ECONOMICS BULLETIN

Super−replicating Bounds on European Option Prices when the Underlying Asset is Illiquid

João Amaro de Matos
School of Economics, Universidade Nova de Lisboa

Paula Antão
School of Economics, Universidade Nova de Lisboa

Abstract

We derive super−replicating bounds on European option prices when the underlying asset is illiquid. Illiquidity is taken as the impossibility of transacting the underlying asset at some points in time, generating market incompleteness. We conclude that option price bounds follow a Black−Scholes partial differential equation where the volatility term is adjusted to reflect different levels of illiquidity.

Citation: Amaro de Matos, João and Paula Antão, (2001) "Super−replicating Bounds on European Option Prices when the Underlying Asset is Illiquid." *Economics Bulletin,* Vol. 7, No. 1 pp. 1−7
Submitted: October 17, 2001. **Accepted:** October 23, 2001.
URL: http://www.economicsbulletin.com/2001/volume7/EB−01G10003A.pdf

1 Introduction

In frictionless complete markets, absence or arbitrage results in a unique price for an option. The option's price is given at any point in time by the value of the continuously rebalanced portfolio that replicates the payoff of the option at maturity. In economies with market frictions, however, classical valuation theories fail and there is no longer an uniquely determined option price. Examples of such market frictions widely studied in the literature include transaction costs, trading restrictions, taxes and borrowing costs. In this work, the imperfection studied is the existence of liquidity constraints.

Liquidity is often defined in terms of the bid-ask spread and/or transaction costs. In this sense, illiquidity is the situation where traders face higher trading costs than at other times or in other markets. However, market participants feel illiquidity in a rather different way. Traders view illiquidity as a restriction on their ability to transact an asset, rather than the existence of transaction costs. This type of illiquidity became more relevant after the recent financial crises when some markets temporarily disappeared. Besides Longstaff (2001), this type of liquidity has not been given much attention, although being an important characteristic of actual financial markets.

We consider illiquidity as the impossibility to transact the underlying asset at given points in time. Therefore, markets are no longer complete and the ability to construct at any point in time a replicating portfolio is constrained. We follow the super-replication approach developed by El Karoui and Quenez (1991,1995) and Karatzas and Kou (1996) to construct the best possible replicating portfolios under this incomplete market structure. This approach determines the minimum (maximum) value that allows a trader selling (buying) options to hedge completely his position. Such values are arbitrage-free bounds for the option price.

In this work, option price bounds are generated *only* as a consequence of illiquidity in the stock market. This way of modeling the bid-ask bounds comes very much in the spirit of the recent paper of Cho and Engle (1999), who empirically explain option spreads by the illiquidity of an underlying market, rather than by the usual imperfections[1].

The note is organized as follows. Section 2 introduces the model and derives the cost of the super-replicating portfolios for both a long and a short position in options. Section 3 derives upper and lower bounds in a more flexible way of defining illiquidity. The last section presents the main conclusions. Proofs are presented in the appendix.

[1] Examples of other imperfections that result in options price bounds are transaction costs studied in Leland (1985), Boyle and Vorst (1992) and Constantinides and Zariphoupoulou (1999), among others, and trading restrictions as in Naik and Uppal (1994) and Jouini and Kallal (1995). The assumption of stochastic volatility also led Frey and Sin (1999) to derive bounds to the prices of European options and Cochrane and Saá-Requejo (2000) developed bounds in a more general setting of incomplete markets.

2 The Model

Let the stock price follow a binomial process over discrete periods. In each period the stock's value evolves according to the rates U and D, where $U > R > D$ and R denotes one plus a constant riskless interest rate over each time period. This way of modelling the evolution of the stock price follows Cox, Ross and Rubinstein (1979).

Consider a European call option with exercise price K and T periods to maturity. At time $t = 0$ the option is traded for a value C. At time $t = T$ the option matures and its value is given by $C_{T,i} = \max(0, U^i D^{T-i} S - K)$ where $i = 0, 1, ..., T$ denotes the number of upward movements of the stock price until maturity.

If there are no arbitrage opportunities, the call option must be worth the same as the cheapest portfolio that exactly replicates the value of the call at each point in time. This portfolio consists of Δ shares of the stock and an amount B in riskless bonds. As time changes, the portfolio is adjusted to continue replicating the final payoff of the call option.

