Susana Raquel Carvalho Ferreira
Mestre

Schottky principal G-bundles over compact Riemann surfaces

Dissertação para obtenção do Grau de Doutor em Matemática

Orientador: Prof. Doutor Carlos Armindo Arango Florentino
Professor Auxiliar
Instituto Superior Técnico da Universidade de Lisboa
Co-orientador: Prof. Doutora Ana Cristina Malheiro Casimiro
Professora Auxiliar
Universidade Nova de Lisboa

Júri:

Presidente: Professor Doutor Jorge Orestes Cerdeira
Arguente(s): Prof. Doutor Peter Beier Gothen
Prof. Doutor André Gama Oliveira

Vogais: Professor Doutor Miguel Tribolet de Abreu
Prof. Doutor João Nuno Gonçalves Faria Martins

FACULDADE DE CIÊNCIAS E TECNOLOGIA
UNIVERSIDADE NOVA DE LISBOA

Setembro de 2014

Schottky principal G-bundles over compact Riemann surfaces

Acknowledgements

I would like to thank my advisors Carlos Florentino and Ana Casimiro, who suggested the subject of this thesis. I thank them for their guidance, encouragement and patience during this work.

I am also grateful to Margarida Mendes Lopes for all the inspiration and enthusiasm in these matters and for her support through the projects that she applied.

I would like to thank all the members of GA-P for useful discussions and pleasant atmosphere in our meetings.

I thank the Polytechnic Institute of Leiria for my job as a mathematics professor and for the effort to reduce my weekly work schedule, which made our research possible.

Many thanks to my colleagues of the Mathematics Department of Management and Technology School of Polytechnic Institute of Leiria. A special thanks to my colleagues and friends Alexandra Baptista and Luís Cotrim for their support and fellowship.

I would like to thank my parents Irene e António, my husband Rui, my sons Pedro and João, my sister Sofia and my parents-in-law Antonieta and Jorge. Their love, support and encouragement provided me with the strength I needed to get through this project.

I thank my friends Andrea, Nuno, Ana, Miguel, Marília and Filipe for their support and friendship which helped me to overcome the dark days.

I gratefully acknowledge the financial support from FCT/Portugal through PhD grant SFRH/BD/37151/2007 and through projects PTDC/MAT/099275/2008; PTDC/MAT/119689/2010; PTDC/MAT/120411/2010; PTDC/MAT-GEO/0675/2012.

Abstract

In this thesis we study Schottky principal G-bundles over a compact Riemann surface X, where G is a connected reductive algebraic group. A Schottky G-bundle is defined as being a principal G-bundle induced by a representention ρ of the fundamental group $\pi_1(X)$ to G, such that, when we use the usual presentation for $\pi_1(X) = \left\{\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g | \prod \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} = 1\right\}$, the images $\rho(\alpha_i)$ are in the center Z of G for all $i = 1, \cdots, g$.

Using analogous methods to those of Ramanathan, we set up a correspondence between the categorical quotient (in the sense of GIT) of Schottky representations $\mathbb{S}$, and the set of equivalence classes of principal G-bundles. This correspondence can be restricted to a non-empty open subset $\mathbb{S}^\sharp$ of $\mathbb{S}$ to obtain a well-defined map $\mathbb{W} : \mathbb{S}^\sharp \to \mathcal{M}_G^{ss}$ (that we call the Schottky moduli map) in which $\mathcal{M}_G^{ss}$ denotes the moduli space of semistable principal G-bundles over X.

One of the main results of the thesis is the proof, based on the description of Ramanathan, that all Schottky G-bundles have trivial topological type. The second main result is the generalisation of the local surjectivity of the Schottky moduli map obtained in [**Flo01**] to the setting of principal G-bundles. More precisely, we show that the map $\mathbb{W} : \mathbb{S}^\sharp \to \mathcal{M}_G^{ss,0}$ is a local submersion around the *unitary* Schottky representations.

Finally, two simpler special cases are addressed: that of principal $\mathbb{C}^*$-bundles over a general Riemann surface, and the case of a general principal G-bundle over an elliptic curve. In both these cases, the Schottky map can be shown to be surjective onto the space of flat bundles.

Keywords: Representations of the fundamental group, character varieties, principal bundles, moduli spaces, compact Riemann surface.

Resumo

Nesta tese investigamos G-fibrados principais de Schottky sobre uma superfície de Riemann compacta X, onde G é um grupo algébrico redutivo e conexo. Um G-fibrado de Schottky é definido como sendo um G-fibrado principal induzido por uma representação do grupo fundamental de X, $\pi_1(X)$, em G tal que, quando usamos a apresentação usual $\pi_1(X) = \left\{\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g \middle| \prod \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} = 1\right\}$, as imagens $\rho(\alpha_i)$ estão contidas no centro Z de G para todo $i = 1, \cdots, g$.

Analogamente ao que foi feito por Ramanathan, construimos uma correspondência entre o quociente categórico (no sentido da Teoria geométrica dos invariantes) das representações de Schottky $\mathbb{S}$ e o conjunto das classes de equivalência de G-fibrados principais. Esta aplicação pode ser restringida a um aberto não vazio $\mathbb{S}^\sharp$ de $\mathbb{S}$ de modo a obter uma correspondência bem definida $\mathbb{W} : \mathbb{S}^\sharp \to \mathcal{M}_G^{ss}$ (a qual denominamos de aplicação moduli de Schottky), onde $\mathcal{M}_G^{ss}$ representa o espaço de moduli de G-fibrados semiestáveis sobre X.
Um dos resultados mais importantes deste trabalho é a demonstração, realizada tendo em conta a descrição de Ramanathan, de que todos os G-fibrados principais de Schottky têm tipo topológico trivial. O segundo resultado mais importante é a generalização do facto, obtido em [**Flo01**], de que a aplicação moduli de Schottky é localmente sobrejectiva, para o caso de G-fibrados principais. Mais precisamente, provamos que a aplicação $\mathbb{W} : \mathbb{S}^\sharp \to \mathcal{M}_G^{ss,0}$ é uma submersão local em torno de representações unitárias de Schottky.

Para finalizar analisamos dois casos particulares. O caso de $\mathbb{C}^*$-fibrados principais de Schottky sobre uma superfície de Riemann compacta geral e o caso de G-fibrados principais sobre a curva elítica. Em ambas as situações, podemos demonstrar que a aplicação de Schottky é sobrejectiva no espaço de fibrados planos.

Palavras Chave: Representações do grupo fundamental, variedade de caracteres, fibrados principais, espaços moduli, superfícies de Riemann compactas.

Contents

Introduction

The uniformization theorem gives a simple uniform parameterization of all Riemann surfaces X of genus $g \geq 2$. It states that each one can be written as a quotient of the upper half-plane by a Fuchsian group Γ, that is, $X \cong \mathbb{H}/\Gamma$ where the group Γ is isomorphic to the fundamental group $\pi_1(X)$. In addition, the 'retrosection theorem' asserts that X can be written as Ω/Γ_s where Γ_s is a Schottky group with a region of discontinuity $\Omega \subset \mathbb{CP}^1$. In particular, Γ_s is a free group F_g of rank g.
Passing from Riemann surfaces to flat bundles over Riemann surfaces, it is natural to question ourselves if we can obtain an analogous parameterization.
Indeed, in their papers [**NS65, NS64**], Narasimhan and Seshadri proved that every semistable vector bundle V over X is induced by a unitary representation $\rho : \pi_1(X) \to U(n) \subset GL(n, \mathbb{C})$, where

$$\pi_1(X) = \left\{\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g \middle| \prod \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} = 1\right\}$$

is the usual presentation of the fundamental group of X. More precisely one has the isomorphism

$$V_\rho \cong \tilde{X} \times \mathbb{C}^n \Big/ \pi_1(X)$$

where the fundamental group $\pi_1(X)$ acts diagonally, the action on the universal cover $\widetilde{X}$ being the natural one, and the action on $\mathbb{C}^n$ comes from the standard action of $GL(n, \mathbb{C})$ on $\mathbb{C}^n$ composed with the representation ρ.
Ramanathan generalised Narasimhan and Seshadri's results to principal G-bundles where G is any reductive algebraic group over $\mathbb{C}$. Following Mumford, he began by adjusting the notion of stability for this case and then constructed the moduli space of stable principal G-bundles over a compact Riemann surface. For principal G-bundles

E_ρ coming from representations ρ of the fundamental group, Ramanathan established that if ρ is unitary (and irreducible), $E_\rho \cong \tilde{X} \times G \big/ \pi_1(X)$ is semistable (respectively, E_ρ is stable).

Taking the results obtained by Narasimhan and Seshadri into account, Florentino [**Flo01**] hoped to describe the space of vector bundles coming from Schottky representations, which corresponds to homomorphisms ρ from the fundamental group $\pi_1(X)$ to $GL(n, \mathbb{C})$ such that the images of the generators α_i, $i = 1, \cdots, g$, $\rho(\alpha_i)$, are sent to the identity element of $GL(n, \mathbb{C})$. In fact, he showed the existence of an open set of the moduli space of flat vector bundles which consists of Schottky vector bundles.

In this thesis, we consider the results achieved by Ramanathan [**Ram75, Ram96**] for principal G-bundles and the ones obtained by Florentino [**Flo01**], and our aim is to extend the results of the later one to the case where G is a connected reductive algebraic group. We study the corresponding structures when Schottky representations are considered as homomorphisms of $\pi_1(X)$ into G such that the g generators α_i's of $\pi_1(X)$ are sent to the center Z of G. Denoting the set of all Schottky representations by $\mathcal{S}$, we prove that the algebraic variety $\mathcal{S}$ is isomorphic to the variety of homomorphisms of the free group F_g into the algebraic group $G \times Z$ (Proposition 2.4). Since conjugated representations define isomorphic G-bundles, we consider the corresponding categorical quotient

$$\mathbb{S} = \mathcal{S} /\!/ G \cong \mathrm{Hom}(F_g, G \times Z) /\!/ G$$

which has the structure of an affine complex algebraic variety. By associating a principal G-bundle to a representation as above, we can define the generalised Schottky map as a map $\mathbb{W} : \mathbb{S} \to M_G$, where M_G denotes the set of isomorphism classes of principal G-bundles.

Using Ramanathan's characterisation of the topological type of a principal G-bundles, we show that all Schottky G-bundles have trivial topological type (Theorem5.11).

The isomorphism between the tangent space to $\mathbb{S}$ at a good representation (def. 1.24(3)) and the first cohomology group with coefficients on a F_g-module

$$T_{[\rho]}\mathbb{S} \cong H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) \oplus \mathfrak{z}^g,$$

where $\mathfrak{g}_{\mathrm{Ad}_{\rho_1}}$ represents the F_g-module in the Lie algebra of G and $\mathfrak{z}$ is the Lie algebra of Z, allows us to compute the dimension of $\mathbb{S} = \mathcal{S}/\!\!/G$.

Now, let $\mathcal{M}_G^{ss}$ denote the moduli space of semistable principal G-bundles on X. To describe the subspace of $\mathcal{M}_G^{ss}$ consisting of Schottky G-bundles, consider the map

$$\mathbb{W} : \mathbb{S}^\sharp \to \mathcal{M}_G^{ss},$$

obtained from the Schottky map, where $\mathbb{S}^\sharp := \mathbb{W}^{-1}\left(\mathcal{M}_G^{ss}\right)$. As before, this map assigns to each equivalence class $[\rho]$ the corresponding class $[E_\rho]$ of principal G-bundle. We prove the following Theorem, which generalises the case of vector bundles.

THEOREM. *7.7 Let G be a connected reductive algebraic group and let ρ be a good and unitary Schottky representation. Then, the derivative of the Schottky map $d\left(\mathbb{W}\right)_\rho : T_{[\rho]}\mathbb{S} \to \mathcal{M}_G^{ss}$ has maximal rank. In particular, the Schottky map $\mathbb{W} : \mathbb{S}^\sharp \to \mathcal{M}_G^{ss}$ is a local submersion. This means that locally around ρ, the map is a projection with* $\dim\left(\mathbb{W}^{-1}\left([E_\rho]\right)\right) = g \dim Z^\circ$.

In the last part of this thesis, we analyse two special cases, principal bundles with group $G = GL(1, \mathbb{C}) = \mathbb{C}^*$ over a Riemann surface X and principal bundles over an elliptic curve. For elliptic curves we obtain an analog result to the one of Florentino which is the following one.

THEOREM. *8.9 Let X be an elliptic curve and let G be a connected reductive algebraic group. Then E is a flat principal G-bundle over X if and only if E is Schottky.*

In the case of Schottky $\mathbb{C}^*$-bundles, the Schottky moduli map, has a different description from Florentino's. However, we obtain the same correspondence between flatness and Schottky property, as we can see in to following Proposition.

Proposition. *8.1 Given a principal $\mathbb{C}^*$-bundle E over a compact Riemann surface X then E is flat if and only if it is Schottky.*

We can outline the contents of the thesis as follows.

Chapter 1 contains a review of some basic definitions and results that will be needed in the forthcoming chapters. In this chapter there are no original results and all relevant references are provided. In section one, we give a background about algebraic groups. Section two is dedicated to the Geometric Invariant Theory and to the character varieties. Most of these results come from Mumford's work. In the third section we introduce the basic definitions related to cohomology groups. Our purpose in the remaining sections of this chapter, is to give all basic notions related to fibre bundles theory and moduli spaces of semistable bundles, focusing mainly on principal bundles. We prove some of the basic results that we had not found in the literature.

In Chapter 2 we define the concept of Schottky representation and we prove that the set of all Schottky representations $\mathcal{S}$ coincides with the variety $\mathrm{Hom}\,(F_g,\, G \times Z)$ consisted of all representations from the free group into the algebraic group $G \times Z$. Moreover, we prove the existence of a geometric quotient $\mathcal{S}/\!\!/G$ and, since this quotient is usually not irreducible, we study its connected components. We finish this chapter with the proof of the existence of good and unitary Schottky representations.

Chapter 3 provides the definition of the Schottky principal bundle and the proof of the fact that, when we consider associated bundles, the property of being Schottky is transferred between each other under certain conditions.

In Chapter 4 we explore properties of the Schottky map. We start by proving that isomorphic Schottky bundles correspond to analytic equivalent representations. Then we define the Schottky moduli map from the categorical quotient of a restricted Schottky representations set to the moduli space of semistable G-bundles. Furthermore in this chapter we make our first approach to obtain our main goal, that is to prove that there is an open subset of equivalent classes of principal bundles induced by Schottky representations.

In Chapter 5 it is defined the concept of the topological type of a G-bundle. It is shown that this topological invariant is related with the universal cover $\widetilde{G}$ of G and the concept of representation type concerning a homomorphism $\rho : \pi_1 \to G$ such that $E \cong E_\rho$. This chapter ends with an important result stating that every Schottky G-bundle over a compact Riemann surface X ($g \geq 2$) is on the connected component containing the trivial bundle of the set consisted of isomorphic classes of G-bundles over X.

In Chapter 6, we compute the dimension of the categorical quotients $\mathbb{S} := \mathcal{S}/\!\!/G$ (resp. $\mathbb{G} := \mathrm{Hom}\left(\pi_1\left(X\right),\, G\right)/\!\!/G$) using the relation between tangent spaces $T_{[\rho]}\mathbb{S}$ (resp. $T_{[\rho]}\mathbb{G}$) with the first cohomology group $H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}\rho_1} \oplus \mathfrak{z}\right)$ (resp. $H^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}\rho}\right)$) where $\mathfrak{g}_{\mathrm{Ad}\rho}$ denotes a F_g- module (resp. π_1-module). It was therefore necessary to compute the dimensions of the cohomology groups $H^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}\rho}\right)$ and $H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}\rho_1} \oplus \mathfrak{z}\right)$.

Using the fact that the subset $\mathcal{M}_G^{sm}$ of the smooth part of the moduli space of equivalent classes of semistable G-bundles is nonempty and open in $\mathcal{M}_G^{ss}$, we prove that the inverse images $\mathbb{G}^\sharp := \mathbb{E}^{-1}\left(\mathcal{M}_G^{sm}\right)$ and $\mathbb{S}^\sharp := \mathbb{W}^{-1}\left(\mathcal{M}_G^{sm}\right)$ are both nonempty. Moreover, we prove that the (Schottky moduli) maps $\mathbb{E}. : \mathbb{G}^* \to \mathcal{M}_G^{sm}$ and $\mathbb{W}. : \mathbb{S}^* \to \mathcal{M}_G^{sm}$ are well defined and that, in particular situations, it is surjective.

Chapter 7 describes procedures for computing the local derivative of the Schottky moduli map at a good representation. We define an hermitian inner product over $H^0\left(X,\, \mathrm{Ad}E_\rho \otimes \Omega_X^1\right)$ which allow us to prove that the local derivative of the Schottky moduli map is an isomorphism when G is semisimple. In addition, if G is reductive, we prove that the Schottky moduli map is a submersion.

In Chapter 8, we comment two particular cases of Schottky principal bundles. One of them is related with the elliptic curve ($g = 1$) which is excluded in the most of the chapters of this thesis. The other one is about principal G-bundles where G is a one dimensional reductive group.

CHAPTER 1

Preliminaries

This chapter introduces some basic concepts that are needed in the forthcoming chapters. First we provide some notions related with the representations of algebraic groups and character varieties. Thereafter, we bring up some fibre bundle theory, mainly, vector and principal bundles over a compact Riemann surface are analysed in more detail.

The definitions and the properties displayed in this chapter are not original. Thus, all the appropriate references are given.

1.1. Algebraic groups

In this section we introduce some definitions and relations concerning to algebraic groups and the corresponding Lie algebras. More details about this topic are available in [**TY05, OV90, Hum75**] among others.
We start by giving general definitions related with algebraic groups over complex numbers, despite of many of them being valid for any algebraically closed field.

Definition 1.1. A complex **algebraic group** G is an algebraic variety over $\mathbb{C}$ and a group such that the following maps

- $\mu : G \times G \to G$ which corresponds to $\mu(g,h) = gh$;
- $i : G \to G$ which corresponds to $i(g) = g^{-1}$;

are morphisms of algebraic varieties. An algebraic group G has an element e such that $\mu(e,g) = g,\ \forall g \in G$ and $i(e) = e$, which is called the identity element of G.

Example 1.2. The special linear group $SL(n,\mathbb{C})$ is a subset of $n \times n$ matrices A with complex entrances satisfying $\det(A) = 1$. This condition is a polynomial equation and $SL(n,\mathbb{C})$ is a group with multiplication and inversion as operations which are polynomial maps. Thus, $SL\,(n,\mathbb{C})$ is an algebraic group.

DEFINITION 1.3. A **homomorphism** $\phi : G \to H$ of algebraic groups is a morphism of varieties which is simultaneously a group homomorphism. Moreover, the map ϕ is an **isomorphism** of algebraic groups if there exists a homomorphism $\psi : H \to G$ such that $\phi \circ \psi = id_H$ and $\psi \circ \phi = id_G$. Over a base field with zero characteristic (like $\mathbb{C}$) any bijective homomorphism of algebraic varieties is an isomorphism.

There is an important type of homomorphisms between groups that leads us to the notion of rational representation.

DEFINITION 1.4. A homomorphism $\phi : G \to GL(V)$ where V is a complex vector space is called a **rational representation** of an algebraic group G in the space V. In this case, the space V is a **finite dimensional rational G-module**. A rational representation is **faithful** if $\ker(\phi) = e$.

THEOREM 1.5. *Every algebraic group G over the complex numbers is isomorphic to a closed subgroup of some $GL\,(n,\ \mathbb{C})$.*

According to this Theorem, all complex algebraic groups are linear. In this way, we can use the matrix notation whenever we have to do some computations, which allows us to simplify some proofs related with algebraic groups.

DEFINITION 1.6. A **left action** (resp. **right action**) of the algebraic group G on a variety V is a morphism $\phi : G \times V \to V$ such that

$$(g,\, v) \quad \mapsto \quad \phi\,(g,v) = g \cdot v$$

(resp. $v \cdot g$) satisfying

(1) $v \cdot e = v$ (resp. $e \cdot v = v$) , $\quad \forall v \in V$

(2) $(v \cdot h) \cdot g = v \cdot (hg)$ (resp. $g \cdot (h \cdot v) = (gh) \cdot v$) , $\quad \forall v \in V, \quad g, h \in G$.

If such action exists, we say that G **acts** on V or that V is a **G-variety**.

Let $\phi : G \times V \to V$ be an action of the algebraic group G on the variety V. The set $\{g \in G | \, g \cdot v = v\}$ is called **stabiliser** or **isotropy group** of $v \in V$ and is denoted by G_v or by $Z_G(v)$. The set $\{g \cdot v | \; g \in G\}$ is called **orbit** of v and it is denoted by Gv or by $\mathcal{O}_v$.

An action ϕ is said to be:

- **transitive** if for any v, $w \in V$ there exists $g \in G$ such that $w = g \cdot v$.
- **free** if for each $g \in G$ and for any $v \in V$ with $g \cdot v = v$ then $g = e$.

We may consider an action $\phi : G \times H \to H$ where H is a subgroup of G. According to the previous notions, we have the following definitions.

Definitions 1.7. *The subgroup given by $\{g \in G| \, gh = hg, \forall h \in H\}$ is called* ***centraliser*** *of the subset H on G and the subgroup $Z(G)$ defined by $\{g \in G| \, hg = gh, \;\; \forall h \in G\}$ is the* ***centre*** *of G.*

The ***connected component of the identity*** *of G is denoted by G° and analogously, $Z(G)^\circ$ stands for the* ***connected component of the identity of the centre*** *of G.*

An algebraic group G is said to be ***connected*** *if $G = G^\circ$.*

Proposition 1.8. *Let G be a nontrivial connected algebraic group. Then it contains a unique maximal closed connected normal solvable subgroup called* ***radical*** *of G and denoted by $R(G)$, and contains a unique maximal closed connected unipotent normal subgroup which is called* ***unipotent radical*** *and denoted by $R_u(G)$:*

(1) if $R(G)$ is trivial, G is said to be ***semisimple****.*

(2) if $R(G) = Z(G)^\circ = (\mathbb{C}^*)^n$ or, equivalently, if $R_u(G)$ is trivial, G is said to be ***reductive****.*

Definitions 1.9. *A subgroup $T \subset G$ is called* **n*-dimensional torus*** *if it is isomorphic to $(\mathbb{C}^*)^n$. The torus T is called maximal if it is a torus and there is no other torus T' with $T \subset T' \subset G$. A subgroup $B \subset G$ is called* ***Borel*** *if it is maximal among the (Zariski) closed connected solvable subgroups of G. A subgroup P of G is called* ***parabolic subgroup*** *of G if it is a closed subgroup such that G/P is a complete variety.*

Parabolic subgroups are all closed subgroups between Borel subgroups and the group G. A subgroup K of G is called **compact** *if it is an algebraic subgroup whose topology is compact Hausdorff.*

If G is a reductive algebraic group we have the following theorem asserting that there exists always a **maximal compact subgroup** K of G such that it is Zariski dense in G and such that its complexification $K_{\mathbb{C}}$ coincides with G. As we will see latter, this fact allows us to write the elements of G in a particular form, usually called polar decomposition.

THEOREM 1.10. *A complex algebraic group G is reductive if and only if it is the complexification $K_{\mathbb{C}}$ of a compact Lie group K.*

EXAMPLES 1.11.
- $G = GL(n, \mathbb{C}) = \left\{(a_{ij}, b) \in \mathbb{C}^{n^2+1} : b \det(a_{ij}) = 1\right\}$ *is a reductive algebraic group.*
  The centre of G is $Z(GL(n, \mathbb{C})) = \{\lambda I_n : \lambda \in \mathbb{C}^*\}$. The subgroup of diagonal matrices in G is a maximal torus of G. And the maximal compact subgroup K of $GL(n, \mathbb{G})$ is $U(n)$, the group of unitary matrices.
- $G = SL(n, \mathbb{C}) = \left\{(a_{ij}) \in \mathbb{C}^{n^2} : \det(a_{ij}) = 1\right\}$ *is a semisimple algebraic group. The centre is $Z(SL(n, \mathbb{C})) = \{\lambda I_n : \lambda^n = 1$ and $\lambda \in \mathbb{C}^*\}$ and the maximal compact subgroup K is $SU(n)$, the group of special unitary matrices.*

THEOREM 1.12. *Let G be an algebraic group. The following conditions are equivalent:*

1. *the group G is reductive;*
2. *the radical $R(G)$ is a torus;*
3. *the connected component $G^\circ = T \cdot G'$ where T is a torus and G' is a connected semisimple subgroup;*
4. *any finite dimensional rational representation of G is completely reducible;*
5. *the group G admits a faithful finite dimensional completely reducible rational representation.*

REMARK. If the algebraic group G is connected and reductive we can write G as an almost direct product $G = Z^\circ \rtimes G'$ where $G' = [G, G]$ is a semisimple algebraic group and the connected component of the center of G, Z°, is a torus.

THEOREM 1.13. *Any complex algebraic group is a complex Lie group with the same dimension.*

The previous theorem allows us to interchange from the category of algebraic groups to the category of Lie groups.

DEFINITION 1.14. Let G be an algebraic group over $\mathbb{C}$. We define the **Lie algebra** of G as the tangent space of G at the identity e, $\mathfrak{g} := T_e(G)$ endowed with a skew-symmetric bilinear map $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity $([X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0)$.

THEOREM 1.15. *Let $\psi : G \to H$ be a homomorphism of algebraic groups. Then its differential at identity e is given by*

$$d\psi_e : T_eG := \mathfrak{g} \to T_eH := \mathfrak{h}$$

and it is a homomorphism of Lie algebras.

For more details about this theorem, see for example [chap. III, [**Hum75**]].

DEFINITION 1.16. Given a maximal compact subgroup K of G then the Lie algebra $\mathfrak{g}$ of G can be written in the following way $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ where $\mathfrak{k}$ denotes the Lie algebra of K. Each element g of the connected reductive group G can be decomposed in the following way

$$g = k.\exp(Y) = k \cdot p \tag{1.1.1}$$

where $k \in K$, $p = \exp(Y)$ with $Y \in i\mathfrak{k}$. This decomposition is called **polar decomposition** of g.

There is an important action of an algebraic group G into itself which is called **conjugation** on G and it is defined as follows

$$\begin{array}{rccl} c: & G \times G & \to & G \\ & (g,h) & \mapsto & c_g h := g \cdot h = ghg^{-1}. \end{array}$$

For each $g \in G$ we define the following morphism

$$\begin{array}{rccl} c_g: & G & \to & G \\ & h & \mapsto & c_g h = ghg^{-1}. \end{array}$$

If we compute the differential of c_g at the identity, we obtain the adjoint map, that is, the map given by

$$\begin{array}{rccl} Ad_g := d_e(c_g): & T_eG & \to & T_gG \cong T_eG \\ & X & \mapsto & Ad_g X. \end{array} \tag{1.1.2}$$

We can write this map as an action of the algebraic group G in its corresponding Lie algebra

$$\begin{array}{rccl} \mathrm{Ad}: & G \times \mathfrak{g} & \to & \mathfrak{g} \\ & (g,X) & \mapsto & \mathrm{Ad}_g X = gXg^{-1}. \end{array} \tag{1.1.3}$$

Proposition 1.17. *Let G be a connected algebraic group and let $\mathfrak{g}$ be the corresponding Lie algebra of G. The* ***adjoint representation*** $\mathrm{Ad}: G \to GL(\mathfrak{g})$*, which associates to each $g \in G$ a map Ad from the Lie algebra $\mathfrak{g}$ of G to itself, is a morphism of algebraic groups and it satisfies the following relations*

1. *If $g \in G$,* $\mathrm{Lie}\,(Z_G(g)) = \{X \in \mathfrak{g} | \mathrm{Ad}_g(X) = X\}$.
2. *Its kernel,* $\ker \mathrm{Ad}$*, coincides with the center $Z(G)$ and its Lie algebra is* $\ker \mathrm{ad} = \mathrm{Lie}(Z(\mathfrak{g}))$ *where $Z(G)$ denotes the centre of G and* ad *defines the differential of* Ad *at the identity.*

1.2. Character variety

The main goal of this section is to introduce the notions and properties of character varieties. Hence, it is provided some important results involving these objects which will be needed in the forthcoming chapters.

Throughout this section, Γ and G denote a finitely generated group and a connected reductive complex algebraic group, respectively, and $\rho:\Gamma\to G$ means a representation (homomorphism) from Γ to G.

DEFINITION 1.18. The set of all representations from Γ to G is called **representation variety** and it is denoted by $R(\Gamma,\ G) := \mathrm{Hom}(\Gamma, G)$. This set has the structure of an affine algebraic variety.

Let us start by seeing that $R(\Gamma, G)$ has the structure of algebraic variety. Any representation $\rho\in R(\Gamma,G)$ is defined by the image of each element that generates Γ. More precisely, we can define the following embedding

$$\begin{array}{ccc} R(\Gamma,\ G) & \hookrightarrow & G^N \\ \rho & \mapsto & (\rho(\gamma_1),\ \cdots,\ \rho(\gamma_N)) \end{array} \tag{1.2.1}$$

where $\Gamma=\langle\gamma_1,\cdots,\gamma_N|\mathcal{W}\rangle$ and $\mathcal{W}$ is a collection of restrictions involving the elements γ_i's. The set $R(\Gamma,G)$ is a subset of G^N and each restriction $\mathcal{W}$ provides, by above map, algebraic constrains in G^N. Thus, $R(\Gamma,G)$ is an algebraic subvariety of G^N. Furthermore, we can define an action of G on $R(\Gamma,G)$ by conjugation as follows

$$\begin{array}{ccc} G\times R(\Gamma,G) & \to & R(\Gamma,G) \\ (g,\rho) & \mapsto & g\cdot\rho = g\rho g^{-1}. \end{array}$$

As it was defined in the previous section, the set given by $Z_G(\rho)=\{g\in G|g\cdot\rho=\rho\}$ is an algebraic subgroup of G and it is called **stabiliser** of ρ.

DEFINITION 1.19. The set defined by $O_\rho=\{\,g\cdot\rho|\,g\in G\}$ is a subvariety of $\mathrm{Hom}(\Gamma,G)$ and it is called **orbit** of ρ or **conjugacy class** of ρ.

When X is a closed orientable surface of genus g (for example a compact Riemann surface), the fundamental group $\pi_1(X)$ is a finitely generated group generated by $2g$ elements that satisfies a particular condition, that is,

$$\pi_1(X)=\left\{\alpha_1,\cdots,\alpha_g,\beta_1,\cdots,\beta_g|\prod\alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}=1\right\}.$$

Hence, we can consider previous constructions replacing Γ by $\pi_1(X)$.

EXAMPLE 1.20. Let us denote by π_1 the fundamental group of X. Putting $\Gamma = \pi_1$ and $G = GL\,(n, \mathbb{C})$, we can construct representations $\rho : \pi_1 \to GL(n, \mathbb{C})$ such that assigns to each element $\gamma \in \pi_1$ a matrix $\rho\,(\gamma)$. The group $GL(n, \mathbb{C})$ acts by conjugation on the set of representations $R\,(\pi_1,\, GL\,(n,\, \mathbb{C}))$. More precisely, given $\rho \in R\,(\pi_1, GL\,(n,\, \mathbb{C}))$ and for each $g \in GL\,(n, \mathbb{C})$, the action $g \cdot \rho$ is given by

$$g \cdot \rho(\gamma) = g\rho(\gamma)g^{-1}.$$

The quotient space $R\,(\pi_1, GL\,(n,\, \mathbb{C}))/\, GL\,(n,\, \mathbb{C})$ consists of equivalence classes of representations. Namely, $\rho_2 \in [\rho_1]$ means

$$\rho_1 \sim \rho_2 \Leftrightarrow \exists g \in GL\,(n,\, \mathbb{C}) :\; \rho_2 = g\rho_1 g^{-1}.$$

In general, the quotient $\mathrm{Hom}(\Gamma, G)/\, G$ is **not Hausdorff**. This means that it may happen that $\rho_1 \nsim \rho_2$ and the respective neighbourhoods are not disjoint ($\overline{\mathcal{O}_{\rho_1}} \cap \overline{\mathcal{O}_{\rho_2}} \neq \varnothing$).

As an example, let us look to the following case given by Goldman [**Gol84b**].

EXAMPLE 1.21. Let π_1 represent the fundamental group of a surface of genus 2 and $G = SL(2, \mathbb{R})$. Consider representations $\rho \in \mathrm{Hom}\,(\pi_1,\, SL(2,\, \mathbb{R}))$ such that $\rho(\alpha_2) = \rho(\beta_1) = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} =: g$ for a fixed $a > 1$ and $\rho(\beta_2) = \rho(\alpha_1)$. Take two representations ρ_1 and ρ_2 from this set and consider $\rho_1(\alpha_1) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\rho_2(\alpha_1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. It is clear that they belong to different conjugacy classes but they cannot be separated. Consider the following sequence of representations ρ_n

$$\rho_n(\alpha_1) = \begin{bmatrix} (1 + a^{-2n})^{\frac{1}{2}} & a^{-2n} \\ 1 & (1 + a^{-2n})^{\frac{1}{2}} \end{bmatrix}.$$

It is clear that, as $n \to +\infty$, $\rho_n(\alpha_1) \to \rho_1$ and $g^n \rho_n(\alpha_1) \to \rho_2$ meaning that ρ_1 and ρ_2 cannot be in disjoint neighbourhoods.

Despite the problem of $R(\Gamma, G)/G$ not being Hausdorff, it is possible to construct a nicer quotient of $R(\Gamma, G)$ by G with better properties like being Hausdorff. The construction of this quotient follows from **[New78, MFK94]**. Explaining briefly, the action of G on the variety $R(\Gamma, G)$ induces a homomorphism of groups $G \to GL(k[R(\Gamma, G)])$ where $k[R(\Gamma, G)] = \{f : R(\Gamma, G) \to \mathbb{C} \text{ regular }\}$ is the ring of regular functions. This is equivalent to define the following action

$$\begin{array}{rcl} k[R(\Gamma, G)] \times G & \to & k[R(\Gamma, G)] \\ f, g & \mapsto & f^g = f(g^{-1} \cdot \rho). \end{array}$$

Let $k[R(\Gamma, G)]^G = \{f \in k[R(\Gamma, G)] \,|\, f^g = f, \forall g \in G\}$ denote the **ring of invariants**. Since G is reductive, Nagata's theorem asserts that the ring $k[R(\Gamma, G)]^G$ is finitely generated. This corresponds to the existence of a variety Ysuch that $k[\mathrm{Hom}(\Gamma, G)]^G = k[Y]$. If such variety Y exists then it is called **geometric quotient**.

Let us, now, give the formal definitions related with geometric quotients.

Definition 1.22. **[Ses72]** A **categorical quotient** is a pair (Y, p) where Y is a variety and $p : X \to Y$ is a G-invariant morphism that satisfies the universal property for quotients. This means that if there exists an invariant G-morphism $\varphi : X \to Z$ then there exists a unique morphism $\psi : Y \to Z$ (up to isomorphism) such that $\varphi = \psi \circ p$.

The pair (Y, p) is called a **good quotient** if

i. $p : X \to Y$ is a surjective G-invariant morphism;
ii. $p_* \left(\mathbb{C}[X]^G\right) = \mathbb{C}[Y]$
iii. if W is closed G-stable subset of X, then $p(W)$ is closed in Y.

A pair (Y, p) is called a **geometric quotient** if

i. $p : X \to Y$ is a good quotient;
ii. $\forall x_1, x_2 \in X, \; p(x_1) = p(x_2) \Leftrightarrow \mathcal{O}_{x_1} = \mathcal{O}_{x_2}$ (i.e., Y is an orbit space).

As previously stated, geometric quotient is an important attribute since this ensures that if two orbits intersects then they are the same (Hausdorff).

Given a G-variety X, Rosenlicht's theorem asserts the existence of an open G-stable subset of X such that it has a geometric quotient.

Definition 1.23. The geometric quotient Y of $R(\Gamma, G)$ by G, usually denoted by $R(\Gamma, G) /\!\!/ G$, has a natural structure of an affine algebraic set. This variety is called G-**character variety** of Γ and it is, usually, denoted by $C(\Gamma, G)$ or $X_G(\Gamma)$.

