculating probabilities which may lead to erroneous conclusions, but are easy to use, because they either are associated with a simplified view of the underlying process that generates the probabilities (Rabin, 2002), or dispense the use of all available information (Gennaioli & Shleifer, 2010). Then, we move to two papers which assume that agents prefer to choose from smaller choice sets (Ortoleva, 2013, Lu *et al.*, 2005). Three limits on the human mind that can affect the way parameters

are viewed, namely limited memory (Rabin, 2002), costly attention (Gabaix, 2014) and finite capacity to process information (Sims, 2003) are also presented. We end the section with a quick reference to a model which assumes that people may have a distorted view of the decision variables, as they are unable to understand what their choice means (Lipman, 1999).

Rabin (2002) and Gennaioli & Shleifer (2010) are two examples of changes introduced in the uncertainty operator, in order to make the resolution of uncertainty easier, given the limits on the human mind. They do not necessarily reduce the dimension of the information dealt with or the number of mental operations carried on, but differ from the classic rationality paradigm in how probabilities are calculated. And this influences the way an uncertain objective function is maximized. While Rabin (2002) assumes that people are unable to understand the independence of draws from an i.i.d. process, Gennaioli & Shleifer (2010) model a Bayesian updating that does not use all the information available, but just the more salient one, given the hypothesis being evaluated. These models are useful in explaining some biases documented in probability assessment experiments. While the former helps understand the *gambler's fallacy* and the *hot-hand fallacy*, the latter provides intuition for the *conjunction fallacy*, *disjunction fallacy* and *base-rate neglect*. Common to them is the existence of a parameter that sets the degree of bounded rationality, which implies they are generalizations of the classic rationality case.

In Rabin (2002), time is represented by  $t \in \{1, \dots\}$  and, in each period, a signal  $s^t$  is extracted from a distribution which takes the value  $a$  with probability  $\theta \in [0, 1]$ , and  $b$  with probability  $1 - \theta$ . The signals are i.i.d., which means the drawing from one period does not affect the probabilities of drawings in the future. The probability  $\theta$ , labeled rate, is random, and distributed according to  $\pi$ . The set of rates that occur with positive probability is  $\Theta$ . Boundedly rational agents, which are said to believe in the *law of large numbers* have the correct prior  $\pi$  and know how to do Bayesian updating, but fail to recognize the independence of i.i.d. draws, assuming that the fact that, in one period, one specific value is drawn reduces the probability of it being drawn again in the following period. This is the reasoning of a roulette



player who believes that, after black has been drawn several times, the probability of drawing red is higher than its true value. To model this belief, a method of signal extraction from an i.i.d. process that is conceptually simpler than the real one is designed. The agent believes that the signals are extracted, without replacement, from an urn with  $N \in \mathbb{N}$  signals, where the probability of extracting  $a$  in the first draw is  $\theta$ . This method, simple as it is, raises some technical issues, which demand some extra assumptions. The number of  $a$  and  $b$  signals in the urn must be integer, hence  $\forall \theta \in \Theta, \theta N \in \mathbb{Z}$ . Whatever  $N$  is, an urn with  $N$  balls is unable to explain the drawing of  $N + 1$  signals. To solve this problem, it is assumed that the urn is replaced every two periods by another with the same signal distribution. In this way, if the perceived  $\theta$  is not updated between extractions, probabilities are correctly assessed in odd periods, but biased in even periods. Finally, to prevent the agent from finding some specific signal extraction impossible, there must exist at least one  $\theta$  in  $\Theta$  such that its corresponding urn has at least two  $a$  and two  $b$  signals. Note that, as  $N$  increases, probabilities are less and less biased until, in the limit, the agent is unboundedly rational. The gambler's fallacy, the belief that long sequences of the same signal reduce the probability of its extraction in the future, results directly from this setting. For example, if  $\theta < 1$ , after  $a$  is extracted in the first period, the agent believes that, in the second period, the probability of extracting  $a$  is  $\frac{N\theta-1}{N-1} < \theta$  and the probability of extracting  $b$  is  $\frac{(1-\theta)N}{N-1} > 1 - \theta$ .

The history of signal extraction at period  $t$  is  $h^t$ . The agent, after each extraction, updates his beliefs about  $\theta$  using the Bayesian logic in a correct way, but with biased probabilities. If  $Pr_N^t(h^t|\theta = \hat{\theta})$  is the (biased) probability that an agent who believes in an urn with  $N$  signals attributes to observing history  $h^t$  when the true rate is  $\hat{\theta}$ , then the probability the agent attributes to  $\theta$  being in fact  $\hat{\theta}$  is:

$$Pr_N^t(\theta = \hat{\theta}|h^t) = \frac{\pi(\hat{\theta}) Pr_N^t(h^t|\theta = \hat{\theta})}{\sum_{\tilde{\theta} \in \Theta} (\pi(\tilde{\theta}) Pr_N^t(h^t|\theta = \tilde{\theta}))}$$

Some interesting conclusions are obtained. When observing extractions from different sources, each with its possibly different rate, the agent exaggerates the

dispersion of rates, because, when he observes different sequences of the same signal, he cannot conceive that they are originated by the same rate. If a long sequence of rare signals is observed by the agent, he may update his belief about the true rate, judging the rare signals to be more probable than they really are, which is what the hot-hand fallacy predicts. Also, a firm searching for good employees and firing bad ones underestimates the quality of the workforce, because it fires good employees before finding that they are in fact good.

[Gennaioli & Shleifer \(2010\)](#) studies a way to assess some features of an object through others, but in a biased way that looks only to what is more salient during the reasoning process. It is then close to [Kőszegi & Szeidl \(2013\)](#), which state that a higher focus is put on what distinguishes an option from the alternatives. To better understand the way the reasoning is modeled, let us illustrate it with an example. A firm wants to hire a manager. There are three informations the firm would like to know about each candidate: whether he passes or fails a theory exam, whether he is a good or bad manager, and whether the firm has losses or a positive result if it hires him. A random variable  $X = \times_{i=1}^{N_x} (X_i)$ , with  $N_x$  features, is distributed according to  $\pi$ . Feature  $i$  has  $N_i$  possible realizations, that is,  $X_i = \{x_i^l\}_{l=1}^{N_i}$ . In the example,  $N_x = 3$ , with  $X_1 = \{0, 1\}$ ,  $X_2 = \{B, G\}$  and  $X_3 = \{L, P\}$  where 0, 1, B, G, L, P represent, respectively, Pass, Fail, Bad, Good, Loss and Positive Result. The set of possible realizations of  $X$  after receiving some information is  $d \subseteq X$ . If the firm is told that a candidate passed the exam,  $d = \{1\} \times X_2 \times X_3$ . The agent forms a set of  $N_q$  hypothesis about one or more of the features. These hypothesis need to be exhaustive, in the sense that they cover all that can happen, but not mutually exclusive. The subset of  $X$  defined by hypothesis  $q$  is  $h_q$ . In the example, hypothesis 1 may be ‘the candidate is good’. A hypothesis 2 stating ‘the candidate is bad’ is enough to close the set of hypothesis. In this case,  $h_1 = X_1 \times \{G\} \times X_3$ . When hypothesis are constructed, it is possible that they, together with the information received, do not restrict one or more of the features. The set of such features for hypothesis  $q$  is  $F_q$ . In the example,  $F_1 = \{X_2\}$ , because the information received is about the exam result and both hypothesis restrict only the candidate’s quality, imposing no restrictions on the firm’s result. After a hypothesis is formed, the

agent builds scenarios on the unrestricted features. The subset of  $X$  defined by the scenario associated to hypothesis  $q$  in which the free features assume the values  $\{\hat{x}_i\}_{i \in F_q}$  is  $s_q^{\{\hat{x}_i\}_{i \in F_q}}$ . In the example, the possible scenarios for hypothesis 1 may be ‘the firm has losses’ and ‘the firm has a positive result’. Thus,  $\forall \Pi \in \{L, P\}, s_1^{\{\Pi\}} = X_1 \times X_2 \times \{\Pi\}$ . Each scenario, linked to a hypothesis, has a level of salience, labeled *representativeness*. It is defined as  $r\left(s_q^{\{\hat{x}_i\}_{i \in F_q}}\right)$ , the probability that the hypothesis is verified, when the information received is true and the scenario is occurring. In the example, we get that, for all  $\Pi \in \{L, P\}$ ,

$$r\left(s_1^{\{\Pi\}}\right) = \frac{Pr(1, G, \Pi)}{Pr(1, G, \Pi) + Pr(1, B, \Pi)}$$

The level of bounded rationality of the agent is defined by the number of most representative scenarios he takes into account,  $b$ . If he takes all possible scenarios into account, he is a classic Bayesian agent. Supposing  $b = 1$  and that, when a candidate passes the test, the probability that he is good is higher when the firm makes a positive result instead of losses, which seems a natural supposition, the most representative scenario for hypothesis 1 is ‘the firm makes a positive result’ and that is the only scenario considered. If  $S_q$  is the set of possible scenarios associated with hypothesis  $q$ , the set of scenarios considered by an agent who only focuses on the  $b$  most representative scenarios is  $\tilde{S}_q^b$ . And this agent evaluates the probability of hypothesis  $q$  occurring, when the information received is true, as:

$$Pr^b(h_q|d) = \frac{Pr\left(\cup_{s \in \tilde{S}_q^b} (h_q \cap d \cap s)\right)}{Pr\left(\cup_{q=1}^{N_q} \left(\cup_{s \in \tilde{S}_q^b} (h_q \cap d \cap s)\right)\right)}$$

If we also assume that, when a candidate passes the test, the probability that he is bad is higher when the firm makes losses instead of a positive result, we get that:

$$Pr^1(h_1|d) = \frac{Pr(1, G, P)}{Pr(1, G, P) + Pr(1, B, L)}$$

Note that, if  $b = 2$ , the agent is Bayesian in the classic sense, and:

$$Pr^2(h_1|d) = \frac{Pr(1, G, P) + Pr(1, G, L)}{Pr(1, G, P) + Pr(1, G, L) + Pr(1, B, L) + Pr(1, B, P)}$$

That is, a *local thinker*, as the authors call an agent who does not take all scenarios into account, does not use all the pieces of information he has available, but only the ones in which hypothesis and scenarios are linked. In the example, this agent ignores the facts that a good manager can have the firm making losses and a bad manager can have the firm making positive results. That is, he uses what more easily comes to his mind and disregard the rest. And, if a representative scenario is unlikely, the impact it has on the probability calculation is exaggerated, resulting in highly biased estimates. This type of reasoning may help explain the conjunction fallacy, that consists in attributing a higher probability to a set than to another which contains it, the disjunction fallacy, which results in considering that the union of two sets is less probable than each of the sets that contain it and the base-rate neglect, which happens when people's estimates focus on how a description fits a certain class of subjects, but the number of subjects in that class is ignored.

Lu *et al.* (2005) and Ortoleva (2013) are two papers which study the implications of the size of the choice set for a boundedly rational agent, who pays a thinking cost increasing in the number of alternatives he has to choose from.<sup>4</sup> While Ortoleva (2013) proposes an axiomatic approach in which agents have preferences for lotteries over subsets, depending on the utility they can derive from them, but also on the size of the subsets they contain, Lu *et al.* (2005) assumes directly that there is a cost proportional to the number of alternatives from a choice set that are considered. Common to both is the idea that the choice set is definable by the agent, which chooses it in an optimal way, given the cost they imply.

Ortoleva (2013) constructs an axiomatic framework, where two types of preferences between lotteries of menus are distinguished: the *genuine*  $\geq^*$  and the *observed*  $\geq$ . While the former reflect what is called the *genuine evaluation* of a lottery, that is, the utility the agent expects to extract from it, the latter reflects not only the

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<sup>4</sup>In this sense, these papers could also be cited in Section 1.2.3, devoted to costly thinking, but we choose to refer to them here, for their specific focus on the choice set.

genuine evaluation, but also the cost of thinking in the options the lottery has. Imposing axioms that relate  $\geq^*$  and  $\geq$  in intuitive ways that reflect the effect the thinking cost should have on the decisions made, a representation of preferences that allows to disentangle the two aspects is obtained.

Formally,  $X$  is the set that contains all possible objects. The power set of  $S$ , that is, the set of subsets of  $X$ , is represented by  $2^X$ . Excluding the empty set from  $2^X$ , we get the set of menus from which the agent can choose, labeled  $\mathbb{X}$ . The set of lotteries over  $\mathbb{X}$  is  $\Delta(\mathbb{X})$ . Singletons, or menus with one single object, are important in this context, as an agent who faces them has a costless choice to make, because there is only one alternative to choose from. The subset of lotteries that attribute positive probabilities only to singletons is  $\tilde{\Delta}(\mathbb{X}) \subseteq \Delta(\mathbb{X})$ . The support of lottery  $a \in \Delta(\mathbb{X})$  is the set of menus in  $\mathbb{X}$  to which it attributes positive probability. Excluding from this set the singletons it possibly has, we define  $s^*(a)$ . The agent has to choose among all lotteries in  $\Delta(\mathbb{X})$ . When he chooses a lottery, he knows the probability of having to choose from each of the menus in the support of the lottery, but not which menu will be drawn. Hence, the process of choosing lotteries implies forming a *contingent plan*, that is, fixing which object will be chosen in case each menu is drawn. The relation between  $\geq^*$  and  $\geq$  is defined in the following way, for all  $a, b \in \Delta(\mathbb{X})$ :

$$a \geq^* b \Leftrightarrow \exists \alpha, \beta \in ]0, 1[ : \begin{cases} \alpha > \beta \\ \alpha a + (1 - \alpha) b \geq \beta a + (1 - \beta) b \end{cases} \quad (1.4)$$

The intuition for (1.4) is the following. If the agent genuinely prefers lottery  $a$  to lottery  $b$ , and has the option between two lotteries  $c$  and  $d$ , both over  $a$  and  $b$ , in which the weight attributed to  $a$  is higher in  $c$ , then that is the one he chooses. In fact,  $c$  and  $d$  imply choosing from the same menus and, thus, create the same thinking cost, but the probabilities in  $c$  are more favorable to  $a$ , the lottery which provides a higher basic utility.

The paper proceeds by defining some axioms that relate  $\geq^*$  and  $\geq$  in ways that make sense according to the idea of costly thinking. For example, *Thinking*

*Aversion* means that, if a lottery containing only singletons is genuinely preferred to a general lottery, mixing each of them with a third lottery produces the same ranking in observed preferences, because such mix changes the thinking cost in the same way. *Weak independence* is an adaptation of the independence axiom in rational choice. It states that genuine preferences, and observed preferences restricted to lotteries with the same support (and, thus, with the same thinking cost implied) satisfy independence, and that unrestricted observed preferences satisfy a version of independence that compensates for the fact that mixing two lotteries with a degenerate singleton lottery can change the observed preference ranking, as the thinking cost remains the same, but the advantage in genuine preferences one of them has may be reduced. *Content Monotonicity* implies that a menu is genuinely preferred to any of its subsets, because it has all the objects they have and possibly more.

Gathering some of the axioms presented, it is possible to prove that the observed preferences are representable by a function  $v : \mathbb{X} \rightarrow \mathbb{R}$ , defined in the following way:

$$v(a) = \sum_{A \in \mathbb{X}} \left( \sum_{s \in S} \left( \mu(s) \left( \max_{x \in A} u(x, s) \right) \right) \right) - C(s^*(a)) \quad (1.5)$$

In (1.5),  $S$  is a set,  $\mu$  is a probability measure on  $S$ ,  $u : X \times S \rightarrow \mathbb{R}$  is a real function and  $C : 2^{\mathbb{X}} \rightarrow 0$  is non-negative real function that is zero valued in  $\emptyset$  and is not lower in a set than in any of its subsets. There is a very intuitive explanation for this representation. If  $S$  is a state space that represents the uncertainty the agent has about his own preferences, we can think of  $u$  as the function that gives the utility derived from each object in each state. And  $C$  can be thought of as the function that attributes to each lottery the cost of thinking in the menus of its support. It is monotonic, in the sense of generating a higher thinking cost when more menus are added to the support of a lottery. In this perspective, the observed preferences are representable by the difference between the expected utility and thinking cost associated to a lottery. An extra axiom may be added, assuring that if two menus are genuinely indifferent and the choice from one of them is not more difficult than from the other, then the latter is observationally preferred to the latter. This makes

possible an even more specific interpretation of  $C$ . Given that the agent is uncertain about its own preferences before knowing what the true state is, he may construct a partition on  $S$  that allows him to know in which states is each object an optimal choice. However, there are several ways he can do it. For example, the partition that separates all elements of  $S$ , the finest one, is always a possibility but probably not the more efficient. Given the extra axiom and assuming that finer partitions imply a higher cost,  $C$  can be thought of the cost of the least costly partition of  $S$  that allows the agent to always make an optimal choice.

In [Lu \*et al.\* \(2005\)](#), there is a general choice set,  $X$ , composed of  $N \in \mathbb{N}$  objects, and an utility function  $u : X \rightarrow \mathbb{R}$ . An agent makes a two-stage choice. In the first stage, he chooses the number of objects he will take into consideration,  $n$ , and randomly selects  $n$  objects from  $X$ . The cost per object considered is  $c$ . In the second stage, he picks the best element among the ones preselected. He is aware of this choice process, hence, in the first stage, he chooses  $n$  to maximize the expected value of utility net of thinking cost. That is, he maximizes  $v : \mathbb{N} \rightarrow \mathbb{R}$ , whose expression is  $v(n) = E(u(x^*(n)))$ , where  $x^*(n)$  is the best choice among the  $n$  considered alternatives, and depends on the randomly chosen combination of objects. The function  $v$  is shown to be increasing with  $n$ , which means the problem reduces to a classic economic one: the agent has to ponder the benefits and costs of looking to one more object, and select the number of objects which equals marginal benefit and marginal cost. This number need not to be  $N$ , hence it is possible that an agent optimally decides to reduce the original choice set. Studying the case in which  $u$  depends only on the ranking of the objects in  $X$  and not on their intrinsic value, the authors establish that  $v$  is lower when one extra object is added to  $X$ . Consequently, that it is possible that the agent is better off without such addition. That is, the fact that choosing from large sets is costly may make the agent prefer to lose one option from the original choice set.

[Mullainathan \(2002\)](#) and [Gabaix \(2014\)](#) are two different approaches to how parameters are viewed by boundedly rational agents, one based on memory and the other on attention. On both of them, agents do not take into account the

whole environment in which their decisions are made, but for different reasons. If, in [Mullainathan \(2002\)](#), events in the past should influence their estimates in each period, but their imperfect memory prevents them from remembering everything, in [Gabaix \(2014\)](#), they choose not to pay full attention to some parameters, because the cost of doing so is not compensated by the benefit of making better decisions.

In [Mullainathan \(2002\)](#), an agent does not know what his income is in each period, and needs to estimate it. The income is cumulative, in the sense that it depends from what has happened in the past. However, the agent's memory is imperfect and he can only remember some past events, forgetting others. More specifically, he remembers more easily those events that he was recently reminded of, or that are similar to the event he observes while trying to remember. This means that positive news in one period make the agent remember more positive than negative events in the past, thus overreacting and judging the income in that period to be higher than it really is.

Time is indexed by  $t \in \{1, \dots\}$ . In period  $t$ , an event  $e^t = (x^t, n^t)$  happens with probability  $p$ . This event is jointly normally distributed, with expected value  $(0, 0)$ , and a variance matrix composed of  $\sigma_x^2$ ,  $\sigma_n^2$  and  $\sigma_{x,n}$  for  $x$ 's variance,  $n$ 's variance and covariance between  $x$  and  $n$ , respectively. If there is no event in period  $t$ ,  $e^t = (0, 0)$ . In this period, there is a permanent shock in income,  $\nu^t = x^t + z^t$ , with  $z^t$  normally distributed with a zero mean and a variance of  $\sigma_z^2$ . Income in period  $t$  is given by  $y^t = \sum_{k=1}^t (\nu^k) + \epsilon^t$ , with  $\epsilon^t$  with  $z^t$  normally distributed with a zero mean and a variance of  $\sigma_\epsilon^2$ . Income in period  $i$  is then composed by the sum of all permanent shocks until  $t$  and a transitory shock  $\epsilon^t$ . With this structure, we can see that all current and past  $x$  values are informative about  $y^t$ , as they influence it, whereas  $n^t$  is non-informative about income. However, it plays a role in the model, by helping creating links between similar events.

The agent has imperfect memory and, at period  $t$ , may or not remember period  $k$ 's event. This originates the binary value  $R_k^t$ , which assumes value 1 if event  $k$  is remembered at period  $k$ , and 0 otherwise. The probability that  $R_k^t = 1$  is  $r_k^t \in [0, 1]$  and is defined in a way that is line with psychological observations about memory.



It is assumed to depend on three factors. First, there is a baseline recall probability,  $\underline{m}$ , that represents the agent's general ability to remember events. Second, based on the notion of *rehearsal*, which consists on the easy remembering of memories that were brought to memory in a recent past,  $\rho$  enhances the probability of remembering period  $k$ 's event at period  $t$  if it was remembered at period  $t-1$ . Finally, the concept of *associativeness*, i.e., the easier remembering of memories close in some way to what is observed in the present, motivates the creation of a similarity function  $a_k^t$ , which evaluates how similar events from period  $k$  and  $t$  are. This function is assumed to be a non-weighted average of a negative transformation of the distances between the informative and non-informative parts of the two events, and to be contained in  $[0, 1]$ . This should make clear why the non-informative part of an event has any impact: it influences which events are remembered at each period. How associative an agent is is defined by  $\chi$ . The remembering probability is then defined in the following way:

$$r_k^t = \underline{m} + \rho R_k^{t-1} + \chi a_k^t \quad (1.6)$$

To ensure that (1.6) is indeed a probability, it is assumed that  $\underline{m} + \rho + \chi < 1$ . If an event is not remembered, it is viewed as  $(0, 0)$ . The memory of period  $k$ 's event in period  $t$  is  $e_t^k (R_k^t)$ . The agent either remembers it perfectly, hence  $e_t^k (1) = e^k$ , or forgets it entirely, viewing it as a shockless event. Therefore,  $e_t^k (0) = (0, 0)$ . Although the agent remembers events in an imperfect way, he is assumed to be able to correctly remember all incomes. Then, while the true history of observables at period  $t$  is  $h^t = (e^k, y^k)_{k \in \{1, \dots, t-1\}}$ , the remembered history at that period is  $h_r^t = (e_t^k (R_k^t), y^k)_{k \in \{1, \dots, t\}}$ . For the case in which memory is perfect, an optimal estimation rule of  $y^t$ ,  $f(e^t, h^t)$ , is obtained, and depends on the current event and true history. The imperfect memory agent is assumed to be naive and not aware of his own imperfect memory, thus making estimations as if the history he remembers is the true one. That is, his estimate of income at period  $t$  is  $\hat{y}_r^t = f(e^t, h_r^t)$ . Rehearsal and associativeness contribute to a persistent overestimation of income. In each period, associativeness biases the memory selection towards the events closer to the current

one, thus exaggerating its impact, while rehearsal helps the remembered events to be remembered again in the subsequent periods. An application to consumer theory allows to conclude that the adding of imperfect memory to a Permanent Income setup allows consumption changes to be predicted using lagged information, mainly because of associativeness. Also, if there are different sources of income, it is stated that the sources whose prediction relies more heavily on memory are the ones which display the highest marginal propensities to consume.

Gabaix (2014) deals with problems of choice that are affected by a set of parameters. It is assumed that, although it is useful to know the true value of the parameters, it is also costly to do so, which may lead people to pay only partial or no attention to them. They have a prior, labeled *default*, about each parameter and the adjustment they make, starting from it and going in the direction of the true value, depends on the attention they decide to pay. The attention level is endogenous and is the result of the weighting of benefits (small utility loss relative to the full attention case) and costs (keeping track of parameters' values). In this sense, this is a model of perfect imperfection, like the ones in Section 1.2.3, as the problem of how to decide is optimally solved. However, there is no true optimality at the second level of thinking (thinking on how to think), as the loss in utility from not paying attention to parameters that is minimized is not the real one, but an approximation to it.

The base problem is a very general one. There is an utility function  $u : \mathbb{R}^{N+M} \rightarrow \mathbb{R}$ , which depends on  $N$  decision variables and  $M$  parameters. The decision variables are contained in  $x \in \mathbb{R}^N$  and the parameters in  $\alpha \in \mathbb{R}^M$ . There are  $K$  constraints, which possibly involve  $x$  and  $\alpha$ . The constraint function is then  $g : \mathbb{R}^{N+M} \rightarrow \mathbb{R}^K$  and the set of constraints is  $g(x, \alpha) \leq 0$ . Parameters are jointly distributed with a zero mean and a variance matrix, in which the variance of parameter  $i$  is  $\sigma_i^2$  and the covariance between parameters  $i$  and  $j$  is  $\sigma_{i,j}$ . The true value of parameter  $i$  is  $\alpha_i^t$ . The boundedly rational agent reveals *sparsity* in his behavior, economizing on the attention paid to parameters. He has a default value for parameter  $i$ ,  $\alpha_i^d$ , which is assumed to be 0 to simplify the analysis. The level of attention devoted to

parameter  $i$  is  $a_i \in [0, 1]$  and the perceived value of this parameter is  $\alpha_i^p(a_i) = a_i \alpha_i^t$ . The cost of paying attention  $a_i$  to parameter  $i$  is  $c(a_i) = \kappa a_i^\theta$ , with  $\kappa$  and  $\theta$  being two non-negative parameters that calibrate the difficulty of the agent in paying attention to parameters. The total cost of an attention level  $a$  is  $C(a) = \sum_{i=1}^M (c(a_i))$ . An optimal imperfection model would use a backwards induction reasoning to determine the optimal attention level. For each attention level, the optimal utility would be calculated, which would allow to find the value of  $a$  that maximizes a function that depends positively on the optimal utility and negatively on the attention cost. However, arguing that this implies that the agent must know the optimal utility for each attention level, which is a more difficult task than the one faced by an unboundedly rational agent, the author proposes that the agent only knows an approximation of the loss in utility caused by less than full attention,  $L(a)$ . The agent then solves the problem in two steps. First, he chooses the attention level that minimizes the sum of  $L$  with  $C$ ,  $a^*$ . Then, he chooses an object  $x^*$ , using the parameter values that result from the previously defined attention level. If the base problem has no constraints, this method presents no problems. Otherwise, it is possible that, by perceiving the parameters to be different from what they really are, the agent focus on a choice set that is not the real one, and ends up wanting to choose a non-available option. This is solved by assuming that the agent chooses, among the objects  $x$  in the true choice set in which  $u'_x(x, \alpha^p(a^*))$  is a multiple of  $g'_x(x, \alpha^p(a^*))$ , the one which maximizes  $u$ . This procedure assures that, for any two features  $q$  and  $r$  from the chosen object  $x^*$ ,  $\frac{u'_{xq}(x^*, \alpha^p(a^*))}{u'_{xr}(x^*, \alpha^p(a^*))} = \frac{g'_{xq}(x^*, \alpha^p(a^*))}{g'_{xr}(x^*, \alpha^p(a^*))}$ , a condition verified in constrained maximization problems, but that is here applied to perceived and not true parameters.

A very intuitive result is the one that states that, if the parameters are uncorrelated (or at least, if the agent sees them like that), i.e., if, for all  $i, j \in \{1, \dots, M\}$ ,  $\sigma_{i,j} = 0$ , then the optimal attention for parameter  $i$  is increasing in its volatility (if the parameter varies much, keeping track of it requires a high attention level), in its influence in the object choice (more important parameters deserve higher attention) and in the utility loss in making sub-optimal choices (more important decisions deserve higher attention levels), and decreasing in  $\kappa$  (if attention

is costly, it is avoided). The application of the sparsity concept to microeconomic theory provides some interesting insights. There is money illusion, as the raise of prices and wealth by the same proportion may change the consumer's choice, because, by not paying the same attention to all parameters, he may believe that his choice set has changed. The Slutsky matrix, which contains the derivatives of the Hicksian demand function, is not symmetric, as different attention paid to the prices of two goods makes the agent have different compensated reactions to a price change of one of them. In a general equilibrium model, a competitive equilibrium is not necessarily Pareto efficient, because the perceived relative prices may differ across agents. In the Edgeworth box, the offer curve is not one-dimensional, because the possibly different attention levels in prices make all prices relevant and not just their ratios, and the equilibrium allocation depends on the price level, because of money illusion.

[Sims \(2003\)](#) may also fit in the class of models that study how boundedly rational agents view parameters, as it deals with the problem of observing data through a communication channel with finite capacity. Based on information theory, the author studies the problem of making decisions when the observation of exogenous data can be done only imperfectly. Agents have some uncertainty about the optimal action they should take, as it depends on a random exogenous variable. They can reduce this uncertainty by observing realizations of the exogenous variable, but the uncertainty reduction that this can generate is limited. The uncertainty of a random variables is measured by its *entropy*, the level of informativeness of each of its draws. The application of the concept to a dynamic macroeconomic model generates reactions to market data that are smoothed and specific to each agent, as their way to process information differs.

Finally, let us mention [Lipman \(1999\)](#) as a paper that deals with the decision variables of an problem. Its premise is that agents may be unable to know all the logical consequences of what they know, implying that, when they take an action, they may not be aware of what it means. An axiomatic structure is constructed with the objective of accounting for the possibility that an agent does not realize

that two pieces of information are logically equivalent. In order to represent the information-dependent preferences of such agent with an expected utility function, a set of *impossible possible worlds*, states which are logically impossible but considered possible by the agent, are added to the true state space. The model helps us understand why people make incorrect choices, even though they have all the information they need to behave otherwise. Their bounded rationality makes them believe that the consequences of taking an action are different from the real ones.