The way we defined illiquidity in the underlying asset makes the construction of such a replicating portfolio impossible, in the sense that it can not be readjusted at all the points in time.

Assume in this section that the portfolio constructed at time $t = 0$ can not be adjusted until time $t = T$ and consider a financial institution selling a call option at $t = 0$. In order to be hedged maximizing its wealth, the institution must minimize the cost of replicating the payoff of the option at maturity. In other words, the problem of the intermediary is

$$\begin{array}{l} Min\ \Delta S + B \\ \{\Delta, B\} \end{array}$$

subject to the terminal conditions:

$$\Delta U^i D^{T-i} S + B R^T \geq C_{T,i} \qquad \text{for } i = 0, ..., T$$

The solution to this problem is obtained following El Karoui and Quenez (1991, 1995) and is given by

$$\overline{C} = \frac{1}{R^T} \left[\frac{R^T - D^T}{U^T - D^T} C_{T,T} + \frac{U^T - R^T}{U^T - D^T} C_{T,0} \right]. \tag{1}$$

This discrete time model can be used to derive a continuous time valuation equation, following Cox, Ross and Rubinstein (1979). Notice that in this context, the expected rate of increase of the underlying asset per unit time is $pU + (1-p) D$, where p denotes the probability that the rate is U. Also per unit time, the variance of the value of the underlying asset is then given by $\sigma^2 = S^2 p (1-p) (U - D)^2$. Suppose now that each original time period is divided into $1/h$ smaller periods. The issue in order to take the limit $h \to 0$ is to suitably characterize the evolution rates for each of these small time intervals. To preserve the expected rate of increase and variance of the value of the underlying asset per unit time in the limit above, Cox, Ross and Rubinstein (1979) show that the rates of increase per time interval h may be adjusted as $U_h = e^{\sigma\sqrt{h}}$, $D_h = e^{-\sigma\sqrt{h}}$ and $R_h = R^h$. Substituting them in equation (1), it follows that

$$\overline{C}(S,\tau) = \frac{1}{R^{hT}}\pi_h C(e^{T\sigma\sqrt{h}}S, \tau - Th) + (1-\pi_h)\, C(e^{-T\sigma\sqrt{h}}S, \tau - Th)$$

where

$$\pi_h = \frac{e^{\log RT} - e^{-T\sigma\sqrt{h}}}{e^{T\sigma\sqrt{h}} - e^{-T\sigma\sqrt{h}}}$$

and τ denotes the time to maturity. Then, expanding the function C around (S,τ) and then each exponential around $h = 0$, in the limit when $h \to 0$ the equation above becomes the Partial Differential Equation (PDE)

$$\frac{1}{2}\frac{\partial^2 C}{\partial S^2}T\sigma^2 S^2 + \frac{\partial C}{\partial S}S(\log R) - C(\log R) - \frac{\partial C}{\partial \tau} = 0. \tag{2}$$

This is simply the Black-Scholes PDE changed by the fact that the volatility σ is replaced by $\sigma\sqrt{T}$.

On the other hand, if the financial institution is concerned about the cost of replicating a long call option on the same underlying asset, its problem is

$$\begin{array}{l} Max\ \Delta S + B \\ \{\Delta, B\} \end{array}$$

subject to the terminal conditions:

$$\Delta U^i D^{T-i} S + BR^T \leq C_{T,i} \qquad \text{for } i = 0, ..., T.$$

The solution to this problem depends on the relation between R^T and the value of the asset at each rebalancing point in time. Let x be defined as the integer satisfying $U^{T-(x+1)}D^{x+1} < R^T < U^{T-x}D^x$, and $0 \leq x \leq T-1$. Following Karatzas and Kou (1996), the lower bound to the price of the call option, is given by

$$\underline{C} = \frac{1}{R^T}\left[\frac{U^{T-(x+1)}D^{x+1}}{U^{T-x}D^x - U^{T-(x+1)}D^{x+1}}C_{T,T-x} + \frac{U^{T-x}D^x - R^T}{U^{T-x}D^x - U^{T-(x+1)}D^{x+1}}C_{T,T-x-1}\right]$$

Proceeding as before, in the continuous-time limit the bound above can be shown to satisfy the PDE

$$\frac{1}{2}\frac{\partial^2 C}{\partial S^2}\frac{\sigma^2}{T}S^2 + \frac{\partial C}{\partial S}S(\log R) - C(\log R) - \frac{\partial C}{\partial \tau} = 0 \tag{3}$$

which is, once again, the Black-Scholes PDE where the volatility σ is replaced by $\sigma/\sqrt{T}$.