The inclusion $k[R(\Gamma, G)]^G \hookrightarrow k[R(\Gamma, G)]$ induces a morphism $\pi_\Gamma : \mathrm{Hom}(\Gamma, G) \twoheadrightarrow C(\Gamma, G)$ such that it is surjective. According to the universal property of quotients, if there is a G-invariant morphism ϕ from $\mathrm{Hom}(\Gamma, G)$ to a variety Z, $\phi(g\rho) = \phi(\rho)$, for all $g \in G$ and all $\rho \in \mathrm{Hom}(\Gamma, G)$, then there exists a unique $\varphi : C(\Gamma, G) \to Z$ such that $\phi = \varphi \pi_\Gamma$.

$$\begin{array}{ccc} \mathrm{Hom}(\Gamma, G) & \xrightarrow{\ \phi\ } & Z \\ \downarrow{\scriptstyle \pi_\Gamma} & \nearrow{\scriptstyle \varphi} & \\ C(\Gamma, G) & & \end{array}$$

The morphism π_Γ maps closed invariant sets of $\mathrm{Hom}(\Gamma, G)$ to closed sets of $C(\Gamma, G)$ and given a point of $C(\Gamma, G)$, $\pi_\Gamma(\rho)$, we have that $\pi_\Gamma^{-1}(\pi_\Gamma(\rho))$ is a closed set of conjugacy classes. This ensures the closure property, although this variety may still have singular or not reduced points. In this way, it is important to define some representations attributes in order to obtain nonsingular or irreducible subvarieties.

Definition 1.24. The representation $\rho : \Gamma \to G$ is called

(1) **irreducible** if $\rho(\Gamma)$ is not contained in any proper parabolic subgroup of G. The set of irreducible representations will be denoted by $\mathrm{Hom}(\Gamma, G)^i$ and the corresponding image in $C(\Gamma, F)$ will be denoted by $C(\Gamma, G)^i$.

(2) **unitary** if $\rho(\Gamma) \subset K$ where K is a maximal compact subgroup of G.

(3) **good** if ρ is irreducible and $Z_G(\rho) = Z(G)$. The set of good representations will be denoted by $\mathrm{Hom}(\Gamma, G)^{\mathfrak{g}}$ and the corresponding image in $C(\Gamma, F)$ will be denoted by $C(\Gamma, G)^{\mathfrak{g}}$.

(4) **stable** if there is a Zariski open neighbourhood of ρ on $\mathrm{Hom}(\Gamma, G)$ preserved by G on which the G action is closed and $Z_G(\rho)/\, Z(G)$ is finite. That is, a

representation ρ is said to be stable if its orbit in $\mathrm{Hom}(\Gamma, G)/G$ is closed and $Z_G(\rho)/\,Z(G)$ is finite.

1.3. Group Cohomology for finitely generated groups

Let $\Gamma = \langle \{\beta_1, \cdots, \beta_n | \, r_1, \cdots, r_k\} \rangle$ be a finitely generated group. Suppose that Γ acts on a vector space V, in the following way

$$\Gamma \times V \to V$$
$$\gamma, v \mapsto \gamma \cdot v.$$

Hence, V gets a structure of Γ-module and we may define the following sets:

- the set $C^i(\Gamma, V) = \{f : \Gamma^i \to V\}$ of functions, from $\Gamma^i = \underbrace{\Gamma \times \cdots \times \Gamma}_{i \text{ times}}$ to V, is called the group of i-**cochains** of Γ with coefficients in V. If $i = 0$ we put $C^0(\Gamma, V) = V$.
- the cochain map $d^i : C^i(\Gamma, V) \to C^{i+1}(\Gamma, V)$ defined by

$$\begin{aligned} d^i(f)\,(\gamma_0, \cdots, \gamma_i) = {} & \gamma_0 \cdot f(\gamma_1, \cdots, \gamma_i) \\ & + \sum_{j=1}^{i} (-1)^j f\,(\gamma_0, \cdots, \gamma_{j-2}, \gamma_{j-1}\gamma_j, \gamma_{j+1}, \cdots, \gamma_i) \\ & + (-1)^{i+1} f(\gamma_0, \cdots, \gamma_{i-1}). \end{aligned}$$

  With some computations it can be shown that $d^{i+1} \circ d^i = 0$.
- the set $Z^i(\Gamma, V) = \ker d^i$ is the group of i-**cocycles** of Γ with coefficients in V.
- the set $B^i(\Gamma, V)$ defined by

$$B^i(\Gamma, V) = \begin{cases} 0, & i = 0 \\ \mathrm{Im}\ d^{i-1}, & i \geq 1 \end{cases}$$

  is the group of i-**coboundaries** of Γ with coefficients in V. Since $d^{i+1} \circ d^i = 0$, it is clear that $B^i(\Gamma, V) \subset Z^i(\Gamma, V)$.
- the quotient $H^i(\Gamma, V) = Z^i(\Gamma, V)/\,B^i(\Gamma, V)$ is called i**th cohomology group** of Γ with coefficients in V.

EXAMPLE 1.25. Let us compute the cases in which $i=0$ and $i=1$.

$$\begin{aligned}
C^1(\Gamma,V) &= \{f:\Gamma\to V\} \text{ and } d^0f(\gamma)=\gamma\cdot v-v\\
Z^0(\Gamma,V) &= \ker d^0=\left\{f\in C^0(\Gamma,V)|d^0f(\gamma)=0\Leftrightarrow\gamma\cdot v=v\right\}=V^\Gamma\\
B^1(\Gamma,V) &= \{f:\Gamma\to V|\exists a\in V:\quad f(\beta)=\beta\cdot a-a,\ \forall\beta\in\Gamma\}\\
C^2(\Gamma,V) &= \{f:\Gamma\times\Gamma\to V\} \text{ and } d^1f(\gamma_0,\gamma_1)=\gamma_0\cdot f(\gamma_1)-f(\gamma_0\gamma_1)+f(\gamma_0)\\
Z^1(\Gamma,V) &= \{f:\Gamma\to V|\gamma_0\cdot f(\gamma_1)-f(\gamma_0\gamma_1)+f(\gamma_0)=0\}\\
&= \{f:\Gamma\to V|f(\gamma_0\gamma_1)=\gamma_0\cdot f(\gamma_1)+f(\gamma_0)\}
\end{aligned}$$

LEMMA 1.26. *(1) The 0-cohomology group $H^0(\Gamma,V)$ is equal to V^Γ, the group of Γ-invariants of V.*

(2) If V is a trivial Γ-module, then $H^1(\Gamma,V)=\mathrm{Hom}(\Gamma,V)$.

(3) In general, the first cohomology group is given by

$$H^1(\Gamma,V)=\frac{\{f:\Gamma\to V|\,f(\alpha\beta)=\alpha\cdot f(\beta)+f(\alpha)\quad\forall\alpha,\beta\in\Gamma\}}{\{f:\Gamma\to V|\,\exists a\in V:\quad f(\beta)=\alpha\cdot a-a\quad\forall\beta\in\Gamma\}}.$$

EXAMPLE 1.27. Let Γ be a finite generated group and let G be a connected reductive algebraic group. Given a representation $\rho:\Gamma\to G$ we may consider the composition of ρ with the adjoint representation

$$\mathrm{Ad}_\rho:\Gamma\to\mathrm{End}(\mathfrak{g})\subset GL(\mathfrak{g}).$$

Let us represent by $\mathfrak{g}_{\mathrm{Ad}_\rho}\subset\mathfrak{g}$ the Γ-module induced by the above representation. This representation can be seen as an action of Γ on $\mathfrak{g}_{\mathrm{Ad}_\rho}$ in the following way

$$\begin{aligned}
\Gamma\times\mathfrak{g}_{\mathrm{Ad}_\rho} &\to \mathfrak{g}_{\mathrm{Ad}_\rho}\\
\gamma,v &\mapsto \gamma\cdot v=\mathrm{Ad}^{-1}_{\rho(\gamma)}v.
\end{aligned}$$

Consider now the cohomology groups of Γ with coefficients in $\mathfrak{g}_{\mathrm{Ad}_\rho}$

$$H^i\left(\Gamma,\mathfrak{g}_{\mathrm{Ad}_\rho}\right)=Z^i\left(\Gamma,\mathfrak{g}_{\mathrm{Ad}_\rho}\right)/B^i\left(\Gamma,\mathfrak{g}_{\mathrm{Ad}_\rho}\right)$$

$$H^0\left(\Gamma,\mathfrak{g}_{\mathrm{Ad}_\rho}\right)=Z^0\left(\Gamma,\mathfrak{g}_{\mathrm{Ad}_\rho}\right)=\left(\mathfrak{g}_{\mathrm{Ad}_\rho}\right)^{\pi_1(X)}$$

$$H^1\left(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) = Z^1\left(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)/B^1\left(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$$
$$= \frac{\left\{\phi : \Gamma \to \mathfrak{g} \middle| \phi(\gamma_0\gamma_1) = \mathrm{Ad}^{-1}_{\rho(\gamma_0)}\phi(\gamma_1) + \phi(\gamma_0) \quad \forall \gamma_0, \gamma_1 \in \Gamma\right\}}{\left\{\phi : \Gamma \to \mathfrak{g} \middle| \exists a \in \mathfrak{g}, \quad \phi(\gamma_0) = \mathrm{Ad}^{-1}_{\rho(\gamma_0)}a - a \quad \forall \gamma_0 \in \Gamma\right\}}.$$

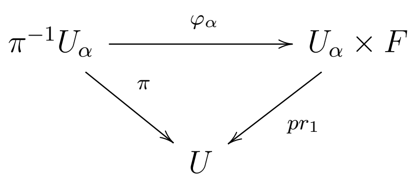
1.4. Principal G-bundles

The aim of this section is to provide all the important notions and results, in the context of this thesis, concerned with fibre bundles. Our attention is particularly focus on principal bundles. For more details see for example [**Hus94, Ste51, Nar76, Ram75**]

Following Steenrood [**Ste51**], we start by introducing the following definition of fibre bundle.

DEFINITION 1.28. A fibre bundle consists on

i. a topological space E called bundle space;
ii. a topological space B named base space;
iii. a projection $\pi : E \to B$;
iv. a topological space $F := \pi^{-1}(x)$ called fibre over x;
v. an effective group G of morphisms between fibres,
vi. a local trivialisation $\{U_i, \varphi_i\}$ where $\{U_i\}$ designates an open cover of the base space B and the morphisms φ_i verify the following commuting diagram



If there is no confusion, we simply denote a fibre bundle by E.

REMARK. The maps $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$ are called transition functions and they verify the cocycle relations, that is, $\varphi_{ij}\varphi_{jl}\varphi_{li} = \mathrm{id}$. Without loss of generality, since $\varphi_{ij} : \pi^{-1}(U_i \cap U_j) \to (U_i \cap U_j) \times F$ can be thought as a map from F to itself and the set of these maps provides the structure group G.

Examples 1.29. *Consider E, B smooth complex manifolds.*

*(1) If F is a complex vector space then the transition maps are linear maps and the bundle E is called a **vector bundle**.*
*(2) Let G be a Lie group (or an algebraic group) acting transitively and freely on F then the maps between fibres are G-equivariant maps and E is a **principal bundle**.*

Since we are mainly interested in principal G-bundles over a compact Riemann surface X, it is important to deal within holomorphic context. In this way, let us give a more concrete definition of holomorphic principal G-bundle over X.

Definition 1.30. Let G be a complex reductive algebraic group. A **holomorphic principal G-bundle** E_G over a compact Riemann surface X is a smooth complex variety with a free right G-action such that the projection $\pi : E_G \to X$ is G-invariant ($\pi(y \cdot g) = \pi(y), \forall y \in E_G$).

Example. If we take the fibre $G = GL(n, \mathbb{C})$ (the general linear group) then we obtain a principal $GL(n, \mathbb{C})$-bundle E over X.

Another fundamental definition is the concept of maps between bundles.

Definition 1.31. Given two bundles $E \to X$ and $E' \to X'$, a **bundle morphism** between E and E' is a pair of maps (ϕ, f) such that the following diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ \downarrow \pi & & \downarrow \pi' \\ X & \xrightarrow{f} & X' \end{array} \tag{1.4.1}$$

commutes. Namely, the morphism ϕ sends fibres of E to fibres of E', that is, $\phi(E_x) \subset E'_{f(x)}, \forall x \in X$. The map f is completely determined by the map ϕ (since π is surjective). The bundles E and E' are **isomorphic** if, additionally, there exists a pair

$(\phi' : E' \to E,\ f' : X' \to X)$ such that $\phi \circ \phi' = \mathrm{id}_{E'}$, $\phi' \circ \phi = \mathrm{id}_E$, $f \circ f' = \mathrm{id}_{X'}$ and $f' \circ f = \mathrm{id}_X$.

If we are dealing with bundles over the same base space then we must have $\pi(y) = (\pi' \circ \phi)(y)$, $\forall y \in E$.

Remark 1.32. According to the above definitions, if we are dealing with vector bundles then ϕ must be a linear map. On the other hand, if we are working with principal bundles then ϕ has to be a G-equivariant homomorphism.

The following Theorem asserts an important property of morphism over G-bundles. More details can be found in [**Hus94**].

Theorem 1.33. *Any morphism of principal G-bundles over the same base space is an isomorphism.*

Definition 1.34. A morphism $s : X \to E$ such that $\pi \circ s = \mathrm{id}_X$ is called a **global section** (**or section**) of E. The set of all global sections is represented by $\Gamma(E)$.

In general, smooth global sections may not exist, it may happen that they cannot be defined in all base space X. For example, in the case of principal G-bundles we have the following Proposition.

Proposition 1.35. *A principal G-bundle $E_G \to X$ is trivial if and only if it admits a section.*

According to this, a non-trivial principal G-bundle E_G does not have any global section. On the other hand, a vector bundle always has, at least, one section (the zero section).

More details about this matter can be found, for example, in [**Hus94**] or [**Ste51**].

1.5. Associated bundles

Herein, we explain how to produce bundles from others. We begin with the idea of pulling back bundles and after this, following [**Nar76**], we give some notions of factors of automorphy and its relationship with a particular type of isomorphic principal bundles. This section ends with a general construction of associated bundles.

Further details as well as the proofs of the results listed in this section can be found in [**Hus94, Ste51**].

DEFINITION 1.36. Let X and Y be topological spaces. Consider a G-bundle E over X and a continuous map $f : Y \to X$. We define the **pull-back of the bundle** E and denote it by $f^*(E)$. The transition functions of the bundle $f^*(E)$ over Y are defined by

$$\varphi_{ij}(y) = \phi_{ij}(f(y))$$

where ϕ_{ij} are the transition functions of the bundle E. In this sense, the bundle $f^*(E)$ is a G-bundle over the space Y and f induces a bundle morphism $\tilde{f} : f^*(E) \to E$. Hence, we have the following commutative diagram

$$\begin{array}{ccc} f^*(E) & \xrightarrow{\tilde{f}} & E \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X. \end{array}$$

Let Γ be an algebraic group and $E_\Gamma \to X$ be a principal Γ-bundle. A **factor of automorphy** on $E_\Gamma \times \Gamma$ with values in an algebraic group G is a holomorphic function $f : E_\Gamma \times \Gamma \to G$ satisfying

$$f(y, \gamma_1\gamma_2) = f(y, \gamma_1) f(y\gamma_1, \gamma_2)$$

for $y \in E_\Gamma$ and $\gamma_1, \gamma_2 \in \Gamma$.

Narasimhan worked about automorphy factors and he established the following Proposition.

PROPOSITION 1.37. [**Nar76**] *Let Γ, G be complex algebraic groups and let E_Γ and E_G be principal bundles over a compact Riemann surface X with fibres Γ and G respectively. If the pull-back of E_G by the projection $\pi_\Gamma : E_\Gamma \to X$, denoted by $\pi^*(E_G)$, is trivial over E_Γ then there exists a map $i_\sigma : E_\Gamma \to E_G$ such that*

$$i_\sigma(y \cdot \gamma) \;\; = \;\; i_\sigma(y) f(y, \gamma)$$

for all $y \in E_\Gamma$ and $\gamma \in \Gamma$, where the map f is a factor of automorphy.

REMARK. If the pull-back of E_G, by the projection $\pi : E_\Gamma \to X$, $\pi^*(E_G)$ is trivial over E_Γ then this means that there exists a section $\sigma : E_\Gamma \to \pi^*(E_G)$. The map i_σ is such that

$$\begin{array}{ccc} \pi^*(E_G) & \xrightarrow{\tilde{\pi}} & E_G \\ \sigma \uparrow \downarrow pr & \nearrow i_\sigma & \downarrow pr \\ E_\Gamma & \xrightarrow{\pi} & X \end{array}$$

$i_\sigma = \tilde{\pi} \circ \sigma$ where $\tilde{\pi} : \pi^*(E_G) \to E_G$ is the natural projection.

DEFINITION 1.38. Two factors of automorphy f, $\tilde{f}$ are said to be **equivalent** if exists a holomorphic function $h : E_\Gamma \to G$ satisfying $\tilde{f}(y, \gamma) = h(y)^{-1} f(y, \gamma) h(y \cdot \gamma)$ for all $y \in E_\Gamma$ and all $\gamma \in \Gamma$.

Given any factor of automorphy $f : E_\Gamma \times \Gamma \to G$ we may construct a holomorphic bundle E_f associated to E_Γ in the following way. First, we define the map

$$\begin{aligned} \psi : \quad E_\Gamma \times G \times \Gamma &\to \quad E_\Gamma \times G \\ (y, g, \gamma) &\mapsto \left(y \cdot \gamma,\, f(y, \gamma)^{-1} g\right) \end{aligned} \tag{1.5.1}$$

for every $y \in E_G$, $\gamma \in \Gamma$, $g \in G$. The properties of the factor of automorphy f allow us to define a right action of Γ on $E_\Gamma \times G$. In this way, we can consider the orbit space $E_f := (E_\Gamma \times G)/\,\Gamma$ and E_f gets the structure of a G-bundle if we consider a right G-action only on the second coordinate

$$\begin{aligned} \varphi : \quad E_f \times G &\to \quad E_f \\ (y, g') &\mapsto \left[x \cdot \gamma,\, f(x, \gamma)^{-1} g g'\right] \end{aligned}$$

where the element $y \in E_f$ represents a class of the form $\left[\left(x \cdot \gamma,\, f(x, \gamma)^{-1} g\right)\right]$.

DEFINITION 1.39. The bundle obtained by the above construction, E_f, is called **associated bundle** to E_Γ by f. Moreover, given two factors of automorphy f, $\tilde{f} : E_\Gamma \times \Gamma \to G$, the G-bundles E_f and $E_{\tilde{f}}$ are **isomorphic** if the factors of automorphy f, $\tilde{f}$ are **equivalent** (Def. 1.38).

PROPOSITION 1.40. *[Proposition A.8,* **[Nar76]** *] Let $E_\Gamma \to M$ be a principal Γ-bundle and G a group. Then the set of isomorphism classes of principal G-bundles on M, whose pull-backs on E_Γ are trivial, is in canonical bijective correspondence with the set of equivalence classes of factors of automorphy on $E_\Gamma \times \Gamma$ with values in G.*

REMARK. When we have a homomorphism of complex algebraic groups $\rho : \Gamma \to G$, this map defines a factor of automorphy that does not depend on $y \in E_\Gamma$.

EXAMPLE 1.41. Let π_1 denotes the fundamental group of a compact Riemann surface X and let G be a complex algebraic group. Considering that a universal cover $\widetilde{X}$ of X is a π_1-bundle over X, given a homomorphism $\rho : \pi_1 \to G$ we construct the following associated principal G-bundle

$$E_\rho := \widetilde{X} \times_\rho G = \left(\widetilde{X} \times G\right) \Big/ \pi_1 \tag{1.5.2}$$

where we have the following identifications

$$(\widetilde{x}, g) \sim (\widetilde{x}, g) \cdot \gamma = \left(\widetilde{x} \cdot \gamma, \rho(\gamma)^{-1} \cdot g\right).$$

DEFINITION 1.42. Let H and G be algebraic groups and let $\rho : H \to G$ be a homomorphism. The function ρ is one factor of automorphy that does not depend on any point of the bundle. In this way, given a H-bundle E, we can construct the G-bundle $E\,(G)$ using the method of (1.5.1) and obtain

$$E(G) = (E \times G)/\,H$$

where we identify the points $(y,\, g)$ and $\left(y \cdot h,\, \rho\,(h)^{-1}\, g\right)$ for all $h \in H$.
The bundle $E(G)$ is said to be an **extension of the structure group** by E (specially when H is a subgroup of G) and E is said to be a **reduction of structure group** by $E(G)$.

The next definition gives a generalisation of the above constructions.

DEFINITION 1.43. Let G be an algebraic group and V a variety. Let $\phi : G \times V \to V$ be a left action of G on V defined by $\phi\,(g, v) = g^{-1} \cdot v$. If E_G is a principal G-bundle, we construct the following quotient

$$E_G(V) = (E_G \times V)/\, G \tag{1.5.3}$$

where we identify the points of the form $(y,\, v)$ and $(y \cdot g,\, g^{-1} \cdot v)$ for all $y \in E_G,\ v \in V$ and $g \in G$. This quotient $E_G(V)$ is called **associated fibre bundle** to the principal G-bundle $E_G \xrightarrow{\pi} X$ induced by the action ϕ. The projection $\widetilde{\pi} : E_G(V) \to X$ assigns to each $\tilde{y} \in E_G\,(V)$ the following element

$$\widetilde{\pi}(\widetilde{y}) = \widetilde{\pi}((y,\, v) \cdot G) = \pi(y), \ \ \forall \widetilde{y} \in E_G(V).$$

Notice that this map is well defined in the sense that it depends only on the class of $\widetilde{y}$. In fact, $\widetilde{\pi}(\widetilde{y} \cdot g) = \widetilde{\pi}((yg,\, g^{-1}v) \cdot G) = \pi(yg) = \pi(y)$ since π is G-invariant.

Remark. As it was stated previously, above action can also be given as a representation $\rho : G \to GL(V)$. If V is a vector space and $G \to GL(V)$ is a representation, $E(V)$ is a vector bundle.

Example 1.44. Given that $\widetilde{X}$ is a $\pi_1(X)$-bundle, we may construct a vector bundle associated to a representation of the fundamental group $\rho : \pi_1(X) \to GL(n, \mathbb{C})$. The quotient space $V_\rho := \widetilde{X} \times \mathbb{C}^n \big/ \!\sim = X \times_\rho \mathbb{C}^n$ is a vector bundle over X.

Proposition 1.45. *Let $E_G(V)$ be the fibre bundle associated to the G-bundle E_G as constructed in definition* 1.43*. For each $x \in X$, the fibre V is homeomorphic to $\widetilde{\pi}^{-1}(x)$.*

Example 1.46. Given the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$ and a G-bundle E_G, we can construct the **adjoint bundle**

$$\mathrm{Ad}(E_G) := E_G \times_{\mathrm{Ad}} \mathfrak{g} \tag{1.5.4}$$

associated to the principal G-bundle E_G. This bundle is a vector bundle with fibre the vector space $\mathfrak{g}$, the Lie algebra of G. The bundle $\mathrm{Ad}(E_G)$ is constructed from $E_G \times \mathfrak{g}$ through the following equivalence relation

$$(y,\, Y) \sim (y,\, Y) \cdot g = (y \cdot g, \mathrm{Ad}_g^{-1} \cdot Y)$$

for all $y \in E_G$, $Y \in \mathfrak{g}$ and $g \in G$. Observe that, according to the way that we define $\mathrm{Ad}(E_G)$, $\pi^{-1}(y)$ is homeomorphic to the Lie algebra $\mathfrak{g}$.

Theorem 1.47. *Let E_G be a principal G-bundle over X and V a left G-variety. The set of sections $\Gamma(X, E_G(V))$ of the fibre bundle $E_G(V)$ are in bijective correspondence with maps $f : E_G \to V$ satisfying $f(y \cdot g) = g^{-1} f(y)$.*

Given two isomorphic principal G-bundles E and E' then there exists an isomorphism ϕ between them (definition 1.31). Accordingly, we may construct isomorphisms between corresponding associated fibre bundles.

$$\begin{array}{ccc} E(V) & \xrightarrow{\phi_V} & E'(V) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ X & \xrightarrow{f} & X' \end{array}$$

Proposition 1.48. *The functions $E_G \mapsto E_G(V)$ and $(\phi, f) \mapsto (\phi_V, f)$ define a functor from the category of principal G-bundles to the category of bundles, admitting the structure of a fibre bundle with fibre V and structure group G.*

1.6. Stability notion

The notions of stable (or semistable) points are extremely important to construct a (coarse) moduli space as was stated by Mumford [**Mum63, MF82**]. Restricting varieties to stable or semistable points allow us to arrange the set of certain objects in order to get an algebraic variety or a scheme (moduli space) that parameterises equivalence classes of these objects.
In this section, we describe notions and properties related with the semistability (stability) of vector and principal bundles. In order to avoid any confusion, we denote by V a vector bundle and by E a principal G-bundle.

Definitions 1.49. *Given a vector bundle V over X, we define* ***slope*** *of V as the number given by*

$$\mu(V) = \frac{\deg V}{\operatorname{rk} V}$$

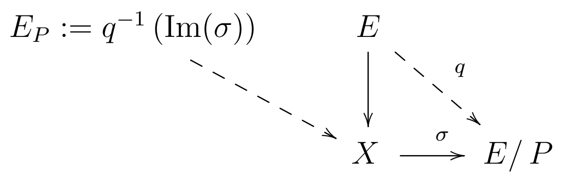
where $\deg(V) = \deg(\det V) = \deg(\wedge^n V)$ *represents the degree of the vector bundle* V *and* $\operatorname{rk} V = n$ *is the rank of* V *(it corresponds to the dimension of the fibre which is a vector space).*

A vector bundle V *in* X *is said to be* ***stable*** *(resp.* ***semistable****) if and only if for every proper sub-bundle* W *of* V *we have the following inequality*

$$\mu(W) := \frac{\deg W}{\mathrm{rk} W} < \mu(V) := \frac{\deg V}{\mathrm{rk} V} \qquad (\textit{resp.}\ \ \mu(W) \leq \mu(V))\,.$$

In order to explain the condition of stability (semistability) for principal G-bundles we must remind the notion of reduction of structure group. In spite of the fact that we had already given this notion (in def. 1.42), we redefine it in order to emphasise some aspects of this construction.

Definition 1.50. Let E be a G-bundle over X and let $P \subset G$ be a closed subgroup of G. One section $\sigma : X \to E/P$ over the bundle E/P corresponds to a **reduction of the structure group** to a P-bundle $E_P := q^{-1}(\mathrm{Im}(\sigma))$ over X.



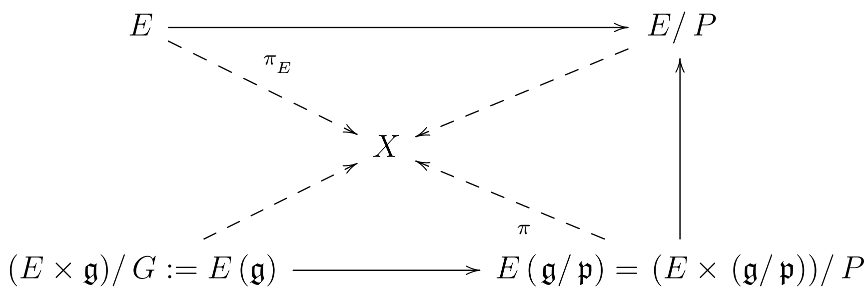
If P is a parabolic subgroup of a connected reductive algebraic group G then G/P is a complete variety and G acts on this variety (def. 1.43). Then we may construct the fibre bundle $E/P := E(G/P) = (E \times G/P)/G$ associated to the G-bundle E.

Now, we are in conditions to give Ramanathan's definitions of stability of a principal G-bundle.

Definition 1.51. [**Ram75**] A holomorphic G-bundle E over X is **stable** (resp. **semistable**) if for every reduction $\sigma : X \to E/P$ to maximal parabolic subgroups P of G we have

$$\deg \sigma^*\left(T_{E/P}\right) > 0 \qquad (\text{resp.} \quad \geq 0)$$

where $T_{E/P}$ is the tangent bundle along fibres of $E/P \to X$. More accurately, $T_{E/P}$ is the vector bundle $E(\mathfrak{g}/\mathfrak{p}) := (E \times (\mathfrak{g}/\mathfrak{p}))/\,P$ over $E/\,P$ associated to the P-bundle $E \to E/\,P$ and to the action of P on $\mathfrak{g}/\mathfrak{p}$ induced by the adjoint action $((p, v) \mapsto \mathrm{Ad}_p\,(v))$.



Let us, now, give the definition of semistability for G-bundles E using the semistability of the corresponding adjoint bundle $\mathrm{Ad}\,(E)$.

Definition 1.52. A G-bundle E is **semistable** if for every reduction E' of the structure group to a maximal parabolic subgroup P we have

$$\deg\left(\mathrm{Ad}(E)/\mathrm{Ad}(E')\right) \geq 0.$$

Taking into consideration that G is reductive, there exists a G-invariant non-degenerate bilinear form on $\mathfrak{g}$ and, consequently, $\mathrm{Ad}(E)$ is isomorphic to its dual. Thus, $-\deg(\mathrm{Ad}(E)^*) = \deg(\mathrm{Ad}(E))$ and we obtain $\deg(\mathrm{Ad}(E)) = 0$.

Proposition 1.53. *[Prop.2.10,* [**AB01**]*] Let G be a connected reductive algebraic group and M a compact Kähler manifold. A principal G bundle E_G over M is semistable if and only if the adjoint bundle* $\mathrm{Ad}(E_G)$ *is semistable.*

Remark 1.54. Ramanathan in [**Ram96**] established the above result for a Riemann surface X of genus $g \geq 2$.

1.7. Flat bundles over a compact Riemann surface

Flat bundles are those induced by representations of the fundamental group of the surface X (base space). Narasimhan and Seshadri proved that stable/semistable vector bundles over a compact Riemann surface X are exactly those which are induced by certain types of representations. In addition, Ramanathan established a similar result for the corresponding structures in the case of principal G-bundles over X where G denotes a connected reductive algebraic group. In what follows, we are going to summarise some of these results due mostly to the authors cited above.

1.7.1. Connections on fibre bundles. Let $E \xrightarrow{\pi} X$ be a fibre bundle over a compact Riemann surface X. For each $y \in E$, consider the following subspace of T_yE

$$V_y = \{Y \in T_yE \ : \ d\pi_y(Y) = 0\}$$

which is designated by **vertical subspace** at y.

A **connection** D on the fibre bundle E is a collection of vector subspaces $H_y \subset T_yE$, called **horizontal spaces**, such that, for each $y \in E$, we have

$$T_yE = H_y \oplus V_y.$$

Every smooth fibre bundle admits a connection. However, since we are dealing with compact Riemann surfaces, it makes sense to work with holomorphic functions and correspondingly, define the concept of holomorphic connection on a fibre bundle.

Moreover, particular characteristics of each fibre bundle have its influence on the properties of the corresponding connection. For example, if we consider a vector bundle over a compact Riemann surface maps are required to be linear, and if we consider principal G-bundles we require smoothness (or holomorphy) and equivariance under the G-action.

Let us first define the concept of holomorphic connection over a holomorphic vector bundle.

Definition 1.55. Let E be a holomorphic vector bundle over a compact Riemann surface X, a **holomorphic connection** D on $E \to X$ is a first order holomorphic differential operator $D : \Gamma(E) \to \Gamma(T^*X \otimes E)$ satisfying the Leibniz rule $D(fs) = df \otimes s + fD(s)$.

Here f is a (locally) holomorphic function on X and s is a (locally) holomorphic section of E. There exists a unique extension of D to a linear operator

$$D : \Omega^p(E) \to \Omega^{p+1}(E)$$

satisfying $D(\omega \otimes s) = d\omega \otimes s + (-1)^p \omega \wedge Ds$ for all p-forms $\omega \in \Omega^p(M)$ and $s \in \Gamma(E)$.

DEFINITION 1.56. A connection D is said to be **flat** if the curvature operator $\Theta := D^2$ vanishes identically. Given that $\Theta := D^2$ is a holomorphic differential 2-form and since over a compact Riemann surfaces there are no holomorphic two forms, we can conclude that every holomorphic connection on $E \to X$ is flat.

A vector bundle $E \to X$ together with a flat connection is called a **flat vector bundle**.

In the case of principal G-bundles, we define the concept of connection on E in the following way.

DEFINITION 1.57. A **connection on a principal G-bundle** E is a **$\mathfrak{g}$-valued 1-form** $\omega_y : T_yE \to \mathfrak{g}$ satisfying the following conditions

- $\omega_y\left(\tilde{Y}(y)\right) = Y, \ \forall\, Y \in \mathfrak{g}$, where $\tilde{Y}(y) = \left.\dfrac{d}{dt}\left(y \exp\left(tY\right)\right)\right|_{t=0}, \quad y \in E.$
- $R_g^* \omega_y = \mathrm{Ad}_{g^{-1}} \omega_y, \ \ \forall\, g \in G,$

where $R_g : E \to E$, $y \to y \cdot g$, represents the right G-action on E. These properties can be translated over each horizontal space in the following way

$$H_{R_g(y)}E = \left(R_g\right)_* H_yE, \ \forall g \in G, \forall y \in E.$$

1.7.2. Flat vector bundles. Many authors such as Weil, Narasinham, Seshadri, Biswas and others studied flat vector bundles over compact Riemann surfaces. In what follows, we bring out some definitions and results obtained by these authors.

DEFINITION 1.58. A vector bundle V over a compact Riemann surface X is **flat** if there exists a representation $\rho \in \mathrm{Hom}(\pi_1(X), GL(n, \mathbb{C}))$ such that $V \cong \tilde{X} \times_\rho \mathbb{C}^n := V_\rho$ (see example 1.44).

There is an important theorem that relates the property of being flat with the degree of the corresponding vector bundle (Weil's Theorem 1.60) but first we need to remind some definitions.

DEFINITION 1.59. A vector bundle V is

- **reducible** if contains a proper subbundle; otherwise V is called **irreducible**.
- **decomposable** if is the direct sum of two vector subbundles V_1 and V_2, i.e, $V = V_1 \oplus V_2$. If there is not any such decomposition V is called **indecomposable**.

THEOREM 1.60. *[Thm 10, [**?, Wei38**]] Let V be a vector bundle over an algebraic curve and let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_l$ be a direct sum of indecomposable bundles V_i. Then V arises from a representation of the fundamental group if and only if* $\deg V_i = 0,\ i = 1, 2, \cdots, l$.

COROLLARY 1.61. *A holomorphic vector bundle V over X admits a holomorphic connection if and only if all indecomposable components of V have degree zero.*

These results allow us to relate the property of being flat with the existence of a holomorphic connection when we are working over compact Riemann surfaces.

COROLLARY 1.62. *A vector bundle V over X is flat if and only if it admits a flat connection (or holomorphic connection).*

1.7.3. Flat principal bundles. According to the previous subsection, the term flat on vector bundles over compact Riemann surfaces X is related with representations of the fundamental group of X and with the existence of holomorphic connections. We can always define connections on vector or principal bundles over X but they could not be holomorphic. Herein, we are going to describe some similar results, but for principal G-bundles case.

Since a holomorphic connection D on a principal G-bundle E_G induces a holomorphic connection on any associated bundle to it, in particular on the adjoint bundle $\mathrm{Ad}(E_G)$

(see for example [**Bis96**]), we will start with the idea of a connection on the adjoint bundle and then we will state a necessary and sufficient condition that relates the existence of a holomorphic connection on the principal bundle and on the associated adjoint bundle.

The connection D, as a holomorphic connection in the vector bundle $\mathrm{Ad}\,(E_G)$, can be defined as a linear operator from the set of (holomorphic) sections of the vector bundle $\mathrm{Ad}\,(E_G)$ to (holomorphic) sections of the cotangent bundle Ω^1_X with values in $\mathrm{Ad}\,(E_G)$

$$D : \Gamma\left(\mathrm{Ad}(E_G)\right) \to \Gamma\left(\mathrm{Ad}(E_G) \otimes \Omega^1_X\right) := \Omega^1_X(\mathrm{Ad}(E_G))$$

satisfying the Leibniz rule $D(f \cdot s) = \partial f.s + fD(s)$ where f is a smooth function on X. Usually we denote the corresponding induced connection by the same letter D.

Proposition 1.63. *[Prop.2.2,* [**AB03**]*] Let G be a connected semisimple linear algebraic group and X a compact Riemann surface. A principal G-bundle E_G over X admits a flat connection if and only if the adjoint bundle $\mathrm{Ad}(E_G)$ admits one.*

We may drop the requirement of G to be semisimple if an extra condition is fulfilled.

Proposition 1.64. *[Prop.3.1,* [**AB03**]*] Let G be a connected reductive linear algebraic group and X a compact Riemann surface. A holomorphic G-bundle E_G over X admits a flat connection if and only if the following conditions hold:*

(1) the adjoint bundle $\mathrm{Ad}(E_G)$ admits a flat connection;
(2) for every character χ of G, the line bundle $(E_G \times \mathbb{C})/\,G$ associated to E_G for χ is of degree zero.