### 1.3 Strategic Interaction

In this section, we give a quick overview of the papers in which the bounds on rationality are not on the way each player decides, but on how he understands the strategic interaction with other players. We first present some examples of models in which players choose machines instead of strategies, and the preprogrammed machines play against each other, but more complex or sophisticated machines are more costly (Rubinstein, 1986, Eliaz, 2003). Then, we move to models where best responding is hard or costly and is replaced by simpler strategies (Rosenthal, 1993, Osborne & Rubinstein, 1998). We also analyze models in which the view of a player about other players or the way they form strategies are simplified (Jehiel, 2005, Eyster & Rabin, 2005, Nagel, 1995, Camerer *et al.*, 2004). Finally, we refer one model in which players are unable to know the consequences of the strategies they choose (McKelvey & Palfrey, 1995).

In both Rubinstein (1986) and Eliaz (2003), the preference of players by simple strategies are modeled by finite-state machines. The players entering in a game, instead of choosing strategies, pick one of the available machines and the combination of machines selected determines the entire set of actions that occur over time. But, while Rubinstein (1986) chooses this modeling specification in order to account for the fact that using sophisticated strategies should be costly, Eliaz (2003) focus on each player's forecasts of the actions of the other players, stating that an equilibrium should only occur if each player decision would not be the same in response to a

simpler forecast than the one that actually originates the equilibrium decision.

A finite-state machine is an instrument that decides all the moves of a player in a game that extends for several periods. It states what is the action in the first period, and how to choose an action in the subsequent periods, depending on what has happened until then. For this, it uses the notion of specific period states, an artificial concept that indicates what should be done in that period given the opponent's action and the state in the previous period. Both papers study  $2 \times 2$  games in infinitely repeated games, hence the set of players, which are indexed by  $i$ , is  $\{1, 2\}$ , and time is represented by  $t \in \{1, \dots\}$ . The set of possible actions for player  $i$  is  $A_i$  and his payoff is represented by  $u_i : A_1 \times A_2 \rightarrow \mathbb{R}$ . The set of states is  $S$ . The set of reaction functions for player  $i$ , which indicate the action to choose for each different state, is  $R_i$ . And the set of transition functions for player  $i$ , which gives the change in states after a choice of a specific action by player  $j \neq i$ , is  $H_i$ . A machine available for player  $i$  is then  $m_i^k = \langle S_i^k, s_i^{0k}, r_i, h_i \rangle$ , where  $S_i^k \subseteq S$ ,  $s_i^{0k} \in S_i^k$ ,  $(r_i^k : S_i^k \rightarrow A_i) \in R_i$  and  $(h_i^k : S_i^k \times A_j \rightarrow S_i^k) \in H_i$ . The set of  $K_i$  machines available to player  $i$  is  $M_i$ . For example, in the infinitely repeated prisoner's dilemma, in which the available actions in each period are  $C$  and  $D$ , the tit-for-tat strategy is followed by player  $i$  if he chooses a machine  $k$  in which  $S_i^k = \{B, G\}$ , where  $B$  and  $G$  represent bad and good mood, respectively,  $s_i^{0k} = G$ ,  $r_i^k(B) = D$ ,  $r_i^k(G) = C$ , and, for all  $s_i^k \in S_i^k$ ,  $h_i^k(s_i^k, C) = G$  and  $h_i^k(s_i^k, D) = B$ . Intuitively, the player initially cooperates and, from then on, cooperation induces a good mood which, in turn, induces cooperation, while defection induces a bad mood which, in turn, induces defection.

The idea in [Rubinstein \(1986\)](#) is to associate strategy complexity with the number of states of a machine. That is, the complexity of machine  $m_i^k$  is  $c(m_i^k) = \#S_i^k$ . Knowing that each machine has a finite number of states, it can be proved that, after some initial periods, the states will cycle in intervals of  $T$  periods. The utility player  $i$  derives from the combination of machines  $m_i$  and  $m_j$ , chosen by himself and by player  $j$ , respectively, is represented by  $\pi_i(m_1, m_2)$ , and assumed to be the average utility such combination generates for him during each cycle. Player  $i$  orders

the possible combination of machines by lexicographically comparing the utility he gets from them and the complexity of the machine chosen by himself, with priority given to utility. That is, if  $>_L$  represents a lexicographic ordering which prioritizes the first argument and  $>_i$  represents player  $i$ 's preferences over combinations of machines, then:

$$(m_1, m_2) >_i (m'_1, m'_2) \Leftrightarrow (\pi_i(m_i, m_j), -c(m_i)) >_L (\pi_i(m'_i, m'_j), -c(m'_i))$$

A Nash equilibrium is then defined in the usual way: a pair of machines is a Nash equilibrium if no player has an incentive to change to another machine, given the machine chosen by its opponent. A stronger concept is also defined. It is labeled *semi-perfect equilibrium*, and it requires that the incentives for machine replacing are non-existent not just at the beginning of the game, when machines are chosen, but also in each period, if it were possible to replace machines during the game.

In [Eliaz \(2003\)](#), the action set of each player is restricted to  $\{C, D\}$  and it is machine simplicity and not complexity that is defined. There are two indicators that define the level of simplicity of a machine  $m_i^k$ . The first,  $x(m_i^k)$ , is the number of pairs of different states which are connected by the transition function. The second,  $y(m_i^k)$ , is the number of states which imply a transition to a new state that depends on the other player's action. Intuitively, if either  $x(m_j^k)$  or  $y(m_j^k)$  are large, then player  $i$ , knowing player  $j$  chooses machine  $m_j^k$ , has a hard time in forecasting player  $j$ 's strategy, either because it involves a high number of states or because it depends strongly on player  $i$ 's actions. Machine  $m$  is defined to be simpler than machine  $m'$  whenever its  $x$  and  $y$  values are not higher than the corresponding values of  $m'$  and at least one of them is lower.

As for the utility implied by each combination of machines, it is defined as the time-discounted sum of payoffs generated by the actions that result from the matching of machines. The equilibrium concept introduced is stronger than the Nash equilibrium. A *Nash equilibrium with stable forecasts* is a combination of machines  $(m_1^*, m_2^*)$  such that for all  $i \in \{1, 2\}$ ,  $m_i^*$  is a best response to  $m_j^*$  (the Nash part) and

there is no  $m_j$  simpler than  $m_j^*$ , which can be best responded with  $m_i^*$  (the stable forecasts part). This last requirement simply means that the machine chosen by player  $i$  in equilibrium cannot be a best response to a simpler  $j$  machine than  $m_j^*$ , because, if this was the case, player  $i$  could make the same machine choice, imagining a much simpler strategy for its opponent than the one he is actually employing. The fact that players dislike complex forecasts provides the motivation this equilibrium concept.

Rosenthal (1993) and Osborne & Rubinstein (1998) are two examples of papers in which players choose actions in an easier way than what is assumed by the concept of Nash equilibrium. In fact, instead of forming a belief about the strategy of their opponents, which necessarily corresponds to the strategy they actually follow, and best respond to it, the players in these models choose actions in a much less strategic way. In Rosenthal (1993), they use one fixed strategy, labeled *rule of thumb* and use it, independently of what their opponents are doing and, in Osborne & Rubinstein (1998), they choose actions based on what they observe when they sample them.

Rosenthal (1993) studies a set of  $K$  games,  $G = \{G_k\}_{k \in \{1, \dots, K\}}$ . Game  $k$  is played with probability  $p_k$ . They all are  $2 \times 2$  games, with the same action spaces for the row and column players. The action space for the  $l$  player is  $A^l = \{a_i^l\}_{i \in \{1, \dots, N^l\}}$ , where  $l \in \{r, c\}$ , and  $r$  and  $c$  stand for row and column, respectively. The utility function of player  $l$  in game  $k$  is  $u_k^l : A^r \times A^c \rightarrow \mathbb{R}$ . There are two populations of players, one with  $r$  and the other with  $c$  players. When one game is drawn, one  $r$  and one  $c$  player are randomly matched. Each player has to choose a rule which dictates a fixed action for each of the games he may play. This means that an  $l$  player has  $N^{lK}$  possible rules. A rule  $i^l \in \{1, \dots, N^{lK}\}$  for the  $l$  player is then  $s_{i^l}^l = (a_{i^l}^l(k))_{k \in \{1, \dots, K\}}$ , where  $a_{i^l}^l(k) \in A^l$ . Rules are costly, and the cost of using rule  $s_{i^l}^l$  is  $c_{i^l}$ . Although no restriction is imposed on this cost in the general version of the model, in the examples presented it is assumed to be increasing in the number of subsets of games to which the same action is attached. With this specification, rules which preclude the same action for all games and a different action for each game<sup>5</sup> are, respectively, the least and most costly ones. The expected net utility of

<sup>5</sup>For an  $l$  player, this is only possible if  $N^l \geq K$ . Otherwise, we mean a rule in which all actions



rule  $s_{i^l}^l$ , given that the  $h \neq l$  player is using rule  $s_{i^h}^h$  is then:

$$v^l(s_{i^l}^l, s_{i^h}^h) = \sum_{k=1}^K (p_k u_k^l(a_{i^l}^l(k), a_{i^h}^h(k))) - c_{i^l}$$

When players select their rules, two distributions of rules,  $\sigma^r$  and  $\sigma^c$  are generated, with  $\sigma^l = (\sigma_{i^l}^l)_{i^l \in \{1, \dots, N^{lK}\}}$ , and  $\sigma_{i^l}^l$  representing the share of  $l$  players that choose rule  $i^l$ . Hence, the expected net utility of rule  $s_{i^l}^l$ , given that the rule distribution among  $h \neq l$  players is  $\sigma^h$ , is:

$$\pi^l(s_{i^l}^l, \sigma^h) = \sum_{i^h=1}^{N^{hK}} (\sigma_{i^h}^h v^l(s_{i^l}^l, s_{i^h}^h))$$

A set of distribution rules,  $\sigma^* = (\sigma^{r*}, \sigma^{c*})$  is a *population equilibrium* if all players are happy with the rule they choose. That is, if, given the distribution of rules among the opposing population, no player gets a higher expected net utility by changing its rule. And this is only true if all  $l$  rules that are selected by a positive share of the  $l$  population generate the same expected net utility, not lower than the one generated by the unused rules.

Osborne & Rubinstein (1998) study  $2 \times 2$  symmetric games. In their model, agents, instead of playing according to their forecasts of their opponents' strategy, try each possible action once and register the payoff generated. An equilibrium exists if the probability of choosing each action (which is the same for both players) is equal to the probability that that action produces the highest payoff in the sampling.

The set of actions for both players is  $A = \{a_1, \dots, a_N\}$ . Their utility function is  $u : A \times A \rightarrow \mathbb{R}$ . Players choose each action according to the distribution  $\sigma = (\sigma_i)_{i \in \{1, \dots, N\}}$ . When a player samples action  $i$ , the utility he gets depends on the action chosen by its opponent. Hence,  $V_i^\sigma$ , the utility that results from a sampling of action  $i$  when actions are chosen according to  $\sigma$  is a random variable, and  $\forall j \in \{1, \dots, N\}, Pr(V_i^\sigma = u(a_i, a_j)) = \sigma_j$ . After the player samples all his actions once, he registers the one which generates the highest payoff, labeled the *winner*, with ties are assigned to at least one game.

solved arbitrarily. The probability that, after this sampling, and with the opponent following distribution  $\sigma$ , action  $i$  is the *winner* is  $w_i^\sigma$ . A distribution  $\sigma^*$  is an  $S(1)$  *equilibrium* if the probability it implies for each action matches the probability that this action is the *winner* after the sampling. That is, if  $\forall i \in \{1, \dots, N\}, w_i^{\sigma^*} = \sigma_i^*$ . The idea is generalized for any number  $k$  of samplings of each action, making the way for the concept of an  $S(k)$  *equilibrium*. Interestingly,  $K$  may be viewed as a measure of unbounded rationality, because a higher number of samplings implies a more accurate evaluation of each alternative. In fact, it is proved that, if a game has a unique Nash equilibrium, it is the limit, when  $k \rightarrow +\infty$ , of its  $S(k)$  *equilibrium*.

In Jehiel (2005) and Eyster & Rabin (2005), the forecasts a player makes about his opponents' moves are simplified, although in different contexts. Jehiel (2005) assumes that players, when making forecasts about what their opponents' actions are, do it in a simplified way that reduces the set of possible opponents' movements a player can face. This is achieved by gathering different possible scenarios of opponents' moves into a set, labeled *analogy class*, and reacting only to what the player expects to be the average behavior in these scenarios. Eyster & Rabin (2005) proposes a similar concept, but in the context of incomplete information. In the game he proposes, players, instead of best responding taking into account the possibility that different types of opponents take different actions, simply assume that all types of opponents take the same action. However, a consistency requirement imposes that this unique action is indeed the average action played by all types of opponents, given the choice profile of each type and their weight in the population. Both papers assume that players are unable (or find it costly) to consider all the possibilities that may arise from their opponents' behavior, and opt to react to a condensed version of it.

In Jehiel (2005), a game, represented in its extensive form, has a set of  $N$  players. The preferences of player  $i$  on the different possible outcomes of the game are represented by  $u_i$ . The expected value of  $u_i$  when actions are chosen stochastically, is  $v_i$ . The set of nodes in the game is  $H$  and the set of  $M_i$  nodes in which player  $i$  moves is  $H_i$ . The action space for player  $i$  when he is in node  $h_i^{m_i}$  is  $A_i(h_i^{m_i})$ . The actual deci-

sions of all players are contained in  $\sigma = (\sigma_i)_{i \in \{1, \dots, N\}}$ , where  $\sigma_i = (\sigma_i(h_i^{m_i}))_{m_i \in \{1, \dots, M_i\}}$  and  $\sigma_i(h_i^{m_i})$  is a distribution over the possible actions player  $i$  can take in node  $h_i^{m_i}$ .

The set of nodes in which  $i$ 's opponents move is  $S_i$ . Player  $i$  is assumed to make a partition on  $S_i$ , labeled an *analogy partition*, in such a way that two nodes can only be a part of the same subset if their action space is the same. An *analogy partition* for player  $i$  is  $An_i = \{s_i^k\}_{k \in \{1, \dots, K\}}$ , where  $K$  is the number of subsets it comprises and each subset is an *analogy class*. Instead of forming an expectation about the moves in all the nodes in  $s_i^k$ , player  $i$  looks to  $s_i^k$  as if it were a single node. The beliefs he forms in this way about his opponents' moves are represented by his *analogy-based expectation*  $\theta_i$ . Specifically, his beliefs about the probability that each action in  $s_i^k$  is taken are represented by  $\theta_i(s_i^k)$ . This means that his beliefs concerning the move of player  $j \neq i$  in a node  $h_j^{m_j}$  that belongs to  $s_i^k$  are represented by  $\phi_j^{s_i^k}(h_j^{m_j}) = \theta_i(s_i^k)$ . The beliefs of all players are  $\theta = (\theta_i)_{i \in \{1, \dots, N\}}$ .

The actual decision of player  $i$ ,  $\sigma_i$ , is a *sequential best response* to  $\theta_i$  if, at any node in which player  $i$  moves, no other decision guarantees a higher utility to player  $i$ , given that his beliefs about his opponents' moves from that node on are contained in  $\theta_i$ . An analogy-based expectation  $\theta_i$  is *weakly consistent* with  $\sigma$  if the probability it assigns to each action in an analogy class  $s_i^k$  that contains at least one node reached with positive probability matches the average frequency of actual play of that action in the nodes that belong to  $s_i^k$ . That is, although the player simplifies the set of opponents' moves when forming beliefs, he has to be right, on average, given what is actually being played. A profile  $(\sigma^*, \theta^*)$  is a *self-confirming analogy based equilibrium* if, for each player  $i$ , his decision  $\theta_i^*$  is a sequential best response to his belief  $\theta_i^*$ , and this is weakly consistent with decision  $\sigma^*$ . The equilibrium is then strengthened with a notion of consistency that focuses on all analogy classes, even the ones which consist on nodes that are actually never played.

In [Eyster & Rabin \(2005\)](#), there are  $N$  players, each of which is of a type unknown to other players. The set of possible types of player  $i$  is  $T_i$  and  $T_0$  denotes the set of possible nature types. is indexed by 0. The set of actions for player  $i$  is  $A_i$ . When players form their decisions, they have to choose the probability

of playing each possible action for all their possible types. In this sense,  $\sigma_i|t_i$  is the probability distribution over  $A_i$  defined by player  $i$  if he is of type  $t_i \in T_i$ . A strategy of player  $i$  is then  $\sigma_i = (\sigma_i|t_i)_{t_i \in T_i}$ . More generally, a strategy profile is  $\sigma = (\sigma_i)_{i \in \{1, \dots, N\}}$ . The subscript  $-i$  refers to all  $i$ 's opponents. Hence, the set of type profiles for  $i$ 's opponents is  $T_{-i} = \times_{j \in \{1, \dots, N\} \setminus i} (T_j)$  and the set of action profiles for  $i$ 's opponents is  $A_{-i} = \times_{j \in \{1, \dots, N\} \setminus i} (A_j)$ . The utility obtained by player  $i$  when his type is  $t_i$ , he chooses action  $a_i$ , and his opponents have types  $t_{-i}$  and play actions  $a_{-i}$  is  $u_i(a_i, a_{-i}, t_{-i}|t_i)$ . The distribution of types (including nature's),  $\theta$ , is common knowledge. The probability that player  $i$  is of type  $t_i$  and his opponents are of types  $t_{-i}$  is  $\theta(t_i, t_{-i})$ . Once they know their type, players update their beliefs about their opponents' types. If, for player  $i$ , the updated probability that his opponents are of type  $\hat{t}_{-i}$ , given that his type is  $t_i$  is  $p_i(\hat{t}_{-i}|t_i)$ , then, according to Bayes' rule:

$$p_i(\hat{t}_{-i}|t_i) = \frac{\theta(t_i, \hat{t}_{-i})}{\sum_{t_{-i} \in T_{-i}} (\theta(t_i, t_{-i}))}$$

When a player  $i$  of type  $t_i$  chooses action  $a_i$ , he has two sources of uncertainty regarding the utility this choice confers. He does not know which actions his opponents are choosing, because of their possible use of mixed strategies, and what are the types of his opponents. However, for each set of beliefs about his opponents' choices,  $b^{t_i}$ , using  $p_i$ , he can find the expected utility of choosing  $a_i$ . It is defined as  $v_i(a_i, b^{t_i}|t_i)$  and has the following expression:

$$v_i(a_i, b^{t_i}|t_i) = \sum_{t_{-i} \in T_{-i}} \left( p_i(t_{-i}|t_i) \sum_{a_{-i} \in A_{-i}} (b^{t_i}(a_{-i}|t_{-i}) u_i(a_i, a_{-i}, t_{-i}|t_i)) \right)$$

A classic Bayesian equilibrium predicts that the beliefs of each player coincide with the exact choices of their opponents, that is,  $b^{t_i} = \sigma_{-i}$ . However, in this model, the beliefs are simplified. Given the actual choices of player  $i$ 's opponents and the fact that player  $i$  is of type  $t_i$ , the average probability, across types, that  $a_{-i}$  is chosen by player  $i$ 's opponents, as seen by player  $i$ , is:

$$\bar{\sigma}_{-i}^{t_i}(a_{-i}) = \sum_{t_{-i} \in T_{-i}} (p_i(t_{-i}|t_i) \sigma_{-i}(a_{-i}|t_{-i}))$$

If the game is *fully cursed*, player  $i$  believes that, whatever the types his opponents have, they always play  $a_{-i}$  with this probability. However, the level of bounded rationality is made flexible with the introduction of a parameter  $\alpha \in [0, 1]$ . With this parameter, player  $i$  of type  $t_i$  believes that the probability of his opponents playing  $a_{-i}$  when their types are  $t_{-i}$  is an average of  $\bar{\sigma}_{-i}^{t_i}(a_{-i})$  and their true probability, weighted by  $\alpha$ . These beliefs, represented by  $\phi^{t_i}$ , then have the characteristic that:

$$\phi^{t_i}(a_{-i}|t_{-i}) = \alpha \bar{\sigma}_{-i}^{t_i}(a_{-i}) + (1 - \alpha) \sigma_{-i}(a_{-i}|t_{-i})$$

A strategy profile is an  $\alpha$ -*cursed equilibrium* when, given this type of beliefs, all actions played with positive probability by all types of all players result in the same expected utility, not lower than the expected utility an action never chosen can provide. That is,  $\sigma^*$  is an  $\alpha$ -*cursed equilibrium* if, for all  $i \in \{1, \dots, N\}$ , all  $t_i \in T_i$ , all  $a_i \in A_i$  such that  $\sigma_i^*(a_i|t_i) > 0$ , and all  $a'_i \in A_i$ ,  $v_i(a_i, \phi^{t_i}|t_i) \geq v_i(a'_i, \phi^{t_i}|t_i)$ .

Still on the theme of beliefs about other players' choices, it is worthwhile to mention the concept of *cognitive hierarchy*, present, for instance, in [Stahl & Wilson \(1995\)](#), [Nagel \(1995\)](#) and [Camerer \*et al.\* \(2004\)](#). It assumes that players have different depths of reasoning, represented by a level  $k \in \mathbb{N}_0$ . Level 0 players choose randomly one of the available strategies, without any strategic reasoning. Level 1 players assume all others are level 0 and best respond to them. In general, level  $k$  players assume all other players are at most level  $k - 1$  (and they form a belief about the distribution of these levels) and best respond to it. This type of reasoning relaxes the assumption of *common knowledge of rationality* traditionally present in game theory, thus simplifying the way players think about strategic interaction.

A different proposal for modeling bounded rationality in games is the one in [McKelvey & Palfrey \(1995\)](#), which assumes that players observe the payoffs resulting from their choices in an imperfect way. In this sense, its main idea is similar to [Kőszegi & Szeidl \(2013\)](#) which, in an individual decision context, proposes that an

agent's comparisons between choices are biased by the differences they present. For an equilibrium to exist, it is required that the choices made are consistent with the observation errors. That is, for each player, the probability that each action is chosen must match the probability that it is viewed as the best action to take, given his observational error and the choices of other players.

There are  $N$  players. The action set of player  $i$  is  $A_i$ , which contains  $M_i$  actions. The action set of all players is  $A = \times_{i \in \{1, \dots, N\}} (A_i)$ . The utility function of player  $i$  depends on all players' actions:  $u_i : A \rightarrow \mathbb{R}$ . A strategy for player  $i$  is  $\sigma_i = (\sigma_i(a_i))_{a_i \in A_i}$ , where  $\sigma_i(a_i)$  is the probability that he chooses  $a_i$ . A strategy profile is  $\sigma = (\sigma_i)_{i \in \{1, \dots, N\}}$ . The subscript  $-i$  stands for all players except  $i$ . If  $a \in A$  is an action profile and  $a(j)$  is player  $j$ 's action in this profile, the expected utility of player  $i$ , when the strategy profile is  $\sigma$ , is  $v_i(\sigma_i, \sigma_{-i}) = \sum_{a \in A} \left( \prod_{j \in \{1, \dots, N\}} (\sigma_j(a(j))) u_i(a) \right)$ . Let us denote a player  $i$ 's strategy that consists on choosing  $a_i$  with probability 1 as  $s_i^{a_i}$ . This player's expected utility when he plays  $s_i^{a_i}$  and his opponents choose  $\sigma_{-i}$  is  $w_i^{a_i}(\sigma_{-i}) = v_i(s_i^{a_i}, \sigma_{-i})$ .

Player  $i$ 's bounded rationality is manifested in the fact that he is unable to correctly observe  $w_i^{a_i}(\sigma_{-i})$ . His perception of it is:

$$\hat{w}_i^{a_i}(\sigma_{-i}) = w_i^{a_i}(\sigma_{-i}) + \varepsilon_i^{a_i} \quad (1.7)$$

In (1.7),  $\varepsilon_i^{a_i}$  represents player  $i$ 's error when observing action  $a_i$ 's payoff. The errors for all the actions in player  $i$ 's set,  $\varepsilon_i = (\varepsilon_i^{a_i})_{a_i \in A_i}$  are jointly distributed with a zero mean according to a probability density function  $f_i$ . Given the choices of his opponents and the errors he make, player  $i$  finds that action  $a_i$  is the best one when his perception of the payoff resulting from this action is not lower than that of all other options. Hence, the set of values for the errors of player  $i$  that makes him belief action  $a_i$  is the best one, when his opponents choose  $\sigma_{-i}$ , is  $R_i^{a_i}(\sigma_{-i}) = \left\{ \varepsilon_i \in \mathbb{R}^{M_i} : \forall a'_i \in A_i, \hat{w}_i^{a_i}(\sigma_{-i}) \geq \hat{w}_i^{a'_i}(\sigma_{-i}) \right\}$ . This implies that  $a_i$  is player  $i$ 's perceived best response to  $\sigma_{-i}$  with the following probability:

$$p_i^{a_i}(\sigma_{-i}) = \int_{\varepsilon_i \in R_i^{a_i}(\sigma_{-i})} (f(\varepsilon_i) d\varepsilon_i) \quad (1.8)$$

A *quantal response equilibrium* is attained if (1.8) is the probability that player  $i$  indeed chooses action  $i$ . That is,  $\sigma^*$  is a quantal response equilibrium if, for all  $i \in \{1, \dots, N\}$  and all  $a_i \in A_i$ ,  $p_i^{a_i}(\sigma_{-i}^*) = \sigma_i^*(a_i)$ .

## 1.4 Conclusion

Since Herbert Simon launched the concept of satisficing in 1955, many researchers took interest in the subject and proposals for modeling the behavior of people who find it difficult to solve an economic problem started to arise. The success of some of the theories in explaining actual phenomena that classic rationality could not predict reinforced the interest of investigators in this area. On the other hand, the documentation of several biases in human behavior was an important motivation and a guidance for new theories of bounded rationality. With the growing of neuroeconomics, a better understanding of the functioning of the human brain in contexts of economic decisions can also provide some cues on which directions to follow. Whatever happens in the future, it seems certain that there will be no turning back in the attempt to know what rationality actually means. If Economics is the science of choices, and choices are made by people, improving the understanding of human behavior seems only natural. But we should not expect this to be an error-free process. After all, we are just boundedly rational.

# Chapter 2

## Costly Thinking or Default Choosing: An Application to Cournot Duopoly

### 2.1 Introduction

Economics is a social science, which studies the way people make decisions, when they have limited resources to fulfill their needs. Not having an exact idea of how the human brain works, it has traditionally assumed unbounded rationality. That is, it assumes that people, in any situation, given the information they possess, are able to find and take the optimal decision. However, numerous evidence that people fail to decide optimally have accumulated through time (see [Conlisk \(1996b\)](#) for a review of some of the studies pointing these failures). In response to this issue, bounded rationality came to existence.

Over the years, bounded rationality spread in many different directions. For example, [Simon \(1955\)](#) introduces the concept of *satisficing*, a way to solve a problem, not by optimizing an objective function, but by selecting an alternative that guarantees a minimum level of satisfaction, called the *aspiration level*. [Tversky \(1972\)](#) proposes a way of comparing alternatives, by looking to their characteristics, elim-



inating them in the way until only one is left. [Arthur \(1991\)](#) studies a method of choosing alternatives randomly according to the quality attached to them, which is adjusted through time according to the success they have when chosen. [Kőszegi & Szeidl \(2013\)](#) claim that people are unable to correctly compare the utility of different alternatives, overfocusing on what distinguishes them and not paying enough attention to their similar characteristics.

These examples show ways to model human decisions that may fail to achieve an optimum. But, even though they implicitly assume that optimization is too hard to be accomplished, they do not quantify how difficult it is. And this is a matter of importance, as the fact that thinking is costly can significantly influence the decisions made. On the contrary, a strand of literature on bounded rationality, which we may label *costly thinking*, focus on this issue. It states that people have the possibility to analyze and solve problems, even difficult ones, but the fact that they have to perform hard mental operations to do so forces them to pay a cost, which is reflected on their final utility. [Conlisk \(1980\)](#) defines a society where people are either *optimizers* or *imitators*, in which the former group is able to optimize, but at a cost, while the latter avoids this cost by imitating average behavior. [Rubinstein \(1986\)](#) analyzes game theory in the perspective of finite-state machines chosen by players, and assumes that the complexity of these machines, reflected in the number of states they have, is costly. [Evans & Ramey \(1992\)](#) presents a macroeconomic model, in which agents are able to form rational expectations, but only if they pay a cost for doing so. In the context of a Cournot oligopoly, firms in [Conlisk \(1996a\)](#) select a convex combination between a rule of thumb and an approximation to the optimal quantity, and pay a cost for reducing the uncertainty as to what is the optimal quantity. [Gabaix \(2014\)](#) studies the consumer problem, introducing a two-stage decision process, which consists of first choosing how much costly attention to devote to each parameter, and then optimize the objective function resulting of this selection.

Our model fits in this strand of literature. And it introduces costly thinking in a very simple way. It assumes that people have to choose between thinking or not.

