NOVA School of Business and Economics

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Three Essays on Structural Breaks

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To my parents and Sandra

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INTRODUCTION

This dissertation consists of three essays that propose robust statistical procedures for testing hypotheses on the slope of the trend function. It includes statistics to test and estimate the number and timing of breaks in the slope of the deterministic trend from a univariate time series and from a multivariate time series robust to stationary, non-stationary and cointegrated environments and robust tests for general linear restrictions in the coefficients of trend, given the estimated regimes.

Structural changes are pervasive in economics: Changes in economic policy, evolving technological progress or specific events with a strong impact in the World economy such as wars or oil price shocks can give rise to structural breaks in any econometric model used to explain the behavior of certain economic variables. On the other hand, the presence of at least one structural break leads to inconsistent estimates and poor forecasts if that break is not properly modeled. Naturally, this fact has led to a large amount of interest on the literature about this topic. Different statistical procedures were proposed to test for the existence of structural breaks and estimate both the number and timing of the change points. The problem is that the majority of these tests are valid only when the data are stationary. This fact restricts the applicability of these tests as, in practice, it is rarely known as to whether the data are stationary or not. However, the literature on multiple structural breaks valid in both I(0) and I(1) environments is relatively scarce. This is a very important problem since in fact formal testing of whether a time series contains structural breaks or not depend on whether the stochastic part is stationary or not.

Hence, in the first chapter of my thesis, we propose new tests for the presence of multiple breaks in the slope of the deterministic trend of a univariate time-series where the number and dates of the breaks are unknown and that are valid in the presence of stationary or unit root shocks. These tests can also be used to sequentially estimate the number of breaks. After developing the asymptotic theory and showing that the tests work well for finite samples, we illustrate the applicability of the proposed tests to various U.S. historical macroeconomic time series. Here we show how important it is to take into account both the (non) stationarity of the data and the possible presence of multiple breaks. We conclude that many macroeconomic variables are characterized by having multiple breaks in the deterministic trend and not only one break as is very popularly advocated in the literature.

The second chapter extends these ideas to the multivariate framework. This extension is important for many reasons: first, intuitively many factors that may be responsible for the presence of a structural break in a univariate time series may, by contagion, result in structural changes in other economic variables. Second, the same circular problem between unit root and trend break testing can also be encountered within the cointegration testing framework. Finally, it has been shown that we can expect substantial payoffs in identifying, precisely, the dates in which breaks have occurred if we estimate them in a multivariate system. Hence, in this chapter, we develop the first procedure which delivers tests for the presence of common broken trends in multivariate time series which do not require knowledge of the form of serial correlation in the data and are robust as to whether the shocks are stationary, non stationary, cointegrated or not cointegrated. The setup is a VAR process for cointegrated variables. We propose tests to detect and estimate the number of change points occurring at known and unknown dates in a system of equations. These tests are simple to implement and can be used to specify the deterministic component of VAR Models. We present Monte Carlo simulation results which suggest that the proposed tests perform well in small samples. The proposed methodology is used to study the existence of trend breaks in data related to economic inequality. In particular, we use a recently compiled database on the concentration of wealth in the richest individuals. Here we identify those international economic events that were responsible for a change in the historical trend of concentration of wealth in various groups of countries close to each other geographically and culturally.

The third chapter of this dissertation contributes to one of the prevalent topics in the economic growth literature: the choice of the growth model more compatible with what we observe in the data. An important part of this discussion can be summarized in three mutually exclusive hypotheses: the "constant trend", "level shift" and "slope shift" hypothesis. The objective of this chapter is to classify countries according to each of these hypotheses and to analyze which of the growth theories seems to be favored. We approach this problem in two-steps: first, the number and the timing of trend breaks are estimated using the approach from the first chapter; and second, conditional on the estimated number of breaks, break dates, and coefficients, a statistical framework is introduced to test for general linear restrictions on the coefficients of the linear disjoint broken trend model. Here, we prove a general result that, under certain conditions, a standard F statistic to test the additional restrictions, given the first step estimated partition, converges asymptotically in distribution to the usual chi-square distribution. We further show how the aforementioned hypotheses can be formulated as linear restrictions on the parameters of the breaking trend model and apply the methodology to per capita output of an extensive list of countries. All of our tests are robust as to whether the data are I(0) or I(1) surpassing technical and methodological concerns on previous empirical evidence. We find evidence favoring the "constant trend" hypothesis for nine countries: Austria, Germany, Switzerland, Canada, United States, Chile, Sweden, Australia and New Zealand. The results of our tests support the "level shift" hypothesis for six countries: France, Netherlands, Brazil, Denmark, Japan and Italy. Finally, there is a third group of eight countries where statistical evidence favors the "growth shift" hypothesis: Belgium, Uruguay, Finland, Norway, United Kingdom, Sri Lanka, Portugal and Spain.

All in all, this dissertation contributes to the literature of structural breaks by proposing a different set of procedures that can be used in a univariate or in a multivariate framework to detect both the number and timing of significant structural changes in the trend function of one or multiple economic variables. This framework also allows to test general linear restrictions on the trend, given the estimated partition. The advantage of this approach is that the empirical practitioner does not need to pre-test for the presence of a unit root or specify the number of cointegrating relations to solve this statistical inference problem. Empirical applications show that it is important to take into account these breaks as they are common in macroeconomic time series.

1. TESTS FOR MULTIPLE BREAKS IN THE TREND WITH STATIONARY OR INTEGRATED SHOCKS

With Luis C. Nunes[1]

1.1 Introduction

Many macroeconomic time series are characterized by a clear tendency to grow over time, that is, as having a deterministic time trend component. There has been a large debate in the literature regarding the appropriate methods to infer about the linearity and stability of the trend function and the nature of the shocks affecting a time series. This is a particularly important issue when it comes to make accurate economic forecasts or test economic hypothesis. In fact, there are many interesting economic applications that involve statistical inference on the parameters of the trend function, namely, in the continuous time macroeconomic modeling (see Bergstrom et al., 1992, Nowman, 1998), in international trade, for example, with the Prebish-Singer hypothesis testing (see Bunzel and Vogelsang, 2005), in the empirical debate regarding regional convergence in per capita income (see Sayginsoy and Vogelsang, 2004), or in environmental economics on the future consequences of global warming (see Vogelsang and Franses, 2005).

The stationarity properties of the shocks have important implications on the appropriate methods to make inferences about the trend function. In particular, the correct approach to make inferences about the stability or the existence of breaks in the trend

[1]We are grateful to participants in seminars at Universiteit Van Amsterdam and Nova School of Business and Economics, in the QED Conference (Amsterdam, May 2009), in the Econometric Society European meeting (joint congress with the European Economic Association; Barcelona, August 2009) and in the XXXV Simposio de la Asociación Española de Economía (Madrid, December 2010)for helpful comments and suggestions on earlier versions of this paper. We also thank Pierre Perron and Josep Lluís Carrion-i-Silvestre for providing us with their Gauss programs. Financial support from *Fundação para a Ciência e Tecnologia* is also acknowledged.

depends on whether the shocks are I(0) or I(1). In the first case one should use regressions on the levels, while for the latter the correct approach is to model the first-differences of the series. However, it is often not known a priori whether the shocks are stationary or contain a unit-root. Moreover, stationarity or unit-root tests also suffer from similar problems since their properties are in turn affected by the stability of the trend function.

Only recently have some solutions to this dilemma been proposed in the literature. These resort to statistical tests of the null hypothesis of a constant linear trend against the alternative of a one break at some unknown date that do not require a priori knowledge of whether the noise is I(0) or I(1). Sayginsoy and Vogelsang (2004) proposed a Mean Wald and a Sup Wald statistic scaled by a factor based on unit root tests to smooth the discontinuities in the asymptotic distributions of the test statistics as the errors go from I(0) to I(1). The scaling factor approach is based on Vogelsang (1998) who proposed test statistics for general linear hypothesis regarding the parameters of the trend function which do not require knowledge as to whether the innovations are I(0) or I(1). Perron and Yabu (2009) proposed a Feasible Quasi Generalized Least Squares approach to estimate the slope of the trend function. By truncating the estimate of the sum of the autoregressive coefficients of the disturbance term to take the value of one whenever the estimate is in a neighborhood of one, they have shown that the limiting distribution of the t-statistic becomes Normal regardless of the persistence of the error term. Kejriwal and Perron (2010) proposed a sequential testing procedure based on Perron and Yabu (2009). Harvey et al. (2009) (hereafter HLT) employed a weighted average of the appropriate regression t-statistics used to test the existence of a broken trend when the errors are I(0) and I(1). However, as Lumsdaine and Papell (1997) point out with an example of Jones (1995), allowing for only one break is not always the best characterization of a macroeconomic variable, specially when analyzing long historical time series.

This paper extends the results from HLT by providing tests of the null hypothesis of no trend breaks against the alternative of one or more breaks in the trend slope which do not require knowledge of the form of serial correlation in the data and are robust as to whether the underlying shocks are stationary or have a unit-root. We build on the framework proposed by HLT for the case of a single break, and construct test statistics that are weighted averages of the appropriate F-statistics to test the existence of multiple trend breaks when the disturbance term is $I(0)$ and $I(1)$ .We adopt the weight function used in HLT and prove that it has the same large sample properties regardless of the number of trend breaks being tested.

We start by considering the case where the true break fractions are known and prove that the proposed statistics converge in distribution to a chi-square distribution under the null. Next, we consider the case where the trend break fractions are unknown and need to be estimated. We transform our statistic in the same spirit as Andrews (1993) and Bai and Perron (1998) and take the supremum of the F statistic over all possible break fractions except those that are actively restricted by the trimming parameter. Here, the weight function is evaluated at the estimated break fractions and we prove that its large sample behavior is similar regardless of the number of break fractions estimated and the number of structural breaks in the trend function. However, the asymptotic null distributions of the appropriate F-statistics for $I(0)$ and $I(1)$ environments are different and so, following Vogelsang (1998), we provide a scaling factor that makes the asymptotic critical values invariant to the degree of persistence of the shocks. Finally, we propose double maximum tests and a sequential test procedure that can be used to estimate the number of trend breaks and that are also robust to the order of integration of the error term. In both the known and unknown break dates settings, our proposed tests are made robust to short memory serial correlation in the shocks via the use of standard non-parametric estimators of the long run variance of the errors.

The outline of this article is as follows. Section 1.2 describes the multiple breaks in the trend model, presents the test statistics for both known and unknown break fractions and establishes the asymptotic behavior of these statistics. The sequential testing procedure to estimate the number of breaks is also described. In Section 1.3 we extend the model to allow for simultaneous shifts in the intercept and slope of the trend functions and develop test procedures for this case. In Section 1.4 we discuss practical issues related to the application of the test statistics proposed, namely, the critical values and the choice of the scaling constants. Size and power properties in finite samples from applying these procedures are also discussed in this section. Section 1.5 provides an empirical application to various U.S. macroeconomic time series data. Section 1.6 concludes the paper with a discussion of some issues raised by our analysis and suggests possible paths for future research. All our key results are proved in a Mathematical Appendix.

1.2 Joint Broken Trend Model

We start by considering a time-series process $\{y_t\}$ with a first-order linear trend and m possible time changes in the slope such that the trend function is always joined at the time of the break, which we call "Model A":

$$y_t = \alpha + \beta t + \sum_{j=1}^{m} \gamma_j DT_t\left(\tau_j^*\right) + u_t, \qquad t = 1, \ldots, T, \tag{1.1}$$

and

$$u_t = \rho u_{t-1} + \varepsilon_t, \qquad t = 2, \ldots, T, \qquad u_1 = \varepsilon_1, \tag{1.2}$$

where $DT_t\left(\tau_j^*\right) := \mathbb{1}\left(t > T_j^*\right)\left(t - T_j^*\right)$ captures the eventual j^{th} break in the slope occurring at date $T_j^* := \lfloor \tau_j^* T \rfloor$ with associated break fraction $\tau_j^* \in (0,1)$ and $0 < \tau_1 < \ldots < \tau_m < 1$. The slope coefficient changes from β to $\beta + \gamma_1$ at time T_1^*, from $\beta + \gamma_1$ to $\beta + \gamma_1 + \gamma_2$ at time T_2^* and, in general, from $\beta + \sum_{i=1}^{j-1} \gamma_i$ to $\beta + \sum_{i=1}^{j} \gamma_i$ at time T_j^* for $j = 1, \ldots, m$. However, notice that the trend function is continuous in every period including the dates at which the slope changes occur. The discontinuous case is considered in Section 1.3.

We assume that ε_t in (1.2) satisfies Assumption 1 of Sayginsoy and Vogelsang (2004, pp. 2-3):

Assumption 1. *The stochastic process ε_t is such that:*

$$\varepsilon_t = C(L)\eta_t, \qquad C(L) = \sum_{i=0}^{\infty} c_i L^i$$

with $C(1)^2 > 0$ *and* $\sum_{i=0}^{\infty} i|c_i| < \infty$*, and where* η_t *is a martingale difference sequence with unit conditional variance and* $\sup_t E\left(\eta_t^4\right) < \infty$*.*

The error term u_t can have one unit root or none. If $|\rho| < 1$, u_t is an I(0) process. But if $\rho = 1$ then u_t turns out to be an I(1) process. We are interested in testing if there are trend breaks in y_t and in estimating the number of breaks in the time series process, independently of whether u_t is $I(0)$ or $I(1)$. Therefore, we would like to test the null hypothesis $H_0 : \gamma_1 = \gamma_2 = \ldots = \gamma_m = 0$ against the two sided alternative: $H_1 : \gamma_1 \neq 0 \vee \gamma_2 \neq 0 \vee \ldots \vee \gamma_m \neq 0$.

Remark 1. Under the conditions of Assumption 1, the long run variance of ε_t is given by $\omega_\varepsilon^2 := \lim_{T\to\infty} T^{-1} E\left(\sum_{t=1}^{T} \varepsilon_t\right)^2 = C\left(1\right)^2$. In the I(0) case, the long run variance of u_t is given by $\omega_u^2 := \lim_{T\to\infty} T^{-1} E\left(\sum_{t=1}^{T} u_t\right)^2 = \omega_\varepsilon^2 / \left(1-\rho\right)^2$.

1.2.1 Known Break Fractions

We start by considering the case where the vector of true break fractions $\tau^* = (\tau_1^*, \tau_2^*, \ldots, \tau_m^*)'$ and hence all the eventual dates when the slope changes occur are known. The number of breaks m is also known.

Similarly to HLT, we partition H_1 into two local alternatives $H_{1,0} : \gamma = \kappa T^{-3/2}$ when u_t is $I\left(0\right)$ and $H_{1,1} : \gamma = \kappa T^{-1/2}$ when u_t is $I\left(1\right)$ where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)'$ and κ is a k-dimensional vector of finite non negative constants, $\kappa = \left(\kappa_1, \kappa_2, \ldots, \kappa_m\right)'$.

Suppose one knows that u_t is $I\left(0\right)$, with $\rho = 0$ and ε_t is Gaussian white noise. Then, to test the null hypothesis H_0, we should use the standard F-statistic. Let $\left(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}_1\left(\tau^*\right), \ldots, \widehat{\gamma}_m\left(\tau^*\right)\right)$ be the OLS estimators of the coefficients in equation (1.1) and $\widehat{u}_t\left(\tau^*\right) := y_t - \widehat{\alpha} - \widehat{\beta} t - \sum_{j=1}^{m} \widehat{\gamma}_j\left(\tau^*\right) DT_t\left(\tau_j^*\right)$ be the corresponding OLS residuals. Also define $x_{DT,t}\left(\tau^*\right) := \{1, t, DT_t\left(\tau_1^*\right), DT_t\left(\tau_2^*\right), \ldots, DT_t\left(\tau_m^*\right)\}'$ as the vector of regressors. The $F_0\left(\tau^*\right)$ statistic is given by [2]:

[2]The notation $[.]_{(i:j,i:j)}$ ($[.]_{(j)}$) is used to denote a submatrix (scalar) formed by rows and columns i until j (the j'th element) from the matrix (vector) within the squared brackets

$$\digamma_0(\tau^*) = \widehat{\gamma}(\tau^*)' \left\{ \widehat{\sigma}^2(\tau^*) \left[\sum_{t=1}^{T} x_{DT,t}(\tau^*) x_{DT,t}(\tau^*)' \right]^{-1}_{(3:m+2,3:m+2)} \right\}^{-1} \widehat{\gamma}(\tau^*)/m \qquad (1.3)$$

where $\widehat{\gamma}(\tau^*) = (\widehat{\gamma}_1(\tau^*), \widehat{\gamma}_2(\tau^*), \ldots, \widehat{\gamma}_m(\tau^*))'$ with

$$\widehat{\gamma}_j(\tau^*) = \left[\left(\sum_{t=1}^{T} x_{DT,t}(\tau^*) x_{DT,t}(\tau^*)' \right)^{-1} \sum_{t=1}^{T} x_{DT,t}(\tau^*) y_t \right]_{(j+2)}, \; j = 1, \ldots, m,$$

and $\widehat{\sigma}^2(\tau^*) := T^{-1} \sum_{t=1}^{T} \widehat{u}_t(\tau^*)^2$.

Now suppose that u_t is known to be $I(1)$, with $\rho = 1$ and Δu_t is a Gaussian white noise process. To test if the slope of the trend function is constant against the alternative of m breaks over time we should use the F-statistic after differentiating the data. So by applying first-differences to equation (1.1) we have:

$$\Delta y_t = \beta + \sum_{j=1}^{m} \gamma_j DU_t(\tau_j^*) + v_t, \qquad t = 2, \ldots, T \qquad (1.4)$$

where $DU_t(\tau_j^*) := \mathbb{1}(t > T_j^*)$ and $v_t = \Delta u_t$. Let $\left(\widetilde{\beta}, \widetilde{\gamma}_1(\tau^*), \widetilde{\gamma}_2(\tau^*), \ldots, \widetilde{\gamma}_m(\tau^*)\right)$ denote the OLS estimators of the parameters from (1.4) and $\widetilde{v}_t(\tau^*) = \Delta y_t - \widetilde{\beta} - \sum_{j=1}^{m} \widetilde{\gamma}_j(\tau^*) DU_t(\tau_j^*)$ the resulting residuals. Also let $x_{DU,t}(\tau^*) := \{1, DU_t(\tau_1^*), DU_t(\tau_2^*), \ldots, DU_t(\tau_m^*)\}'$ denote the vector of regressors. The $\digamma_1(\tau^*)$ statistic is given by:

$$\digamma_1(\tau^*) = \widetilde{\gamma}(\tau^*)' \left\{ \widetilde{\sigma}^2(\tau^*) \left[\sum_{t=2}^{T} x_{DU,t}(\tau^*) x_{DU,t}(\tau^*)' \right]^{-1}_{(2:m+1,2:m+1)} \right\}^{-1} \widetilde{\gamma}(\tau^*)/m \qquad (1.5)$$

where $\widetilde{\gamma}(\tau^*) = (\widetilde{\gamma}_1(\tau^*), \widetilde{\gamma}_2(\tau^*), \ldots, \widetilde{\gamma}_m(\tau^*))'$ with

$$\widetilde{\gamma}_j(\tau^*) = \left[\left(\sum_{t=2}^{T} x_{DU,t}(\tau^*)\, x_{DU,t}(\tau^*)'\right)^{-1} \sum_{t=2}^{T} x_{DU,t}(\tau^*)\,\Delta y_t\right]_{(j+1)}, \; j = 1, \ldots, m,$$

and $\widetilde{\sigma}^2(\tau^*) := (T-1)^{-1} \sum_{t=2}^{T} \widetilde{v}_t(\tau^*)^2$.

Remark 2. This paper is focusing its attention on the existence of multiple structural breaks in the trend function. However, it is straightforward to adapt the test statistic to other hypothesis of interest, for example, to test if the magnitude of the breaks was the same in two different periods, or even non-linear hypothesis.

To accommodate more general forms of autocorrelation of the error terms as allowed in Assumption 1, we simply substitute $\widehat{\sigma}^2(\tau^*)$ and $\widetilde{\sigma}^2(\tau^*)$ by non-parametric estimators of the long-run variances. Following Newey and West (1987), the following estimators can be used:

$$\widehat{\omega}^2(\tau^*) := \widehat{\gamma}_0(\tau^*) + 2\sum_{j=1}^{l} h(j/l)\,\widehat{\gamma}_j(\tau^*), \qquad \widehat{\gamma}_j(\tau^*) = T^{-1} \sum_{t=j+1}^{T} \widehat{u}_t(\tau^*)\,\widehat{u}_{t-j}(\tau^*), \quad (1.6)$$

and

$$\widetilde{\omega}^2(\tau^*) := \widetilde{\gamma}_0(\tau^*) + 2\sum_{j=1}^{l} h(j/l)\,\widetilde{\gamma}_j(\tau^*), \qquad \widetilde{\gamma}_j(\tau^*) = (T-1)^{-1} \sum_{t=j+1}^{T} \widetilde{v}_t(\tau^*)\,\widetilde{v}_{t-j}(\tau^*), \quad (1.7)$$

where the weights are given by $h(j/l) := 1 - j/(l+1)$ with lag truncation $l = O(T^{1/4})$. In the sequel, unless otherwise stated, any reference to $\mathcal{F}_0(\tau^*)$ and $\mathcal{F}_1(\tau^*)$ will be taken to imply those based on these long run variance estimators.

We now establish the asymptotic distribution of the $\mathcal{F}_0(\tau^*)$ and $\mathcal{F}_1(\tau^*)$ statistics.

Theorem 1. *Let the time series process be generated by (1.1) and (1.2), and let Assumption 1 hold.*

(i) If u_t is $I(0)$ ($|\rho| < 1$) then, under $H_{1,0}$: (a) $\mathcal{F}_0(\tau^*) \xrightarrow{d} \frac{1}{m} J_0(\tau^*, \kappa)$, and (b) $\mathcal{F}_1(\tau^*) = O_p\left(\frac{l}{T}\right)$, where

$$J_0(\tau^*, \kappa) \sim \chi^2_m(\mu_0), \mu_0 = \kappa' \left[Q_0(\tau^*)/\omega_u^2\right] \kappa, Q_0(\tau^*) = \int_0^1 RT(r, \tau^*) RT(r, \tau^*)' dr$$

(ii) If u_t is $I(1)$ ($|\rho| = 1$) then, under $H_{1,1}$: (a) $\mathcal{F}_0(\tau^*) = O_p\left(\frac{T}{l}\right)$, and (b) $\mathcal{F}_1(\tau^*) \xrightarrow{d} \frac{1}{m} J_1(\tau^*, \kappa)$, where

$$J_1(\tau^*, \kappa) \sim \chi^2_m(\mu_1), \mu_1 = \kappa' \left[Q_1(\tau^*)/\omega_\varepsilon^2\right] \kappa, Q_1(\tau^*) = \int_0^1 RU(r, \tau^*) RU(r, \tau^*)' dr$$

The $\chi^2_m(\mu)$ denotes the non-central chi-square distribution with m degrees of freedom and $RT(r, \tau^*) = (RT(r, \tau_1^*), RT(r, \tau_2^*), \ldots, RT(r, \tau_m^*))'$ where $RT(r, \tau_i^*)$ is the continuous time residual from the projection of $(r - \tau_i^*)\mathbb{1}(r > \tau_i^*)$ onto the space spanned by $\{1, r\}$ and $RU(r, \tau^*) = (RU(r, \tau_1^*), \ldots, RU(r, \tau_m^*))'$ where $RU(r, \tau_i^*)$ is the continuous time residual from the projection of $\mathbb{1}(r > \tau_i^*)$ onto $\{1\}$.

Remark 3. From Theorem 1 we can easily conclude that, under $H_0 : \gamma = \mathbf{0}_{m\times 1}$ (or $\kappa = \mathbf{0}_{m\times 1}$), we have $m \cdot \mathcal{F}_0(\tau^*) \xrightarrow{d} \chi^2_m$ if u_t is I(0) and also $m \cdot \mathcal{F}_1(\tau^*) \xrightarrow{d} \chi^2_m$ if u_t is I(1). If we knew all the true potential break dates and also the order of integration of the error term u_t, we could use the appropriate F-statistic to test if the potential m changes in slope are statistically significant or not using critical values from the chi-square distribution with m degrees of freedom. Also note that in the particular case of only one break, $m = 1$, Theorem 1 is basically equivalent to Theorem 1 in HLT by the equivalence between the F-statistic and the squared t-statistic when testing only one coefficient.

Remark 4. From the results of part (i) of Theorem 1 it is seen that when u_t is I(0), $\mathcal{F}_1(\tau^*)$ converges in probability to zero, regardless of the value of κ. Similarly, from the results in part (ii) of Theorem 1 it is seen that when u_t is I(1), $\mathcal{F}_0(\tau^*)$ diverges irrespective of the value of κ.