Given that $\mathrm{Ad}(E_G)$ has degree zero, by definition of semistable vector bundle, $\mathrm{Ad}(E_G)$ is semistable if and only if any subbundle (in particular any indecomposable component) has degree zero and, by Theorem 1.60, this is equivalent to admit a flat connection.

Proposition 1.65. *[Prop 14, [?]] Let E be a principal G-bundle over a compact Riemann surface X where G is a complex Lie group. Then the following properties are equivalent:*

(1) E admits a flat connection.

(2) E admits constant transition functions.

(3) E is isomorphic to E_ρ for a representation $\rho : \pi_1(X) \to G$.

*If a principal G-bundle E over X fulfils the above conditions, E is called **flat**.*

Definition 1.66. Let G be a complex algebraic group and let $\pi_1(X)$ be the fundamental group of X. Two representations $\rho, \sigma : \pi_1(X) \to G$ are called **conjugate** or **equivalent** if there exists an element $w \in G$ such that $\rho(\gamma) = w\sigma(\gamma) w^{-1}$ for all $\gamma \in \pi_1(X)$.

The following proposition establishes which type of relationship exists between principal bundles induced by conjugate representations.

Proposition 1.67. *Given two conjugate representations $\rho, \sigma : \pi_1(X) \to G$ then the corresponding induced G-bundles are isomorphic (or equivalent).*

Proof. By construction, $E_\rho \cong \widetilde{X} \times_\rho G$, $E_\sigma \cong \widetilde{X} \times_\sigma G$ and in order to obtain a morphism of principal bundles the following diagram

$$\begin{array}{ccc} E_\rho & \xrightarrow{\varphi} & E_\sigma \\ & \searrow_{\pi_\rho} \quad \swarrow_{\pi_\sigma} & \\ & X & \end{array}$$

has to be commutative. Hence, we define the map $\varphi : E_\rho \to E_\sigma$ such that φ assigns to each element $v = [\tilde{x}, g] \in E_\rho$ an element $\varphi(v) = w \in E_\sigma$ such that $\pi_\rho(v) = \pi_\sigma(\varphi(v)) = \pi_\sigma(w) = x \in X$. Taking in account the above universal cover $\pi : \widetilde{X} \to X$, we fix $\tilde{x} \in \pi^{-1}(x)$. Now, we can consider the map $E_\rho \xrightarrow{\varphi} E_\sigma$ defined in the ensuing way.

$$\varphi(v) \;=\; \varphi([\widetilde{x}, g]) := [\tilde{x}, g'] = w.$$

where $g' = \overline{\varphi}(g) = hgh^{\cdot 1}$ for $h \in G$ such that h also verifies $\rho(\gamma) = h\sigma(\gamma)h^{-1}$ for all $\gamma \in \pi_1(X)$. The map $\overline{\varphi}$ is an endomorphism of G.

The bundles E_ρ and E_σ are induced by the representations ρ and σ, respectively. This means that each element $\bar{g}$ of G acts on the points v of E_ρ via γ the corresponding element of $\pi_1(X)$ via ρ. Let us prove that φ is well defined, that is, given two elements $(\tilde{x}, g) \sim \left(\tilde{x}\cdot\gamma,\, \rho(\gamma)^{-1}\cdot g\right)$ on E_ρ then the corresponding images in E_σ coincides.

$$\varphi\left(\left[\tilde{x}\cdot\gamma,\, \rho(\gamma)^{-1}\cdot g\right]\right) = \left[\tilde{x}\cdot\gamma,\, \overline{\varphi}\left(\rho(\gamma)^{-1}\cdot g\right)\right]$$

Using the fact that ρ and σ are conjugate we obtain $\rho(\gamma) = h\sigma(\gamma)h^{-1}$ for some $h \in G$, so

$$\begin{aligned}
&= \varphi\left(\left[\tilde{x}\cdot\gamma,\, \left(h\sigma(\gamma)h^{-1}\right)^{-1}\cdot g\right]\right)\\
&= \left[\tilde{x}\cdot\gamma,\, \overline{\varphi}\left(h^{-1}\sigma(\gamma)^{-1}hgh^{-1}\sigma(\gamma)h\right)\right].\\
&= \left[\tilde{x}\cdot\gamma,\, hh^{-1}\sigma(\gamma)^{-1}hgh^{-1}\sigma(\gamma)hh^{-1}\right]\\
&= \left[\tilde{x}\cdot\gamma,\, \sigma(\gamma)^{-1}\left(hgh^{-1}\right)\sigma(\gamma)\right]\\
&= \left[\tilde{x}\cdot\gamma,\, \sigma(\gamma)^{-1}\cdot g'\right]\\
&= \left[\tilde{x},\, g'\right]
\end{aligned}$$

since the elements $w \in E_\sigma$ are defined by $(\tilde{x}, g') \sim \left(\tilde{x}\cdot\gamma,\, \sigma(\gamma)^{-1}\cdot g'\right)$. □

Now, let us prove that associated vector bundles to flat G-bundles are isomorphic to vector bundles induced by representations of $\pi_1(X)$ into $GL(V)$.

PROPOSITION 1.68. *Let G be a connected reductive algebraic group and E a G-bundle over a compact Riemann surface X. Suppose that $E \cong E_\rho$ where $\rho : \pi_1(X) \to G$ is a surjective representation and that $\tau : G \to GL(V)$ is a linear representation of G in the complex vector space V. Then the associated vector bundle $E(V)$ is isomorphic to $E_{\tau\rho}$ where $E_{\tau\rho}$ is the vector bundle associated to $\tau \circ \rho : \pi_1(X) \to GL(V)$.*

Proof. First let us look to the linear representation $\tau : G \to GL(V)$. This is equivalent to give an action of G on V, that is, $G \times V \to V$ such that each pair (g, v) corresponds to $g \cdot v = \tau(g)^{-1} \cdot v$.

The vector bundle $E(V)$ is obtained by identifying the following elements of $E \times V$

$$(y, v) \sim \left(y \cdot g, \tau^{-1}(g) \cdot v\right).$$

In addition, the G-bundle E is induced by a representation $\rho : \pi_1(X) \to G$, namely, $E = \widetilde{X} \times_\rho G$. Each element y of this G-bundle can be written as $[\tilde{x}, h]$ where we are identifying elements of the form

$$(x,\, h) \sim \left[\tilde{x} \cdot \gamma, \rho(\gamma)^{-1} \cdot h\right]$$

where $\gamma \in \pi_1(X)$ and $g = \rho(\gamma)$.

In order to prove that $E(V)$ is isomorphic to $E_{\tau\rho}$, let us denote by $\pi_{\tau\rho} : E_{\tau\rho} \to X$ and $\pi_V : E(V) \to X$ the corresponding projections. Now, we have to construct a map ϕ between these vector bundles such that ϕ is an isomorphism. This corresponds to show that the map $\phi|_{\pi^{-1}(x)} : \pi_V^{-1}(x) \to \pi_{\tau\rho}^{-1}(x)$ is a vector space isomorphism for each $x \in X$.

Each element in $E_{\tau\rho} = \widetilde{X} \times_{\tau\rho} V$ is, locally, of the form $[(\tilde{x},\, v)] = [(\tilde{x}\gamma,\, (\tau \circ \rho)(\gamma)^{-1} v)]$ where $\gamma \in \pi_1(X)$ is such that $(\tau \circ \rho)(\gamma) = w$. In this way, we may define the following bundle morphism

$$\phi : E(V) \to E_{\tau\rho}$$

that to each $[(y, v)] \in \pi_V^{-1}(x)$ assigns an element $[(\tilde{x},\, v)] \in \pi_{\tau\rho}^{-1}(x)$ where $\pi_V([(y, v)]) = \pi_{\tau\rho}[(\tilde{x},\, v)] = p(\tilde{x}) = x \in X$ where $p : \widetilde{X} \to X$ is the projection map of a universal cover $\widetilde{X}$ of X.

The morphism ϕ is well defined since ϕ does not depend on the element consider in the class $[y, v]$

$$\begin{aligned}
&\phi([(y, v) \cdot g]) = \phi\left[\left(y \cdot g,\ \tau(g)^{-1} v\right)\right] = \phi\left[\left((\tilde{x},\, h) \cdot g,\ \tau(g)^{-1} v\right)\right] \\
&= \phi\left[\left([\tilde{x},\, hg],\ \tau(g)^{-1} v\right)\right] := \left[\tilde{x},\, \tau(hg)\,\tau(g)^{-1} v\right] = [\tilde{x},\, \tau(h)\, v] := \phi\,[(\tilde{x},\, v)]
\end{aligned}$$

Furthermore, over each fibre we have the following isomorphisms

$$\begin{array}{ccc} \pi_V^{-1}(x) & \xrightarrow{\phi} & \pi_{\tau\rho}^{-1}(x) \\ \downarrow \cong & & \downarrow \cong \\ V & \xrightarrow{\underline{\phi}} & V. \end{array}$$

This means that if $(y_1, v_1), (y_2, v_2) \in \pi_V^{-1}(x)$ then we have the following operations

$$\pi_V(z(y_1, v_1)) = \pi_V(y_1, v_1), \quad \forall z \in \mathbb{C}$$

$$\pi_V((y_1, v_1) + (y_2, v_2)) = \pi_V(y_1 + y_2, v_1 + v_2) = \pi_V(y_1, v_1) = \pi_V(y_2, v_2)$$

and the same happens to $\pi_{\tau\rho}$. Hence, we may see that $\underline{\phi}$ is an isomorphism of vector spaces for each $x \in X$. In fact,

$$\begin{aligned} \phi[z_1(y_1, v_1) + z_2(y_2, v_2)] &= \phi[(y_1 + y_2, z_1v_1 + z_2v_2)] \\ &= \phi[((\tilde{x}_1 + \tilde{x}_2, g_1 + g_2), z_1v_1 + z_2v_2)] \\ &= (\tilde{x}_1 + \tilde{x}_2, z_1v_1 + z_2v_2) = z_1(\tilde{x}_1, v_1) + z_2(\tilde{x}_2, v_2) \\ &= z_1\phi[(y_1, v_1)] + z_2\phi[(y_2, v_2)] \end{aligned}$$

$\forall z_1, z_2 \in \mathbb{C}$, $\forall (y_1, v_1), (y_2, v_2) \in \pi_V^{-1}(x)$, $\forall x \in X$. Thus, we may conclude that ϕ is an isomorphism of vector bundles. □

PROPOSITION 1.69. *Let G and H be algebraic groups and $\varphi : G \to H$ a group homomorphism. Additionally, let E_H be a H-bundle obtained from a G-bundle E_G over a compact Riemann surface X by extension of structure group by φ. If E_G is flat then E_H is also flat.*

PROOF. Given a flat principal bundle E_G, this corresponds to be induced from the universal covering bundle by a representation $\rho : \pi_1(X) \to G$, that is, $E_G = \widetilde{X} \times_\rho G$. In this way, we obtain the diagram

$$\begin{array}{ccc} \pi_1(X) & \xrightarrow{\rho} & G \\ & \searrow^{\varphi\circ\rho} & \downarrow \varphi \\ & & H \end{array}$$

and correspondingly we have the ensuing relations

$$E_H = E_G \times_\varphi H \cong \left(\widetilde{X} \times_\rho G\right) \times_\varphi H$$

which identify the following points

$$(\tilde{x}, g, h) \sim (\tilde{x}, g, h) \cdot g' = \left(\tilde{x} \cdot \gamma, gg', \varphi(\rho(\gamma))^{-1} h\right)$$

where $\gamma \in \pi_1(X)$ such that $g' = \rho(\gamma)$.

In this sense, we may consider that $E_H \cong \widetilde{X} \times_{\bar{\rho}} H$ where the representation $\bar{\rho} : \pi_1(X) \to H$ is obtained from the composition of φ with ρ. Therefore, the principal H-bundle E_H is flat. □

1.8. Moduli spaces of flat bundles

In this section, we describe the moduli space of flat bundles over a compact Riemann surface of genus $g \geq 2$. These important results and definitions were given, mostly, by Mumford, Narasimhan, Seshadri and Ramanathan. Essentially, they relate stable points on the set of equivalence classes of bundles to representations of the fundamental group of a compact Riemann surface X.
Hereinafter, when we use the term moduli space we mean coarse moduli space. In both cases (vectorial and principal) we cannot usually construct a (fine) moduli space, namely, a family that parametrises the objects (bundles) and such that the family has a nice structure, like being a variety.

Throughout this section X and G will denote, respectively, a compact Riemann surface with genus g and a connected reductive algebraic group over complex numbers. In order to simplify notation, we use π_1 to denote the fundamental group of X, $\pi_1(X)$.

1.8.1. Moduli space of vector bundles. Mumford, Narasimhan and Seshadri proved, in [**NS64, NS65, Mum63**], the existence of a nice orbit space (coarse moduli space) parameterising the set of equivalence classes of stable (semistable) vector bundles over X. Moreover, they proved important properties of this space, such as, it is an algebraic set, and they, also, compute its dimension. In what follows we are going to highlight some properties and definitions that will be useful to understand the structure and properties of the corresponding moduli space.

Using the notation introduced in [**Flo01**], let

$$G_n := \mathrm{Hom}\,(\pi_1, GL(n,\mathbb{C}))/\,GL(n,\mathbb{C})$$

denotes the **set of equivalence classes of representations** of the fundamental group π_1 in $GL(n,\mathbb{C})$ and consider the following map

$$\begin{array}{rccc} V.: & G_n & \to & H^1(X,\, GL\,(n,\,\mathbb{C})) \\ & [\rho] & \mapsto & [V_\rho] \end{array} \qquad (1.8.1)$$

assigning to each equivalence class of a representation ρ the equivalence class of the flat vector bundle $V_\rho = \widetilde{X} \times_\rho \mathbb{C}^n$.

Since to each representation we may associate a flat vector bundle over X, it is natural to ask when we have equivalent representations corresponding to isomorphic vector bundles. Narasimhan and Seshadri gave the answer to this question. They proved that this really happens when the representations are unitary (and irreducible).

PROPOSITION 1.70. *[Prop. 4.2,* [**NS64**]*] Let ρ_1 and ρ_2 be n-dimensional unitary representations of the fundamental group $\pi_1(X)$ of a compact Riemann surface X. Then the holomorphic vector bundles V_{ρ_1} and V_{ρ_2} are isomorphic if and only if the representations ρ_1 and ρ_2 are equivalent.*

With this notion of equivalence it is natural to ask if, when we restrict to stable flat vector bundles over X, we obtain a nice space (moduli space) of equivalence classes of these vector bundles.

THEOREM 1.71. *[Thm 2, Cor. 1,* [**NS65**]*] Let X be a Riemann surface of genus $g \geq 2$. Then*

(1) A holomorphic bundle V with rank n and degree zero is stable if and only if $V \cong V_\rho$ for some irreducible unitary representation ρ of the fundamental group of X.

(2) The correspondence $\rho \mapsto V_\rho$ establishes a categories equivalence between the category of irreducible unitary representations of the fundamental group and the category of stable holomorphic vector bundles of degree zero over X.

DEFINITION 1.72. [**NS65**] A **holomorphic family** of vector bundles on X parametrised by a complex space T, $\mathcal{W} := \{\mathcal{W}_t\}_{t\in T}$, is a holomorphic vector bundle $\mathcal{W}$ on $T \times X$. If we consider the inclusion

$$\begin{array}{rcl} i : X & \to & T \times X \\ x & \mapsto & (t, x) \end{array}$$

each $\mathcal{W}_t = i^*\mathcal{W}$ is the pull-back of image of $\mathcal{W}$ by the inclusion i.

THEOREM 1.73. *[Theorem 2,* [**NS65**]*] Let T be a complex space (resp. algebraic space) parameterising a holomorphic (resp. an algebraic) family $\{\mathcal{W}_t\}$, $t \in T$, of vector bundles of rank n on X. Then the following subsets of T*

$$T^s = \{t \in T : \ \mathcal{W}_t \ \textit{is stable}\}$$

$$T^{ss} = \{t \in T : \ \mathcal{W}_t \ \textit{is semistable}\}$$

are open sets (resp. Zariski open sets) of T.

Considering the concept of unitary representation introduced in definition 1.24 (2), in the case $\rho : \pi_1 \to GL(n, \mathbb{C})$, the maximal compact subgroup of $GL(n, \mathbb{C})$ is $U(n)$, consequently one unitary representation is one that can be reduced to $\rho : \pi_1 \to U(n)$.

Let $\mathbb{U}_\tau(n)$ (respectively $\mathbb{U}^\circ_\tau(n)$) denote the set of n-dimensional unitary (resp. irreducible and unitary) representations from π_1 to $GL(n, \mathbb{C})$ of type τ.

DEFINITION 1.74. A representation $\rho : \pi_1 \to U(n)$ is of **type** τ if $\rho(\gamma) = \tau(\gamma) I_{n\times n}$, $\forall \gamma \in \pi_{\tilde{x}}$ where

- $\tilde{x} \in \tilde{X}$, a universal cover of X,
- the map $p : \widetilde{X} \to X$ is the projection associated to $\widetilde{X}$ with $p(\tilde{x}) = x$,
- the group $\pi_{\tilde{x}}$ is the stabiliser of $\tilde{x}$,
- the map τ is a character of $\pi_{\tilde{x}}$.

The group $PU(n) = U(n)/\,Z(U(n))$ acts freely on the right on $\mathbb{U}^{\circ}_{\tau}(n)$, $\rho T = T^{-1}\rho T$ where $T \in PU(n)$ and $\rho \in \mathbb{U}^{\circ}_{\tau}(n)$.

Proposition 1.75. **[NS64]** *The set $\mathbb{U}^{\circ}_{\tau}(n)$ has a natural structure of a real analytic manifold of dimension $2(n^2(g-1)+1)+n^2-1$. Moreover, equivalent classes of n-dimensional irreducible unitary representations of type τ, $U^{\circ}_{\tau}(n)/\,PU(n)$, form a real analytic manifold of dimension $2(n^2(g-1)+1)$.*

Let $M^s(n,d)$ (respectively, $M^{ss}(n,d)$) denote the set of equivalence classes of stable (resp. semistable) vector bundles with rank n and degree d, over a compact Riemann surface X of $g \geq 2$.

Corollary 1.76. *[**[NS65, MFK94]**] There exists a complex manifold $M^s(n,0)$ that parameterises the isomorphism classes of stable holomorphic bundles of rank n and degree 0. Furthermore, $M^s(n,0)$ has the following properties:*

(1) it is connected and it has (complex) dimension $n^2(g-1)+1$.

(2) it has a natural complex structure, that is, if $\{\mathcal{W}_t\}_{t\in T}$ is a family of stable holomorphic bundles of rank n and degree 0, parameterised by a complex manifold T, then the classifying map

$$\begin{array}{ccc} T & \to & M^s(n,0) \\ t & \mapsto & [\mathcal{W}_t] \end{array} \tag{1.8.2}$$

is holomorphic.

1.8.2. Moduli space of principal bundles. Generalising concepts of Narasimhan and Seshadri, Ramanathan proved in **[Ram96, Ram75]** that, for a Riemann surface X of genus $g \geq 2$ and a connected reductive algebraic group G over $\mathbb{C}$, the set of isomorphism classes of stable G-bundles of a given topological type has a natural structure of a connected normal complex space. In what follows, we review some of the most important results of Ramanathan that will be needed in the forthcoming sections.

Let us start with a generalisation of Narasimhan and Seshadri's Theorem (Theorem 1.71), made by Ramanathan, to principal G-bundles where G is a reductive algebraic group.

PROPOSITION 1.77. *[Prop.2.2,* **[Ram75]***] If $\rho : \pi_1(X) \to G$ is a unitary representation then E_ρ is semistable. Further, if ρ is also irreducible then E_ρ is stable.*

There exists an important group related to a G-bundle E over X, which is called **automorphisms group** of E and it is denoted by $\mathrm{Aut}\,(E)$. It is convenient that this group is as small as possible in order to obtain smoothness (as we will see later). In this case it has to be reduced to the center of the group G, $Z := Z\,(G)$.

Given a principal G-bundle E over X, we consider the corresponding adjoint bundle $\mathrm{Ad}\,E$. Each section $\sigma : X \to \mathrm{Ad}\,E$ induces a G-equivariant map $\underline{\sigma} : E \to G$ such that $\underline{\sigma}\,(y \cdot g) = g^{-1} \cdot \underline{\sigma}\,(y) = g^{-1}\underline{\sigma}\,(y)\,g$.

Consider an automorphism $\phi_\sigma : E \to E$ defined by $\phi_\sigma\,(y) = y\,\underline{\sigma}\,(y)$. According to G-equivariance of $\underline{\sigma}$, we obtain the same property on ϕ_σ, that is,

$$\phi_\sigma\,(y \cdot g) = yg\underline{\sigma}\,(y \cdot g) = ygg^{-1}\underline{\sigma}\,(y)\,g = \phi_\sigma\,(y) \cdot g.$$

In this way, sections of the adjoint bundle corresponds exactly to automorphisms of E (for more details see, for example, **[Hus94]**). In this way, we obtain the following equalities

$$\mathfrak{Lie}\,(\mathrm{Aut}\,(E)) \cong H^0\,(X, E\,(\mathfrak{g})) = H^0\,(X,\,\mathrm{Ad}\,(E))\,.$$

REMARK. The group of automorphism $\mathrm{Aut}\,(E)$ of a G-bundle E has a physics interpretation related with the particles movement and is called **gauge group**.

PROPOSITION 1.78. *[Prop. 3.2,* **[Ram75]** *] Let E be a stable G-bundle then*

$$H^0\,(X,\,\mathrm{Ad}\,(E)) = \mathfrak{z}$$

where $\mathfrak{z}$ is the centre of $\mathfrak{g}$. In particular, if G is semisimple then $H^0\,(X,\,\mathrm{Ad}\,(E)) = 0$ and $\mathrm{Aut}\,(E)$ is finite.

Remark. Given that $H^0(X, \mathrm{Ad}(E)) = \mathfrak{Lie}(\mathrm{Aut}(E))$ and considering that Z is always contained in $\mathrm{Aut}(E)$ as a normal subgroup, we have always $\mathfrak{z} = Lie(Z) \subset H^0(X, \mathrm{Ad}(E))$.

If E is stable then $\mathrm{Aut}(E)$ is a finite extension of the centre Z, that is, $\mathrm{Aut}(E)/Z$ is finite. Furthermore, when G is semisimple, its centre is finite thus we obtain the fact of $\mathrm{Aut}(E)$ being finite.

Proposition 1.79. *[Thm. 4.1,* **[Ram75]***] Let $\{E_t\}_{t\in T}$ be an analytic family of G-bundles parameterised by the complex space T. Then the set $\mathcal{F} = \{t \in T : E_t \text{ is stable}\}$ is a Zariski open subset of T.*

When we are dealing with vector bundles we define some types of invariants in order to index these objects. The main goal of this approach is to "catalog" the set of similar objects and look for a space with specific characteristics (moduli space). In the case of flat G-bundles, usually, we set the following topological invariant

$$c(\rho) = c(E_\rho) \in \pi_1(G).$$

This object is called **topological type** of E_ρ and it is related with the representation ρ. More accurately, this element is given through the exact sequence

$$1 \to \pi_1(G) \to \widetilde{G} \to G \to 1$$

where $\widetilde{G}$ is a universal cover of G. Since G is a connected algebraic group over $\mathbb{C}$, there always exists a universal cover $\widetilde{G}$ such that $\ker p = \pi_1(G)$ is finite. Moreover, $Z(G) \subset \pi_1(G)$.

Remark. If we consider the case of vector bundles over X, the topological type of V is exactly the degree of the vector bundle V, $\deg(V)$.

The set of isomorphism classes of G-bundles M_G is indexed by the elements of $\pi_1(G)$, that is,

$$M_G = \coprod_{d\in\pi_1(G)} M_G^d.$$

Let us denote by $M^{s,d}$ the set of isomorphisms classes of stable G-bundles over X with topological type $d \in \pi_1(G)$. The following theorem is a generalisation of Theorem 1.76 and establishes that the moduli space $M^{s,d}$ is a connected complex space.

THEOREM 1.80. *[Thm. 4.3 and Prop. 5.1,* **[Ram75]***] The set of isomorphism classes of stable G-bundles of a given topological type $d \in \pi_1(G)$, $M^{s,d}$ has the structure of a connected normal (Hausdorff) complex space. Moreover, if $\{E_s\}_{s\in S}$ is any holomorphic family of bundles, of the given topological type d, the natural map $S \to M^{d,s}$ is holomorphic.*

We are, now, in conditions to introduce the Ramanathan's Theorem analogous to Narasimhan and Seshadri's Proposition 1.70 but, in this case, applied to principal G-bundles.

THEOREM 1.81. *[Thm 7.1,* **[Ram75]***] A holomorphic G-bundle E on a compact Riemann surface X $(g \geq 2)$ is stable if and only if it is of the form E_ρ, for some irreducible unitary representation ρ of the fundamental group. Moreover, the topological type and the equivalence class of ρ under conjugation by elements of K are uniquely determined by E.*

The next Proposition sets the existence of irreducible and unitary representations of any type when G is semisimple. In this way we can always guarantee the existence of stable G-bundles on X. We actually can generalise a little more the following proposition to the case where G is reductive as we will discuss later on the Proposition 2.18.

PROPOSITION 1.82. *[Proposition 7.7,* **[Ram75]***] Let X be a compact Riemann surface of genus greater or equal to 2. For any $c \in \pi_1(G)$, G semisimple, there is an irreducible unitary representation $\rho : \pi_1(X) \to K$ such that $\chi(E_\rho) = c$.*

THEOREM 1.83. *[Thm. 5.9,* **[Ram96]** *] Let G be a connected reductive algebraic group and X be a compact Riemann surface with genus $g \geq 2$. The coarse moduli space $\mathcal{M}^{ss,d}$ of isomorphism classes of semistable principal G-bundles of topological type d*

is irreducible, projective, normal and Cohen-Macaulay variety, and its dimension is given by $(g-1)\mathrm{dim}G + \mathrm{dim}Z(G)$. *The subset* $\mathcal{M}^{s,d}$ *of* $\mathcal{M}^{ss,d}$ *corresponding to stable* G*-bundles is open (and dense).*

CHAPTER 2

Schottky Representations

Given a compact Riemann surface X of genus $g \geq 1$, we may write it as a quotient Ω_Σ / Σ where Σ is a Schottky group and Ω_Σ is the corresponding region of discontinuity. The Schottky group Σ is a group generated by the image of a representation of the free group F_N, with N generators, into $PSL(2,\mathbb{C})$ (up to conjugation). These ideas lead Florentino (in [**Flo01**]) to define the concept of Schottky representation, that is, a homomorphism $\rho : F_g \to GL(n,\mathbb{C})$ and to relate it with the notion of vector bundles induced by representations of a free group.

In this chapter we extend the definition of Schottky representation in [**Flo01**] to representations $\rho : F_g \to G \times Z$ from the free group with g generators to the product $G \times Z$ where G is a connected reductive algebraic group and Z the corresponding centre. We also prove some properties of the set consisted by all representations of this type.

Let us begin by fixing some notation. We denote by X a compact Riemann surface of genus $g \geq 2$ and by $\pi_1 = \pi_1(X)$ the corresponding fundamental group with the usual presentation

$$\pi_1(X) = \left\{ \alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g : \prod_{i=1}^{g} [\alpha_i, \beta_i] = 1 \right\}$$

and let F_g denote the free group in g generators $\gamma_1, \cdots, \gamma_g$.

2.1. Free group representations

In order to define Schottky representation for a general reductive group, we begin by recalling the definitions for the case of $GL(n,\mathbb{C})$. Here, we denote by $B_1, \cdots, B_g$ the g generators of F_g. Consider a group homomorphism $p : \pi_1 \to F_g$ with the following properties

$$p(\alpha_i) = \mathbf{1} \qquad \text{and} \qquad p(\beta_i) = B_i$$

for $i = 1, \cdots, g$, and where $\mathbf{1}$ denotes the identity element of F_g. This homomorphism lead us to the following short exact sequence of groups

$$1 \to N \to \pi_1(X) \xrightarrow{p} F_g \to 1 \tag{2.1.1}$$

where N denotes the smallest normal subgroup of $\pi_1(X)$ containing the elements $\{\alpha_1, \cdots, \alpha_g\}$. For each homomorphism $\rho : \pi_1 \to GL(n, \mathbb{C})$, the map p induces one homomorphism $\tilde{\rho} : F_g \to GL(n, \mathbb{C})$ defined by $\tilde{\rho}(B_i) = \rho(p(\beta_i))$ where $B_i = p(\beta_i)$. According to this construction, Florentino defined the concept of **Schottky representations** as homomorphisms $\tilde{\rho} : F_g \to GL(n, \mathbb{C})$ that are obtained in this way, i.e., all homomorphisms $\rho : \pi_1(X) \to GL(n, \mathbb{C})$ such that $\rho(\alpha_i) = 1, \forall \alpha_i \in \pi_1(X)$. In this context, $\mathcal{S}_n$ denotes the set of all Schottky representations constructed in the above way.

We generalise this concept for representations into an arbitrary connected reductive algebraic group G (over $\mathbb{C}$) in the following definition.

DEFINITION 2.1. A representation $\rho : \pi_1(X) \to G$ is called a **Schottky representation** if $\rho(\alpha_i) \in Z$ for all $\alpha_i \in \pi_1(X)$, where Z denotes the center of the connected reductive algebraic group G. The set of all Schottky representations is denoted by $\mathcal{S}$.

The set of all Schottky representations

$$\mathcal{S} = \{\rho \in \mathrm{Hom}(\pi_1(X), G) : \rho(\alpha_i) \in Z(G)\},$$

as a subset of $\mathrm{Hom}(\pi_1(X), G)$ has a nice structure, as we can see in the following Lemma. The proof of this lemma is a similar result to the one for the case of $\mathrm{Hom}(\pi_1(X), GL(n, \mathbb{C}))$ (definition 1.18), the set $\mathcal{S}$ is also an algebraic variety.

LEMMA 2.2. *The set of Schottky representations $\mathcal{S}$ is an algebraic variety.*

PROOF. If we consider the map $e_v : \mathcal{S} \to G^{2g}$ defined by

$$e_v(\rho) = (\rho(\beta_1), \rho(\alpha_1), \cdots, \rho(\beta_g), \rho(\alpha_g))$$

it is clear that each representation $\rho \in \mathcal{S}$ is completely defined by the image of the generators $\{\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g\}$. Thus, as in the case of $\mathrm{Hom}(\pi_1(X), G)$, this map is an embedding in G^{2g} and consequently $\mathcal{S}$ gets the structure of an algebraic subvariety of G^{2g}. □

Recall that we have fixed a set of generators $\{B_1, \cdots, B_g\}$ of the free group F_g and $\{\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g\}$ of the fundamental group $\pi_1(X)$. As above, given a Schottky representation ρ, since $\rho(\alpha_i) \in Z$ and as a representation of $\pi_1(X)$, we can consider this representation as a representation from the free group F_g. In order to obtain this correspondence, we define a representation

$$\tilde{\rho} : F_g \to G \times Z \tag{2.1.2}$$

as a pair of representations $(\tilde{\rho}_1, \tilde{\rho}_2)$ where $\tilde{\rho}_1 : F_g \to G$ maps each $B_i \in F_g$ to $\rho(\beta_i) \in G$ and $\tilde{\rho}_2 : F_g \to Z$ maps each $B_i \in F_g$ to $\rho(\alpha_i) \in Z$.

In what follows, we aim to prove that we can identify the varieties $\mathcal{S}$ and $\mathrm{Hom}(F_g, G \times Z)$. Afterwards, if we consider a G-action over $\mathcal{S}$, we prove that this action coincides with the analogous G-action on $\mathrm{Hom}(F_g, G \times Z)$. With these identifications it will be simpler and more intuitive to work with the set of Schottky representations $\mathcal{S}$. We just have to think on it as a set of representations of the free group in $G \times Z$, $\mathrm{Hom}(F_g, G \times Z)$. Let us start by defining an action of the algebraic group $G \times Z$ on $\mathrm{Hom}(F_g, G \times Z)$ by conjugation in the following way

$$(g, h) \cdot \tilde{\rho} = (g, h) \cdot (\tilde{\rho}_1, \tilde{\rho}_2) = \left(g\tilde{\rho}_1 g^{-1}, h\tilde{\rho}_2 h^{-1}\right).$$

According to the fact that $h \in Z$, this action reduces to

$$(g, h) \cdot \tilde{\rho} = \left(g\tilde{\rho}_1 g^{-1}, \tilde{\rho}_2\right).$$

Hence, this is actually a G-action on $\mathrm{Hom}(F_g, G \times Z)$ where each element $g \in G$ acts only in the first coordinate

$$g \cdot \tilde{\rho} = \left(g\tilde{\rho}_1 g^{-1}, \tilde{\rho}_2\right)$$

for all $g \in G$.

LEMMA 2.3. *If G is a reductive algebraic group, there exists a categorical quotient $\mathrm{Hom}\,(F_g, G\times Z)\,/\!\!/G$, that is, a surjective G-morphism*

$$p:\mathrm{Hom}\,(F_g, G\times Z)\to \mathrm{Hom}\,(F_g, G\times Z)\,/\!\!/G.$$

PROOF. Nagata's Theorem states that the ring of invariants $\mathbb{C}\left[\mathrm{Hom}\,(F_g, G\times Z)\right]^G$ is finitely generated when G is reductive. Additionally, [Theorem 3.5, [**New78**]] or [Thm. 6.1, [**Dol03**]], stating that if a reductive group acts on a G-variety then there exists a categorical quotient $\mathrm{Hom}\,(F_g, G\times Z)\,/\!\!/G$. □

In the next theorem we prove that any Schottky representation can be defined as an element of the variety $\mathrm{Hom}(F_g, G\times Z)$ and vice versa.

PROPOSITION 2.4. *Let G be a complex connected (reductive) algebraic group and let Z denotes its center, then the following algebraic G-varieties are isomorphic*

$$\mathcal{S}\cong \mathrm{Hom}(F_g, G\times Z)\cong (G\times Z)^g.$$

PROOF. Since all representations of $\mathcal{S}$ or $\mathrm{Hom}\,(F_g, G\times Z)$ are completely defined by the corresponding image of its generators, we start by defining two morphims of varieties, $\phi:\mathcal{S}\to\mathrm{Hom}\,(F_g, G\times Z)$ and $\psi:\mathrm{Hom}\,(F_g, G\times Z)\to\mathcal{S}$, in terms of the respective generators. Since we are working over $\mathbb{C}$, it is enough to show that ϕ is a bijective homomorphism and that $\psi\circ\phi = id_{\mathcal{S}}$ and $\phi\circ\psi = id_{\mathrm{Hom}(F_g,G\times Z)}$.

Each representation $\rho:\pi_1(X)\to G$ of $\mathcal{S}$ fulfils $\rho(\alpha_i)\in Z$ for all $i = 1,\cdots,g$. Hence, we define the morphism ϕ in the following way

$$\begin{array}{rccc} \phi: & \mathcal{S} & \to & \mathrm{Hom}\,(F_g, G\times Z) \\ & \rho & \mapsto & \phi(\rho)=\tilde{\rho} \end{array} \tag{2.1.3}$$

where the element $\tilde{\rho}$ is a pair of representations $(\tilde{\rho}_1,\tilde{\rho}_2)\in\mathrm{Hom}\,(F_g, G\times Z)$ with

$$\begin{array}{rccc} \tilde{\rho}_1: & F_g & \to & G \\ & B_i & \mapsto & \rho(\beta_i) \end{array} \quad \text{and} \quad \begin{array}{rccc} \tilde{\rho}_2: & F_g & \to & Z \\ & B_i & \mapsto & \rho(\alpha_i)\,. \end{array} \tag{2.1.4}$$

The elements B_i's, α_i's and β_i's denote, respectively, the g generators of the free group F_g and the $2g$ the generators of $\pi_1(X)$.

Similarly, we define the map

$$\psi : \operatorname{Hom}(F_g, G \times Z) \rightarrow \mathcal{S} \tag{2.1.5}$$

assigning to each representation $\tilde{\rho}$ in $\operatorname{Hom}(F_g, G \times Z)$ the representation $\psi(\tilde{\rho}) = \psi(\tilde{\rho}_1, \tilde{\rho}_2) = \rho$ in $\mathcal{S}$ in the following way

$$\begin{aligned} \psi(\tilde{\rho})(\beta_i) &= \tilde{\rho}_1(B_i) \in G \\ \psi(\tilde{\rho})(\alpha_i) &= \tilde{\rho}_2(B_i) \in Z \end{aligned} \tag{2.1.6}$$

thinking on $\tilde{\rho}(B_i)$ as $(\tilde{\rho}_1(B_i), \tilde{\rho}_2(B_i)) = (\tilde{\rho}_1(B_i), e) \cdot (e, \tilde{\rho}_2(B_i))$.