If they do, they pay a fixed thinking cost and behave just like in the classical unbounded rationality models. Otherwise, they do not pay any thinking cost, but can only choose what we call their default choice. It is a choice they can pick without mental effort, either because they are familiar with it, because it is a focal point, or simply because it comes to mind. [Kahneman \(2003\)](#) provides some support to the notion of default choice. They propose a framework for human decisions, based in three operations: perception, intuition and reasoning. The first one is a response to stimuli and cannot be materialized, whereas the other two can be assessed and verbalized. However, while intuition is based on simple and fast processes, which can be affected by emotions and are guided by patterns and associations, reasoning is slower, and treats information in a controlled and structured way. More importantly, the former is labeled effortless and the latter effortful. Our notion of choosing between costless default choosing and costly thinking represents this difference in effort between intuition and reasoning. [Choi \*et al.\* \(2003\)](#) studies the 401(k) enrollment decision of firms' employees, assuming that, when they are hired, their 401(k) plan has a default saving rate. If they want to change it, they have to incur in a cost and, in that case, select their optimal saving rate. This goes in line with our concept of costly thinking to avoid sticking with the default decision. [Gabaix \(2014\)](#) main focus is on parameters, and the way agents perceive them. However, they make use of the concept of default action, the optimal action when the parameters perceived by agents are the default ones.

Our model is close in spirit to the work of John Conlisk, especially in [Conlisk \(1980\)](#) and [Conlisk \(1996a\)](#), but there are some important differences. In [Conlisk \(1980\)](#), people who do not optimize, the imitators, choose an option that depends on the average choice society makes, whereas our non-optimizers simply choose a default option, which spares them from the effort of observing other people's choices, a mechanism more in line with the framework suggested by [Kahneman \(2003\)](#). Also, in [Conlisk \(1980\)](#), people become optimizers or imitators in their childhood and they act as such as adults, but in our model, they select endogenously whether to optimize or not.<sup>1</sup> In this way, the flexibility people have to decide for themselves

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<sup>1</sup>Notice however that, in [Conlisk \(1996a\)](#), the decision of a child to become an imitator or

is accounted for in our model. In [Conlisk \(1996a\)](#), firms are assumed to choose a convex combination of the rule of thumb (equivalent to our default choice) and an approximation to the optimal quantity. However, this way of choosing quantities is exogenously imposed and not derived from the model's fundamentals. In contrast, our model, applied to the Cournot duopoly, predicts that firms choose the default or optimal quantity, without imposing anything other than optimality in the decision to think.

In models such as the ones just mentioned, and also in ours, an important conceptual problem arises: the *infinite regress*. If, in a given problem, agents have to pay a cost for being able to find the optimal choice, why would they be able to costlessly solve the thinking problem in an optimal way? But, if we assume that this second problem is also costly solved, we need to define how to solve the way to solve the original problem. And this reasoning continues, level after level, creating a spiral problem, which seems to have no solution. We acknowledge the existence of such problem but, nonetheless, assume optimality at the second level. Though it may seem an unjustified solution, it has, as [Conlisk \(1996a\)](#) affirms, the merit of allowing us to modify the classic choice problem in a way that does not ignore the fact that not all problems are easily solved by everyone, and, with that, to get a problem complexity explanation of observed phenomena. On the other hand, as [Conlisk \(1996a\)](#) also argues, people are not familiar with every problem they are faced with, but are familiar with problem solving with general, and the difficulties they have with optimization. This implies that, although optimizing a specific problem is costly, the comparison of the benefits and costs of thinking about it may be costless.

After formally introducing our concept in an individual decision context, we apply it to a Cournot duopoly. Our motivation for doing so is a series of experimental papers ([Huck \*et al.\*, 1999](#), [Rassenti \*et al.\*, 2000](#), [Huck \*et al.\*, 2002](#), [Bosch-Domènech & Vriend, 2003](#)), which have different objectives, but report some evidence that we think may be explained with a costly thinking model. They all consist on experi-  

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optimizer is not arbitrary, as it depends on the share of optimizers in the population and their relative performance at the date of choice.

ments in which subjects acted as firms on a Cournot oligopoly with a linear demand, having to select, in each period, the quantity to sell. One main result of all these papers is that there is not a unique strategy followed by all subjects. Some of the papers test different strategies, such as best responding and imitating, but all of them fail to select one single strategy as a good representative of subjects' behavior. In our model, strategy heterogeneity is obtained even with ex-ante identical firms. In fact, we conclude that, even if firms have the same default quantity, an equilibrium in which some decide not to think and others best respond to them is possible. For this to be true, the thinking cost has to be neither too high nor too low. Also, the individual behavior these papers report seems to reveal some degree of stickiness, in the sense that the number of players not adjusting their choices from one period to the other or, at least, not reacting to their opponents' choices is considerable. Our hypothesis that players, when not best responding, are sticking to their default, seems to be in line with this fact. We develop a simple dynamic extension to our model that gets inter temporal stickiness and eventual stabilization of quantities, except in very special circumstances. [Bosch-Domènech & Vriend \(2003\)](#), who focus their analysis on an imitation behavior, are intrigued by the fact that, when decisions are harder to make, the imitation of successful behavior is not more prevalent and, at the same time, declare that the stay-put rule, which consists of a player imitating himself from one period to the other, is the most successful among the ones they suggest. Our default choice logic can explain both these facts, and suggest a change of perspective for the study they make: the alternative to best responding may not be imitation, as they intend and [Conlisk \(1980\)](#) proposes, but instead the selection of the default choice. Within the four papers analyzed, [Bosch-Domènech & Vriend \(2003\)](#) is the one which captures most of our attention, because it deals with the difficulty in making decisions, represented by the time limit to make a choice and the complexity in the way information is made available to subjects.<sup>2</sup> Associating

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<sup>2</sup>[Huck et al. \(1999\)](#) and [Rassenti et al. \(2000\)](#) do not address the issue of decision complexity, because they provide different information sets to their treatment groups, whereas, in [Bosch-Domènech & Vriend \(2003\)](#), all subjects receive the same information, but the way it is presented differs across treatment groups.

the decision complexity with the thinking cost present in our model, we are able to explain why decisions are more dispersed when thinking is harder. The intuition for this is that, if we allow default quantities to vary between firms, the number of firms best responding in equilibrium is decreasing in the thinking cost, making the equilibrium evolve from total symmetry (all firms choosing the symmetric Nash quantity), to partial symmetry (the best responding firms making the same decision) to maximum dispersion (all firms choosing their default quantities).

The rest of the paper is organized in the following way. In [Section 2.2](#), we formally present the idea of costly thinking in individual decision, and apply it to a simple problem of consumer choice. [Section 2.3](#) analyzes a Cournot oligopoly with costly thinking firms, comparing the resulting equilibria with the Nash equilibrium, and studying the impact of an increase in the thinking cost. The results obtained are compared to some conclusions of the experimental Cournot literature in [Section 2.4](#). [Section 2.5](#) concludes. All the proofs are relegated to the [Appendix](#).

## 2.2 Individual Choice

Our model applies to any problem in which at least one individual has to make a choice to maximize some objective function. Let us say that the choice set available to an individual is  $A$ , with  $a$  representing its general element. One of the elements of this set is the default choice of the individual,  $a^d$ . It may be something he is familiar with, something he sees being chosen by other people, a focal point, or simply something that is intuitive to him. What defines this element is that the individual knows it, as well as the consequences of its choice, without the bearing of any cost. The objective function the individual wants to maximize is  $u : A \rightarrow \mathbb{R}$  and we call it the *basic utility* function. The novelty relative to the classic rationality model is that the individual cannot simply maximize  $u$ , choosing one of its maximizers in  $A$ . If he is to do so, he incurs in a fixed thinking cost,  $F \geq 0$ . That means that the individual decision has an extra dimension in this model, as, besides needing to select one element from the choice set, the individual also has to choose  $t$  from

$\{0, 1\}$ , with 0 representing not thinking, and 1 thinking. If he chooses  $t = 0$ , he does not expand his knowledge of  $A$  and selects  $a^d$ , the only mentally available option to him. If, on the contrary,  $t = 1$ , he is able to analyze  $A$  and select the best available option. To make things clear, we define the strategy space of the individual as  $S$ :

$$S = (\{0\} \times \{a^d\}) \cup (\{1\} \times A) \quad (2.1)$$

The new function to be maximized, when the thinking decision is accounted for is  $v : S \rightarrow \mathbb{R}$ , and we call it the *final utility* function. Its expression is the following:

$$v(a, t) = \begin{cases} v^d = u(a^d) & , \quad t = 0 \\ v^t(a) = u(a) - F & , \quad t = 1 \end{cases} \quad (2.2)$$

Let us use an example to illustrate the way the model works. The individual is a consumer who wants to choose the quantity to buy of goods  $x_1$  and  $x_2$ . His wealth is  $M$  and the price of good  $x_l$ , with  $l \in \{1, 2\}$ , is  $p_l$ . Both prices are assumed to be positive. That means his choice set is:

$$A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0 \wedge x_2 \geq 0 \wedge p_1 x_1 + p_2 x_2 \leq M\}$$

His default choice is  $x^d = (x_1^d, x_2^d) \in A$ . The individual has an utility function that depends on the quantity consumed of both goods,  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ , which general expression is  $u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$ , with  $0 < \alpha < 1$ . This function is the consumer's basic utility function.

In the classic rationality model, the individual chooses, among the alternatives in  $A$ , the one that maximizes  $u$ . We index all variables relative to this situation, in which no thinking costs exists, with  $c$ . The solution of the classic consumer's problem is  $x^c = \left(\frac{\alpha}{p_1}, \frac{1-\alpha}{p_2}\right) M$ , which implies  $u^c = \left(\frac{\alpha}{p_1}\right)^\alpha \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} M$ .

Now, suppose the individual has to pay a fixed mental cost of  $F \geq 0$  to think the problem through and be able to choose  $x^c$ . If  $F \geq u^c$ , he trivially decides not to think, regardless of what  $x^d$  is. This is because the cost of thinking is higher than the maximum utility he could possibly get by analyzing  $A$ , meaning that the

benefit from thinking can not possibly outweigh the cost it implies. So, let us focus on the case in which  $F \in [0, u^c[$ . Given his default choice, the individual has to decide whether to think or not. If he does, he maximizes  $u$  in  $A$ , but pays a mental cost of  $F$ , thus getting a final utility of  $u^c - F$ . Otherwise, he consumes his default choice, pays no mental cost, and gets a final utility of  $u^d = u(x^d)$ . Let us index the situation in which the thinking cost is possibly present with  $*$ . The optimal choice of the individual is then, in this case (assuming that the individual does not think when he is indifferent between doing it or not):

$$x^* = \begin{cases} x^c & , \quad F < u^c - u^d \\ x^d & , \quad F \geq u^c - u^d \end{cases}$$

A consumer whose default choice does not guarantee him an acceptable utility level has an incentive to search for a better alternative and does it, finding  $x^c$ , at the expense of a mental cost of  $F$ . On the contrary, a consumer whose default choice is good enough does not find it interesting to analyze the choice set and remains idle, choosing  $x^d$ . This means that a consumer in the former situation gets the same basic utility as a classic rational consumer, whereas if he is in the latter situation, his basic utility is lower than  $u^c$  or equal to it, if he is fortunate enough to have  $x^c$  as his default choice. In what regards final utility, a consumer is almost always worse than in the classic situation, as he either gets a basic utility which is lower than  $u^c$ , or he has to deduct a mental cost of  $F$  to the basic utility of  $u^c$  he gets by choosing  $x^c$ . The exceptions are the cases in which  $F = 0$ , which corresponds to classic rationality, or  $x^d = x^c$ , in which a consumer can choose effortlessly the optimal choice, basic utility-wise.

If we fix  $F$ , it is  $x^d$  which determines in which situation a consumer is. With this in mind, we define the concept of isofin curves. For a given  $F$ , an isofin curve of level  $v$  is a set of default points in the choice set that generate the same final utility for a consumer. Formally:

$$IF_F^v = \{x^d \in A : v^*(F, x^d) = v\} \tag{2.3}$$

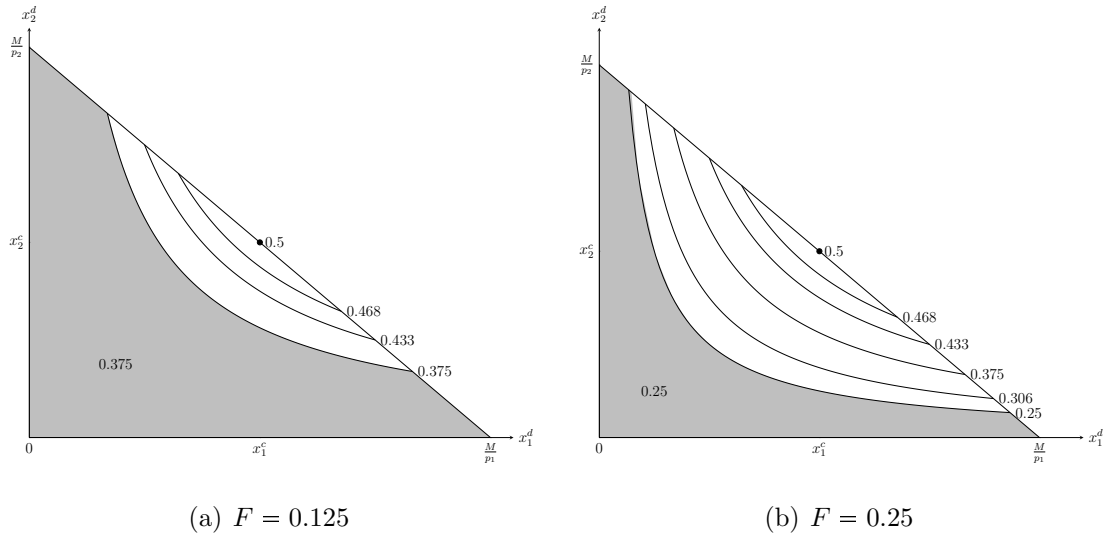


Figure 2.1: Isofin curves, when  $\alpha = \frac{1}{2}$  and  $p_1 = p_2 = M = 1$

We represent some isofin curves in [Figure 2.1](#). The numbers next to the lines or inside the grey areas represent the final utility of a consumer whose default choice belongs to that line or region. It is important to distinguish these curves from indifference curves, which consist of sets of consumption baskets which generate the same basic utility to a consumer. In fact, indifference curves are useful in generating isofin curves, as the knowledge of an isofin curve implies the ability to solve the classic utility maximization problem. An obvious difference between isofin and indifference curves in this case is the possibility that an isofin curve is not an actual curve, but a region. This happens because all consumers whose default choices guarantee them a basic utility lower than a certain threshold take the same action and get the same final utility. Observing [Figure 2.1\(a\)](#), we can see that all consumers get a final utility lower than the classic basic utility, which is 0.5, with the exception of a consumer whose default choice is  $(\frac{1}{2}, \frac{1}{2})$ , the classic optimal choice. On the other hand, no consumer ever gets a final utility lower than 0.375. This happens because  $F = 0.125$  and  $u^c = 0.5$  and all consumers whose default choice guarantees a basic utility lower than  $u^c - F = 0.375$  decide to analyze  $A$  and choose  $x^c$ . In [Figure 2.1\(b\)](#), everything is the same, with the exception of  $F$ , which increases from 0.125 to 0.25. The consequence of this increase is a contraction of the gray area, as a higher thinking cost induces more consumers to stick to their default choices. Thinking



is now less valuable and some consumers who analyzed  $A$  in search for a better alternative before now consider their default choice acceptable. As a consequence, some consumers get the same basic utility as before, specifically the ones which were already sticking to their default, and others are worse off, either because they give up on thinking and accept a lower final utility, or because they maintain their decision to think, but pay a higher mental cost for doing it.

## 2.3 Cournot symmetric duopoly

In this section, we apply our model to the problem of a Cournot duopoly, with symmetric fixed thinking costs, default quantities and linear costs and a linear demand. We present the classic game in [Section 2.3.1](#) and introduce the behavioral version of the game in [Section 2.3.2](#). We then analyze its thinking, default and mixed equilibria, respectively, in [Section 2.3.3](#), [Section 2.3.4](#) and [Section 2.3.5](#), and gather them in [Section 2.3.6](#), devoted to comparative statics relative to the thinking cost.

### 2.3.1 The classic game

There are two firms, indexed by  $i \in N = \{1, 2\}$ . For a fixed  $i \in N$ ,  $j$  is the index of the firm different from firm  $i$ . Firms simultaneously choose the quantity they produce of a homogeneous product in a single period. The quantity produced by firm  $i$  is  $q_i$ . The cost function is linear and the same for both firms:  $\forall i \in N, C_i(q_i) = cq_i$ , with  $c \geq 0$ . Inverse demand is linear and depends negatively on the total quantity produced,  $Q = \sum_{i=1}^2 (q_i)$ :  $P(Q) = \max\{0, a - bQ\}$ , with  $a > c \geq 0$  and  $b > 0$ . To facilitate reading, we define  $\phi = \frac{a-c}{b}$ . Firms can choose to produce any non-negative quantity, however, any market quantity larger than  $\frac{a}{b}$  generates a zero price, which means there is never an incentive to produce it. Hence, we define that, for all  $i \in N$ ,  $A_i = [0, \frac{a}{b}]$ . Each firm wants to maximize its profit, the basic utility in this context. Then, for all  $i \in N$ :

$$u_i(q_i, q_j) = \pi_i(q_i, q_j) = bq_i(\phi - (q_i + q_j)) \quad (2.4)$$

The best response function of firm  $i$  has the following expression:  $r_i(q_j) = \max\{0, \frac{1}{2}(\phi - q_j)\}$ . As firms are symmetric, their best response functions are equivalent, and we can lose the subscript, and use  $r(q) = \max\{0, \frac{1}{2}(\phi - q)\}$ . The only equilibrium is  $(\frac{1}{3}, \frac{1}{3})\phi$ , which we call  $c$ . We then have  $q_1^c = q_2^c = q^c = \frac{1}{3}\phi$  and  $\pi_1^c = \pi_2^c = \pi^c = \frac{1}{9}b\phi^2$ .

| Variable | Expression           |
|----------|----------------------|
| $q^c$    | $\frac{1}{3}\phi$    |
| $Q^c$    | $\frac{2}{3}\phi$    |
| $P^c$    | $\frac{2c+a}{3}$     |
| $\pi^c$  | $\frac{1}{9}b\phi^2$ |
| $\Pi^c$  | $\frac{2}{9}b\phi^2$ |
| $S^c$    | $\frac{2}{9}b\phi^2$ |
| $W^c$    | $\frac{4}{9}b\phi^2$ |

Table 2.1: Classic equilibrium variables

We define  $\Pi^e = \sum_{i=1}^2 (\pi_i^e)$ ,  $S^e = \int_0^{Q^e} ((P(Q) - Q^e) dQ)$  and  $W^e = \Pi^e + S^e$  as, respectively, total profits, consumer surplus and social welfare. Throughout the paper, we remove the subscripts from individual variables, when they have the same value for both firms. The values of all equilibrium variables are in [Table 2.1](#). They are not important per se, but as a reference for comparison with the situation in which the thinking cost is present. Even so, it is worthwhile noticing that both firms make exactly the same choice and get the same profit, which is not surprising, as they are symmetric.

### 2.3.2 Duopoly with thinking cost

If thinking is costly, firms need to choose, besides the quantity to produce, whether to think or not, that is, firm  $i$  has to select  $t_i$  from  $\{0, 1\}$ . If it chooses to think, it has to pay a fixed cost of  $F \geq 0$ .<sup>3</sup> Otherwise, its only possible quantity choice is the default quantity,  $q^d$ , which we assume to be the same for both firms. We say the default quantity is small whenever  $q^d < q^c$  and large if  $q^d > q^c$ , and define  $\Delta = |q^d - q^c|$ . Although any quantity in firm  $i$ 's choice set is, in theory, a possible default quantity, we restrict  $q^d$  to be in  $[0, q^m]$ , where  $q^m = \frac{1}{2}\phi$  is the monopoly quantity in this market.<sup>4</sup> With this assumption, the relevant expression for the demand function is  $P(Q) = a - bQ$  and the expression of the best response function can be simplified in the following way:<sup>5</sup>

$$r(q) = \frac{1}{2}(\phi - q) \quad (2.5)$$

The default market quantity,  $Q^d = \sum_{i=1}^2 (q^d) = 2q^d$ , is the total quantity produced by the firms when both are sticking to the default quantity. Firm  $i$ 's pure strategy space is  $S = (\{0\} \times \{q^d\}) \cup (\{1\} \times [0, q^m])$ . This firm has to choose an element from this set to maximize its final utility,  $v_i$ , defined in the following way:

$$v_i(q_i, t_i, q_j) = \begin{cases} bq^d(\phi - (q^d + q_j)) & , \quad t_i = 0 \\ bq_i(\phi - (q_i + q_j)) - F & , \quad t_i = 1 \end{cases} \quad (2.6)$$

There is an equilibrium whenever both firms are maximizing their final utilities. We have four possible equilibria: both firms thinking and choosing the classic game quantities, both firms not thinking and sticking to their defaults, and each of the firms sticking to its default and the other best responding to it. We study each of them in the following subsections.

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<sup>3</sup>If  $F = 0$ , we are back to the classic situation.

<sup>4</sup>This is justified by the fact that no quantity larger than  $q^m$  is ever a best response. Also, notice that the monopoly quantity is the largest quantity a firm operating in this market with any number of competitors produces.

<sup>5</sup>Moreover, this means that market quantity is never larger than  $\phi$ , and individual profits are never negative, as the equilibrium price is never higher than  $c$ .

### 2.3.3 Cournot thinking equilibrium

In a thinking equilibrium, both firms are best responding. As the classic model has only one equilibrium, there is at most one thinking equilibrium. This is because, if this equilibrium exists, it involves both firms choosing the same quantity they do in the classic equilibrium  $c = \frac{1}{3}\phi$ . Given that firm  $j$  is producing  $\frac{1}{3}\phi$ , firm  $i$  has to be comfortable with its decision of best responding to this quantity. If it best responds, it gets the same profits as in the classic equilibrium:  $\frac{1}{9}b\phi^2$ . If it sticks to the default quantity, its profits are  $bq^d (\frac{2}{3}\phi - q^d)$ . Firm  $i$  is only happy with its decision to best respond, if  $F$  is lower than the difference between these two profits. For the equilibrium to exist, both players must be optimally thinking, which means there is a thinking equilibrium if and only if:

$$F < \bar{F}^T = b\Delta^2 \quad (2.7)$$

The cutoff value for  $F$  which separates the cases in which there is and there is not a thinking equilibrium,  $\bar{F}^t$ , is the square of the distance between the default and the classic equilibrium quantities. If these quantities are close, then only low values of  $F$  make this equilibrium possible, because the default quantity is very attractive and hardly justifies looking for a better alternative. If, on the contrary, these quantities are significantly apart, then this equilibrium can be sustained with a high  $F$ , because best responding is, profit-wise, a very attractive strategy, when compared with sticking to the default. Note that the  $\bar{F}_T$  is never negative, but is 0 when  $\Delta = 0$ . This means that, if the default quantity is exactly the classic equilibrium one, the thinking equilibrium is not possible, since, by sticking to the default quantity, firms can obtain the classic equilibrium profit.

The equilibrium variables are presented in [Table 2.2](#). In comparison with [Table 2.1](#), the new variables are  $v_i$ , final utility of firm  $i$ ,  $V = \sum_{i=1}^2 (v_i)$ , total final utility of firms, and  $Z = W + V$ , net (of thinking costs) social welfare. As the quantities are the same as the classic equilibrium ones, so are price, profits, consumer surplus and social welfare. However, as both firms think, we get that individual profits

| Variable | Expression   | Infimum | Supremum |
|----------|--------------|---------|----------|
| $q^*$    | $q^c$        | $q^c$   | $q^c$    |
| $Q^*$    | $Q^c$        | $Q^c$   | $Q^c$    |
| $P^*$    | $P^c$        | $P^c$   | $P^c$    |
| $\pi^*$  | $\pi^c$      | $\pi^c$ | $\pi^c$  |
| $\Pi^*$  | $\Pi^c$      | $\Pi^c$ | $\Pi^c$  |
| $v^*$    | $\pi^c - F$  | 0       | $\pi^c$  |
| $V^*$    | $\Pi^c - 2F$ | 0       | $\Pi^c$  |
| $S^*$    | $S^c$        | $S^c$   | $S^c$    |
| $W^*$    | $W^c$        | $W^c$   | $W^c$    |
| $Z^*$    | $W^c - 2F$   | $S^c$   | $W^c$    |

Table 2.2: Thinking equilibrium variables

are cut by  $F$  to obtain final utilities, whereas total profits and social welfare are reduced in  $2F$  to obtain total final utility and net social welfare. If this equilibrium exists and is indeed the one attained, then there are bad news relative to the classic equilibrium: it predicts the right quantities, but it is less rewarding for firms than normally assumed.

### 2.3.4 Cournot default equilibrium

In a default equilibrium, both firms decide not to think and produce their default quantities, leaving best responding totally abandoned. When they do this, they simply accept the profit that the combination of these quantities produce. Firm  $i$ , by doing this, gets a profit of  $bq^d(\phi - Q^d)$ , where  $Q^d$ , whereas, if it best responded, its profit would be  $\frac{1}{4}b(\phi - q^d)^2$ . For this to be an equilibrium,  $F$  has to be higher than the difference between the latter and the former. And this is true for both firms, hence the default equilibria exists if and only if:

$$F \geq \underline{F}^D = \frac{9}{4}b\Delta^2 \quad (2.8)$$

As happens in the thinking equilibrium, the cutoff value for  $F$  depends positively on  $\Delta$ . If  $q^d$  and  $q^c$  are close, not thinking is attractive, because the profits obtained in doing so are similar to the classic equilibrium one, so only a very small thinking cost can lead firms to abandon their defaults. Otherwise, only high levels of thinking cost can induce the firms not to best respond, as sticking to the default quantity guarantees a low profit. Contrary to what happens in the thinking equilibrium, there is always a value for  $F$  that assures the existence of the default equilibrium. As  $F$  has no upper bound, even in the case of a low quality (profit-wise) default quantity, both firms accepting the profits implied by the simultaneous choice of the default quantity is possible, if  $F$  is high enough.

| Variable | Expression           | Infimum | Supremum           |
|----------|----------------------|---------|--------------------|
| $q^*$    | $q^d$                | 0       | $\frac{3}{2}q^c$   |
| $Q^*$    | $Q^d$                | 0       | $3Q^c$             |
| $P^*$    | $a - bQ^d$           | $c$     | $a$                |
| $\pi^*$  | $bq^d (\phi - Q^d)$  | 0       | $\frac{9}{8}\pi^c$ |
| $\Pi^*$  | $bQ^d (\phi - Q^d)$  | 0       | $\frac{9}{4}\Pi^c$ |
| $v^*$    | $bq^d (\phi - Q^d)$  | 0       | $\frac{9}{8}\pi^c$ |
| $V^*$    | $bQ^d (\phi - Q^d)$  | 0       | $\frac{9}{4}\Pi^c$ |
| $S^*$    | $\frac{1}{2}bQ^{d2}$ | 0       | $\frac{9}{4}S^c$   |
| $W^*$    | $bQ^d (\phi - q^d)$  | 0       | $\frac{9}{8}W^c$   |
| $Z^*$    | $bQ^d (\phi - q^d)$  | 0       | $\frac{9}{8}W^c$   |

Table 2.3: Default equilibrium variables

As for the implications of this equilibrium on the market, there is a wide range of possible scenarios, as individual quantities are the same as the default one, and it can be anything in  $[0, q^m]$  (or, at least, anything that, together with  $F$ , makes this

equilibrium possible). If  $q^d$  is large, we have large individual and total quantities and a low price, and vice-versa. As the production cost is the same for both firms, individual profits are half the profit a monopolist would have, if he produced  $Q^d$ . This means that, when  $Q^d$  is the monopoly quantity,  $q^m$ , or, equivalently, when  $q^d = \frac{1}{2}q^m\phi$ , individual profits are maximized, and each firm earns half the monopoly profit, which is higher than the classic individual profit. If  $q^d \in \{0, q^m\}$ , either market quantity or price is 0, hence the possibility of null individual profits. Total profit is simply the double of individual profit, so the same reasoning applies. Consumer surplus, which is increasing in the quantity produced, benefits from a large default quantity, and is higher than the classic one if and only if  $Q^d$  is larger than  $Q^c$ . It attains its maximum when  $q^d$  is at its largest value,  $q^m$ . Social welfare is increasing in the quantity produced, until the market price equals marginal cost. The restriction we put on the possible range of values for the default quantity means that the market price is never below marginal cost, which implies that this variable is again maximized when  $q^d = q^m$  and is higher than the classic social welfare when  $q^d > q^c$ . Finally, individual and total final utilities and net social welfare are exactly the same as their profit and social welfare counterparts, as no firm bears any thinking cost in this equilibrium.

### 2.3.5 Cournot mixed equilibria

In a mixed equilibrium, one of the firms sticks to its default and the other best responds to it. An  $i$ -mixed equilibrium is one in which firm  $i$  best responds to firm  $j$ . This is an interesting equilibrium, as it predicts that ex-ante identical firms choose different quantities.