Since, in practice, the order of integration is not known we would like to find a procedure that, at least asymptotically, converges to the asymptotic distribution of $\mathcal{F}_0(\tau^*)$ when u_t is I(0) and to the asymptotic distribution of $\mathcal{F}_1(\tau^*)$ when u_t is I(1). More specifically, we would like to find a weight function, call it $\lambda(.)$, such that $\lambda(.) \xrightarrow{p} 1$ if u_t is $I(0)$ and $\lambda(.) \xrightarrow{p} 0$ if u_t is $I(1)$ ensuring that the appropriate statistic with non-degenerate distribution is selected. We employ the solution proposed by HLT and let $\lambda(.)$ be a function of the KPSS statistic of the original data $S_0(\tau^*)$ and of the differenced data $S_1(\tau^*)$:

$$S_0(\tau^*) := \frac{\sum_{t=1}^{T}\left(\sum_{i=1}^{t}\widehat{u}_i(\tau^*)\right)^2}{T^2\widehat{\omega}^2(\tau^*)}, \qquad S_1(\tau^*) := \frac{\sum_{t=2}^{T}\left(\sum_{i=1}^{t}\widetilde{v}_i(\tau^*)\right)^2}{(T-1)^2\widetilde{\omega}^2(\tau^*)} \tag{1.8}$$

Lemma 1. *Let the conditions of Theorem 1 hold:*

(i) If u_t is $I(0)$, then: (a) $S_0(\tau^*) = O_p(1)$, and (b) $S_1(\tau^*) = O_p(l/T)$.

(ii) If u_t is $I(1)$, then: (a) $S_0(\tau^*) = O_p(T/l)$, and (b) $S_1(\tau^*) = O_p(1)$.

Since by Lemma 1 the KPSS statistics, $S_0(\tau^*)$ and $S_1(\tau^*)$, have the same asymptotic rates of convergence for a single or more trend breaks we can use the same weight function from HLT:

$$\lambda(S_0(\tau^*), S_1(\tau^*)) := \exp\left[-\left\{gS_0(\tau^*)S_1(\tau^*)\right\}^v\right] \tag{1.9}$$

where g and v are positive constants. Now we are able to form the $\mathcal{F}_\lambda^*$ statistic and study its asymptotic distribution:

$$\mathcal{F}_\lambda(\tau^*) := \{\lambda(S_0(\tau^*), S_1(\tau^*)) \times \mathcal{F}_0(\tau^*)\} + \{[1-\lambda(S_0(\tau^*), S_1(\tau^*))] \times \mathcal{F}_1(\tau^*)\} \tag{1.10}$$

Notice that a higher g gives more weight to $\mathcal{F}_1$ keeping everything else constant. Using the results from Theorem 1 and Lemma 1, we get the following result.

Corollary 1. *Let the conditions of Theorem 1 hold.*

(i) If u_t is $I(0)$, then: $\lambda(S_0(\tau^*), S_1(\tau^*)) \xrightarrow{p} 1$ under both H_0 and $H_{1,0}$, and $\mathcal{F}_\lambda(\tau^*) = \mathcal{F}_0(\tau^*) + o_p(1) \xrightarrow{d} \frac{1}{m}J_0(\tau^*, \kappa)$.

(ii) If u_t *is* $I(1)$, *then:* $\lambda\left(S_0\left(\tau^*\right), S_1\left(\tau^*\right)\right) \xrightarrow{p} 0$, *under both* H_0 *and* $H_{1,1}$, *and* $\mathcal{F}_\lambda\left(\tau^*\right) = \mathcal{F}_1\left(\tau^*\right) + o_p(1) \xrightarrow{d} \frac{1}{m} J_1\left(\tau^*, \kappa\right)$.

Remark 5. From Corollary 1 we observe that we have constructed a test statistic to test the presence of m candidate trend breaks at known break dates that is valid regardless of the order of integration of the errors. If u_t is I(0), $\mathcal{F}_\lambda\left(\tau^*\right)$ is asymptotically equivalent to $\mathcal{F}_0\left(\tau^*\right)$, while if u_t is I(1), $\mathcal{F}_\lambda\left(\tau^*\right)$ becomes asymptotically equivalent to $\mathcal{F}_1\left(\tau^*\right)$. Since, given these conditions, both $m \cdot \mathcal{F}_0\left(\tau^*\right)$ and $m \cdot \mathcal{F}_1\left(\tau^*\right)$ converge in distribution to a chi-square distribution with m degrees of freedom under the null we can use the critical values of the central chi-square distribution for $\mathcal{F}_\lambda\left(\tau^*\right)$ irrespective of whether the disturbances, u_t, are I(0) or I(1).

1.2.2 Unknown Break Fractions

In this section, we consider tests of multiple structural changes in the trend function with unknown change points. Suppose that the true break fractions τ^* are unknown but the number of breaks, m, is known. Proceeding in the same way as Andrews (1993) and Bai and Perron (1998) we can form F type statistics to test the null hypothesis of no trend breaks against the alternative hypothesis that there are m trend breaks. Let $\tau^m := (\tau_1, \ldots, \tau_m)$ and $\Lambda_m = \{(\tau_1, \ldots, \tau_m) : |\tau_{i+1} - \tau_i| \geq \eta, \tau_1 \geq \eta, \tau_m \leq 1 - \eta\}$ and assume throughout that $\tau^* \in \Lambda_m$. If we knew that u_t was I(0) the F-statistic would be defined as:

$$\mathcal{F}_0^*(m|0) := \sup_{\tau^m \in \Lambda_m} \mathcal{F}_0\left(\tau^m\right) \tag{1.11}$$

and if we knew that u_t was I(1) the statistic would be given by:

$$\mathcal{F}_1^*(m|0) := \sup_{\tau^m \in \Lambda_m} \mathcal{F}_1\left(\tau^m\right), \tag{1.12}$$

where the associated vectors of estimated break fractions of τ^* are given by

$$\widehat{\tau}^m := \arg\sup_{\tau^m \in \Lambda_m} \mathcal{F}_0\left(\tau^m\right) \tag{1.13}$$

and

$$\widetilde{\tau}^m := \underset{\tau^m \in \Lambda_m}{\arg\sup} \mathcal{F}_1(\tau^m), \tag{1.14}$$

respectively, such that $\mathcal{F}_0^*(m|0) = \mathcal{F}_0(\widehat{\tau}^m)$ and $\mathcal{F}_1^*(m|0) = \mathcal{F}_1(\widetilde{\tau}^m)$. To solve the problem of an unknown order of integration of the error term we follow the same strategy as in the known break fraction case and write the analogue of the $\mathcal{F}_\lambda(\tau^*)$ statistic:

$$\mathcal{F}_\lambda^*(m|0) := \{\lambda(\widehat{\tau}^m, \widetilde{\tau}^m) \times \mathcal{F}_0^*(m|0)\} + b_\xi^m \{[1 - \lambda(\widehat{\tau}^m, \widetilde{\tau}^m)] \times \mathcal{F}_1^*(m|0)\} \tag{1.15}$$

where $\lambda(\widehat{\tau}^m, \widetilde{\tau}^m) := \lambda(S_0(\widehat{\tau}^m), S_1(\widetilde{\tau}^m))$ and b_ξ^m is a positive finite constant such that, as will be explained below, for any significance level ξ, the critical value of $\mathcal{F}_\lambda^*(m|0)$ is the same regardless of whether u_t is I(0) or I(1). The following Theorem states the asymptotic distribution of $\mathcal{F}_0^*(m|0)$ and $\mathcal{F}_1^*(m|0)$ under the null hypothesis $\gamma = 0$ when the innovation sequence $\{u_t\}$ is either I(0) or I(1).

Theorem 2. *Let the time series process be generated by (1.1) and (1.2) under $H_0 : \gamma = \mathbf{0}_{m\times 1}$ and let Assumption 1 hold.*

(i) If u_t is $I(0)$, then: (a) $\mathcal{F}_0^*(m|0) \xrightarrow{d} \frac{1}{m} \sup_{\tau^m \in \Lambda_m} J_0(\tau^m, 0)$, and (b) $\mathcal{F}_1^*(m|0) = O_p\left(\frac{l}{T}\right)$.

(ii) If u_t is $I(1)$, then: (a) $\mathcal{F}_0^*(m|0) = O_p\left(\frac{T}{l}\right)$, and (b) $\mathcal{F}_1^*(m|0) \xrightarrow{d} \frac{1}{m} \sup_{\tau^m \in \Lambda_m} J_1(\tau^m, 0)$.

Remark 6. HLT established the divergence rates for the 1 break case under a fixed alternative $H_1 : \gamma \neq 0$ using $\sup t$ instead of $\sup \mathcal{F}$ statistics. Since $\mathcal{F}_i(\tau_1) = (t_i(\tau_1))^2$ and $\mathcal{F}_i(\tau_1) \leqslant 2\mathcal{F}_i(\tau_1, \tau_2) \leqslant m\mathcal{F}_i(\tau_1, \ldots, \tau_m), i = 0, 1$, the consistency of $\mathcal{F}_0^*$ and $\mathcal{F}_1^*$ follow immediately from Theorem 3 from HLT.

Next, we establish the large sample behavior of the weight function $\lambda(S_0(\widehat{\tau}^m), S_1(\widetilde{\tau}^m))$. For this purpose, we need to know the asymptotic behavior of the KPSS statistics $S_0(\tau^m)$ and $S_1(\tau^m)$ when the disturbances u_t are either I(0) or I(1) and the vector of break points, τ, is estimated, i.e., for the cases $\tau^m = \widehat{\tau}^m$ and $\tau^m = \widetilde{\tau}^m$.

Lemma 2. *Let the conditions of Theorem 1 hold.*

(i) If u_t is $I(0)$, then: (a) $S_0(\widehat{\tau}^m) = O_p(1)$, and (b) $S_1(\widetilde{\tau}^m) = O_p(l/T)$.

(ii) If u_t is $I(1)$, then: (a) $S_0(\widehat{\tau}^m) = O_p(T/l)$, and (b) $S_1(\widetilde{\tau}^m) = O_p(1)$.

From Lemma 2 it is seen that the results from Lemma 1 are unchanged and so the large sample behavior of the KPSS statistics is the same regardless of whether the trend break dates are known or unknown. We conjecture that Lemma 2 holds independently of assuming the null hypothesis $H_0 : \gamma_1 = \gamma_2 = \ldots = \gamma_m = 0$ or the alternative $H_1 : \gamma_j \neq 0$, $j = 1, \ldots, m$, as shown in HLT for the 1 break case. This implies that we can continue to use the same $\lambda(.)$ function as defined above for the case of known break dates since if u_t is I(0) then $\lambda(\widehat{\tau}^m, \widetilde{\tau}^m) \xrightarrow{p} 1$ while if u_t is I(1) we have $\lambda(\widehat{\tau}^m, \widetilde{\tau}^m) \xrightarrow{p} 0$, under both H_0 and H_1, and so the F statistic that we would like to be chosen depending on the order of integration of u_t is actually selected asymptotically. Therefore we can state the following corollary:

Corollary 2. *Let the conditions of Theorem 2 hold.*

(i) If u_t is $I(0)$, then: $F_\lambda^*(m|0) = F_0^*(m|0) + o_p(1) \xrightarrow{d} \dfrac{1}{m} \sup_{\tau^m \in \Lambda_m} J_0(\tau^m, 0)$.

(ii) If u_t is $I(1)$, then: $F_\lambda^*(m|0) = b_\xi^m F_1^*(m|0) + o_p(1) \xrightarrow{d} b_\xi^m \dfrac{1}{m} \sup_{\tau^m \in \Lambda_m} J_1(\tau^m, 0)$.

Notice that contrary to the known break fraction case, the asymptotic distribution of $F_0^*(m|0)$ is different from $F_1^*(m|0)$ and both no longer converge to a chi-square distribution with m degrees of freedom. In this case using the same reasoning as HLT, we can choose a constant b_ξ^m such that the critical values become the same for both I(0) and I(1) errors.

1.2.3 Double Maximum Tests

The tests discussed above require the specification of the number of trend breaks, m, under the alternative hypothesis. However, in most applications, one is not sure about the number of breaks. Therefore, we consider tests of the null of no trend break against the alternative hypothesis of an unknown number of breaks in the trend slope up to some maximum M. Following Bai and Perron (1998), we use the class of double maximum tests which are generally written as:

$$D\max \mathcal{F}_0^* := \max_{1\leq m\leq M} a_{0,m}\mathcal{F}_0^*(m|0) = \max_{1\leq m\leq M} a_{0,m} \sup_{\tau^m\in\Lambda_m} \mathcal{F}_0(\tau^m) \tag{1.16}$$

and

$$D\max \mathcal{F}_1^* := \max_{1\leq m\leq M} a_{1,m}\mathcal{F}_1^*(m|0) = \max_{1\leq m\leq M} a_{1,m} \sup_{\tau^m\in\Lambda_m} \mathcal{F}_1(\tau^m) \tag{1.17}$$

with $(a_{0,1},\ldots,a_{0,M})$ and $(a_{1,1},\ldots,a_{1,M})$ fixed weights that may be chosen in a way that reflects some prior knowledge regarding the likelihood that the data has a certain number of trend breaks. We use the same weight function to obtain a double maximum test that is valid for both I(0) and I(1) errors:

$$D\max \mathcal{F}_\lambda^* := \left\{\lambda\left(\widehat{\tau}^M, \widetilde{\tau}^M\right) \times D\max \mathcal{F}_0^*\right\} + b_\xi^M \left\{\left[1-\lambda\left(\widehat{\tau}^M, \widetilde{\tau}^M\right)\right] \times D\max \mathcal{F}_1^*\right\} \tag{1.18}$$

The b_ξ^M denote a constant that can be chosen, as before, in a way that guarantees the same critical values for both I(0) and I(1) cases. From Theorem 2 and the Continuous Mapping Theorem we may easily find the asymptotic distribution of the $D\max \mathcal{F}_\lambda^*$ test statistic.

Corollary 3. *Let the conditions of Theorem 2 hold.*

(i) If u_t is $I(0)$, then:

$$D\max \mathcal{F}_\lambda^* = \max_{1\leq m\leq M} a_{0,m}\ \mathcal{F}_0^*(m|0) + o_p(1) \xrightarrow{d} \max_{1\leq m\leq M} a_{0,m} \frac{1}{m} \sup_{\tau^m\in\Lambda_m} J_0(\tau^m, 0).$$

(ii) If u_t is $I(1)$, then:

$$D\max \mathcal{F}_\lambda^* = b_\xi^M \max_{1\leq m\leq M} a_{1,m}\mathcal{F}_1^*(m|0) + o_p(1) \xrightarrow{d} b_\xi^M \max_{1\leq m\leq M} a_{1,m} \frac{1}{m} \sup_{\tau^m\in\Lambda_m} J_1(\tau^m, 0).$$

We consider as in Bai and Perron (1998) two cases: the $UD\max \mathcal{F}$ type of test where the weights are chosen uniformly across all possible number of breaks, $a_{d,1} = \ldots = a_{d,M} = 1$, $d = 0, 1$, and the $WD\max \mathcal{F}$ where the weights are defined in such a way that the marginal p-values are equal across values of m, i.e., $a_{d,1} = 1$ and for $m > 1$,

$a_{d,m} = \dfrac{C_d(\xi, 1)}{C_d(\xi, m)}$ where $C_d(\xi, m)$ is the asymptotic critical value of the test $\mathcal{F}_d^*$ for a significance level ξ and m breaks.

1.2.4 Sequential Tests and Estimation of the Number of Breaks

As in Bai and Perron (1998), we also extend our methodology to a test of the null hypothesis of l breaks in the trend against the alternative of $l+1$ breaks. Let $\widehat{\tau}^l = (\widehat{\tau}_1, \ldots, \widehat{\tau}_l)'$ and $\widetilde{\tau}^l = (\widetilde{\tau}_1, \ldots, \widetilde{\tau}_l)'$ denote the vectors of estimated break fractions assuming l breaks in the I(0) and I(1) cases, respectively, as defined in equations (1.13) and (1.14). Let $\mathcal{F}_0(\widehat{\tau}_1, \ldots, \widehat{\tau}_{i-1}, \zeta, \widehat{\tau}_i, \ldots, \widehat{\tau}_l)$ be the standard F-statistic for testing $H_0 : \gamma_{l+1} = 0$ versus the alternative $H_1 : \gamma_{l+1} \neq 0$ in the Model:

$$y_t = \alpha + \beta t + \sum_{j=1}^{l} \gamma_j DT_t(\widehat{\tau}_j) + \gamma_{l+1} DT_t(\zeta) + u_t$$

Similarly, let $\mathcal{F}_1(\widetilde{\tau}_1, \ldots, \widetilde{\tau}_{i-1}, \zeta, \widetilde{\tau}_i, \ldots, \widetilde{\tau}_l)$ be the standard F-statistic for testing $H_0 : \gamma_{l+1} = 0$ versus the alternative $H_1 : \gamma_{l+1} \neq 0$ in the Model :

$$\Delta y_t = \beta + \sum_{j=1}^{l} \gamma_j DU_t(\widetilde{\tau}_j) + \gamma_{l+1} DU_t(\zeta) + v_t$$

When the break dates are not known, we use the $\mathcal{F}_0^*(l+1|l)$ and $\mathcal{F}_1^*(l+1|l)$ test statistics defined as $\mathcal{F}_0^*(1|0) := \sup_{\tau^1 \in \Lambda_1} \mathcal{F}_0(\tau)$, $\mathcal{F}_1^*(1|0) := \sup_{\tau^1 \in \Lambda_1} \mathcal{F}_1(\tau)$ for $l = 0$; and for $l > 0$ as

$$\mathcal{F}_0^*(l+1|l) := \max_{1 \leq i \leq l+1} \sup_{\zeta \in \Lambda_{0,i}} \mathcal{F}_0(\widehat{\tau}_1, \ldots, \widehat{\tau}_{i-1}, \zeta, \widehat{\tau}_i, \ldots, \widehat{\tau}_l)$$

$$\mathcal{F}_1^*(l+1|l) := \max_{1 \leq i \leq l+1} \sup_{\zeta \in \Lambda_{1,i}} \mathcal{F}_1(\widetilde{\tau}_1, \ldots, \widetilde{\tau}_{i-1}, \zeta, \widetilde{\tau}_i, \ldots, \widetilde{\tau}_l)$$

where the possible eligible break fractions ζ are contained in the following sets in which η is the trimming parameter:

$$\Lambda_{0,i} = \{\zeta : \widehat{\tau}_{i-1} + (\widehat{\tau}_i - \widehat{\tau}_{i-1})\,\eta \leq \zeta \leq \widehat{\tau}_i - (\widehat{\tau}_i - \widehat{\tau}_{i-1})\,\eta\} \tag{1.19}$$

and

$$\Lambda_{1,i} = \{\zeta : \widetilde{\tau}_{i-1} + (\widetilde{\tau}_i - \widetilde{\tau}_{i-1})\,\eta \leq \zeta \leq \widetilde{\tau}_i - (\widetilde{\tau}_i - \widetilde{\tau}_{i-1})\,\eta\}\,. \tag{1.20}$$

with $\widehat{\tau}_0 = 0$ and $\widehat{\tau}_{l+1} = 1$. The next Theorem establishes the asymptotic behaviour of $\mathcal{F}_0^*(l+1|l)$ and $\mathcal{F}_1^*(l+1|l)$ for different orders of integration of the error term u_t.

Theorem 3. *Let the time series process y_t be generated according to (1.1) and (1.2) with $m = l$ breaks and let Assumption 1 hold.*

(i) If u_t is $I(0)$, then: (a) $\lim_{T\to\infty} P\left(\mathcal{F}_0^*(l+1|l) \leq x\right) = G_0(x)^{l+1}$, where $G_0(x)$ is the distribution function of $\sup_{\tau^m\in\Lambda_m} J_0(\tau^m, 0)$ for $m = 1$, and (b) $\mathcal{F}_1^*(l+1|l) = O_p(l/T)$.

(ii) If u_t is $I(1)$, then: (a) $\mathcal{F}_0^*(l+1|l) = O_p(T/l)$, and (b) $\lim_{T\to\infty} P\left(\mathcal{F}_1^*(l+1|l) \leq x\right) = G_1(x)^{l+1}$, where $G_1(x)$ is the distribution function of $\sup_{\tau^m\in\Lambda_m} J_1(\tau^m, 0)$ for $m = 1$.

Remark 7. The results in the previous Theorem show that critical values for the sequential tests can be computed from the quantiles of the asymptotic distributions of the $\mathcal{F}_0^*$ and $\mathcal{F}_1^*$ test statistics for the case of just one break ($m = 1$).

The $\mathcal{F}_\lambda^*(l+1|l)$ statistic is then given by:

$$\begin{aligned}\mathcal{F}_\lambda^*(l+1|l) := &\left\{\lambda\left(\widehat{\tau}^{l+1}, \widetilde{\tau}^{l+1}\right) \times \mathcal{F}_0^*(l+1|l)\right\} \\ &+ b_\xi^{l+1|l}\left\{\left[1 - \lambda\left(\widehat{\tau}^{l+1}, \widetilde{\tau}^{l+1}\right)\right] \times \mathcal{F}_1^*(l+1|l)\right\}\end{aligned} \tag{1.21}$$

where $\widehat{\tau}^{l+1} = (\widehat{\tau}_1, \ldots, \widehat{\tau}_{l+1})'$ and $\widetilde{\tau}^{l+1} = (\widetilde{\tau}_1, \ldots, \widetilde{\tau}_{l+1})'$ and $b_\xi^{l+1|l}$ is a constant that ensures that for a given significance level ξ and null hypothesis of l trend breaks the critical values of the asymptotic distribution of $\mathcal{F}_\lambda(l+1|l)$ is the same in both I(0) and I(1) cases.

Using Lemma 2 and the fact that the order of probability of the KPSS statistics $S_0\left(\widehat{\tau^{l+1}}\right)$ and $S_1\left(\widehat{\tau^{l+1}}\right)$ under I(0) or I(1) errors is unchanged both under the null and the alternative hypothesis, it is readily seen that the weight function has the same asymptotic behavior as in Corollary 1 and so we may state the following corollary:

Corollary 4. *Let the conditions of Theorem 3 hold.*

(i) If u_t is $I(0)$, then $\lambda\left(\widehat{\tau}^{l+1}, \widetilde{\tau}^{l+1}\right) \xrightarrow{p} 1$, $\mathcal{F}^*_\lambda(l+1|l) = \mathcal{F}^*_0(l+1|l) + o_p(1)$ *and* $\lim_{T\to\infty} P\left(\mathcal{F}^*_\lambda(l+1|l) \le x\right) = G_0(x)^{l+1}$.

(ii) If u_t is $I(1)$, then $\lambda\left(\widehat{\tau}^{l+1}, \widetilde{\tau}^{l+1}\right) \xrightarrow{p} 0$, $\mathcal{F}^*_\lambda(l+1|l) = b_\xi^{l+1|l}\mathcal{F}^*_1(l+1|l) + o_p(1)$ *and* $\lim_{T\to\infty} P\left(b_\xi^{l+1|l}\mathcal{F}^*_\lambda(l+1|l) \le x\right) = G_1(x)^{l+1}$.

The $\mathcal{F}^*_\lambda(l+1|l)$ can be used to estimate the number of breaks in the trend slope without making any assumption about the errors being I(0) or I(1). The procedure starts with $l = 0$, by using the $\mathcal{F}^*_\lambda(1|0)$ to test for the presence of one break. If the null hypothesis is rejected, we set $l = 1$ and perform the $\mathcal{F}^*_\lambda(2|1)$ test. The procedure is repeated until the $\mathcal{F}^*_\lambda(l+1|l)$ test cannot reject the null hypothesis of l breaks.

Remark 8. In small samples, for some particular combinations of the breaks in the trend slope, this sequential procedure may not perform well. For instance, in the presence of two breaks of opposite signs, the $\mathcal{F}^*_\lambda(1|0)$ may have low power in identifying the two breaks, causing the sequential estimation procedure to stop too soon. A simple modification of this sequential procedure that is able to obviate to this problem consists in using the $\mathcal{F}^*_\lambda$ with $m = 2$ or a double maximum test $D\max\mathcal{F}^*_\lambda$ whenever the $\mathcal{F}^*_\lambda(1|0)$ does not reject the null hypothesis of no break. If the $\mathcal{F}^*_\lambda$ with $m = 2$ or the double maximum test does not reject H_0 then we conclude that there are no trend breaks. Otherwise we proceed to $\mathcal{F}^*_\lambda(3|2)$. We call these sequential procedures $Seq\mathcal{F}^*_\lambda(1|0)$, $Seq\mathcal{F}^*_\lambda(2|0)$, $SeqUD\max\mathcal{F}^*_\lambda$ and $SeqWD\max\mathcal{F}^*_\lambda$. Figure 1.16 summarizes the steps to implement in each type of sequential test presented.

Remark 9. The sequential procedure to estimate the number of breaks can be made consistent by letting the significance level of the $\mathcal{F}^*_\lambda(l+1|l)$ test converge to zero slowly enough as explained in Proposition 8 from Bai and Perron (1998). However, for a given sample, this has no practical implications and the usual significance levels can be used.

1.3 Disjoint Broken Trend Model

The analysis of the previous section can be generalized to the case of a model with m disjoint broken trends where the level may also change at the same time as the slope.