Now, we prove that ϕ is bijective. According to the way we had defined ϕ, the injectivity $\phi(\rho) = \phi(\rho') \Leftrightarrow \rho = \rho'$ is obvious. Concerning to surjectivity, we take an arbitrary element $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2) \in \operatorname{Hom}(F_g, G \times Z)$. We intend to prove that we can define one representation $\rho \in \mathcal{S}$ such that $\phi(\rho) = \tilde{\rho}$.

The representation $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2) : F_g \rightarrow G \times Z$ is such that $\tilde{\rho}_1(B_i) \in G$ and $\tilde{\rho}_2(B_i) \in Z$ and thereby we construct the representation $\rho \in \operatorname{Hom}(\pi_1, G)$ in the ensuing way

$$\rho(\beta_i) = \tilde{\rho}_1(B_i) \in G \ \text{ and } \ \rho(\alpha_i) = \tilde{\rho}_2(B_i) \in Z.$$

Clearly, this representation ρ is Schottky and

$$\begin{cases} \phi(\rho)(\beta_i) = & \tilde{\rho}_1(B_i) \\ \phi(\rho)(\alpha_i) = & \tilde{\rho}_2(B_i) \end{cases}$$

that is, $\phi(\rho) = \tilde{\rho}$.

In order to prove the remaining part of the proposition we have to show that $\psi \circ \phi = id_{\mathcal{S}}$ and $\phi \circ \psi = id_{\operatorname{Hom}(F_g, G \times Z)}$.

Given any element ρ of $\mathcal{S}$ the image of the map $(\psi \circ \phi)(\rho)$ is given by the composition of laws in (2.1.4) and (2.1.6). This means that, for each generators α_i and β_i of $\pi_1(X)$, we have the following equalities

$$(\psi \circ \phi)(\rho)(\alpha_i) = \psi((e, \tilde{\rho}_2(B_i))) = \tilde{\rho}_2(B_i) = \rho(\alpha_i)$$

and

$$(\psi \circ \phi)(\rho)(\beta_i) = \psi((\tilde{\rho}_1(B_i), e)) = \tilde{\rho}_1(B_i) = \rho(\beta_i).$$

Hence we obtain, for an arbitrary $\rho \in \mathcal{S}$, $(\psi \circ \phi)(\rho) = \rho$. Thus, $\psi \circ \phi = id_{\mathcal{S}}$.

Remains to prove that $\phi \circ \psi = id_{\mathrm{Hom}(F_g, G\times Z)}$. In an analogous way, for any element $\tilde{\rho}$ of $\mathrm{Hom}(F_g, G \times Z)$, let us compute the image of the map $(\phi \circ \psi)(\tilde{\rho})$. First, by (2.1.6), the image $\rho = \psi(\tilde{\rho}) = \psi(\tilde{\rho}_1, \tilde{\rho}_2)$ is defined by $\rho(\alpha_i) = \psi(e, \tilde{\rho}_2(B_i))$ and $\rho(\beta_i) = \psi(\tilde{\rho}_1(B_i), e)$. According to the fact that ψ is a homomorphism, we write the image $\psi(\tilde{\rho})(B_i)$ in the following way $\psi(\tilde{\rho})(B_i) = \psi(\tilde{\rho}_1(B_i), e)\, \psi(e, \tilde{\rho}_2(B_i))$. Now, the composition $(\phi \circ \psi)(\tilde{\rho}(B_i))$ is equal to

$$\phi(\psi(\tilde{\rho}_1(B_i), e)\, \psi(e, \tilde{\rho}_2(B_i)))$$

and using the fact that ϕ is a homomorphism, we obtain

$$\begin{aligned}(\phi \circ \psi)(\tilde{\rho})(B_i) &= \phi(\psi(\tilde{\rho}_1(B_i), e))\, \phi(\psi(e, \tilde{\rho}_2(B_i))) \\ &= \phi(\rho(\beta_i))\, \phi(\rho(\alpha_i)) \\ &= (\tilde{\rho}_1(B_i), e)\, (e, \tilde{\rho}_2(B_i)) \\ &= \tilde{\rho}(B_i).\end{aligned}$$

Then, we can conclude that, $\phi \circ \psi = id_{\mathrm{Hom}(F_g, G\times Z)}$.

At this point, we demonstrate that ϕ is an isomorphism of algebraic varieties. To finish the Proposition's proof we just have to determine that the map ϕ is G-equivariant, that is, $\phi(g \cdot \rho) = g \cdot \phi(\rho)$ for all $g \in G$. As above, it is enough to verify this property over the generators of $\pi_1(X)$ and F_g.

$$g \cdot \phi(\rho(\alpha_i)) = g \cdot (e, \tilde{\rho}_2(B_i)) = g\tilde{\rho}_2(B_i)\, g^{-1} = \tilde{\rho}_2(B_i) = \phi(\rho(\alpha_i)) \overset{\rho(\alpha_i) \in Z(G)}{=} \phi(g \cdot \rho(\alpha_i))$$

Now, let us consider that $g \cdot \rho = \sigma$ and $\phi(\sigma) = \tilde{\sigma} = (\tilde{\sigma}_1, \tilde{\sigma}_2)$

$$g \cdot \phi(\rho(\beta_i)) = g \cdot (\tilde{\rho}_1(B_i), e) = \left(g\tilde{\rho}_1(B_i)g^{-1}, e\right) = (\tilde{\sigma}_1(B_i), e),$$

additionally,

$$\phi(g \cdot \rho(\beta_i)) = \phi(\sigma(\beta_i)) = (\tilde{\sigma}_1(\beta_i), e).$$

Then we obtain the G-equivariance of ϕ, that is, $g \cdot \phi(\rho) = \phi(g \cdot \rho)$. In conclusion, we have the isomorphism of the G-varieties $\mathcal{S} \cong \mathrm{Hom}(F_g, G \times Z)$ and, according to the map defined on (1.2.1), the isomorphism $\mathrm{Hom}(F_g, G \times Z) \cong (G \times Z)^g$ follows immediately. □

Having the set of Schottky representations $\mathcal{S}$ identified with the set of representations of the free group $\mathrm{Hom}(F_g, G \times Z)$, it becomes natural to wonder how can we write the corresponding categorical quotient $\mathbb{S} := \mathcal{S}/\!\!/G$.
According to the Proposition 2.4, both $\mathcal{S}/\!\!/G$ and $\mathrm{Hom}(F_g, G \times Z)/\!\!/G$ are good quotients of $\mathcal{S}$ by G. Good quotients are categorical, namely, there exists a unique morphism (up to isomorphism) $\underline{\phi}$ such that the following diagram

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\phi} & \mathrm{Hom}(F_g, G \times Z(G)) \\ {\scriptstyle \pi_{\mathcal{S}}}\downarrow & \searrow^{\pi_\phi} & \downarrow{\scriptstyle \pi_{\mathrm{Hom}}} \\ \mathcal{S}/\!\!/G & \xrightarrow{\underline{\phi}} & \mathrm{Hom}(F_g, G \times Z(G))/\!\!/G \end{array}$$

commutes, i.e., $\underline{\phi} \circ \pi_{\mathcal{S}} = \pi_\phi$. Furthermore, since ϕ is an isomorphism of G-varieties, it is obvious that $\underline{\phi}$ is an isomorphism of categorical quotients.
Having all this into account, we state the following Lemma.

Lemma 2.5. *The isomorphism of affine G-varieties $\phi : \mathcal{S} \to \mathrm{Hom}(F_g, G \times Z)$ induces a unique isomorphism (up to isomorphism) between the corresponding categorical quotients*

$$\underline{\phi} : \mathcal{S}/\!\!/G \to \mathrm{Hom}(F_g, G \times Z)/\!\!/G.$$

An important feature over varieties is irreducibility, the next proposition establishes when we obtain on these varieties such property.

PROPOSITION 2.6. *Let G be a connected reductive algebraic group. The categorical quotient $\mathbb{S} = \mathcal{S} /\!\!/ G$ is irreducible if and only if the center of G, Z, is connected.*

PROOF. If G is connected then the character variety $C(F_g, G) = \mathrm{Hom}(F_g, G) /\!\!/ G$ is irreducible [Lemma 6.1, [**Mar00**]]. The group $G \times Z$ is connected if and only if Z is connected (since G is connected). In this way, the geometric quotient $\mathbb{S} \cong \mathrm{Hom}(F_g, G \times Z) /\!\!/ (G \times Z)$ is irreducible since the connected reductive algebraic group $G \times Z$ acts on the connected algebraic variety $\mathrm{Hom}(F_g, G \times Z)$. Moreover, since the action of Z is trivial, $\mathrm{Hom}(F_g, G \times Z) /\!\!/ (G \times Z)$ coincides with $\mathrm{Hom}(F_g, G \times Z) /\!\!/ G$. $\square$

In Proposition 2.6, we were forced to add the connectedness condition on the center Z of the algebraic group G. Since we are considering the case of a connected reductive algebraic group G and, usually, its center is non connected, we want to consider this possibility. In this way, let us assume that the center of G, Z, can be non connected and correspondingly $G \times Z$ might be non connected reductive group.

Given the center Z of a connected reductive algebraic group G, the connected componente Z° is an algebraic torus and the quotient $Z_f = Z/Z^\circ$ is finite. Moreover there is a short exact sequence $1 \to Z^\circ \to Z \to Z_f \to 1$ and thus we can write the algebraic group Z as a semidirect product,

$$Z = Z^\circ \rtimes Z_f.$$

This decomposition corresponds to a surjective homomorphism with finite kernel: $Z^\circ \times Z_f \to Z$ such that to each (z°, z_f) we assign the element $z = z^\circ z_f$.

REMARK. The finite group Z_f is the kernel of the projection $p : G \to G/[G, G] \to Z^\circ(G)$.

As a variety, we can write Z as a cartesian product of above subgroups, $Z = Z^\circ \times Z_f$. Having these facts in mind we establish the following Proposition which gives us a procedure to compute the number of irreducible components of the categorical quotient $\mathbb{S}$ and moreover, it gives a description of each connected component .

PROPOSITION 2.7. *Let G be a connected reductive algebraic group and let Z denote the center of G. All irreducible components of the categorical quotient $\mathbb{S}$ are isomorphic to*

$$\mathrm{Hom}\,(F_g, G \times Z^\circ) /\!\!/ G \cong (G^g /\!\!/ G) \times (Z^\circ)^g$$

where Z° is the identity component of Z. Moreover, the number of irreducible components of the categorical quotient $\mathbb{S} = \mathcal{S} /\!\!/ G$ is given by the number of elements of the quotient $Z_f = Z/\,Z^\circ$, to the power of g, that is, $|Z_f|^g$.

PROOF. As it was mentioned before the center of G, Z, can be written, thinking as an algebraic variety, as $Z \cong Z^\circ \times Z_f$. In this way, as varieties, we get the following isomorphism

$$\begin{aligned} \mathrm{Hom}\,(F_g, G \times Z) &\cong \mathrm{Hom}\,(F_g, G) \times \mathrm{Hom}\,(F_g,\, Z) \\ &\cong G^g \times Z^g \\ &\cong (G \times Z^\circ)^g \times (Z_f)^g \end{aligned}$$

Let us now consider a G-action on the above varieties. The former is defined by $g \cdot \rho(\gamma) = g\rho(\gamma)\, g^{-1}$. In the latter one we define the G-action on $(G \times Z^\circ)^g$(and on $(Z_f)^g$) as $g \cdot (h, z) = (ghg^{-1}, gzg^{-1}) = (ghg^{-1}, z)$ (respectively, $g \cdot z_f = g z_f g^{-1} = z_f$). It is obvious that G acts trivially on Z° and on Z_f. Hence we obtain the following isomorphic varieties $\mathrm{Hom}\,(F_g, G \times Z) /\!\!/ G \cong (G^g /\!\!/ G) \times Z^g$. If we denote by $\mathbb{S}^\circ = (G \times Z^\circ)^g /\!\!/ G = (G^g /\!\!/ G) \times (Z^\circ)^g$, this corresponds to a connected component of $\mathbb{S}$. Taking all this into account, we obtain the following isomorphism

$$\mathbb{S} \cong \mathbb{S}^\circ \times (Z_f)^g\,.$$

□

Now that we have analysed the connected components of the categorical quotient $\mathbb{S}$, we want to compute the corresponding geometric quotient. By Rosenlicht theorem, there exists an open G-stable subset U of $\mathcal{S}$ such that there exists the geometric quotient $U /\!\!/ G$. According to Mumford's Theorem [1.10, [**MFK94**]] the set of all stable points (definition 1.24(4)) of $\mathcal{S}$, $\mathcal{S}^s$, is such that the quotient $\mathbb{S}^s = \mathcal{S}^s /\!\!/ G$ is a geometric quotient of $\mathcal{S}^s \subset \mathcal{S}$.

PROPOSITION 2.8. *Let $\mathcal{S}^s$ denote the subset of $\mathcal{S}$ consisted by all stable (irreducible) representations of $\mathcal{S}$. Then $\mathcal{S}^s$ is a (open) dense subset of $\mathcal{S}$ (in complex topology). Moreover, $\mathbb{S}^s = \mathcal{S}^s /\!\!/ G$ is a geometric quotient of $\mathcal{S}^s$.*

PROOF. First let us remind a Sikora's Corollary [Cor. 31(2),[**Sik10**]] stating that any representation ρ, of a free group (with $N \geq 2$ elements) on a reductive group G, is a stable representation if and only if ρ is irreducible. According to another Sikora's result [Prop. 29 (1), [**Sik10**]], the set of irreducible representations is dense in $\mathcal{S}$. To finish, we remind a Theorem [Thm. 1.10, [**MFK94**]] (or [Thm 3.14, [**New78**]]) which assert that the quotient of the set of stable points is a geometric quotient, and we apply to the set of stable representations $\mathcal{S}^s$ and we conclude that $\mathbb{S}^s = \mathcal{S}^s /\!\!/ G$ is a geometric quotient of $\mathcal{S}^s$. ☐

The following corollary establishes the analogy of working with the geometric quotients $\mathbb{S}^s$ and $\mathrm{Hom}\,(F_g, G \times Z)^s /\!\!/ G$.

PROPOSITION 2.9. *Let $\mathcal{S}^s$ and $\mathrm{Hom}\,(F_g,\, G \times Z)^s$ denote the subsets of stable representations of $\mathcal{S}$ and $\mathrm{Hom}\,(F_g,\, G \times Z)$ respectively. Then $\mathcal{S}^s \cong \mathrm{Hom}\,(F_g,\, G \times Z)^s$ and there exist the geometric quotients $\mathbb{S}^s = \mathcal{S}^s /\!\!/ G$ and $\mathrm{Hom}\,(F_g, G \times Z)^s /\!\!/ G$. Moreover, $\mathbb{S}^s \cong \mathrm{Hom}\,(F_g, G \times Z)^s /\!\!/ G$.*

PROOF. By Proposition 2.8, since $\mathcal{S}^s$ and $\mathrm{Hom}\,(F_g,\, G \times Z)^s$ are open (and dense) and $\phi : \mathcal{S} \to \mathrm{Hom}\,(F_g,\, G \times Z)$ is an isomorphism of G-varieties then the restriction of ϕ to stable points, $\phi|_{\mathcal{S}^s}$, maps isomorphically $\mathcal{S}^s$ to $\mathrm{Hom}\,(F_g,\, G \times Z)^s$. Considering the diagram of the proof of the Lemma 2.5 and restricting the corresponding maps to the subsets of stable representations we obtain the diagram

$$\begin{array}{ccc} \mathcal{S}^s & \xrightarrow{\ \phi\ } & \mathrm{Hom}^s\,(F_g,\, G \times Z) \\ {\scriptstyle \pi_{\mathcal{S}^s}}\downarrow & \searrow^{\pi_\phi} & \downarrow{\scriptstyle \pi_{\mathrm{Hom}}} \\ \mathcal{S}^s /\!\!/ G & \xrightarrow{\ \underline{\phi}\ } & \mathrm{Hom}^s\,(F_g, G \times Z) /\!\!/ G \end{array}$$

Since geometric quotients are categorical, the morphism $\underline{\phi}$ is unique (up to isomorphism). $\square$

According to the above Proposition, from now on, we make the following identifications

$$\mathbb{S}^s \equiv \mathrm{Hom}^s\,(G \times Z)\,/\!\!/G.$$

2.2. Good representations

In the definition 1.24(3) we refereed the concept of a good representation, this notion is quite important if we pretend, in particular, to handle with tangent spaces. These points have the particularity of being smooth points of the geometric quotient, as we will see throughout this section. Our main goal of this section is to study the set of these points and to state some properties that will be useful in the forthcoming sections.

Let us begin by defining the concept of good representation applied to the case of Schottky representations.

Definition 2.10. A representation $\rho \in \mathcal{S} \subset \mathrm{Hom}\,(\pi_1,\, G)$ is said to be **good** if ρ is irreducible as an element of $\mathrm{Hom}\,(\pi_1,\, G)$ and if the stabiliser of the representation ρ, $Z(\rho)$, coincides with the center Z of the group G. The set of all **good Schottky representations** is denoted by $\mathcal{S}^{\mathrm{g}}$.

As in the case of stable representations, the set of good representations is open in $\mathcal{S}$ thus it is very important to have the guarantee of non emptiness of the set $\mathcal{S}^{\mathrm{g}}$. The following Lemma allows us to assert the existence of good Schottky representations .

Lemma 2.11. *[Lemma 4.6,* **[Mar00]***] If Γ has a finitely generated group with N elements, F_N, as a quotient, for some $N \geq 2$, and if G is a connected reductive algebraic group then there exists a good representation in* $\mathrm{Hom}\,(\Gamma, G)$.

REMARK 2.12. For the case of compact Riemann surfaces X of genus $g \geq 2$, the fundamental group π_1 is a quotient of a free group F_N where $N \geq 2$, so above Lemma confirms directly the existence of good representations on $\mathrm{Hom}\,(\pi_1,\, G)$.

Now, we want to prove that the set of good Schottky representations is nonempty. In order to obtain this assertion, we first have to prove the following Proposition which claims that a unitary Schottky representation ρ in $\mathrm{Hom}\,(\pi_1,\, G)$ is good if and only if, when we consider it as an element $(\rho_1,\, \rho_2)$ in $\mathrm{Hom}\,(F_g,\, G \times Z)$, we have ρ_1 good.

PROPOSITION 2.13. *Let ρ be a representation of $\mathcal{S}$. Writing ρ as before, $\rho = (\rho_1, \rho_2)$: $F_g \to G \times Z$, where $\rho_1 : F_g \to G$ and $\rho_2 : F_g \to Z$ are homomorphisms, we have:*

(a) $Z(\rho) = Z(\rho_1)$,

(b) If ρ_1 is irreducible if and only if ρ is irreducible.
In particular, ρ is a good Schottky representation as an element of $\mathrm{Hom}\,(\pi_1\,(X)\,, G)$ if and only if $\rho_1 : F_g \to G$ is good too.

PROOF. Let ρ be a unitary Schottky representation, as a representation in $\mathrm{Hom}\,(\pi_1, G)$.

(a) Let us begin by proving that $Z\,(\rho) = Z\,(\rho_1)$. Since the representation ρ is completely defined by the image of the generators of π_1, the intersection of the corresponding stabilisers, in G, gives the stabiliser of ρ. Then,

$$\begin{aligned} Z\,(\rho) &= \bigcap_{i=1}^{g} Z\,(\rho\,(\beta_i)) \bigcap_{i=1}^{g} Z\,(\rho\,(\alpha_i)) \\ &= \bigcap_{i=1}^{g} Z\,(\rho_1\,(B_i)) \bigcap_{i=1}^{g} Z\,(\rho_2\,(\gamma_i)) \\ &= \bigcap_{i=1}^{g} Z\,(\rho_1\,(B_i)) \bigcap G \end{aligned}$$

since $\rho_2\,(B_i) \in Z$ then its centraliser consists in all elements of G. Then we obtain

$$Z\,(\rho) = \bigcap_{i=1}^{g} Z\,(\rho_1\,(B_i)) = Z\,(\rho_1)\,.$$

(b) Now we want to prove that ρ is reducible if and only if ρ_1 is reducible.

Let us suppose that ρ is reducible, this means that, the image $\rho(\pi_1)$ is contained in a proper parabolic subgroup of G

$$\rho(\pi_1) \subset P.$$

This is equivalent to have, in particular, all images of the generators of π_1 contained in P,

$$\Leftrightarrow \rho(\alpha_i), \rho(\beta_i) \in P, \ \forall i = 1, \cdots, g$$

This implies that

$$\rho(\beta_i) = \rho_1(B_i) \in P, \ \forall i \Leftrightarrow \rho_1(F_g) \subset P.$$

With this, we prove that ρ_1 being irreducible implies that ρ is irreducible.

Reciprocally, let us see that if ρ_1 is reducible then we obtain the reducibility of ρ.

If ρ_1 is reducible, then there is a proper parabolic subgroup of G such that $\rho_1(F_g) \subset P$. Since any parabolic P contains the center of G, we can conclude that

$$\begin{array}{l} \rho_1(F_g) \subset P \Leftrightarrow \rho_1(B_i) \in P, \ \forall i = 1, \cdots, g \Leftrightarrow \rho(\beta_i) \in P, \ \forall i = 1, \cdots, g \\ \rho_2(F_g) \subset Z \Leftrightarrow \rho_2(B_i) \in P, \ \forall i = 1, \cdots, g \Leftrightarrow \rho(\alpha_i) \in P, \ \forall i = 1, \cdots, g \end{array},$$

equivalently, $\rho(\pi_1) \subset P$. Thus ρ is reducible.

To conclude, we have reached to ρ is irreducible if and only if ρ_1 is irreducible and that $Z(\rho_1) = Z(\rho)$, i.e., ρ is good if and only if ρ_1 is good.

□

LEMMA 2.14. *If X is a Riemann surface of genus $g \geq 2$, the subsets of* ***good representations*** $\mathrm{Hom}^{\mathrm{g}}(\pi_1, G)$ *and* $\mathcal{S}^{\mathrm{g}}$ *are dense in* $\mathrm{Hom}(\pi_1, G)$ *and* $\mathcal{S}$*, respectively.*

PROOF. By Lemma 2.11 and remark 2.12, it is obvious that $\mathrm{Hom}(\pi_1, G)^{\mathrm{g}} \neq \varnothing$. Since the set $\mathrm{Hom}(\pi_1, G)^{\mathrm{g}}$ is open on $\mathrm{Hom}(\pi_1, G)$ (Prop.33, [**Sik10**]) it implies that $\mathrm{Hom}(\pi_1, G)^{\mathrm{g}}$ is dense on $\mathrm{Hom}(\pi_1, G)$.

Let us construct a good Schottky representation. In this way, we start by defining one representation $\rho = (\rho_1, \rho_2)$ where $\rho_1 : F_g \to G$, $\rho_2 : F_g \to Z$ and such that the homomorphism ρ_2 is constant and equal to e the identity element of G. According to Lemma 2.11, there always exists a good $\rho_1 : F_g \to G$ since F_g is the free group with at least $g = 2$ and since G is a connected reductive group. The representation $\rho = (\rho_1,\, \rho_2)$, where ρ_1 and ρ_2 are above representations, is a good Schottky representation. In fact, by Proposition 2.13, ρ_1 being good is equivalent to ρ being good. Now, since the set of good Schottky representations is open in $\mathcal{S}$ (again Prop.33, [**Sik10**]), we can conclude that it is dense on $\mathcal{S}$. $\square$

Let us denote by $\mathbb{S}^{\mathrm{g}} = \mathcal{S}^{\mathrm{g}} /\!\!/ G$ and by $\mathbb{G}^{\mathrm{g}} = \mathrm{Hom}^{\mathrm{g}}(\pi_1,\, G) /\!\!/ G$ the geometric quotients of $\mathcal{S}^{\mathrm{g}}$ and of $\mathrm{Hom}^{\mathrm{g}}(\pi_1,\, G)$ by G, respectively. If we take into consideration the following Proposition due to Sikora, we will be able to guarantee that the geometric quotients $\mathbb{G}^{\mathrm{g}}$ and $\mathbb{S}^{\mathrm{g}}$ are dense on the sets of irreducible representations $\mathbb{G}^i$ and $\mathbb{S}^i$, respectively.

Proposition 2.15. *[Cor.50, [**Sik10**]] For every reductive group G and every surface group or free group Γ, the geometric quotient of good representations $\mathrm{Hom}^{\mathrm{g}}(\Gamma, G) /\!\!/ G$ is an open subset of the categorical quotient of irreducible representations $\mathrm{Hom}^i(\Gamma, G) /\!\!/ G$ and a smooth complex manifold.*

Applying above Proposition to the cases $\mathrm{Hom}(\pi_1(X), G)$ and $\mathrm{Hom}(F_g, G \times Z)$, we obtain the following Theorem

Theorem 2.16. *For every reductive group G and for a compact Riemann surface of genus $g \geq 2$, the geometric quotients of good representations $\mathbb{S}^{\mathrm{g}} := \mathcal{S}^{\mathrm{g}} /\!\!/ G$ and $\mathbb{G}^{\mathrm{g}} := \mathrm{Hom}^{\mathrm{g}}(\pi_1(X), G) /\!\!/ G$ are nonempty open subsets of, respectively, $\mathbb{S}^i$ and $\mathbb{G}^i$. Furthermore, they are smooth complex manifolds.*

Proof. First, Lemma 2.14 assures that $\mathrm{Hom}^{\mathrm{g}}(\pi_1(X), G)$ and $\mathcal{S}^{\mathrm{g}}$ are nonempty and so they are dense in $\mathrm{Hom}^i(\pi_1(X), G)$ and $\mathcal{S}^i$, respectively. The Theorem comes directly by applying Proposition 2.15. $\square$

2.3. Unitary representations

Narasimhan, Seshadri and Ramanathan found the interest in the study of unitary and irreducible representations from a surface group to G. In the same context, it would be of great interest to find unitary and good Schottky representations when G is a connected reductive group over $\mathbb{C}$ and X is a compact Riemann surface with genus $g \geq 2$ with fundamental group $\pi_1 := \pi_1(X)$.
To start, we need to remind two relevant properties related to compact groups. One of them is, for a connected reductive algebraic group G over the complex numbers, there always exists a maximal compact connected real Lie group K such that its complexification $K_{\mathbb{C}}$ coincides with G. Another property is reminded in the following Theorem, which states that any connected compact (real) Lie group can be generated by two elements.

THEOREM 2.17. *[**[H.34]**] Let K be a connected compact Lie group. Then there exist two elements $g,\, h \in K$ such that $\overline{\langle g,h\rangle} = K$. Moreover, the set of pairs in K $\Gamma(K) := \left\{(g,\, h) : \overline{\langle g,h\rangle} = K\right\}$ is dense in $K \times K$.*

Now we are in conditions to establish the following Proposition.

PROPOSITION 2.18. *Let X denote a compact Riemann surface with genus $g \geq 2$ and let G be a connected reductive algebraic group. Then there exist unitary Schottky representations of π_1 into G, such that $\rho(\pi_1)$ is dense in K, a maximal compact of G.*

PROOF. Let K be a maximal compact subgroup of G. Let us construct a Schottky representation into K, such that ρ is unitary and irreducible, that is, $\rho(\pi_1) \subset K$ and is not contained in any proper parabolic subgroup of G. We define the Schottky representation

$$\rho(\alpha_i) = e, \quad \forall i = 1, \cdots, g \\ \rho(\beta_1) = g \\ \rho(\beta_2) = h \\ \rho(\beta_i) = e, \quad \forall i = 3, \cdots, g. \tag{2.3.1}$$

with $g, h \in K$ such that $\overline{\langle g, h\rangle} = K$, which is guaranteed by Theorem 2.17. It is obvious that the homomorphism ρ, constructed in (2.3.1), is Schottky (since all images of α_i by ρ are in the center). Moreover, ρ is unitary since $\rho(\pi_1) \subset \overline{\langle g, h\rangle} = K$. □

Proved the existence of unitary Schottky representations, we would like to be sure of the existence of unitary and good Schottky representations when the compact Riemann surface has genus $g \geq 2$. According to definition 2.10, a representation $\rho \in \mathcal{S}$ is good if it is good as an element of $\operatorname{Hom}(\pi_1(X), G)$, now we want to translate this definition thinking on ρ as a representation in $\operatorname{Hom}(F_g, G \times Z)$.

PROPOSITION 2.19. *Let X be a compact Riemann surface of genus $g \geq 2$ and let G be a connected reductive algebraic group then there is always a good and unitary Schottky representation of $\pi_1(X)$ in G.*

PROOF. Let us begin with a unitary Schottky representation $\rho : \pi_1(X) \to K \subset G$ constructed as in Proposition 2.18 such that the closure of the subgroup generated by g and h is K. So, $\overline{\rho(\pi_1(X))} = K$ and since K is Zariski dense in G, this implies that the Zariski closure of $\rho(\pi_1(X))$ is not contained in any proper parabolic subgroup of G. In this way, we obtain a unitary and irreducible representation. In order to prove that it is also good we have to prove that $Z_G(\rho) = Z(G)$.
First let us prove that $Z_G(\rho)$ coincides with the centraliser of K in G, $Z_G(K)$. The stabiliser of ρ consists in all elements of $a \in G$ such that $a \cdot \rho = \rho$. If an element $a \in Z_G(\rho)$ then, by construction a commutes with h and g.
Let $a \in G$ denote an element such that it commutes with g and h, that is, $ag = ga$ and $ah = ha$. Now for any $k \in K$ we want to prove that $ak = ka$. As an element of

K, k can be written as a limit of elements of $\langle g, h\rangle$, $k = \lim_n k_n$, with each k_n equals to an expression of the type $g^{j_1}h^{m_1}\cdots g^{j_l}h^{n_l}$, thus

$$ak = a\lim_n g^{j_1}h^{m_1}\cdots g^{j_l}h^{n_l} = \lim_n ag^{j_1}h^{m_1}\cdots g^{j_l}h^{n_l}$$

and since a commutes with g and h,

$$= \lim_n g^{j_1}h^{m_1}\cdots g^{j_l}h^{n_l}a = ka.$$

In this way, $a \in Z_G(K)$, namely, $Z_G(\rho) \subset Z_G(K)$.

Now we want to show that $Z_G(K) = Z_G$. Clearly, $Z_G \subset Z_G(K)$. Let us consider an arbitrary element a in $Z_G(K)$ and let us consider the homomorphism

$$\begin{array}{rccc} c_a: & G & \to & G \\ & g & \mapsto & c_a(g) = aga^{-1} \end{array}$$

This map c_a defined over K coincides with the identity map since $a \in Z_G(K)$.
Now, let us compute the differential of c_a in the identity e of G.

$$\begin{array}{rccc} d(c_a)_e: & T_eG & \to & T_{c_a(e)}G \\ & \| & & \| \\ & \mathfrak{g} & \to & \mathfrak{g} \\ & Y & \mapsto & aYa^{-1}. \end{array}$$

Each element Y of $\mathfrak{g}$ can be written as a sum

$$Y = Y_1 + iY_2$$

where $Y_1, Y_2 \in \mathfrak{k} = \mathfrak{Lie}(K)$. The image of this element by the linear map $d(c_a)_e$ becomes

$$d(c_a)_e(Y) = d(c_a)_e(Y_1 + iY_2) = d(c_a)_e(Y_1) + i\ d(c_a)_e(Y_2) = Y_1 + iY_2 = Y.$$

This means that $d(c_a)_e = \mathrm{id}$.

On the other hand, the linear map $d\left(c_a\right)_e$ coincides with the adjoint map Ad_a. Additionally, if we consider the adjoint map as an action of G into $GL\left(\mathfrak{g}\right)$, $\mathrm{Ad} : G \to GL\left(\mathfrak{g}\right)$, we have that $\ker \mathrm{Ad} = Z_G$. Having all these into account, since $d\left(c_a\right)_e = id_{\mathfrak{g}}$, we can conclude that $a \in Z_G$. $\square$

CHAPTER 3

Schottky Principal Bundles

In this chapter we introduce the concept of Schottky principal G-bundle E over a compact Riemann surface. We explain the relationship between E and its adjoint bundle AdE, in particular, with regard to semistability, existence of a flat connection and Schottky property.

In terms of notation, through this chapter X denotes a compact Riemann surface (with genus g), $\pi_1 = \pi_1(X)$ its fundamental group, G represents a connected reductive algebraic group and Z its center. Reminding the notation introduced in the previous chapter, $\mathcal{S} \equiv \mathrm{Hom}(F_g, G \times Z)$ is the set of Schottky representations, $\mathbb{S}$ denote the respective categorical quotient $\mathcal{S} /\!\!/ G$ and $\mathbb{G} = \mathrm{Hom}(\pi_1, G) /\!\!/ G$.

3.1. Schottky bundles

Throughout this section we give the definitions of Schottky bundles and we prove some results generalising others given by Florentino in [**Flo01**].

Let $\pi_1 = \pi_1(X)$ denote the fundamental group of a compact Riemann surface X of genus $g \geq 2$ with the following presentation

$$\pi_1(X) = \left\{ \alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g : \prod_{i=1}^{g} [\alpha_i, \beta_i] = 1 \right\}$$

and let F_N denote the free group with N generators $\gamma_1, \cdots, \gamma_N$.

Consider the sheaf of germs of holomorphic functions $\mathcal{G}$ from X to $GL(n, \mathbb{C})$ and the following maps

$$\nu : H^1(X, GL(n, \mathbb{C})) \to H^1(X, \mathcal{G})$$

that sends a flat $GL(n,\mathbb{C})$-bundle into the corresponding holomorphic vector bundle of rank n over X and

$$\begin{array}{cccc} V.: & G_n & \to & H^1(X,\, GL(n,\mathbb{C})) \\ & \rho & \mapsto & V_\rho := \widetilde{X} \times_\rho \mathbb{C}^n \end{array} \tag{3.1.1}$$

assigning to each representations $\rho \in G_n := \operatorname{Hom}(\pi_1,\, GL(n,\mathbb{C}))$ the corresponding induced flat vector bundle V_ρ. Florentino defines Schottky vector bundle as a bundle that is isomorphic to V_ρ where ρ is a Schottky representation, that is, a representations from the free group F_g to $GL(n,\mathbb{C})$.

Similarly to above construction, given a representation of the fundamental group $\pi_1(X)$ into a complex algebraic group G, $\rho : \pi_1(X) \to G$, we can construct a principal G-bundle $E_\rho := \widetilde{X} \times_\rho G$ (see example 1.44) where each point is an equivalence class given by the following relation

$$(\tilde{x},\ g) \sim \left(\tilde{x}\cdot\gamma,\ \rho(\gamma)^{-1}\cdot g\right), \qquad \forall\gamma\in\pi_1.$$

The principal G-bundle E_ρ is said to be induced by the representation $\rho : \pi_1 \to G$.

Definition 3.1. A principal G-bundle E_G over the Riemann surface X is a **Schottky principal G-bundle** if E_G is isomorphic to a bundle E_ρ where $\rho : \pi_1 \to G$ is a Schottky representation, that is, $\rho(\alpha_i) \in Z$ for all $i = 1, \cdots, g$.

Now, let us see some simple examples of Schottky principal bundles over a compact Riemann surface with genus $g > 1$.

Example 3.2. If $G = GL(n,\mathbb{C})$ then $Z \cong \mathbb{C}^*$ is the set of scalar matrices. We define $\rho : \pi_1 \to GL(n,\, \mathbb{C})$ in the following way

$$\rho(\alpha_i) = \lambda_i I_n, \quad \rho(\beta_i) \in U(n)$$

where I_n is the identity $n\times n$-matrix, $\lambda_i \in \mathbb{C}^*$ for $i = 1, \cdots, g$ and $U(n)$ denotes the unitary group.

Any $\gamma \in \pi_1(X)$ can be written as a combination of the generators elements (α_i's and β_i's). In this way, we can think in the equivalence class only in terms of the generators of $\pi_1(X)$.

$$(\tilde{x},\ g) \sim \left(\tilde{x} \cdot \alpha_i,\ \rho(\alpha_i)^{-1} \cdot g\right) = (\tilde{x} \cdot \alpha_i,\ g) \quad \text{and} \quad (\tilde{x},\ g) \sim \left(\tilde{x} \cdot \beta_i,\ \rho(\beta_i)^{-1} \cdot g\right).$$

In this case, the Schottky $GL(n,\ \mathbb{C})$-bundle is known as the frame bundle (the effect of each element $U(n)$ corresponds to changing basis).