In an  $i$ -mixed equilibrium,  $q_j^* = q^d$  and  $q_i^* = r(q^d) = \frac{1}{2}(\phi - q^d)$ , implying  $\pi_i^* = \frac{1}{4}b(\phi - q^d)^2$  and  $\pi_j^* = \frac{1}{2}bq^d(\phi - q^d)$ . If firm  $i$  and  $j$  were to change their thinking decisions, the former would just choose the default quantity, while the latter would best respond to the quantity actually chosen by firm  $i$ , which is the best response to  $q^d$ . This means that firm  $j$  would choose to produce  $r(r(q^d)) = \frac{1}{4}(\phi + q^d)$ . Their profits would then be  $bq^d(\phi - Q^d)$  and  $\frac{1}{16}b(\phi + q^d)^2$ , respectively. For this

to actually be an equilibrium,  $F$  has to be such that no firm wants to change its decision. Hence, it has to be greater or equal to the difference between the profit firm  $j$  would have if it best responded and the one it actually has, and, at the same time, lower than the difference between the profit made by firm  $i$  and the one it would have if it abandoned thinking. We then have that an  $i$ -mixed equilibrium exists if and only if:

$$\underline{F}^M = \frac{9}{16}b\Delta^2 \leq F < \frac{9}{4}b\Delta^2 = \overline{F}^M \quad (2.9)$$

As firms are symmetric, whenever an 1-mixed equilibrium exists, so does a 2-mixed equilibrium. These two equilibria are possible for some values of  $F$  if and only if . If  $F$  is neither too high nor too low,<sup>6</sup> the two mixed equilibria are possible and they involve symmetric firms making different thinking decisions and producing different quantities. In fact, note that the best response function is strictly decreasing in the quantity chosen by the opponent and has a fixed point in  $q^c$ , which means that, if one of the firms is sticking to a small (respectively large) default quantity, the other one is best responding to it, choosing a larger (respectively smaller) quantity than  $q^c$ .

To understand why this asymmetric equilibria are possible when firms are symmetric, let us focus on [Figure 2.2](#), which represents the profit functions of both firms when their opponents are playing the equilibrium quantities in an  $i$ -mixed equilibrium. The dashed line is the graph of firm  $i$ 's profit function, facing  $q^d$  as the quantity chosen by its opponent. As firm  $i$  is best responding to this quantity, it chooses  $r(q^d)$ , the quantity that guarantees it the maximum profit in these conditions. The dotted line then represents the graph of firm  $j$ 's profit function, when the quantity chosen by firm  $i$  is  $r(q^d)$ . The quantity that maximizes this function is  $r(r(q^d))$ , but it is not chosen by firm 2, which, in this equilibrium, prefers not to think and just chooses  $q^d$ . For this to be an equilibrium, the thinking cost must be higher or equal to the potential gain in becoming a best responder for firm  $j$  and

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<sup>6</sup>Notice that these two equilibria are only possible if, in this type of equilibrium, if  $\Delta > 0$ , because otherwise the best responding firm produces  $q^c$ , which it can do without thinking.



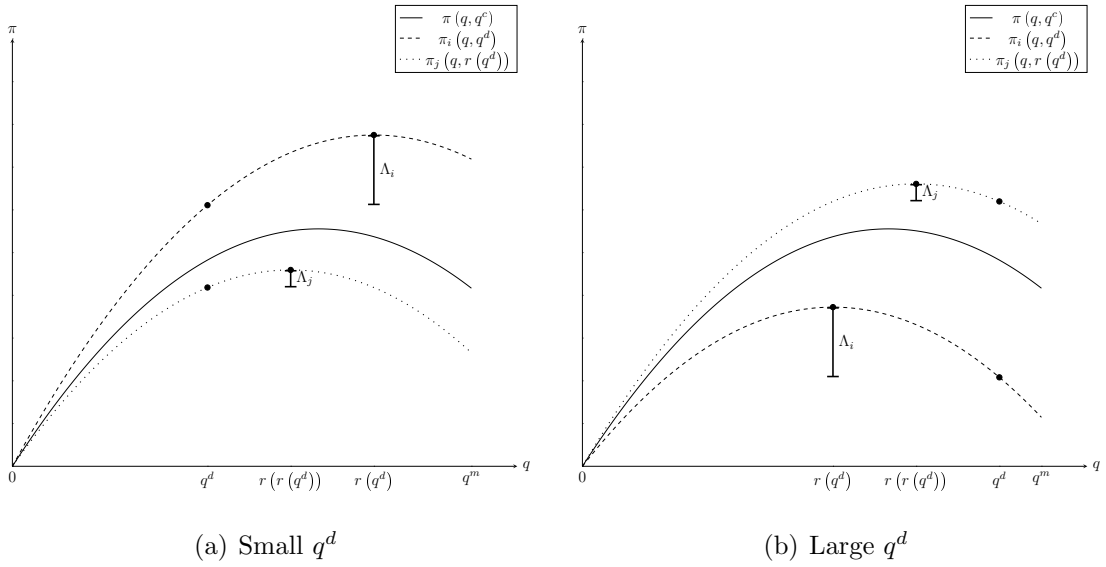


Figure 2.2: Profit functions of firms  $i$  and  $j$  in an  $i$ -mixed equilibrium

lower than the potential loss in abandoning best responding for firm  $i$ . And this is only possible if the latter is higher than the former. And, as we next explain, this is always true in this model.

Firm  $j$  would increase its profit by  $\Lambda_j$ , the vertical distance drawn next to the dotted line, if it changed its thinking decision. Firm  $i$ , in turn, would have a loss of  $\Lambda_i$ , the vertical distance drawn next to the dashed line, in its profit, if it did the same. And it is clear from the figure that, if  $q^d$  is either small or large,  $\Lambda_i > \Lambda_j$ . Before discussing further why this is the case, let us state a technical result that allows us to focus on quantities instead of profits, when comparing potential profit gains and losses:

**Proposition 1.** Regardless of the quantity chosen by its opponent, the difference in the profit a firm makes when choosing the best response to it and any other quantity is increasing in the distance between these two quantities. More specifically,  $\forall i \in N, \forall (q_i, q_j) \in [0, q^m]^2, \pi_i(r(q_j), q_j) - \pi_i(q_i, q_j) = b(r(q_j) - q_i)^2$ .

*Proof.* See [Appendix](#). □

[Proposition 1](#) states that we can order differences in the profit functions between the maximum profit and any other one, by simply looking to the distance between

the maximizer quantity and the other one being compared. That is, the loss in one firm's profit resulting from changing production from the optimal level to another quantity is larger than the correspondent loss of the other firm if and only if the distance between quantities is larger in the first case. Why does this happen? Independently of the quantity faced by each firm, its profit function has a constant second derivative,  $2b$ . As we are supposed to confront the value of each profit function at its maximum, where the first derivative is 0, and at one other quantity, it does not matter which profit function we are looking at, just how much this other point is distant from the maximizer. [Figure 2.2](#) shows this: the dashed and the dotted graphs behave in the same way when  $q$  departs from their maximizers, even if at different heights.

To understand why  $\Lambda_i > \Lambda_j$ , note that quantities in the Cournot model are strategic substitutes.<sup>7</sup> This explains the relative position of the three graphs in each subfigure of [Figure 2.2](#). The solid line represents the graph of the profit function of a firm facing  $q^c$  as the quantity chosen by its opponent. Focusing on [Figure 2.2\(a\)](#), the fact that  $q^d$  is smaller than  $q^c$  makes firm  $i$ 's profit function (which is affected by the choice of firm  $j$ ) to have a graph which is above the solid line and to be maximized with a quantity larger than  $q^c$ . This, in turn, implies that the graph of firm  $j$ 's profit function (which is affected by the choice of firm  $i$ ) is below the solid line and to be maximized with a quantity smaller than  $q^c$ . All this implies that  $q^d$  is closer to the quantity that maximizes firm  $j$ 's profit function than the one that maximizes firm  $i$ 's profit function. And as, when comparing  $\Lambda_i$  and  $\Lambda_j$ , we can simply focus on the relative position of quantities, we can conclude that firm  $i$  has more to lose in giving up thinking than firm  $j$  has to gain in starting doing so, and the equilibrium is possible, if  $F$  allows it. In [Figure 2.2\(b\)](#), where  $q^d > q^c$ , there is an analogous reasoning that leads to the same conclusion: firm  $i$ 's profit function is lower than  $\pi_i^c$  and so is its choice, which makes firm  $j$ 's profit function to be maximized with a quantity larger than  $q^c$ , but smaller than  $q^d$ ,<sup>8</sup> and so,  $q^d$  is closer to  $r(q^d)$  than to

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<sup>7</sup>In fact, an increase in the quantity produced by one firm decreases the marginal revenue of its opponent, making it respond with a reduction of its own quantity.

<sup>8</sup>This quantity is smaller than  $q^d$  because it is  $r(r(q^d))$ , and the more times you apply the best

$r((q^d))$ .

Intuitively, if a firm expects its rival to stick to the default quantity, and  $F$  is not too high, it decides to best respond to it. By doing so, it makes thinking less valuable for its rival than for itself, as the strategic substitutability of quantities makes the best response of the rival closer to the default quantity than its own. If  $F$  is not too low, the rival indeed prefers not to think and an equilibrium is achieved.

| Variable  | Expression                                  | Infimum              | Supremum             |
|-----------|---|----------------------|----------------------|
| $q_i^*$   | $\frac{1}{2}(\phi - q^d)$                   | $\frac{3}{4}q^c$     | $\frac{3}{2}q^c$     |
| $q_j^*$   | $q^d$                                       | 0                    | $\frac{3}{2}q^c$     |
| $Q^*$     | $\frac{1}{2}(\phi + q^d)$                   | $\frac{3}{4}Q^d$     | $\frac{9}{8}Q^d$     |
| $P^*$     | $\frac{c+a-bq^d}{2}$                        | $\frac{3c+a}{4}$     | $\frac{c+a}{2}$      |
| $\pi_i^*$ | $\frac{1}{4}b(\phi - q^d)^2$                | $\frac{9}{16}\pi^c$  | $\frac{9}{4}\pi^c$   |
| $\pi_j^*$ | $\frac{1}{2}bq^d(\phi - q^d)$               | 0                    | $\frac{9}{8}\pi^c$   |
| $\Pi^*$   | $\frac{1}{4}b(\phi^2 - q^{d2})$             | $\frac{27}{32}\Pi^c$ | $\frac{9}{8}\Pi^c$   |
| $v_i^*$   | $\frac{1}{4}b(\phi - q^d)^2 - F$            | 0                    | $\frac{27}{16}\pi^c$ |
| $v_j^*$   | $\frac{1}{2}bq^d(\phi - q^d)$               | 0                    | $\frac{9}{8}\pi^c$   |
| $V^*$     | $\frac{1}{4}b(\phi^2 - q^{d2}) - F$         | 0                    | $\frac{27}{26}\Pi^c$ |
| $S^*$     | $\frac{1}{8}b(\phi + q^d)^2$                | $\frac{9}{16}S^c$    | $\frac{81}{64}S^c$   |
| $W^*$     | $\frac{1}{8}b(3\phi - q^d)(\phi + q^d)$     | $\frac{27}{32}W^c$   | $\frac{135}{128}W^c$ |
| $Z^*$     | $\frac{1}{8}b(3\phi - q^d)(\phi + q^d) - F$ | $\frac{9}{32}W^c$    | $\frac{45}{44}W^c$   |

Table 2.4:  $i$ -mixed equilibrium variables, with  $i \in N$

The values of all the model variables are presented in [Table 2.4](#). The non-thinking firm, choosing its default, can produce anything from 0 to  $q^m$ , whereas the other firm, best responding to it, can act as a monopolist (if firm  $j$  chooses a 0 quantity), but never decides not to produce (as even the choice of the monopoly response function to a quantity, the closer it gets to  $q^c$ ).

quantity by one of the firms leaves room for profit for the other one). Total quantity depends positively on  $q^d$ , even though  $q_i^*$  is decreasing on it. This happens because the linearity of cost and demand imply a smooth best response, which reacts to an increase in the opponent's quantity with a smaller quantity decrease. Hence, the total effect of an increase in  $q^d$  is positive. And, if total quantity is increasing in  $q^d$ , price is decreasing.

Firm  $i$ 's profit is also decreasing in  $q^d$ , as the quantity produced by firm  $j$  decreases the price obtained by each unit sold. As  $q_j^d \leq q^m$ , firm  $j$ 's profit, on the other hand, is increasing in  $q^d$ . In fact, if  $q^d = 0$ , total quantity is relatively small and market price relatively high, which means that an increase in the quantity produced by firm  $j$  allows it to increase its profit. There is a point in which the increase in quantity does not compensate the reduction in price, and it is precisely  $q^m$ . Final utility of firm  $j$  behaves in the same way, because there is no thinking for this firm in this equilibrium. As for the comparison of the two firms' profits, it depends on the fact that  $q^d$  is large or small. If  $q^d$  is small, we know, as quantities are strategic substitutes in this model, that firm  $i$  decides to produce a quantity larger than  $q^c$  and is better off, profit-wise, than in the classic equilibrium, whereas firm  $j$ , producing too little, is taken advantage of, and gets a lower profit than in the classic equilibrium. Otherwise, the opposite happens. Total profit, which is the same a monopolist operating in this market would have if it produced  $Q^*$ , is decreasing in  $q^d$ , because  $Q^*$  is the monopoly quantity when  $q^d = 0$  and gets further away from it as  $q^d$  increases.

If  $q^d \in \{0, q^m\}$ , firm  $i$ 's final utility can be arbitrarily close to 0, when  $F$  approaches  $\bar{F}^M$ . The fact that this happens at  $q^d = \frac{1}{2}$  is not surprising, because this is the default quantity which generates the lowest profit for firm  $i$ . When  $q^d = 0$ , although firm  $i$ 's profits are high, so can be  $F$ , because the default quantity is as distant as can be from  $q^c$ , implying that firm  $i$  is willing to support a great thinking cost to be able to best respond. More intuitively, the highest value of  $v_i^*$  is attained when  $q^d = 0$  and  $F = \underline{F}^M$ . As for the comparison between  $v_i^*$  and  $\pi^c$ , it is the subject of [Proposition 2](#).

**Proposition 2.** In an  $i$ -mixed equilibrium, firm  $i$  attains a higher final utility than its profit in the classic equilibrium if the default quantity is small and the thinking cost is low. More specifically,  $v_i^* > \pi^c \Leftrightarrow q^d < q^c \wedge F < \frac{1}{4}b\Delta (5q^c - q^d)$ .

*Proof.* See [Appendix](#). □

[Proposition 2](#) gives conditions which guarantee that the thinking firm has an increase in profit relative to the classic situation which more than compensates the thinking effort needed to obtain it.

Total final utility has its infimum at 0 as, when  $q^d = 0$  and  $F$  is very close to  $\bar{F}^M$ , the final utility of firm  $i$  and  $j$  is, respectively, very close to 0 and exactly 0. On the other hand, its supremum (and maximum) is higher than  $\Pi^c$ , but, incidentally, it is not achieved when total profit is maximized. This is because, when  $q^d = 0$ , total profit is indeed maximized, but, as  $\Delta$  is relatively high, firm  $j$  has a high incentive to think, which means that  $\underline{F}^M$  is high, and firm  $i$  cannot escape from a large thinking cost. When  $q^d$  increases from 0,  $\Delta$  decreases and so does  $\underline{F}^M$ , but the same happens to total profit. There is then a trade-off between reducing the thinking cost and reducing profit. The first effect dominates when  $q^d$  is close to 0, because total quantity is close to  $q^m$ , where the total profit function is relatively flat, but eventually this relation is reversed.<sup>9</sup>

Consumer surplus and social welfare are increasing in total quantity,<sup>10</sup> which means that both consumers and a utilitarian social planner who ignores thinking costs would endorse, if possible, the largest possible default quantity. As the total quantity may be larger than  $Q^c$ , both these variables can attain higher values than in the classic equilibrium. Finally, net social welfare is never 0, as, even when firms get no final utility whatsoever, consumer surplus is positive. Its maximum must be attained when  $q^d$  is large because, when  $q^d < q^c$ , a small increase in  $q^d$

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<sup>9</sup>Notice that the reverse point must be lower than  $q^c$  because, if  $q^d$  increases from  $q^c$ , total profit decreases, but  $\Delta$  increases and so does  $\underline{F}^M$ . In fact, some algebra shows that total final utility is maximized when  $q^d = \frac{9}{13}q^c$  and  $F = \frac{9}{169}\pi^c$ , resulting in  $V^* = \frac{27}{26}\Pi^c$ .

<sup>10</sup>Social welfare is increasing in total quantity as the maximum total quantity ever produced is  $\phi$ , which means that price is never below marginal cost.

increases social welfare and reduces  $\Delta$  and, therefore,  $\underline{F}^m$ . It is natural that it is a large default quantity that maximizes it, because social welfare benefits from high production levels. However, the maximum is not attained when  $q^d = q^m$ , the largest possible quantity, because this would imply a default quantity too distant from  $q^c$ , forcing the thinking firm to exert a high thinking effort. Its maximum is higher than  $W^c$ , which means that, even when thinking costs are accounted for, society may be better than what the classic model predicts.<sup>11</sup>

### 2.3.6 The impact of thinking cost

Having studied all possible equilibria, we can analyze what happens in the model as  $F$  increases. The thinking cost represents the difficulty firms have to understand the problem they are faced with, and it would be interesting to know the impact of an increase in the problem difficulty or, equivalently, a decrease in the ability of firms to deal with the problem at hand. Let us label the thinking, mixed and default equilibria by  $T$ ,  $M$  and  $D$ , respectively. We already know that, if  $\Delta = 0$ , the only possible equilibrium is  $D$ , regardless of  $F$ , so we focus on the case in which  $\Delta > 0$ .

Observing (2.7), (2.8) and (2.9), we can tell that  $\underline{F}^M < \overline{F}^T < \overline{F}^M = \underline{F}^D$ . Hence, when  $F \geq \underline{F}^D$ , the only possible equilibrium is  $D$ . On the other hand, if  $F \in [\underline{F}^M, \overline{F}^T]$ , three equilibria are simultaneously possible: thinking, 1-mixed and 2-mixed. However, assuming that one of the existing equilibria is selected in each situation, we can be sure of one thing: when  $F$  increases from 0, the equilibrium in play goes from  $T$  to  $M$  to  $D$ . The transition from thinking to mixed can be made either when  $F$  is  $\underline{F}^M$  or  $\overline{F}^T$ , but the equilibrium ordering is this for sure. In this transition, one of the firms best respond and the other sticks to its default in the  $M$  equilibrium. Let us index the former by  $a$  and the latter by  $p$ , where  $a$  and  $p$  stand for active and passive, respectively.

The decision variable, quantity, defines all the other variables, so let us focus on how it changes when  $F$  increases. Table 2.5 shows, in a graphical manner, this evolution, for the quantities chosen by the active and passive firm and market

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<sup>11</sup>Net social welfare is maximized when  $q^d = \frac{15}{11}q^c$  and  $F = \frac{9}{121}\pi^c$ , resulting in  $Z^* = \frac{45}{44}W^c$ .

| Variable | Low $q^d$ |     |     | High $q^d$ |     |     |
|----------|-----------|-----|-----|------------|-----|-----|
|          | $T$       | $M$ | $D$ | $T$        | $M$ | $D$ |
| $q_a^*$  | ●         | ●   | ●   | ●          | ●   | ●   |
| $q_p^*$  | ●         | ●   | ●   | ●          | ●   | ●   |
| $Q^*$    | ●         | ●   | ●   | ●          | ●   | ●   |

Table 2.5: Equilibrium quantity sizes in each type of equilibrium

quantity. The greater the circle area, the larger the quantity it represents. What happens to the passive firm is obvious: if  $q^d$  is small, it is smaller than what it selects in the thinking equilibrium, and so the quantity it chooses in equilibrium is reduced when the equilibrium goes from thinking to mixed and remains the same in the default equilibrium. If  $q^d$  is large, the first change is an increase instead of a reduction. The quantity chosen by the active firm does not change monotonically with  $F$ , in opposition to  $q_p$ . If  $q^d$  is small, the quantity this firm chooses in the mixed equilibrium is larger than  $q^d$ , because of the strategic substitutability of quantities and the fact that  $r(q^c) = q^c$ . On the other hand, the quantity it chooses in the default equilibrium,  $q^d$ , is, by definition, larger than  $q^c$ . If  $q^d$  is large, the reverse happens. This generates [Proposition 3](#):

**Proposition 3.** If the default quantity is not the classic equilibrium one, total quantity is decreasing (respectively, increasing) with  $F$ , if  $q^d$  is small (respectively, large).

*Proof.* See [Appendix](#). □

To see why [Proposition 3](#) holds, suppose  $q^d$  is small. When  $F$  is low enough, the thinking equilibrium occurs and total quantity is  $Q^c = 2q^c$ . When  $F$  increases and the equilibrium turns into a mixed one, the passive firm reduces its quantity to  $q^d$  and the active firm best responds to it, as it also does in the thinking equilibrium.

We already know that this best response is smooth, in the sense that a decrease of one unit in the quantity it is applied to generates an increase smaller than one unit in the quantity it produces. Hence, the increase in  $q^a$  less than compensates the reduction in  $q^p$  and the market quantity goes down. If  $F$  continues to increase and the equilibrium changes to  $D$ , the active firm reduces its quantity to  $q^d$ , resulting in a further diminishing of the market quantity. The same logic applied to a large  $q^d$  shows that total quantity increases with  $F$ . Of course, as price is decreasing in total quantity,  $F$  influences  $P^*$  in the opposite way it influences  $Q^*$ .

| Variable  | Low $q^d$   | High $q^d$ |
|-----------|---|------------|
| $\pi_a^*$ | $M$   | $T$        |
| $\pi_p^*$ | $T \rightarrow D^a$                                 | $M$        |
| $v_a^*$   | $M$   | $T$        |
| $v_p^*$   | $T \rightarrow D$                                   | $M$        |
| $\Pi^*$   | $M \rightarrow D$                                   | $T$        |
| $V^*$     | $T \rightarrow M \rightarrow D / T \rightarrow D^b$ | $T$        |
| $S^*$     | $T$   | $D$        |
| $W^*$     | $T$   | $D$        |
| $Z^*$     | $T$   | $D$        |

<sup>a</sup>The  $\rightarrow$  sign indicates a change as  $q^d$  increases from 0 to  $q^c$ .

<sup>b</sup>The / sign separates the situations in which the transition from  $T$  to  $M$  happens when  $F = \underline{F}^M$  and  $F = \overline{F}^T$ .

Table 2.6: Equilibria which maximize the utility and profit variables

The remaining model variables are the ones that represent the success obtained by firms, consumers or society as a whole, either the thinking cost is accounted for or not. The equilibria in which they attain their maximum values are presented in [Table 2.6](#).

The values different from  $T$  in the rows which do not depend on the thinking cost ( $\pi_a^*$ ,  $\pi_p^*$ ,  $\Pi^*$ ,  $S^*$  and  $W^*$ ) show us that some of the agents in this model can



at least appear to be better when the thinking cost exists and is enough to drive away the equilibrium from  $T$ . In fact, while in the  $T$  equilibrium, all these variables equal their classic equilibrium counterparts, the same does not happen with the  $M$  and  $D$  equilibria. Hence, if either the  $M$  or the  $D$  equilibrium maximize one of these variables, then we can state that the actor (firm, firms, consumers or society) to which it refers is better off (excluding the cost of thinking) than in the classic equilibrium, provided the thinking cost is high enough. For example, if  $q^d$  is small and  $F$  is such that the  $M$  equilibrium is in play, the active firm gets a higher profit than the classic one, because its opponent is being less aggressive than it would be without the presence of the thinking cost. If  $q^d$  is large, both consumers and society as a whole (if we do not account for thinking cost) benefit from a high enough thinking cost to sustain the default equilibrium, because that is the one with the largest total quantity.

As for the remaining variables ( $v_a^*$ ,  $v_p^*$ ,  $V^*$  and  $Z^*$ ), the analysis is not so straightforward. The reason is that the thinking equilibrium is the one that implies the highest thinking cost, possibly in equality with some other. And this means that equilibria different from  $T$  may be preferred just to avoid the cost of thinking. However, these variables are important in understanding which values of  $F$  are preferred by a social planner who takes the cost of thinking into account. This leads us to state the following:

**Proposition 4.** If the default quantity is not the classic equilibrium one, society as a whole benefits from a low (respectively, high) thinking cost, if  $q^d$  is small (respectively, large), either social welfare is net of thinking cost or not.

*Proof.* See [Appendix](#). □

[Proposition 4](#) implies that a market in which small quantities are intuitive benefits from an environment in which decisions are easy to take, so that firms can best respond to one another and go in the direction of the classic Nash equilibrium. However, if intuition favors large quantities, society is better off with a hard decisional environment. In this case, the fact that decisions are hard to take does

not mean that society has to bear the cost of deciding, but either that firms decide endogenously not to react to their competitors and just choose the quantity their intuition suggests.

## 2.4 Experimental Cournot literature

Even though our model makes a very simple addition to classic rationality, it may help shed some light on some results of the experimental Cournot literature. We analyze four papers in this strand of literature, [Huck \*et al.\* \(1999\)](#), [Rassenti \*et al.\* \(2000\)](#), [Huck \*et al.\* \(2002\)](#) and [Bosch-Domènech & Vriend \(2003\)](#), which we think may be better understood at the light of our model.

The main objective of [Huck \*et al.\* \(1999\)](#) is to investigate how well do some individual strategies perform in a Cournot oligopoly. They focus on five individual strategies: *best response dynamics*, which assumes that a firm best responds to the quantity chosen by its opponents in the previous period, *imitate the best*, which states that a firm chooses the quantity produced by the firm with the highest profit in the previous period, *imitate the average*, which leads a firm to produce the average quantity chosen by its opponents in the previous period, and two others, *directional learning* and *error learning*. The experimental setup includes five treatment groups, which differ in the amount and type of information provided to subjects. Their main finding is that no single strategy is able to explain the subjects' behavior and that the most successful strategy depends on the information environment.

[Rassenti \*et al.\* \(2000\)](#) investigate if the repeated play of the Cournot oligopoly converges to the Nash equilibrium. They also consider some individual strategies, one of which is important in the analysis we make here. It is *partial adjustment to best responses* and consists of a weighted average of the best response to the opponents' quantity and the own quantity in the previous period. They have two treatment groups, which again differ in the information received, and conclude that the facts that individual strategies seem to be heterogeneous and that some subjects do not seem to react to their opponents' moves may play a role in the non-convergence

to the Nash equilibrium.

The main research question of [Huck \*et al.\* \(2002\)](#) is whether the theoretical prediction that a linear oligopoly evolves in a very different way when there is inertia (in the sense that firms are not always able to adjust their choices) and not, converging to the Nash equilibrium in the former case and diverging in the latter. Once again, individual strategies are analyzed and, besides the best response dynamic, the imitate the average rule and a mix of these two, they also focus on *fictitious play*. Their two treatment groups differ in the fact that inertia is only present in one of them. They point out that the following of a mixed strategy between best responding and imitate the average, a process which may generate convergence with inertia and without it, helps to explain why convergence is roughly verified in both treatment groups.

[Bosch-Domènech & Vriend \(2003\)](#) checks if an increase in the difficulty of making decisions by firms leads to a higher prevalence of imitation of successful behavior. For this, they define three treatment groups that have access to the same information, although the complexity in its presentation and the time limit to make decisions is different among groups. They focus on seventeen individual strategies, which are basically divided in three groups, according to their logic: best responding, imitation or reinforcement learning. They conclude that, contrary to what they expected, imitation is not more prevalent when decisions are harder to make.

We now present some facts from these papers, which we think can be understood in the light of our model.

**Fact 1.** In each period, subjects adopt different individual strategies.

This fact is obvious in [Huck \*et al.\* \(1999\)](#), which precisely have as a main goal to analyze which individual strategies are used. They perform a regression, trying to explain how the change in quantity from one period to the other is influenced by the best response to the opponents' quantity, the average quantity produced by the opponents and the quantity chosen by the most successful firm. By doing this, they test the explanatory power of the best response dynamics, imitate the average and

imitate the best, and conclude that all three strategies are important in explaining the results, and their relative success depends on the treatment group analyzed. When confronted with the fact that the coefficients for the imitation terms were significant, they state the following (Huck *et al.* , 1999, p. 12): “Either all subjects are to some extent imitators or some subjects primarily imitate and others follow different leaning rules”. On their summary, they conclude that (Huck *et al.* , 1999, p. 14) “Focusing on myopic best reply dynamics and imitation dynamics we find, however, that both adjustment rules play a role for subjects’ decisions provided that they possess the necessary information to apply these rules”. The same authors, in Huck *et al.* (2002), perform again a regression trying to evaluate the explanatory power of individual strategies in quantity differences in consecutive periods and conclude that (Huck *et al.* , 2002, p. 10) “Given the size of the coefficients it seems that subjects played a mixture of best reply and ‘imitate the average’”. Rassenti *et al.* (2000), in turn, regress the quantity chosen by each subject in each period on his own quantity in the previous period and on his opponents’ total quantity in the previous period and the one before, trying to estimate the merits of each individual strategy. They conclude that (Rassenti *et al.* , 2000, p. 17) “Estimation of individual decision rules reveals a great deal of heterogeneity in subjects’ decision-making”. This idea is reinforced when the authors perform a pooled estimation, gathering all the subjects, and comment on it, stating that (Rassenti *et al.* , 2000, p. 16) “The pooled estimation results mask a large amount of heterogeneity in individual decision rules”.