3 General Model

This section develops a more flexible way of characterizing illiquidity. In the former section, the underlying asset could not be transacted after $t = 0$ and before $t = T$. It is now assumed that, in the discrete-time setting, the asset cannot be transacted during f consecutive points in time and then can be transacted for g consecutive points in time. This structure is repeated until the maturity date T. In other words, for all integer $a < \frac{T-f}{f+g}$ the asset is *not* transacted for $t \in \cup_a [a(f+g)+1, a(f+g)+f]$ and may be transacted in the remaining set of points in time. Under this setting, an analogous development of the upper and lower bounds can be made, leading to the following PDE:

$$\frac{1}{2}\frac{\partial^2 C}{\partial S^2}\frac{(f+1)^2+(g-1)}{f+g}\sigma^2 S^2 + \frac{\partial C}{\partial S}S(\log R) - C(\log R) - \frac{\partial C}{\partial \tau} = 0 \tag{4}$$

for the upper bound and

$$\frac{1}{2}\frac{\partial^2 C}{\partial S^2}\frac{g}{f+g}\sigma^2 S^2 + \frac{\partial C}{\partial S}S(\log R) - C(\log R) - \frac{\partial C}{\partial \tau} = 0 \tag{5}$$

for the lower bound.

As before, option price bounds follow a Black-Scholes PDE with an adjustment in the volatility. Notice that when it is possible to trade the underlying asset in every point in time, i.e. when $f = 0$, equations (4) and (5) are exactly equal to the Black-Scholes PDE. Notice also, that equations (2) and (3) can be readily obtained respectively from (4) and (5) by considering that, according to the assumptions in section 2, there is a sequence of $T - 1$ points in time where we cannot transact the underlying asset, i.e. $f = T - 1$, followed by just one point where the asset can be transacted, i.e. $g = 1$.

4 Conclusion

The model presented introduces illiquidity in the sense that it is not be possible to transact the underlying asset in every point in time. This inability to adjust the hedging portfolio will result in additional risk for traders. Therefore, there is no price that guarantees the absence of arbitrage opportunities for both short and long position in the trade of options. In this work, super-replication bounds on European options prices are derived under this assumption.

Super-replicating price bounds follow a Black-Scholes PDE where the volatility comes adjusted to consider different levels of illiquidity. As expected, the Black-Scholes price lies between these arbitrage-free bounds. Moreover, as liquidity in the underlying asset increases, bounds become narrower and collapse in the Black-Scholes price.

References

[1] Boyle, P. and T. Vorst, 1992, Option Replication in Discrete Time with Transactions Costs, Journal of Finance, 47, 271-293.

[2] Cho, Y. and R. Engle, 1999, Modeling the Impacts of Market Activity on Bid-Ask Spreads in the Option Market, discussion paper 99-05, UCSD.

[3] Cox, J.C., S.A. Ross and M. Rubinstein, 1979, Option Pricing: A Simplified Approach, Journal of Financial Economics, 7, 229-263.

[4] Cochrane, J. and J. Saá-Requejo, 2000, Beyond Arbitrage: Good-deal Asset Price Bounds in Incomplete Markets, The Journal of Political Economy, 108, 79-120.

[5] Constantinides, G and T. Zariphoupoulou, 1999, Bounds on Prices of Contingent Claims in an Intertemporal Economy with Proportional Transaction Costs and General Preferences, Finance and Stochastics, 3, 345-369.

[6] El Karoui, N. and M.C. Quenez, 1991, Programation Dynamique et Évaluation des Actifs Contingents en Marché Incomplet, Comptes Rendues de l' Academy des Sciences de Paris, Série I, 313, 851-854

[7] El Karoui, N. and M.C. Quenez, 1995, Dynamic Programming and Pricing of Contingent Claims in an Incomplete Market, SIAM Journal of Control and Optimization, 33, 29-66.

[8] Frey, R. and C. Sin, 1999, Bounds on European Options Prices under Stochastic Volatility, Mathematical Finance, 5, 197-232.