EXAMPLE 3.3. Consider $G = SL(n, \mathbb{C})$, in this case, its center is $Z = \{\lambda I_n :\ \lambda^n = 1\}$. In a similar way, we define $\rho : \pi_1 \to SL(n,\ \mathbb{C})$ in the following way

$$\rho(\alpha_i) = \lambda_i I_n, \quad \rho(\beta_i) \in SL(n,\ \mathbb{C})$$

$\lambda_i \in \mathbb{C}^*$ and $\lambda_i^n = 1$ for $i = 1, \cdots, g$.

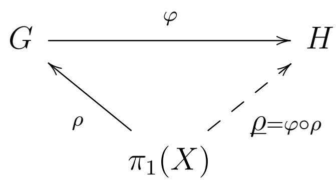
3.2. Associated Schottky bundles

We have treated the construction of bundles associated to initial ones (section 1.5) since this able us to relate properties of one type of bundles to another one.
In the following we intend to describe in which manner the property of being a Schottky bundle is transferred to associated bundles. Throughout this section, G and H denote reductive algebraic groups, Z_G and Z_H denote the corresponding centres.

Let us start by seeing that the Schottky property is stable under the extension of the structure group, this follows from Proposition 1.68.

COROLLARY 3.4. *Let G and H be algebraic groups and $\varphi : G \to H$ a group homomorphism such that $\varphi(Z_G) \subset Z_H$ where Z_G and Z_H are the centres of G and H, respectively. Let $E_H = (E_G \times H)/G$ be the H-bundle obtained from the G-bundle E_G extending the structure group by φ. If E_G is a Schottky G-bundle then E_H is a Schottky H-bundle .*

PROOF. By definition, we have that $E_H = E_G(H) = E_G \times_\varphi H$. Similarly to Proposition 1.68 and performing similar constructions we can get the following diagram



and, accordingly, we obtain $E_H \cong E_{\underline{\rho}}$. Schottky property of the representation $\underline{\rho} : \pi_1(X) \to H$ follows straightforward since by hypothesis $\varphi(Z_G) \subset Z_H$, thus $\underline{\rho}(\alpha_i) = \varphi\left(\rho\left(\alpha_i\right)\right) \in Z(H)$. $\square$

Important facts concerning to semistability criterion lead us to undertake a specific construction: the adjoint bundle associated to a principal G-bundle. For example, this help us to establish certain conditions relating the semistability property between both of them.

In example 1.46 was done the construction of the adjoint vector bundle associated to a G-bundle. It was established the notation $\mathrm{Ad}(E_G) = E_G \times_{\mathrm{Ad}} \mathfrak{g}$ (constructed on example 1.46) where the following points were identified

$$(y,\, v) \sim \left(y \cdot g,\, \mathrm{Ad}_g^{-1} \cdot v\right)$$

for all $g \in G$, where $y \in E_G$ and $v \in \mathfrak{g}$. Now, we use this construction to prove that if E_G is Schottky then its associated vector bundle $\mathrm{Ad}\left(E_G\right)$ is a Schottky vector bundle. Later on we prove the converse statement restricted to certain conditions.

PROPOSITION 3.5. *Let X be a compact Riemann surface and G a connected reductive algebraic group. If E_G is a Schottky G-bundle then the adjoint bundle* $\mathrm{Ad}(E_G)$*, associated to E_G, is a Schottky vector bundle with fibre* $\mathfrak{g}$.

PROOF. As E_G is a Schottky G-bundle , there is a representation $\rho : \pi_1 \to G$ with $\rho\left(\alpha_i\right) \in Z_G$ for all $i = 1, \cdots, g$ such that $E_G \cong E_\rho = \widetilde{X} \times_\rho G$. Each point of this bundle is obtained by the following identification

$$(\tilde{x},\, g) \sim \left(\tilde{x}\cdot\gamma,\, \rho\,(\gamma)^{-1}\cdot g\right).$$

By construction, the vector bundle associated to E_G by the adjoint representation can be seen as

$$\mathrm{Ad}(E_G) = E_G \times_{\mathrm{Ad}} \mathfrak{g} \cong E_\rho \times_{\mathrm{Ad}} \mathfrak{g}$$

and since $E_G \cong E_\rho$, correspondingly we have that

$$\mathrm{Ad}(E_G) \cong \widetilde{X} \times_\rho G \times_{\mathrm{Ad}} \mathfrak{g} \cong \widetilde{X} \times_{\mathrm{Ad}\rho} \mathfrak{g}$$

where the equivalence relation is given by

$$(y,\, v) = ((\tilde{x},\, g)\,,\, v) \sim ((\tilde{x},\, g)\,,\, v)\cdot\gamma = \Big(\left(\tilde{x}\cdot\gamma,\, \rho\,(\gamma)^{-1}\cdot g\right),\, \mathrm{Ad}^{-1}_{\rho(\gamma)}\cdot v\Big).$$

for all $\gamma \in \pi_1\,(X)$.

The G-action relation on $\widetilde{X}\times\mathfrak{g}$ is given by $(\widetilde{x}, v)\cdot g = \left(\widetilde{x}\cdot\gamma, \mathrm{Ad}^{-1}_{\rho(\gamma)}v\right)$, where $g = \rho\,(\gamma)$ and $\mathrm{Ad}_\rho : \pi_1\,(X) \to GL\,(\mathfrak{g})$ are given by the composition of the representations Ad and ρ.

Attended to the fact that $\rho(\alpha_i) \in Z_G$ and since $\ker\,(\mathrm{Ad}) = Z_G$, we obtain $\mathrm{Ad}\rho(\alpha_i) = \mathbf{1}$ for all $i = 1, \cdots, g$. Thus, we achieved to a representation $\mathrm{Ad}_\rho : \pi_1 \to GL(\mathfrak{g})$ that corresponds to a Schottky representation (in the sense of [**Flo01**]). In this way, we conclude that $\mathrm{Ad}(E_G) \cong V_{\mathrm{Ad}\rho} = \widetilde{X} \times_{\mathrm{Ad}\rho} \mathfrak{g}$ is a Schottky vector bundle with fibres isomorphic to the Lie algebra $\mathfrak{g}$. □

At this time, it is natural that we ask ourselves if the reciprocal of Proposition 3.5 works. Starting from a Schottky vector bundle V_ρ, do we obtain a Schottky principal bundle associated to the first?

Let us begin by construct a particular type of principal bundle associated to V_ρ. This bundle is called frame bundle. The following proposition states that, in this particular situation, the principal bundle obtained is Schottky.

Proposition 3.6. *Let X be a compact Riemann surface. If V is a Schottky vector bundle then the associated frame bundle, $GL(V)$, is a Schottky principal $GL(n, \mathbb{C})$-bundle.*

Proof. Since V is a Schottky vector bundle, there is a representation $\rho : \pi_1(X) \to GL(n, \mathbb{C})$ such that $V \cong V_\rho$ and that fulfils $\rho(\alpha_i) = 1$ for all $i = 1, \cdots, g$. We may construct the frame bundle $GL(V)$ associated to $V \cong \widetilde{X} \times_\rho \mathbb{C}^n$

$$GL(V) := \mathcal{I}som\,(\mathcal{O}_X^n, V)$$

consisted by all local isomorphisms between the trivial bundle $\mathcal{O}_X^n$ (of rank n) and V. The bundle $GL(V)$ is a principal $GL(n, \mathbb{C})$-bundle associated to V. $\square$

In order to establish an analogue of the above proposition, for the case of the adjoint bundle, we must add one condition, that is, the existence of a flat connection on the principal G-bundle as we can see in the following Proposition.

Proposition 3.7. *Let X be a compact Riemann surface and let G be a connected reductive algebraic group. Suppose that the G-bundle E_G admits a flat connection and let $\mathrm{Ad}(E_G)$ be the adjoint bundle associated to E_G. If $\mathrm{Ad}(E_G)$ is a Schottky vector bundle then E_G is a Schottky G-bundle.*

Proof. First, if E_G admits a flat connection then it is induced from the universal cover of X by a homomorphism $\rho : \pi_1(X) \to G$, that is,

$$E_G \cong \widetilde{X} \times_\rho G.$$

Consider the adjoint bundle $\mathrm{Ad}(E_G)$ associated to E_G by the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$. By Proposition 1.68, we have that $\mathrm{Ad}(E_G) = E_G(\mathfrak{g}) \cong E_{\mathrm{Ad}\rho}$ where

$$\begin{array}{ccc} \pi_1(X) & \xrightarrow{\quad\rho\quad} & G \\ & {\scriptstyle \mathrm{Ad}\rho}\searrow \quad \swarrow{\scriptstyle \mathrm{Ad}} & \\ & GL(\mathfrak{g}) & \end{array}.$$

Since by hypothesis $\mathrm{Ad}(E_G)$ is a Schottky vector bundle, this means that $\mathrm{Ad}\rho(\alpha_i) = \mathbf{1}, \quad \forall i = 1, \cdots, g$. As $\ker(\mathrm{Ad}) = Z(G)$ (Proposition 1.17), we may conclude that $\rho(\alpha_i) \in Z(G)$ for all $i = 1, \cdots, g$, that is, $E_G \cong E_\rho$ where ρ is a Schottky representation. □

If E_G admits a flat connection then the statements of above Propositions can be reformulated in the following manner.

Corollary 3.8. *Consider a compact Riemann surface X, a connected reductive algebraic group G and a principal G-bundle E_G with a flat connection. Under this conditions, E_G is a Schottky G-bundle if and only if the adjoint bundle $\mathrm{Ad}(E_G)$, associated to E_G, is a Schottky vector bundle with fibre $\mathfrak{g}$.*

Analogously, if G is a connected semisimple algebraic group, Proposition 1.63 ensures that E_G admits a flat connection whenever $\mathrm{Ad}(E_G)$ admits one too. Since this fact actually happens whenever $\mathrm{Ad}(E_G)$ is a Schottky vector bundle then the conditions of Proposition 3.7 are fulfilled.

Corollary 3.9. *Let X be a compact Riemann surface and G a connected semisimple algebraic group. Then E_G is a Schottky G-bundle if and only if the adjoint bundle $\mathrm{Ad}(E_G)$, associated to E_G, is a Schottky vector bundle with fibre $\mathfrak{g}$.*

We conclude that the fact of $\mathrm{Ad}\,(E_G)$ being a Schottky vector bundle is not sufficient to imply that the original principal bundle E_G is Schottky. In fact, we can have a principal G-bundle E_G such that $\mathrm{Ad}\,(E_G) = E_G \times_{\mathrm{Ad}} \mathfrak{g} \cong V_\rho$ but this does not imply that $E_G \cong E_{\underline{\rho}}$ with $\underline{\rho} : \pi_1(X) \to G$. Let us see an example.

Example. Consider any principal G-bundle E_G such that $c_1(E_G) \neq 0$ and with the corresponding adjoint bundle $\mathrm{Ad}\,(E_G)$ a line bundle $L = \widetilde{X} \times \mathbb{C}$. Since any line bundle of degree 0 is Schottky, $\mathrm{Ad}\,(E_G)$ is a Schottky vector bundle with the corresponding principal bundle E_G being non-Schottky.

CHAPTER 4

Schottky Map

In this chapter we intend to generalise the concept of Schottky map, constructed by [**Flo01**], for the case of Schottky representations $\mathcal{S} \subset \mathrm{Hom}(\pi, G)$ where G denotes a connected reductive algebraic group. We want to obtain a relationship between the set of equivalent Schottky representations and the set of isomorphic semistable flat G-bundles over a compact Riemann surface doing a similar approach to Ramanathan's ideas.

Throughout this chapter, G denotes a connected reductive group, X and $\pi_1 = \pi_1(X)$ stand for, respectively, a connected Riemann surface with genus $g \geq 2$ and its corresponding fundamental group, $\underline{M}_G$ denotes the set of isomorphism classes of flat G-bundles and $\mathcal{M}_G$ the moduli space of semistable G-bundles.

4.1. Analytic equivalence

In [**Flo01**], Florentino defined the map $V. : G_n \to H^1(X, GL(n, \mathbb{C}))$ that assigns to each representation ρ the corresponding Schottky vector bundle $V_\rho = \widetilde{X} \times_\rho \mathbb{C}^n$. Through this section we want to give analogous definitions applied in the context of principal bundles.

Let us start by defining the following map

$$\begin{array}{rccc} \mathbf{E}. : & \mathrm{Hom}(\pi_1(X), G) & \to & \underline{M}_G \\ & \rho & \mapsto & E_\rho \end{array} \tag{4.1.1}$$

assigning each representation $\rho : \pi_1(X) \to G$, where G is a connected reductive algebraic group (over the complex numbers), to the corresponding G-bundle induced by the representation ρ, $E_\rho = \widetilde{X} \times_\rho G$.

DEFINITION 4.1. The map $E.$ given by (4.1.1) assigning to each representation ρ the induced principal G-bundle $E_\rho = \widetilde{X} \times_\rho G$, restricted to the set of Schottky representations

$$W. := E.|_{\mathcal{S}} : \quad \mathcal{S} \quad \rightarrow \quad \underline{M}_G$$

is called **general Schottky map**.

The map $W.$ is obviously non injective since two different representations could induce isomorphic flat G-bundles. Even if we consider the above map defined on the set $\mathcal{S}/G$ of equivalence class of Schottky representations in the usual sense, namely, ρ is equivalent to σ when $\rho = g\sigma g^{-1}$ for an element $g \in G$ with $\rho,\, \sigma \in \mathcal{S}$, we obtain the following correspondence

$$\begin{array}{rccc} \mathbf{W}: & \mathcal{S}/\,G & \rightarrow & \underline{M}_G \\ & [\rho] & \mapsto & [E_\rho] \end{array}$$

which is still non-injective. In fact, we may have isomorphic G-bundles without the corresponding representations being equivalent in the usual sense.

On the point of view of Schottky uniformization, there exists another concept of equivalence which gives us a better equivalence relation in this case. To introduce this notion we need to consider the compact Riemann surface X written as a quotient of a domain of discontinuity Ω by a group Γ, that is, $X = \Omega/\,\Gamma$.

DEFINITION 4.2. Let $\rho,\, \sigma \in \mathrm{Hom}\,(\Gamma, G)$. The representations ρ and σ are **analytically equivalent** if there exists a complex analytic map $\psi : \Omega \rightarrow G$ such that $\psi\,(\gamma z)\,\sigma\,(\gamma) = \rho\,(\gamma)\,\psi\,(z)$ for all $\gamma \in \Gamma$, $z \in \Omega$.

Florentino established the way that analytic equivalence of representations in $\mathrm{Hom}\,(\pi_1,\, GL\,(n,\, \mathbb{C}))$ is translated in terms of relations between the corresponding induced vector bundles. In the case of G-bundles, given two analytic equivalent Schottky representations ρ and σ, next Lemma establishes that the corresponding induced bundles E_ρ and E_σ revels the same relationship.

LEMMA 4.3. *Let G be a connected reductive algebraic group, ρ, $\sigma \in \mathrm{Hom}(\pi_1, G)$, $\widetilde{x}_0 \in \widetilde{X}$ and let Ω^1_X be the canonical line bundle of X. Then the following conditions are equivalent:*

(1) $E_\rho \cong E_\sigma$;

(2) there exists a holomorphic function $h : \widetilde{X} \to G$ *such that*

$$h(\tilde{x}\gamma) = \rho(\gamma)^{-1} h(\tilde{x})\sigma(\gamma);$$

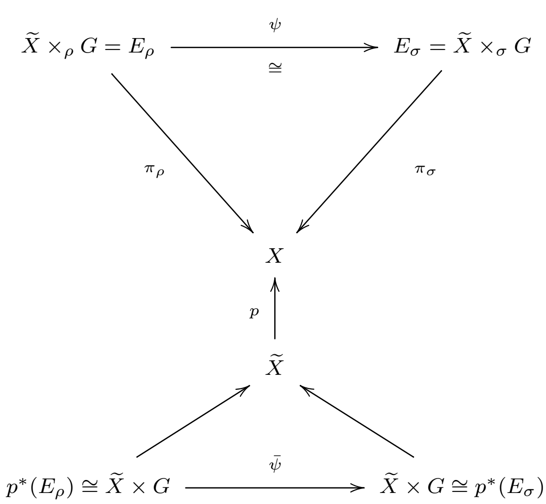
(3) there exists $\omega \in H^0(X, \mathrm{Ad}(E_\rho) \otimes \Omega^1_X)$ *such that*

$$\sigma(\gamma) = h_\omega(\tilde{x})^{-1}\rho(\gamma)h_\omega(\tilde{x}\cdot\gamma)$$

where h_ω *is the unique solution of the differential equation* $h^{-1}dh = \omega$ *with the condition* $h(\tilde{x}_0) = I$.

PROOF. Now let us prove that the above assertions are equivalent.

[(1) $\Leftrightarrow$ (2)] Since the G-bundles E_ρ and E_σ are obtained from $\widetilde{X}$ by extending the structure group by, respectively, ρ and σ to G, an isomorphism of these G-bundles corresponds to give a map between the trivial bundle $\widetilde{X} \times G$ with certain properties. Let us observe the following diagram



Since both $p^*(E_\rho)$ and $p^*(E_\sigma)$ are trivial over $\widetilde{X}$, we can apply Proposition 1.40 which states that isomorphism between E_ρ and E_σ is equivalent to obtain equivalent factors of automorphy on $\widetilde{X} \times \pi_1$ with values in G. Since ρ and σ are equivalent factors of automorphy, by definition (def. 1.38), there exists a holomorphic map $h : \widetilde{X} \to G$ such that $\sigma(\gamma) = h(\tilde{x})^{-1}\rho(\gamma)h(\tilde{x} \cdot \gamma)$ and equivalently

$$h(\tilde{x} \cdot \gamma) = \rho(\gamma)^{-1}h(\tilde{x})\sigma(\gamma).$$

[(2) $\Leftrightarrow$ (3)] To prove this equivalence, let us observe that the vector space $H^0(X, \mathrm{Ad}(E_\rho) \otimes \Omega^1_X)$ is consisted by holomorphic 1-forms on $\widetilde{X}$ with values in the Lie algebra $\mathfrak{g}$ of G,

$$\left\{\omega : \widetilde{X} \to \mathfrak{g} | \omega(\gamma)\gamma' = \gamma \cdot \omega\right\} = \Omega^1_{\mathrm{Ad}(E_\rho)}.$$

The action $\gamma \cdot \omega$ for $\gamma \in \pi_1(X)$ is performed via the adjoint action, that is, $\gamma \cdot \omega = \mathrm{Ad}^{-1}_{\rho(\gamma)}\omega$. Given (2) we have

$$h(\tilde{x} \cdot \gamma) = \rho(\gamma)^{-1}h(x)\sigma(\gamma) \Leftrightarrow \sigma(\gamma) = h(\tilde{x})^{-1}\rho(\gamma)h(\tilde{x} \cdot \gamma).$$

If we differentiate in order the coordinate $\tilde{x}$, we get

$$\begin{aligned} 0 &= d\sigma(\gamma) \\ &= -h(\tilde{x})^{-2}\, dh\rho(\gamma)h(\tilde{x} \cdot \gamma) + h(\tilde{x})^{-1}\rho(\gamma)\gamma' dh(\tilde{x} \cdot \gamma) \end{aligned}$$

equivalently,

$$\begin{aligned} h(\tilde{x})^{-2}\, dh\, \rho(\gamma)h(\tilde{x} \cdot \gamma) &= h(\tilde{x})^{-1}\rho(\gamma)\gamma' dh(\tilde{x} \cdot \gamma) \\ \Leftrightarrow h(\tilde{x})^{-1}\, dh\, \rho(\gamma)h(\tilde{x} \cdot \gamma) &= \rho(\gamma)\gamma' dh(\tilde{x} \cdot \gamma) \\ \Leftrightarrow (\rho(\gamma))^{-1}\; h(\tilde{x})^{-1}\, dh\, \rho(\gamma) &= \gamma' dh(\tilde{x} \cdot \gamma)(h(\tilde{x} \cdot \gamma))^{-1} \end{aligned}$$

Now putting $\eta = h^{-1}dh$, the equation can be rewritten as

$$\mathrm{Ad}^{-1}_{\rho(\gamma)}\eta = \eta(\gamma)\gamma'.$$

This means that η is a section of $\mathrm{Ad}(E_\rho) \otimes \Omega^1_X$ and we get (3).

Conversely, since the solution of differential equation $h\omega = dh$ with the condition $\omega(\tilde{x_0}) = I$ over the simply connected space $\widetilde{X}$ is unique and satisfies the equality (3) then, obviously it satisfies (2). □

This Lemma proves that analytic equivalence is the best way to relate representations and corresponding principal bundles. In fact, it states that isomorphic induced G-bundles corresponds to analytic equivalent representations and vice-versa. The usual equivalence relation between representations ($\rho = g\sigma g^{-1}$ where $g \in G$) only guarantees that the corresponding induced bundles are isomorphic and not the converse one.

4.2. Period map

Previously, we have mentioned how an isomorphism of flat G-bundles reflects in terms of the corresponding representations (Lemma 4.3). These representations are related in a particular way with a 1-form $\omega \in H^0(X, \mathrm{Ad}E_\rho \otimes \Omega^1_X)$. In this section, we want to explore the idea that given an element $\omega \in H^0(X, \mathrm{Ad}E_\rho \otimes \Omega^1_X)$ we obtain the corresponding representations in the context of Lemma 4.3. Actually, this remind us a familiar map called Period map.

DEFINITION 4.4. Let $\omega \in H^0(X, \mathrm{Ad}E_\rho \otimes \Omega^1_X)$ and $\tilde{x}_0 \in \widetilde{X}$ be a fixed point, the map

$$\begin{array}{rcl} \omega_{\tilde{x}_0} : \pi_1(X) & \to & \mathfrak{g} \\ \gamma & \mapsto & \omega_{\tilde{x}_0}(\gamma) = \int_{\tilde{x}_0}^{\tilde{x}_0 \cdot \gamma} \omega \end{array}$$

is called **period map** and it is an element of the group of 1-cochain $C^1(\pi_1, \mathrm{Ad}_\rho)$.

In the framework of our problem, in the next Proposition, we redefine the above map and we show that the way we define this map is precise.

PROPOSITION 4.5. *The **period map** is a well defined correspondence*

$$P_{\mathrm{Ad}\rho} : H^0\left(X, \mathrm{Ad}E_\rho \otimes \Omega^1_X\right) \to H^1\left(\pi_1(X), \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$$

that assigns to each ω *the cohomological class* $P_{\mathrm{Ad}\rho}(\omega) := [\omega]$ *where for each* $\gamma \in \pi_1(X)$, $\omega_{\tilde{x}}(\gamma) = \int_{\tilde{x}}^{\tilde{x}\cdot\gamma} \omega$.

PROOF. Let us fix $\tilde{x}_0 \in \widetilde{X}$ and consider arbitrary elements $\gamma_1, \gamma_2 \in \pi_1(X)$. In order to show that the map $P_{\mathrm{Ad}\rho}$ is well defined we have to prove two things. First, we have to prove that $P_{\mathrm{Ad}\rho}(\omega)$ is an element of the first cohomology group $H^1\left(\pi_1(X), \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$. Then, after, we have to prove that $P_{\mathrm{Ad}\rho}(\omega)$ does not depend on the fixed base point $\tilde{x}_0$.

Let us confirm that $\omega_{\tilde{x}_0}$ is an element of the set of 1-cocycles $Z^1\left(\pi_1(X), \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$. By definition $\omega_{\tilde{x}_0}(\gamma_0\gamma_1) = \int_{\tilde{x}_0}^{\tilde{x}_0\cdot(\gamma_0\gamma_1)} \omega$. Since $\tilde{x}_0 \cdot (\gamma_0\gamma_1) = (\tilde{x}_0 \cdot \gamma_0)\cdot\gamma_1$ we get

$$P_{\mathrm{Ad}\rho}^{\gamma_0\gamma_1}(\omega) = \omega_{\tilde{x}_0}(\gamma_0\gamma_1) = \int_{\tilde{x}_0}^{\tilde{x}_0\cdot\gamma_0} \omega + \int_{\tilde{x}_0\cdot\gamma_0}^{(\tilde{x}_0\cdot\gamma_0)\gamma_1} \omega$$

$$= \omega_{\tilde{x}_0}(\gamma_0) + \int_{\tilde{x}_0}^{\tilde{x}_0\gamma_1} \omega(\gamma_0)(\gamma_0)'$$

Since elements of $H^0(X, \mathrm{Ad}E_\rho \otimes K)$ verifies the property $\omega(\gamma)\gamma' = \gamma\cdot\omega$ for all $\gamma \in \pi_1(X)$, the previous is equal to

$$= \omega_{\tilde{x}_0}(\gamma_0) + \int_{\tilde{x}_0}^{\tilde{x}_0\gamma_1} \gamma_0\cdot\omega = \omega_{\tilde{x}_0}(\gamma_0) + \gamma_0\cdot\omega_{\tilde{x}_0}(\gamma_1).$$

then we obtain

$$P_{\mathrm{Ad}\rho}^{\gamma_0\gamma_1}(\omega) = P_{\mathrm{Ad}\rho}^{\gamma_0}(\omega) + \gamma_0\cdot P_{\mathrm{Ad}\rho}^{\gamma_1}(\omega)$$

This means that $P_{\mathrm{Ad}\rho}(\omega)$ is a 1-cocycle, i.e., $P_{\mathrm{Ad}\rho}(\omega) \in Z^1\left(\pi_1(X), \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$.

Now, we consider this map defined above another fixed base point $\tilde{x}_1 \in \widetilde{X}$ and we are going to prove that the map $P_{\mathrm{Ad}\rho}$ is independent of the base point. This means that we want to prove the class of the images $P_{\mathrm{Ad}\rho,\tilde{x}_1}^{\gamma}$ and $P_{\mathrm{Ad}\rho,\tilde{x}_0}^{\gamma}$ in $H^1\left(\pi_1(X), \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$ coincides.

The image of $P_{\mathrm{Ad}\rho,\tilde{x}_1}^{\gamma}(\omega)$ is defined by

$$P_{\mathrm{Ad}\rho,\tilde{x}_1}^{\gamma}(\omega) = \omega_{\tilde{x}_1}(\gamma) = \int_{\tilde{x}_1}^{\tilde{x}_1\cdot\gamma} \omega.$$

Since $\widetilde{X}$ is simply connected, we put above integral in the following way

$$P_{\mathrm{Ad}\rho,\tilde{x}_1}^{\gamma}(\omega) = \int_{\tilde{x}_1}^{\tilde{x}_0} \omega + \int_{\tilde{x}_0}^{\tilde{x}_0\cdot\gamma} \omega + \int_{\tilde{x}_0\cdot\gamma}^{\tilde{x}_1\cdot\gamma} \omega$$

$$= \int_{\tilde{x}_1}^{\tilde{x}_0} \omega + P_{\mathrm{Ad}_\rho,\tilde{x}_0}^{\gamma}(\omega) + \int_{\tilde{x}_0}^{\tilde{x}_1} \omega(\gamma)\gamma' = -\int_{\tilde{x}_0}^{\tilde{x}_1} \omega + P_{\mathrm{Ad}_\rho,\tilde{x}_0}^{\gamma}(\omega) + \int_{\tilde{x}_0}^{\tilde{x}_1} \gamma\cdot\omega$$

Then we get

$$P^{\gamma}_{\mathrm{Ad}_\rho,\tilde{x}_1}(\omega) = P^{\gamma}_{\mathrm{Ad}_\rho,\tilde{x}_0}(\omega) + \left(\underbrace{\mathrm{Ad}^{-1}_{\rho(\gamma)} \int_{\tilde{x}_0}^{\tilde{x}_1} \omega - \int_{\tilde{x}_0}^{\tilde{x}_1} \omega}_{\in B^1\left(\pi_1(X), \mathfrak{g}_{\mathrm{Ad}_\rho}\right)} \right). \tag{4.2.1}$$

We conclude that $P_{\mathrm{Ad}_\rho,\tilde{x}_1}$ and $P_{\mathrm{Ad}_\rho,\tilde{x}_0}$ differs by a 1-coboundary, that is, the map $P_{\mathrm{Ad}\rho}$ is well defined. □

4.3. Schottky moduli map

We want to achieve to an analogous result of [**NS65**] or [**Ram96**] for vector and principal bundles, respectively, which states that the set of analytic equivalence of special representations gives an open subset of the space moduli. In this way, our main goal is to define a map between the set of equivalent class of representations to the moduli space of semistable principal bundles. In this section, we give our first step in this direction.

Previously we defined the general Schottky map as the map $W. : \mathcal{S} \to \underline{M}_G$ where $\underline{M}_G$ represents the set of isomorphism class of flat G-bundles. Although, we could define this map in order that its image is contained in the moduli space of semistable G-bundles $\mathcal{M}^{ss}_G$, (introduced on subsection 1.8.2). With regard to obtain a well defined map let us begin by setting the following notation, let $\pi_1 := \pi_1(X)$ denote the fundamental group of X and

$$\mathcal{S}^\sharp = W^{-1}\left(\mathcal{M}^{ss}_G\right) \quad \Big(\text{respectively } \mathrm{Hom}\left(\pi_1, G\right)^\sharp = E^{-1}\left(\mathcal{M}^{ss}_G\right)\Big). \tag{4.3.1}$$

PROPOSITION 4.6. *The **Schottky moduli map***

$$\mathbf{W} : \mathcal{S}^\sharp \to \mathcal{M}^{ss}_G \tag{4.3.2}$$

is well defined.

PROOF. In order to prove that this map is well defined, we just have to show that the set $S^\sharp$ is nonempty. According to Proposition 2.18, there always exists unitary

Schottky representations. By Ramanathan's Proposition 1.77, we conclude that $\mathcal{S}^\sharp \neq \varnothing$. So, given $\rho \in \mathcal{S}^\sharp$, the corresponding G-bundle E_ρ is semistable. In this way and according to the definition, the map $\mathbf{W}.$ sending Schottky representations to the corresponding semistable G-bundle is well defined. □

If we consider the family $\mathcal{F} = \{E_\sigma\}_{\sigma\in\mathcal{I}}$ of holomorphic semistable G-bundles, where $\mathcal{I} = \{\sigma \in \mathrm{Hom}\left(\pi_1\left(X\right), G\right) :\ E_\sigma \text{ is semistable}\}$, according to Theorem 1.80, the map $E. : \mathcal{I} \to \mathcal{M}_G^{ss}$ is holomorphic. In this way, we establish the next Proposition.

PROPOSITION 4.7. *For G reductive and $g \geq 2$, the map*

$$\mathbf{E} : \mathrm{Hom}\left(\pi_1, G\right)^\sharp \to \mathcal{M}_G^{ss},$$

is holomorphic. Furthermore, its restriction to $\mathcal{S}$ is also holomorphic.

Considering that conjugate representations $\rho, \sigma \in \mathcal{S}$, induce isomorphic G-bundles $E_\rho \cong E_\sigma$(Proposition 1.67), the map

$$\mathbb{W} :\ \mathbb{S}^\sharp \to \mathcal{M}_G^{ss} \tag{4.3.3}$$

where $\mathbb{S}^\sharp = \mathcal{S}^\sharp /\!\!/ G$ is the categorical quotient of $\mathcal{S}^\sharp\,(\neq \varnothing)$, is well defined. The map $\mathbb{W}.$ assigns to each element $[\rho]$, of the quotient $\mathbb{S}^\sharp$, the corresponding equivalence class of semistable G-bundles $[E_\rho]$. In this context, we can establish the ensuing Proposition.

PROPOSITION 4.8. *The Schottky moduli map $\mathbb{W} :\ \mathbb{S}^\sharp \to \mathcal{M}_G^{ss}$ is well defined.*

REMARK. It is possible to define the Schottky moduli map over the set of analytic equivalence of representations (Lemma 4.3) but this set does not have a nice or known structure.

CHAPTER 5

Topological Type

Ramanathan introduced, in his paper [**Ram75**], the notion of topological type related to a principal G-bundle E over a compact Riemann surface X where G is a connected reductive algebraic group over $\mathbb{C}$. This notion allows us to consider the set of isomorphism classes of flat G-bundles indexed by the elements of the fundamental group of G, $\pi_1(G)$,

$$M_G = \bigsqcup_{\delta \in \pi_1(G)} M_G^{\delta}.$$

In this chapter we are going to prove that the semistable Schottky G-bundles are in the same connected component of M_G, the one related to the identity element of $\pi_1(G)$.

Since each class in the variety $\mathrm{Hom}(\pi_1(X), G)/G$ corresponds to an isomorphism class of flat principal G-bundles, that is, principal bundles with flat connections and, since in some cases, the number of connected components of $\mathrm{Hom}(\pi_1(X), G)/G$ is given by $\pi_1(G)$ (see for example[**Li93**], [**Gol84a**]) it turns out to be important to count the number of elements of $\pi_1(G)$. Mainly, because this gives the number of connected components of the moduli space of semistable flat G-bundles. ([**Oli11, Gol84b**]).

Throughout this chapter, X denotes a compact Riemann surface with genus $g \geq 2$ and $\pi_1 = \pi_1(X)$ represents its fundamental group with its usual presentation

$$\left\langle \alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g : \prod_{i=1}^{g} [\alpha_i, \beta_i] = 1 \right\rangle.$$

In general, G denotes a connected reductive algebraic group over the complex numbers with identity element e and $\widetilde{G}$ represents a universal cover of G.

5.1. Topological type of a representation

Since the moduli space of semistable G-bundles over a Riemann surface is a disjoint union of connected components indexed by elements of $\pi_1(G)$, we are going to analyse $\pi_1(G)$ and set some properties of this group. Inasmuch as one of the constructions of $\pi_1(G)$ results from universal cover, we are going to start with several definitions and properties of the universal cover of G, denoted by $\widetilde{G}$.

DEFINITION 5.1. A covering space $p : \widetilde{G} \to G$ of a connected Lie group G is called a universal cover if $\widetilde{G}$ is a simply connected Lie group and p is a continuous surjective homomorphism.

THEOREM 5.2. *[Thm. 61,* **[Pon46]***] If G is a complex connected topological group then there exists a universal cover $p : \widetilde{G} \to G$ where p is a continuous surjective homomorphism and* $\ker p = \pi_1(G) \subset Z_{\widetilde{G}}$ *is a discrete group.*

The following proposition sets the existence of a universal cover $\widetilde{G}$ in the category of algebraic groups when G is a connected semisimple algebraic group.

PROPOSITION 5.3. *[***[BT72]***, Prop. 2.24] If G is a connected semisimple algebraic group, there exists a simply connected group $\widetilde{G}$ and an isogeny $p : \widetilde{G} \to G$, that is, a surjective homomorphism p with finite kernel. In this case $\widetilde{G}$ is called the* ***universal cover*** *of G and* $\ker p = \pi_1(G) \subset Z_{\widetilde{G}}$ *is the fundamental group of G. Moreover, $\widetilde{G}$ is a simply connected semisimple algebraic group.*

REMARK 5.4. If G is a connected semisimple algebraic group then $\widetilde{G}$ is a simply connected algebraic group, p is an (central) isogeny and the identity element of $\widetilde{G}$, $\tilde{e}$, satisfies $p(\tilde{e}) = e$. However, in general, the universal cover of an algebraic group is not an algebraic group but $\widetilde{G}$ has always a complex Lie group structure.
In some cases where G is a connected reductive algebraic group, it is usual to write G as an almost direct product $G = \mathcal{D}(G) \rtimes Z_G^{\circ}$ where $\mathcal{D}(G)$ is its derived group, which is

semisimple, and Z_G° is the connected component of the identity of the centre of G, Z_G. In this way we can work with the universal cover $\widetilde{G} = \widetilde{\mathcal{D}(G)} \times \widetilde{Z_G^\circ}$ of G.

Since in the context of our work we are not forced to work in the category of algebraic groups, we use Theorem 5.2 to guarantee the existence of a universal cover of G, this means a Lie group $\widetilde{G}$ with a projection $p : \widetilde{G} \to G$.

Let us now give some examples of some universal covers and respective fundamental groups, in the category of classic Lie groups.

Examples 5.5.

- $G = PSL(2,\mathbb{C})$, $\widetilde{G} = SL(2,\mathbb{C})$ *and* $\pi_1(PSL(2,\mathbb{C})) = \mathbb{Z}_2$.
- $G = GL(n,\mathbb{C})$ *and* $\pi_1(G) = \mathbb{Z}$*, the universal cover* $\widetilde{G}$ *is of the form* $\mathbb{C}^r \times SL(n,\mathbb{C})$*, since* $SL(n,\mathbb{C})$ *is simply connected.*
- $G = U(n)$ *(classic compact Lie group) and* $\pi_1(U(n)) = \mathbb{Z}$*, the universal cover of* G *is* $\widetilde{G} = \mathbb{R} \times SU(n)$.
- $G = SO(n,\mathbb{C})$ *and* $\pi_1(G) = \mathbb{Z}_2$*, the universal cover of* G*, for* $n \geq 3$*, is* $\widetilde{G} = \mathrm{Spin}(n)$*, the spin group* .