We present a reason for firms to endogenously choose different individual strategies. In our mixed equilibrium, some firms best respond to their opponents, while others stick to their default quantity, and this behavior heterogeneity is justified by the existence of a thinking cost, even though this cost is the same for all firms. Best responding always guarantees a higher profit than just choosing the default quantity, but it comes at a cost, the thinking cost. The cited papers focus on reactions (or absence of) to quantities from the previous period, because that is what subjects observe, while, in our model, firms best respond (or not) to contemporary quantities. However, the fact in analysis, strategy heterogeneity in each period, is

obtained endogenously in our model: some firms decide to best respond and other stick to their default quantities.

**Fact 2.** One of the most employed strategies by firms in each period is choosing their previous period quantity.

One of the individual strategies studied by [Bosch-Domènech & Vriend \(2003\)](#) is the stay put rule, which states that a subject selects the same quantity he did in the previous period, and this rule seems to be rather successful. In fact, we can read that ([Bosch-Domènech & Vriend, 2003](#), p. 19) “the most successful rule throughout is the stay-put rule, with the only exception being the ‘easy’ triopoly, where it is closely beaten by one other rule”.

Although our model is static and does not predict firms reacting to occurrences in periods different from the one in which they are deciding, we can think of a very simple dynamic extension, that provides some intuition for this inter temporal rigidity. Suppose time is indexed by  $t \in \mathbb{N}$  and the default and chosen quantities of player  $i$  in period  $t$  are, respectively,  $q_i^d(t)$  and  $q_i^*(t)$ . In the first period, the default quantity is the same for both firms:  $\forall i \in N, q_i^d(1) = q$ . In subsequent periods, the default quantity of player  $i$  is the quantity he chose in the last period, that is,  $\forall i \in N, \forall t \geq 2, q_i^d(t) = q_i^*(t-1)$ . As the game is repeated, it is natural that firms find less difficult to think about it. So, let us assume that the thinking cost in period  $t$  is  $F(t) = \delta^{t-1}F$ , where  $F > 0$  is the initial thinking cost and  $\delta \in ]0, 1[$ . Also, bounded rational firms as the ones we model are myopic, and they do not take into account the effect of their decisions for future periods. In this setting, the following result holds:

**Proposition 5.** If the thinking cost decreases at rate  $\delta$ , and firms are myopic, have the same default quantity in the first period, and update the default quantities according to choices they make, repetition of quantities in consecutive periods is assured and may be large and eventually firms learn to play the classic equilibrium and stick to it, expect if  $\delta = \frac{1}{4}$ .

*Proof.* See [Appendix](#). □

The exception in [Proposition 5](#) exists because of the possibility that firms always (with the possible exception of the first period) play a mixed equilibrium. However, for that to happen, the thinking cost may not decrease too quickly, because that implies that firm eventually learn the classic equilibrium, nor too slowly, because similar thinking costs in consecutive periods lead firms to abandon thinking after a period of best responding, because the adjustment they make in default quantities between periods puts them in a comfortable default situation. Moreover, if firms are to always play a mixed equilibrium, they alternately best respond to each other, which brings their default quantities increasingly closer. Hence, for this sequence of equilibria to be sustained, the thinking cost has to be in increasingly narrower intervals, which is only true if  $F$  decreases at the same rate as these intervals' limits, which is found to be  $\frac{1}{4}$ .

Our perspective that firms either best respond or stick to their default can perhaps help to understand why [Bosch-Domènech & Vriend \(2003\)](#), contrary to what had expected, do not find that imitation is more prevalent when decisions are harder to make. They suggest that subjects may use, when facing a difficult decision, an imitation strategy. Our perspective, on the other hand, is that, if thinking is too hard, subjects simply abandon it, not even paying attention to their rivals' choices. A similar experiment to the one in [Bosch-Domènech & Vriend \(2003\)](#), but focused on the prevalence of absence of reaction, instead of imitation, when the decisional environment is complex, seems a promising research avenue.

**Fact 3.** When decisions are harder to make, quantities are more dispersed.

In ([Bosch-Domènech & Vriend, 2003](#), p. 13), the following fact is presented: “As the learning-about-the-environment task becomes more complex, output choices become more spread out”.

The impact of the thinking cost in the equilibrium played in our model can, in some way, explain this fact. We can associate the complexity of the learning-about-the-environment task with our thinking cost. And, as  $F$  increases from 0, the model equilibrium goes from thinking to mixed to default. In a symmetric model, as the one we construct and the one the authors experiment, the thinking

equilibrium is symmetric, having both firms choosing the same quantity. In the mixed equilibrium, one firm best responds and the others sticks to its default, and, in the default equilibrium, both firms choose the default quantity. However, if we introduced more firms in the market and allowed for different default quantities, the equilibrium logic would be the same. If  $F$  is small enough so that all firms want to think, a thinking equilibrium occurs and all firms choose the classic equilibrium. If  $F$  is at an intermediate level, some firms stick to their defaults and the remaining best respond to them. In this case, all the best responders choose the same quantity. Finally, if  $F$  is high enough, no firm wants to think and they all produce their default quantities. Thus, even though the default equilibrium, played at high levels of  $F$ , seems to contradict the fact that quantity dispersion is increasing in the thinking cost, with different default quantities this is generally obtained.

## 2.5 Conclusion

In the line of the bounded rationality models in which the difficulty in decision making is accounted for, we propose a very simple way to model costly thinking. Its application to a consumer choice problem shows immediately the impact the idea can have in traditional Economics. Within the choice set, defined by the budget constraint, the chosen option may not be the one that maximizes utility, provided the thinking cost is high enough. This makes the way to the notion of isofin curves, sets of default choices which guarantee the same utility net of thinking cost. If an agent has a good enough default choice, in the sense that the utility level it implies is close to the optimal utility, he may decide that it is not worthwhile to solve the utility maximization problem and simply select his default choice. On the contrary, all default choices which guarantee an utility level below a certain threshold induce agents to make the effort to think and find the optimal action, utility-wise.

Although costly thinking affects individual decisions, it also has an impact on models of strategic interaction, which gather and confront the decisions of each agent. We put the idea to test in a symmetric Cournot duopoly setting, and analyze

the impact of costly thinking in equilibrium strategies. We find three types of equilibria: thinking, mixed and default where two, one and zero firms best respond, respectively. The mixed equilibria, which occur when the thinking cost is neither too high nor too low, is the more interesting one, as it precludes asymmetric strategies in a symmetric model. The existence of this equilibria is connected to the notion of strategic substitutes. The facts that best responses are negatively sloped imply that a thinking firm has more to lose in giving up thinking than a non-thinking firm would gain if it became a thinker, because the distance between the default quantity and the best response to the opponent's equilibrium choice is larger for the former firm.

Although multiple equilibria are sometimes possible, an increase in the thinking cost reduces the number of firms thinking in equilibrium, as would be expected. The impact of this evolution in the model variables depend on the size of the default quantity. More specifically, if it is smaller or larger than the Nash quantity. A small default quantity means non-thinking firms choose to produce a small quantity. Hence, an increase in the thinking cost reduces market quantity and social welfare. However, if the default quantity is large, we get the reverse effect and social welfare is maximized when a default equilibrium is reached, even if thinking costs are accounted. This means that, if intuition points toward large quantities, an environment in which firms' decisions are hard to make is socially optimal.

The thinking cost concept seems fit to explain some regularities present in the Cournot experimental literature. Within the papers we analyze, the most common result is the adoption of different strategies by subjects, even when markets are totally symmetric. Our model, in which the decision to think is endogenous, shows that it is possible that only some players are responsive to their opponents' choices. The rigid inter temporal behavior registered in one of the papers analyzed is obtained by a dynamic extension of our model, which predicts that, unless in very specific situations, quantities are repeated in some periods and eventually stabilize in the classic Cournot values. Finally, the impact that the thinking cost has in the type of equilibrium played can help explain the fact that quantities become more dispersed



when decisions are harder to make.

Some extensions to the present model can be considered. The dynamic version we presented in [Section 2.4](#) is still at an early stage and it would be interesting to further develop it. The most obvious is the generalization of the number of firms in the market. The symmetry of default quantities and the thinking cost can be relaxed, at the cost of parsimony, but with the benefit of a higher flexibility and new insights resulting from the interaction between people with different abilities to think (different thinking costs) and intuition qualities (different default choices). It is also possible to make the thinking cost more sophisticated and add to it a variable component, positively depending on the number of alternatives considered. More specifically, the agent, besides deciding whether to think or not, would have to define the portion of the choice set to be analyzed, and then compare the options closer to his default choice. The variable thinking cost would be increasing in the size of the constrained choice set generated. The agent's global problem would become more complicated, but the possible quality of his choices would cease to be binary and become (possibly) continuous. Another possibility would be to move the responsibility of choosing quantities from the firm as an institution, to its manager, and assume that he has the incentive to maximize profits by receiving a fixed share of them. A high fixed share would make the manager highly concerned with finding the optimal quantity, counterweighting the thinking cost, which pushes his quantity decision towards the default.

The costly thinking concept we present here seems general enough to have the possibility of being applied to different settings. In Industrial Economics, the Bertrand, Stackelberg and Monopoly models should also change with the introduction of costly thinking and it would be interesting to compare these changes with the ones we get in the Cournot model. We already tried a very simple application to consumer theory, but further, more general, analysis is possible. Agency theory and games like the Prisoner's Dilemma or the Beauty Contest could also benefit from the introduction of costly thinkers.

# Appendix

## Proof of Proposition 1

To facilitate reading, let us define  $r = r(q_j)$ . Using (2.4), we get that  $\pi_i(r, q_j) - \pi_i(q_i, q_j) = b(r - q_i)(\phi - q_j - r - q_i)$ . As (2.5) implies that  $\phi - q_j = 2r$ , we get that  $\pi_i(r, q_j) - \pi_i(q_i, q_j) = b(r - q_i)(2r - r - q_i) = b(r - q_i)^2$ .

## Proof of Proposition 2

Taking the values of  $\pi^c$  and  $v_i^*$  from, respectively, Table 2.1 and Table 2.4, and rearranging, we know that  $v_i^* > \pi^c$  if and only if:

$$F < \frac{1}{4}b(q^c - q^d)(5q^c - q^d) \quad (2.10)$$

On the other hand, for an  $i$ -mixed equilibrium to be possible, (2.9) needs to be verified, which implies:

$$F \geq \frac{9}{16}b(q^d - q^c)^2 \quad (2.11)$$

For (2.10) to be true when (2.11) is, we need the right-hand-side of (2.10) to be greater than the right-hand-side of (2.11). And this happens if and only if  $q^c < q^d$ .

## Proof of Proposition 3

Let us call total quantity in equilibrium  $e$ , with  $e \in \{T, M, D\}$ ,  $Q^e$ . By definition of each type of equilibrium, we know that  $Q^T = Q^d$ ,  $Q^M = q^d + r(q^d)$  and  $Q^D = Q^d$ . From Table 2.1 and (2.5), we know that  $q^c = \frac{1}{3}\phi$  and  $r(q) = \frac{1}{2}(\phi - q)$ . Hence, we get that  $Q^D - Q^M = 3(Q^M - Q^T) = \frac{3}{2}(q^d - q^c)$ . This means that  $Q^M - Q^T$  and  $Q^D - Q^M$  have the same sign as  $q^d - q^c$ . If  $q^d$  is small, this sign is negative and total quantity decreases with  $F$ . If  $q^d$  is large, the opposite happens.

## Proof of Proposition 4

Let us call social welfare in equilibrium  $e \in \{T, M, D\}$ ,  $W^e$ . Table 2.2, Table 2.3 and Table 2.4 show that  $W^T = \frac{4}{9}b\phi^2$ ,  $W^M = \frac{1}{8}b(3\phi - q^d)(\phi + q^d)$  and  $W^D = bQ^d(\phi - q^d)$ . Manipulating these expressions, we get that  $W^M - W^T = \frac{1}{8}b(\frac{5}{3}\phi - q^d)(q^d - q^c)$  and  $W^D - W^M = \frac{15}{8}b(\frac{3}{5}\phi - q^d)(q^d - q^c)$ . As  $q^d \leq \frac{1}{2}\phi$ , the sign of both  $W^M - W^T$  and  $W^D - W^M$  is the same sign of  $q^d - q^c$ . Hence, the highest social welfare occurs in the thinking equilibrium, when  $q^c$  is small, and in the default equilibrium, when  $q^c$  is large.

Net social welfare is negatively influenced by  $F$  in the thinking and mixed equilibria, which means that the highest value of this variable in these equilibria is attained when  $F$  is the lowest possible value that makes each equilibrium possible. This value is 0 in the thinking equilibrium. However, it can either be  $\underline{F}^M$  or  $\overline{F}^T$  in the mixed equilibrium, depending on which value of  $F$  the transition between thinking and mixed equilibria occurs. Let us call the highest net social welfare in equilibrium  $e \in \{T, D\}$ ,  $Z^e$ . The highest net social welfare in equilibrium  $M$  is  $Z_x^M$ , with  $x \in \{L, H\}$ , where  $L$  and  $H$  represent the situations in which the transition is made at  $\underline{F}^M$  and  $\overline{F}^T$ , respectively.

Again observing Table 2.2, Table 2.3 and Table 2.4, we get  $Z^T = \frac{4}{9}b\phi^2$ ,  $Z_L^M = \frac{1}{8}b\left(\frac{5}{2}\phi^2 + 5q^d\phi - \frac{11}{2}q^{d2}\right)$ ,  $Z_H^M = \frac{1}{4}b\left(\frac{19}{18}\phi^2 + \frac{11}{3}q^d\phi - \frac{9}{2}q^{d2}\right)$  and  $Z^D = W^D = bQ^d(\phi - q^d)$ .

Focusing on situation  $L$ , we get  $Z_L^M - Z^T = \frac{11}{16}b\left(\frac{19}{33}\phi - q^d\right)(q^d - q^c)$  and  $Z^D - Z_L^M = \frac{21}{16}b\left(\frac{5}{7}\phi - q^d\right)(q^d - q^c)$ . Hence, both  $Z_L^M - Z^T$  and  $Z^D - Z_L^M$  have the same sign as  $q^d - q^c$ . In this case, net social welfare is maximized when  $F = 0$  and  $F \geq \underline{F}^D$ , if  $q^d$  is small and large, respectively.

Focusing on situation  $H$ , we get  $Z_H^M - Z^T = \frac{9}{8}b\left(\frac{13}{27}\phi - q^d\right)(q^d - q^c)$  and  $Z^D - Z_H^M = \frac{7}{8}b\left(\frac{19}{21}\phi - q^d\right)(q^d - q^c)$ . If  $q^d$  is small, both  $Z_H^M - Z^T$  and  $Z^D - Z_H^M$  are negative, which means net social welfare is maximized when  $F = 0$ . If  $q^d$  is high and not larger than  $\frac{13}{27}\phi$ , both  $Z_H^M - Z^T$  and  $Z^D - Z_H^M$  are positive, which means net social welfare is maximized when  $F \geq \underline{F}^D$ . If  $q^d > \frac{13}{27}\phi$ ,  $Z^D - Z_H^M$  is positive, but  $Z_H^M - Z^T$  is negative. However, the fact that  $Z^D - Z^T = 2b\left(\frac{2}{3}\phi - q^d\right)(q^d - q^c) > 0$  means that net social welfare is also maximized when  $F \geq \underline{F}^D$ .

### Proof of Proposition 5

Let us first state some useful results. Given Proposition 1,  $\pi(r(q_j), q_j) - \pi(q_i, q_j) = b(r(q_j) - q_i)^2$ . Hence,  $F(t) \leq \pi(r(q_j), q_j) - \pi(q_i, q_j)$  if and only if  $\sqrt{\frac{F(t)}{b}} \leq |r(q_j) - q_i|$ . We define

$$\tilde{F}(t) = \sqrt{\frac{F(t)}{b}} \quad (2.12)$$

As  $F(t)$  is decreasing and converges to 0, the same is true for  $\tilde{F}(t)$ . Let us define  $r_l(q)$  as the quantity that results from  $l \in \mathbb{N}_0$  applications of the response function to  $q$ . This means that  $\forall l \in \mathbb{N}, r_l(q) = q^c - \frac{1}{2}r_{l-1}$ , a non-homogeneous recursive relation. Solving it, we get that:

$$r_l(q) = q^c - \left(-\frac{1}{2}\right)^l (q^c - q) \quad (2.13)$$

From (2.13), we can get the three following expressions, which are useful below:

$$|r_1(q) - r_0(q)| = \frac{3}{2}\Delta \quad (2.14)$$

$$|r_l(q) - q^c| = \left(\frac{1}{2}\right)^l \Delta \quad (2.15)$$

$$|r_{l+2}(q) - r_l(q)| = \left(\frac{1}{2}\right)^l \frac{3}{4}\Delta \quad (2.16)$$

Let  $* (t) \in \{T, M1, M2, D\}$  denote the equilibrium played in period  $t$ , where  $Mi$ , with  $i \in N$  represents an  $i$ -mixed equilibrium. If, at any period,  $T$  is played, then it is followed by  $D$  equilibria in all the subsequent periods, with both firms producing the classic equilibrium quantity. This is true because, after a  $T$  equilibrium, both default quantities are set to  $q^c$  and are never altered again, because firms, from then on, always produce  $q^c$  without thinking. If, at any period,  $D$  is played, the default quantities remain the same in the next period, with the possible exception of the thinking cost.

Using (2.7), (2.8), (2.9) and (2.12), we know that, in period 1,  $T$  is possible if  $\tilde{F} < \Delta$ ,  $D$  is possible if  $\tilde{F} \geq \frac{3}{2}\Delta$  and  $M1$  and  $M2$  are possible if  $\frac{3}{4}\Delta \leq \tilde{F} < \frac{3}{2}\Delta$ . If

$T$  is played in this period, the quantity each player produces from then on is  $q^c$ . If it is  $D$  which is played, the default quantities in period 2 do not change, hence the conditions that make each equilibrium possible are the same.

If an  $Mi$  equilibrium is played in period 1, it must be true that  $\frac{3}{4}\Delta \leq \tilde{F} < \frac{3}{2}\Delta$ , and  $q_i^d(2) = r(q)$  and  $q_j^d(2) = q$ . In this period, a thinking equilibrium is possible if the firm whose default quantity is closer to  $q^c$  wants to think. And this firm is firm  $i$ , because, as (2.13) shows, consecutive applications of the response function to  $q$  make it increasingly closer to  $q^c$ . Thus, the thinking equilibrium in this period is only possible if  $\tilde{F}(2) < |r(\cdot) - q^c|$ . From (2.15), this is equivalent to  $\tilde{F}(2) < \frac{1}{2}\Delta$ . A default equilibrium is possible in this period only if the firm whose default quantity is further away from its best response to its opponent default quantity does not want to think. As  $q_i^d(2) = r(q_i^d(2))$  and  $q_j^d(2) \neq r^2(q) = r(q_i^d(2))$ , this firm is firm  $j$ . Hence, this equilibrium is only possible if  $\tilde{F}(2) \geq |r^2(q) - q|$ , which becomes, using (2.16),  $\tilde{F}(2) \geq \frac{3}{4}\Delta$ . Equilibrium  $Mi$  is impossible in period 1 as  $q_i^d(2) = r(q_i^d(2))$ , which means that, regardless of  $\tilde{F}(2)$ , firm 1 does not want to think when firm 2 is producing its default quantity. Equilibrium  $Mj$  is possible if, when firm  $i$  produces  $q_i^d(2) = r(q)$  and firm  $j$  best responds to it, producing  $r^2(q)$ , firm  $j$  wants to think and firm  $i$  does not. This is true if  $|r^3(q) - r(q)| \leq \tilde{F} < |r^2(q) - q|$ . Using (2.16), this is equivalent to  $\frac{3}{8}\Delta \leq \tilde{F} < \frac{3}{4}\Delta$ .

Thus, following equilibrium  $Mi$  in period 1,  $T$ ,  $D$  and  $Mj$  are possible equilibria in period 2. If, in this period,  $T$  is played, quantities remain the same in all subsequent periods, and, if  $D$  is played, there are no changes in the quantities produced, as both firms stick to their defaults, which correspond to their previous period choice. Besides, if  $D$  is played in period 2 and the thinking cost does not decrease too much in period 3, the same equilibrium is played again.

In general, before a thinking equilibrium is played, quantities only change when a mixed equilibrium is played. And, after a  $Mi$  equilibrium, the following mixed equilibrium has to be of type  $Mj$ , because a  $Mi$  equilibrium produces such default quantities that make it impossible to be played again, until the default quantities change. And that only happens with a  $T$  or a  $Mj$  equilibrium. Hence, the pos-

sible history of equilibria include alternate mixed equilibria, possibly separated by any number of default equilibria and, eventually, a thinking equilibrium, which determines the rest of the game. The only situation in which quantities are never repeated is when, in each period, a mixed equilibrium is played, with a possible default equilibrium in period 1. However, this only happens if  $F$  decreases at a very specific rate.

To see this, focus on the case in which mixed equilibria are played every period. In period 1, a  $Mi$  equilibrium is played. Relative to period  $t \in \{1, \dots\}$ , let us define the following two time-dependent functions:

$$\alpha(t) = \begin{cases} i & , t \text{ odd} \\ j & , t \text{ even} \end{cases}$$

$$\beta(t) = \begin{cases} j & , t \text{ odd} \\ i & , t \text{ even} \end{cases}$$

In period  $t$ , we have a  $M\alpha(t)$  equilibrium. Also, by induction, we get  $q_{\alpha(t)}^d = r_t(q)$  and  $q_{\beta(t)}^d = r_{t-1}(q)$ . Hence, for the mixed equilibrium to be possible in period  $t$ , we need that  $|r_{t+2}(q) - r_t| < \tilde{F}(t) < |r_{t+1}(q) - r_{t-1}|$ . Using (2.12) and (2.16), we conclude that, for this sequence of equilibria to be possible, the following must hold:

$$\forall t \geq 1, \left(\frac{1}{4}\right)^{t-1} \frac{9}{16} b\Delta^2 \leq F(t) < \left(\frac{1}{4}\right)^{t-1} \frac{9}{4} b\Delta^2 \quad (2.17)$$

Applying a similar logic to a sequence of equilibria starting with a default equilibrium and alternating mixed equilibria from period 2 on, with the first being a  $Mi$  equilibrium, we conclude that, for it to be possible, the following must hold:

$$F \geq \frac{9}{4} b\Delta^2 \wedge \forall t \geq 2, \left(\frac{1}{4}\right)^{t-2} \frac{9}{16} b\Delta^2 \leq F(t) < \left(\frac{1}{4}\right)^{t-2} \frac{9}{4} b\Delta^2 \quad (2.18)$$

Since  $F(t) = \delta^{t-1} F$ , (2.17) and (2.18) are only true if  $\delta = \frac{1}{4}$ , because that is the rate at which their lower and upper bounds decrease. And they are never true simultaneously, as (2.17) implies  $\frac{9}{16} b\Delta^2 \leq F < \frac{9}{4} b\Delta^2$  and (2.18) requires  $F \geq \frac{9}{4} b\Delta^2$ .



# Chapter 3

## A Self-Delusive Theory of Voter Turnout<sup>0</sup>

### 3.1 Introduction

One of the long-lasting puzzles of rational choice theory is the so-called *turnout paradox*. Given that the probability that one's vote changes the election outcome in large electorates is negligible, even a very minority cost of voting should induce people to abstain.<sup>1</sup> However, in modern democracies, people vote, and do so in large numbers.

One of the reasons that may explain high participation rates is the overestimation of pivotal probabilities. In his seminal book about turnout, [Blais \(2000, p. 65\)](#) writes “*some respondents clearly overestimate  $P$  [the pivotal probability]. (...) It could be*

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<sup>0</sup>This chapter was written in co-authorship with Susana Peralta.

<sup>1</sup>An extensive literature has evolved from the seminal insight by [Downs \(1957\)](#). Later papers propose a game theoretical approach ([Palfrey & Rosenthal, 1983, 1985](#)) to endogenize the pivotal probability of the initial decision theoretical models ([Riker & Ordeshook, 1968](#)). This literature concludes in general that turnout in large elections should be fairly low. Another strand of the literature, starting with [Feddersen & Pesendorfer \(1996, 1999\)](#), assumes away voting costs; the question is then why do people abstain, and the authors provide a theory of rational abstention based on the information of the voters. For surveys, please refer to [Blais \*et al.\* \(2000\)](#), [Dhillon & Peralta \(2002\)](#), [Feddersen \(2004\)](#), [Dowding \(2005\)](#) and [Geys \(2006\)](#).



*that some people vote because they overestimate P.*” Similarly, [Duffy & Tavits \(2008\)](#) claim that *“the overestimation of pivotality may, thus, provide a solution to the paradox of voter turnout – voting happens because people systematically think that their vote counts more than it actually does (...)”* This suggestion is confirmed by experimental evidence: both [Duffy & Tavits \(2008\)](#) and [Blais et al. \(2000\)](#) show that higher estimates of individual decisiveness increases the probability of turnout.

Moreover, there is considerable experimental and survey evidence about the overestimation of the probability of decisiveness in voting contexts ([Blais et al. , 2000](#), [Acevedo & Krueger, 2004](#) and [Duffy & Tavits, 2008](#)). This finding is common to survey and experimental data and robust to different experimental designs. [Blais et al. \(2000\)](#) ask people to report how many of the remaining subjects vote for each party, while [Duffy & Tavits \(2008\)](#) ask subjects to report pivotal probabilities directly. Also, in the former experiment the subjects are informed that they are participating in an election, while in the latter the participation payoffs are induced in a neutral fashion, where the subject decision is whether or not to buy a token (representing “voting”). Finally, it is noteworthy that there is considerable heterogeneity across individuals in the reported pivotal probabilities. [Table 3.1](#) shows the relative frequency of each pivotal probability in a survey conducted before the 1993 federal elections in Canada, reported in [Blais \(2000\)](#) (Canada had a population of about 28.682.000 in 1993).

The literature has little to suggest as regards the way in which individuals compute pivotal probabilities and in particular the reason for their overestimation. In the same survey reported above, [Blais \(2000, p. 74\)](#) has asked the respondents their agreement with a number of statements, including *“My own vote may not count for much, but if all people who think like me vote, it could make a majority difference.”* As much as 94% of the respondents agreed with this sentence. A dummy variable that turns one when the respondent agrees with the sentence comes out significant in a logistic regression of the decision to vote in the Canadian federal election. This leads [Blais \(2000, p. 74\)](#) to conclude for *“the possibility that some people vote because they believe that somehow their own vote counts.”*

Table 3.1: Self-reported pivotal probabilities in survey data (Blais, 2000)

| Probability       | Absolute Frequency | Relative Frequency |
|-------------------|--------------------|--------------------|
| 1/10              | 19                 | 2,25%              |
| 1/100             | 8                  | 0,95%              |
| 1/1.000           | 17                 | 2,02%              |
| 1/10.000          | 16                 | 1,90%              |
| 1/100.000         | 34                 | 4,03%              |
| 1/1.000.000       | 87                 | 10,32%             |
| 1/10.000.000      | 190                | 22,54%             |
| 1/100.000.000     | 261                | 30,96%             |
| Doesn't know      | 201                | 23,84%             |
| Refuses to answer | 10                 | 1,19%              |
| Total             | 843                | 100%               |

As put by the author, this perception about the behavior of like minded others may reflect self-delusion. Acevedo & Krueger (2004) report an experiment similar to Quattrone & Tversky (1984), where subjects are exposed to an election environment and each must report the likelihood of his party outvoting the opponent in case he votes or abstains, respectively. They report that 46% of the subjects “*have a tendency to project their own decisions to similar others.*”. This *projection* measure, when correlated with voting intentions, yields the so-called *voter’s illusion*.<sup>2</sup> This setup has the advantage of modeling the specific bias suggested in the survey data reported by Blais *et al.* (2000).

In this paper, we propose a model that features this sort of reasoning on behalf

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<sup>2</sup>The political science literature has come up with other hypothesis based on some sort of misperception on behalf of the voters about their actual influence in an election. It is worth to mention quasi-magical thinking (Masel, 2007) and belief in personal relevance (Acevedo & Krueger, 2004). Quasi-magical thinking corresponds to acting as if one believes that his action influence the outcome, even though he does not hold that belief, while belief in personal relevance is the classification of one’s action as important to the final outcome, whether it is a good diagnostic of others’ actions or not.

of potential voters. More specifically, each individual is endowed with an exogenous belief that a given proportion of like-minded others (same party supporters) choose the same action as he does. A closely related paper is [Grafstein \(1991\)](#), who also uses a theoretical model where voters overestimate their pivotalness. We depart from [Grafstein \(1991\)](#) in several ways. Firstly, we allow for heterogeneity across individuals in subjective pivotalness, as suggested by [Table 3.1](#). Secondly, while [Grafstein \(1991\)](#) assumes agents project their decisions not only on the supporters of their preferred party, but also on others, we shall assume away the latter. Finally, we assume a continuum of voters, while in [Grafstein \(1991\)](#) the set of voters is discrete, although one of his results is that positive turnout prevails with an infinite population.<sup>3</sup>

Similarly to [Grafstein \(1991\)](#), we show that the voting game has an equilibrium with positive turnout. However, we provide a more thorough characterization of the set of possibly multiple equilibria. This has the interesting empirical prediction that the same fundamentals may lead to different turnout levels and winners of the election, providing a theoretical underpinning for the importance of abstention in election outcomes, which has been documented empirically ([Bernhagen & Marsh, 1997](#)).