Therefore, we consider the following model:

$$y_t = \alpha + \beta t + \sum_{j=1}^{m} \delta_j DU_t\left(\tau_j^*\right) + \sum_{j=1}^{m} \gamma_j DT_t\left(\tau_j^*\right) + u_t \qquad t = 1, \ldots, T, \tag{1.22}$$

and

$$u_t = \rho u_{t-1} + \varepsilon_t, \qquad t = 2, \ldots, T, \qquad u_1 = \varepsilon_1, \tag{1.23}$$

satisfying Assumption 1 and $|\rho| \leq 1$. In what follows we will refer to this model as "Model B". Notice that δ_j and γ_j capture the change, respectively, in the level and slope coefficients of the series at time T_j. The slope coefficient changes from β to $\beta+\gamma_1$ and the level shifts from α to $\alpha+\delta_1$ at time T_1^*. At break point T_2^* the slope coefficient changes from $\beta+\gamma_1$ to $\beta+\gamma_1+\gamma_2$ and the level goes from $\alpha+\delta_1$ to $\alpha+\delta_1+\delta_2$. Generally, in period T_j^* the slope coefficient changes from $\beta + \sum_{i=1}^{j-1} \gamma_i$ to $\beta + \sum_{i=1}^{j} \gamma_i$ while the level shifts from $\alpha + \sum_{i=1}^{j-1} \delta_i$ to $\alpha + \sum_{i=1}^{j} \delta_i$ for $j = 1, \ldots, m$. The trend function is discontinuous at a break date T_j^* if $\delta_j \neq 0$.

The first-differenced form of "Model B" is given by:

$$\Delta y_t = \beta + \sum_{j=1}^{k} \delta_j D_t\left(\tau_j^*\right) + \sum_{j=1}^{k} \gamma_j DU_t\left(\tau_j^*\right) + \Delta u_t, \qquad t = 2, \ldots, T \tag{1.24}$$

where $D_t\left(\tau_j^*\right) := \mathbb{1}\left(t = T_j^* + 1\right)$. Our interest is, as in Model A, to construct a test that is able to test if there are trend breaks in y_t and to develop a procedure to estimate the number of breaks in the trend slope regardless of whether u_t is $I(0)$ or $I(1)$. The null hypothesis of interest continues to be $H_0 : \gamma_1 = \gamma_2 = \ldots = \gamma_m = 0$ against the two sided alternative: $H_1 : \gamma_1 \neq 0 \vee \gamma_2 \neq 0 \vee \ldots \vee \gamma_m \neq 0$. Note that we do not place any restrictions on the values of δ_j and the interest lies only on the breaks in the trend slopes.

1.3.1 Known Break Fractions

Following the same steps as in Model A, we start by assuming that the true break fractions $\tau^* = (\tau_1^*, \tau_2^*, \ldots, \tau_m^*)'$ are known. Let $H_0, H_1, H_{1,0}$ and $H_{1,1}$ be defined as in Section 1.2.

We don't impose any restrictions on the vector of parameters δ_j to derive the asymptotic behaviour of the new test statistics.

Consider first the case when u_t is known to be $I(0)$. We redefine the data matrix, $x_{DT,t}(\tau^*) := \{1, t, DU_t(\tau_1^*), \ldots, DU_t(\tau_m^*), DT_t(\tau_1^*), \ldots, DT_t(\tau_m^*)\}'$.

Now we are able to rewrite $F_0(\tau^*)$ as:

$$\mathcal{F}_0(\tau^*) = \widehat{\gamma}(\tau^*)' \left\{ \widehat{\omega}^2(\tau^*) \left[\sum_{t=1}^{T} x_{DT,t}(\tau^*) x_{DT,t}(\tau^*)' \right]^{-1}_{(m+3:2m+2, m+3:2m+2)} \right\}^{-1} \widehat{\gamma}(\tau^*)/m \tag{1.25}$$

where

$$\widehat{\gamma}_j(\tau^*) = \left[\left(\sum_{t=1}^{T} x_{DT,t}(\tau^*) x_{DT,t}(\tau^*)' \right)^{-1} \sum_{t=1}^{T} x_{DT,t}(\tau^*) y_t \right]_{(j+2+m)}$$

with $\widehat{\gamma}(\tau^*) = (\widehat{\gamma}_1(\tau^*), \widehat{\gamma}_2(\tau^*), \ldots, \widehat{\gamma}_m(\tau^*))'$ and the long run variance $\widehat{\omega}^2(\tau^*)$ computed as before but using the new set of residuals $\widehat{u}_t(\tau^*) = y_t - \widehat{\alpha} - \widehat{\beta}t - \sum_{j=1}^{m} \widehat{\delta}_j DU_t(\tau_j^*) - \sum_{j=1}^{m} \widehat{\gamma}_j(\tau^*) DT_t(\tau_j^*)$. When u_t follows an $I(1)$ process, we use the first-differenced model and the vector of regressors becomes $x_{DU,t}(\tau^*) := \{1, D_t(\tau_1^*), \ldots, D_t(\tau_m^*), DU_t(\tau_1^*), \ldots, DU_t(\tau_m^*)\}'$. The $\mathcal{F}_1(\tau^*)$ statistic is now given by:

$$\mathcal{F}_1(\tau^*) = \widetilde{\gamma}(\tau^*)' \left\{ \widetilde{\omega}^2(\tau^*) \left[\sum_{t=2}^{T} x_{DU,t}(\tau^*) x_{DU,t}(\tau^*)' \right]^{-1}_{(m+2:2m+1, m+2:2m+1)} \right\}^{-1} \widetilde{\gamma}(\tau^*)/m \tag{1.26}$$

where

$$\widetilde{\gamma}_j(\tau^*) = \left[\left(\sum_{t=2}^{T} x_{DU,t}(\tau^*) x_{DU,t}(\tau^*)' \right)^{-1} \sum_{t=2}^{T} x_{DU,t}(\tau^*) \Delta y_t \right]_{(j+1+m)}$$

with $\widetilde{\gamma}(\tau^*) = (\widetilde{\gamma}_1(\tau^*), \widetilde{\gamma}_2(\tau^*), \ldots, \widetilde{\gamma}_m(\tau^*))'$.

The variance estimator $\widetilde{\omega}^2(\tau^*)$, is now computed using the following residuals: $\widetilde{v}_t(\tau^*) = \Delta y_t - \widetilde{\beta} - \sum_{j=1}^{m} \widetilde{\delta}_j D_t(\tau_j^*) - \sum_{j=1}^{m} \widetilde{\gamma}_j(\tau^*) DU_t(\tau_j^*)$:

The next theorem establishes the asymptotic distribution of $\mathcal{F}_0(\tau^*)$ and $\mathcal{F}_1(\tau^*)$ under

$H_{1,0}$ and $H_{1,1}$ with δ unrestricted when the error term is I(0) and I(1).

Theorem 4. *Let the time series process be generated by (1.22) and (1.23) and let Assumption 1 hold.*

(i) If u_t *is* $I(0)$ $(|\rho| < 1)$ *then, under* $H_{1,0}$*, (a)* $F_0(\tau^*) \xrightarrow{d} \frac{1}{m} K_0(\tau^*, \kappa)$*, and (b)* $F_1(\tau^*) = O_p\left(\frac{l}{T}\right)$*, where*

$$K_0(\tau^*, \kappa) \sim \chi^2_m(\mu_0), \mu_0 = \kappa' \left[Q_0(\tau^*)/\omega_u^2\right] \kappa, Q_0(\tau^*) = \int_0^1 RT_U(r, \tau^*) RT_U(r, \tau^*)' dr$$

(ii) If u_t *is* $I(1)$ $(|\rho| = 1)$ *then, under* $H_{1,1}$*, (a)* $F_0(\tau^*) = O_p\left(\frac{T}{l}\right)$*, and (b)* $F_1(\tau^*) \xrightarrow{d} \frac{1}{m} J_1(\tau^*, \kappa)$*, where*

$$J_1(\tau^*, \kappa) \sim \chi^2_m(\mu_1), \mu_1 = \kappa' \left[Q_1(\tau^*)/\omega_\varepsilon^2\right] \kappa, Q_1(\tau^*) = \int_0^1 RU(r, \tau^*) RU(r, \tau^*)' dr$$

where $\chi^2_m(\mu)$ *is the non-central chi-square distribution with* m *degrees of freedom and* $RT_U(r, \tau^*) = (RT_U(r, \tau_1^*), \ldots, RT_U(r, \tau_m^*))'$ *where* $RT_U(r, \tau_i^*)$ *is the continuous time residual from the projection of* $(r - \tau_i^*) \mathbb{1}(r > \tau_i^*)$ *onto the space spanned by* $\{1, r, \mathbb{1}(r > \tau_1^*), \ldots, \mathbb{1}(r > \tau_m^*)\}$ *and* $RU(r, \tau^*)$ *is defined in Theorem 1.*

Remark 10. As in "Model A", notice that under H_0 both $m \cdot F_0(\tau^*)$ and $m \cdot F_1(\tau^*)$ converge in distribution to the chi-square distribution with m degrees of freedom. So, again, if we know the order of integration of the disturbance term we can apply the appropriate F-statistic and use the critical value from the χ^2_m table to see if there is statistical evidence of the existence of m trend breaks.

We now extend our analysis to the case where it is not known if the error is I(0) or I(1). Following the same steps of the proof of Lemma 1 we are able to show that the orders of probability of the redefined KPSS statistics $S_0(\tau^*)$ and $S_1(\tau^*)$ remain the same as presented in that Lemma. Given this fact and Theorem 4 we may state the following corollary:

Corollary 5. *Let the conditions of Theorem 4 hold:*

(i) If u_t is $I(0)$, then $\lambda\left(S_0\left(\tau^*\right), S_1\left(\tau^*\right)\right) \xrightarrow{p} 1$, under both H_0 and $H_{1,0}$, and $\mathcal{F}_\lambda\left(\tau^*\right) = \mathcal{F}_0\left(\tau^*\right) + o_p\left(1\right) \xrightarrow{d} \frac{1}{m} K_0\left(\tau^*, k\right)$.

(ii) If u_t is $I(1)$, then $\lambda\left(S_0\left(\tau^*\right), S_1\left(\tau^*\right)\right) \xrightarrow{p} 0$, under both H_0 and $H_{1,1}$, and $\mathcal{F}_\lambda\left(\tau^*\right) = \mathcal{F}_1\left(\tau^*\right) + o_p\left(1\right) \xrightarrow{d} \frac{1}{m} J_1\left(\tau^*, k\right)$.

1.3.2 Unknown Break Fractions

We now consider the case where the true break fractions τ^* are unknown in Model B. For this purpose we adapt the test statistics to this model in the same way as done in the previous section. We redefine $\mathcal{F}_0^*\left(m|0\right)$ and $\mathcal{F}_1^*\left(m|0\right)$ using expressions with the new $\mathcal{F}_0\left(\tau\right)$ and $\mathcal{F}_1\left(\tau\right)$ presented above as well as $\widehat{\tau}$, $\widetilde{\tau}$ and $\mathcal{F}_\lambda^*\left(m|0\right)$.

Remark 11. Although our objective is only to test for changes in slope, we have to set additionally $\delta = 0$ in order to obtain a pivotal limiting null distribution for our test statistic. Hence, as in HLT the null hypothesis must be restated as $H_0 : \gamma = \delta = 0$.

The following Theorem states the asymptotic distribution of the re-defined $\mathcal{F}_0^*\left(m|0\right)$ and $\mathcal{F}_1^*\left(m|0\right)$ under the restated null hypothesis H_0 when the innovation sequence $\{u_t\}$ is either I(0) or I(1).

Theorem 5. *Let the time series process be generated by (1.22) and (1.23) under $H_0 : \gamma = \delta = \mathbf{0}_{m\times 1}$ and let Assumption 1 hold.*

(i) If u_t is $I(0)$, then: (a) $\mathcal{F}_0^*\left(m|0\right) \xrightarrow{d} \frac{1}{m} \sup_{\tau^m \in \Lambda_m} K_0\left(\tau^m, 0\right)$, and (b) $\mathcal{F}_1^*\left(m|0\right) = O_p\left(\frac{l}{T}\right)$.

(ii) If u_t is $I(1)$, then: (a) $\mathcal{F}_0^*\left(m|0\right) = O_p\left(\frac{T}{l}\right)$, and (b) $\mathcal{F}_1^*\left(m|0\right) \xrightarrow{d} \frac{1}{m} \sup_{\tau^m \in \Lambda_m} J_1\left(\tau^m, 0\right)$.

To establish the asymptotic behavior of the $\mathcal{F}_\lambda^*\left(m|0\right)$ statistic we need to compute the order of probability of the redefined $S_0\left(\widehat{\tau}^m\right)$ and $S_1\left(\widetilde{\tau}^m\right)$ in arbitrarily large samples. Extending in a straightforward way Lemma 2 to Model B we can conclude that the divergence rates are the same as in Model A. This implies that the limit behaviour of the weight function $\lambda\left(S_0\left(\widehat{\tau}^m\right), S_1\left(\widetilde{\tau}^m\right)\right)$ is similar to the cases presented above and so we can finally state the following corollary:

Corollary 6. *Let the conditions of Theorem 5 hold.*

(i) If u_t is $I(0)$, then $F_\lambda^*(m|0) = F_0^*(m|0) + o_p(1) \xrightarrow{d} \frac{1}{m} \sup_{\tau^m \in \Lambda_m} K_0(\tau^m, 0)$.

(ii) If u_t is $I(1)$, then $F_\lambda^*(m|0) = b_\xi^m F_1^*(m|0) + o_p(1) \xrightarrow{d} b_\xi^m \frac{1}{m} \sup_{\tau^m \in \Lambda_m} J_1(\tau^m, 0)$.

As in Model A, the constant b_ξ^m adjusts the critical values of F_λ^* and, hence, delivers a test statistic with asymptotic critical values that are invariant to the order of integration of u_t. Asymptotic results for the double maximum test and the sequential test procedures to estimate the number of trend breaks in Model B can be obtained as straightforward extensions of those obtained for Model A.

Corollary 7. *Let the conditions of Theorem 5 hold.*

(i) If u_t is $I(0)$, then:

$$D\max F_\lambda^* = \max_{1\le m\le M} a_{0,m}\ F_0^*(m|0) + o_p(1) \xrightarrow{d} \max_{1\le m\le M} a_{0,m} \frac{1}{m} \sup_{\tau^m \in \Lambda_m} K_0(\tau^m, 0).$$

(ii) If u_t is $I(1)$, then:

$$D\max F_\lambda^* = b_\xi^M \max_{1\le m\le M} a_{1,m} F_1^*(m|0) + o_p(1) \xrightarrow{d} b_\xi^M \max_{1\le m\le M} a_{1,m} \frac{1}{m} \sup_{\tau^m \in \Lambda_m} J_1(\tau^m, 0).$$

Corollary 8. *Let the time series process $\{y_t\}$ be generated according to (1.22) and (1.23) with $m = l$ breaks and let Assumption 1 hold.*

(i) If u_t is $I(0)$, then $\lambda\left(\widehat{\tau}^{l+1}, \widetilde{\tau}^{l+1}\right) \xrightarrow{p} 1$, $F_\lambda^*(l+1|l) = F_0^*(l+1|l) + o_p(1)$ *and* $\lim_{T\to\infty} P\left(F_\lambda^*(l+1|l) \le x\right) = Q_0(x)^{l+1}$, *where $Q_0(x)$ is the distribution function of* $\sup_{\tau^m \in \Lambda_m} K_0(\tau^m, 0)$ *for $m = 1$.*

(ii) If u_t is $I(1)$, then $\lambda\left(\widehat{\tau}^{l+1}, \widetilde{\tau}^{l+1}\right) \xrightarrow{p} 0$, $F_\lambda^*(l+1|l) = b_\xi^{l+1|l} F_1^*(l+1|l) + o_p(1)$ *and* $\lim_{T\to\infty} P\left(b_\xi^{l+1|l} F_\lambda^*(l+1|l) \le x\right) = G_1(x)^{l+1}$ *where $G_1(x)$ is defined in Theorem 3.*

1.4 Size and Power Simulations

In this section we provide the results of several Monte Carlo simulations. The trimming parameter η was set equal to 0.15. Asymptotic critical values were obtained with discrete approximations (T=1000) of the asymptotic distributions using 5000 simulations and the rndn pseudo random number generator in Gauss. To apply these tests we need to choose constants g and v from the weight function and the bandwidth parameter l from the long run variance estimator. After considering several combinations of the values of g and v constants in the weight function, and truncation lag l in the long run variance estimator we have chosen $g = 500 + 750 \times (m-1)$, $v = 6$, $l = [4(T/100)]^{1/4}$ as these presented the best results in terms of size and power in the range of simulations considered. Hence these are the values which should be chosen in practical applications of these tests. These results apply for both Models A and B. Table 1.1 reports the obtained asymptotic critical values for the class $F_{\lambda}^{*}(m|0)$ statistics for $m = 1, \ldots, 5$ and for the $UD\max F_{\lambda}^{*}$ and $WD\max F_{\lambda}^{*}$ statistics up to a maximum of 3 trend breaks. In Table 1.2 we present critical values for the $F_{\lambda}^{*}(l+1|l)$ statistic for different values of l. Since the values provided are for the unknown break fraction case we also provide the values of b_{ξ}^{m}.To analyze the power and size properties we used 5000 simulations with 150 observations derived from the following DGP based on Model B:

$$y_t = \alpha + \beta t + \sum_{j=1}^{m} \delta_j DU_t\left(\tau_j^*\right) + \sum_{j=1}^{m} \gamma_j DT_t\left(\tau_j^*\right) + u_t \tag{1.27}$$

with the error term given by:

$$(1-\rho L)\, u_t = (1-\theta L)\, \varepsilon_t, t = 2, \ldots, T,\ u_1 = \varepsilon_1,\ \varepsilon_t \backsim NIID\,(0,1) \tag{1.28}$$

We analyzed different levels of persistence on the error term u_t measured by the autoregressive parameter ρ and moving average parameter θ. We use $\rho = 1 - \dfrac{c}{T}$ with $c \in \{0, 10, 20, T\}$ and $\theta \in \{-0.5, 0, 0.5\}$. For the power curves, we generated data from the DGP described by equations (1.27) and (1.28) for a grid of $\gamma_1 = \delta_1/5$ values covering the range $[0, 1]$ with steps of 0.1. Results for the size of the $\mathcal{F}_\lambda^*(m|0)$ and $D\max\mathcal{F}_\lambda^*$ test statistics with the number of breaks under the alternative $m = 1, \ldots, 5$ and upper bound $M = 3$ are presented in Table 1.3 for $T = 150$. In the case of $I(1)$ ($c = 0, \rho = 1$) shocks we see that the $\mathcal{F}_\lambda^*(m|0)$ test is oversized specially when $\theta = -0.5$. Size distortions become specially higher with m if $\theta \in \{-0.5, 0\}$ but in the case of $\theta = 0.5$ the size remains fairly constant regardless of the number of trend breaks set under the alternative hypothesis.

For $\rho \approx 0.93$ (c $= 10$) and $\rho \approx 0.87$ (c $= 20$) the $\mathcal{F}_\lambda^*(m|0)$ test shows reasonable size control for $\theta \in \{-0.5, 0\}$ with a slight size depreciation towards the over-sizing region for $\rho \approx 0.93$ and $\theta = -0.5$.

In the case of $\rho = 0$ (c $=$ T) we observe that for $m = 1$ and $m = 2$ the $\mathcal{F}_\lambda^*(m|0)$ is slightly oversized if $\theta \in \{-0.5, 0\}$ and undersized if $\theta = 0.5$. Since in these cases the size decreases with m we have large degree of under-size with a higher number of trend breaks under H_1.

In general the $UD\max\mathcal{F}_\lambda^*$ and $WD\max\mathcal{F}_\lambda^*$ statistics seem to have similar finite sample size performances for $M = 3$: $D\max\mathcal{F}_\lambda^*$ class is specially under-sized in the case of pure MA shocks with $\theta = 0.5$ and over-sized if the errors follow an I(1) process with $\theta = -0.5$, similarly ro what was observed for the $\mathcal{F}_\lambda^*(m|0)$ statistics. Unreported simulations show that these size distortions become worse with the increase of the number of trend breaks allowed under H_1. However, the $WD\max\mathcal{F}_\lambda^*$ is substantially more sensitive than the $UD\max\mathcal{F}_\lambda^*$ to M.

Consider now Figures 1.1, 1.2 and 1.3 that display the power of the tests for a DGP with 1 change point as a function of the magnitude of the break γ_1 occurring in the middle of the sample, $\tau_1^* = 1/2$, for different values of ρ and θ. The results show that the tests have similar power for the case of $I(1)$ shocks with small differences attributable to unequal finite sample size performances. However, in most cases with $I(0)$ shocks the $\mathcal{F}_\lambda^*(1|0)$ has higher power than all the other tests which is not surprising since our DGP includes only 1 trend break. Also, notice that the power $\mathcal{F}_\lambda^*(m|0)$ definitely decreases as we increase the number of trend breaks set under H_1. This is explained by the fact that, as we increase m, we are allowing for more breaks than necessary to detect the single break in the DGP.

Finally, consider Table 1.4. Here we present the empirical relative frequency at 5% level of the proposed sequential statistics estimating 0,1,2 and more than 2 trend breaks. In our experiment, y_t may have no breaks, 1 trend break and 2 trend breaks with the same magnitude and same sign,$\gamma_1 = \gamma_2$, or with opposite signs, $\gamma_1 = -\gamma_2$. We considered a trend break of magnitudes $\gamma_1 \in \{0.5, 1\}$ occurring in the middle of the sample if there is 1 break, $\tau_1^* = 0.5$, and located at $\tau_1^* = 1/3$ and $\tau_2^* = 2/3$ if there are 2 trend breaks. The usual values of ρ were considered with no moving average effects, $\theta = 0$. All sequential tests have power to efficiently detect the presence one break in trend. For a DGP with 2 trend breaks with same sign and magnitude the tests show similar and reasonable power to detect 2 breaks. This happens specially as we decrease the persistence of the errors, ρ. However, the differences are quite considerable when we look for the 2 opposite breaks case. Here $SeqF_\lambda^*(1|0)$ has very low power to detect breaks and is clearly outperformed by its competitors $SeqF_\lambda^*(2|0)$, $SeqUD\max F_\lambda^*$ and $SeqWD\max F_\lambda^*$. For example, for the highest magnitude considered in the simulations $\gamma_1 = 1, \gamma_2 = -1$ and $I(1)$ shocks the $SeqF_\lambda^*(1|0)$ only estimates 2 breaks with 41% power while the other sequential tests have probability of around 90% to detect 2 change points. Also if u_t is a highly persistent $I(0)$ process ($c = 10, 20$) and for the same magnitudes the $SeqF_\lambda^*(1|0)$ only detects 2 breaks with, at most, 25% probability whereas its competitors display almost full power. On the basis of the results on Table 1.4, we would recommend the use of $SeqF_\lambda^*(2|0)$ when testing the null of no trend break against an unknown number of trend breaks: this sequential test has smaller and only mild size distortions in comparison with the other sequential tests and is able to detect with high power changes in the trend function without suffering the opposite breaks problem. If the empirical researcher is sure about the number of trend breaks under the alternative then it should use the $F_\lambda^*(m|0)$ and specify the number of m trend breaks under H_1. However, it should be cautious if m is quite large (≥ 4) and the number of observations T is small because simulation results show increasing size distortions with m. In that case, we recommend the use of the $D\max F_\lambda^*$ statistics as a pre test to check if there are trend breaks and if the null is rejected use $F_\lambda^*(m|0)$ to estimate the break dates.

1.5 Empirical Application

In this section, we apply our trend break tests to the dataset compiled by Stock and Watson (2008) available on Mark Watson's website. Particularly, we analyze 79 quarterly time series (192 observations) and 108 monthly time series (576 observations) for the United States spanning from 1959 to 2006. A detailed description of each individual variable can be found in that study. The series are only measured in logarithms whenever Stock and Watson (2008) implemented the logarithm transformation in their analysis. Otherwise, we use the original time series. Tables 1.5 to 1.9 present results using Model B for $F_{\lambda}^{*}(m|0)$ for $m = 1, \ldots, 3$, $UD \max F_{\lambda}^{*}$ and $WD \max F_{\lambda}^{*}$ tests and the estimated break dates are provided in square brackets when the null is rejected at 5% significance level. These were obtained as weighted averages of the estimated break dates by F_{0}^{*} and F_{1}^{*}: $\{\lambda(S_0(\widehat{\tau}^m), S_1(\widetilde{\tau}^m)) \times \widehat{\tau}^m\} + \{[1 - \lambda(S_0(\widehat{\tau}^m), S_1(\widetilde{\tau}^m))] \times \widetilde{\tau}^m\}$.

We see that there is evidence for a change in the slope of the trend function at 5% level in more than half of the variables analyzed: at least one of the tests rejects the null of no break in the deterministic trend for 105 or 56% of the series. All tests detect the presence of at least one trend break for 85 from these 105 variables. Hence, all tests seem to be pointing out to the same decision for most of the variables.