[9] Jouini E. and H. Kallal, 1995, Arbitrage in Securities Markets with Short-Sales Constraints,.Mathematical Finance, 9, 97-116.

[10] Karatzas, I. and S. G. Kou, 1996, On the Pricing of Contingent Claims with Constrained Portfolios, Annals of Applied Probability, 6, 321-369.

[11] Leland, H., 1985, Option Pricing and Replication with Transactions Costs, Journal of Finance, 40, 1283-1301.

[12] Longstaff, F., 2001, Optimal Portfolio Choice and the Valuation of Illiquid Securities, The Review of Financial Studies, 14, 407-431.

[13] Naik, V. and R. Uppal, 1994, Leverage Constraints and the Optimal Hedging of Stock and Bond Options, Journal of Financial and Quantitative Analysis, 29, 199-222.

A appendix

A.1 The PDE for the upper bound

Let g be the number of initial points in time with transactions, including $t = 0$; f is the number of points in time between two consecutive transactions, that is, $f + 1$ is the number of periods one have to wait to transact again the underlying asset; $p = \frac{R-D}{U-D}$ and $\pi = \frac{R^T-D^T}{U^T-D^T}$. Then,

1. the term multiplying C is
   $\sum_{i=0}^{g-1} \binom{g-1}{i} p^i (1-p)^{g-1-i} - (1 + (g+f)hr) = -(g+f)hr;$
2. the term multiplying $\frac{\partial C}{\partial \tau}$ is
   $\sum_{i=0}^{g-1} \binom{g-1}{i} p^i (1-p)^{g-1-i} (-(g+f)h) = -(g+f)h;$
3. the term multiplying $\frac{\partial C}{\partial S} \sigma \sqrt{h} S$ is
   $$\sum_{i=0}^{g-1} \binom{g-1}{i} p^i (1-p)^{g-1-i} \left[\pi(2i + f + 2 - g) + (1-\pi)(2i - f - g)\right] = \frac{hr}{\sigma\sqrt{h}}(f+g)$$
   and finally
4. the term multiplying $\frac{\partial^2 C}{\partial S^2} \sigma^2 h S^2$ is
   $\sum_{i=0}^{g-1} \binom{g-1}{i} p^i (1-p)^{g-1-i} \left[\pi(2i+f+2-g)^2 + (1-\pi)(2i-f-g)^2\right]$
   $= (f+1)^2 + g - 1$

Then, the partial differential equation can be rewritten as

$$-rC - \frac{\partial C}{\partial \tau} + \frac{\partial C}{\partial S} Sr + \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 \frac{(f+1)^2 + g - 1}{f+g} = 0$$

A.2 The PDE for the lower bound

Let g be the number of initial points in time with transactions, including $t = 0$; f is the number of points in time between two consecutive transactions, that is, $f + 1$ is the number of periods one have to wait to transact again the underlying asset; $p = \frac{R-D}{U-D}$ and $P' = \frac{R^T-D}{U-D}$. Then,

1. the term multiplying C is
   $\sum_{i=0}^{g-1} \binom{g-1}{i} p^i (1-p)^{g-1-i} - (1 + (g+f)hr) = -(g+f)hr;$

2. the term multiplying $\frac{\partial C}{\partial \tau}$ is

   $\sum_{i=0}^{g-1} \binom{g-1}{i} p^i (1-p)^{g-1-i} (-(g+f)h) = -(g+f)h;$

3. the term multiplying $\frac{\partial C}{\partial S}\sigma\sqrt{h}S$ is

   $$\sum_{i=0}^{g-1} \binom{g-1}{i} p^i (1-p)^{g-1-i} \left[\pi'(2i+2-g) + (1-\pi')(2i-g)\right] = \frac{hr}{\sigma\sqrt{h}}(f+g)$$

   and finally

4. the term multiplying $\frac{\partial^2 C}{\partial S^2}\sigma^2 h S^2$ is

   $\sum_{i=0}^{g-1} \binom{g-1}{i} p^i (1-p)^{g-1-i} \left[\pi'(2i+2-g)^2 + (1-\pi')(2i-g)^2\right] = g.$

Then, the partial differential equation satisfied by the call price is given by

$-rC - \frac{\partial C}{\partial \tau} + \frac{\partial C}{\partial S} Sr + \frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 \frac{g}{f+g} = 0.$