Our model of self-delusion is inspired by [Bravo-Furtado & Côte-Real \(2009a\)](#), who apply it to a majority election among three parties, where non-Duvergerian outcomes arise in equilibrium since voters need not vote strategically. Individuals with such beliefs are said to be rhizomatic ([Bravo-Furtado & Côte-Real, 2009a](#)).<sup>4</sup>

The literature has long acknowledged the limitations of the rational voter paradox and offers several explanations of voter turnout which depart from the rationality

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<sup>3</sup>[Grafstein \(1991\)](#) calls this type of reasoning “*evidential*” as opposed to the causal rationality usually assumed in economics. Causal rationality imposes that, when deciding, one ignores the potential link between his and other’s decisions, even if one believes in its existence. Evidential reasoning, in turn, assumes that people do not ignore this link when they believe it exists.

<sup>4</sup>[Conitzer \(2012\)](#) applies a similar idea to a setup in which of the alternatives is the best one but some voters may make mistakes and vote for the opponent. The probabilistic voting behavior of individuals incorporates the idea that they “*are likely to vote similarly to their neighbors*”.

hypothesis, besides the self-delusion based ones as the above mentioned Grafstein (1991). Sieg & Schulz (1995) and Conley *et al.* (2001) use evolutionary game theory tools to explain the emergence of positive turnout. In Conley *et al.* (2001), voters use mixed strategies and increase their voting probability if this increases their expected utility. More recently, Coate & Conlin (2004) and Feddersen & Sandroni (2006) put forward group-based theories of voter turnout, where the individual decisions depend on the collective payoff. Li & Majumdar (2010), in turn, suppose that voters experience regret when they fail to vote, more so if their preferred candidate loses the election. A recent paper by Aaron *et al.* (Edlin *et al.* , 2007) shows that voters with *altruistic* preferences may have considerable turnout rates. This approach is complementary to ours, in the sense that the amplifying effect of the group is embedded in the expected benefit part of the calculus of voting, rather than the pivotal probability.

In this paper, we model a two-party, first-past-the-post election with a continuum of voters. Parties have asymmetric sizes. The continuum assumption ensures that each individual has zero mass and therefore would not vote, if he were rational in the classical sense, and hence any positive turnout must stem from self-delusion. We model the idea of “people who think like me behave like me” as follows. A self-delusional voter believes that an idiosyncratic exogenous fraction of his party supporters behave like him. The self-delusion parameter is an exogenous type, a *belief*, and is drawn from a given continuous distribution function on the  $[0, 1]$  support. Our main results are as follows. Firstly, there always exist at least one equilibrium with positive turnout. Secondly, for a range of minority party sizes, the game displays multiple equilibria, at least one with the majority party, and at least two with the minority party winning the election. The minority party may win the election, provided party sizes are not too different. We study the impact of *ex-ante* and *ex-post* closeness of the election, as measured by the party sizes and (endogenous) margin of victory, respectively, on turnout. We show that turnout increases with closeness except in some equilibria with a minority-win, including the lowest-turnout one. We interpret this result as a natural consequence of the fact that in this lowest-turnout equilibrium the set of voters displays very high degrees of self-delusion. Our

analysis of closeness provides the first theoretical rationale for the utilization of the margin of victory in empirical studies, which is not grounded on the usual rational expectations hypothesis put forward in the empirical literature. It also explains why some empirical studies obtain non-conclusive results about the impact of closeness, sometimes with a negative sign, which is contrary to the usual intuition. Finally, we provide a rationale for the intentional inducement of perceived social and personal links amongst party supporters performed by political activists and political parties via, e.g., social networks. These may actually increase the degree of self-delusion amongst the party supporters. We show that, given a minority size and a distribution of self-delusion under which the only equilibria entail the a majority-win, changing the distribution of self-delusion of the minority supporters to make them more self-delusion than the majority ones (in the first-order stochastic dominance sense), creates room for the existence of minority-win equilibria. Our model also sheds light on the stylized fact that turnout has an impact on the electoral outcome. Indeed, in the space of parameters for which multiple equilibria exist, any of the two parties may win the election, with different turnout levels arising in each type of equilibrium.

The remainder of the paper is organized as follows. [Section 3.2](#) presents the model and a few useful preliminary results. We show how the minority size determines the possibility of multiple equilibria with both parties winning the election in [Section 3.3](#) and relate turnout, closeness and election results in [Section 3.4](#). We illustrate the advantage for each party of inducing self-delusion among its supporters in [Section 3.5](#) and conclude in [Section 3.6](#). All the proofs are relegated to the [Appendix](#).

## 3.2 The Model

We begin by formalizing the model, and we do it in three steps. In [Section 3.2.1](#), we introduce the model variables and parameters and explain its basic functioning. Then, in [Section 3.2.2](#), we focus on the decision of each individual agent and find

his best reply to the other players' moves. Finally, in [Section 3.2.3](#), we gather all agents' decisions and close the model by defining the equilibrium.

### 3.2.1 Setup

A first-past-the-post election between 2 parties,  $i \in \{A, B\}$  takes place. The most voted party wins the election. In case of a tie, a coin toss decides the winner.

There is a mass of size  $1 + m$ , with  $m \in ]0, 1]$ , of voters.<sup>5</sup> Each agent  $x$  may decide to vote for his preferred party or to abstain and is assumed to strictly prefer one of the parties. The mass of party  $A$  supporters is equal to 1, while that of  $B$  supporters is  $m$ . With a slight abuse of language, we are going to call  $B$  the *minority* party, and  $A$  the *majority* party, even when  $m = 1$ . Moreover,  $a$  and  $b$  represent a general  $A$  and  $B$  supporter, respectively.

Voting entails a cost of  $c > 0$ .<sup>6</sup> Without loss of generality, the payoff that the voter gets when his preferred party wins the election is normalized to 1, and 0 otherwise. Notice that the voter gains  $\frac{1}{2} - c$  when he turns a defeat into a tie or a tie into a win with his vote. In order to ensure that the voter wants to cast his vote in these situations, we follow the usual assumption in the literature ([Dhillon & Peralta, 2002](#)), i.e.  $c < \frac{1}{2}$ .

As explained in the [Introduction](#), we allow the voters to be self-delusive in the following sense. When individual  $x$  decides whether to vote he believes that a fraction  $q_x \in [0, 1]$  of like-minded others – i.e., his party supporters – behave like him.<sup>7</sup> Equivalently,  $q_x$  may represent the probability that  $x$  attaches to the event that each supporter of his own party chooses the same strategy as he does. For party  $i \in \{A, B\}$ , the beliefs  $q_i \in [0, 1]$  are distributed according to the twice continuously

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<sup>5</sup>For a recent paper that studies turnout with a continuum of voters, please refer to [Li & Majumdar \(2010\)](#).

<sup>6</sup>We assume away heterogeneity in voting costs, in order to focus solely on the impact of heterogeneous self-delusion in voter turnout.

<sup>7</sup>We depart from [Grafstein \(1991\)](#) in supposing that individuals do not assume anything as regards the behavior of the opposing party's supporters. For a similar self-delusional assumption applied to strategic voting in a three-candidate election, see [Bravo-Furtado & Côte-Real \(2009b\)](#).

differentiable cumulative function  $F_i$ . For technical reasons, we impose that  $\forall i \in \{A, B\}, F_i(0) = 0$  and  $F'_i(0) > 0$ . Whenever appropriate, we shall illustrate our results with the uniform distribution in both parties. We focus on pure strategy equilibria, and use  $v_x \in \{0, 1\}$ , where 0 represents abstention and 1 voting, to denote agent  $x$ 's decision. These individual decisions induce two party-specific sets, the mass of  $i = A, B$  supporters that vote and that abstain, respectively, which we denote  $V_i$  and  $N_i$ .

### 3.2.2 The decision to vote

We now study the individual decision on whether to vote for one's preferred party or to abstain. Let us look at agent  $a$ . He gets a payoff of 1 if  $A$  wins,  $\frac{1}{2}$  in case of a tie, and 0 if  $B$  wins. Additionally, he incurs a cost of  $c$  if he votes. Suppose that, by voting,  $a$  does not change the election outcome. Then, it is obvious that he is better off by abstaining because this avoids him the voting cost  $c$  and does not change the payoff he gets from the winner of the election. If, however,  $a$  turns a defeat into a tie or win, or a tie into a win, then it pays for him to vote given our assumption on  $c$ . This behavior, namely, that  $a$  votes if and only if he thinks he is pivotal, is standard in instrumental voting games.

To begin with, we state the obvious result that an agent of type  $q_x = 0$  (rational in the usual sense) never votes given our assumption of a continuum of players, for he knows he cannot possibly alter the election outcome. This implies that the positive turnout that may obtain in the model is due solely to the self-delusional beliefs of the voters.

Now take an agent  $a$  with  $q_a > 0$ . Let us emphasize a few important facts regarding  $a$ 's decision. Firstly,  $a$ 's like-minded agents are the party  $A$  supporters, hence he perceives the actual  $V_B$  and  $N_B$ , as his self-delusional belief only concerns his party's supporters. Secondly, as regards the masses of voters and abstainers in party  $A$ ,  $a$  considers that a fraction  $q_a$  of  $A$  supporters choose the same action as he does, independently of what their strategy prescribes. Hence, among the mass of people supporting party  $A$  whose equilibrium strategies prescribe voting, he

believes that a proportion  $q_a$  mimic his behavior, and the same is true for the set of abstainers. As for the remaining agents,  $a$  believes they stick to their equilibrium actions. In algebraic terms, given  $V_A$  and  $N_A$ ,  $a$  believes that a mass of  $(1 - q_a) V_A$  agents definitely vote and a mass of  $(1 - q_a) N_A$  agents definitely abstain. The others do the same as him. If he decides to abstain, the perceived mass of votes for  $A$  is  $(1 - q_a) V_A$ . If, on the contrary, he votes, he believes that the voting mass in  $A$  is  $V_A + q_a N_A$ .

Armed with these preliminary insights, it is straightforward that there are two different situations in which  $a$  believes he is pivotal:

- (i) Party  $A$  loses if he abstains and does not if he votes, i.e.  $(1 - q_a) V_A < V_B \leq V_A + q_a N_A$
- (ii) There is a tie if he abstains and party  $A$  wins if he votes, i.e.  $(1 - q_a) V_A = V_B < V_A + q_a N_A$

Given that  $q_a > 0$ , it is always true that  $(1 - q_a) V_A < V_A + q_a N_A$ , which implies that (i) and (ii) reduce to  $(1 - q_a) V_A \leq V_B \leq V_A + q_a N_A$ . Naturally, analogous conditions exist for party  $B$  supporters. This allows us to establish the following [Lemma](#).

**Lemma 1.** Take a given agent  $x$  who prefers party  $i$  to party  $j$  ( $i, j \in \{A, B\}, i \neq j$ ). His optimal decision, denoted  $v_x^*(q_x, V_A, V_B)$ , is to vote if and only if  $q_x > 0$  and

$$(1 - q_x) V_i \leq V_j \leq V_i + q_x N_i$$

### 3.2.3 Voting equilibrium

A direct implication of [Lemma 1](#) is that the equilibrium is characterized by a pair of cut-off levels,  $(q_A^*, q_B^*)$ , such that  $x$  votes if and only if he has a type above or equal to  $q_i^*$ . To see why, notice that agent  $x$  believes that, if he abstains, the actual voting mass in his preferred party,  $V_i$ , is reduced by  $q_x V_i$ , and, if he votes, it increases by



$q_x N_i$ . These differences are both increasing in  $q_x$ , implying that whenever type  $q_x$  votes, so do higher types, who consider themselves more influential in the election outcome. On the other hand, if an agent with type  $q_x$  has abstaining as an optimal decision, then every supporter of party  $i$  with a lower type also abstains. The following [Lemma](#) formalizes this idea.

**Lemma 2.** An equilibrium of the voting game is characterized by the vector  $(q_A^*, q_B^*) \in [0, 1]^2$  such that,  $\forall i \in \{A, B\}$  and  $\forall x$  supporting party  $i$ ,

$$q_x \geq q_i^* \wedge q_x > 0 \Rightarrow v_x = 1, \quad \text{and} \quad q_x < q_i^* \vee q_x = 0 \Rightarrow v_x = 0$$

Obviously, an equilibrium of the game entails either a win of one of the parties or a tie between the two. This latter situation is simple to characterize and is the object of [Lemma 3](#).

**Lemma 3.** One of the (possibly, multiple) equilibria is a tie if and only if the parties' sizes are the same, i.e.,  $m = 1$ .

It is easy to understand why this is the case. On one hand, if both parties have the same size, then everyone (except for the non self-delusional agents, who have 0 mass) voting has to be an equilibrium because, when this happens, the actual voting masses are the same, which leads all self-delusional agents to believe they will turn their party's defeat into a victory by voting. On the other hand, if a tie is an equilibrium, then the voting masses in both parties are the same, which means the optimal decision for everyone (again, except for the non self-delusional agents) is to vote. If all agents from both parties are voting and there is a tie, then it must be true that both parties have the same size, i.e.,  $m = 1$ .

We now undertake a preliminary analysis of the equilibria of the game in which one of the party wins. We begin by showing that one of the two conditions in [Lemma 1](#) is redundant for each party. Let us suppose that the model is in equilibrium, i.e.,  $V_A = V_A^*$  and  $V_B = V_B^*$  and, without loss of generality, that  $A$  is the winner, that is,  $V_A^* > V_B^*$ . As regards party  $A$  supporters, notice that the condition  $V_B^* \leq V_A^* + q_a N_A^*$  in [Lemma 1](#) is verified for all the agents. Actually, even those for whom  $q_a = 0$

acknowledge that the mass of voters in the opponent party is larger than that of their own party if they vote. For the winning party supporters we are, thus, left with the condition  $(1 - q_a) V_A^* \leq V_B^*$ . The self-delusional cut-off level in this party is defined by:

$$(1 - q_a^*) V_A^* = V_B^* \tag{3.1}$$

What is the rationale for this condition? Take an agent with  $q_a = 1$ : this individual believes that all his fellow partisans act like him, hence there are no votes in  $A$  if he abstains, leading party  $A$  to, at most, tie. Now, take an agent with  $q_a$  sufficiently close to zero: for him,  $A$  is the winner even if he abstains, as he believes that the voting mass in his party is very close to the actual one. The marginal  $A$  supporter is the one who believes that he turns party  $A$ 's win into a tie if he abstains. It is instructive to rewrite (3.1) as  $q_a^* V_A^* = V_A^* - V_B^*$  which shows that the marginal voter is the one who believes he eliminates party  $A$ 's margin of victory if he abstains. All party  $A$  supporters with a higher  $q_a$  believe that their action influences the election outcome and vote to guarantee that  $A$  wins. Conversely, those with a lower  $q_a$  believe  $A$ 's win is assured independently of the action they choose.

Let us now look at the decision of a party  $B$  supporter. Since  $B$  actually loses the election, each  $B$  supporter acknowledges that  $B$  loses should he abstain, that is, the condition  $(1 - q_B^*) V_B^* \leq V_A^*$  in Lemma 1 is trivially verified. For the losing party supporters, we are thus left with the condition  $V_A^* \leq V_B^* + q_b N_B^*$ . The agents who vote in equilibrium believe that  $B$  wins (or ties), should they decide to vote. Again, it is straightforward to define a cut-off type  $q_B^*$  below which individuals abstain:

$$V_B^* + q_B^* N_B^* = V_A^*. \tag{3.2}$$

In this case, the marginal voter believes that  $B$  ties if he votes. Rewriting (3.2) as  $V_A^* - V_B^* = q_B^* N_B^*$  highlights the fact that this voter believes that he catches up with the winning party by recovering the margin of victory with his vote.

Recalling the result in Lemma 2, i.e., that only the individuals with levels of  $q$

higher or equal to the marginal ones, the equilibrium voting masses in  $A$  and in  $B$  are, respectively:

$$V_A^* = 1 - F_A(q_A^*) \tag{3.3}$$

$$V_B^* = (1 - F_B(q_B^*)) m \tag{3.4}$$

An equilibrium occurs when (3.1), (3.2), (3.3) and (3.4) are verified simultaneously. An analogous system of equations holds in an equilibrium in which  $B$  wins the election. The next Section characterizes the set of equilibria thus obtained.

### 3.3 Who wins the election?

In this Section, we discuss the election result. We relate it to the size of the minority party,  $m$ , and show that when it is sufficiently large, it may actually win the election to the majority party.

The next Proposition shows how the set of possible equilibria of the game change by varying the size of the minority.

**Proposition 1.** Suppose that voters are self-delusional. Then,

- (i) If the parties have different sizes, there is at least one equilibrium in which the majority party wins the election;
- (ii) If the parties have different sizes, there are at least two equilibria in which the minority party wins the election, provided the difference between the party sizes is not too large;
- (iii) If the parties have the same size, there is at least one equilibrium with each of the parties winning, and one equilibrium in which there is a tie.

*Proof.* See Appendix. □

Therefore, there exists a minimum size of the minority, denoted  $\underline{m}$ , above which some equilibria of the game entail a minority-win. Interestingly, when  $\underline{m} < m < 1$ , there are at least two equilibria in which the minority wins the election. Conversely, there always exists at least one equilibrium in which the majority wins the election. As we often use the uniform case to illustrate results and provide intuition, we present a version of [Proposition 1](#) specific to the case in which the level of self-delusion is uniformly distributed in both parties in [Corollary 1](#).

**Corollary 1.** Suppose that voters are self-delusional and the level of self-delusion is uniformly distributed in both parties. Then,

- (i) If the parties have different sizes, there is one equilibrium in which the majority party wins the election;
- (ii) If the parties have different sizes, there are two equilibria in which the minority party wins the election, provided its size is larger than 87,9% of the size of the majority party;
- (iii) If the parties have the same size, there is one equilibrium with each of the parties winning, and one equilibrium in which there is a tie.

*Proof.* See [Appendix](#). □

To understand this result, notice that, if  $q$  is uniformly distributed in both parties, i.e.,  $\forall i \in \{A, B\}, F_i(q_i) = q_i$ , the system (3.1) – (3.4) boils down to:

$$\begin{cases} q_A &= 1 - \sqrt{(1 - q_B) m} \\ q_B &= 1 - (1 - (1 - q_B) q_B) m \end{cases} \quad (3.5)$$

What about the case in which the minority  $B$  wins? Then, the analogous system of equations reads:

$$\begin{cases} q_B &= 1 - \frac{1 - q_A (1 - q_A)}{m} \\ q_B &= 1 - \sqrt{\frac{1 - q_A}{m}} \end{cases} \quad (3.6)$$

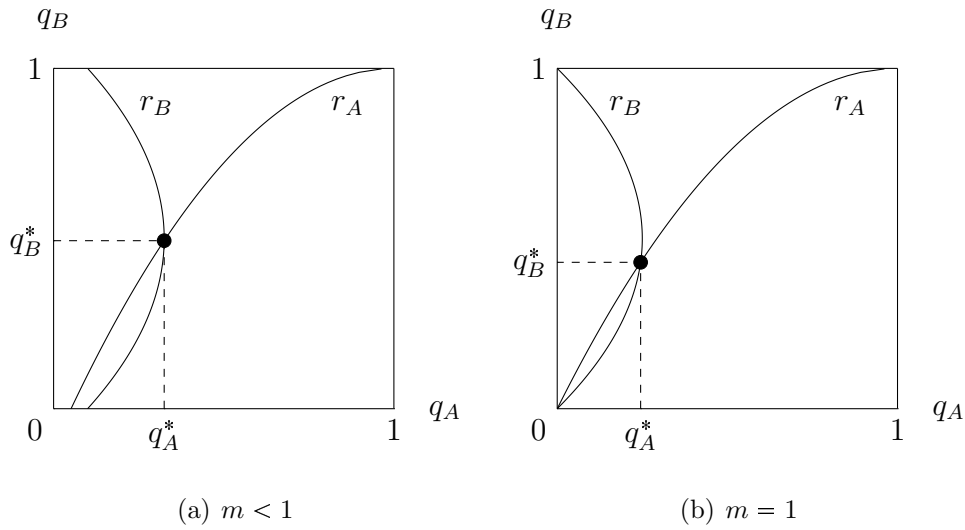


Figure 3.1: Majority-win equilibria with self-delusion distributed uniformly

The solutions to (3.5) and (3.6), the equilibria of the model, are represented in Figure 3.1 and Figure 3.2.

Let us provide an interpretation of these figures. The systems (3.5) and (3.6) define the functions  $r_A$  and  $r_B$ , for the cases in which  $A$  and  $B$  wins the election, respectively. Take the cases in which  $A$ , the majority party, wins, represented in Figure 3.1. The curve  $r_A$  gives, for each possible marginal individual in party  $B$ , the  $q_A$  level of the party  $A$  supporter who believes that there is a tie should he abstain. Conversely,  $r_B$  gives, for each possible marginal individual in party  $A$ , the  $q_B$  level of the party  $B$  supporter who believes that there is a tie should he vote. Obviously, an equilibrium exists when the two intersect. The two curves always intersect once (except for the case in which  $m = 1$ , in which the  $(0, 0)$  intersection represents a tie). In the cases in which  $B$ , the minority party, wins, presented in Figure 3.2, the curves  $r_A$  and  $r_B$  are analogous to the previous ones, and intersect only if  $m \geq \underline{m}$ , once if  $m = \underline{m}$  and twice otherwise (again, when  $m = 1$ , the  $(0, 0)$  intersection represents a tie). Another way to state the result in Proposition 1 is, then, as follows.

**Corollary 2.** Suppose that voters are self-delusional. There is one critical value of  $m$ ,  $\underline{m} < 1$  such that

- (i) When  $m < \underline{m}$ , there is at least one equilibrium in which party  $A$  wins the

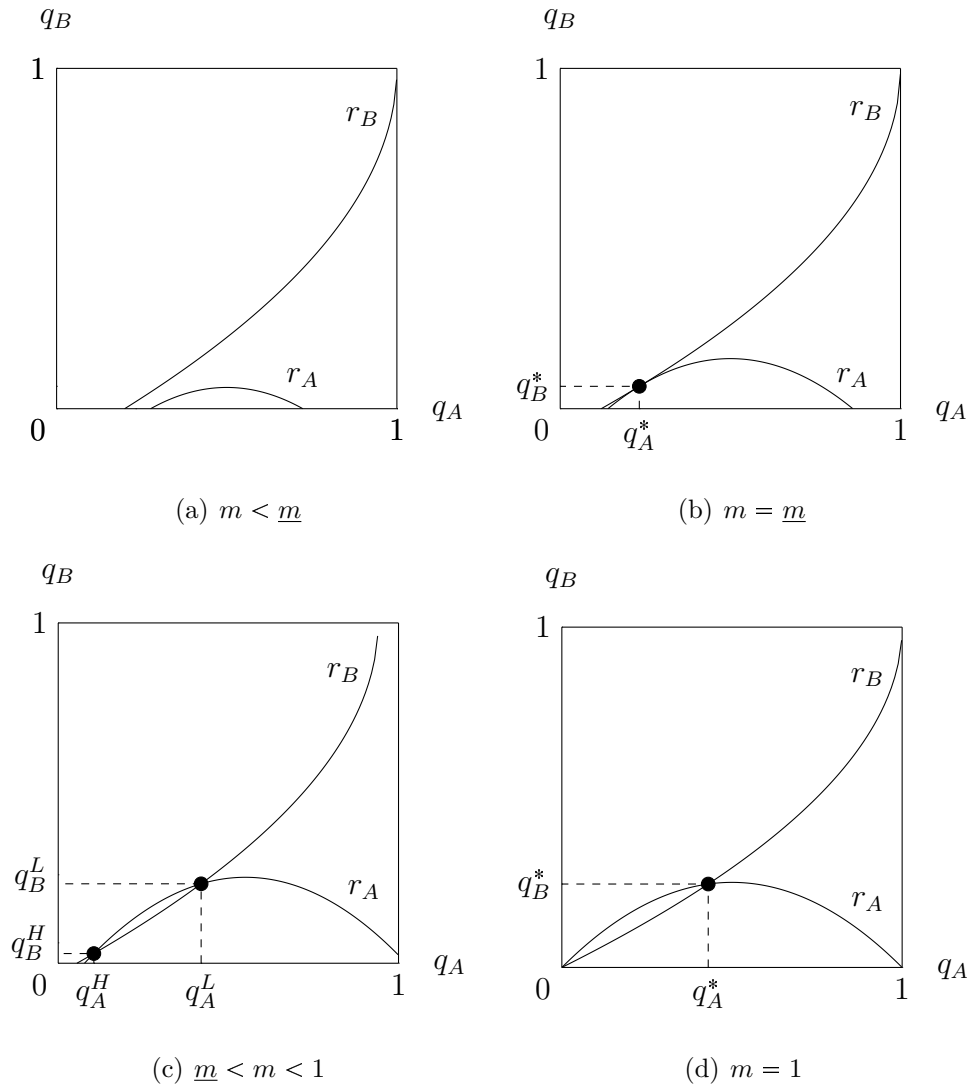


Figure 3.2: Minority-win equilibria with self-delusion distributed uniformly

election;

- (ii) When  $\underline{m} < m < 1$ , there is at least one equilibrium in which party  $A$  wins the election and at least two equilibria in which party  $B$  wins the election;
- (iii) When  $m = 1$ , there are at least three equilibria: a tie, and one in which each of the parties wins the election.

For the uniform case, the corollary goes through, by replacing “at least” with “exactly”. This corollary highlights the fact that this game has multiple equilibria, inducing potentially different election results, provided the party sizes are not too

different. The reason for the multiplicity of equilibria is easily grasped by concentrating on the uniform distribution, and letting  $m = 1$  (Figure 3.1(b) and Figure 3.2(d)). Using (3.5) and (3.6) with  $m = 1$ , one gets as a solution  $q_A^* \approx 0,245$  and  $q_B^* \approx 0,430$ . In such an equilibrium,  $V_A^* \approx 0,755, V_B^* \approx 0,570$  and, conversely,  $N_A^* \approx 0,245, N_B^* \approx 0,430$ . Now, let us check what happens to the marginal individual in party  $A$ : if he decides not to vote, he believes that the voting mass in party  $A$  decreases by  $0,245 \times 0,755$ , i.e., it goes down to  $(1 - 0,245) \times 0,755 = 0,570 = V_B^*$ . In contrast, the marginal individual in party  $B$  believes that if he decides to vote,  $V_B^*$  increases by  $0,430^2$ , i.e., it goes up to  $0,755$ , which is precisely  $V_A^*$ . What about the less self-delusional individuals, that is, the abstainers? Those in party  $A$  believe that  $A$  wins anyway, even without their vote, hence there is no reason to bear the cost of voting; conversely, those in party  $B$  think that  $B$  loses anyway, and the same reasoning applies. As for the more self-delusional individuals, that is, the ones who vote, those of the winning party  $A$  believe that  $A$  loses should they abstain (even if  $A$  actually wins) and wins if they vote, while those of the losing party  $B$  believe that  $B$  loses if they abstain and it wins if they vote (despite the fact that  $B$  actually loses). By symmetry, there is another equilibrium where party  $B$  wins the election, with  $q_A^* \approx 0,430$  and  $q_B^* \approx 0,245$ .

The reasoning of a self-delusional voter actually lies at the heart of the multiplicity result. A voter  $x$  believes that, should he abstain, he subtracts a share  $q_x$  from the actual mass of voters of his party. That is, self-delusion is like a magnifying lens through which the self-delusional individual observes the mass of voters of his party. Now, in the case in which both parties have the same size and the same distribution, the party which wins the election *must* include in its voting mass less self-delusional individuals than the defeated party (otherwise it loses the election). We then have the sort of self-fulfilling prophecy which is common in games with multiple equilibria. The very existence of a larger voting mass ensures that individuals with a low  $q$  still believe themselves to be pivotal. In the defeated party, instead, the low  $q$  types apply their magnifying lens to a small actual voting mass, and this is not enough to make them vote. The voters in this party *must* have a powerful magnifying lens, for otherwise they could not be voting given the relatively small size of the actual

voting mass.

Let us now move to the case where the party asymmetry is not too high, i.e.,  $\underline{m} < m < 1$ , represented by [Figure 3.1\(a\)](#) and [Figure 3.2\(c\)](#). Here, there are three different equilibria: one in which the majority wins the election, and two with the minority winning. Let us index these equilibria by  $e \in \{M, L, H\}$ , where  $M$  is the majority-win,  $L$  is the low-turnout minority win and  $H$  is the high-turnout minority-win one. The same reason that explains the multiplicity of equilibria in the case of symmetric parties applies here for the two minority-win equilibria. Let the marginal self-delusion level in party  $i$  in equilibrium  $e$  be denoted  $q_i^e$ . Then, we have  $q_i^L > q_i^H, \forall i \in \{A, B\}$ .<sup>8</sup> Clearly, the  $L$  equilibrium displays a relatively smaller mass of votes which excludes the less self-delusional ones, while the  $H$  one entails a larger mass of voters which, as such, *must* include the less self-delusional ones. Again, it is the very fact that the mass of voters is large that allows the voters with a less powerful magnifying lens to vote. This same mechanism also explains why the  $r_A$  curve is actually a correspondence, in the case where the minority party  $B$  wins the election. Indeed, for each marginal individual in party  $B$ , which determines a voting mass in this party, there are two possible marginal individuals in party  $A$  who believe that there is a tie, should they vote. These two possible levels of self-delusion are such that (i) the voting mass is relatively large and the marginal individual is not too self-delusional and projects his decision to vote on a small mass of abstainers, or (ii) the voting mass is relatively small and the marginal individual is very self-delusional and projects his decision to vote on a large mass of abstainers. Coming back to the definition of the  $r_A$  locus  $V_A + q_A N_A = V_B$ , it encompasses the actual voting mass  $V_A$  and the projected one (should the marginal individual vote)  $q_A N_A$ . Clearly, in the high-turnout equilibrium it is the former that dominates, while in the low-turnout equilibrium it is the latter. This distinction shows that in the  $L$  equilibrium the set of voters displays high degrees of self-delusion.