If we adopt a more conservative decision rule and increase the significance level to 1% the results are almost unchanged as the null is rejected for 101 or 54% of the series by at least one test and for 64 or 34% of the series, all the tests are unanimous in rejecting the null of no structural change.

The tests referred so far in this section require the specification of the number of breaks (F_{λ}^{*}) or test against an unknown number of breaks but do not specify the break dates ($D \max F_{\lambda}^{*}$). In practice, it is valuable to know not only if a break is present in the data but also when and how often did these changes occurred. Additionally, in some cases there may be some ambiguity on the results, namely, for the same variable some tests result in statistically significant trend breaks while others favor a constant trend function.

For these reasons, it is useful to consider the sequential tests presented in Section 1.2.4 adapted to Model B. The number of estimated break fractions and its respective break dates for each sequential test are presented in Tables 1.10 to 1.14.

We focus on the results of the $SeqF^*_\lambda(2|0)$ procedure as it provided the best overall finite sample results in Section 1.4. An immediate observation that we obtain from the table is that the sequential tests agree on the estimated number of breaks for almost all variables. There are 11 exceptions and 7 out of these 11 series only have a different estimate with the $SeqF^*_\lambda(1|0)$ procedure. A plausible reason for this result is the low power of the $SeqF^*_\lambda(1|0)$ in the presence of multiple breaks in slope with opposite sign. In fact, by simple visual inspection of the plots of the series in Figures 1.4 to 1.15, it seems that the first and last regime estimated under the $SeqF^*_\lambda(2|0)$ procedure have approximately the same slope.

On the other hand, the results naturally confirm the previous finding that a significant portion of the variables analyzed have at least one change in the slope of its trend function: 103 out of 187 series have at least one break in the trend according to the $SeqF^*_\lambda(2|0)$ procedure. Actually, for 52 or 28% of the variables we find statistical evidence for the presence of 2 breaks and for 28 or 15% of the series, the test estimates 3 significant changes in the slope of the trend function.

In the graphs of Figures 1.4 to 1.15 we superimposed the estimated break dates suggested by the sequential procedure $SeqF^*_\lambda(2|0)$ and fitted values of the breaking trend model. We see that the estimated breaks correspond closely to the ones suggested by visual inspection and are mainly dated in the late 1960s, early 1970s, early 1980s and early1990s. Hence, as expected the break dates are focused on periods with important fluctuations in U.S. economic activity like, for example, the collapse of the Bretton Woods system in 1971, the oil price shocks of 1973 and 1979 or the 1980 recession.

1.6 Conclusions

In this paper we presented tests for the presence of multiple structural change in the trend slope of a univariate time series which do not require knowledge of the form of serial correlation in the data and are valid regardless of the shocks being $I(0)$ or $I(1)$. We have considered two Models: a Joint and a Disjoint Broken Trends Model. We have extended the test procedure proposed by Harvey et al. (2009) and constructed a weighted average of two F-statistics, one standardly used when the data is $I(0)$ and the other usually applied for data exhibiting a unit root. We start by considering the case in which the empirical researcher is sure about the break dates if there is any structural change in the trend function. Next, we proposed tests for known number of trend breaks but unknown break dates under the alternative. Here, the break dates estimated are global maximizers of the F statistics over all permissible break fractions. Finally, we analyzed tests for the practitioner who is also not sure about the number of break dates if trend changes have occurred. We analyzed double maximum tests and also 4 sequential procedures that can be used to estimate the number of breaks. We have established the large sample properties of all these tests. Monte Carlo evidence shows that our tests have good size and power properties and recommend the use of a modified sequential approach where the double maximum test or the $F_{\lambda}(2\backslash 0)$ is used to detect breaks of opposite signs. An empirical example illustrated the usefulness of the proposed procedures.

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Appendix

Proof of Theorem 1. (i) (a) From the Frisch-Waugh-Lovell Theorem we can write $m \cdot F_0(\tau^*)$ as:

$$m \cdot F_0(\tau^*) = \left\{ \kappa + T^{3/2} \left[\sum RT_t(\tau^*) RT_t(\tau^*)' \right]^{-1} \sum RT_t(\tau^*) u_t \right\}' \left[\frac{T^{-3} \sum RT_t(\tau^*) RT_t(\tau^*)'}{\widehat{\omega}^2(\tau^*)} \right] \left\{ \kappa + T^{3/2} \left[\sum RT_t(\tau^*) RT_t(\tau^*)' \right]^{-1} \sum RT_t(\tau^*) u_t \right\}$$

with $RT_t(\tau^*) := (RT_t(\tau_1^*), \ldots, RT_t(\tau_m^*))'$ where $RT_t(\tau_j^*)$ is the vector of residuals from the regression of $DT_t(\tau_j^*)$ on $\{1, t\}$. From standard weak convergence results, namely, the Continuous Mapping Theorem (CMT) and the Functional Central Limit Theorem (FCLT), $T^{-\frac{1}{2}} \sum_{t=1}^{\lfloor Tr \rfloor} u_t \xrightarrow{d} \omega_u W(r)$, we can establish that:

$$T^{3/2} \left[\sum RT_t(\tau^*) RT_t(\tau^*)' \right]^{-1} \sum RT_t(\tau^*) u_t \xrightarrow{d} \omega_u \left[\int_0^1 RT(r, \tau^*) RT(r, \tau^*)' \, dr \right]^{-1} \int_0^1 RT(r, \tau^*) \, dW(r) := \omega_u Q_0(\tau^*)^{-1} \int_0^1 V_0(\tau^*, r) \, dr$$

where $W(r)$ is the standard Brownian Motion, $RT(r, \tau)$ is the continuous time residual vector whose j^{th} element is given by the projection of $(r - \tau_j) \mathbb{1}(r > \tau_j)$ onto the space spanned by $\{1, r\}$, $Q_0(\tau^*) := \int_0^1 RT(\tau^*, r) RT(\tau^*, r)' \, dr$ and $\int_0^1 V_0(\tau^*, r) \, dr := \int_0^1 RT(\tau^*, r) \, dW(r)$. It is also well known that the long run variance estimator $\widehat{\omega}^2(\tau^*)$ is consistent, $\widehat{\omega}^2(\tau^*) \xrightarrow{p} \omega_u^2$. With these results, the asymptotic distribution of $m \cdot F_0(\tau^*)$ can be written as :

$$m \cdot F_0(\tau^*) \xrightarrow{d} \left\{ \kappa + \omega_u Q_0(\tau^*)^{-1} \int_0^1 V_0(\tau^*, r) \, dr \right\}' \left[Q_0(\tau^*) / \omega_u^2 \right] \left\{ \kappa + \omega_u Q_0(\tau^*)^{-1} \int_0^1 V_0(\tau^*, r) \, dr \right\}.$$

It is straightforward to show that this corresponds to a non-central chi-square distribution with m degrees of freedom and non centrality parameter $\kappa' \left[Q_0 \left(\tau^* \right) / \omega_u^2 \right] \kappa$.

(b) Again appealing to the Frisch-Waugh-Lovell Theorem it is possible to rewrite $\left(\frac{T}{l} \right) F_1 \left(\tau^* \right)$ as:

$$\left(\frac{T}{l} \right) F_1 \left(\tau^* \right) = \frac{1}{m} \left\{ \kappa T^{-\frac{1}{2}} + \left[T^{-1} \sum RU_t \left(\tau^* \right) RU_t \left(\tau^* \right)' \right]^{-1} \sum RU_t \left(\tau^* \right) \triangle u_t \right\}'$$

$$\left[\frac{T^{-1} \sum RU_t \left(\tau^* \right) RU_t \left(\tau^* \right)'}{l \widetilde{\omega}^2 \left(\tau^* \right)} \right] \left\{ \kappa T^{-\frac{1}{2}} + \left[T^{-1} \sum RU_t \left(\tau^* \right) RU_t \left(\tau^* \right)' \right]^{-1} \sum RU_t \left(\tau^* \right) \triangle u_t \right\} \tag{1.29}$$

with $RU_t \left(\tau^* \right) = \left(RU_t \left(\tau_1^* \right), \ldots, RU_t \left(\tau_m^* \right) \right)'$ where $RU_t \left(\tau_j^* \right)$ is the vector of residuals from the regression of $DU_t \left(\tau_j^* \right)$ on $\{1\}$. Now notice that $RU_t \left(\tau_j^* \right)$ can be simplified to:

$$RU_t(\tau_j^*) = \begin{cases} \tau_j^* - 1 & , \text{if } t \leq T_j^* \\ \tau_j^* & , \text{if } t > T_j^* \end{cases}$$

and so we get that $\sum RU_t \left(\tau_j^* \right) \triangle u_t = \tau_j^* u_T - u_{T_j^*-} \left(1 - \tau_j^* \right) u_1 = O_p \left(1 \right)$ since $u_t \backsim I \left(0 \right)$. Also, Leybourne et al. (2007) proved that $l \widetilde{\omega}^2 \left(\tau^* \right)$ has a finite and positive probability limit provided that $l = o \left(T^{1/2} \right)$ and Assumption 1 from their paper holds. Hence, we get $l \widetilde{\omega}^2 \left(\tau^* \right) \xrightarrow{p} -2 \sum_{s=0}^{\infty} s \gamma_s = O_p \left(1 \right)$, where $\gamma_s = E \left[\triangle u_t \triangle u_{t-s} \right]$. Finally, it is straightforward to see that

$$T^{-1} \sum RU_t \left(\tau^* \right) RU_t \left(\tau^* \right)' \longrightarrow \int_0^1 RU \left(\tau^*, r \right) RU \left(\tau^*, r \right)' dr = O \left(1 \right).$$

So, since all terms from the right hand side of (1.29) are $O_p \left(1 \right)$ we proved that $\left(\frac{T}{l} \right) F_1 \left(\tau^* \right) = O_p \left(1 \right)$.

(ii) (a) We have that $\left(\frac{l}{T}\right) F_0(\tau^*)$ equals:

$$\begin{aligned}\left(\frac{l}{T}\right) F_0(\tau^*) = & \frac{1}{m}\left\{\kappa + T^{-3}\left[\sum RT_t(\tau^*)\, RT_t(\tau^*)'\right]^{-1} T^{-5/2}\sum RT_t(\tau^*)\, u_t\right\}' \\ & \left[\frac{T^{-3}\sum RT_t(\tau^*)\, RT_t(\tau^*)'}{(lT)^{-1}\widehat{\omega}^2(\tau^*)}\right] \\ & \left\{\kappa + T^{-3}\left[\sum RT_t(\tau^*)\, RT_t(\tau^*)'\right]^{-1} T^{-5/2}\sum RT_t(\tau^*)\, u_t\right\}\end{aligned}$$

Using standard weak convergence results we can prove that:

$$T^{-5/2}\sum RT_t(\tau^*)\, u_t \xrightarrow{d} \omega_u \int_0^1 RT(r,\tau^*)\, dW(r) = O_p(1)$$

and

$$T^{-3}\sum RT_t(\tau^*)\, RT_t(\tau^*)' \longrightarrow \int_0^1 RT(r,\tau^*)\, RT(r,\tau^*)'\, dr = O(1).$$

Extending appropriately formula (23) from Kwiatkowski et al. (1992) to Model A, it is possible to show that $(lT)^{-1}\widehat{\omega}^2(\tau^*) \xrightarrow{p} \omega_u^2 \int_0^1 H(r,\tau^*)^2\, dr$ where $H(r,\tau^*)$ is the continuous time residual from the projection of $W(r)$ onto the space spanned by $\{1, r, (r-\tau_1^*)\,\mathbb{1}(r>\tau_1^*), \ldots, (r-\tau_m^*)\,\mathbb{1}(r>\tau_m^*)\}$. Since all terms have non-degenerate distributions we can say that $\left(\frac{l}{T}\right) F_0(\tau^*) = O_p(1)$.

(b) Following the same lines from the proofs of the previous results we can rewrite $m \cdot F_1(\tau^*)$ as:

$$\begin{aligned}m \cdot F_1(\tau^*) = & \left\{\kappa + \left[T^{-1}\sum RU_t(\tau^*)\, RU_t(\tau^*)'\right]^{-1} T^{-1/2}\sum RU_t(\tau^*)\, \varepsilon_t\right\}' \\ & \left[\frac{T^{-1}\sum RU_t(\tau^*)\, RU_t(\tau^*)'}{\widetilde{\omega}^2(\tau^*)}\right] \\ & \left\{\kappa + \left[T^{-1}\sum RU_t(\tau^*)\, RU_t(\tau^*)'\right]^{-1} T^{-1/2}\sum RU_t(\tau^*)\, \varepsilon_t\right\}\end{aligned}$$

From

$$\left[T^{-1}\sum RU_t(\tau^*)\,RU_t(\tau^*)'\right]^{-1}T^{-1/2}\sum RU_t(\tau^*)\,\varepsilon_t \xrightarrow{d}$$
$$\left[\int_0^1 RU(\tau^*,r)\,RU_t(\tau^*,r)'\,dr\right]^{-1}\omega_\varepsilon\int_0^1 RU(\tau^*,r)\,dW(r) := \omega_\varepsilon Q_1(\tau^*)^{-1}\int_0^1 V_1(\tau^*,r)\,dr$$

where $Q_1(\tau^*) := \int_0^1 RU(\tau^*,r)\,RU(\tau^*,r)'\,dr$ and $\int_0^1 V_1(\tau^*,r)\,dr := \int_0^1 RU(\tau^*,r)\,dW(r)$ and using the fact that $\widetilde{\omega}^2(\tau^*) \xrightarrow{p} \omega_\varepsilon^2$, we can establish the asymptotic distribution of $m\cdot F_1(\tau^*)$:

$$m\cdot F_1(\tau^*) \xrightarrow{d} \left\{\kappa+\omega_\varepsilon Q_1(\tau^*)^{-1}\int_0^1 V_1(\tau^*,r)\,dr\right\}'\left[Q_1(\tau^*)^{-1}/\omega_\varepsilon^2\right]$$
$$\left\{\kappa+\omega_\varepsilon Q_1(\tau^*)^{-1}\int_0^1 V_1(\tau^*,r)\,dr\right\}$$

As before, it is straightforward to show that this corresponds to a non-central chi-square distribution with m degrees of freedom and non centrality parameter $\kappa'\left[Q_1(\tau^*)\,\omega_\varepsilon^2\right]\kappa$.

□

Proof of Lemma 1. Results (i)(a) and (i)(b) follow from Kwiatkowski et al. (1992). If the error term $u_t \backsim I(0)$ then $S_0(\tau^*) = O_p(1)$ and converges to a function of the Wiener process. Similarly, if $u_t \backsim I(1)$, then when we differentiate the model the disturbances become stationary and so $S_1(\tau^*) = O_p(1)$.Result (ii)(a) follows from $(lT)^{-1}\widehat{\omega}^2(\tau^*) \xrightarrow{p} \omega_u^2\int_0^1 H(r,\tau^*)^2\,dr$, an extension of expression (22) from Kwiatkowski et al. (1992) to Model A, and

$$T^{-4}\sum_{t=1}^{T}\left(\sum_{i=1}^{t}\widehat{u}_i(\tau^*)\right)^2 \xrightarrow{P} \omega_u^2\int_0^1\left(\int_0^a H(r,\tau^*)\,dr\right)^2 da.$$

Hence, we have that under $u_t \backsim I(1)\,,\ S_0(\tau^*) = O_p(T/l)$. Finally, from results from Leybourne et al. (2007) we get that if $u_t \backsim I(0)$, then $l\widetilde{\omega}^2(\tau^*) \xrightarrow{p} -2\sum_{s=0}^{\infty} s\gamma_s = O_p(1)$ and $T^{-1}\sum_{t=2}^{T}\left(\sum_{i=1}^{t}\widetilde{v}_i(\tau^*)\right)^2 = O_p(1)$ which establishes the result $\left(\frac{T}{l}\right)S_1(\tau^*) = O_p(1)$.

$\square$

Proof of Theorem 2. We start by proving (i)(a) and (ii)(b). From Theorem 1 it is immediate that, for any finite number L of values of the $m-$dimensional vector of break fractions τ^m, $\left(\digamma_0\left(\tau^{m,1}\right), \ldots, \digamma_0\left(\tau^{m,L}\right)\right)'$ and $\left(\digamma_1\left(\tau^{m,1}\right), \ldots, \digamma_1\left(\tau^{m,L}\right)\right)'$ weakly converge to $\left(J_0\left(\tau^{m,1}, 0\right), \ldots, J_0\left(\tau^{m,L}, 0\right)\right)'$ and $\left(J_1\left(\tau^{m,1}, 0\right), \ldots, J_1\left(\tau^{m,L}, 0\right)\right)'$, respectively, and so we establish the finite dimensional convergence of these test statistics. Also from the proof of Theorem 1 we observe that $\digamma_0\left(.\right)$ is a functional of $\left(T^{-3}\sum RT_t\left(.\right)RT_t\left(.\right)', T^{-3/2}\sum RT_t\left(.\right)u_t, \widehat{\omega}^2\left(.\right)\right)$ $\digamma_1\left(.\right)$ is a functional of the process $\left(T^{-1}\sum RU_t\left(.\right)RU_t\left(.\right)', T^{-1/2}\sum RU_t\left(.\right)\varepsilon_t, \widetilde{\omega}^2\left(.\right)\right)$. Using similar arguments from Zivot and Andrews (1992) we can show the joint weak convergence of these processes:

$$\left(T^{-3}\sum RT_t\left(.\right)RT_t\left(.\right)', T^{-3/2}\sum RT_t\left(.\right)u_t, \widehat{\omega}^2\left(.\right)\right)' \Rightarrow$$
$$\Rightarrow \left(\int_0^1 RT\left(.,r\right)RT\left(.,r\right)'dr, \int_0^1 RT\left(.,r\right)dW\left(r\right), \omega_u^2\left(.\right)\right)',$$

$$\left(T^{-1}\sum RU_t\left(.\right)RU_t\left(.\right)', T^{-1/2}\sum RU_t\left(.\right)\varepsilon_t, \widetilde{\omega}^2\left(.\right)\right)' \Rightarrow$$
$$\Rightarrow \left(\int_0^1 RU\left(.,r\right)RU\left(.,r\right)'dr, \int_0^1 RU\left(.,r\right)dW\left(r\right), \omega_\varepsilon^2\left(.\right)\right)'$$

The sup function is continuous a.s. with respect to

$$\left(\int_0^1 RT\left(.,r\right)RT\left(.,r\right)'dr, \int_0^1 RT\left(.,r\right)dW\left(r\right), \omega_u^2\left(.\right)\right)'$$

and

$$\left(\int_0^1 RU\left(.,r\right)RU\left(.,r\right)'dr, \int_0^1 RU\left(.,r\right)dW\left(r\right), \omega_\varepsilon^2\left(.\right)\right)',$$

which implies that $\digamma_0^*(m|0) \Rightarrow \dfrac{1}{m}\sup_{\tau^m\in\Lambda_m} J_0\left(\tau^m, 0\right)$ and $\digamma_1^*(m|0) \Rightarrow \dfrac{1}{m}\sup_{\tau^m\in\Lambda_m} J_1\left(\tau^m, 0\right)$ by the CMT, following the same lines from the proof of the Theorem from Zivot and Andrews (1992). From Theorem 1(i)(b) and the fact that $l\widetilde{\omega}^2\left(\tau^m\right) \xrightarrow{p} -2\sum_{s=0}^{\infty} s\gamma_s$ uniformly in τ^m it follows that if $u_t \backsim I\left(0\right)$, then $\digamma_1\left(\tau^m\right) = O_p\left(\dfrac{l}{T}\right)$ uniformly in τ^m and so the result in (i)(b) is proved. Finally, from Theorem 1(ii)(a) and the fact that $(lT)^{-1}\widehat{\omega}^2(\tau^m) \xrightarrow{d} \omega_u^2 \int_0^1 H(r,\tau^m)^2 dr$ uniformly in τ^m we can show that if $u_t \backsim I(1)$, then $\digamma_0(\tau^m) = O_p\left(\frac{T}{l}\right)$ uniformly in τ^m and so the result in (ii)(a) is proved. $\square$

Proof of Theorem 3. Throughout the proof we employ the following additional notation: Let $RSSR_0\left(\tau_{i-1}^*, \tau_i^*\right)$ and $USSR_0\left(\tau_{i-1}^*, \zeta, \tau_i^*\right)$ denote, respectively, the restricted and unrestricted sum of squared residuals for testing $H_0 : \gamma = 0$ in the model:

$$y_t = \alpha + \beta\left(t - T_{i-1}^*\right) + \gamma DT_t(\zeta) + u_t, t = T_{i-1}^* + 1, \ldots, T_i^*$$

with $T_i^* := \lfloor \tau_i^* T \rfloor$.

Similarly, denote by $RSSR_1\left(\tau_{i-1}^*, \tau_i^*\right)$ and $USSR_1\left(\tau_{i-1}^*, \zeta, \tau_i^*\right)$, respectively, the restricted and unrestricted sum of squared residuals for testing $H_0 : \gamma = 0$ in the model:

$$y_t = \beta + \gamma DU_t(\zeta) + v_t, t = T_{i-1}^* + 1, \ldots, T_i^*$$

(i) (a) Notice that:

$$\digamma_0\left(\tau_1^*, \ldots, \tau_{i-1}^*, \zeta, \tau_{i+1}^*, \ldots, \tau_l^*\right) = \frac{RSSR_0\left(\tau_{i-1}^*, \tau_i^*\right) - USSR_0\left(\tau_{i-1}^*, \zeta, \tau_i^*\right)}{\widehat{\omega}^2\left(\tau_1^*, \ldots, \tau_{i-1}^*, \zeta, \tau_{i+1}^*, \ldots, \tau_l^*\right)} + o_p(1)$$

Since,under H_0 there are l trend breaks occurring at dates $(T_1^*, \ldots, T_l^*)$, it is well known that $\widehat{\omega}^2\left(\tau_1^*, \ldots, \tau_{i-1}^*, \zeta, \tau_{i+1}^*, \ldots, \tau_l^*\right) \xrightarrow{p} \omega_u^2$. Moreover, similar arguments from the proof of Theorem 2 can be used to prove that, under H_0, for each $i = 1, \ldots, l+1$:

$$\sup_{\zeta \epsilon \Lambda_{0,i}^*} \frac{RSSR_0\left(\tau_{i-1}^*, \tau_i^*\right) - USSR_0\left(\tau_{i-1}^*, \zeta, \tau_i^*\right)}{\widehat{\omega}^2\left(\tau_1^*, \ldots, \tau_{i-1}^*, \zeta, \tau_{i+1}^*, \ldots, \tau_l^*\right)} \xrightarrow{d} \sup_{\zeta \epsilon \Lambda_{0,i}^*} J_0\left(\frac{\zeta - \tau_{i-1}^*}{\tau_i^* - \tau_{i-1}^*}, 0\right) \quad (1.30)$$

where $\Lambda_{0,i}^*$ is as defined in (1.19)with $\widehat{\tau}_{i-1}$ and $\widehat{\tau}_i$ replaced by τ_{i-1}^* and τ_i^*, respectively. Since $(\widehat{\tau}_1, \ldots, \widehat{\tau}_l)$ have been obtained by the $\digamma_0^*(l|0)$ statistic we get that, under H_0, $\widehat{\tau}_i - \tau_i^* = O_p\left(T^{-\frac{3}{2}}\right)$ from Theorem 3 of Perron and Zhu (2005). This implies that $\widehat{T}_i = T_i^* + O_p\left(T^{-\frac{1}{2}}\right)$ and so it is possible to show that (1.30) holds with $\left(\tau_1^*, \ldots, \tau_{i-1}^*, \tau_i^*, \ldots, \tau_l^*\right)$ replaced by $(\widehat{\tau}_1, \ldots, \widehat{\tau}_{i-1}, \widehat{\tau}_i, \ldots, \widehat{\tau}_l)$. Now notice that $RSSR_0\,(.,.)$ and $USSR_0\,(.,.)$ are computed on different and non overlapping regimes which implies independence of the weak limits in (1.30). Since we are taking the maximum over $l+1$ independent random variables we get that:

$$\lim_{T\to\infty} P\left(\digamma_0^*\,(l+1|l) \leq x\right) = G_0\,(x)^{l+1}$$

where $G_0\,(x)$ is the distribution function of $\sup\limits_{\tau^1\in\Lambda_1} J_0\left(\tau^1, 0\right)$ where we employed the change in variable $\tau^1 = \left(\zeta - \tau_{i-1}^*\right) / \left(\tau_i^* - \tau_{i-1}^*\right)$.