Given a connected reductive group G and a corresponding universal cover $\widetilde{G}$, we want to analyse the lifts to $\widetilde{G}$ of elements $a_i,\ b_i$ in G such that $\prod_{i=1}^{g} [a_i,\ b_i] = e$. We want to understand how the lifts of a_i's and b_i's in $\widetilde{G}$ behave when $\prod_{i=1}^{g} [a_i,\ b_i] = \prod_{i=1}^{g} [\rho(\alpha_i), \rho(\beta_i)] = e$ since any representation $\rho : \pi_1(X) \to G$ verifies this property.

Let us consider the following exact sequence (of Lie groups)

$$1 \to \ker(p) \to \widetilde{G} \xrightarrow{p} G \to 1$$

where $\ker p = \pi_1(G)$ and every element $p^{-1}\left(\prod_{i=1}^{g} [a_i,\ b_i]\right) = p^{-1}(e)$ is obviously on $\ker(p)$.

When we lift the elements $a_1 = \rho(\alpha_1)$, $b_1 = \rho(\beta_1), \cdots, a_g = \rho(\alpha_g)$, $b_g = \rho(\beta_g) \in G$ to elements of $\widetilde{G}$, in general, we get that

$$\prod_{i=1}^{g} \left[\tilde{a}_i,\, \tilde{b}_i\right] = \delta \neq \tilde{e}$$

where $\tilde{e}$ is the identity of $\widetilde{G}$ and $\tilde{a}_1, \tilde{b}_1, \cdots, \tilde{a}_g, \tilde{b}_g$ is a choice of the corresponding lifts ($\tilde{a}_i \in p^{-1}(a_i)$ and $\tilde{b}_i \in p^{-1}(b_i)$). The element δ belongs to $p^{-1}(e) = \ker p = \pi_1(G)$ and it measures the obstruction to lift any representation ρ to a representation $\tilde{\rho} : \pi_1(X) \to \widetilde{G}$ (for more details see for example [**HL03**]) .

Definition 5.6. Let $\widetilde{G}$ be a universal cover of G with $p : \widetilde{G} \to G$ the corresponding homomorphism and let $\rho : \pi_1(X) \to G$ be a representation. Let $\tilde{a}_i, \tilde{b}_i \in \widetilde{G}$ denote corresponding (choice of) lifts by p of $\rho(\alpha_i)$ and $\rho(\beta_i)$, that is, $\tilde{a}_i \in p^{-1}(\rho(\alpha_i))$ and $\tilde{b}_i \in p^{-1}(\rho(\beta_i))$ for each $i = 1, \cdots, g$. The **type** of the homomorphism ρ is defined to be the element obtained by

$$\delta = \prod_{i=1}^{g} \left[\tilde{a}_i,\, \tilde{b}_i\right] \in \ker p \cong \pi_1(G)\,.$$

Note that, in fact, the element δ is on the kernel of p, since

$$p(\delta) = p(\prod_{i=1}^{g} \left[\tilde{a}_i,\, \tilde{b}_i\right]) = \prod_{i=1}^{g} \left[p(\tilde{a}_i),\, p(\tilde{b}_i)\right] = \prod_{i=1}^{g} \left[\rho(\alpha_i),\, \rho(\beta_i)\right] = \rho(\prod_{i=1}^{g} [\alpha_i,\, \beta_i]) = e.$$

As we know, the short exact sequence $1 \to \pi_1(G) \to \widetilde{G} \xrightarrow{p} G \to 1$ induces the following long exact sequence between the corresponding cohomology groups

$$1 \to H^0(X, \pi_1(G)) \to H^0\left(X,\, \widetilde{G}\right) \to H^0(X,\, G) \xrightarrow{\delta} H^1(X, \pi_1(G))$$
$$\to H^1\left(X,\, \widetilde{G}\right) \to H^1(X,\, G) \xrightarrow{\delta} H^2(X, \pi_1(G)) \cong \pi_1(G) \to \cdots$$

where the map $\delta^* : H^*(X,\, G) \to H^{*+1}(X, \pi_1(G))$ is called connecting homomorphism. Equivalently, we may rewrite above sequence as

$$1 \to H^1(X, \pi_1(G)) \to H^1\left(X,\, \widetilde{G}\right) \to H^1(X,\, G) \xrightarrow{\delta} H^2(X, \pi_1(G)) \cong \pi_1(G)\,.$$

The map $\delta : H^1(X,\, G) \to H^2(X,\, \pi_1(G))$ associates to each G-bundle E a cohomology class $\delta(E)$, called **characteristic class** of E. By the isomorphism $H^2(X, \pi_1(G)) \cong$

$\pi_1(G)$ and the definition 5.6, the element $\delta(E) \in \pi_1(G)$ coincides with the type of the representation ρ when $E \cong E_\rho$.

DEFINITION 5.7. Given a flat principal G-bundle E over a compact Riemann surface X, we assign to it the **topological type** given by the following characteristic class

$$\delta(E) \in H^2(X, \pi_1(G)) \cong \pi_1(G),$$

where $E \cong E_\rho$ for a representation ρ from $\pi_1(X)$ to G.

The following theorem due to Ramanathan states that the above topological invariants, topological type of the G-bundle E_ρ and the type of the homomorphism of ρ, coincides.

THEOREM 5.8. [**Ram75**] *The type of a representation $\rho : \pi_1(X) \to G$ corresponds to the topological type of a principal G-bundle E_ρ over X. More precisely, ρ has type $\delta \in \pi_1(G)$ if and only if E_ρ has topological type δ.*

PROOF. Let us consider the exponential map $\exp : \widetilde{Z}^\circ \to Z^\circ$ where $\widetilde{Z}^\circ$ is a universal cover of Z° and an element $\mathbf{c} \in \widetilde{Z}^\circ$ such that $\exp(\mathbf{c}) = e$. Ramanathan defined the topological type of E_ρ as the element obtained by $\chi(E_\rho) = \delta - \mathbf{c}$ and he remarked that we can consider that $\mathbf{c} = 0$ (rmk 6.2, [**Ram75**]) and, in this way, we obtain

$$\chi(E_\rho) = \delta$$

where δ is the element defined above, that is, the type of the representation ρ. □

EXAMPLE 5.9. In the case of Schottky vector bundles, $a_i = \rho(\alpha_i) = \mathbf{1} \in GL(n, \mathbb{C})$ for all $i = 1, \cdots, g$. Let us consider the following universal covering of $GL(n, \mathbb{C})$

$$p : SL(n, \mathbb{C}) \times \mathbb{C} \to GL(n, \mathbb{C}).$$

Since $\tilde{a}_i \in p^{-1}(\mathbf{1}) = \ker p = \pi_1(GL(n, \mathbb{C})) = \mathbb{Z} \subset Z\left(\widetilde{GL}(n, \mathbb{C})\right)$, we get that

$$\delta = \prod_{i=1}^{g} \left[\tilde{a}_i, \tilde{b}_i\right] = \tilde{e}$$

This means that all Schottky vector bundles have trivial topological type.

5.2. Topological triviality of Schottky G-bundles

In this section we want to prove that all Schottky principal G-bundle over a compact Riemann surface, where G is a connected reductive algebraic group, has trivial topological type.

Let us start by proving that, when we consider a universal covering, we obtain the following property on the respective centres.

PROPOSITION 5.10. *Let $\widetilde{G} \xrightarrow{p} G$ be a universal covering of a connected reductive Lie group G then*

$$p\left(Z_{\widetilde{G}}\right) = Z_G \text{ and } Z_{\widetilde{G}} = p^{-1}\left(Z_G\right).$$

PROOF. Given any element $\tilde{z} \in Z_{\widetilde{G}}$ we have the following equality

$$\tilde{z}\tilde{g} = \tilde{g}\tilde{z}, \ \forall \tilde{g} \in \widetilde{G}.$$

Applying the map p and denoting $p(\tilde{z})$ by z we obtain

$$\begin{cases} p(\tilde{z}\tilde{g}) = p(\tilde{z})\, p(\tilde{g}) = zp(\tilde{g}) \\ p(\tilde{g}\tilde{z}) = p(\tilde{g})\, p(\tilde{z}) = p(\tilde{g})\, z \end{cases}.$$

Thus, $zp(\tilde{g}) = p(\tilde{g})\, z$ for all $\tilde{g} \in \widetilde{G}$ and since p is surjective this equality is verified for every $g \in G$. So, $p(\tilde{z}) = z \in Z(G)$.

Conversely, consider $g \in Z_G$ this means that $gh = hg, \quad \forall h \in G$, or equivalently, $ghg^{-1}h^{-1} = e$, $\forall h \in G$. For any $\tilde{g} \in p^{-1}(g)$, the element given by

$$\delta_{\tilde{h}} = \tilde{h}\tilde{g}\tilde{h}^{-1}\tilde{g}^{-1}$$

for an arbitrary element $\tilde{h} \in \widetilde{G}$ is such that $\delta_{\tilde{h}} \in \ker p$, since p is a homomorphism of Lie groups. Thus

$$\delta_{\tilde{h}} \in p^{-1}\left(hgh^{-1}g^{-1}\right) = p^{-1}(e).$$

Now, let us consider the following holomorphic map

$$\begin{array}{rccc} \psi_{\delta_{\tilde{h}}}: & \widetilde{G} & \to & \ker p \\ & u & \mapsto & u\delta_{\tilde{h}}u^{-1} \end{array}$$

This map is well defined for every $u \in \widetilde{G}$, since $p\left(u\delta_{\tilde{h}}u^{-1}\right) = p\left(u\right)p\left(\delta_{\tilde{h}}\right)p\left(u\right)^{-1} = p\left(u\right)ep\left(u\right)^{-1} = e$.

Considering that $\ker p \cong \pi_1(G) \subset Z_{\widetilde{G}}$, by Theorem 5.2, $\ker p$ is a discrete subgroup of $\widetilde{G}$, this implies that the image of the total space $\tilde{G}$ under $\psi_{\delta_{\tilde{h}}}$ is a single point in $\ker p$. A priori, this point depends on our choice of $\tilde{h}$. Although, since $\widetilde{G}$ is simply connected, there is a continuous path $\lambda : [0,1] \to \widetilde{G}$ with $\lambda(0) = \tilde{e}$ and $\lambda(1) = \tilde{h}$, where $\tilde{e}$ is the identity element of $\widetilde{G}$ and such that the homotopy

$$\begin{array}{rcl} \Psi : \widetilde{G} \times [0,1] & \to & \ker p \\ (u,t) & \mapsto & u\lambda(t)\tilde{g}\lambda(t)^{-1}\tilde{g}^{-1}u^{-1} \end{array}$$

is also continuous. Thus, we conclude that $\Psi(\widetilde{G} \times [0,1])$ reduces to a single point. But $\Psi(u,0) = u\lambda(0)\tilde{g}\lambda\left(0\right)^{-1}\tilde{g}^{-1}u^{-1} = u\tilde{e}\tilde{g}\tilde{e}^{-1}\tilde{g}^{-1}u^{-1} = \tilde{e}$. This means that $\psi_{\delta_{\tilde{h}}}(u) = \tilde{e}, \quad \forall u \in \widetilde{G}$. In particular, $\tilde{e} = \psi_{\delta_{\tilde{h}}}(\tilde{e}) = \delta_{\tilde{h}} = \tilde{h}\tilde{g}\tilde{h}^{-1}\tilde{g}^{-1}$ for all $\tilde{h} \in p^{-1}\left(h\right)$. Since $\tilde{h}$ was arbitrary, we conclude that $\tilde{g} \in Z_{\widetilde{G}}$ and so $Z_G \subset p\left(Z_{\widetilde{G}}\right)$.

It remains to prove the second equality, that is, $Z_{\widetilde{G}} = p^{-1}\left(Z_G\right)$.

The inclusion $Z_{\widetilde{G}} \subseteq p^{-1}\left(Z_G\right)$ is obvious since $Z_{\widetilde{G}} \subseteq p^{-1}\left(p\left(Z_{\widetilde{G}}\right)\right) = p^{-1}\left(Z_G\right)$ by above equality.

Conversely, let $\tilde{z} \in p^{-1}\left(Z_G\right)$ then $p\left(\tilde{z}\right)g = gp\left(\tilde{z}\right)$ for all $g \in G$. Consider an arbitrary element $\tilde{g} \in p^{-1}\left(g\right) \subset \widetilde{G}$ and denote by δ the element $\delta = \tilde{z}\tilde{g}\tilde{z}^{-1}\tilde{g}^{-1}$. Using the same argument as above, the image $\Psi(\widetilde{G} \times [0,1])$ reduces to a single point, that is, $\delta = \tilde{e}$. Considering that the element $\tilde{g} \in \widetilde{G}$ was arbitrary for all $g \in G$, we can conclude that $\tilde{z} \in Z_{\widetilde{G}}$. □

THEOREM 5.11. *If G is a connected reductive algebraic group then any Schottky G-bundle E_ρ has trivial topological type.*

PROOF. Consider any Schottky G-bundle E, where G is a connected reductive algebraic group. By definition there exists a Schottky representation $\rho : \pi_1\left(X\right) \to G$ such that all $a_i = \rho\left(\alpha_i\right) \in Z_G$ for all $i = 1, \cdots, g$ and $E \cong E_\rho$.

By Ramanathan's Theorem 5.8, the topological type of E_ρ, is given by $\delta \in \pi_1(G)$, where

$$\delta = \prod_{i=1}^{g} \left[\tilde{a}_i, \tilde{b}_i\right],$$

for an arbitrary choice of lifts $\tilde{a}_i \in p^{-1}(a_i) \subset p^{-1}(Z_G)$ and $\tilde{b}_i \in p^{-1}(b_i)$ of a universal cover $p : \widetilde{G} \to G$. By Proposition 5.10, we have that $\tilde{a}_i \in p^{-1}(Z_G) = Z_{\widetilde{G}}$, for all $i = 1, ..., g$, and therefore

$$\delta = \prod_{i=1}^{g} \left[\tilde{a}_i, \tilde{b}_i\right] = \prod_{i=1}^{g} \tilde{e} = \tilde{e},$$

where $\tilde{e}$ is the identity element of $\widetilde{G}$. In this way, we prove that a Schottky G-bundle E_ρ , where G is a connected reductive algebraic group, has trivial topological type. □

By Theorem 1.83 and by Proposition 1.82, the components of the moduli space of semistable principal G-bundles $\mathcal{M}_G$ are normal projective varieties and, moreover, each one of these components is indexed by the topological type of the G-bundles over X, that is, elements of $\pi_1(G)$. This means that we can write the moduli space $\mathcal{M}_G$ as the following disjoint union

$$\mathcal{M}_G = \bigsqcup_{\delta \in \pi_1(G)} \mathcal{M}_G^\delta.$$

Taking into consideration these facts and the above theorem, we are in conditions to conclude the following Corollary.

Corollary 5.12. *The moduli space of semistable Schottky G-bundles over a compact Riemann surfaces with $g \geq 2$ is contained inside the connected component of the trivial G-bundle $\mathcal{M}_G^{ss,0}$ in $\mathcal{M}_G^{ss}$.*

In the section 4.3, we defined the Schottky moduli map $\mathbb{W} : \mathbb{S}^\sharp \to \mathcal{M}_G^{ss}$ where $\mathbb{S}^\sharp$ represents the categorical quotient of the set of equivalence classes of Schottky representations such that the isomorphism classes of the induced bundles belongs to $\mathcal{M}_G^{ss}$. According to Corollary 5.12 we can rewrite the Schottky moduli map in the following way

$$\mathbb{W} : \mathbb{S}^\sharp \to \mathcal{M}_G^{ss,0}.$$

CHAPTER 6

Tangent Spaces

In chapter 4, we characterised the Schottky moduli map as the map defined by

$$\begin{aligned} W. : \mathcal{S}^{\sharp} &\to \mathcal{M}_G^{ss} \\ \rho &\mapsto [E_\rho] \end{aligned}$$

where $\mathcal{S}^{\sharp} = W^{-1}(\mathcal{M}_G^{ss})$. The corresponding map, defined over the categorical quotient space $\mathbb{S}^{\sharp} = \mathcal{S}^{\sharp} /\!\!/ G$ of the Schottky representations, is denoted by

$$\begin{aligned} \mathbb{W} : \quad \mathbb{S}^{\sharp} &\to \mathcal{M}_G^{ss} \\ [\rho] &\mapsto [E_\rho]\,. \end{aligned}$$

Both maps are well defined as seen in chapter 4.
Since some properties of the derivative of this map induce special features in the original one (like surjectivity), we are now interested in the computation of the local derivative of the Schottky moduli map. In order to obtain this, we are going to characterise the corresponding Zariski tangent spaces of the set of Schottky representations $\mathcal{S}$ and of the moduli space of flat bundles over X.

Throughout this chapter, G denotes a connected reductive algebraic group, $Z = Z_G$ its corresponding center and by F_g the free group generated by g elements. In order to simplify the notation, we use π_1 instead of $\pi_1(X)$ when there is no risk of confusion.

6.1. Tangent spaces of representations spaces

In this section, we study tangent spaces of the categorical quotients of representations $\mathbb{G} := \mathrm{Hom}(\pi_1(X), G) /\!\!/ G$ and of its subspace of Schottky representations $\mathbb{S} = \mathcal{S} /\!\!/ G$. By a classic result, dimensions of $\mathbb{G}$ and $\mathbb{S}$ are given by the dimension of the corresponding tangent space on smooth points. Additionally, Proposition 6.1 allows us to switch elements from the tangent space to elements of the first cohomology group. Throughout this section, Γ denotes any finitely generated group (in particular, sometimes Γ will denote the fundamental group of a compact Riemann surface), F_g represents a free group with g elements and $\mathfrak{g}_{\mathrm{Ad}_\rho}$ the Γ-module induced by $\mathrm{Ad}\rho : \Gamma \to GL(\mathfrak{g})$, the adjoint representation composed with a representation ρ of $\mathrm{Hom}(\Gamma, G)$, with coefficients in the Lie algebra of G, $\mathfrak{g}$.

Let us begin by recalling the following important relationship between the Zariski tangent space of the character variety of representations at a good representation ρ and the first cohomology group $H^1\left(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$. This result was proved by Weil [**Wei38**], Lubotzky and Magid [**LM85**] and generalised by many authors, like Goldman [**Gol84b**] and Martin [**Mar00**],

Proposition 6.1. *Let Γ be a finitely generated group and let G be a connected reductive algebraic group then for a good representation $\rho \in \mathrm{Hom}(\Gamma, G)$ we have the following isomorphism*

$$T_{[\rho]}\left(\mathrm{Hom}(\Gamma, G) /\!\!/ G\right) \cong H^1\left(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho}\right).$$

Remark. This proposition requires that G is connected, although, we will see that for some particular cases, the isomorphism $H^1\left(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) \cong T_{[\rho]}\left(\mathrm{Hom}(\Gamma, G) /\!\!/ G\right)$ could also be valid for G non connected.

According to Proposition 6.1, we can compute the dimension of the tangent space to the character variety $\mathrm{Hom}(\Gamma, G) /\!\!/ G$ at an equivalence class of good representations by computing the dimension of the corresponding first cohomology group. In order to obtain the latter one, it will be important to remember concepts and examples related with group cohomology that were given in section 1.3. In particular the example 1.27, where we analysed the first cohomology groups of a finitely generated group, Γ, with coefficients in the Γ-module $\mathfrak{g}_{\mathrm{Ad}_\rho}$, $H^1\left(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) = Z^1\left(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) / B^1\left(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$.

The following Lemma establishes the way to compute dimension of the first cohomology group , $H^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$, of the fundamental group $\pi_1(X)$ with coefficients in $\mathfrak{g}_{\mathrm{Ad}_\rho}$.

LEMMA 6.2. *[Lemma 6.2,[**Mar00**]] Let G be a connected reductive algebraic group with centre Z and X be a compact Riemann surface of genus g. For $\rho \in \operatorname{Hom}(\pi_1, G)$ irreducible we have*

$$\dim Z^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) = (2g-1)\dim G + \dim Z,$$
$$\dim H^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) = (2g-2)\dim G + 2\dim Z.$$

This allows us, according to Proposition 6.1, to figure out the corresponding dimension of the Zariski tangent space $T_{[\rho]}\mathbb{G}$. By definition 1.24(3), we know that a good representation is irreducible and so, we can establish the following Proposition as a consequence of Proposition6.1, for $\Gamma = \pi_1(X)$ and Lemma 6.2.

PROPOSITION 6.3. *If ρ is a good representation of* $\operatorname{Hom}(\pi_1, G)$ *and let $\mathbb{G}$ be the categorical quotient of representations on* $\operatorname{Hom}(\pi_1, G)$ *by G then*

$$T_{[\rho]}\mathbb{G} \cong H^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$$

and, moreover,

$$\dim T_{[\rho]}\mathbb{G} = (2g-2)\dim G + 2\dim Z(G).$$

By Theorem 2.16, the set of good Schottky representations, $\mathcal{S}^{\mathrm{g}}$, is a nonempty open variety and $\mathbb{S}^{\mathrm{g}}$ is the corresponding geometric quotient. Using Lemma 2.5, where we proved that $\mathbb{S} \cong \operatorname{Hom}(F_g, G \times Z) /\!\!/ G$, and Proposition 6.1, we are able to compute the dimension of the tangent space of $\mathbb{S}$ over good Schottky representations and correspondingly we calculate the dimension of $\mathbb{S}$.
In order to use Proposition 6.1, we have to, first of all, compute the dimension of the first cohomology group $H^1\left(F_g, \mathfrak{Lie}(G\times Z)_{\mathrm{Ad}_\rho}\right)$ where $\mathfrak{Lie}(G\times Z)_{\mathrm{Ad}_\rho} = (\mathfrak{g}\oplus\mathfrak{z})_{\mathrm{Ad}_\rho}$ denotes the F_g module with coefficients in the Lie algebra $\mathfrak{Lie}(G\times Z) = \mathfrak{g}\oplus\mathfrak{z}$.

Let us start by reminding a basic property of the first cohomology group that will help us to compute the dimension of $H^1\left(F_g, \mathfrak{Lie}(G\times Z)_{\mathrm{Ad}_\rho}\right)$. In spite of the elementary nature of this result, we give a proof for the convenience of the reader.

Lemma 6.4. *Given a finitely generated group* Γ*and two* Γ*-modules* A *and* B*, we have the following isomorphism*

$$H^1(\Gamma,\, A\oplus B)\cong H^1(\Gamma,\, A)\oplus H^1(\Gamma,\, B)\,.$$

Proof. The first cohomology group $H^1(\Gamma,\, A\oplus B)$ is given by the quotient of the set of 1-cocycles

$$Z^1(\Gamma,\, A\oplus B)=\{\varphi:\Gamma\to A\oplus B:\ \varphi(\gamma_1\gamma_2)=\varphi(\gamma_1)+\gamma_1\cdot\varphi(\gamma_2)\}$$

by the set of 1coboundaries

$$B^1(\Gamma, A\oplus B)=\{\varphi:\Gamma\to A\oplus B:\ \varphi(\gamma)=\gamma\cdot v-v,\text{ for some } v\in A\oplus B\}\,.$$

Now for each 1-cocycle φ, since $\varphi:\Gamma\to A\oplus B$, we can consider φ as (φ_A,φ_B) where $\varphi_A:\Gamma\to A$ and $\varphi_B:\Gamma\to B$. The map φ, as an element of $Z^1(\Gamma, A\oplus B)$, verifies the following equality

$$\varphi(\gamma_1\gamma_2)=\varphi(\gamma_1)+\gamma_1\cdot\varphi(\gamma_2)=(\varphi_A(\gamma_1)\,,\varphi_B(\gamma_1))+\gamma_1\cdot(\varphi_A(\gamma_2)\,,\varphi_B(\gamma_2))\,.$$

Since A and B are Γ-modules, Γ acts on $A\oplus B$ over each component, that is,

$$\gamma_1\cdot(\varphi_A(\gamma_2)\,,\varphi_B(\gamma_2))=(\gamma_1\cdot\varphi_A(\gamma_2)\,,\gamma_1\cdot\varphi_B(\gamma_2))\,.$$

In this way, we get a 1-cocycle relation in each component

$$(\varphi_A(\gamma_1\gamma_2)\,,\varphi_B(\gamma_1\gamma_2))=\varphi(\gamma_1\gamma_2)=(\varphi_A(\gamma_1)+\gamma_1\cdot\varphi_A(\gamma_2)\,,\ \varphi_B(\gamma_1)+\gamma_1\cdot\varphi_B(\gamma_2))\,.$$

We conclude that $Z^1(\Gamma,\, A\oplus B)=Z^1(\Gamma,\, A)\oplus Z^1(\Gamma,\, B)$.

Analogously, the 1-coboundaries are of the form

$$(\varphi_A(\gamma)\,,\varphi_B(\gamma))=\varphi(\gamma)=\gamma\cdot v-v$$

for some $v\in A\oplus B$, here $v=(v_A,v_B)$, so we obtain

$$(\varphi_A(\gamma)\,,\varphi_B(\gamma))=\gamma\cdot(v_A,v_B)-(v_A,v_B)=(\gamma\cdot v_A-v_A,\gamma\cdot v_B-v_B)\,.$$

With this, we prove that $B^1(\Gamma,\, A\oplus B)=B^1(\Gamma,\, A)\oplus B^1(\Gamma,\, B)$.

Returning to the first cohomology group

$$H^1(\Gamma,\, A\oplus B) = Z^1(\Gamma,\, A)\oplus Z^1(\Gamma,\, B)\big/\, B^1(\Gamma,\, A)\oplus B^1(\Gamma,\, B)\,,$$

let us consider two arbitrary elements of the same class φ, ψ in $H^1(\Gamma,\, A\oplus B)$. By definition, $(\varphi-\psi)$ is 1-coboundary, that is, $(\varphi-\psi)(\gamma) = \gamma\cdot v - v$. This implies that

$$\gamma.v - v = \varphi(\gamma) - \psi(\gamma) = (\varphi_A(\gamma) - \psi_A(\gamma)\,, \varphi_B(\gamma) - \psi_B(\gamma))\,,$$

and this proves that $\varphi_A - \psi_A$ and $\varphi_B - \psi_B$ are 1-coboundaries of $B^1(\Gamma,\, A)$ and $B^1(\Gamma,\, B)$, respectively.

Conversely, given two 1-cocycles φ_A and ψ_A of the same equivalence class of $H^1(\Gamma, A)$ and two 1-cocycles φ_B and ψ_B of the same equivalence class of $H^1(\Gamma, B)$, we can construct two elements $\varphi = (\varphi_A, \varphi_B)$ and $\psi = (\psi_A, \psi_B)$ of $Z^1(\Gamma,\, A\oplus B)$ where by definition

$$(\varphi_A - \psi_A, \varphi_B - \psi_B) \in B^1(\Gamma,\, A\oplus B)\,.$$

Immediately, we can conclude that $H^1(\Gamma,\, A\oplus B) = H^1(\Gamma,\, A)\oplus H^1(\Gamma,\, B)$. □

LEMMA 6.5. *Let Z denote the center of a reductive algebraic group G and suppose that the free group F_g, with g elements, acts trivially on Z then the first cohomology group of the F_g-module $\mathfrak{z} = \mathfrak{Lie}(Z)$, $H^1(F_g,\, \mathfrak{z})$, is isomorphic to $\mathbb{C}^{g\dim Z}$.*

PROOF. Since F_g acts trivially on Z, $H^1(F_g,\, \mathfrak{z}) \cong \mathfrak{z}^g$.

Given a reductive algebraic group G, its connected component of the center is an algebraic torus, that is, $Z^\circ = (\mathbb{C}^*)^{\dim Z^\circ}$. Then, $\mathfrak{Lie}(Z) = \mathfrak{Lie}(Z^\circ)$, $\mathfrak{Lie}(\mathbb{C}^*) = \mathbb{C}$ and $\dim Z = \dim Z^\circ = \dim \mathfrak{z}$.
Consequently, we obtain $H^1(F_g,\, \mathfrak{z}) \cong \mathfrak{z}^g = (\mathfrak{Lie}(Z^\circ))^g \cong \left(\mathbb{C}^{\dim Z}\right)^g$. □

Now we are in conditions to establish the following Lemma.

THEOREM 6.6. *Let G be a connected reductive algebraic group and let Z be its center. As F_g-module $\mathfrak{Lie}(G\times Z)_{\mathrm{Ad}_\rho}$ coincides with $\mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\oplus\mathfrak{z}$ where $\rho = (\rho_1, \rho_2)$ with $\rho_1 : F_g \to G$*

and $\rho_2 : F_g \to Z$ *homomorphisms. Then*

$$H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}} \oplus \mathfrak{z}\right) \cong H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) \oplus H^1\left(F_g, \mathfrak{z}\right)$$

and, if ρ *is good, its dimension is given by*

$$\dim H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}} \oplus \mathfrak{z}\right) = (g-1)\dim G + (g+1)\dim Z.$$

Proof. Given a $\rho \in \mathrm{Hom}\left(F_g, G \times Z(G)\right)$ it can be written as a pair of representations

$$\begin{array}{rcl} F_g & \to & G \times Z(G) \\ B_i & \mapsto & \rho(B_i) = (\rho_1(B_i), \rho_2(B_i)) \end{array}$$

where $\rho_1 : F_g \to G$ and $\rho_2 : F_g \to Z$ are also homomorphisms. Since $\mathfrak{Lie}\,(G \times Z) = \mathfrak{g} \oplus \mathfrak{z}$ we obtain the corresponding equality for F_g-modules $\mathfrak{Lie}(G \times Z)_{\mathrm{Ad}_\rho} = \mathfrak{g}_{\mathrm{Ad}_{\rho_1}} \oplus \mathfrak{z}_{\mathrm{Ad}_{\rho_2}}$ and since F_g acts trivially on $\mathfrak{z}$, we achieve to $\mathfrak{Lie}(G \times Z)_{\mathrm{Ad}_\rho} = \mathfrak{g}_{\mathrm{Ad}_{\rho_1}} \oplus \mathfrak{z}$.
If we use Lemma 6.4, we obtain

$$H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}} \oplus \mathfrak{z}\right) \cong H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) \oplus H^1\left(F_g, \mathfrak{z}\right).$$

Now, by Lemma 6.5, the previous relation is equivalent to

$$H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}} \oplus \mathfrak{z}\right) \cong H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) \oplus \left(\mathbb{C}^g\right)^{\dim \mathfrak{z}}.$$

In this way, the dimension of cohomology group $H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}} \oplus \mathfrak{z}\right)$ is given by

$$\dim H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) + \dim \left(\mathbb{C}^g\right)^{\dim \mathfrak{z}} = \dim H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) + g \dim \mathfrak{z}.$$

We have reduced the computation of the dimension of $H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}} \oplus \mathfrak{z}\right)$ to that of $H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right)$. By definition, the dimension of $H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right)$ is given by

$$\dim Z^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) - \dim B^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right).$$

To compute the dimension of the group consisted by 1-cocycles of the F_g-module with coefficients in $\mathfrak{g}_{\mathrm{Ad}_{\rho_1}}$, we begin by analysing the cocycle condition, that is,

$$Z^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) = \left\{\phi : F_g \to \mathfrak{g} \middle|\ \phi(B_iB_j) = \mathrm{Ad}_\rho^{-1}(B_i)\phi(B_j) + \phi(B_i),\ \forall B_i, B_j \in F_g\right\}.$$

Since F_g is the free group, any 1-cocycle ϕ is completely defined by the image of its generators, this means that there is no cocycle condition and that

$$\dim Z^1(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}) = g\dim(\mathfrak{g}).$$

The set $B^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right)$ is consisted by 1-coboundaries, that is,

$$B^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}} \oplus \mathfrak{z}_{\mathrm{Ad}_{\rho_2}}\right) = \left\{\phi : F_g \to \mathfrak{g}\oplus\mathfrak{z} \mid \phi\left(B_i\right) = B_i \cdot a - a\right\}.$$

In order to compute the dimension of this space, let us consider the following map between vector spaces

$$\begin{array}{rccl} \psi_\rho : & \mathfrak{g} & \to & (\mathfrak{g})^g \\ & v & \mapsto & \left(\left(\rho_1\left(B_1\right)\right)^{-1} v\rho_1(B_1) - v, \cdots, \left(\rho_g\left(B_g\right)\right)^{-1} v\rho(B_g) - v\right). \end{array}$$

Thus, the dimension of the group of 1-coboundaries is given by

$$\begin{aligned} \dim B^1(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}} \oplus \mathfrak{z}_{\mathrm{Ad}_{\rho_2}}) &= \dim \mathrm{Im}\psi_\rho \\ &= \dim\mathfrak{g} - \dim\ker\psi_\rho \\ &= \dim\mathfrak{g} - \dim Z_{\rho_1} \end{aligned}$$

where $Z_{\rho_1} = \{v \in \mathfrak{g} | v\rho_1(B_i) = \rho_1(B_i)v, \forall i = 1, \cdots, g\}$ is the stabiliser of ρ_1.
Subsequently, the dimension of the first cohomology group $H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right)$ is given by

$$\dim H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) = g\dim(\mathfrak{g}) - (\dim\mathfrak{g} - \dim Z_{\rho_1}) = g\dim G - \dim G + \dim Z_{\rho_1}.$$

Since by hypothesis $\rho = (\rho_1, \rho_2)$ is good and according to Proposition 2.13, this is equivalent to ρ_1 being good, which implies that $Z\left(\rho_1\right) = Z$. Thus the previous expression is equal to

$$\dim H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) = (g-1)\dim G + \dim Z.$$

In conclusion, we obtain $\dim H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}} \oplus \mathfrak{z}\right) = (g-1)\dim G + \dim Z + g\dim Z$. □

COROLLARY 6.7. *Let G be a connected reductive algebraic group and suppose that the center Z of G is also connected. Then the dimension of the categorical quotient of Schottky representations $\mathbb{S} = \mathcal{S}/\!\!/G$ is given by $(g-1)\dim G + (g+1)\dim Z$.*

PROOF. Since good representations define smooth points of the quotient, the dimension of the categorical quotient is given by the dimension of the tangent space over a smooth point

$$\dim \operatorname{Hom}(F_g, G \times Z) /\!\!/ G \;\; = \dim T_{[\rho]} \left(\operatorname{Hom}(F_g, G \times Z) /\!\!/ G\right).$$

Using Proposition 6.1, this is equal to $\dim H^1\left(F_g,\, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}} \oplus \mathfrak{z}\right)$. Then by Theorem 6.6, we obtain $\dim \operatorname{Hom}(F_g, G \times Z) /\!\!/ G = (g-1)\dim G + (g+1)\dim Z$. □

In a more general setting, the reductive group $G \times Z$ could be non-connected, and in this fashion, we are not in conditions of the Proposition 6.1. In spite of the categorical quotient of the set of Schottky representations being non connected, Proposition 2.7 allows us to use the fact that each connected component of $\operatorname{Hom}(F_g, G \times Z) /\!\!/ G$ is isomorphic to $\operatorname{Hom}(F_g, G \times Z^\circ) /\!\!/ G \cong (G^g /\!\!/ G) \times (Z^\circ)^g$. Hence, we can compute the dimension of each connected component using this isomorphism and we obtain the following Theorem which is a generalisation of Corollary 6.7.

THEOREM 6.8. *Consider a connected reductive algebraic group G and its centre Z. Let $\mathbb{S}^\circ$ denote a connected component of the categorical quotient of Schottky representations. Then the tangent space to $\mathbb{S}^\circ$ at a good representation ρ satisfies the following*

$$T_{[\rho]}\mathbb{S}^\circ \cong H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) \oplus \mathfrak{z}^g$$

where $\mathfrak{g}_{\mathrm{Ad}_{\rho_1}}$ represents the F_g-module in the Lie algebra of G and $\mathfrak{z}$ is the Lie algebra of Z. Moreover, the dimension of $\mathbb{S}^\circ$ is given by

$$\dim \mathbb{S}^\circ = (g-1)\dim G + (g+1)\dim Z.$$

PROOF. Let us begin by noting that each connected component of $\mathcal{S}$ is isomorphic to the identity component $\mathcal{S}^\circ$. So, without loss of generality, we denote by $\mathcal{S}^\circ$ an arbitrary connected component of $\mathcal{S}$ and by $\mathbb{S}^\circ$ the corresponding categorical quotient, $\mathbb{S}^\circ := \mathcal{S}^\circ /\!\!/ G$.