Having extensively characterized the equilibria of the model, we can go further and use their properties to explain some known facts about elections. That is the

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<sup>8</sup>Notice that an equilibrium with a higher turnout corresponds to a lower marginal  $q$ .



purpose of the next [Section](#).

## 3.4 Empirical implications of self-delusion

There are some empirical regularities relating turnout and other variables in elections that have become evident over time. We focus on two of them: closeness in the voting share and the winner of the election. First, in [Section 3.4.1](#), we search for a relation between turnout and closeness, using both an *ex-ante* measure (the relative size of the parties) and an *ex-post* (the margin of victory of the winning party) one. We find that turnout can either increase or decrease with closeness and that the self-delusional reasoning of voters plays an important role in the decreasing of turnout, when it occurs. Then, in [Section 3.4.2](#), we compare the turnout level of the minority-win and majority-win equilibria and get the intuitive result that turnout is higher when the minority wins the election.

### 3.4.1 Election closeness

It seems relevant to address the relationship between turnout and closeness of the election. As put by [Geys \(2006\)](#) in his meta-analysis, “*Closeness is by far the most analyzed element in the turnout literature.*” We shall define closeness using two distinct perspectives. One is the *ex-ante* size of the parties, and the other is the *ex-post* realized margin of victory, using the terminology in [Geys \(2006\)](#). The latter is widely used in empirical studies ([Geys, 2006](#)), but has been criticized on the grounds of its endogeneity. [Geys \(2006\)](#) lists a number of *ex-ante* measures of closeness which have been used in the literature, including previous election results, opinion polls, and newspaper reports. For a recent example, refer to [Fauvelle-Aymar & François \(2006\)](#). In our model, the natural *ex-ante* measure to use is the party size, which is given.

Firstly, we look at the impact of the relative party sizes on turnout, an *ex-ante* measure of closeness. To simplify matters, we are going to present the results when

the degree of self-delusion is uniformly distributed within each party, since this allows us to specify the exact number of equilibria. Moreover, in order to avoid the uninteresting case of no equilibrium and the non-generic cases of  $m = 1$  or  $m = \underline{m}$ , we shall concentrate on the interval  $]\underline{m}, 1[$ , in which there are exactly three equilibria,  $M$ ,  $L$  and  $H$ . [Proposition 2](#) summarizes the effects of an approximation of the relative parties' sizes in this case.

**Proposition 2.** Suppose that voters are self-delusional, the level of self-delusion is uniformly distributed in both parties and  $\underline{m} < m < 1$ . If the relative size of the minority,  $m$ , increases, then,

- (i) Turnout increases in both parties in the  $M$  equilibrium;
- (ii) In the two equilibria in which the minority wins the election, turnout levels diverge, that is, turnout decreases in the  $L$  equilibrium and increases in the  $H$  equilibrium, in both parties.

*Proof.* See [Appendix](#). □

Closeness increases turnout in the  $M$  and in the  $H$  equilibrium. Indeed, the  $H$  equilibrium converges to the tie equilibrium when  $m = 1$ , that is, everyone votes in both parties. However, in the  $L$  equilibrium, decreasing the disadvantage of the minority leads to a lower turnout. What is the intuition for this result? Observe [Figure 3.3](#) and recall that the reasoning of the marginal individual in the defeated party  $A$  is “if I vote, I turn the defeat into a tie”, which implies that the voting mass in party  $A$ , as perceived by him when he votes,  $V_A + q_A N_A$ , is equal to  $V_B$ . In the  $H$  equilibrium it is the actual voting mass  $V_A$  that dominates, while in the  $L$  equilibrium it is the projected one,  $q_A N_A$ . The former is decreasing, while the latter is increasing, in  $q_A$ . If the size of the minority party,  $m$ , increases, then, for a given marginal individual in party  $B$ , the mass of voters in this party increases. Now, the defeated party  $A$  is facing a larger voting mass from the opponent party. Given that in the  $L$  equilibrium it is the projected voting mass  $q_A N_A$ , which is increasing in  $q_A$ , that dominates, the marginal individual must increase, further decreasing turnout.

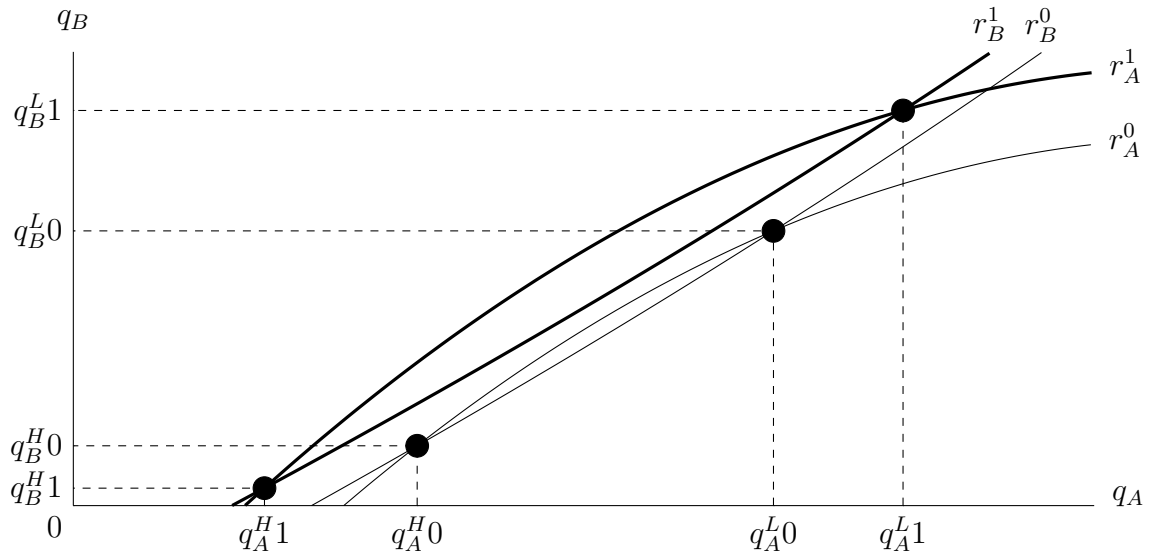


Figure 3.3: Effect of an increase in  $m$  in the minority-win equilibria

Conversely, in the  $H$  equilibrium, it is the actual voting mass  $V_A$ , which is decreasing in  $q_A$ , that dominates, hence the marginal individual must decrease, thus increasing turnout. This explains why the turnout levels diverge.

We thus obtain the unexpected result that closeness of the election may actually decrease turnout. Interestingly, the meta-analysis by [Geys \(2006\)](#) identifies several studies which conclude that closeness either decreases turnout or does not have any conclusive effect. Our approach suggests that this apparent anomaly may actually stem from self-delusion.

The result above is obtained under the hypothesis of a uniform distribution. With alternative distributions, more equilibria may exist, but, in the majority-win situation, the same comparative statics apply to the lowest equilibrium; for the minority-win equilibria, the lowest and highest turnout equilibria behave like the  $L$  and  $H$  equilibrium, respectively. As for the other (possibly, many) equilibria, the impact of  $m$  alternates between increasing and decreasing turnout levels.

We now analyze the relationship between the margin of victory, which is an *ex-post* measure of closeness, and turnout. In the uniform case, when the three equilibria exist, it is possible to link in a very intuitive way these two concepts. The next [Proposition](#) establishes that the higher the turnout, the lower the margin of

victory (and the closer the electoral race).

**Proposition 3.** Suppose that voters are self-delusional, the level of self-delusion is uniformly distributed in both parties and  $m \geq \underline{m}$ . Then, the level of turnout in each party and the margin of victory move in opposite directions.

*Proof.* See [Appendix](#). □

We therefore provide a theoretical underpinning for the utilization of this ex-post endogenous measure of closeness which is unrelated to rational expectations on behalf of the voters, the traditional justification put forward in the literature ([Geys, 2006](#)). The intuition is different depending on the winner of the election. When party  $B$  wins, the reasoning of the marginal individual in party  $A$  is such that he believes to cancel the actual margin of victory by voting: in fact, he believes he is adding his projected voting mass,  $q_A^* N_A^*$ , to the actual voting mass in  $A$ , leading to a tie. It is then clear that, when the margin of victory increases,  $q_A^*$  also increases, both because self-delusion and the mass of abstainers increase. In other words, one has to be more self-delusional to believe that he can cancel a higher margin of victory by dragging a part of the population of abstainers with him in his decision to vote. When party  $A$  wins, the intuition is not so clear. In this case, the marginal individual in party  $A$  believes that, if he didn't vote, the margin of victory would disappear and there would be a tie. More precisely, he believes that  $q_A^* V_A^*$  would be removed from the actual voting in mass in  $A$  if he were to abstain. Here, it is not so clear that a higher margin of victory occurs when  $q_A^*$  is higher, as this implies a lower  $V_A^*$ . However, when the three equilibria exist,  $m \geq \underline{m}$  and high levels of  $m$ , in the  $M$  equilibrium, result in high voting masses and low self-delusion marginal levels. Therefore, in order for  $q_A^* V_A^*$  to increase,  $q_A^*$  should also increase, so that the marginal individual in party  $A$  believes to be able to remove a higher percentage of the large actual voting mass by abstaining.

### 3.4.2 Election result

Let us now shed some light on the well-established empirical fact that turnout levels seem to have an impact on election results. Earlier contributions ([Pacek & Radcliff, 1995](#)) use turnout rates as explanatory variables and show that the impact on election results can be considerable. This has the obvious flaw that turnout rates are endogenous to the voting behavior that also generates the election outcome. A more recent body of literature analyzes the impact of turnout on election outcomes by simulating a counter-factual where non-voters in a given election would actually vote and obtain more modest, although significant, impacts of turnout on election outcomes ([Citrin \*et al.\*, 2003](#), [Bernhagen & Marsh, 1997](#), [Saglie \*et al.\*, 2012](#)). [Fowler \(2013\)](#) uses the adoption of compulsory voting in Australia as a natural experiment and obtains considerable impact of turnout on the vote share of the labor party and on policy outcomes at the national level.

It is straightforward that in the minority-win equilibria, turnout of minority supporters is higher than that of majority supporters. However, nothing is said about the total turnout level, that is, across the two parties' supporters. It turns out that, in the uniform case, when  $\underline{m} < m < 1$ , we may rank the turnout levels between the three possible equilibria. This is established in the next [Proposition](#).

**Proposition 4.** Suppose that voters are self-delusional, the level of self-delusion is uniformly distributed in both parties and  $\underline{m} < m < 1$ . Then, turnout is the highest in the  $H$  equilibrium, followed by the  $L$  and then by the  $M$  one.

*Proof.* See [Appendix](#). □

This proposition provides a theoretical explanation for the fact that variation in turnout levels is associated with different election results. In particular, higher turnout levels occur in minority-win equilibria. This is, to the best of our knowledge, the first paper to derive this endogenous relationship, which is often times used in the empirical literature. Importantly, this is not a causal relation.

The intuition is easy to grasp by focusing on [Proposition 2](#). When  $m = 1$ , the parties have the same size and the  $M$  and  $L$  equilibria have exactly the same total

turnout, as one is the symmetric of the the other. When  $m$  decreases from 1, turnout in each party increases in the  $M$  equilibrium and decreases in the  $L$  equilibrium. Hence, it is clear that, when the parties have different sizes, total turnout is higher in the  $M$  than in the  $L$  equilibrium. Besides, it is obvious that total turnout is higher in the  $H$  than in the  $L$  equilibrium, as turnout in both parties is always higher in the former.

The analysis undertaken so far does not say anything regarding the impact of self-delusion in a given party's chances of winning the election. In order to do so, one must have an operational way to compare the self-delusion levels of different distributions. We do so in the following [Section 3.5](#), assuming specific distributions of the degree of self-delusion for each party and stating that the one with more self-delusion voters is the one with the distribution of self-delusional level that first-order stochastically dominates that of the other party.

### 3.5 The advantage of self-delusion

The concept of self-delusion we introduced needs not to be equally present in all parties participating in an election. This difference in self-delusion amongst the voters of each party may result from connectedness in social networks such as Facebook and Twitter. There is a growing body of literature (see, e.g., [Cameron \*et al.\* \(2013\)](#), [Cogburn & Espinoza-Vasquez \(2011\)](#) and [Dimitrova \*et al.\* \(2011\)](#)) testing the impact of the parties' presence in social networks on election results, which suggests a statistically significant impact (albeit small in some references). The link between horizontal community linkages and political participation has been studied by political scientists. For instance, [Fennema & Tillie \(1999\)](#) show that political participation of migrant communities in the city politics in Amsterdam is related to the degree of *civic community*, as measured by the frequency of reading ethnic newspapers, the frequency of watching the Amsterdam special television for migrants, and, more importantly, the number of community organizations existing in the city, as well as their connectedness as reflected in the fact that the same person(s)

may sit simultaneously in the governing board of two or more organizations.

Let us formalize this difference in the level of self-delusion by introducing two new variables,  $\alpha_A$  and  $\alpha_B$ , and defining that, for all  $i \in \{A, B\}$ ,  $F_i(\alpha_i, q_i) = q_i^{\alpha_i}$ , with  $\alpha_i > 0$ . This implies that  $\forall i \in \{A, B\}$  and  $\forall q \in ]0, 1[$ ,  $F_{i\alpha_i}'(\alpha_i, q_i) < 0$ . In other words,  $F_i$  first-order stochastically dominates  $F_j$  if and only if  $\alpha_i > \alpha_j$ . Hence,  $\alpha_i$  may be thought as a measure of the level of self-delusion of party  $i$  and a higher  $\alpha_i$  corresponds to a more self-delusional party. [Proposition 1](#) remains true, so the majority may always win and, provided the parties' sizes are not too different, so does the minority. But now we can enrich [Proposition 2](#), as we have two new parameters which may influence turnout:  $\alpha_A$  and  $\alpha_B$ . [Proposition 5](#) establishes the effect of an increase in the self-delusion level of each party in the range of party  $B$  sizes which allow both parties to win the election.

**Proposition 5.** If  $F_i(\alpha_i, q_i) = q_i^{\alpha_i}$ , with  $\alpha_i > 0$ , which means  $\alpha_i$  measures the level of self-delusion in party  $i$ , then  $\underline{m}$  is increasing in  $\alpha_A$  and decreasing in  $\alpha_B$ .

*Proof.* See [Appendix](#). □

This result is quite intuitive. In fact, suppose  $\alpha_A$  increases, but  $m$  does not change. This means that although party  $A$  has the same support mass, its supporters are now more self-delusional. In other words, they feel that a higher proportion of their fellow partisans decide in the same way they do, and so, believe themselves to be pivotal more often than before. This represents a kind of empowerment of party  $A$ . In this case,  $\underline{m}$  goes up, implying that there is a range of minority sizes which allowed  $B$  to be a possible winner of the election, but do not allow it anymore. Facing a more self-deluded opponent in the elections, party  $B$  needs to get a higher relative support than before in order to win the election. Something very similar occurs when  $\alpha_B$  goes up. Now, it is party  $B$  which becomes more self-deluded and  $\underline{m}$  decreases. In this case, it is in party  $B$  we observe an increased propensity to vote, and so this party can win the election with less support than before.

With this result, we can see that a party's support is not the only factor which helps it win an election. The degree to which its supporters believe to be able to be

connected to their fellow partisans may also help to explain why some parties win elections and others do not. With this in mind, it would be worthwhile to study elections where the winning party was not the most supported one and investigate if, for any reason, there was, amongst the supporters of this party, a higher link or feeling of group identity which could induce a kind of self-delusional reasoning as the one we present here.

### 3.6 Conclusion

This paper proposes a theory of voter turnout based on the rhizomatic Nash equilibrium concept proposed by [Bravo-Furtado & Côte-Real \(2009a\)](#). In our setup with a continuum of agents, no one would vote if they were rational in the classical sense, for each voter has zero mass and cannot influence the election outcome. Rhizomatic thinking allows one to obtain positive turnout under very general conditions. In addition, the game has multiple equilibria, at least one of them with the majority party winning the election, and at least two with the minority party winning. Closeness of the election, as measured by the inverse of the party size difference, may decrease voter turnout in the equilibrium where the minority party wins the election. We also show that there is a trade-off between size and rhizomatic thinking, in the sense that the minority party can afford to become minorityer, while still securing itself a victory, provided its supporters become more rhizomatic, in a precise sense defined in the paper.

The existence of multiple equilibria stems from the very nature of rhizomatic thinking. Rhizomatic thinking is like a magnifying lens, and what the rhizomatic individual is doing is observing the mass of voters with this lens. Now the party which wins the election *must* include in its voting mass less rhizomatic individuals than the defeated party (otherwise it loses the election). We then have the sort of self-fulfilling prophecy which is common in games with multiple equilibria. The very existence of a majority voting mass ensures that individuals with low rhizomacy levels still believe themselves to be pivotal. In the defeated party, instead, the low



rhizomatic types apply their magnifying lens to a minority actual voting mass, and this is not enough to make them vote. The voters in this party *must* have a powerful magnifying lens, for otherwise they could not be voting given the relatively minority size of the actual voting mass.

This paper is a first step into the application of a new concept to the voter turnout paradox. There are a number of ways in which one may envisage extending this analysis. The most natural one is allowing for mixed strategies; abandoning the continuum hypothesis is another interesting avenue for future research. Parties are quite passive in our setup: letting them choose platforms or, perhaps more interestingly, invest in the invisible links that make their supporters rhizomatic using, e.g., the so-called social networks such as Facebook seems a promising step for future research.

## Appendix

### Proof of Proposition 1

First of all, we define, for  $i \in \{A, B\}$ , party  $i$ 's mass of supporters, voting rate and abstention rate, defined, respectively, by:

$$s_i = \begin{cases} 1 & i = A \\ m & i = B \end{cases} \quad (3.7)$$

$$v_i(q_i) = 1 - F_i(q_i) \quad (3.8)$$

$$n_i(q_i) = F_i(q_i) \quad (3.9)$$

Generalizing (3.1), (3.2), (3.3) and (3.4) and using (3.7), (3.8) and (3.9), one can see that an equilibrium where party  $i$  is the winner and party  $j$  is the loser solves the following system of equations:

$$\begin{cases} ((1 - q_i) v_i(q_i)) s_i = v_j(q_j) s_j \\ (v_j(q_j) + q_j n_j(q_j)) s_j = v_i(q_i) s_i \end{cases}$$

Rearranging, you get:

$$\begin{cases} q_i = \frac{q_j n_j(q_j)}{v_j(q_j) + q_j n_j(q_j)} \\ (v_j(q_j) + q_j n_j(q_j)) s_j - v_i(q_i) s_i = 0 \end{cases}$$

So, if we introduce the functions defined by:

$$h_i(q_j) = \frac{q_j n_j(q_j)}{v_j(q_j) + q_j n_j(q_j)} \quad (3.10)$$

$$g_i(q_j) = (v_j(q_j) + q_j n_j(q_j)) s_j - v_i(h_i(q_j)) s_i \quad (3.11)$$

the problem reduces to finding, for each  $m \in ]0, 1[$ , the zeros in  $[0, 1]$  of  $g_i$ , as defined in (3.11). For all  $i \in \{A, B\}$ ,  $F_i$  is twice continuously differentiable, which means  $h_i$  and  $g_i$  also are. Note that  $g_i(0) = s_j - s_i$  and  $g_i(1) = s_j > 0$ .

To prove (i), just notice that  $g_A$  is continuous,  $g_A(0) = m - 1 < 0$  and  $g_A(1) = m > 0$ . Hence, by the intermediate value theorem, there is at least one zero of  $g_A$  in  $]0, 1[$ .

The proof of (ii) is not so straightforward, as  $g_B(0) = 1 - m > 0$  and  $g_B(1) = 1 > 0$ . To move on, let us invoke the Weierstrass theorem, which helps us prove that there is a solution to the problem  $\min_{q_A \in [0,1]} g_B(q_A)$ , as  $g_B$  is a continuous function and  $[0, 1]$  is a compact set. Let us define, for each  $(i, j) \in \{A, B\}^2 : i \neq j$  and  $m \in [0, 1]$ , the following functions:

$$q_i^m(m) = \arg \min_{q_i \in [0,1]} g_j(q_i) \quad (3.12)$$

$$p_i(m) = \min_{q_j \in [0,1]} g_i(q_j) = g_i(q_j^m(m)) \quad (3.13)$$

Before we proceed, we establish the following result:

**Result 1.**  $\forall i \in \{A, B\}, \forall m \in [0, 1], g_i$  is strictly decreasing in 0 and strictly increasing in 1.

*Proof.* We know that, for each  $i \in \{A, B\}$  and  $m \in [0, 1]$ ,  $g_i$  is differentiable in  $[0, 1]$ , as  $F_A$  and  $F_B$  also are. Therefore,  $\forall (i, j) \in \{A, B\}^2 : i \neq j$ ,  $g'_i$  exists and is defined by:

$$g'_i(q_j) = ((1 - q_j) v'_j(q_j) + n_j(q_j)) s_j - v'_i(h_i(q_j)) h'_i(q_j) s_i \quad (3.14)$$

A simple calculation shows that  $\forall (i, j) \in \{A, B\}^2 : i \neq j$ :

$$h'_i(q_j) = \frac{n_j(q_j) v_j(q_j) - q_j v'_j(q_j)}{(v_j(q_j) + q_j n_j(q_j))^2} \quad (3.15)$$

We thus get that  $g'_i(0) = v'_j(0) s_j = -F'_j(0) s_j < 0$  and  $g'_i(1) = s_j + v'_i(1) v'_j(1) s_i = s_j + F'_i(1) F'_j(1) s_i > 0$ , which means that  $g_i$  is strictly decreasing in 0 and strictly increasing in 1.  $\square$

Since, for each  $(i, j) \in \{A, B\}^2 : i \neq j$ , and  $m \in [0, 1]$ ,  $g_i$  is strictly decreasing in 0, and  $g_i(1) = s_j > s_j - s_i = g_i(0)$ ,  $q_j^m \in ]0, 1[$ . As  $g_i$  is continuous and  $[0, 1]$  is constant, and therefore continuous, in  $m$ ,  $p_i$  is continuous, by the maximum theorem. Also, as  $g_i$  is differentiable, by the envelope theorem, we get that:

$$\begin{aligned} p'_A(m) &= g'_{A_m}(m, q_B^m) = v_B(q_B^m) + q_B^m n_B(q_B^m) \\ p'_B(m) &= g'_{B_m}(m, q_A^m) = -v_B(h_B(q_A^m)) \end{aligned} \quad (3.16)$$

As  $q_A^m < 1$ , (3.16) is negative or, equivalently,  $p_B$  is strictly decreasing. Now, on one hand, when  $m = 0$ ,  $g_B(q_A) = v_A(q_A) + q_A n_A(q_A) > 0$ ,  $\forall q_A \in [0, 1]$ , which means  $p_B(0) > 0$ . On the other hand, when  $m = 1$ ,  $g_B(0) = 0$  and  $g_B$  is decreasing in 0, and so  $p_B(1) < 0$ . Summing all up and using again the intermediate value theorem, we conclude that  $p_B$  has exactly one zero in  $]0, 1[$ , which we denote  $\underline{m}$ . We are then faced with three possibilities:

- (i) If  $0 < m < \underline{m}$ ,  $p_B(m) > 0$ , which means  $g_B > 0$ ,  $\forall q_A \in [0, 1]$ , and there are no equilibria in which  $B$  wins the election.

- (ii) If  $m = \underline{m}$ ,  $p_B(m) = 0$ , which means  $g_B(q_A^m) = 0$ , and there is at least one equilibrium in which  $B$  wins the election (there may be more than one, as  $q_A^m$  may be a correspondence).
- (iii) If  $m > \underline{m}$ ,  $p_B(m) < 0$ , which means  $g_B$  has at least one zero in  $]0, q_A^m[$  and at least one other in  $]q_A^m, 1[$ , by the intermediate value theorem, and there are at least two equilibria in which  $B$  wins the election.

To show (iii) is now easy. To prove the existence of an equilibrium in which party  $i = \{A, B\}$  wins the election, we need to show that, when  $m = 1$ ,  $g_i$  has at least one zero in  $[0, 1]$ . In this case, we get  $g_i(0) = 0$  and  $g_i(1) = 1 > 0$ . As  $g_i$  is decreasing in  $0$ ,  $p_i(1) < 0$ . Therefore, by the intermediate value theorem,  $g_i$  has at least one zero in  $]0, 1[$ . Finally, Lemma 3 tells us that, when  $m = 1$ , a tie with all self-delusional agents voting is an equilibrium.

### Proof of Corollary 1

Let us begin by proving an useful intermediate Result.

**Result 2.** If  $\forall i \in \{A, B\}, F_i(q_i) = q_i$ , then,  $\forall (i, j) \in \{A, B\}^2 : i \neq j, \forall m \in [0, 1]$ ,  $q_i^m(m)$  is unique and  $g_i$  is strictly decreasing in  $[0, q_j^m(m)]$  and strictly increasing in  $[q_j^m(m), 1]$ .

*Proof.* Let us fix  $(i, j) \in \{(A, B), (B, A)\}$  and  $m \in [0, 1]$ . If we derivate (3.14) and (3.15), we get the following (omitting the argument  $q_j$  from every function for an easier reading):

$$g_i''(q_j) = ((1 - q_j) v_j'' - 2v_j') s_j - \left( v_i''(h_i) h_i'^2 + v_i'(h_i) h_i'' \right) s_i \quad (3.17)$$

$$h_i''(q_j) = - \frac{(2v_j v_j' + q_j v_j'') (v_j + q_j n_j) + 2 (n_j v_j - q_j v_j') ((1 - q_j) v_j' + n_j)}{(v_j + q_j n_j)^3} \quad (3.18)$$

Now, as  $g_i$  is differentiable, it must be the case that  $g_i'(q_j^m) = 0$ . Using (3.14), (3.15) and the fact that  $\forall i \in \{A, B\}, F_i(q_i) = q_i$ , we find that:

$$(1 - 2q_j^m) m = \frac{q_j^m (2 - q_j^m)}{1 - (1 - q_j^m) q_j^m}$$

The right-hand side of this equation is positive, which means the left-hand side must also be, hence,  $q_j^m < \frac{1}{2}$ . Then, using (3.17) and (3.18), we get the following expression for  $g_i''$  evaluated at  $q_j^m$ :

$$g_i''(q_j^m) = 2 \left( s_j + \frac{q_j^m (1 - (1 - q_j^m) q_j^m) + q_j^m (2 - q_j^m) (1 - 2q_j^m)}{(1 - (1 - q_j^m) q_j^m)^3} \right)$$

Since  $q_j^m < \frac{1}{2}$ , this expression is positive. And this means that  $q_j^m$  is unique and  $g_i$  is strictly decreasing in  $[0, q_j^m(m)]$  and strictly increasing in  $[q_j^m(m), 1]$ . To see why, imagine that there is more than one  $q_j \in [0, 1]$  such that  $g_i'(q_j) = 0$ . Call the smallest  $q_1$  and the second smallest  $q_2$ . As  $g_i'$  is 0 and  $g_i''$  is positive in both of them, we know that  $g_i'$  must be positive in some interval to the right of  $q_1$  and negative in some interval to the left of  $q_2$ . Therefore, as  $g_i'$  is continuous, by the intermediate value theorem, there must be a  $q_j \in ]q_1, q_2[$ , such that  $g_i'(q) = 0$ . But then  $q_2$  is not the second smallest, which is a contradiction. Hence, there is only one  $q_j \in [0, 1]$  such that  $g_i'(q_j) = 0$  and it has to be  $q_j^m$ . As  $g_i$  is continuous and, by Result 1,  $g_i'(0) < 0$  and  $g_i'(1) > 0$ , by the intermediate value theorem,  $g_i'$  must be negative in  $[0, q_j^m[$  and positive in  $[q_j^m, 1[$ , which completes the proof.  $\square$

Knowing Result 2, the proof of (i) becomes easy. As, for all  $m \in [0, 1]$ ,  $g_A(0) < 0$ ,  $g_A(1) > 0$  and  $g_A$  is strictly decreasing in  $[0, q_B^m(m)]$  and strictly increasing in  $[q_B^m(m), 1]$ ,  $g_A$  has exactly one zero in  $[0, 1]$ .