(b) Notice that:

$$\sup_{\zeta\epsilon\Lambda_{1,i}^*} \digamma_1\left(\tau_1^*, \ldots, \tau_{i-1}^*, \zeta, \tau_{i+1}^*, \ldots, \tau_l^*\right) = \sup_{\zeta\epsilon\Lambda_{1,i}^*} \frac{RSSR_1\left(\tau_{i-1}^*, \tau_i^*\right) - USSR_1\left(\tau_{i-1}^*, \zeta, \tau_i^*\right)}{\widetilde{\omega}^2\left(\tau_1^*, \ldots, \tau_{i-1}^*, \zeta, \tau_{i+1}^*, \ldots, \tau_l^*\right)} \tag{1.31}$$

where $\Lambda_{1,i}^*$ is as defined in (1.20) with $\widetilde{\tau}_{i-1}$ and $\widetilde{\tau}_i$ replaced by τ_{i-1}^* and τ_i^*, respectively. Similar arguments from the proof of Theorem 2 allow us to establish that, under the null, $l\widetilde{\omega}^2\left(\tau_1^*, \ldots, \tau_{i-1}^*, \zeta, \tau_{i+1}^*, \ldots, \tau_l^*\right) = O_p\,(1)$ and, furthermore, $T\left(RSSR_1\left(\tau_{i-1}^*, \tau_i^*\right) - USSR_1\left(\tau_{i-1}^*, \zeta, \tau_i^*\right)\right) = O_p\,(1)$ uniformly in ζ, given that $u_t \backsim I\,(0)$ and the fact that there are no breaks between observations $T_{i-1}^* + 1$ and T_i^*. Hence, the F-statistic in (1.31) is $O_p\left(\frac{l}{T}\right)$ uniformly. Since $\widetilde{\tau}_i - \tau_i^*$ is $O_p\left(T^{-\frac{1}{2}}\right)$ by the proof of Theorem 3 from HLT, this is enough to establish that, for each $i = 1, \ldots, l+1$:

$$\sup_{\zeta\epsilon\Lambda_{1,i}} \frac{RSSR_1\left(\widetilde{\tau}_{i-1}, \widetilde{\tau}_i\right) - USSR_1\left(\widetilde{\tau}_{i-1}, \zeta, \widetilde{\tau}_i\right)}{\widetilde{\omega}^2\left(\widetilde{\tau}_1, \ldots, \widetilde{\tau}_{i-1}, \zeta, \widetilde{\tau}_{i+1}, \ldots, \widetilde{\tau}_l\right)} = O_p\left(\frac{l}{T}\right)$$

uniformly in ζ. Since, asymptotically, we are taking the maximum over $l+1$ i.i.d random variables that are $O_p\left(\frac{l}{T}\right)$ we obtain the desired result.

(ii) (a) Similar arguments from the proof of Theorem 2 can be employed to show that, under the null of no trend breaks, if $u_t \backsim I\,(1)$, then the left hand side of (1.30) is $O_p\left(\frac{T}{l}\right)$ uniformly over all ζ. Since $\widehat{\tau}_i - \tau_i^* = O_p\left(T^{-\frac{1}{2}}\right)$ from Theorem 3 of

Perron and Zhu (2005) the rate of convergence remains the same when we replace $\left(\tau_1^*, \ldots, \tau_{i-1}^*, \tau_i^*, \ldots, \tau_l^*\right)$ by $(\widehat{\tau}_1, \ldots, \widehat{\tau}_{i-1}, \widehat{\tau}_i, \ldots, \widehat{\tau}_l)$. Hence, we have that, for each $i = 1, \ldots, l+1$:

$$\sup_{\zeta \epsilon \Lambda_{0,i}} \frac{RSSR_0\left(\widehat{\tau}_{i-1}, \widehat{\tau}_i\right) - USSR_0\left(\widehat{\tau}_{i-1}, \zeta, \widehat{\tau}_i\right)}{\widehat{\omega}^2\left(\widehat{\tau}_1, \ldots, \widehat{\tau}_{i-1}, \widehat{\tau}_i, \ldots, \widehat{\tau}_l\right)} = O_p\left(\frac{T}{l}\right)$$

Since, asymptotically, we are taking the maximum over $l+1$ i.i.d random variables that are $O_p\left(\frac{T}{l}\right)$ we obtain the desired result.

(b) Using the fact, under H_0, $\widehat{T}_i = T_i^* + O_p(1)$ from Bai and Perron (1998), the same arguments from (i)(a) can be used to show that:

$$\lim_{T\to\infty} P\left(F_1^*\left(l+1|l\right) \leq x\right) = G_1(x)^{l+1}$$

where $G_1(x)^{l+1}$ is the distribution function of $\sup_{\tau^1 \in \Lambda^1} J_1\left(\tau^1, 0\right)$.

$\square$

Tab. 1.1: Asymptotic critical values and b_ξ^m values for the multiple trend breaks F_λ^* and Double Maximum $UD\max F_\lambda^*$ and $WD\max F_\lambda^*$ tests.

Model A

	$F_\lambda^*(m\|0)$													
	$m=1$		$m=2$		$m=3$		$m=4$		$m=5$		$UD\max F_\lambda^*(M=3)$		$WD\max F_\lambda^*(M=3)$	
ξ	c.v.	b_ξ^1	c.v	b_ξ^2	c.v	b_ξ^3	c.v	b_ξ^4	c.v	b_ξ^5	c.v	b_ξ^3	c.v	b_ξ^3
0.10	4.87	0.69	3.97	0.63	3.36	0.64	2.81	0.64	2.31	0.67	5.18	0.69	5.87	0.72
0.05	6.18	0.72	4.84	0.67	3.98	0.67	3.31	0.66	2.71	0.69	6.39	0.72	7.22	0.73
0.01	9.44	0.77	6.71	0.72	5.50	0.72	4.45	0.72	3.60	0.73	9.52	0.77	10.87	0.79

Model B

	$F_\lambda^*(m\|0)$													
	$m=1$		$m=2$		$m=3$		$m=4$		$m=5$		$UD\max F_\lambda^*(M=3)$		$WD\max F_\lambda^*(M=3)$	
ξ	c.v.	b_ξ^1	c.v	b_ξ^2	c.v	b_ξ^3	c.v	b_ξ^4	c.v	b_ξ^5	c.v	b_ξ^3	c.v	b_ξ^3
0.10	7.68	1.09	7.39	1.18	6.40	1.23	5.52	1.25	4.14	1.19	8.46	1.13	8.73	1.11
0.05	9.41	1.10	8.44	1.17	7.17	1.20	6.06	1.22	4.57	1.17	9.86	1.11	10.54	1.11
0.01	12.91	1.05	10.66	1.14	8.81	1.16	7.29	1.18	5.56	1.13	13.02	1.05	14.4	1.07

Tab. 1.2: Asymptotic critical values and b_ξ^l values of the sequential trend breaks test $F_\lambda^*(l+1|l)$

Model A

	$F_\lambda(l\|l+1)$									
	$l=0$		$l=1$		$l=2$		$l=3$		$l=4$	
ξ	Critical value	$b_\xi^{1\|0}$	Critical value	$b_\xi^{2\|1}$	Critical value	$b_\xi^{3\|2}$	Critical value	$b_\xi^{4\|3}$	Critical value	$b_\xi^{5\|4}$
0.10	4.87	0.69	6.15	0.72	6.79	0.72	7.41	0.74	7.86	0.74
0.05	6.18	0.72	7.48	0.74	8.19	0.74	8.83	0.75	9.41	0.77
0.01	9.44	0.77	10.99	0.79	12.07	0.82	12.86	0.84	13.73	0.87

Model B

	$F_\lambda(l+1\|l)$									
	$l=0$		$l=1$		$l=2$		$l=3$		$l=4$	
ξ	Critical value	$b_\xi^{1\|0}$	Critical value	$b_\xi^{2\|1}$	Critical value	$b_\xi^{3\|2}$	Critical value	$b_\xi^{4\|3}$	Critical value	$b_\xi^{5\|4}$
0.10	7.68	1.09	9.33	1.10	10.38	1.10	10.92	1.09	11.27	1.07
0.05	9.41	1.10	10.97	1.08	11.80	1.06	12.42	1.05	12.90	1.05
0.01	12.91	1.05	15.16	1.09	16.16	1.09	16.89	1.11	17.28	1.10

Tab. 1.3: Empirical size of $F^*_\lambda(m|0)$ and $D\max F^*_\lambda$ tests, 5% nominal level, T = 150.

ρ	θ	$F^*_\lambda(m\|0)$					$UD\max F^*_\lambda$	$WD\max F^*_\lambda$
		$m=1$	$m=2$	$m=3$	$m=4$	$m=5$		
0	-0.5	0.069	0.062	0.041	0.010	0.003	0.020	0.023
	0.0	0.068	0.067	0.041	0.012	0.002	0.019	0.023
	0.5	0.015	0.005	0.001	0.000	0.000	0.001	0.001
0.87	-0.5	0.052	0.045	0.044	0.045	0.035	0.034	0.038
	0.0	0.065	0.056	0.058	0.049	0.034	0.044	0.048
	0.5	0.145	0.130	0.117	0.068	0.028	0.090	0.100
0.93	-0.5	0.040	0.050	0.071	0.097	0.097	0.040	0.050
	0.0	0.046	0.052	0.068	0.089	0.078	0.042	0.050
	0.5	0.105	0.102	0.098	0.082	0.045	0.082	0.088
1	-0.5	0.133	0.187	0.253	0.297	0.312	0.173	0.185
	0.0	0.114	0.156	0.210	0.250	0.258	0.143	0.149
	0.5	0.109	0.108	0.113	0.114	0.086	0.105	0.105

Tab. 1.4: Size and Power of Sequential Tests, Model B, T=150

			$SeqF^*_\lambda(1\|0)$				$SeqF^*_\lambda(2\|0)$				$SeqUD\max F^*_\lambda$				$SeqWD\max F^*_\lambda$			
γ_1	γ_2	ρ	0 br	1 br	2 br	>2br	0 br	1 br	2 br	>2br	0 br	1 br	2 br	>2br	0 br	1 br	2 br	>2br
0	0	0	0.93	0.07	0.00	0.00	0.88	0.07	0.05	0.00	0.92	0.07	0.01	0.00	0.91	0.07	0.02	0.00
0.5	0		0.00	0.98	0.02	0.00	0.00	0.98	0.02	0.00	0.00	0.98	0.02	0.00	0.00	0.98	0.02	0.00
1	0		0.00	0.97	0.03	0.00	0.00	0.97	0.03	0.00	0.00	0.97	0.03	0.00	0.00	0.97	0.03	0.00
0.5	0.5		0.00	0.01	0.98	0.01	0.00	0.01	0.98	0.01	0.00	0.01	0.98	0.01	0.00	0.01	0.98	0.01
1	1		0.00	0.00	0.99	0.01	0.00	0.00	0.99	0.01	0.00	0.00	0.99	0.01	0.00	0.00	0.99	0.01
0.5	-0.5		1.00	0.00	0.00	0.00	0.00	0.00	0.99	0.01	0.00	0.00	0.99	0.01	0.00	0.00	0.99	0.01
1	-1		0.31	0.00	0.67	0.01	0.00	0.00	0.98	0.02	0.00	0.00	0.98	0.02	0.00	0.00	0.98	0.02
0	0	0.87	0.94	0.06	0.00	0.00	0.89	0.06	0.04	0.00	0.90	0.06	0.03	0.00	0.89	0.06	0.04	0.00
0.5			0.02	0.95	0.03	0.00	0.02	0.95	0.03	0.00	0.01	0.95	0.03	0.00	0.02	0.95	0.03	0.00
1	0		0.00	0.97	0.03	0.00	0.00	0.97	0.03	0.00	0.00	0.97	0.03	0.00	0.00	0.97	0.03	0.00
0.5	0.5		0.00	0.79	0.20	0.01	0.00	0.79	0.20	0.01	0.00	0.79	0.20	0.01	0.00	0.79	0.20	0.01
1	1		0.00	0.00	0.98	0.02	0.00	0.00	0.98	0.02	0.00	0.00	0.98	0.02	0.00	0.00	0.98	0.02
0.5	-0.5		0.99	0.00	0.00	0.00	0.26	0.00	0.71	0.02	0.52	0.00	0.46	0.02	0.40	0.00	0.57	0.02
1	-1		0.78	0.00	0.21	0.01	0.00	0.00	0.96	0.04	0.00	0.00	0.96	0.04	0.00	0.00	0.96	0.04
0	0	0.93	0.95	0.04	0.00	0.00	0.91	0.04	0.04	0.00	0.92	0.04	0.03	0.01	0.91	0.04	0.04	0.01
0.5	0		0.09	0.88	0.03	0.00	0.07	0.88	0.05	0.00	0.08	0.88	0.04	0.00	0.07	0.88	0.05	0.00
1	0		0.00	0.96	0.03	0.00	0.00	0.96	0.03	0.00	0.00	0.96	0.03	0.00	0.00	0.96	0.03	0.00
0.5	0.5		0.00	0.87	0.12	0.01	0.00	0.87	0.12	0.01	0.00	0.87	0.12	0.01	0.00	0.87	0.12	0.01
1	1		0.00	0.02	0.94	0.04	0.00	0.02	0.94	0.04	0.00	0.02	0.94	0.04	0.00	0.02	0.94	0.04
0.5	-0.5		0.97	0.01	0.02	0.00	0.33	0.01	0.63	0.03	0.53	0.01	0.43	0.03	0.41	0.01	0.55	0.03
1	-1		0.74	0.00	0.25	0.01	0.00	0.00	0.95	0.05	0.00	0.00	0.95	0.05	0.00	0.00	0.95	0.05
0	0	1	0.89	0.10	0.01	0.00	0.79	0.10	0.10	0.00	0.82	0.10	0.07	0.01	0.79	0.10	0.10	0.01
0.5	0		0.23	0.71	0.06	0.00	0.17	0.71	0.12	0.01	0.19	0.71	0.10	0.01	0.17	0.71	0.11	0.01
1	0		0.00	0.91	0.09	0.01	0.00	0.91	0.09	0.01	0.00	0.91	0.09	0.01	0.00	0.91	0.09	0.01
0.5	0.5		0.01	0.85	0.13	0.01	0.00	0.85	0.13	0.01	0.00	0.85	0.13	0.01	0.00	0.85	0.13	0.01
1	1		0.00	0.08	0.85	0.07	0.00	0.08	0.85	0.07	0.00	0.08	0.85	0.07	0.00	0.08	0.85	0.07
0.5	-0.5		0.78	0.11	0.10	0.01	0.30	0.11	0.56	0.03	0.42	0.11	0.43	0.03	0.33	0.11	0.52	0.04
1	-1		0.54	0.01	0.41	0.04	0.00	0.01	0.91	0.08	0.01	0.01	0.90	0.07	0.00	0.01	0.91	0.08

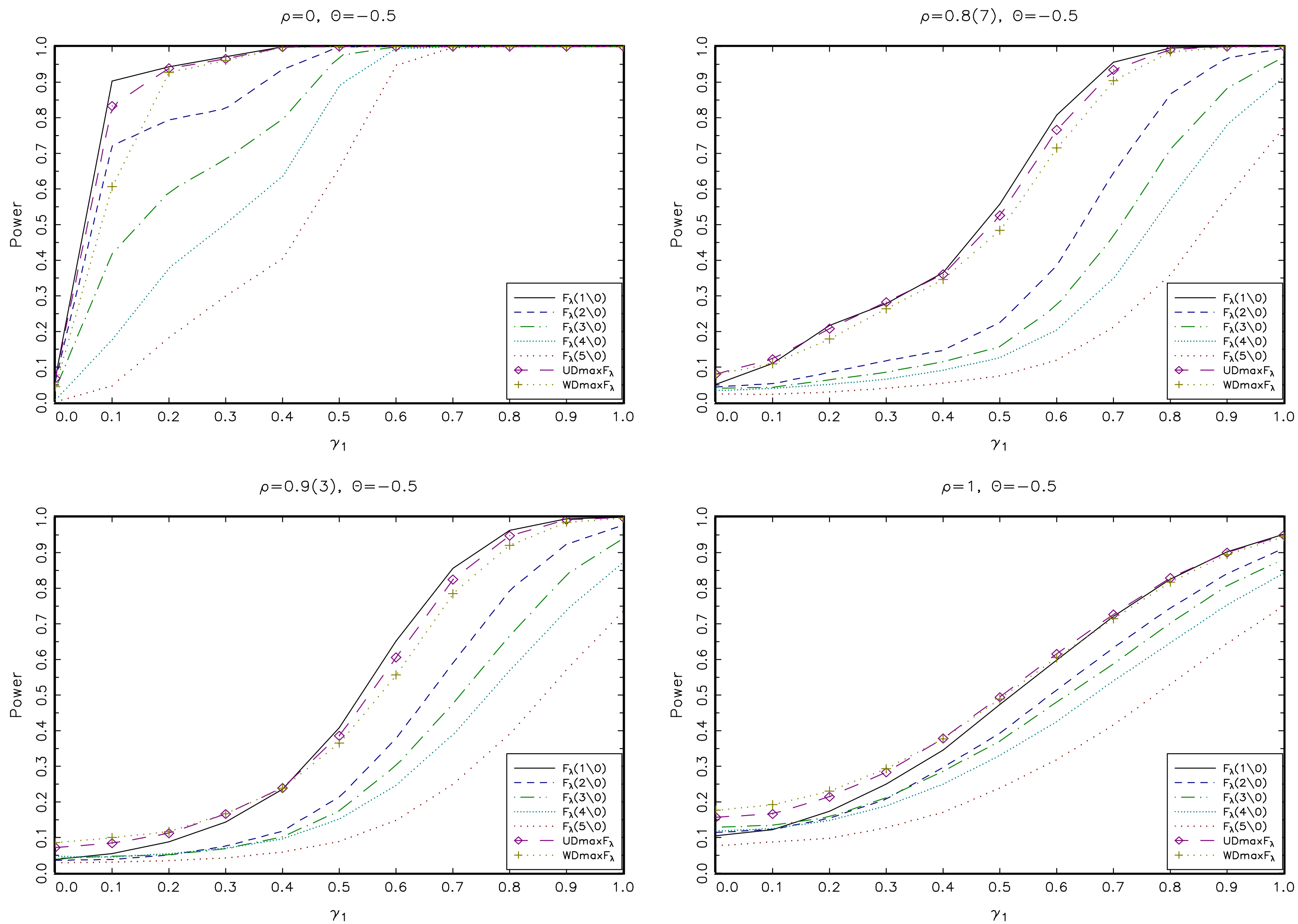

Fig. 1.1: Power (1 break) of F^*_λ and $D\max F^*_\lambda$ tests, Model B, T=150

$$y_t = \delta_1 DU_t(\tau_1^*) + \gamma_1 DT_t(\tau_1^*) + u_t, (1-\rho L)\, u_t = (1-\theta L)\, \varepsilon_t,$$
$$\delta_1 = 5\gamma_1, \tau_1^* = 0.5.$$

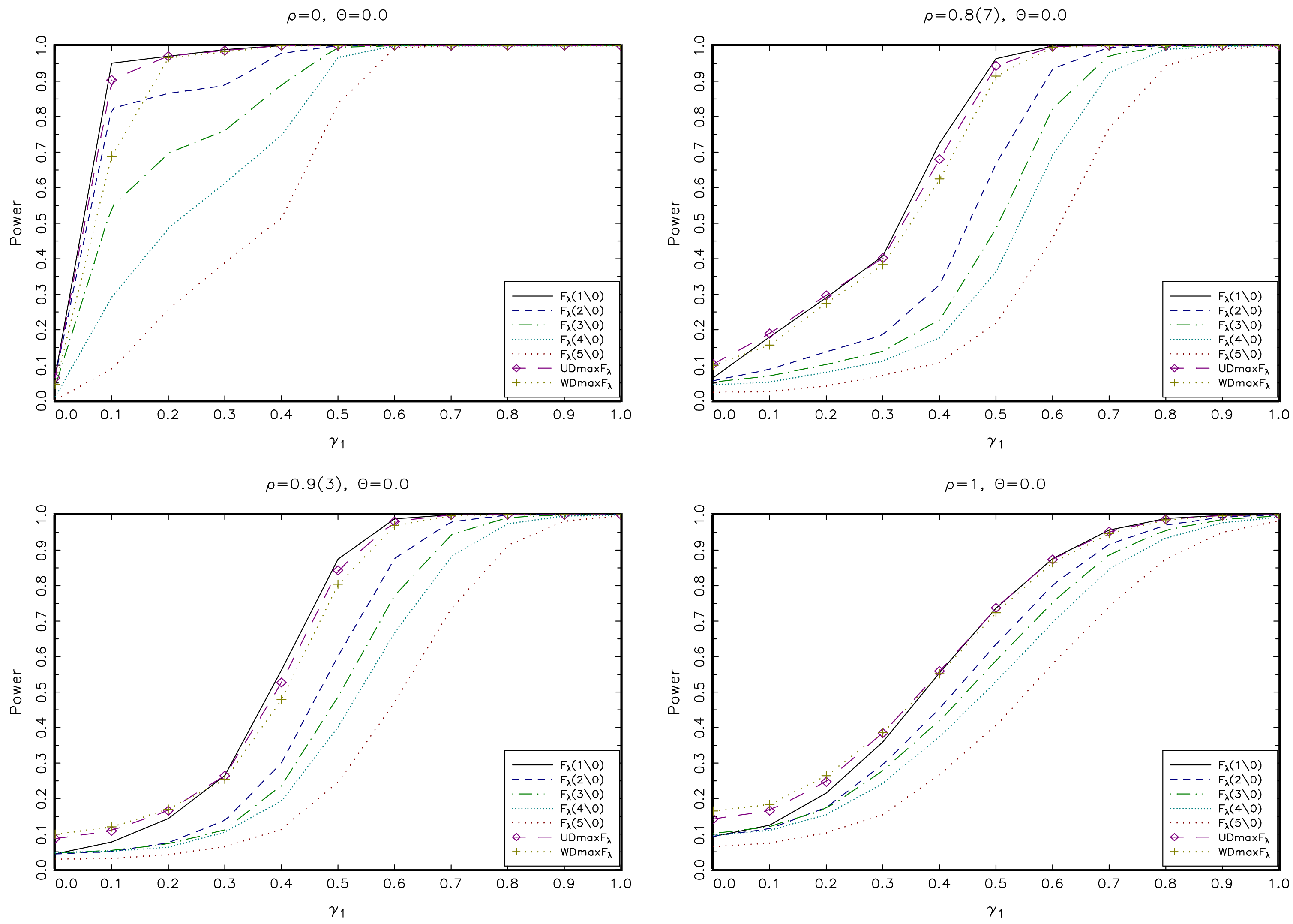

Fig. 1.2: Power (1 break) of $F_\lambda^*(m|0)$ and $D\max F_\lambda^*$ tests, Model B, T=150

$$y_t = \delta_1 DU_t(\tau_1^*) + \gamma_1 DT_t(\tau_1^*) + u_t, (1-\rho L) u_t = (1-\theta L)\varepsilon_t,$$
$$\delta_1 = 5\gamma_1, \tau_1^* = 0.5$$

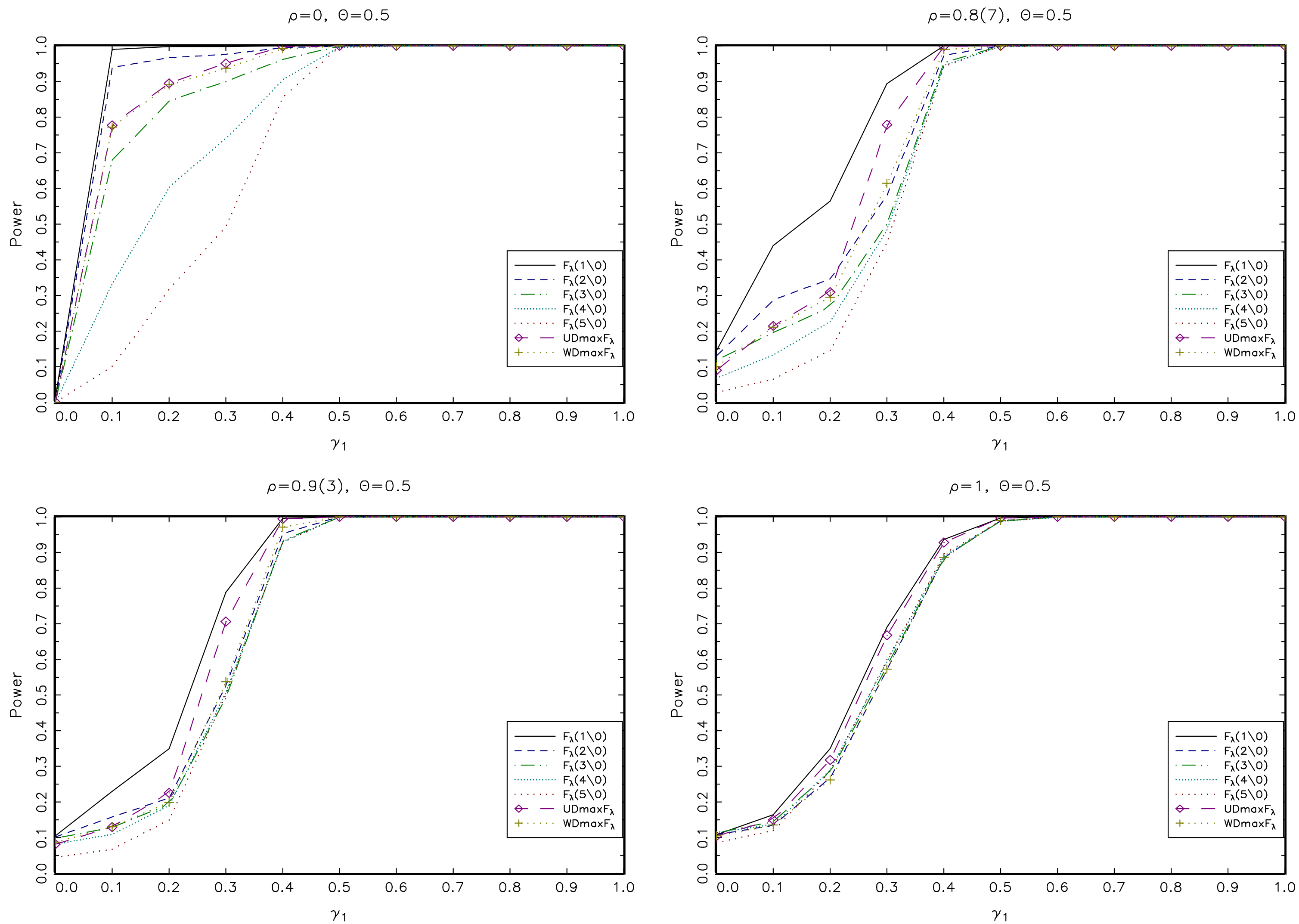

Fig. 1.3: Power (1 break) of $F_{\lambda}^{*}(m|0)$ and $D\max F_{\lambda}^{*}$ tests, Model B, T=150

$$y_t = \delta_1 DU_t(\tau_1^*) + \gamma_1 DT_t(\tau_1^*) + u_t, (1-\rho L)\, u_t = (1-\theta L)\, \varepsilon_t,$$
$$\delta_1 = 5\gamma_1, \tau_1^* = 0.5$$