In each $\mathcal{S}^\circ$ there exists a good representation since the set of good representations is dense and nonempty in $\mathcal{S}^i$ (Theorem 2.16). As it was mentioned previously, each (good) representations $\rho \in \mathcal{S} \cong \mathrm{Hom}\,(F_g,\, G \times Z)$ can be seen as $\rho = (\rho_1, \rho_2)$ where $\rho_1 : F_g \to G$ (good) and $\rho_2 : F_g \to Z$ are representations of F_g in G and in Z, respectively. So, we obtain the following isomorphism of vector spaces

$$T_{[\rho]}\mathbb{S}^{\circ} \cong T_{[\rho_1]}\left(\mathrm{Hom}\left(F_g, G\right) /\!\!/ G\right) \oplus T_{[\rho_2]}\left(Z^{\circ}\right)^g.$$

If ρ is good then, by Proposition 2.13, ρ_1 is good. Then, applying Proposition 6.1, the tangent space

$$T_{[\rho_1]}\left(\mathrm{Hom}\left(F_g, G\right) /\!\!/ G\right) \cong H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right).$$

In addition $T_{[\rho_2]}\left(Z^{\circ}\right)^g \cong T_{[\rho_2]}\left(\mathrm{Hom}\left(F_g,\, Z^{\circ}\right)\right) \cong \mathfrak{z}^g$ where $\mathfrak{z}$ is the Lie algebra of the center Z of the algebraic group G. Thus, we achieve to the following isomorphism of vector spaces

$$T_{[\rho]}\mathbb{S}^{\circ} \cong H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) \oplus \mathfrak{z}^g.$$

In term of dimensions, we get

$$\dim T_{[\rho]}\mathbb{S}^{\circ} = \dim H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) + g \dim Z$$

Applying the same ideas of Theorem 6.6, we compute the dimension of the first cohomology group $H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right)$

$$\begin{aligned} \dim H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) &= \dim Z^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) - \dim B^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) \\ &= g\left(\dim \mathfrak{g}\right) - \left(\dim \mathfrak{g} - \dim Z\left(\rho_1\right)\right) \\ &= \left(g-1\right)\dim G + \dim Z\left(\rho_1\right). \end{aligned}$$

Since ρ is a good representation, this implies, by Proposition 2.13, that ρ_1 is also a good representation, then $Z\left(\rho_1\right) = Z$. Taking all this into account, we obtain

$$\dim T_{[\rho]}\mathbb{S}^{\circ} = (g-1)\dim G + \dim Z + g\dim Z.$$

To finish, we just have to remind that a good representation ρ is such that $[\rho]$ is a smooth point of the corresponding connected component $\mathbb{S}^{\circ}$ of the categorical quotient $\mathbb{S}$ and hence $\dim \mathbb{S}^{\circ} = \dim T_{[\rho]}\mathbb{S}^{\circ}$. □

REMARK 6.9. According to the fact that $\mathbb{S}$ decomposes into a union of (connected) varieties all isomorphic to $\mathbb{S}^{\circ}$, its tangent space over each element $[\rho]$, where ρ is a good representation, $T_{[\rho]}\mathbb{S}$ coincides with $T_{[\rho]}\mathbb{S}^{\circ}$.

Therefore, we can state the following Corollary.

COROLLARY 6.10. *Let $\mathbb{S}$ be the categorical quotient of isomorphic Schottky representations then*

$$\dim \mathbb{S} = (g-1)\dim G + (g+1)\dim Z.$$

6.2. Smooth points of $\mathcal{M}_G$

In section 4.3, we have introduced the notion of Schottky moduli map as the map $\mathbb{W} : \mathbb{S}^{\sharp} \to \mathcal{M}_G^{ss}$ that assigns to each equivalence class $[\rho] \in \mathbb{S}^{\sharp} = \mathbb{W}^{-1}(\mathcal{M}_G^{ss})$ the corresponding equivalence class of semistable G-bundles $[E_\rho]$.
Correspondingly, we can also consider the map

$$\begin{array}{rccc} \mathbb{E}: & \mathbb{G}^{\sharp} & \to & \mathcal{M}_G^{ss} \\ & [\rho] & \mapsto & [E_\rho] \end{array}$$

where $\mathbb{G}$ denotes the categorical quotient $\mathrm{Hom}(\pi_1(X), G)/\!\!/G$ and $\mathbb{G}^{\sharp} = \mathbb{E}^{-1}(\mathcal{M}_G^{ss})$.
Our main goal in this chapter is to compute the local derivatives of $\mathbb{E}$ and $\mathbb{W}$. In this way, we have to consider the restriction of the above maps to the subset of smooth points. These points corresponds to principal G-bundles E with the smallest possible number of automorphisms $\mathrm{Aut}(E)$. There exists a type of G-bundles such that corresponds to smooth points on the moduli space, this notion is defined as follows.

DEFINITION 6.11. A stable principal $G-$bundle E over X is **regularly stable** if its automorphism group and the centre of G, Z_G, coincides.

Biswas and Hofmann (Lemma 2.2, [**BH**]) proved that if E_ρ is stable then $\mathrm{Aut}(E)$ coincides with the stabiliser of ρ, $Z(\rho)$. If additionally, ρ is good then $Z(\rho) = Z(G)$, that is, E_ρ is a regularly stable.

According to the following Proposition, the set of regularly stable bundles coincides with the set of smooth points if these bundles are defined over a compact Riemann surfaces with $g \geq 3$. In the remaining cases, the set of G-bundles that corresponds to smooth points on $\mathcal{M}_G^{ss}$ could be bigger.

REMARK 6.12. Biswas and Hoffmann [**BH**] proved that the notion of regularly stable is of extreme importance for the case where X is a compact Riemann surface of genus $g \geq 3$ since the smooth locus of $\mathcal{M}_G^{ss}$ consists precisely of the moduli points $[E] \in \mathcal{M}_G^{ss}$ of regularly stable principal G-bundles E over X.

In spite of all that, there is an important theorem related with varieties that proves the existence of smooth points. Moreover, the set of these points is an open (nonempty) Zariski variety.

PROPOSITION 6.13. *[Thm. 5.3, [?]] Let Y be a variety. Then the set $\mathrm{Sing}\, Y$ of singular points of Y is a proper closed subset of Y.*

REMARK 6.14. In this way, using Theorem 1.83, we have that $\mathcal{M}_G^{ss}$ is a projective variety then

$$\mathcal{M}_G^{sm} := \mathcal{M}_G^{ss} \backslash \mathcal{M}_G^{sing} \tag{6.2.1}$$

is a Zariski (non empty) open subset of $\mathcal{M}_G^{ss}$. In this way, the inverse image $\mathbb{E}^{-1}\left(\mathcal{M}_G^{sm}\right) = \mathbb{G}^{\sharp}$ is non empty.

6.3. Derivative of the Schottky moduli map

Throughout this section, we want to analyse and study more properties of the Schottky map. According to Lemma 4.3, we have that $E_\sigma \cong E_\rho$ in $\mathcal{M}_G^{ss}$ if and only if the corresponding representations ρ and σ satisfy the analytic equivalence. This equivalence is defined by the following relation

$$\sigma(\gamma) = h_\omega(\tilde{x})^{-1} \rho(\gamma)\, h_\omega(\tilde{x} \cdot \gamma) \tag{6.3.1}$$

where $\omega \in H^0(X, \mathrm{Ad}E_\rho \otimes \Omega^1_X)$, $h_\omega : \widetilde{X} \to G$ is the unique solution of the differential equation $\omega = h^{-1}dh$ with $h(\tilde{x}_0) = e$ and $\tilde{x}_0$ is a fixed point in $\widetilde{X}$. This means that the map $\mathbb{E}$ is not injective since we may have non conjugate representations which have the same image on $\mathcal{M}_G^{ss}$. However, we may cluster these representations in such a way that the preimage by $\mathbb{E}$ of a point $[E_\rho]$ is represented by a special set. In this way, let us begin by giving the following definition.

Definition 6.15. Given a representation $\rho \in \mathrm{Hom}(\pi_1, G)^\sharp = \mathbf{E}^{-1}(\mathcal{M}_G^{ss})$ we define the following correspondence

$$\begin{array}{rccc} Q_\rho : & H^0(X, \mathrm{Ad}E_\rho \otimes \Omega^1_X) & \to & \mathbb{G} \\ & \omega & \mapsto & Q_\rho(\omega), \end{array}$$

where we define

$$Q_\rho(\omega) := [\sigma],$$

where $\sigma \in \mathrm{Hom}(\pi_1, G)$ is the representation given by (6.3.1) and the notation $[\sigma]$ represents the equivalence class of σ in the categorical quotient $\mathbb{G}$.

As a consequence of Lemma 4.3 we can establish the following property for the map Q_ρ.

Lemma 6.16. *The fibre* $\mathbb{E}^{-1}([E_\rho])$ *coincides with the image* $Q_\rho(H^0(X, \mathrm{Ad}E_\rho \otimes \Omega^1_X))$.

Proof. Since the map $\mathbb{E} : \mathbb{G}^\sharp \to \mathcal{M}_G^{ss}$ is defined by $\mathbb{E}([\rho]) = [E_\rho]$ and according to Lemma 4.3 we can write

$$\mathbb{E}^{-1}([E_\rho]) = \left\{\sigma \in \mathrm{Hom}(\pi_1, G) : \sigma(\gamma) = h_\omega^{-1}(\tilde{x})\rho(\gamma)h_\omega(\tilde{x}\cdot\gamma)\right\}.$$

By definition 6.15, this is exactly $Q_\rho(H^0(X, \mathrm{Ad}E_\rho \otimes \Omega^1_X))$. □

The Schottky map $\mathbb{W}$ is obtained by the composition

$$\mathbb{S}^\sharp \overset{i}{\hookrightarrow} \mathbb{G}^\sharp \overset{\mathbb{E}}{\longrightarrow} \mathcal{M}_G^{ss}. \tag{6.3.2}$$

In this way, we obtain an analogous result for the case of Schottky representations, that is, $\mathbb{W}^{-1}([E_\rho]) = Q_\rho(H^0(X, \mathrm{Ad}E_\rho \otimes \Omega^1_X))$ when $\rho \in \mathcal{S}^\sharp$.

Let us, now, compute the local derivative of $\mathbb{W}$. According to (6.3.2), this derivative can be obtained by the following composition

$$T_{[\rho]}\mathbb{S} \overset{i}{\hookrightarrow} T_{[\rho]}\mathbb{G}^{\sharp} \overset{d\mathbb{E}_{\rho}}{\longrightarrow} T_{[E_{\rho}]}\mathcal{M}_{G}^{ss} \tag{6.3.3}$$

where $d\mathbb{E}_{\rho}$ is the local derivative of $\mathbb{E} : \mathbb{G}^{\sharp} \to \mathcal{M}_{G}^{ss}$ over a good representation ρ.

If $\rho = (\rho_1, \rho_2) \in \mathrm{Hom}\,(F_g, G \times Z) \cong \mathcal{S}$ is a good representation, by Proposition 6.3, Theorem 6.8 and remark 6.9, we can rewrite (6.3.3) as

$$H^1\left(F_g,\, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) \oplus \mathfrak{z}^g \overset{i}{\hookrightarrow} H^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_{\rho}}\right) \overset{d\mathbb{E}_{\rho}}{\longrightarrow} T_{[E_{\rho}]}\mathcal{M}_{G}^{ss}. \tag{6.3.4}$$

In section 4.2, we defined the period map as the correspondence given by

$$\begin{aligned} P_{\mathrm{Ad}\rho} : H^0\left(X, \mathrm{Ad}E_{\rho} \otimes \Omega_X^1\right) &\to H^1\left(\pi_1,\, \mathfrak{g}_{\mathrm{Ad}_{\rho}}\right) \\ \omega &\mapsto P_{\mathrm{Ad}\rho}\left(\omega\right) \end{aligned}$$

where $P_{\mathrm{Ad}\rho}\left(\omega\right)\left(\gamma\right) = \int_{\gamma} \omega$. In the following Lemma, we relate this map with Q_{ρ} and subsequently, we connect the cohomology group $H^1\left(\pi_1,\, \mathfrak{g}_{\mathrm{Ad}_{\rho}}\right)$ of the diagram (6.3.4) to $H^0\left(X, \mathrm{Ad}E_{\rho} \otimes \Omega_X^1\right)$ highlighting the relations of Lemma 4.3.

Lemma 6.17. *For each good representation ρ on $\mathrm{Hom}\,(\pi_1(X), G)$, the derivative of the map Q_{ρ} at the identity, $d(Q_{\rho})_0$, coincides with $P_{\mathrm{Ad}_{\rho}}$.*

Proof. First we want to compute the local derivative of Q_{ρ} at 0, that is,

$$d\left(Q_{\rho}\right)_0 : \quad H^0\left(X, \mathrm{Ad}E_{\rho} \otimes \Omega_X^1\right) \to T_{[\rho]}\mathbb{G} \cong H^1\left(\pi_1(X), \mathfrak{g}_{\mathrm{Ad}_{\rho}}\right)$$

This derivative is defined by

$$d\left(Q_{\rho}\right)_0(\eta) = \frac{d}{dt} Q_{\rho}\left(0 + \eta t\right)\bigg|_{t=0}.$$

Since G is a connected reductive group over complex numbers, there exists a homomorphism $\phi : G \to GL\,(n, \mathbb{C})$ such that to each representation $\rho \in \mathrm{Hom}\,(\pi_1(X), G)$

we obtain the corresponding representation $\phi \circ \rho = \tilde{\rho} \in \mathrm{Hom}\left(\pi_1(X), GL(n, \mathbb{C})\right)$. Summarising, we have the following diagram

$$\begin{array}{ccc} T_{[\rho]}\mathbb{G} & \xrightarrow{\;d_{[\rho]}\bar{\phi}\;} & T_{[\phi(\rho)]}\mathbf{G}_n \\ {\scriptstyle d(Q_\rho)_0}\uparrow & & {\scriptstyle d\left(Q_{\tilde{\rho}}\right)_0}\uparrow \\ H^0\left(X, \mathrm{Ad}(E_\rho)\otimes\Omega^1_X\right) & \longrightarrow & H^0\left(X, \mathrm{End}\left(E_{\tilde{\rho}}\right)\otimes\Omega^1_X\right) \end{array}$$

For t small, we define $\rho_t(\gamma) = h_{t\eta}^{-1}(\tilde{x})\rho(\gamma)h_{t\eta}(\tilde{x}\cdot\gamma)$, where $h_{t\eta}$ is the unique solution of the diferential equation $t\eta = h^{-1}dh$ with $h(\tilde{x}_0) = I$. Recall that, by definition $Q_\rho(t\eta) = [\rho_t]$ (similarly for $Q_{\tilde{\rho}}(t\tilde{\eta})$).

Without loss of generality and in order to simplify notation, we denote $\tilde{\rho}_t$ by the same letters ρ_t. The benefit of this argument is that we can use matrix notation and simplify all computations.

Now, let us return to our problem and consider the solution $h_{t\eta}$ of the above differential equation. We can write the solution $h_{t\eta}$ of the differential equation $t\eta = h^{-1}dh$ with the initial condition $h(\tilde{x}_0) = I$, as what is called a path ordered exponential

$$h_{t\eta}(\tilde{x}) = P\exp\int_{\tilde{x}_0}^{\tilde{x}} t\eta.$$

However, the only properties that we need of this solution is the leading order terms, as follows. Assume, without loss of generality, that $G \subset GL(n, \mathbb{C}) \subset M_n(\mathbb{C})$, for some n. Then, we can write this solution using matrix notation and the corresponding exponential expansion as

$$h_{t\eta}(\tilde{x}) = I + t\int_{\tilde{x}_0}^{\tilde{x}}\eta + O\left(t^2\right) \text{ and } h_{t\eta}^{-1}(\tilde{x}) = I - t\int_{\tilde{x}_0}^{\tilde{x}}\eta + O\left(t^2\right), \; \tilde{x} \in \widetilde{X}.$$

For t small, $\rho_t \in Q_\rho(t\eta)$ becomes

$$\rho_t(\gamma) = \left(I - t\int_{\tilde{x}_0}^{\tilde{x}}\eta + O\left(t^2\right)\right)\rho(\gamma)\left(I + t\int_{\tilde{x}_0}^{\tilde{x}\cdot\gamma}\eta + O\left(t^2\right)\right), \quad \gamma \in \pi_1(X)$$

$$= \rho(\gamma) + t\left(\rho(\gamma)\left(\int_{\tilde{x}_0}^{\tilde{x}\cdot\gamma}\eta\right) - \left(\int_{\tilde{x}_0}^{\tilde{x}}\eta\right)\rho(\gamma)\right) + O\left(t^2\right).$$

We want to compute the derivative $\left.\frac{d}{dt}Q_\rho(0+\eta t)\right|_{t=0}$ and, in some sense, this corresponds to determine the derivative of a path of representations ρ_t. The derivative

$$\dot{\rho}_t(\gamma) := \frac{d}{dt}\rho_t(\gamma) = \lim_{t\to 0}\frac{\rho_t(\gamma)-\rho(\gamma)}{t},$$

is a tangent vector to the group G at the point $\rho(\gamma)$. At the Lie algebra (tangent space at the identity), this element corresponds to $\rho(\gamma)^{-1}\dot{\rho}_t(\gamma)$. Now, we can conclude that the derivative $\left.\frac{d}{dt}Q_\rho(0+\eta t)\right|_{t=0}$ is equal to

$$\rho(\gamma)^{-1}\left(\rho(\gamma)\left(\int_{\tilde{x}_0}^{\tilde{x}\cdot\gamma}\eta\right)-\left(\int_{\tilde{x}_0}^{\tilde{x}}\eta\right)\rho(\gamma)\right)=\int_{\tilde{x}_0}^{\tilde{x}\cdot\gamma}\eta-\left(\rho(\gamma)^{-1}\int_{\tilde{x}_0}^{\tilde{x}}\eta\rho(\gamma)\right)$$

$$=\int_{\tilde{x}_0}^{\tilde{x}\cdot\gamma}\eta-\left(\int_{\tilde{x}_0}^{\tilde{x}}\eta(\gamma)\gamma'\right)=\int_{\tilde{x}_0}^{\tilde{x}\cdot\gamma}\eta-\int_{\tilde{x}_0\cdot\gamma}^{\tilde{x}\cdot\gamma}\eta=\int_{\tilde{x}_0}^{\tilde{x}\cdot\gamma}\eta+\int_{\tilde{x}\cdot\gamma}^{\tilde{x}_0\cdot\gamma}\eta=\int_{\tilde{x}_0}^{\tilde{x}_0\cdot\gamma}\eta=P_{\mathrm{Ad}}(\eta(\gamma)).$$

With this, we have finished the proof that $d(Q_\rho)_0(\eta)$ coincides with $P_{\mathrm{Ad}}(\eta)$ in $H^1\left(\pi_1(X),\mathfrak{g}_{\mathrm{Ad}_\rho}\right)$. □

According to the above Lemma, the following sequence

$$H^0\left(X,\mathrm{Ad}E_\rho\otimes\Omega_X^1\right)\xrightarrow{d(Q_\rho)_0}T_{[\rho]}\mathbb{G}\xrightarrow{d\mathbb{E}_\rho}T_{[E_\rho]}\mathcal{M}_G \tag{6.3.5}$$

is exact if $\mathrm{Im}\left(d(Q_\rho)_0\right)=\ker(dE_\rho)$.

Lemma 6.18. *For each good representation $\rho\in\mathrm{Hom}(\pi_1(X),G)$, the image of $d(Q_\rho)_0$ coincides with the kernel of the map $d\mathbb{E}_\rho$.*

Proof. First let us consider two representations σ and ρ such that $E_\sigma\cong E_\rho$. In this way, we have that $\sigma\in Q_\rho\left(H^0\left(X,\mathrm{Ad}(E_\rho)\otimes\Omega_X^1\right)\right)$.

Given any $\eta\in H^0\left(X,\mathrm{Ad}E_\rho\otimes\Omega_X^1\right)$ and $t\in\mathbb{C}$, we have that

$$Q_\rho(t\eta)\subset\mathbb{E}^{-1}([E_\sigma])\Leftrightarrow\mathbb{E}(Q_\rho(t\eta))\subset[E_\sigma]$$

and this means that the representation σ satisfies $\sigma(\gamma)=h_{t\eta}^{-1}(\tilde{x})\,\rho(\gamma)h_{t\eta}(\tilde{x}\cdot\gamma)$ for a unique function $h_{t\eta}t\eta=d(h_{t\eta})$ with $h_{t\eta}(\tilde{x}_0)=I$.

Considering that the image of $\mathbb{E}\left(Q_\rho\left(0\right)\right)$ is the equivalence class $[E_{\hat\rho}]$ where the representation $\hat\rho$ satisfies $\hat\rho(\gamma) = g^{-1}\rho(\gamma)g$ for a unique element g of G (solution of $dh = 0$). Hence

$$\sigma(\gamma) = h_{t\eta}^{-1}\left(\tilde x\right)\rho(\gamma)h_{t\eta}\left(\tilde x\cdot\gamma\right) = h_{t\eta}^{-1}\left(\tilde x\right)gg^{-1}\rho(\gamma)gg^{-1}h_{t\eta}\left(\tilde x\cdot\gamma\right),$$

If we denote by $\hat h = g^{-1}h_{t\eta}$, this new function $\hat h$ is the unique solution of $\hat h t\eta = d\hat h$ satisfying $\hat h(\tilde x_0) = g^{-1}$, then we obtain

$$\sigma\left(\gamma\right) = \hat h^{-1}\left(\tilde x\right)\hat\rho\left(\gamma\right)\hat h\left(\tilde x\cdot\gamma\right). \tag{6.3.6}$$

By Lemma 4.3, we have founded one function $\hat h : \widetilde X \to G$ satisfying (6.3.6) and this implies that $E_\sigma \cong E_{\hat\rho}$. Equivalently, $\mathbb{E}\left(Q_\rho\left(t\eta\right)\right) = \mathbb{E}\left(Q_\rho\left(0\right)\right)$ (as points in $\mathcal{M}$) and, in this isomorphism, taking $t \to 0$, we obtain the differential of $\mathbb{E}\circ Q_\rho$ at the origin

$$d\left(\mathbb{E}\circ Q_\rho\right)_0\left(\eta\right) = d\left(\mathbb{E}\right)_{Q_\rho(0)}\left(\eta\right)\circ d(Q_\rho)_0(\eta) = 0$$

for $\eta \in T_0H^0\left(X, \mathrm{Ad}(E_\rho)\otimes\Omega^1_X\right) \cong H^0\left(X, \mathrm{Ad}(E_\rho)\otimes\Omega^1_X\right)$. We have proved that $d\left(Q_\rho\right)_0$ belongs to the kernel of $d\left(E_\rho\right)$.

Conversely, let us consider $\phi \in \ker d\mathbb{E}_\rho$, this means that $d\mathbb{E}_\rho\left(\phi\right) = 0$ then ϕ is tangent to the fibre of the map $\mathbb{E}$ at $[\rho]$, which means, by Lemma 6.16, that it is tangent to the image of Q_ρ at 0, so there is an $\eta \in H^0\left(X, \mathrm{Ad}E_\rho\otimes K\right)$ such that $\phi = d(Q_\rho)_0(\eta)$. □

If ρ is a good representation of $\mathrm{Hom}\left(\pi_1\left(X\right), G\right)$, according to Lemma 6.17, we have the ensuing diagram

$$\begin{array}{ccccc} H^0\left(X, \mathrm{Ad}E_\rho\otimes\Omega^1_X\right) & \xrightarrow{d(Q_\rho)_0} & T_{[\rho]}\mathbb{G}^\sharp & \xrightarrow{d\mathbb{E}_\rho} & T_{[E_\rho]}\mathcal{M}^{ss}_G \\ & \searrow^{P_{\mathrm{Ad}_\rho}} & \downarrow\cong & & \\ & & H^1\left(\pi_1(X), \mathfrak{g}_{\mathrm{Ad}_\rho}\right). & & \end{array} \tag{6.3.7}$$

Since we have the composition $T_{[\rho]}\mathbb{S} \overset{i}{\hookrightarrow} T_{[\rho]}\mathbb{G} \overset{d(\mathbb{E}_\rho)_0}{\longrightarrow} T_{[E_\rho]}\mathcal{M}_G$, if ρ is also a Schottky representation, we obtain the following diagram

$$\begin{array}{ccccc} H^0\left(X, \mathrm{Ad}E_\rho\otimes\Omega^1_X\right) & \xrightarrow{d(Q_\rho)_0} & T_{[\rho]}\mathbb{S}^\sharp & \xrightarrow{d\mathbb{W}_\rho} & T_{[E_\rho]}\mathcal{M}^{ss}_G \\ & \searrow & \downarrow\cong & & \\ & & H^1\left(\pi_1(X), \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right)\oplus\mathfrak{z}^g. & & \end{array} \tag{6.3.8}$$

Using above relations and previous results, we obtain the following Theorem which establishes the way to determine the kernel of the Schottky moduli map.

THEOREM 6.19. *Let $\mathbb{W}.:\mathbb{S}\to\mathcal{M}_G^{ss}$ be the Schottky moduli map and ρ a unitary and good Schottky representation then we have the following isomorphism*

$$\ker \mathrm{d}\left(\mathbb{W}_\rho\right)\cong \mathrm{Im}\left(\mathrm{H}^1\left(\mathrm{F_g},\ \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right)\oplus\mathfrak{z}^{\mathrm{g}}\right)\bigcap \mathrm{Im} d\left(Q_\rho\right)_0. \tag{6.3.9}$$

PROOF. The kernel of $d\left(\mathbb{W}_\rho\right)$ is given by

$$\ker d\left(\mathbb{W}_\rho\right)=\mathrm{Im}\left(T_{[\rho]}\mathbb{S}\right)\bigcap\ker d\left(\mathbb{E}_\rho\right)$$

where $\mathrm{Im}\left(T_{[\rho]}\mathbb{S}\right)$ represents the image of $T_{[\rho]}\mathbb{S}$ on $T_{[\rho]}\mathbb{G}$ by the inclusion i.
Theorem 6.8 states that $T_{[\rho]}\mathbb{S}\cong H^1\left(F_g,\ \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right)\oplus\mathfrak{z}^g$, using this fact we get

$$\ker d\left(\mathbb{W}_\rho\right)=\mathrm{Im}\left(H^1\left(F_g,\ \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right)\oplus\mathfrak{z}^g\right)\bigcap\ker d\left(\mathbb{E}_\rho\right)$$

where $\mathrm{Im}\left(H^1\left(F_g,\ \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right)\oplus\mathfrak{z}^g\right)$ represents the image of $H^1\left(F_g,\ \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right)\oplus\mathfrak{z}^g$ on $H^1\left(\pi_1\left(X\right),\ \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$.
Finally, we use Lemma 6.18 to conclude that $\ker d\left(\mathbb{W}_\rho\right)$ is given by $\mathrm{Im}\left(H^1\left(F_g,\ \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right)\oplus\mathfrak{z}^g\right)\bigcap\mathrm{Im} d\left(Q_\rho\right)_0$. □

Latter, we will come back to the description of this kernel and we will see that, if we consider that ρ is a unitary Schottky representation, the map $d\left(\mathbb{W}_\rho\right)$ will have maximal rank.

To finish this section, let us make a synopsis, using the following diagram, of what we achieved until this moment.

$$\begin{array}{ccccc}
H^0\left(X,\mathrm{Ad}E_\rho\otimes\Omega_X^1\right) & \xrightarrow{P_{\mathrm{Ad}}\equiv d(Q_\rho)_0} & H^1\left(\pi_1(X),\mathfrak{g}_{\mathrm{Ad}_\rho}\right) & \xrightarrow{d\mathbb{E}_\rho} & T_{[E_\rho]}\mathcal{M}_G^{ss} \\
 & & \uparrow{\scriptstyle di} & \nearrow{\scriptstyle d\mathbb{W}_\rho} & \\
 & & H^1\left(F_g,\mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right)\oplus\mathfrak{z}^g & &
\end{array}.$$

where we have computed the dimension of each one of the above sets. By Lemma 6.2, we have that the dimension of $H^1\left(\pi_1\left(X\right), \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$ is given by

$$\dim H^1\left(\pi_1\left(X\right), \mathfrak{g}_{\mathrm{Ad}_\rho}\right) = 2\left(g-1\right)\dim G + 2\dim Z(G)$$

and by Theorem 6.8, the dimension of $H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) \oplus \mathfrak{z}^g$ is obtained by

$$\dim\left(H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) \oplus \mathfrak{z}^g\right) = \left(g-1\right)\dim G + \left(g+1\right)\dim Z(G).$$

Thus, the dimension of the kernel of $d\left(\mathbb{W}_\rho\right)$ is given by

$$\dim\ker d\left(\mathbb{W}_\rho\right) = \dim\left(\left(\mathrm{H}^1\left(\mathrm{F_g}, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) \oplus \mathfrak{z}^{\mathrm{g}}\right) \bigcap \mathrm{Im} d\left(Q_\rho\right)_0\right)$$

(6.3.10)

$$\dim\ker d\left(\mathbb{W}_\rho\right) = \dim\left(\mathrm{Im}\left(H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) \oplus \mathfrak{z}^g\right) \cap \mathrm{Im}\left(H^0\left(X, \mathrm{Ad}E_\rho \otimes \Omega^1_X\right)\right)\right)$$

If we prove, under certain conditions, that $H^0\left(X, \mathrm{Ad}E_\rho \otimes \Omega^1_X\right) \oplus H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$ gives all $H^1\left(\pi_1\left(X\right), \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$ then the dimension of the above intersection will be exactly $\dim \mathfrak{z}^g = g\dim Z$. This will be our main goal in Chapter 7.

CHAPTER 7

Schottky Moduli Map

In previous chapters we have considered the Schottky moduli map

$$\mathbb{W}.:\mathbb{S}^{\sharp}\rightarrow\mathcal{M}_{G}^{ss},$$

from the categorical quotient of Schottky representations, more accurately, $\mathbb{S}^{\sharp}=\mathbb{W}^{-1}\left(\mathcal{M}_{G}^{ss}\right)$, to the moduli space of semistable principal G-bundles.

In this chapter, we want to study better the nature of the image of the above map. For this purpose, we return to the computing of the derivative of the Schottky moduli map over a good (and sometimes unitary) Schottky representation ρ, that is,

$$d\left(\mathbb{W}_{\rho}\right)_{0}:T_{[\rho]}\mathbb{S}^{\sharp}\rightarrow T_{[E_{\rho}]}\mathcal{M}_{G}^{ss}. \tag{7.0.11}$$

Now, let us recall that there are two relevant subspaces of the tangent space $T_{[\rho]}\mathbb{G}\cong H^{1}\left(\pi_{1},\,\mathfrak{g}_{\mathrm{Ad}_{\rho}}\right)$ (at a smooth point) where π_1 represents the fundamental group of a compact Riemann surface X $(g\geq 2)$. One of them is

$$\mathrm{Im}\left(H^{1}\left(F_{g},\,\mathfrak{g}_{\mathrm{Ad}_{\rho_{1}}}\right)\oplus\mathfrak{z}^{g}\right)\subset H^{1}\left(\pi_{1},\,\mathfrak{g}_{\mathrm{Ad}_{\rho}}\right)$$

where $\rho=(\rho_1,\,\rho_2)$ is a Schottky representation, written as an element of $\mathrm{Hom}\left(F_g,\,G\times Z\right)$, and $H^{1}\left(F_{g},\,\mathfrak{g}_{\mathrm{Ad}_{\rho_{1}}}\right)$ is the first cohomology group of the free group with coefficients in the F_g-module $\mathfrak{g}_{\mathrm{Ad}_{\rho_{1}}}$ (introduced in section 1.3). The other is

$$\mathrm{Im}d\left(Q_{\rho}\right)_{0}\subset H^{1}\left(\pi_{1},\,\mathfrak{g}_{\mathrm{Ad}_{\rho}}\right)$$

where Q_ρ is the map $Q_{\rho}:H^{0}\left(X,\mathrm{Ad}E_{\rho}\otimes\Omega_{X}^{1}\right)\rightarrow\mathbb{G}$ sending each 1-form ω, with coefficients in the Lie algebra $\mathfrak{g}$ of G, to an equivalence class of a representation (chapter 6).

If G is a semisimple algebraic group, from the proof of Theorem 7.6 (7.2.1), since in this case $\mathfrak{z}=0$, we obtain

$$\ker \mathrm{d}\left(\mathbb{W}_{\rho}\right) \cong \mathrm{Im}\left(H^{1}\left(F_{g},\, \mathfrak{g}_{\mathrm{Ad}\rho 1}\right)\right) \bigcap \mathrm{Im} d\left(Q_{\rho}\right)_{0}.$$

In this case, the categorical quotient of Schottky representations and the moduli space have the same dimension: $(g-1)\dim G$ (recall Theorem 6.8). In this context, we want to show that $\ker d\left(\mathbb{W}_{\rho}\right)=0$, at a class $[\rho]\in\mathbb{S}$, such that $\rho:\pi_1(X)\to K$ is a Schottky unitary representation (recall that K is a fixed maximal compact subgroup of G).

In order to simplify notation throughout this chapter, we denote by π_1 the fundamental group $\pi_1(X)$ of a compact Riemann surface X of genus $g\geq 2$ and Z denotes the center of the algebraic group G.

7.1. Bilinear relations

Consider a maximal compact subgroup K of a complex connected reductive algebraic group G and let us fix an hermitian structure on the complex Lie algebra $\mathfrak{g}$ of G which is invariant under the adjoint action of K on $\mathfrak{g}$. Let us denote the corresponding non-degenerate hermitian bilinear form on $\mathfrak{g}$ by $\langle\, ,\, \rangle:\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$. Under these conditions, we are going to define the following hermitian inner product

$$H^{0}\left(X,\mathrm{Ad}\left(E_{\rho}\right)\otimes\Omega_{X}^{1}\right)\times H^{0}\left(X,\mathrm{Ad}\left(E_{\rho}\right)\otimes\Omega_{X}^{1}\right)\overset{(\,,\,)}{\to}\mathbb{C}. \tag{7.1.1}$$

To obtain this, we use the Period map $P_{\mathrm{Ad}_{\rho}}:H^{0}\left(X,\,\mathrm{Ad}E_{\rho}\otimes\Omega_{X}^{1}\right)\to H^{1}\left(\pi_1,\,\mathfrak{g}_{\mathrm{Ad}_{\rho}}\right)$ (defined on section 4.2) to transport elements of the vector space $H^{0}\left(X,\,\mathrm{Ad}E_{\rho}\otimes\Omega_{X}^{1}\right)$ to $H^{1}\left(\pi_1,\,\mathfrak{g}_{\mathrm{Ad}_{\rho}}\right)$ and after, we consider a pairing defined on the first cohomology group $H^{1}\left(\pi_1,\,\mathfrak{g}_{\mathrm{Ad}_{\rho}}\right)\times H^{1}\left(\pi_1,\,\mathfrak{g}_{\mathrm{Ad}_{\rho}}\right)\to\mathbb{C}$ using the previous fixed bilinear form on $\mathfrak{g}$.

Example 7.1. If $G=GL(n,\mathbb{C})$ then we can use $\langle A,\,B\rangle=\mathrm{tr}\left(AB^{*}\right),\ \forall A,\,B\in G$ where $*$ means conjugate transpose.

Let us begin by consider a universal cover $\widetilde{X}$ of the compact Riemann surface X of genus $g \geq 2$ and let D denote a fundamental domain for the quotient $X = \widetilde{X}\big/\pi_1$. In concrete terms, we can consider $\widetilde{X}$ as the upper half plane $\mathbb{H}$ and D a closed subset bounded by $4g$ closed geodesic segments inside $\mathbb{H}$.

In terms of these structures, we can give the following definition.

Definition 7.2. Given two arbitrary holomorphic 1-forms $\omega_1,\, \omega_2 \in H^0\left(X, \mathrm{Ad}E_\rho \otimes \Omega^1_X\right)$, we define the following hermitian inner product

$$(\omega_1,\, \omega_2) := \int_X \langle \omega_1, \omega_2 \rangle := i \int_D \langle h_1(z), h_2(z) \rangle \, dz \wedge d\bar{z} \tag{7.1.2}$$

where $\omega_i = h_i(z)\, dz$ for $z \in \widetilde{X}$ and D is a fundamental domain for $X = \widetilde{X}\big/\pi_1(X)$.

Remark. The above integral is independent of the choice of the fundamental domain D and the corresponding local coordinates. On the other hand, it only depends on the choice of the bilinear form on the Lie algebra $\mathfrak{g}$.