To prove (ii), let us first find  $\underline{m}$  for the case in which  $\forall i \in \{A, B\}, F_i(q_i) = q_i$ . We know, from the Proof of Proposition 1, that  $\underline{m}$  is the the only zero of  $p_B$ , as defined in (3.13), with  $i = B$  and  $j = A$ . If this is the case, when  $m = \underline{m}$ ,  $g_B(q_A^m) = 0$  and, as happens with every  $m$ ,  $g_B'(q_A^m) = 0$ . Gathering these informations, we get the following system of equations:

$$\begin{cases} 1 - (1 - q_A) q_A - \left(1 - \frac{q_A^2}{1 - (1 - q_A) q_A}\right) = 0 \\ 2q_A - 1 + \frac{q_A(2 - q_A)}{(1 - (1 - q_A) q_A)^2} = 0 \end{cases}$$

The solution of this system of equations is  $(q_A, m) = \left(\frac{5 - \sqrt{13}}{6}, \frac{26\sqrt{13} - 70}{27}\right)$ , which means that  $\underline{m} = \frac{26\sqrt{13} - 70}{27} \approx 87,9\%$ .

Now, if  $m > \underline{m}$ ,  $p_B(m)$  is negative. This, together with the facts that  $g_B(0) > 0$ ,  $g_A(1) > 0$  and  $g_A$  is strictly decreasing in  $[0, q_A^m(m)]$  and strictly increasing in  $[q_A^m(m), 1]$  proves that  $g_B$  has exactly two zeros in  $[0, 1]$ .

At last, (iii) is already proved. The proofs of (i) and (ii) cover the case in which  $m = 1$ , which means that, when the parties have the same size, there is a single equilibrium with party  $A$  winning and one other with party  $B$  winning the election. As for the tie, number (iii) of Proposition 1 remains true in the uniform case.

## Proof of Proposition 2

Let us consider here that,  $\forall \{i, j\} \in \{A, B\} : i \neq j$ , the function  $g_i$ , as defined in (3.11), depends on  $m$  and  $q_j$ .  $g_i$  is a  $C^1$  function, as it is twice continuously differentiable. Fixing  $m = m^* > \underline{m}$ , both  $A$  and  $B$  may win the election in equilibrium, and so  $\forall i \in \{A, B\}, \exists q_j^* \in ]0, 1[ : g_i(m^*, q_j^*) = 0$ . From the Proof of Corollary 1, we know that  $g_i(m^*, q_j^m) < 0$  and, in  $q_j$ ,  $g_i$  is strictly decreasing in  $[0, q_j^m]$  and strictly increasing in  $[q_j^m, 1]$ , which means that  $g_{i q_j}'(m^*, q_j^*) \neq 0$ . The conditions for the application of the implicit function theorem are thus met and we can say that there is, in a neighborhood of  $m^*$ , a function  $f_{q_j^*}$  which gives, for each  $m$ , the  $q_j$  that equals  $g_i$  to 0, that is, the equilibrium self-delusion level in party  $j$ . From the same theorem, we also know that  $f_{q_j^*}$  is a  $C^1$  function and that, for all  $m$  in the domain of  $f_{q_j^*}$ :

$$f_{q_j^*}'(m) = - \frac{g_{i m}'(m, f_{q_j^*}(m))}{g_{i q_j}'(m, f_{q_j^*}(m))} \quad (3.19)$$

Before going to each specific equilibrium, let us just note that  $h_i$ , as defined in

(3.10), gives, for each  $q_j$ , the equilibrium level of  $q_i$ . As  $q_j^* \in ]0, 1[$ , (3.15) is positive in  $q_j^*$ , which means that  $q_A^*$  and  $q_B^*$  move in the same direction.

Now, the proof of (i) is easy. An easy derivation shows that  $g_{A_m}'(m, q_B) = v_B(q_B) + q_B n_B(q_B)$ . As  $q_B^* \in ]0, 1[$ , we know that  $g_{A_m}'(m^*, q_B^*) > 0$ . From the Proof of Corollary 1,  $g_{A_{q_B}'}(m^*, q_B^*) > 0$ . And so, (3.19) is negative in  $m^*$ , and, in the  $M$  equilibrium,  $q_A^*$  and  $q_B^*$  decrease when  $m$  increases, which means that turnout increases in both parties.

As for (ii), the proof is similar. After finding that  $g_{B_m}'(m, q_A) = -v_B(h_A(q_A))$  and remembering that  $q_A^* \in ]0, 1[$ , we conclude that  $g_{B_m}'(m^*, q_A^*) < 0$ . The Proof of Corollary 1 tells us that  $g_{B_{q_A}'}(m^*, q_A^H) < 0$  and  $g_{B_{q_A}'}(m^*, q_A^L) > 0$ . Hence, for the  $H$  equilibrium, (3.19) is negative in  $m^*$ , and for the  $L$  equilibrium, (3.19) is positive in  $m^*$ . Therefore, in both parties, turnout levels diverge when  $m$  increases.

### Proof of Proposition 3

First, let us look at an equilibrium in which  $A$  is the winner. According to (3.1), it is true that  $V_A^* - V_B^* = q_A^* V_A^* = q_A^* (1 - q_A^*)$ , which is strictly increasing in  $[0, \frac{1}{2}]$  and strictly decreasing in  $[\frac{1}{2}, 1]$ . In the Proof of Proposition 2, we find that, in the  $M$  equilibrium,  $f_{q_B}^*(m) < 0$  and  $h_A(q_B) > 0$ . This means that both  $q_A^*$  and  $q_B^*$  decrease with  $m$ . Solving the model with  $m = \underline{m}$ , we get that  $q_A^* \approx 0,340$ , which implies that, if  $m \geq \underline{m}$ ,  $q_A^* < \frac{1}{2}$ . Hence, in this range of  $m$ , a higher margin of victory occurs when  $q_A^*$  and  $q_B^*$  are higher or, in other words, when turnout is lower.

Now, an adaptation of (3.2) to the case in which  $B$  is the winner shows that  $V_B^* - V_A^* = q_A^* N_A^* = q_A^{*2}$ , which is strictly increasing in  $[0, 1]$ . This implies, in the same way as in the  $M$  equilibrium, that higher margins of victory are associated with lower turnout levels.

### Proof of Proposition 4

Let us start by defining the general turnout rate in equilibrium  $e$ , for each  $m \in ]\underline{m}, 1]$  (remember that  $f_{q_i}^e$  is the function that gives, for each  $m$ , the marginal  $q_i$  in

equilibrium  $e$ ):

$$T^e(m) = \frac{V_A^e(m) + V_B^e(m)}{1+m} = \frac{\sum_{i \in \{A, B\}} v_i \left( f_{q_i^*}^e(m) \right) s_i(m)}{1+m} \quad (3.20)$$

Our goal is to prove that  $\forall m \in ]\underline{m}, 1]$ ,  $T^M(m) < T^L(m) < T^H(m)$ .

Now, in an equilibrium in which party  $i$  wins and party  $j$  loses the election ( $i, j \in \{A, B\}$ ,  $i \neq j$ ), we know that (3.11) is 0, which means that  $v_i \left( f_{q_i^*}^e(m) \right) s_i(m) = \left( v_j \left( f_{q_j^*}^e(m) \right) + f_{q_j^*}^e(m) n_j \left( f_{q_j^*}^e(m) \right) \right) s_j(m)$ . Introducing this and the fact that  $\forall i \in \{A, B\}$ ,  $F_i(q_i) = q_i$  into (3.20) and deriving it, we get (omitting the argument from every function for an easier reading):

$$T^{e'}(m) = \frac{\left( -2f_{q_j^*}^{e'} v_j s_j + (2v_j + n_j^2) s_j' \right) (1+m) - (2v_j + n_j^2) s_j}{(1+m)^2} \quad (3.21)$$

In the  $M$  equilibrium,  $j = B$  and so (3.21) becomes:

$$T^{M'}(m) = \frac{-2f_{q_B^*}^{M'} v_B m (1+m) + 2v_B + n_B^2}{(1+m)^2}$$

As stated in the [Proof of Proposition 2](#),  $f_{q_B^*}^{M'} < 0$ , which means that  $T^{M'} > 0$  and  $T^M$  is strictly increasing. In the  $L$  equilibrium,  $j = A$  and so (3.21) becomes:

$$T^{L'}(m) = -\frac{2v_A \left( (1+m) f_{q_A^*}^{L'} + 1 \right) + n_A^2}{(1+m)^2}$$

The [Proof of Proposition 2](#) tells us that  $f_{q_A^*}^{L'} > 0$ , hence  $T^{L'} < 0$  and  $T^L$  is strictly decreasing. Solving the model, we find out that  $T^M(\underline{m}) \approx 0,582 < 0,846 \approx T^M(\underline{m})$  and that  $T^M(1) = T^L(1) \approx 0,662$ . Since for all  $e \in \{M, L, H\}$ ,  $T^e$  is continuous (because for all  $e \in \{M, L, H\}$  and  $i \in \{A, B\}$ ,  $f_{q_i^*}^e$  is continuous), we conclude that  $\forall m \in ]\underline{m}, 1[$ ,  $T^M(m) < T^L(m)$ .

Finally, just note that  $\forall m \in ]\underline{m}, 1[$ ,  $\forall i \in \{A, B\}$ ,  $f_{q_i^*}^L(m) > f_{q_i^*}^H(m)$ , which is enough to prove that, in this interval,  $T^L(m) < T^H(m)$ .



## Proof of Proposition 5

In this setting, the problem of finding an equilibrium where  $B$  wins the election still consists on finding the zeros of  $g_B$ , as defined in (3.11). The functions  $q_i^m$  and  $p_i$ , defined in (3.12) and (3.13), respectively, depend on  $m$ ,  $\alpha_A$  and  $\alpha_B$ , in this context. But  $p_B$  remains strictly decreasing in  $m$ ,  $p_B(0, \alpha_A, \alpha_B) > 0$  and  $p_B(1, \alpha_A, \alpha_B) < 0$ , by an analogous reasoning to the one applied in the Proof of Proposition 1, hence  $\underline{m}$  still exists.

By the envelope theorem, for all  $\delta \in \{m, \alpha_A, \alpha_B\}$ , we know that  $p_{B\delta}'(m, \alpha_A, \alpha_B) = g_{B\delta}'(m, \alpha_A, \alpha_B, q_A^m(m, \alpha_A, \alpha_B))$ . Removing the arguments for an easier reading, we get:

$$p_{Bm}'(m, \alpha_A, \alpha_B) = -v_B(h_B) < 0 \quad (3.22)$$

$$p_{B\alpha_A}'(m, \alpha_A, \alpha_B) = (1 - q_A^m) v_{A\alpha_A}' - v_{Bq_B}'(h_B) h_{B\alpha_A}' \quad (3.23)$$

$$p_{B\alpha_B}'(m, \alpha_A, \alpha_B) = \ln(h_B) h_b^{\alpha_A} m < 0 \quad (3.24)$$

Excluding the case  $\underline{m} = 1$ , in which  $q_B^m = 0$ , and noticing that  $g_{Bq_A}'(m, \alpha_A, \alpha_B, 1) = 1 + \alpha_A \alpha_B m > 0$ , we can conclude that  $g_{Bq_A}'(m, \alpha_A, \alpha_B, q_a^m) = 0$ . Using (3.14), this implies that:

$$v_{Bq_B}'(h_B) = \frac{(1 - q_A^m) v_{Aq_A}' + n_A}{h_{Bq_A}'} \quad (3.25)$$

Plugging (3.25) into (3.23), and doing some manipulations, we get that:

$$p_{B\alpha_A}'(m, \alpha_A, \alpha_B) = -\ln(q_A^m) (q_A^m)^{\alpha_A} \frac{1 - (q_A^m)^{\alpha_A} + (q_A^m)^{\alpha_A+1}}{1 - (q_A^m)^{\alpha_A} + q_A^m} > 0 \quad (3.26)$$

Now, fixing  $(\alpha_A, \alpha_B)$ ,  $\underline{m}$  is such that  $p_B(\underline{m}, \alpha_A, \alpha_B) = 0$ . Applying the implicit function theorem to this equation, we can say it defines a function  $f(\alpha_A, \alpha_B)$  which gives the value of  $\underline{m}$  for each for each  $(\alpha_A, \alpha_B)$ . The same theorem tells us that, for all  $\delta \in \{\alpha_A, \alpha_B\}$ :

$$f'_\delta(\alpha_A, \alpha_B) = -\frac{p_{B\delta}'(f(\alpha_A, \alpha_B), \alpha_A, \alpha_B)}{p_{Bm}'(f(\alpha_A, \alpha_B), \alpha_A, \alpha_B)} \quad (3.27)$$

Using (3.22), (3.24), (3.26) and (3.27), we get  $f'_{\alpha_A}(\alpha_A, \alpha_B) > 0$  and  $f'_{\alpha_B}(\alpha_A, \alpha_B) < 0$ , thus proving the result.



# Bibliography

- Acevedo, Melissa, & Krueger, Joachim L. 2004. Two Egocentric Sources of the Decision to Vote: The Voter's Illusion and the Belief in Personal Relevance. *Political Psychology*, **25**(1).
- Arthur, W. Brian. 1991. Designing Economic Agents that Act like Human Agents: A Behavioral Approach to Bounded Rationality. *The American Economic Review*, **81**(2), 353–359.
- Arthur, W. Brian. 1994. Inductive Reasoning and Bounded Rationality. *The American Economic Review*, **84**(2), 406–411.
- Bernhagen, Patrick, & Marsh, Michael. 1997. The Partisan Effects of Low Turnout: Analyzing Vote Abstention as a Missing Data Problem. *Electoral Studies*, **26**(3), 548–560.
- Blais, André. 2000. *To Vote or Not to Vote? The Merits and Limits of Rational Choice Theory*. University of Pittsburgh Press.
- Blais, André, Young, Robert, & Lapp, Miriam. 2000. The Calculus of Voting: An Empirical Test. *European Journal of Political Research*, **37**, 181–201.
- Bosch-Domènech, Antoni, & Vriend, Nicolaas J. 2003. Imitation of Successful Behaviour in Cournot Markets. *The Economic Journal*, **113**(487), 495–524.
- Bravo-Furtado, João, & Côte-Real, Paulo P. 2009a. Rationality, Behavior and Rhizomatic Thinking: Revisiting the Self-Goal Choice Assumption in Games.
- Bravo-Furtado, João, & Côte-Real, Paulo P. 2009b. Rhizomatic Thinking and Voting Equilibria in Large Multi-Candidate Elections under Plurality Rule.

- C. Mueller, Dennis, & Stratmann, Thomas. 2003. The economic effects of democratic participation. *Journal of Public Economics*, **87**(9-10), 2129–2155.
- Camerer, Colin. 1998. Bounded Rationality in Individual Decision Making. *Experimental Economics*, **1**(2), 163–183.
- Camerer, Colin F., Ho, Teck-Hua, & Chong, Juin-Kuan. 2004. A Cognitive Hierarchy Model of Games. *The Quarterly Journal of Economics*, **119**(3), 861–898.
- Cameron, Michael P., Barrett, Patrick, & Stewardson, Bob. 2013. Can Social Media Predict Election Results? Evidence from New Zealand.
- Choi, James J., Laibson, David, Madrian, Brigitte C., & Metrick, Andrew. 2003. Optimal Defaults. *The American Economic Review*, **93**(2), 180–185.
- Citrin, Jack, Schickler, Eric, & Sides, John. 2003. What if Everyone Voted? Simulating the Impact of Increased Turnout in Senate Elections. *American Journal of Political Science*, **47**(1), 75–90.
- Coate, Stephen, & Conlin, Michael. 2004. A Group-Rule Utilitarian Approach to Voter Turnout: Theory and Evidence. *The American Economic Review*, **94**(5), 1476–1504.
- Cogburn, Derrick L., & Espinoza-Vasquez, Fatima K. 2011. From Networked Nominee to Networked Nation: Examining the Impact of Web 2.0 and Social Media on Political Participation and Civic Engagement in the 2008 Obama Campaign. *Journal of Political Marketing*, **10**(1-2), 189–213.
- Conitzer, Vincent. 2012. Should Social Network Structure Be Taken into Account in Elections? *Mathematical Social Sciences*, **64**(1), 100–102.
- Conley, John P., Toossi, Ali, & Wooders, Myrna. 2001. Evolution and voting: how Nature makes us public spirited. *Warwick Economic Research Papers*.
- Conlisk, John. 1980. Costly Optimizers versus Cheap Imitators. *Journal of Economic Behavior & Organization*, **1**(3), 275–293.

- Conlisk, John. 1996a. Bounded Rationality and Market Fluctuations. *Journal of Economic Behavior & Organization*, **29**(2), 223–250.
- Conlisk, John. 1996b. Why Bounded Rationality? *Journal of Economic Literature*, **34**(2), 669–700.
- Cox, James C., & Walker, Mark. 1998. Learning to Play Cournot Duopoly Strategies. *Journal of Economic Behavior & Organization*, **36**(2), 141–161.
- de Palma, Andre, Myers, Gordon M., & Papageorgiou, Yorgos Y. 1994. Rational Choice under an Imperfect Ability to Choose. *The American Economic Review*, **84**(3), 419–440.
- DeMichelis, Stefano, & Dhillon, Amrita. 2001. Learning in elections and voter turnout equilibria. *Warwick Economic Research Papers*.
- Dhillon, Amrita, & Peralta, Susana. 2002. Economic Theories of Voter Turnout. *The Economic Journal*, **112**, 332–352.
- Dimitrova, Daniela V., Shehata, Adam, Strömbäck, Jesper, & Nord, Lars W. 2011. The Effects of Digital Media on Political Knowledge and Participation in Election Campaigns: Evidence from Panel Data. *Communication Research*, **20**(10), 1–24.
- Dowding, Keith. 2005. Is it Rational to Vote? Five Types of Answer and a Suggestion. *The British Journal of Politics & International Relations*, **7**(3), 442–459.
- Downs, Anthony. 1957. An Economic Theory of Political Action in a Democracy. *The Journal of Political Economy*, **65**(2), 135–150.
- Duffy, John, & Tavits, Margit. 2008. Beliefs and Voting Decisions: A Test of the Pivotal Voter Model. *American Journal of Political Science*, **52**(3), 603–618.
- Edlin, Aaron, Gelman, Andrew, & Kaplan, Noah. 2007. Voting as a Rational Choice: Why and How People Vote to Improve the Well-Being of Others. *Rationality and Society*, **19**(3), 293–314.
- Eliasz, Kfir. 2003. Nash Equilibrium when Players Account for the Complexity of Their Forecasts. *Games and Economic Behavior*, **44**(2), 286–310.

- Evans, George W., & Ramey, Garey. 1992. Expectation Calculation and Macroeconomic Dynamics. *The American Economic Review*, **82**(1), 207–224.
- Eyster, Erik, & Rabin, Matthew. 2005. Cursed Equilibrium. *Econometrica*, **73**(5), 1623–1672.
- Fauvelle-Aymar, Christine, & François, Abel. 2006. The Impact of Closeness on Turnout: An Empirical Relation Based on a Study of a Two-Round Ballot. *Public Choice*, **127**(3), 461–483.
- Feddersen, Timothy, & Sandroni, Alvaro. 2006. The Calculus of Ethical Voting. *International Journal of Game Theory*, **35**(1), 1–25.
- Feddersen, Timothy J. 2004. Rational Choice Theory and the Paradox of Not Voting. *Journal of Economic Perspectives*, **18**(1), 99–112.
- Feddersen, Timothy J., & Pesendorfer, Wolfgang. 1996. The Swing Voter’s Curse. *The American Economic Review*, **86**(3), 408–424.
- Feddersen, Timothy J., & Pesendorfer, Wolfgang. 1999. Abstention in elections with asymmetric information and diverse preferences. *American Political Science Review*, **93**(2), 381–398.
- Fennema, Meindert, & Tillie, Jean. 1999. Political Participation and Political Trust in Amsterdam: Civic Communities and Ethnic Networks. *Journal of Ethnic and Migration Studies*, **25**(4), 703–726.
- Fowler, Anthony. 2013. Electoral and Policy Consequences of Voter Turnout: Evidence from Compulsory Voting in Australia. *Quarterly Journal of Political Science*, **8**(2), 159–182.
- Gabaix, Xavier. 2014. A Sparsity-Based Model of Bounded Rationality. *The Quarterly Journal of Economics*, **129**(4), 1661–1710.
- Gabaix, Xavier, Laibson, David, Moloche, Guillermo, & Weinberg, Stephen. 2006. Costly Information Acquisition: Experimental Analysis of a Boundedly Rational Model. *The American Economic Review*, **96**(4), 1043–1068.

- Gennaioli, Nicola, & Shleifer, Andrei. 2010. What Comes to Mind. *The Quarterly Journal of Economics*, **125**(4), 1399–1433.
- Geys, Benny. 2006. Explaining Voter Turnout: A Review of Aggregate-Level Research. *Electoral Studies*, **25**(4), 637–663.
- Gigerenzer, Gerd, & Goldstein, Daniel G. 1996. Reasoning the Fast and Frugal Way: Models of Bounded Rationality. *Psychological Review*, **103**(4), 650–669.
- Goldberg, Jeffrey, Markoczy, Livia, & Zahn, G. Lawrence. 2005. Symmetry and the Illusion of Control as Bases for Cooperative Behavior. *Rationality and Society*, **17**(2), 243–270.
- Grafstein, Robert. 1991. An Evidential Decision Theory of Turnout. *American Journal of Political Science*, **35**(4), 989–1010.
- Guttman, Joel M. 1996. Rational Actors, Tit-for-Tat Types, and the Evolution of Cooperation. *Journal of Economic Behavior & Organization*, **29**(1), 27–56.
- Halpern, Joseph Y., & Pass, Rafael. 2015. Algorithmic Rationality: Game theory with Costly Computation. *Journal of Economic Theory*, **156**, 246–268.
- Huck, Steffen, Normann, Hans-Theo, & Oechssler, Jörg. 1999. Learning in Cournot Oligopoly – An Experiment. *The Economic Journal*, **109**(454), 80–95.
- Huck, Steffen, Normann, Hans-Theo, & Oechssler, Jörg. 2002. Stability of the Cournot Process – Experimental Evidence. *International Journal of Game Theory*, **31**(1), 123–136.
- Huck, Steffen, Normann, Hans-Theo, & Oechssler, Jörg. 2004. Two Are Few and Four Are Many: Number Effects in Experimental Oligopolies. *Journal of Economic Behavior & Organization*, **53**(4), 435–446.
- Jehiel, Philippe. 2005. Analogy-Based Expectation Equilibrium. *Journal of Economic Theory*, **123**(2), 81–104.
- Kahneman, Daniel. 2003. Maps of Bounded Rationality - Psychology for Behavioral Economics. *The American Economic Review*, **93**(5), 1449–1475.



- Kahneman, Daniel, & Tversky, Amos. 1979. Prospect Theory: An Analysis of Decision under Risk. *Econometrica*, **47**(2), 263–291.
- Kőszegi, Botond, & Szeidl, Adam. 2013. A Model of Focusing in Economic Choice. *The Quarterly Journal of Economics*, **128**(1), 53–104.
- Laibson, David. 1997. Golden Eggs and Hyperbolic Discounting. *The Quarterly Journal of Economics*, **112**(2), 443–477.
- Levine, David K. 1998. Modeling Altruism and Spitefulness in Experiments. *Review of Economic Dynamics*, **1**(3), 593–622.
- Li, Ming, & Majumdar, Dipjyoti. 2010. A Psychologically Based Model of Voter Turnout. *Journal of Public Economic Theory*, **12**(5), 979–1002.
- Lijphart, Arend. 1997. Unequal Participation: Democracy’s Unresolved Dilemma. *The American Political Science Review*, **91**(1), 1–14.
- Lipman, Barton L. 1991. How to Decide How to Decide How to...: Modeling Limited Rationality. *Econometrica*, **59**(4), 1105–1125.
- Lipman, Barton L. 1995. Information Processing and Bounded Rationality: A Survey. *Canadian Journal of Economics*, **28**(1), 42–67.
- Lipman, Barton L. 1999. Decision Theory without Logical Omniscience: Toward an Axiomatic Framework for Bounded Rationality. *The Review of Economic Studies*, **66**(2), 339–361.
- Lu, Huei-Chung, Chen, Mingshen, & Chang, Juin-Jen. 2005. Are More Alternatives Better for Decision-Makers? A Note on the Role of Decision Cost. *Theory and Decision*, **58**(3), 283–304.
- March, James G. 1978. Bounded Rationality, Ambiguity, and the Engineering of Choice. *The Bell Journal of Economics*, **9**(2), 587–608.
- Masel, Joanna. 2007. A Bayesian Model of Quasi-Magical Thinking Can Explain Observed Cooperation in the Public Good Game. *Journal of Economic Behavior & Organization*, **64**(2), 216–231.

- McKelvey, Richard D., & Palfrey, Thomas R. 1995. Quantal Response Equilibria for Normal Form Games. *Games and Economic Behavior*, **10**(1), 6–38.
- Morris, Michael W., & Sim, Damien L. H. 1998. Distinguishing Sources of Cooperation in the One-Round Prisoner’s Dilemma: Evidence for Cooperative Decisions Based on the Illusion of Control. *Journal of Experimental Social Psychology*, **32**, 494–512.
- Mullainathan, Sendhil. 2002. A Memory-Based Model of Bounded Rationality. *The Quarterly Journal of Economics*, **117**(3), 735–774.
- Nagel, Rosemarie. 1995. Unraveling in Guessing Games: An Experimental Study. *The American Economic Review*, **85**(5), 1313–1326.
- Ortoleva, Pietr. 2013. The Price of Flexibility: Towards a Theory of Thinking Aversion. *Journal of Economic Theory*, **148**(3), 903–934.
- Osborne, Martin J., & Rubinstein, Ariel. 1998. Games with Procedurally Rational Players. *The American Economic Review*, **88**(4), 834–847.
- Pacek, Alexander, & Radcliff, Benjamin. 1995. Turnout and the Vote for Left-of-Centre Parties: A Cross-National Analysis. *British Journal of Political Science*, **25**(1), 137–143.
- Palfrey, Thomas R., & Rosenthal, Howard. 1983. A Strategic Calculus of Voting. *Public Choice*, **41**, 7–53.
- Palfrey, Thomas R., & Rosenthal, Howard. 1985. Voter Participation and Strategic Uncertainty. *The American Political Science Review*, **79**(1), 62–78.
- Quattrone, George A., & Tversky, Amos. 1984. Causal versus Diagnostic Contingencies: On Self-Deception and on the Voter’s Illusion. *Journal of Personality and Social Psychology*, **40**(2), 237–248.
- Rabin, Matthew. 1993. Incorporating Fairness into Game Theory and Economics. *The American Economic Review*, **83**, 1281–1302.

- Rabin, Matthew. 2002. Inference by Believers in the Law of Small Numbers. *The Quarterly Journal of Economics*, **117**(3), 775–816.
- Radner, Roy, & Rothschild, Michael. 1975. On the Allocation of Effort. *Journal of Economic Theory*, **10**(3), 358–376.
- Rassenti, Stephen, Reynolds, Stanley S., Smith, Vernon L., & Szidarovszky, Ferenc. 2000. Adaptation and Convergence of Behavior in Repeated Experimental Cournot Games. *Journal of Economic Behavior & Organization*, **41**(2), 117–146.
- Reis, Ricardo. 2006a. Inattentive Consumers. *Journal of Monetary Economics*, **53**(8), 1761–1800.
- Reis, Ricardo. 2006b. Inattentive Producers. *The Review of Economic Studies*, **73**(3), 793–821.
- Riker, William H., & Ordeshook, Peter C. 1968. A Theory of the Calculus of Voting. *The American Political Science Review*, **62**(1), 25–42.
- Roemer, John E. 2010. Kantian Equilibrium. *The Scandinavian Journal of Economics*, **112**(1), 1–24.
- Rosenthal, Robert W. 1993. Rules of Thumb in Games. *Journal of Economic Behavior & Organization*, **22**(1), 1–13.
- Rubinstein, Ariel. 1986. Finite Automata Play the Repeated Prisoner's Dilemma. *Journal of Economic Theory*, **39**(1), 83–96.
- Saglie, Jo, Bergh, Johannes, & Bjørklund, Tor. 2012. Do Labour Parties Suffer from Declining Turnout? Evidence from Norwegian Local Elections. *Local Government Studies*, **38**(2), 249–270.
- Sieg, Gernot, & Schulz, Christof. 1995. Evolutionary dynamics in the voting game. *Public Choice*, **85**, 157–172.
- Simon, Herbert A. 1955. A Behavioral Model of Rational Choice. *The Quarterly Journal of Economics*, **69**(1), 99–118.

- Simon, Herbert A. 1956. Rational Choice and the Structure of the Environment. *Psychological Review*, **63**(2), 129–138.
- Sims, Christopher A. 2003. Implications of Rational Inattention. *Journal of Monetary Economics*, **50**(3), 665–690.
- Smith, Vernon L., & Walker, James M. 1993. Monetary Rewards and Decision Cost in Experimental Economics. *Economic Inquiry*, **31**(2), 245–261.
- Spiegler, Ran. 2011. *Bounded Rationality and Industrial Organization*. Oxford University Press.
- Stahl, Dale O., & Wilson, Paul W. 1995. On Players' Models of Other Players: Theory and Experimental Evidence. *Games and Economic Behavior*, **10**(1), 218–254.
- Tversky, Amos. 1972. Elimination by Aspects: A Theory of Choice. *Psychological Review*, **79**(4), 281–299.
- Tversky, Amos, & Kahneman, Daniel. 1991. Loss Aversion in Riskless Choice: A Reference-Dependent Model. *The Quarterly Journal of Economics*, **106**(4), 1039–1061.
- Tversky, Amos, & Kahneman, Daniel. 1992. Advances in Prospect Theory: Cumulative Representation of Uncertainty. *Journal of Risk and Uncertainty*, **5**(4), 297–323.