Tab. 1.5: Empirical Application of F^*_λ and $D\max F^*_\lambda$ tests

Variables\Test	$F^*_\lambda(m\|0)$ $m=1$	$F^*_\lambda(m\|0)$ $m=2$	$F^*_\lambda(m\|0)$ $m=3$	$UD\max F^*_\lambda$	$WD\max F^*_\lambda$
Real Gross Domestic Product	66.71*** (1969Q4)	4.17	3.98	4.61	5.07
Real Personal Consumption Expenditures	5.69	6.17	5.35	5.86	6.81
Real Personal Consumption Expenditures - Durable Goods	1.43	4.07	3.32	3.87	4.40
Real Personal Consumption Expenditures - NonDur	2.89	4.32	4.07	4.11	5.18
Real Personal Consumption Expenditures - Services	26.92*** (1972Q4)	17.68*** (1973Q1,1988Q1)	16.16*** (1972Q4,1981Q2,1988Q3)	27.25***	26.10***
Real Gross Private Domestic Investment	0.74	1.78	11.22*** (1983Q4,1991Q1,1999Q3)	11.20**	14.71***
Real Gross Private Domestic Investment - FixedInv	1.13	2.56	3.04	2.81	3.88
Real Gross Private Domestic Investment - NonRes	2.46	3.24	3.81	3.52	4.85
Real Gross Private Domestic Investment - NonRes - struct	4.78	3.94	3.52	4.84	4.64
Real Gross Private Domestic Investment - NonRes - Equip	2.08	4.07	4.10	3.87	5.22
Real Gross Private Domestic Investment - Residential	1.46	2.94	2.50	2.79	3.18
Real Exports	1.99	2.80	3.51	3.24	4.47
Real Imports	2.07	3.70	3.25	3.51	4.14
Real Government Consumption Expenditures+Gross Investment	9.66** (1967Q4)	11.08*** (1967Q4,1976Q3)	9.51*** (1967Q4,1976Q3,1986Q1)	10.52**	12.10**
Real Government Consumption Expenditures+Gross Investment-Fed.	2.91	9.58** (1967Q3,1975Q1)	9.69*** (1967Q3,1974Q4,1987Q3)	9.10*	12.33**
Real Government Consumption Expenditures+Gross Investment-State/Loc.	19.40*** (1968Q2)	14.35*** (1975Q3,1983Q1)	15.01*** (1968Q2,1975Q3,1983Q1)	19.64***	19.10***
Real Final Sales of Domestic Product	71.57*** (1969Q4)	5.03	4.77	6.11	6.07
Real Gross Domestic Purchases	3.07	3.57	3.10	3.39	3.94
Real Final Sales to Domestic Purchasers	3.96	4.31	3.86	4.09	4.92
Real Gross National Product	71.46*** (1969Q4)	4.15	3.86	4.72	4.91
Gross Domestic Product	13.94*** (1983Q4)	68.56*** (1967Q3,1982Q2)	88.65*** (1966Q1,1973Q2,1982Q1)	81.82***	112.83**
Personal Consumption Expenditures	11.48** (1990Q3)	64.04*** (1972Q3,1981Q3)	61.24*** (1972Q3,1981Q3,1991Q3)	60.83***	77.95***
Personal Consumption Expenditures - Durable Goods	32.26*** (1994Q3)	47.97*** (1973Q2,1981Q4)	90.30*** (1973Q1,1981Q3,1994Q4)	83.34***	114.93**
Personal Consumption Expenditures - Nondurable Goods	9.81** (1981Q2)	39.42*** (1972Q2,1980Q4)	30.15*** (1972Q2,1980Q4,1990Q2)	37.44***	42.62***
Personal Consumption Expenditures - Services	12.45** (1967Q2)	68.05*** (1972Q4,1983Q2)	84.82*** (1966Q1,1973Q2,1983Q2)	78.28***	107.95**
Gross Private Domestic Investment	18.39*** (1982Q1)	67.89*** (1972Q3,1981Q3)	75.96*** (1966Q1,1973Q2,1981Q3)	70.11***	96.68***
Gross Private Domestic Investment - Fixed Investment	18.46*** (1982Q1)	68.86*** (1972Q3,1981Q3)	80.53*** (1966Q1,1973Q2,1981Q3)	74.33***	102.49**
Gross Private Domestic Investment - NonRes.	23.29*** (1982Q1)	65.11*** (1972Q4,1981Q4)	66.65*** (1966Q1,1973Q2,1981Q4)	61.85***	84.83***
Gross Private Domestic Investment - NonRes - struct	8.66* (1967Q2)	25.95*** (1968Q1,1981Q4)	29.98*** (1968Q1,1981Q4,1999Q2)	27.67***	38.15***
Gross Private Domestic Investment - NonRes. - Equip.	25.75*** (1982Q2)	60.61*** (1973Q2,1981Q4)	57.38*** (1973Q2,1981Q3,1990Q3)	57.57***	73.03***
Gross Private Domestic Investment - Res.	12.93*** (1966Q2)	48.55*** (1970Q2,1981Q1)	44.52*** (1966Q2,1973Q3,1981Q1)	46.11***	56.66***
Exports	14.94*** (1980Q4)	30.43*** (1972Q1,1980Q4)	22.20*** (1972Q1,1980Q4,1999Q2)	28.90***	32.90***
Imports	10.62** (1980Q4)	23.22*** (1971Q3,1980Q4)	17.26*** (1971Q3,1980Q4,1998Q4)	22.05***	25.10***

Tab. 1.6: Empirical Application of F^*_λ and $D\max F^*_\lambda$ tests (continued)

Variables\Test	$F^*_\lambda(m\|0)$			$UD\max F^*_\lambda$	$WD\max F^*_\lambda$
	$m=1$	$m=2$	$m=3$		
Government Consumption Expenditures & Gross Investment	17.24*** (1983Q4)	72.49*** (1967Q4,1981Q4)	55.96*** (1967Q4,1982Q1,1998Q4)	68.85***	78.38***
Government Consumption Expenditures & Gross Investment - Federal	17.63*** (1983Q4)	68.31*** (1967Q4,1981Q4)	50.63*** (1967Q4,1981Q4,1999Q2)	64.89***	73.86***
Government Consumption Expenditures & Gross Investment - State/local	16.56*** (1982Q2)	52.28*** (1966Q1,1982Q1)	42.72*** (1966Q1,1973Q2,1981Q4)	49.66***	56.53***
Final Sales of Domestic Product	13.92*** (1983Q4)	69.12*** (1967Q3,1982Q2)	90.41*** (1966Q1,1973Q2,1982Q1)	83.44***	115.06**
Gross Domestic Purchases	13.53*** (1982Q2)	64.42*** (1972Q2,1981Q3)	76.44*** (1966Q1,1973Q2,1981Q3)	70.55***	97.29***
Final Sales to Domestic Purchasers	13.57*** (1982Q2)	64.91*** (1972Q2,1981Q4)	77.79*** (1966Q1,1973Q2,1981Q4)	71.79***	99.00***
Gross National Product	13.98*** (1983Q4)	68.66*** (1967Q3,1982Q2)	88.80*** (1966Q1,1973Q2,1982Q1)	81.96***	113.02**
Output per hour all persons: Business Sector	7.09	7.81* (1972Q4,1995Q2)	6.67* (1972Q4,1982Q4,1995Q1)	7.42	8.49
Real Compensation per hour,Employees:Nonfarm Business	10.35** (1972Q4)	13.16*** (1972Q4,1996Q4)	10.73*** (1972Q4,1985Q4,1996Q4)	12.50**	14.22**
Hours of all persons: NonFarm Business Sector	18.17*** (1998Q1)	2.64	2.46	3.16	3.13
Unit Labor Cost: Non farm Business Sector	11.28** (1982Q2)	39.60*** (1972Q3,1981Q4)	38.50*** (1966Q1,1973Q2,1981Q4)	37.62***	49.00***
Gross domestic product Price Index	13.63*** (1983Q4)	67.66*** (1967Q3,1982Q2)	87.20*** (1966Q1,1973Q2,1981Q4)	80.49***	110.99**
Personal consumption expenditures Price Index	11.38** (1990Q3)	63.98*** (1972Q3,1981Q3)	60.94*** (1972Q3,1981Q3,1991Q3)	60.77***	77.55***
Durable goods Price Index	32.49*** (1994Q3)	47.90*** (1973Q2,1981Q4)	90.80*** (1973Q1,1981Q3,1994Q4)	83.80***	115.56**
Motor vehicles and parts Price Index	12.13** (1995Q3)	38.43*** (1973Q3,1981Q3)	46.78*** (1973Q2,1981Q3,1995Q3)	43.18***	59.54***
Furniture and household equipment Price Index	68.04*** (1994Q1)	62.18*** (1982Q3,1994Q3)	138.05** (1973Q1,1982Q1,1994Q3)	127.42**	175.70**
Other Price Index	21.76*** (1991Q2)	27.75*** (1973Q2,1980Q4)	37.16*** (1973Q2,1980Q4,1991Q2)	34.30***	47.30***
Nondurable goods Price Index	9.78** (1981Q2)	39.31*** (1972Q2,1980Q4)	30.05*** (1972Q2,1980Q4,1990Q2)	37.34***	42.51***
Food Price Index	9.62** (1981Q2)	43.63*** (1972Q1,1980Q4)	33.53*** (1972Q1,1980Q4,1990Q3)	41.44***	47.18***
Clothing and shoes Price Index	60.84*** (1991Q2)	44.79*** (1966Q1,1991Q2)	57.13*** (1966Q1,1977Q2,1991Q2)	61.59***	72.71***
Gasoline, fuel oil, and other energy goods Price Index	2.39	6.80	8.20** (1972Q3,1980Q4,1998Q4)	7.57	10.44*
Other Price Index	11.48** (1992Q1)	37.81*** (1973Q1,1982Q4)	43.59*** (1973Q1,1982Q4,1992Q1)	40.24***	55.48***
Services Price Index	12.56** (1967Q2)	68.03*** (1972Q4,1983Q2)	84.97*** (1966Q1,1973Q2,1982Q4)	78.42***	108.14**
Housing Price Index	20.69*** (1968Q2)	66.58*** (1974Q1,1986Q1)	82.34*** (1967Q1,1974Q2,1986Q1)	76.00***	104.80**
Household operation Price Index	13.20*** (1984Q2)	48.30*** (1969Q4,1983Q4)	38.79*** (1966Q1,1973Q2,1982Q4)	45.88***	52.23***
Electricity and gas Price Index	6.30	28.93*** (1972Q4,1982Q3)	27.03*** (1972Q4,1982Q4,1999Q2)	27.48***	34.40***
Other household operation Price Index	22.09*** (1985Q4)	40.21*** (1967Q2,1985Q2)	30.69*** (1967Q2,1978Q3,1985Q4)	38.20***	43.48***
Transportation Price Index	8.16* (1984Q1)	23.31*** (1973Q3,1980Q4)	21.74*** (1966Q1,1973Q3,1980Q4)	22.14***	27.67***
Medical care Price Index	19.76*** (1992Q3)	40.23*** (1973Q2,1983Q4)	48.66*** (1973Q2,1982Q4,1992Q4)	44.91***	61.93***
Recreation Price Index	15.12*** (1991Q1)	25.24*** (1966Q4,1991Q1)	23.08*** (1973Q1,1981Q3,1991Q1)	23.97***	29.37***
Other Price Index	10.26** (1986Q2)	33.88*** (1972Q3,1981Q2)	30.24*** (1972Q4,1981Q2,1991Q3)	32.18***	38.48***
Gross private domestic investment Price Index	17.67*** (1982Q1)	64.24*** (1972Q3,1981Q3)	70.44*** (1966Q1,1973Q2,1981Q3)	65.02***	89.66***
Fixed investment Price Index	17.81*** (1982Q1)	65.40*** (1972Q3,1981Q3)	74.84*** (1966Q1,1973Q2,1981Q3)	69.08***	95.25***
Nonresidential Price Index	22.92*** (1982Q1)	63.78*** (1972Q4,1981Q4)	64.95*** (1966Q1,1973Q2,1981Q4)	60.59***	82.66***
Structures	8.60* (1967Q2)	25.19*** (1968Q1,1981Q4)	29.39*** (1968Q1,1981Q4,1999Q2)	27.12***	37.40***
Equipment and software Price Index	25.45*** (1982Q2)	60.10*** (1973Q2,1981Q4)	56.32*** (1973Q2,1981Q3,1990Q3)	57.09***	71.69***
Residential Price Index	13.05*** (1966Q2)	45.06*** (1970Q2,1981Q1)	40.59*** (1966Q2,1973Q3,1981Q1)	42.80***	51.65***

Tab. 1.7: Empirical Application of F_λ^* and $D\max F_\lambda^*$ tests (continued)

Variables\Test	$F_\lambda^*(m\|0)$ $m=1$	$F_\lambda^*(m\|0)$ $m=2$	$F_\lambda^*(m\|0)$ $m=3$	$UD\max F_\lambda^*$	$WD\max F_\lambda^*$
Exports Price Index	14.93*** (1980Q4)	30.42*** (1972Q1,1980Q4)	22.20*** (1972Q1,1980Q4,1999Q2)	28.89***	32.89***
Goods Price Index	13.98*** (1980Q4)	25.83*** (1972Q1,1980Q4)	19.21*** (1972Q1,1980Q4,1999Q2)	24.54***	27.93***
Services Price Index	14.33*** (1982Q4)	43.92*** (1970Q2,1982Q2)	35.56*** (1970Q2,1982Q1,1990Q2)	41.72***	47.49***
Imports Price Index	10.71** (1980Q4)	23.32*** (1971Q3,1980Q4)	17.27*** (1971Q3,1980Q4,1998Q4)	22.15***	25.21***
Goods Price Index	10.14** (1980Q4)	21.66*** (1971Q2,1980Q4)	16.07*** (1971Q2,1980Q4,1998Q4)	20.58***	23.42***
Services Price Index	7.98* (1980Q2)	24.82*** (1970Q2,1980Q2)	18.18*** (1970Q2,1980Q2,1990Q2)	23.57***	26.83***
Government consumption expenditures and gross investment Price Index	16.59*** (1983Q4)	69.63*** (1967Q4,1981Q4)	54.86*** (1967Q4,1982Q1,1999Q1)	66.14***	75.28***
Federal Price Index	17.21*** (1983Q4)	66.92*** (1967Q4,1981Q4)	50.27*** (1967Q4,1981Q4,1999Q2)	63.57***	72.36***
State and local Price Index	15.82*** (1982Q2)	49.81*** (1966Q1,1982Q1)	40.56*** (1966Q1,1973Q2,1981Q4)	47.32***	53.86***
Industrial Production Index - Total Index	5.12	4.73	4.12	5.19	5.24
Industrial Production Index - Products, Total	6.58	4.25	3.83	6.66	6.38
Industrial Production Index - Final Products	7.62	5.13	4.34	7.72	7.39
Industrial Production Index - Consumer Goods	10.69** (1973M1)	7.90* (1973M1,1982M11)	6.05	10.83**	10.37*
Industrial Production Index - Durable Consumer Goods	2.96	4.41	3.79	4.19	4.82
Industrial Production Index - Nondurable Consumer Goods	19.41*** (1973M1)	11.37*** (1967M3,1978M2)	8.59** (1967M3,1978M2,1985M10)	19.65***	18.82***
Industrial Production Index - Business Equipment	2.17	2.25	2.51	2.32	3.20
Industrial Production Index - Materials	3.78	4.43	3.74	4.20	4.78
Industrial Production Index - Durable Goods Materials	2.98	4.55	3.78	4.32	4.92
Industrial Production Index - Nondurable Goods Materials	14.34*** (1973M12)	8.54** (1973M12,1982M11)	6.67* (1973M12,1982M11,1999M9)	14.52***	13.91**
Industrial Production Index - Manufacturing	3.85	4.02	3.66	3.89	4.66
Industrial Production Index - Residential Utilities	2232.62* (1973M9)	111.47** (1973M9,1994M4)	878.90** (1973M9,1983M3,1988M5)	2592.92*	2592.90*
Industrial Production Index - Fuels	5.59	5.78	4.57	5.69	6.28
Napm Production Index (Percent)	4.73	1.79	1.59	1.57	2.02
Capacity Utilization - Manufacturing	1.81	3.07	2.53	2.91	3.32
Avg Hrly Earnings, Prod Wrkrs, Nonfarm - Goods-Producing	78.08*** (1982M6)	205.14** (1967M4,1982M4)	173.79** (1967M1,1974M3,1982M1)	194.86**	221.80**
Avg Hrly Earnings, Prod Wrkrs, Nonfarm - Construction	96.10*** (1981M11)	86.16*** (1967M1,1981M11)	63.16*** (1967M1,1975M2,1982M6)	97.29***	93.19***
Avg Hrly Earnings, Prod Wrkrs, Nonfarm - Mfg	60.42*** (1982M5)	134.08** (1967M8,1982M1)	128.00** (1967M1,1974M3,1981M12)	127.36**	162.91**
Real Avg Hrly Earnings, Prod Wrkrs, Nonfarm - Goods-Producing	41.22*** (1972M12)	30.37*** (1972M12,1993M4)	27.58*** (1972M12,1981M11,1992M11)	41.74***	39.98***
Real Avg Hrly Earnings, Prod Wrkrs, Nonfarm - Construction	82.53*** (1972M12)	60.76*** (1972M12,1993M8)	47.63*** (1972M12,1981M11,1993M8)	83.56***	80.04***
Real Avg Hrly Earnings, Prod Wrkrs, Nonfarm - Mfg	24.75*** (1978M11)	19.58*** (1978M11,1992M10)	14.50*** (1978M11,1986M1,1993M4)	25.06***	24.00***
Employees, Nonfarm - Total Private	5.73	3.84	3.84	5.80	5.56
Employees, Nonfarm - Goods-Producing	5.03	3.36	3.70	5.09	4.88
Employees, Nonfarm - Mining	6.76	12.73*** (1971M9,1981M10)	13.81*** (1971M9,1981M10,1989M6)	12.75**	17.58***
Employees, Nonfarm - Construction	3.68	3.43	3.35	3.73	4.27
Employees, Nonfarm - Mfg	8.94* (1979M5)	7.37	5.51	9.05*	8.67
Employees, Nonfarm - Durable Goods	7.07	5.10	4.44	7.16	6.85
Employees, Nonfarm - Nondurable Goods	22.47*** (1998M1)	16.39*** (1969M7,1998M2)	11.95*** (1966M12,1974M11,1998M2)	22.75***	21.79***
Employees, Nonfarm - Service-Providing	20.22*** (1990M1)	16.50*** (1979M5,1999M9)	12.18*** (1979M5,1992M7,1999M9)	20.47***	19.61***
Employees, Nonfarm - Trade, Transport, Utilities	15.41*** (1989M4)	10.50** (1989M2,1999M9)	8.15** (1971M8,1979M2,1999M9)	15.61***	14.95***

Tab. 1.8: Empirical Application of F_λ^* and $D\max F_\lambda^*$ tests (continued)

Variables\Test	$F_\lambda^*(m\|0)$			$UD\max F_\lambda^*$	$WD\max F_\lambda^*$
	$m=1$	$m=2$	$m=3$		
Employees, Nonfarm - Wholesale Trade	16.30*** (1980M1)	11.51*** (1980M1,1999M9)	9.77*** (1971M4,1979M5,1999M9)	16.50***	15.81***
Employees, Nonfarm - Retail Trade	28.21*** (1989M4)	17.35*** (1989M4,1999M8)	12.88*** (1979M2,1989M4,1999M8)	28.56***	27.36***
Employees, Nonfarm - Financial Activities	45.38*** (1987M6)	29.01*** (1987M8,1995M6)	23.31*** (1966M8,1987M6,1995M6)	45.94***	44.01***
Employees, Nonfarm - Government	67.82*** (1975M3)	42.31*** (1968M6,1978M5)	32.54*** (1968M6,1978M3,1985M5)	68.66***	65.77***
Index Of Help-Wanted Advertising In Newspapers	5.46	4.02	4.01	5.53	5.30
Employment: Ratio; Help-Wanted Ads:No. Unemployed Clf	5.99	5.75	4.57	6.06	6.22
Civilian Labor Force: Employed, Total (Thous.)	5.72	5.96	4.78	5.79	6.44
Civilian Labor Force: Employed, Nonagric.Industries (Thous.)	7.95* (1979M1)	6.23	4.97	8.05	7.71
Unemployment Rate: All Workers, 16 Years & Over (%)	3.90	5.40	5.53	5.13	7.04
Unemploy.By Duration: Average(Mean)Duration In Weeks	3.08	3.88	4.25	3.92	5.41
Unemploy.By Duration: Persons Unempl.Less Than 5 Wks (Thous.)	3.96	4.54	3.91	4.31	4.97
Unemploy.By Duration: Persons Unempl.5 To 14 Wks (Thous.)	2.45	5.16	3.97	4.90	5.58
Unemploy.By Duration: Persons Unempl.15 Wks + (Thous.)	4.09	6.67	5.39	6.33	7.21
Unemploy.By Duration: Persons Unempl.15 To 26 Wks (Thous.)	2.92	4.87	4.19	4.63	5.34
Unemploy.By Duration: Persons Unempl.27 Wks (Thous)	5.07	7.31	5.64	6.94	7.90
Avg Wkly Hours, Prod Wrkrs, Nonfarm - Goods-Producing	1.07	2.57	2.25	2.44	2.87
Avg Wkly Overtime Hours, Prod Wrkrs, Nonfarm	1.90	2.60	3.20	2.96	4.08
Housing Authorized: Total New Priv Housing Units (Thous.)	1.73	2.40	2.24	2.28	2.85
Housing Starts:Nonfarm(1947-58)	1.60	2.09	1.91	1.99	2.44
Housing Starts:Northeast (Thous.U.)	1.37	2.05	2.16	2.00	2.75
Housing Starts:Midwest(Thous.U.)	1.68	2.76	2.66	2.62	3.38
Housing Starts:South (Thous.U.)	1.01	2.66	2.65	2.52	3.37
Housing Starts:West (Thous.U.)	2.32	2.70	2.43	2.56	3.10
Interest Rate: Federal Funds (Effective) (% Per Annum)	4.14	4.07	4.04	4.19	5.14
Interest Rate: U.S.Treasury Bills,Sec Mkt,3-Mo.(% Per Annum)	3.99	4.15	4.10	4.04	5.22
Interest Rate: U.S.Treasury Bills,Sec Mkt,6-Mo.(% Per Annum)	4.03	4.15	4.06	4.08	5.17
Interest Rate: U.S.Treasury Const Maturities,1-Yr.(% Per Annum)	4.59	4.64	4.40	4.64	5.60
Interest Rate: U.S.Treasury Const Maturities,5-Yr.(% Per Annum)	6.85	5.58	5.11	6.93	6.64
Interest Rate: U.S.Treasury Const Maturities,10-Yr.(% Per Annum)	8.66* (1981M8)	6.81	6.06	8.77*	8.40
Bond Yield: Moody'S Aaa Corporate (% Per Annum)	12.92*** (1981M8)	9.38** (1973M3,1981M8)	7.96** (1973M3,1981M8,1993M9)	13.08***	12.53**
Bond Yield: Moody'S Baa Corporate (% Per Annum)	13.57*** (1981M12)	10.03** (1981M12,1993M9)	8.97*** (1973M3,1981M12,1993M9)	13.74***	13.16**
Fygm6-Fygm3	1.14	2.20	2.19	2.09	2.79
Fygt1-Fygm3	1.57	2.34	2.49	2.32	3.18
Fygt10-Fygm3	0.82	1.74	1.79	1.66	2.28
Fyaaac-Fygt10	1.79	1.59	5.58	5.47	7.27
Fybaac-Fygt10	1.62	1.68	1.64	1.64	2.09
Money Stock: M1	28.05*** (1994M1)	28.75*** (1966M12,1994M1)	22.32*** (1966M12,1980M2,1993M10)	28.40***	31.09***