In order to define a pairing on $H^1\left(\pi_1,\, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$, we need to extend 1-cocycles $\phi : \pi_1 \to \mathfrak{g}_{\mathrm{Ad}_\rho}$ to the group ring $\mathbb{Z}[\pi_1]$ (see **[Flo01, Gol84b]**). This allows us to use the ensuing Fox calculus notation. Since the boundary ∂D can be considered as the boundary of a $4g$ polygon, we can ordered its vertices in the following way

$$\left\{z_0, z_0\alpha_1, z_0\alpha_1\beta_1, z_0\alpha_1\beta_1\alpha_1^{-1}, z_0R_1, z_0R_1\alpha_2, \cdots, z_0R_g = z_0\right\}$$

where $R_k = \prod_{i=1}^{k} \alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}$. Now, consider the following notations

- $\dfrac{\partial R}{\partial \alpha_i} := R_{i-1} - R_i\beta_i$
- $\dfrac{\partial R}{\partial \beta_i} := R_{i-1}\alpha_k - R_i$

and an involution $\sharp$ on $\mathbb{Z}[\pi_1]$ defined by $\sharp\left(\sum n_i\gamma_i\right) = \sum n_i\gamma_i^{-1}$.

According to the foregoing notations,

$$\sharp\frac{\partial R}{\partial \alpha_i} = R_{i-1}^{-1} - \beta_i^{-1}R_i^{-1} \quad \text{and} \quad \sharp\frac{\partial R}{\partial \beta_i} = \alpha_i^{-1}R_{i-1}^{-1} - R_i^{-1}. \tag{7.1.3}$$

DEFINITION 7.3. With the previous notation, we define the pairing on $H^1\left(\pi_1(X), \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$ by

$$\langle\langle \phi_1, \phi_2 \rangle\rangle = \sum_{i=1}^{g} -\left\langle\left\langle \phi_1\left(\sharp\frac{\partial R}{\partial \alpha_i}\right), \phi_2(\alpha_i)\right\rangle\right\rangle + \left\langle\left\langle \phi_1\left(\sharp\frac{\partial R}{\partial \beta_i}\right), \phi_2(\beta_i)\right\rangle\right\rangle,$$

for any $\phi_1, \phi_2 \in Z^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$.

REMARK. This pairing coincides with the composition of ensuing cup product followed by contraction using the bilinear form $\langle\,,\rangle$ of $\mathfrak{g}$ on the coefficients and the evaluation on the fundamental 2-cycle

$$H^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) \times H^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) \xrightarrow{\cup} H^2\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right) \xrightarrow{\langle\,,\rangle} H^2(\pi_1, \mathbb{C}) \cong \mathbb{C}. \tag{7.1.4}$$

Moreover, we can see that this bilinear form defines a holomorphic sympletic form on the complex manifold $\mathbb{G}^{sm}$ which consists of the smooth points of the algebraic variety $\mathbb{G} = \mathrm{Hom}(\pi_1, G) /\!\!/ G$ (see **[Gol84b]**).

Let us establish the following Proposition, which gives us a way to compute the hermitian inner product defined on (7.1.2) considering the above pairing (for details about cup product see for example [**?**]).

PROPOSITION 7.4. *Let $\rho : \pi_1 \to K \subset G$ be a unitary representation and let* Ad *denote the adjoint action of G. Then, for all 1-forms $\omega_1, \omega_2 \in H^0\left(X, \mathrm{Ad}(E_\rho) \otimes \Omega^1_X\right)$, the Hermitian inner product of these forms is given by*

$$(\omega_1, \omega_2) = \left\langle\left\langle P_{\mathrm{Ad}_\rho}(\omega_1), P_{\mathrm{Ad}_\rho}(\omega_2)\right\rangle\right\rangle$$

where $\left\langle\left\langle P_{\mathrm{Ad}_\rho}(\omega_1), P_{\mathrm{Ad}_\rho}(\omega_2)\right\rangle\right\rangle = \sum_{i=1}^{g} -\left\langle\left\langle \phi_1\left(\sharp\frac{\partial R}{\partial \alpha_i}\right), \phi_2(\alpha_i)\right\rangle\right\rangle + \left\langle\left\langle \phi_1\left(\sharp\frac{\partial R}{\partial \beta_i}\right), \phi_2(\beta_i)\right\rangle\right\rangle.$

PROOF. First let us fix some notation. In terms of local coordinates $z \in \widetilde{X}$, if we consider the function $f_i : \widetilde{X} \to \mathfrak{g}$ defined by $f_i(z) = \int_{z_0}^{z} \omega_i$ then $\omega_i = df_i$. Observe that $f_i(z\gamma) = \int_{z_0}^{z\cdot\gamma} \omega_i$ and, equivalently, $f_i(z\gamma) = \int_{z_0}^{z_0\cdot\gamma} \omega_i + \int_{z_0\cdot\gamma}^{z\cdot\gamma} \omega_i$. Considering $\phi_i(\gamma) =$

$\int_{z_0}^{z_0\cdot\gamma}\omega_i$ a 1-cocycle representing $P_{\mathrm{Ad}_\rho}^{\gamma}(\omega_i)$, we obtain

$$f_i(z\gamma) = \phi_i(\gamma) + \int_{z_0}^{z}\omega_i(\gamma)\gamma' = \phi_i(\gamma) + \int_{z_0}^{z}\gamma\cdot\omega_i = \phi_i(\gamma) + \mathrm{Ad}_{\rho(\gamma)}^{-1} f_i(z) \tag{7.1.5}$$

since $\omega_i(\gamma)\gamma' = \gamma\cdot\omega_i$ and this corresponds to an adjoint action on f_i, that is, $\int_{z_0}^{z}\gamma\cdot\omega_i = \mathrm{Ad}_{\rho(\gamma)}^{-1} f_i(z)$.

According to definition 7.2 and applying Stokes' theorem to (7.1.2), we obtain the following expression for the hermitian inner product of two 1-forms $\omega_1,\, \omega_2 \in H^0\left(X, \mathrm{Ad}E_\rho \otimes \Omega_X^1\right)$

$$(\omega_1,\, \omega_2) = \int_{\partial D}\langle f_1(z), h_2(z)\rangle\, d\bar{z}. \tag{7.1.6}$$

Using the fact that the boundary ∂D can be considered as the boundary of a $4g$ polygon and fixing z_0 as a base point of $\widetilde{X}$, we write the previous expression (7.1.6) in the following way

$$\begin{aligned}(\omega_1,\, \omega_2) &= \int_{z_0}^{z_0\alpha_1}\langle f_1(z), h_2(z)\rangle\, dz + \int_{z_0\alpha_1}^{z_0\alpha_1\beta_1}\langle f_1(z), h_2(z)\rangle\, dz + \\ &\cdots + \int_{z_0R_{g-1}\alpha_g\beta_g\alpha_g^{-1}}^{z_0R_g}\langle f_1(z), h_2(z)\rangle\, dz.\end{aligned} \tag{7.1.7}$$

In order to simplify the above expression, let us compute the integrals over the paths α_i and α_i^{-1}.

$$\int_{z_0R_{i-1}}^{z_0R_{i-1}\alpha_i}\langle f_1(z), h_2(z)\rangle\, dz + \int_{z_0R_{i-1}\alpha_i\beta_i}^{z_0R_{i-1}\alpha_i\beta_i\alpha_i^{-1}}\langle f_1(z), h_2(z)\rangle\, dz =$$

$$= \int_{z_0}^{z_0\alpha_i}\langle f_1(zR_{i-1}), h_2(zR_{i-1})\rangle (R_{i-1})'\, dz + \int_{z_0R_i\beta_i\alpha_i}^{z_0R_i\beta_i}\langle f_1(zR_i\beta_i), h_2(z)\rangle\, dz$$

where $z_0R_{i-1}\alpha_i\beta_i\alpha_i^{-1}$corresponds to $z_0R_i\beta_i$ and doing change of variables in the first integral. Doing the same thing on the second integral, we obtain

$$= \int_{z_0}^{z_0\alpha_i}\langle f_1(zR_{i-1}), h_2(zR_{i-1})\rangle (R_{i-1})'\, dz - \int_{z_0}^{z_0\alpha_i}\langle f_1(zR_i\beta_i), h_2(zR_i\beta_i)\rangle (R_i\beta_i)'\, dz$$

$$= \int_{z_0}^{z_0\alpha_i}\left\langle f_1(zR_{i-1}), h_2\left(zR_{i-1}^{-1}\right)\right\rangle \left(R_{i-1}^{-1}\right)' - \langle f_1(zR_i\beta_i), h_2(zR_i\beta_i)\rangle \left(R_i^{-1}\beta_i\right)'\, dz.$$

Using property (7.1.5) of f_1 and defining $\phi_i(\gamma) = \int_{z_0}^{z_0\cdot\gamma} \omega_i$ a 1-cocycle representing $P^{\gamma}_{\mathrm{Ad}_\rho}(\omega_i)$, we obtain

$$\begin{aligned} = & \int_{z_0}^{z_0\alpha_i} \Big\langle \phi_1(R_{i-1}) + \mathrm{Ad}^{-1}_{\rho(R_{i-1})} f_1(z)\,,\, h_2(zR_{i-1}) \Big\rangle (R_{i-1})' \\ - & \Big\langle \phi_1(R_i\beta_i) + \mathrm{Ad}^{-1}_{\rho(R_i\beta_i)} f_i(z)\,,\, h_2(zR_i\beta_i) \Big\rangle (R_i\beta_i)'\, dz \end{aligned}$$

$$\begin{aligned} = & \int_{z_0}^{z_0\alpha_i} \langle \phi_1(R_{i-1})\,,\, h_2(zR_{i-1})\rangle (R_{i-1})' + \Big\langle \mathrm{Ad}^{-1}_{\rho(R_{i-1})} f_1(z)\,,\, h_2(zR_{i-1}) \Big\rangle (R_{i-1})' \\ - & \langle \phi_1(R_i\beta_i)\,,\, h_2(zR_i\beta_i)\rangle (R_i\beta_i)' - \Big\langle \mathrm{Ad}^{-1}_{\rho(R_i\beta_i)} f_1(z)\,,\, h_2(zR_i\beta_i) \Big\rangle (R_i\beta_i)'\, dz \end{aligned}$$

Since the non-degenerate hermitian bilinear form $\langle\, ,\, \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is G-invariant, namely, it is invariant for the adjoint action, the above expression can be simplify to

$$\begin{aligned} = & \int_{z_0}^{z_0\alpha_i} \langle \phi_1(R_{i-1})\,,\, h_2(zR_{i-1})\rangle (R_{i-1})' + \langle f_1(z)\,, h_2(z)\rangle + \\ - & \langle \phi_1(R_i\beta_i)\,,\, h_2(zR_i\beta_i)\rangle (R_i\beta_i)' - \langle f_1(z)\,, h_2(z)\rangle\, dz \end{aligned}$$

$$= \int_{z_0}^{z_0\alpha_i} \langle \phi_1(R_{i-1})\,,\, h_2(zR_{i-1})\rangle (R_{i-1})' - \langle \phi_1(R_i\beta_i)\,,\, h_2(zR_i\beta_i)\rangle (R_i\beta_i)'\, dz$$

Now, we consider the product $\langle\langle\ ,\ \rangle\rangle$ defined over elements of $Z^1\left(\pi_1(X)\,, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$

$$= \Big\langle\Big\langle \phi_1(R_{i-1})\,,\, \mathrm{Ad}^{-1}_{\rho(R_{i-1})} \cdot \phi_2(\alpha_i) \Big\rangle\Big\rangle - \Big\langle\Big\langle \phi_1(R_i\beta_i)\,,\, \mathrm{Ad}^{-1}_{\rho(R_i\beta_i)} \cdot \phi_2(\alpha_i) \Big\rangle\Big\rangle\ . \tag{7.1.8}$$

Simple computations lead us to the following equality $\phi_1(R_{i-1}) = -\mathrm{Ad}_{\rho(R_{i-1})} \cdot \phi_1\left(R_{i-1}^{-1}\right)$ and according to the fact that the above inner product is invariant to the adjoint action, (7.1.8) is equivalent to

$$\begin{aligned} = \quad & -\left\langle\left\langle \phi_1\left(R_{i-1}^{-1}\right),\, \phi_2(\alpha_i) \right\rangle\right\rangle + \left\langle\left\langle \phi_1\left(\beta_i^{-1}R_i^{-1}\right),\, \phi_2(\alpha_i) \right\rangle\right\rangle \\ = - \quad & \left\langle\left\langle \phi_1\left(R_{i-1}^{-1}\right) - \phi_1\left(\beta_i^{-1}R_i^{-1}\right),\, \phi_2(\alpha_i) \right\rangle\right\rangle . \end{aligned} \tag{7.1.9}$$

Using the fact that ϕ_1 is linear as a Fox calculus notation, (see [**Gol84b, Flo01**]), since $\sharp\frac{\partial R}{\partial \alpha_i} = R_{i-1}^{-1} - \beta_i^{-1} R_i^{-1}$, the integrals over the paths α_i and α_i^{-1} become

$$\begin{aligned}\int_{z_0 R_{i-1}}^{z_0 R_{i-1}\alpha_i} \langle f_1(z), h_2(z)\rangle\, dz + \int_{z_0 R_{i-1}\alpha_i\beta_i}^{z_0 R_{i-1}\alpha_i\beta_i\alpha_i^{-1}} \langle f_1(z), h_2(z)\rangle\, dz = \\ = -\left\langle\left\langle \phi_1\left(\sharp\tfrac{\partial R}{\partial \alpha_i}\right),\, \phi_2(\alpha_i)\right\rangle\right\rangle.\end{aligned} \tag{7.1.10}$$

Doing analogous computations for the integrals over the paths β_i and β_i^{-1} on (7.1.7) and using again Fox calculus notation $\left(\sharp\frac{\partial R}{\partial \beta_i} = \alpha_i^{-1} R_{i-1}^{1} - R_i^{-1}\right)$, we obtain

$$\begin{aligned}\int_{z_0 R_{i-1}\alpha_i}^{z_0 R_{i-1}\alpha_i\beta_i} \langle f_1(z), h_2(z)\rangle\, dz + \int_{z_0 R_{i-1}\alpha_i\beta_i\alpha_i^{-1}}^{z_0 R_i} \langle f_1(z), h_2(z)\rangle\, dz = \\ = \left\langle\left\langle \phi_1\left(\sharp\tfrac{\partial R}{\partial \beta_i}\right),\, \phi_2(\beta_i)\right\rangle\right\rangle.\end{aligned} \tag{7.1.11}$$

To finish the proof of this proposition, we just have to use expressions (7.1.10) and (7.1.11) on (7.1.7). □

7.2. Local derivative at unitary representations

From Theorem 6.19, we know that the kernel of the local derivative of the Schottky map at an unitary representation $\rho \in \mathrm{Hom}(\pi_1, G)$ is given by

$$\ker \mathrm{d}(\mathbb{W}_\rho) \cong \left(H^1(F_g, \mathfrak{g}_{\mathrm{Ad}\rho_1}) \oplus \mathfrak{z}^g\right) \bigcap \mathrm{Im} d(Q_\rho)_0$$

where the Schottky representation is written as $\rho = (\rho_1, \rho_2) : F_g \to G \times Z$ such that $\rho_1(B_i) \in G$ and $\rho_2(B_i) \in Z$. According to Lemma 6.17, since the derivative of the map Q_ρ at the identity, $d(Q_\rho)_0$, coincides with P_{Ad_ρ}, we can write the kernel as the following intersection

$$\ker \mathrm{d}(\mathbb{W}_\rho) \cong \left(H^1(F_g, \mathfrak{g}_{\mathrm{Ad}\rho_1}) \oplus \mathfrak{z}^g\right) \bigcap \mathrm{Im} P_{\mathrm{Ad}_\rho}.$$

Note that we are identifying the cohomology space $H^1(F_g, \mathfrak{g}_{\mathrm{Ad}\rho_1}) \oplus \mathfrak{z}^g$ with its image under the natural inclusion

$$H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) \oplus \mathfrak{z}^g \subset H^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right).$$

Let us remind that the period map $P_{\mathrm{Ad}_\rho} : H^0\left(X, \mathrm{Ad}\left(E_\rho\right) \otimes \Omega_X^1\right) \to H^1\left(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$ is given by $P_{\mathrm{Ad}_\rho}(\omega) := [\omega]$ with $\omega_z(\gamma) = \int_{z_0}^{z_0\cdot\gamma} \omega$ for each $\gamma \in \pi_1(X)$ and for a fixed point $z_0 \in \tilde{X}$.

Proposition 7.5. *Let ρ be a Schottky unitary representation. Suppose that $\omega \in H^0\left(X, \mathrm{Ad}\left(E_\rho\right) \otimes \Omega_X^1\right)$ is such that $P_{\mathrm{Ad}_\rho}(\omega) \in H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right)$ (in particular, the component of $P_{\mathrm{Ad}_\rho}(\omega)$ in $\mathfrak{z}^g$ vanishes). Then $\omega = 0$.*

Proof. By hypothesis, for any $\omega \in H^0\left(X, \mathrm{Ad}\left(E_\rho\right) \otimes \Omega_X^1\right)$ we have $P_{\mathrm{Ad}_\rho}(\omega) \in H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right)$. According to Proposition 7.4, the hermitian inner product of ω is defined by

$$(\omega, \omega) = \left\langle\left\langle P_{\mathrm{Ad}_\rho}(\omega), P_{\mathrm{Ad}_\rho}(\omega)\right\rangle\right\rangle.$$

In this case the cup product of this class with itself is

$$P_{Ad_\rho}(\omega) \cup P_{Ad_\rho}(\omega) \in H^2\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right)$$

Since for a free group F_g, $H^2\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) = 0$, we obtain $P_{Ad_\rho}(\omega) \cup P_{Ad_\rho}(\omega) = 0$ and by proposition 7.4, $\omega = 0$ since the Hermitian product is non-degenerate. □

Theorem 7.6. *For G a semisimple algebraic group and for any good and unitary Schottky representation ρ, the local derivative of the Schottky map $d(\mathbb{W})_\rho : T_{[\rho]}\mathbb{S} \to \mathcal{M}_G^{ss}$ is an isomorphism.*

Proof. First of all notice that the dimension of both spaces is the same. Indeed, since G is semisimple, $\dim Z = 0$. Now, applying Theorem 6.8 to $T_{[\rho]}\mathbb{S}$ and Theorem 1.83 to $\mathcal{M}_G^{ss}$ we achieve to the following equalities

$$\dim T_{[\rho]}\mathbb{S} = (g-1)\dim G$$
$$\dim \mathcal{M}_G^{ss} = (g-1)\dim G.$$

Thus, it is suffices to determine the kernel of $d(\mathbb{W})_\rho$, that is,

$$\ker \mathrm{d}\left(\mathbb{W}_\rho\right)_0 = \mathrm{H}^1\left(\mathrm{F_g}, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) \bigcap \mathrm{Im} d\left(Q_\rho\right)_0 = H^1\left(F_g, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) \bigcap \mathrm{Im} P_{Ad_\rho} \tag{7.2.1}$$

and show that it is zero. Note that we are identifying the cohomology space $H^1\left(F_g,\, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right)$ with its image under the natural inclusion

$$H^1\left(F_g,\, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) \subset H^1\left(\pi_1,\, \mathfrak{g}_{\mathrm{Ad}_\rho}\right).$$

Let $[\phi] \in H^1\left(\pi_1,\, \mathfrak{g}_{\mathrm{Ad}_\rho}\right)$ be an element in $\ker d\left(\mathbb{W}\right)_\rho$, by (7.2.1), we have that $\phi \in H^1\left(F_g,\, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right)$ and $\phi \in \mathrm{Im}P_{Ad_\rho}$. So, there is $\omega \in H^0\left(X, \mathrm{Ad}\left(E_\rho\right) \otimes \Omega^1_X\right)$ such that $\phi = P_{Ad_\rho}(\omega)$ and by Proposition 7.5, we conclude that $\omega = 0$. Then $\phi = P_{Ad_\rho}(0) = 0$. Since $\phi \in \ker d\left(\mathbb{W}\right)_\rho$ was arbitrary we have $\ker d\left(\mathbb{W}\right)_\rho = 0$ as intended. □

We now consider the general case for reductive G.

THEOREM 7.7. *Let G be a connected reductive algebraic group and let ρ be a good and unitary Schottky representation. Then, the derivative of the Schottky map $d\left(\mathbb{W}\right)_\rho : T_{[\rho]}\mathbb{S} \to \mathcal{M}_G^{ss}$ has maximal rank. In particular, the Schottky map $\mathbb{W} : \mathbb{S}^\sharp \to \mathcal{M}_G^{ss}$ is a local submersion. This means that locally around ρ, the map is a projection with* $\dim\left(\mathbb{W}^{-1}\left([E_\rho]\right)\right) = g \dim Z^\circ$.

PROOF. In section 6.3, we proved that

$$\ker d\left(\mathbb{W}_\rho\right) = \left(H^1\left(F_g,\, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) \oplus H^1\left(F_g,\, \mathfrak{z}\right)\right) \bigcap \mathrm{Im}\left(P_{\mathrm{Ad}_\rho}\right)$$

by an elementary result of linear algebra we obtain

$$\ker d\left(\mathbb{W}_\rho\right) \cong \left(H^1\left(F_g,\, \mathfrak{g}_{\mathrm{Ad}_{\rho_1}}\right) \bigcap \mathrm{Im}\left(P_{\mathrm{Ad}_\rho}\right)\right) \oplus \left(H^1\left(F_g,\, \mathfrak{z}\right) \bigcap \mathrm{Im}\left(P_{\mathrm{Ad}_\rho}\right)\right).$$

The left side of the direct sum is zero by the proof of Theorem 7.6. It is easy to see that

$$H^1\left(F_g,\, \mathfrak{z}\right) \bigcap \mathrm{Im}\left(P_{\mathrm{Ad}_\rho}\right) = H^1\left(F_g,\, \mathfrak{z}\right).$$

So, we conclude that

$$\dim \ker d\left(\mathbb{W}_\rho\right) = \dim H^1\left(F_g,\, \mathfrak{z}\right) = g\, \dim \mathfrak{z}.$$

Since, by Theorem 6.8 $\dim T_{[\rho]}\mathbb{S} = (g-1)\dim G + (g+1)\dim Z$ and by Theorem 1.83, $\dim \mathcal{M}_G^{ss} = (g+1)\dim G + \dim Z$, thus

$$\dim T_{[\rho]}\mathbb{S} = \dim \mathcal{M}_G^{ss} + \dim \ker d\left(\mathbb{W}_\rho\right),$$

and we conclude that $d\left(\mathbb{W}_\rho\right)$ is a local submersion at ρ. $\square$

CHAPTER 8

Particular Cases

Since in the previous chapters it has been excluded the case of compact Riemann surfaces with $g = 1$, that is, elliptic curves, in this chapter our main purpose is to study this case. Note that the case of $\mathbb{P}^1$ $(g = 0)$ is not relevant for Schottky bundles, since there are no Schottky representations (other than the trivial one) as its fundamental group is trivial.

We begin by analysing another special case, when we have a principal Schottky $\mathbb{C}^*$-bundles over a compact Riemann surface with $g \geq 2$. In this situation the group $G = \mathbb{C}^*$ is abelian, so it is an elementary example but still interesting.

8.1. Principal $\mathbb{C}^*$-bundle

Let us fix the group $G = GL(1, \mathbb{C}) \cong \mathbb{C}^*$ and let E denotes a principal $\mathbb{C}^*$-bundle over a compact Riemann surface X. It is well known that $\mathbb{C}^*$-bundle are equivalent to line bundles, i.e., vector bundles of rank one. According to [**?, Flo01**], every line bundle with degree 0 is Schottky.
The following is the analogous result in the principal bundle setting.

PROPOSITION 8.1. *Given a principal $\mathbb{C}^*$-bundle E over a compact Riemann surface X then E is Schottky if and only if it is flat.*

PROOF. If E_G is Schottky then $E \cong E_\rho$ where $\rho : \pi_1 \to G$ (with $\rho(\alpha_i) = 1$ as usual), equivalently, E_G is flat.

To prove the converse statement, suppose that E_G is flat. Since $\mathbb{C}^*$ is an abelian group, the conjugation is trivial, therefore the associated adjoint line bundle $\mathrm{Ad}(E)$ is, in fact, a trivial line bundle over X. So, $\mathrm{Ad}(E)$ is a Schottky line bundle. We obtain the result applying Corollary 3.8 which states that if $\mathrm{Ad}\,(E)$ is Schottky and E is flat, then E is Schottky. □

REMARK 8.2. In our context, the space of Schottky representations is

$$\mathbb{S} = \mathrm{Hom}\,(F_g,\ (\mathbb{C}^* \times \mathbb{C}^*))$$

since the center of $\mathbb{C}^*$ is itself. Moreover the conjugation action is trivial, so there is no need to take the GIT quotient. Note that this is not the same space considered in [**Flo01**] for the case of line bundles.

REMARK 8.3. For $G = \mathbb{C}^*$, it is well known that all G-bundles are semistable. Thus, the moduli space of semistable $\mathbb{C}^*$-bundles coincides with the space of all $\mathbb{C}^*$-bundles, which is the first cohomology group $H^1(X, \mathcal{O}_X^*)$. So, we have

$$\mathcal{M}_{\mathbb{C}^*}^{ss} \cong H^1(X, \mathcal{O}_X^*).$$

It is well known that this sits in an exact sequence

$$H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to \mathbb{Z},$$

whose last morphism is the degree, or first Chern class. So, the space of flat $\mathbb{C}^*$-bundles coincides with the kernel of the degree map, that is, with

$$H^1(X, \mathcal{O}_X^*)^0 \cong J(X),$$

where $J(X)$ is the Jacobian of X.

In this context the Schottky map looks as follows

$$\mathbb{W} : \mathrm{Hom}\left(F_g, (\mathbb{C}^*)^2\right) \to J(X).$$

Then, Proposition 8.1 implies that this map is onto. Also note that $\dim J\,(X) = g$ whereas $\dim \mathbb{S} = 2g$, so this description in terms of principal bundles is not exactly the same as the description of the line bundle case in [**Flo01**], although, as we have seen, the main results still hold.

8.2. Schottky G-bundles over elliptic curves

In spite of Ramanthan's Theorem 1.83 requires that the genus of the compact Riemann surface $g \geq 2$, it holds for the case of $g = 1$. In this section, we consider principal Schottky bundles over an elliptic curve X since we have been excluding this case in the previous chapters.

The methodology that we will use throughout this section is the following. First of all, we consider the case of vector bundles over an elliptic curve and we remind some results relating flatness, semistability and the Schottky property. Then, we relate principal G-bundles with the corresponding adjoint bundle in order to translate some of the previous properties to this case.

Let us begin by reminding the following theorem, due to Atiyah and Tu [**Ati57, Tu93**], which relates semistability with the indecomposable property.

THEOREM 8.4. [**Tu93**]*/ Every indecomposable vector bundle over an elliptic curve is semistable; it is stable if and only if its rank and degree are relatively prime.*

Using Weil's Theorem (Thm. 1.60), we show in the next Proposition that, if we are dealing with vector bundles over elliptic curves, flatness and semistability are related.

PROPOSITION 8.5. *Let V be a vector bundle over an elliptic curve X. Then V is flat if and only if V is semistable of degree zero.*

PROOF. Let V be a flat vector bundle over X and consider the decomposition of V in a direct sum of indecomposable subbundles

$$V = \bigoplus V_i.$$

We can assure that this decomposition always exists by Krull-Remak-Schmidt Theorem. Theorem 1.60 states that $\deg(V_i) = 0$ and, by Theorem 8.4, each one of V_i's are semistable. Since the sum of semistable vector bundles of the same slope $(\mu(V_i) = 0)$ is a semistable one and with the same slope, we conclude that V is semistable and $\deg(V) = 0$.

Conversely, let V be a semistable vector bundle with degree zero. If V is indecomposable then, by Corollary 1.61, V has a holomorphic connection, that is, V is flat. Otherwise, let us, again, consider the direct sum of indecomposable subbundles $V = \bigoplus V_i$. We have that $0 = \deg(V) = \sum \deg(V_i)$, thus, if there exists a V_i such that $\deg(V_i) < 0$ then it must exists, at least, one V_j such that $\deg(V_j) > 0 = \deg(V)$. This contradicts the hypothesis that V is semistable. Therefore, all of these V_i's must have degree zero which implies, by Theorem 1.60, that V is flat. □

More recently, Florentino proved the following theorem which establishes that flat vector bundles over elliptic curves are all Schottky.

THEOREM 8.6. *[Theorem 6,***[Flo01]***] Every flat vector bundle over a Riemann surface of genus 1 is Schottky.*

Now, we want to use the above results to establish similar conclusions for principal G-bundle over an elliptic curve. In order to obtain this, we have to consider the associated adjoint bundle. This purpose is achieved in the next Proposition.

PROPOSITION 8.7. *Let X be an elliptic curve, G a semisimple algebraic group and E_G a G-bundle over X. Then the following are equivalent:*

(1) E_G is semistable;
(2) $\mathrm{Ad}(E_G)$ is semistable with degree zero;
(3) $\mathrm{Ad}(E_G)$ admits a flat connection;
(4) E_G admits a flat connection.

PROOF. The proof is done by using several of the previous results. Proposition 1.53 states that E_G is semistable if and only if $\mathrm{Ad}(E_G)$ has also this property, thus we obtain the equivalence between the two first assertions.

The statements (2) and (3) are equivalent by the above Proposition 8.5.

Finally, we can use Proposition 1.63, since G is semisimple, to conclude that E_G admits a flat connection if and only if $\mathrm{Ad}E_G$ admits one. $\square$

REMARK 8.8. In the above Proposition, we may consider G reductive although the equivalence (3) $\Leftrightarrow$ (4) is not valid in this case. By Proposition 1.64, we may just guarantee (3) $\Longleftarrow$ (4).

According to the above results, we may conclude the following Theorem for the case of a compact Riemann surface of genus $g = 1$.

THEOREM 8.9. *Let X be an elliptic curve and let G be a connected reductive algebraic group. Then E is a flat principal G-bundle over X if and only if E is Schottky.*

PROOF. If the G-bundle E admits a flat connection then it induces a flat connection in $\mathrm{Ad}\,(E)$. Using Theorem 8.6, $\mathrm{Ad}\,(E)$ is Schottky as it is a flat vector bundle with degree 0. By Proposition 3.7, since $\mathrm{Ad}\,(E)$ is Schottky and E is flat we obtain that E is a Schottky principal G-bundle.

The converse one is obvious since a Schottky G-bundles E is, by definition, induced by a representations of $\pi_1 \to G$ so E is flat. $\square$

The following Corollary follows straightforward by Proposition 8.7 and the above Theorem.

COROLLARY 8.10. *Let X be an elliptic curve and let G be a semisimple algebraic group. Then every semistable principal G-bundle over X is Schottky.*

REMARK 8.11.
(1) In the case of compact Riemann surface with genus $g = 1$, the fundamental group is a free abelian group $\pi_1(X) = \{\alpha, \beta : \alpha\beta = \beta\alpha\}$. Given any representation $\rho : \pi_1(X) \to G$, if we denote by $a = \rho(\alpha) \in Z_G$ and $b = \rho(\beta) \in G$, we obtain

$$ab = \rho(\alpha\beta) = \rho(\beta\alpha) = ba.$$

This means that a and b are commutative elements, then $\rho(\pi_1(X))$ is abelian. Consequently, if G is not abelian then there is no irreducible representations (neither good representations).

(2) The fact of the nonexistence of irreducible representations does not imply that there is no moduli space. In fact, Friedman et al. described, in [**FMW98**], that the moduli space of semistable vector bundles over an elliptic curve is a weighted projective space.

(3) Therefore, we can define the moduli Schottky map $\mathbb{W} : \mathbb{S} \to \mathcal{M}^{ss}$ and, according to Corollary 8.10, this is indeed (globally) surjective.

Bibliography

[AB01] B. Anchouche and I. Biswas, *Einstein-Hermitian connections on polystable principal bundles over a compact Kahler manifold*, Amer. Jour. Math. **123** (2001), no. 2, 207–228.

[AB03] H. Azad and I. Biswas, *On holomorphic principal bundles over a compact Riemann surface admitting a flat connection, ii*, Bull. London Math. Soc. **35** (2003), 440–444.

[Ati57] M.F. Atiyah, *Vector bundles over an elliptic curve*, Proc. London Math. Soc. **7** (1957).

[BH] I. Biswas and N. Hoffman, *A Torelli theorem for moduli spaces of principal bundles over a curve*, arXiv:10003.4061v2.

[Bis96] I. Biswas, *Principal bundles admitting a holomorphic connection*, Int. J. Math. **07** (1996).

[BT72] A. Borel and J. Tits, *Compléments à l'article "groupes réductifs"*, Publications Mathématiques de l'IHÉS **41** (1972), 253–276.

[Dol03] I. Dolgachev, *Lectures on invariant theory*, Cambrigde University Press **296** (2003).

[Flo01] C. Florentino, *Schottky uniformization and vector bundles over Riemann surface*, Manuscripta Math. **105** (2001), 69–83.

[FMW98] R. Friedman, J. Morgan, and E. Witten, *Principal g-bundles over elliptic curves*, Math. Research Letters **5** (1998), 97–118.

[Gol84a] W. Goldman, *Representations of fundamental groups of surfaces*, Proceedings of the Special Year in Topology (1983-1984).

[Gol84b] ———, *The sympletic nature of fundamental groups of surfaces*, Advances in Mathematics **54** (1984), 200–225.

[H.34] Auerbach H., *Sur les groupes lineaires bornes (iii)*, Studia Math. **5** (1934), 43–45.

[HL03] N. Ho and M. Liu, *Connected components of the space of surface group representations*, Inter. Math. Res. Notices **44** (2003), 2359–2372, arXiv:math/0303255v1.

[Hum75] J. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics, vol. 21, Sringer-Verlag New York, 1975.

[Hus94] D. Husemöller, *Fibre bundles*, no. 20, Graduate Texts in Mathematics, 1994.

[Li93] J. Li, *The space of surface group representations*, Manuscripta Math. **78** (1993), 223–243.

[LM85] A. Lubotzky and A. Magid, *Varieties of representations of finitely generated groups*, Memoirs of the AMS **336** (1985), 2205–2213.

[Mar00] B. Martin, *Restrictions of representations of surface group to a pair of free subgroups*, Journal of Algebra **225** (2000), 231–249.

[MF82] D. Mumford and J. Fogarty, *Geometric invariant theory*, Spinger, 1982.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Springer-Verlag Berlin Heidelberg, 1994.

[Mum63] D. Mumford, *Projective invarients of projective structures and applications*, Proc. Inter. Congress Mathematics 1962 (1963), 526–530.

[Nar76] M.S. Narasimhan, *Vector bundles on compact riemann surface*, in Complex Analysis and its Applications, IAEA **3** (1976).

[New78] P. Newstead, *Introduction to moduli problems and orbit spaces*, Tata Institute of fundamental Research, Bombay, 1978.

[NS64] M.S. Narasimhan and C.S. Seshadri, *Holomorphic vector bundles on a compact Riemann surface*, Mathematische Annalen **155** (1964), no. 1, 69–80.

[NS65] ______, *Stable unitary vector bundles on compact Riemann surface*, Annals of Mathematics Second Series **82** (1965), no. 3, 540–567.

[Oli11] A. Oliveira, *Representations of surface groups in the projective general linear group*, Int. J. Math. **22** (2011), no. 223.

[OV90] A.L. Onishichik and E.B. Vinberg, *Lie groups and algebraic groups*, Springer-Verlag Berlin Heidelberg, 1990.

[Pon46] L. Pontrjagin, *Topological groups*, Princeton University Press, 1946.

[Ram75] A. Ramanathan, *Stable principal bundles on a compact Riemann surface*, Math. Ann. **213** (1975), 129–152.

[Ram96] ______, *Moduli for principal bundles over algebraic curves: I, ii*, Proc. Indian Acad. Sci. **106** (1996).

[Ses72] C.S. Seshadri, *Quotient spaces modulo reductive algebraic groups*, Annals of Mathematics **Second Series, Vol. 95** (1972), no. 3, 511–556.

[Sik10] A. Sikora, *Character varieties*, Transactions of the American Mathematical Society **364** (2010), no. 10, 5173–5208.

[Ste51] N. Steenrod, *The topology of fibre bundles*, Princeton University Press, Princeton, New Jersey, 1951.

[Tu93] L. Tu, *Semistable bundles over an elliptic curve*, Advances in Mathematics **98** (1993), 1–26.

[TY05] Patrice Tauvel and Rupert Yu, *Lie algebras and algebraic groups*, Springer-Verlag Berlin Heidelberg New York, 2005.

[Wei38] A. Weil, *Généralization des fonctions abéliennes*, J. Math Pure App. **17** (1938), 47–87.