Tab. 1.9: Empirical Application of F_{λ}^{*} and $D\max F_{\lambda}^{*}$ tests (continued)

Variables\Test	$F_{\lambda}^{*}(m\|0)$			$UD\max F_{\lambda}^{*}$	$WD\max F_{\lambda}^{*}$
	$m=1$	$m=2$	$m=3$		
Mzm Frb St. Louis	5.68	9.50** $(1980M2,1987M4)$	8.14** $(1980M2,1987M4,1995M3)$	9.03*	10.36*
Money Stock:M2	37.62*** $(1986M11)$	33.84*** $(1970M5,1986M11)$	33.68*** $(1970M5,1986M11,1995M3)$	38.08***	42.87***
Monetary Base, Adj For Reserve Requirement Changes	34.00*** $(1967M5)$	28.08*** $(1967M5,1994M10)$	21.27*** $(1966M11,1975M2,1994M6)$	34.43***	32.98***
Depository Inst Reserves:Total	18.64*** $(1993M11)$	16.45*** $(1982M5,1993M11)$	12.01*** $(1966M10,1982M5,1993M11)$	18.87***	18.07***
Depository Inst Reserves:Nonborrowed	9.22* $(1993M11)$	13.64*** $(1984M6,1993M11)$	9.96*** $(1974M8,1984M6,1993M11)$	12.96**	14.75***
Commercial And Industrial Loans At All Commercial Banks	19.69*** $(1985M1)$	13.26*** $(1986M11,1994M1)$	9.91*** $(1975M1,1986M11,1994M1)$	19.94***	19.10***
Consumer Credit Outstanding - Nonrevolving	9.19* $(1986M8)$	17.82*** $(1986M8,1993M10)$	14.46*** $(1976M11,1986M8,1993M10)$	16.92***	19.26***
Personal Consumption Expenditures, Price Index	22.16*** $(1990M11)$	106.56** $(1972M11,1981M10)$	99.01*** $(1966M2,1973M4,1981M10)$	101.21**	126.01**
Personal Consumption Expenditures - Durable Goods, Price Index	56.02*** $(1995M2)$	74.30*** $(1973M10,1981M12)$	107.79** $(1973M10,1981M10,1995M2)$	99.49***	137.19**
Personal Consumption Expenditures - Nondurable Goods, Price Index	16.13*** $(1981M2)$	54.52*** $(1972M10,1981M2)$	40.73*** $(1972M9,1981M2,1990M7)$	51.79***	58.95***
Personal Consumption Expenditures - Services, Price Index	25.45*** $(1967M7)$	120.39** $(1973M3,1982M12)$	137.55** $(1966M2,1973M5,1982M12)$	126.95**	175.06**
CPI All Items	16.95*** $(1967M4)$	93.09*** $(1972M11,1981M8)$	82.07*** $(1966M2,1973M4,1981M8)$	88.42***	104.46**
CPI Less Food And Energy	19.71*** $(1966M2)$	85.03*** $(1973M6,1982M6)$	90.27*** $(1966M2,1973M10,1982M5)$	83.31***	114.88**
PCE Price Index Less Food And Energy	31.48*** $(1992M2)$	98.53*** $(1973M4,1982M12)$	115.23** $(1973M5,1982M9,1992M2)$	106.35**	146.65**
Producer Price Index: Finished Goods	13.54*** $(1981M11)$	67.60*** $(1972M9,1981M5)$	48.08*** $(1972M9,1981M3,1990M7)$	64.21***	73.09***
Producer Price Index:Finished Consumer Goods	9.88** $(1981M4)$	49.94*** $(1972M9,1981M3)$	36.45*** $(1972M9,1981M3,1999M1)$	47.44***	54.00***
Producer Price Index:Intermed Mat.Supplies & Components	9.07* $(1981M6)$	34.83*** $(1972M6,1981M3)$	27.09*** $(1972M6,1981M3,1999M1)$	33.08***	37.66***
Producer Price Index:Crude Materials	3.89	4.87	5.85	5.40	7.45
Real Producer Price Index:Crude Materials	6.34	5.36	4.80	6.41	6.14
Spot Market Price Index:Bls & Crb: All Commodities	2.38	6.16	5.88	5.85	7.49
Real Spot Market Price Index:Bls & Crb: All Commodities	3.26	5.01	5.10	4.76	6.50
Producer Price Index: Crude Petroleum	4.94	6.39	6.08	6.07	7.74
PPI Crude	6.06	6.48	5.58	6.16	7.10
NAPM Commodity Prices Index (Percent)	0.58	1.59	1.14	1.05	1.45
Effective Exchange Rate: United States	4.18	6.66	7.06* $(1977M11,1985M1,1992M7)$	6.51	8.98*
Foreign Exchange Rate: Switzerland	2.02	5.76	5.18	5.48	6.59
Foreign Exchange Rate: Japan	4.21	6.66	4.97	6.33	7.20
Foreign Exchange Rate: United Kingdom	5.93	5.47	5.35	6.00	6.80
Foreign Exchange Rate: Canada	9.98** $(1998M10)$	8.89** $(1991M8,1998M10)$	8.00** $(1984M5,1991M8,1998M10)$	10.11**	10.18*
S&P'S Common Stock Price Index: Composite	3.86	5.40	4.23	5.13	5.83
S&P'S Common Stock Price Index: Industrials	3.39	5.63	4.39	5.35	6.09
S&P'S Composite Common Stock: Dividend Yield (% Per Annum)	4.44	4.46	4.25	4.49	5.41
S&P'S Composite Common Stock: Price-Earnings Ratio (%)	3.35	3.77	2.90	3.58	4.07
Common Stock Prices: Dow Jones Industrial Average	6.56	6.81	5.32	6.64	7.36
U. Of Mich. Index Of Consumer Expectations	1.36	2.13	1.99	2.02	2.53
Purchasing Managers' Index	2.37	1.69	1.66	1.60	2.12
NAPM New Orders Index (Percent)	2.12	1.92	1.61	1.49	2.06
NAPM Vendor Deliveries Index (Percent)	1.69	1.64	1.62	1.56	2.06
NAPM Inventories Index (Percent)	0.66	1.29	1.13	1.22	1.43
New Orders (Net) - Consumer Goods & Materials	3.79	3.53	3.68	3.84	4.68
New Orders, Nondefense Capital Goods	3.68	3.06	2.54	3.72	3.56

Tab. 1.10: Empirical Application of Sequential tests to various U.S. macroeconomic time series

Variables\Test	$SeqF_{\lambda}^{*}(1\backslash 0)$	$SeqF_{\lambda}^{*}(2\backslash 0)$	$SeqUD\max F_{\lambda}^{*}$	$SeqWD\max F_{\lambda}^{*}$
Real Gross Domestic Product	1 (1969Q4)	1 (1969Q4)	1 (1969Q4)	1 (1969Q4)
Real Personal Consumption Expenditures	0	0	0	0
Real Personal Consumption Expenditures - Durable Goods	0	0	0	0
Real Personal Consumption Expenditures - NonDur	0	0	0	0
Real Personal Consumption Expenditures - Services	1 (1972Q4)	1 (1972Q4)	1 (1972Q4)	1 (1972Q4)
Real Gross Private Domestic Investment	0	0	2 (1972Q4,1982Q3)	2 (1972Q4,1982Q3)
Real Gross Private Domestic Investment - FixedInv	0	0	0	0
Real Gross Private Domestic Investment - NonRes	0	0	0	0
Real Gross Private Domestic Investment - NonRes - struct	0	0	0	0
Real Gross Private Domestic Investment - NonRes - Equip	0	0	0	0
Real Gross Private Domestic Investment - Residential	0	0	0	0
Real Exports	0	0	0	0
Real Imports	0	0	0	0
Real Government Consumption Expenditures+Gross Investment	0	2 (1967Q4,1976Q3)	2 (1967Q4,1976Q3)	2 (1967Q4,1976Q3)
Real Government Consumption Expenditures+Gross Investment-Fed.	0	2 (1967Q3,1975Q1)	0	2 (1967Q3,1975Q1)
Real Government Consumption Expenditures+Gross Investment-State/Loc.	1 (1968Q2)	1 (1968Q2)	1 (1968Q2)	1 (1968Q2)
Real Final Sales of Domestic Product	1 (1969Q4)	1 (1969Q4)	1 (1969Q4)	1 (1969Q4)
Real Gross Domestic Purchases	0	0	0	0
Real Final Sales to Domestic Purchasers	0	0	0	0
Real Gross National Product	1 (1969Q4)	1 (1969Q4)	1 (1969Q4)	1 (1969Q4)
Gross Domestic Product	3 (1966Q1,1973Q2,1982Q1)	3 (1966Q1,1973Q2,1982Q1)	3 (1966Q1,1973Q2,1982Q1)	3 (1966Q1,1973Q2,1982Q1)
Personal Consumption Expenditures	3 (1972Q3,1981Q3,1991Q3)	3 (1972Q3,1981Q3,1991Q3)	3 (1972Q3,1981Q3,1991Q3)	3 (1972Q3,1981Q3,1991Q3)
Personal Consumption Expenditures - Durable Goods	3 (1973Q1,1981Q3,1994Q4)	3 (1973Q1,1981Q3,1994Q4)	3 (1973Q1,1981Q3,1994Q4)	3 (1973Q1,1981Q3,1994Q4)
Personal Consumption Expenditures - Nondurable Goods	2 (1972Q2,1980Q4)	2 (1972Q2,1980Q4)	2 (1972Q2,1980Q4)	2 (1972Q2,1980Q4)
Personal Consumption Expenditures - Services	3 (1966Q1,1973Q2,1983Q2)	3 (1966Q1,1973Q2,1983Q2)	3 (1966Q1,1973Q2,1983Q2)	3 (1966Q1,1973Q2,1983Q2)
Gross Private Domestic Investment	2 (1972Q3,1981Q3)	2 (1972Q3,1981Q3)	2 (1972Q3,1981Q3)	2 (1972Q3,1981Q3)
Gross Private Domestic Investment - Fixed Investment	2 (1972Q3,1981Q3)	2 (1972Q3,1981Q3)	2 (1972Q3,1981Q3)	2 (1972Q3,1981Q3)
Gross Private Domestic Investment - NonRes.	2 (1972Q4,1981Q4)	2 (1972Q4,1981Q4)	2 (1972Q4,1981Q4)	2 (1972Q4,1981Q4)
Gross Private Domestic Investment - NonRes - struct	0	3 (1968Q1,1981Q4,1999Q2)	3 (1968Q1,1981Q4,1999Q2)	3 (1968Q1,1981Q4,1999Q2)
Gross Private Domestic Investment - NonRes. - Equip.	2 (1973Q2,1981Q4)	2 (1973Q2,1981Q4)	2 (1973Q2,1981Q4)	2 (1973Q2,1981Q4)
Gross Private Domestic Investment - Res.	2 (1970Q2,1981Q1)	2 (1970Q2,1981Q1)	2 (1970Q2,1981Q1)	2 (1970Q2,1981Q1)
Exports	2 (1972Q1,1980Q4)	2 (1972Q1,1980Q4)	2 (1972Q1,1980Q4)	2 (1972Q1,1980Q4)
Imports	2 (1971Q3,1980Q4)	2 (1971Q3,1980Q4)	2 (1971Q3,1980Q4)	2 (1971Q3,1980Q4)

Tab. 1.11: Empirical Application of Sequential tests to various U.S. macroeconomic time series (continued)

Variables\Test	$SeqF^*_\lambda(1\backslash 0)$	$SeqF^*_\lambda(2\backslash 0)$	$SeqUD\max F^*_\lambda$	$SeqWD\max F^*_\lambda$
Government Consumption Expenditures & Gross Investment	2 (1967Q4,1981Q4)	2 (1967Q4,1981Q4)	2 (1967Q4,1981Q4)	2 (1967Q4,1981Q4)
Government Consumption Expenditures & Gross Investment - Federal	2 (1967Q4,1981Q4)	2 (1967Q4,1981Q4)	2 (1967Q4,1981Q4)	2 (1967Q4,1981Q4)
Government Consumption Expenditures & Gross Investment - State/local	2 (1966Q1,1982Q1)	2 (1966Q1,1982Q1)	2 (1966Q1,1982Q1)	2 (1966Q1,1982Q1)
Final Sales of Domestic Product	3 (1966Q1,1973Q2,1982Q1)	3 (1966Q1,1973Q2,1982Q1)	3 (1966Q1,1973Q2,1982Q1)	3 (1966Q1,1973Q2,1982Q1)
Gross Domestic Purchases	2 (1972Q2,1981Q3)	2 (1972Q2,1981Q3)	2 (1972Q2,1981Q3)	2 (1972Q2,1981Q3)
Final Sales to Domestic Purchasers	2 (1972Q2,1981Q4)	2 (1972Q2,1981Q4)	2 (1972Q2,1981Q4)	2 (1972Q2,1981Q4)
Gross National Product	3 (1966Q1,1973Q2,1982Q1)	3 (1966Q1,1973Q2,1982Q1)	3 (1966Q1,1973Q2,1982Q1)	3 (1966Q1,1973Q2,1982Q1)
Output per hour all persons: Business Sector	0	0	0	0
Real Compensation per hour,Employees:Nonfarm Business	2 (1972Q4,1996Q4)	2 (1972Q4,1996Q4)	2 (1972Q4,1996Q4)	2 (1972Q4,1996Q4)
Hours of all persons: NonFarm Business Sector	1 (1998Q1)	1 (1998Q1)	1 (1998Q1)	1 (1998Q1)
Unit Labor Cost: Non farm Business Sector	2 (1972Q3,1981Q4)	2 (1972Q3,1981Q4)	2 (1972Q3,1981Q4)	2 (1972Q3,1981Q4)
Gross domestic product Price Index	3 (1966Q1,1973Q2,1981Q4)	3 (1966Q1,1973Q2,1981Q4)	3 (1966Q1,1973Q2,1981Q4)	3 (1966Q1,1973Q2,1981Q4)
Personal consumption expenditures Price Index	3 (1972Q3,1981Q3,1991Q3)	3 (1972Q3,1981Q3,1991Q3)	3 (1972Q3,1981Q3,1991Q3)	3 (1972Q3,1981Q3,1991Q3)
Durable goods Price Index	3 (1973Q1,1981Q3,1994Q4)	3 (1973Q1,1981Q3,1994Q4)	3 (1973Q1,1981Q3,1994Q4)	3 (1973Q1,1981Q3,1994Q4)
Motor vehicles and parts Price Index	3 (1973Q2,1981Q3,1995Q3)	3 (1973Q2,1981Q3,1995Q3)	3 (1973Q2,1981Q3,1995Q3)	3 (1973Q2,1981Q3,1995Q3)
Furniture and household equipment Price Index	3 (1973Q1,1982Q1,1994Q3)	3 (1973Q1,1982Q1,1994Q3)	3 (1973Q1,1982Q1,1994Q3)	3 (1973Q1,1982Q1,1994Q3)
Other Price Index	3 (1973Q2,1980Q4,1991Q2)	3 (1973Q2,1980Q4,1991Q2)	3 (1973Q2,1980Q4,1991Q2)	3 (1973Q2,1980Q4,1991Q2)
Nondurable goods Price Index	2 (1972Q2,1980Q4)	2 (1972Q2,1980Q4)	2 (1972Q2,1980Q4)	2 (1972Q2,1980Q4)
Food Price Index	0	2 (1972Q1,1980Q4)	2 (1972Q1,1980Q4)	2 (1972Q1,1980Q4)
Clothing and shoes Price Index	1 (1991Q2)	1 (1991Q2)	1 (1991Q2)	1 (1991Q2)
Gasoline, fuel oil, and other energy goods Price Index	0	0	0	0
Other Price Index	3 (1973Q1,1982Q4,1992Q1)	3 (1973Q1,1982Q4,1992Q1)	3 (1973Q1,1982Q4,1992Q1)	3 (1973Q1,1982Q4,1992Q1)
Services Price Index	3 (1966Q1,1973Q2,1982Q4)	3 (1966Q1,1973Q2,1982Q4)	3 (1966Q1,1973Q2,1982Q4)	3 (1966Q1,1973Q2,1982Q4)
Housing Price Index	3 (1967Q1,1974Q2,1986Q1)	3 (1967Q1,1974Q2,1986Q1)	3 (1967Q1,1974Q2,1986Q1)	3 (1967Q1,1974Q2,1986Q1)
Household operation Price Index	2 (1969Q4,1983Q4)	2 (1969Q4,1983Q4)	2 (1969Q4,1983Q4)	2 (1969Q4,1983Q4)
Electricity and gas Price Index	0	2 (1972Q4,1982Q3)	2 (1972Q4,1982Q3)	2 (1972Q4,1982Q3)
Other household operation Price Index	2 (1967Q2,1985Q2)	2 (1967Q2,1985Q2)	2 (1967Q2,1985Q2)	2 (1967Q2,1985Q2)
Transportation Price Index	0	2 (1973Q3,1980Q4)	2 (1973Q3,1980Q4)	2 (1973Q3,1980Q4)
Medical care Price Index	3 (1973Q2,1982Q4,1992Q4)	3 (1973Q2,1982Q4,1992Q4)	3 (1973Q2,1982Q4,1992Q4)	3 (1973Q2,1982Q4,1992Q4)
Recreation Price Index	2 (1966Q4,1991Q1)	2 (1966Q4,1991Q1)	2 (1966Q4,1991Q1)	2 (1966Q4,1991Q1)
Other Price Index	2 (1972Q3,1981Q2)	2 (1972Q3,1981Q2)	2 (1972Q3,1981Q2)	2 (1972Q3,1981Q2)
Gross private domestic investment Price Index	2 (1972Q3,1981Q3)	2 (1972Q3,1981Q3)	2 (1972Q3,1981Q3)	2 (1972Q3,1981Q3)
Fixed investment Price Index	2 (1972Q3,1981Q3)	2 (1972Q3,1981Q3)	2 (1972Q3,1981Q3)	2 (1972Q3,1981Q3)
Nonresidential Price Index	2 (1972Q4,1981Q4)	2 (1972Q4,1981Q4)	2 (1972Q4,1981Q4)	2 (1972Q4,1981Q4)
Structures	0	3 (1968Q1,1981Q4,1999Q2)	3 (1968Q1,1981Q4,1999Q2)	3 (1968Q1,1981Q4,1999Q2)
Equipment and software Price Index	2 (1973Q2,1981Q4)	2 (1973Q2,1981Q4)	2 (1973Q2,1981Q4)	2 (1973Q2,1981Q4)
Residential Price Index	2 (1970Q2,1981Q1)	2 (1970Q2,1981Q1)	2 (1970Q2,1981Q1)	2 (1970Q2,1981Q1)

Tab. 1.12: Empirical Application of Sequential tests to various U.S. macroeconomic time series (continued)

Variables\Test	$SeqF^*_\lambda(1\backslash 0)$	$SeqF^*_\lambda(2\backslash 0)$	$SeqUD\max F^*_\lambda$	$SeqWD\max F^*_\lambda$
Exports Price Index	2 $_{(1972Q1,1980Q4)}$	2 $_{(1972Q1,1980Q4)}$	2 $_{(1972Q1,1980Q4)}$	2 $_{(1972Q1,1980Q4)}$
Goods Price Index	2 $_{(1972Q1,1980Q4)}$	2 $_{(1972Q1,1980Q4)}$	2 $_{(1972Q1,1980Q4)}$	2 $_{(1972Q1,1980Q4)}$
Services Price Index	2 $_{(1970Q2,1982Q2)}$	2 $_{(1970Q2,1982Q2)}$	2 $_{(1970Q2,1982Q2)}$	2 $_{(1970Q2,1982Q2)}$
Imports Price Index	2 $_{(1971Q3,1980Q4)}$	2 $_{(1971Q3,1980Q4)}$	2 $_{(1971Q3,1980Q4)}$	2 $_{(1971Q3,1980Q4)}$
Goods Price Index	2 $_{(1971Q2,1980Q4)}$	2 $_{(1971Q2,1980Q4)}$	2 $_{(1971Q2,1980Q4)}$	2 $_{(1971Q2,1980Q4)}$
Services Price Index	0	2 $_{(1970Q2,1980Q2)}$	2 $_{(1970Q2,1980Q2)}$	2 $_{(1970Q2,1980Q2)}$
Government consumption expenditures and gross investment Price Index	2 $_{(1967Q4,1981Q4)}$	2 $_{(1967Q4,1981Q4)}$	2 $_{(1967Q4,1981Q4)}$	2 $_{(1967Q4,1981Q4)}$
Federal Price Index	2 $_{(1967Q4,1981Q4)}$	2 $_{(1967Q4,1981Q4)}$	2 $_{(1967Q4,1981Q4)}$	2 $_{(1967Q4,1981Q4)}$
State and local Price Index	2 $_{(1966Q1,1982Q1)}$	2 $_{(1966Q1,1982Q1)}$	2 $_{(1966Q1,1982Q1)}$	2 $_{(1966Q1,1982Q1)}$
Industrial Production Index - Total Index	0	0	0	0
Industrial Production Index - Products, Total	0	0	0	0
Industrial Production Index - Final Products	0	0	0	0
Industrial Production Index - Consumer Goods	1 $_{(1973M1)}$	1 $_{(1973M1)}$	1 $_{(1973M1)}$	1 $_{(1973M1)}$
Industrial Production Index - Durable Consumer Goods	0	0	0	0
Industrial Production Index - Nondurable Consumer Goods	1 $_{(1973M1)}$	1 $_{(1973M1)}$	1 $_{(1973M1)}$	1 $_{(1973M1)}$
Industrial Production Index - Business Equipment	0	0	0	0
Industrial Production Index - Materials	0	0	0	0
Industrial Production Index - Durable Goods Materials	0	0	0	0
Industrial Production Index - Nondurable Goods Materials	1 $_{(1973M12)}$	1 $_{(1973M12)}$	1 $_{(1973M12)}$	1 $_{(1973M12)}$
Industrial Production Index - Manufacturing	0	0	0	0
Industrial Production Index - Residential Utilities	1 $_{(1973M9)}$	1 $_{(1973M9)}$	1 $_{(1973M9)}$	1 $_{(1973M9)}$
Industrial Production Index - Fuels	0	0	0	0
Napm Production Index (Percent)	0	0	0	0
Capacity Utilization - Manufacturing	0	0	0	0
Avg Hrly Earnings, Prod Wrkrs, Nonfarm - Goods-Producing	3 $_{(1967M1,1974M3,1982M1)}$	3 $_{(1967M1,1974M3,1982M1)}$	3 $_{(1967M1,1974M3,1982M1)}$	3 $_{(1967M1,1974M3,1982M1)}$
Avg Hrly Earnings, Prod Wrkrs, Nonfarm - Construction	2 $_{(1967M1,1981M11)}$	2 $_{(1967M1,1981M11)}$	2 $_{(1967M1,1981M11)}$	2 $_{(1967M1,1981M11)}$
Avg Hrly Earnings, Prod Wrkrs, Nonfarm - Mfg	3 $_{(1967M1,1974M3,1981M12)}$	3 $_{(1967M1,1974M3,1981M12)}$	3 $_{(1967M1,1974M3,1981M12)}$	3 $_{(1967M1,1974M3,1981M12)}$
Real Avg Hrly Earnings, Prod Wrkrs, Nonfarm - Goods-Producing	1 $_{(1972M12)}$	1 $_{(1972M12)}$	1 $_{(1972M12)}$	1 $_{(1972M12)}$
Real Avg Hrly Earnings, Prod Wrkrs, Nonfarm - Construction	2 $_{(1972M12,1993M8)}$	2 $_{(1972M12,1993M8)}$	2 $_{(1972M12,1993M8)}$	2 $_{(1972M12,1993M8)}$
Real Avg Hrly Earnings, Prod Wrkrs, Nonfarm - Mfg	1 $_{(1978M11)}$	1 $_{(1978M11)}$	1 $_{(1978M11)}$	1 $_{(1978M11)}$
Employees, Nonfarm - Total Private	0	0	0	0
Employees, Nonfarm - Goods-Producing	0	0	0	0
Employees, Nonfarm - Mining	0	3 $_{(1971M9,1981M10,1989M6)}$	3 $_{(1971M9,1981M10,1989M6)}$	3 $_{(1971M9,1981M10,1989M6)}$
Employees, Nonfarm - Construction	0	0	0	0
Employees, Nonfarm - Mfg	0	0	0	0
Employees, Nonfarm - Durable Goods	0	0	0	0
Employees, Nonfarm - Nondurable Goods	1 $_{(1998M1)}$	1 $_{(1998M1)}$	1 $_{(1998M1)}$	1 $_{(1998M1)}$
Employees, Nonfarm - Service-Providing	1 $_{(1990M1)}$	1 $_{(1990M1)}$	1 $_{(1990M1)}$	1 $_{(1990M1)}$
Employees, Nonfarm - Trade, Transport, Utilities	1 $_{(1989M4)}$	1 $_{(1989M4)}$	1 $_{(1989M4)}$	1 $_{(1989M4)}$

Tab. 1.13: Empirical Application of Sequential tests to various U.S. macroeconomic time series (continued)

Variables\Test	$SeqF_\lambda^*(1\backslash 0)$	$SeqF_\lambda^*(2\backslash 0)$	$SeqUD\max F_\lambda^*$	$SeqWD\max F_\lambda^*$
Employees, Nonfarm - Wholesale Trade	1 $_{(1980M1)}$	1 $_{(1980M1)}$	1 $_{(1980M1)}$	1 $_{(1980M1)}$
Employees, Nonfarm - Retail Trade	1 $_{(1989M4)}$	1 $_{(1989M4)}$	1 $_{(1989M4)}$	1 $_{(1989M4)}$
Employees, Nonfarm - Financial Activities	1 $_{(1987M6)}$	1 $_{(1987M6)}$	1 $_{(1987M6)}$	1 $_{(1987M6)}$
Employees, Nonfarm - Government	1 $_{(1975M3)}$	1 $_{(1975M3)}$	1 $_{(1975M3)}$	1 $_{(1975M3)}$
Index Of Help-Wanted Advertising In Newspapers	0	0	0	0
Employment: Ratio; Help-Wanted Ads:No. Unemployed Clf	0	0	0	0
Civilian Labor Force: Employed, Total (Thous.)	0	0	0	0
Civilian Labor Force: Employed, Nonagric.Industries (Thous.)	0	0	0	0
Unemployment Rate: All Workers, 16 Years & Over (%)	0	0	0	0
Unemploy.By Duration: Average(Mean)Duration In Weeks	0	0	0	0
Unemploy.By Duration: Persons Unempl.Less Than 5 Wks (Thous.)	0	0	0	0
Unemploy.By Duration: Persons Unempl.5 To 14 Wks (Thous.)	0	0	0	0
Unemploy.By Duration: Persons Unempl.15 Wks + (Thous.)	0	0	0	0
Unemploy.By Duration: Persons Unempl.15 To 26 Wks (Thous.)	0	0	0	0
Unemploy.By Duration: Persons Unempl.27 Wks (Thous)	0	0	0	0
Avg Wkly Hours, Prod Wrkrs, Nonfarm - Goods-Producing	0	0	0	0
Avg Wkly Overtime Hours, Prod Wrkrs, Nonfarm	0	0	0	0
Housing Authorized: Total New Priv Housing Units (Thous.)	0	0	0	0
Housing Starts:Nonfarm(1947-58)	0	0	0	0
Housing Starts:Northeast (Thous.U.)	0	0	0	0
Housing Starts:Midwest(Thous.U.)	0	0	0	0
Housing Starts:South (Thous.U.)	0	0	0	0
Housing Starts:West (Thous.U.)	0	0	0	0
Interest Rate: Federal Funds (Effective) (% Per Annum)	0	0	0	0
Interest Rate: U.S.Treasury Bills,Sec Mkt,3-Mo.(% Per Annum)	0	0	0	0
Interest Rate: U.S.Treasury Bills,Sec Mkt,6-Mo.(% Per Annum)	0	0	0	0
Interest Rate: U.S.Treasury Const Maturities,1-Yr.(% Per Annum)	0	0	0	0
Interest Rate: U.S.Treasury Const Maturities,5-Yr.(% Per Annum)	0	0	0	0
Interest Rate: U.S.Treasury Const Maturities,10-Yr.(% Per Annum)	0	0	0	0
Bond Yield: Moody'S Aaa Corporate (% Per Annum)	2 $_{(1973M3,1981M8)}$	2 $_{(1973M3,1981M8)}$	2 $_{(1973M3,1981M8)}$	2 $_{(1973M3,1981M8)}$
Bond Yield: Moody'S Baa Corporate (% Per Annum)	3 $_{(1973M3,1981M12,1993M9)}$	3 $_{(1973M3,1981M12,1993M9)}$	3 $_{(1973M3,1981M12,1993M9)}$	3 $_{(1973M3,1981M12,1993M9)}$
Fygm6-Fygm3	0	0	0	0
Fygt1-Fygm3	0	0	0	0
Fygt10-Fygm3	0	0	0	0
Money Stock: M1	2 $_{(1966M12,1994M1)}$	2 $_{(1966M12,1994M1)}$	2 $_{(1966M12,1994M1)}$	2 $_{(1966M12,1994M1)}$

Tab. 1.14: Empirical Application of Sequential tests to various U.S. macroeconomic time series (continued)

Variables\Test	$SeqF^*_\lambda(1\backslash 0)$	$SeqF^*_\lambda(2\backslash 0)$	$SeqUD\max F^*_\lambda$	$SeqWD\max F^*_\lambda$
Mzm Frb St. Louis	0	2 (1980M2,1987M4)	0	0
Money Stock:M2	3 (1970M5,1986M11,1995M3)	3 (1970M5,1986M11,1995M3)	3 (1970M5,1986M11,1995M3)	3 (1970M5,1986M11,1995M3)
Monetary Base, Adj For Reserve Requirement Changes	2 (1967M5,1994M10)	2 (1967M5,1994M10)	2 (1967M5,1994M10)	2 (1967M5,1994M10)
Depository Inst Reserves:Total	1 (1993M11)	1 (1993M11)	1 (1993M11)	1 (1993M11)
Depository Inst Reserves:Nonborrowed	0	2 (1984M6,1993M11)	2 (1984M6,1993M11)	2 (1984M6,1993M11)
Commercial And Industrial Loans At All Commercial Banks	1 (1985M1)	1 (1985M1)	1 (1985M1)	1 (1985M1)
Consumer Credit Outstanding - Nonrevolving	0	2 (1986M8,1993M10)	2 (1986M8,1993M10)	2 (1986M8,1993M10)
Personal Consumption Expenditures, Price Index	3 (1966M2,1973M4,1981M10)	3 (1966M2,1973M4,1981M10)	3 (1966M2,1973M4,1981M10)	3 (1966M2,1973M4,1981M10)
Personal Consumption Expenditures - Durable Goods, Price Index	3 (1973M10,1981M10,1995M2)	3 (1973M10,1981M10,1995M2)	3 (1973M10,1981M10,1995M2)	3 (1973M10,1981M10,1995M2)
Personal Consumption Expenditures - Nondurable Goods, Price Index	2 (1972M10,1981M2)	2 (1972M10,1981M2)	2 (1972M10,1981M2)	2 (1972M10,1981M2)
Personal Consumption Expenditures - Services, Price Index	3 (1966M2,1973M5,1982M12)	3 (1966M2,1973M5,1982M12)	3 (1966M2,1973M5,1982M12)	3 (1966M2,1973M5,1982M12)
CPI All Items	2 (1972M11,1981M8)	2 (1972M11,1981M8)	2 (1972M11,1981M8)	2 (1972M11,1981M8)
CPI Less Food And Energy	3 (1966M2,1973M10,1982M5)	3 (1966M2,1973M10,1982M5)	3 (1966M2,1973M10,1982M5)	3 (1966M2,1973M10,1982M5)
PCE Price Index Less Food And Energy	3 (1973M5,1982M9,1992M2)	3 (1973M5,1982M9,1992M2)	3 (1973M5,1982M9,1992M2)	3 (1973M5,1982M9,1992M2)
Producer Price Index: Finished Goods	2 (1972M9,1981M5)	2 (1972M9,1981M5)	2 (1972M9,1981M5)	2 (1972M9,1981M5)
Producer Price Index:Finished Consumer Goods	2 (1972M9,1981M3)	2 (1972M9,1981M3)	2 (1972M9,1981M3)	2 (1972M9,1981M3)
Producer Price Index:Intermed Mat.Supplies & Components	0	2 (1972M6,1981M3)	2 (1972M6,1981M3)	2 (1972M6,1981M3)
Producer Price Index:Crude Materials	0	0	0	0
Real Producer Price Index:Crude Materials	0	0	0	0
Spot Market Price Index:Bls & Crb: All Commodities	0	0	0	0
Real Spot Market Price Index:Bls & Crb: All Commodities	0	0	0	0
Producer Price Index: Crude Petroleum	0	0	0	0
PPI Crude	0	0	0	0
NAPM Commodity Prices Index (Percent)	0	0	0	0
Effective Exchange Rate: United States	0	0	0	0
Foreign Exchange Rate: Switzerland	0	0	0	0
Foreign Exchange Rate: Japan	0	0	0	0
Foreign Exchange Rate: United Kingdom	0	0	0	0
Foreign Exchange Rate: Canada	1 (1998M10)	1 (1998M10)	1 (1998M10)	1 (1998M10)
S&P'S Common Stock Price Index: Composite	0	0	0	0
S&P'S Common Stock Price Index: Industrials	0	0	0	0
S&P'S Composite Common Stock: Dividend Yield (% Per Annum)	0	0	0	0
S&P'S Composite Common Stock: Price-Earnings Ratio (%)	0	0	0	0
Common Stock Prices: Dow Jones Industrial Average	0	0	0	0
U. Of Mich. Index Of Consumer Expectations	0	0	0	0
Purchasing Managers' Index	0	0	0	0
NAPM New Orders Index (Percent)	0	0	0	0
NAPM Vendor Deliveries Index (Percent)	0	0	0	0
NAPM Inventories Index (Percent)	0	0	0	0
New Orders (Net) - Consumer Goods & Materials	0	0	0	0
New Orders, Nondefense Capital Goods	0	0	0	0

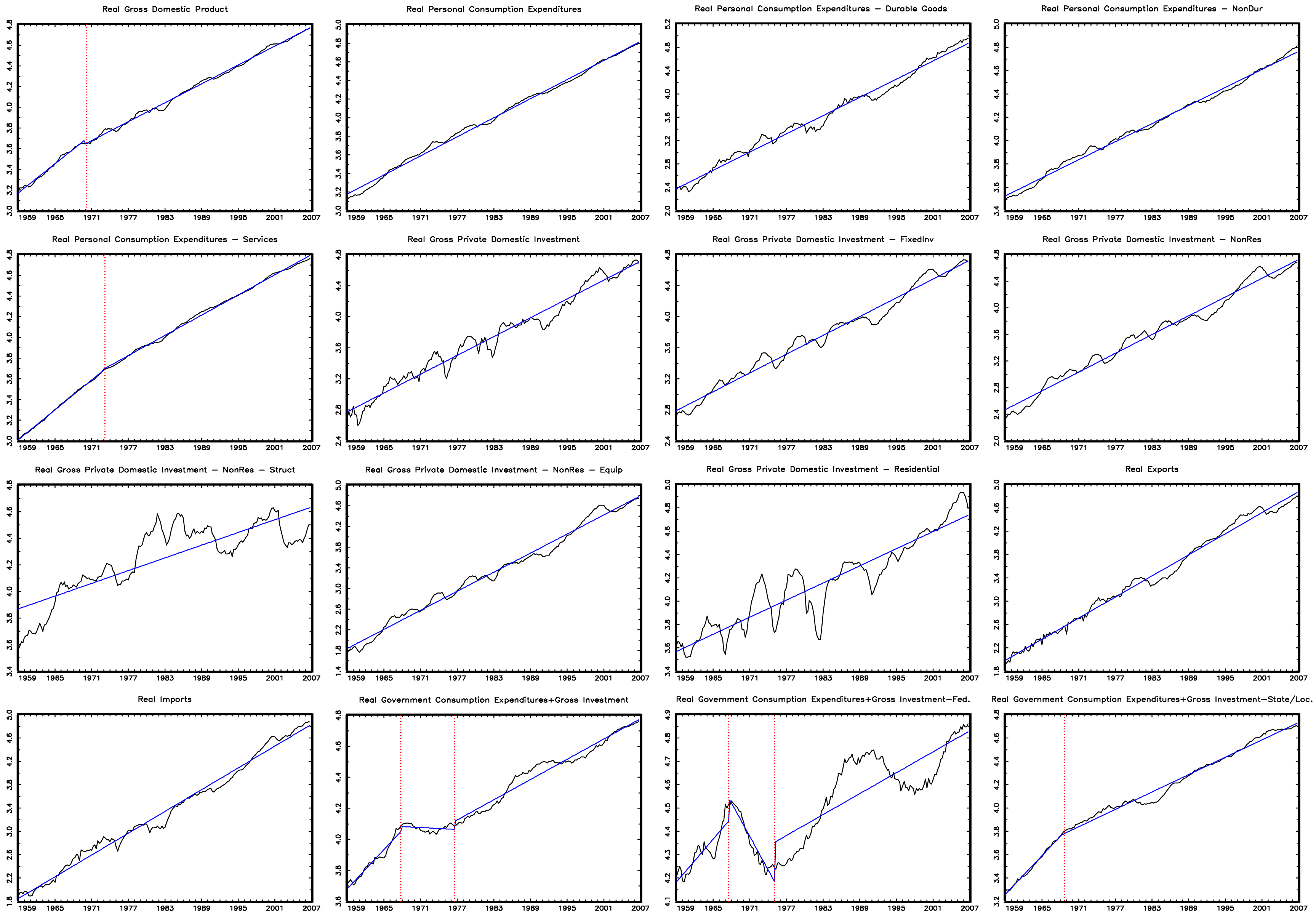

Fig. 1.4: Various U.S. macroeconomic time series and estimated Model B break dates

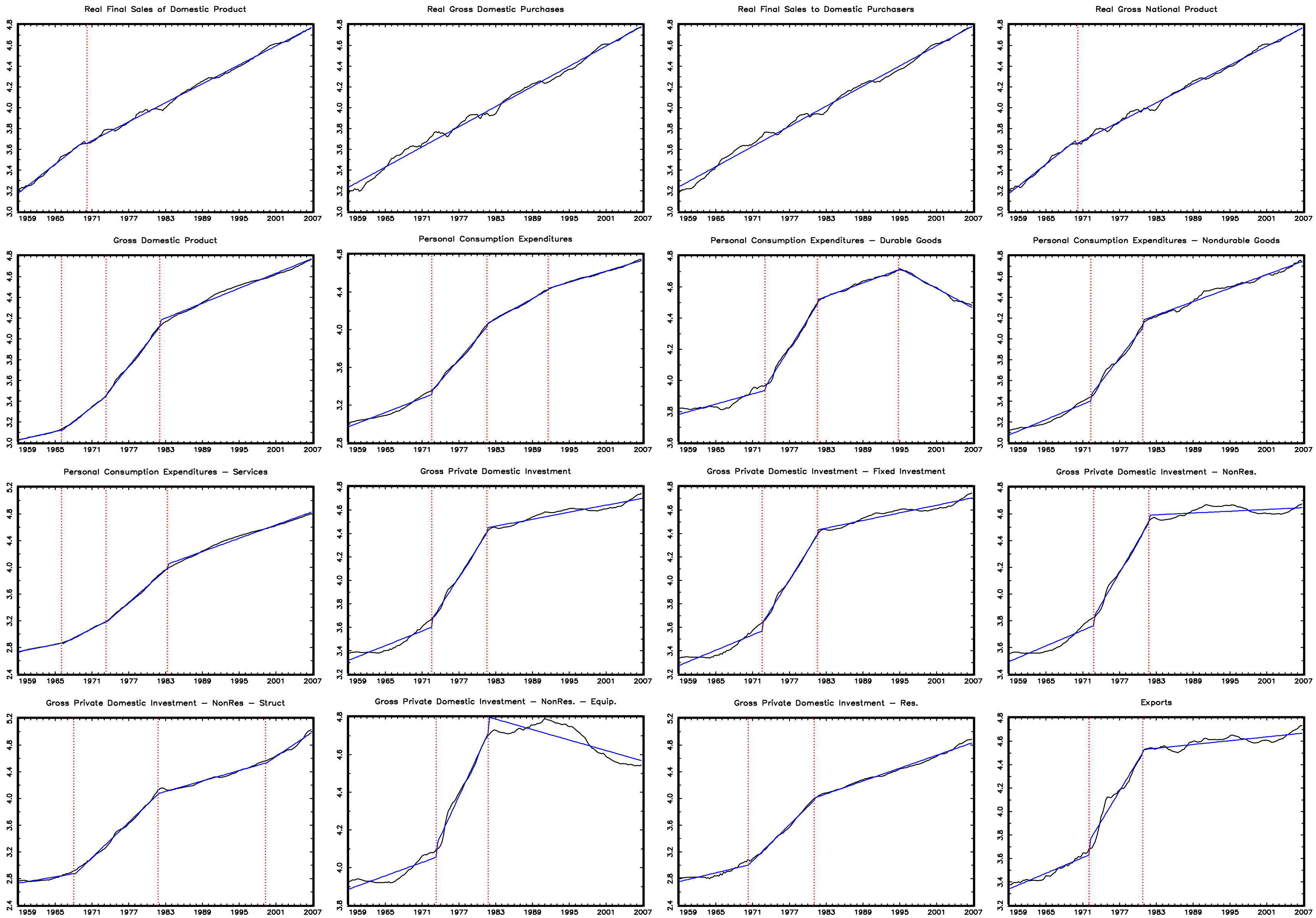

Fig. 1.5: Various U.S. macroeconomic time series and estimated Model B break dates (continued)

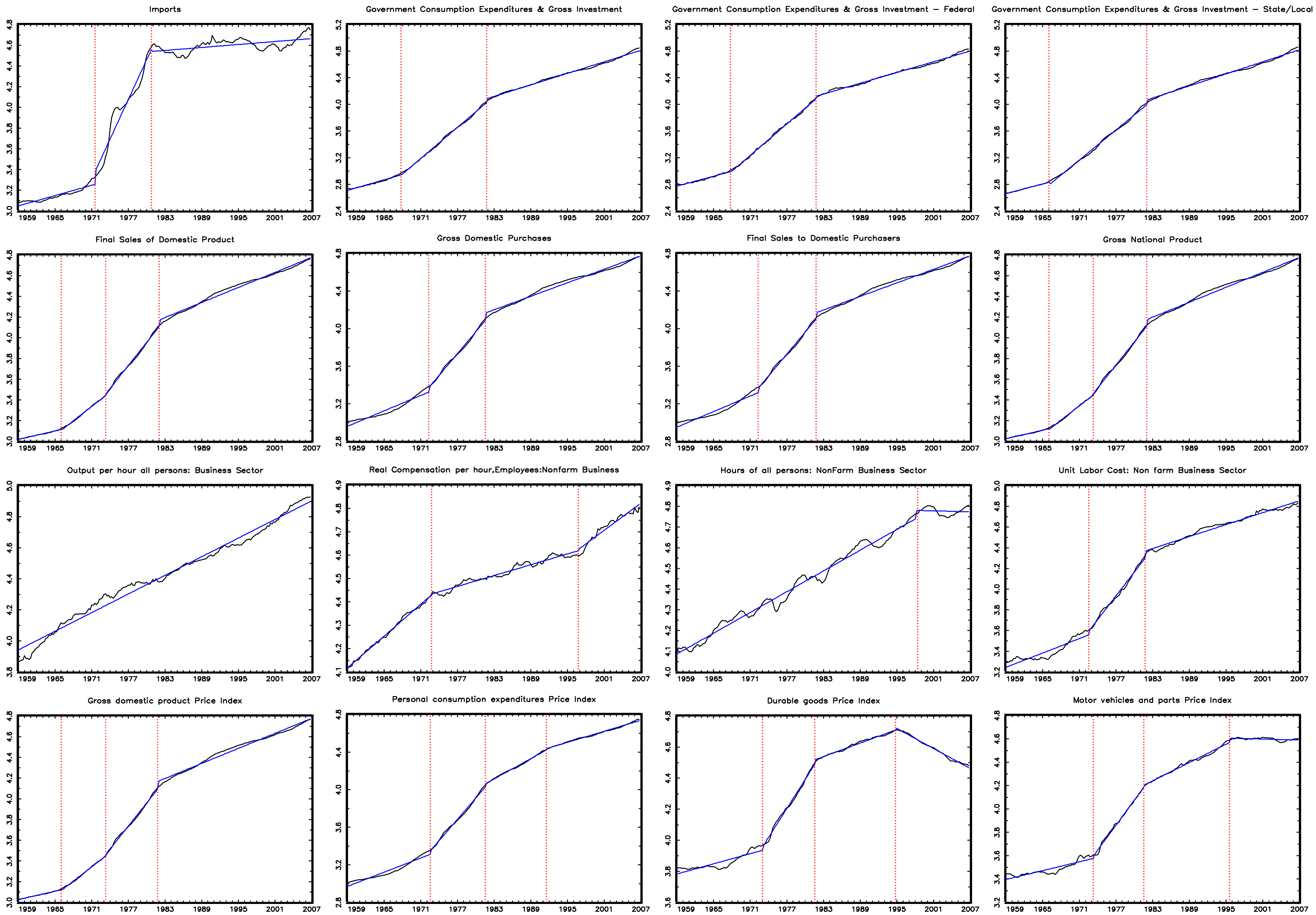

Fig. 1.6: Various U.S. macroeconomic time series and estimated Model B break dates (continued)

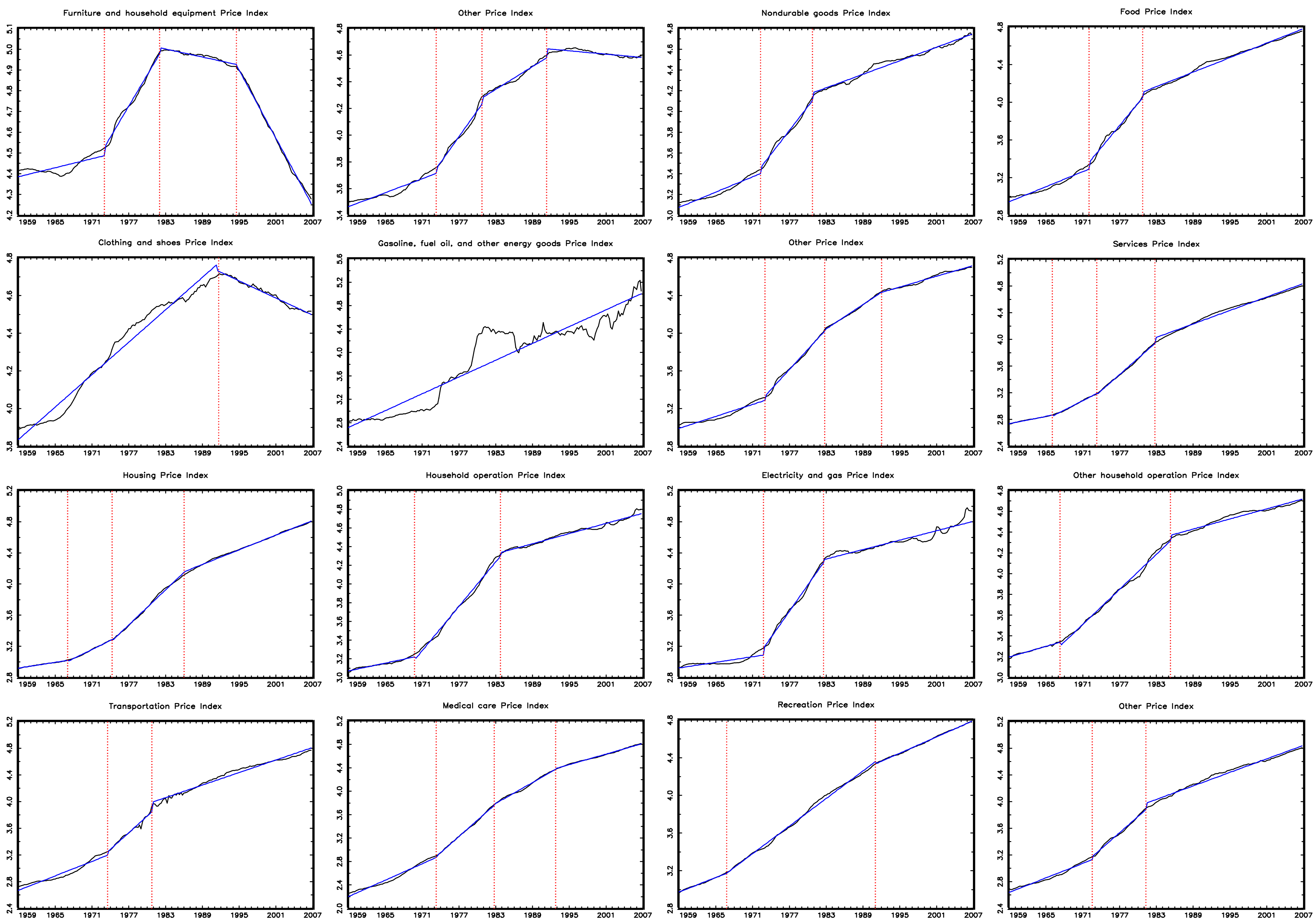

Fig. 1.7: Various U.S. macroeconomic time series and estimated Model B break dates (continued)

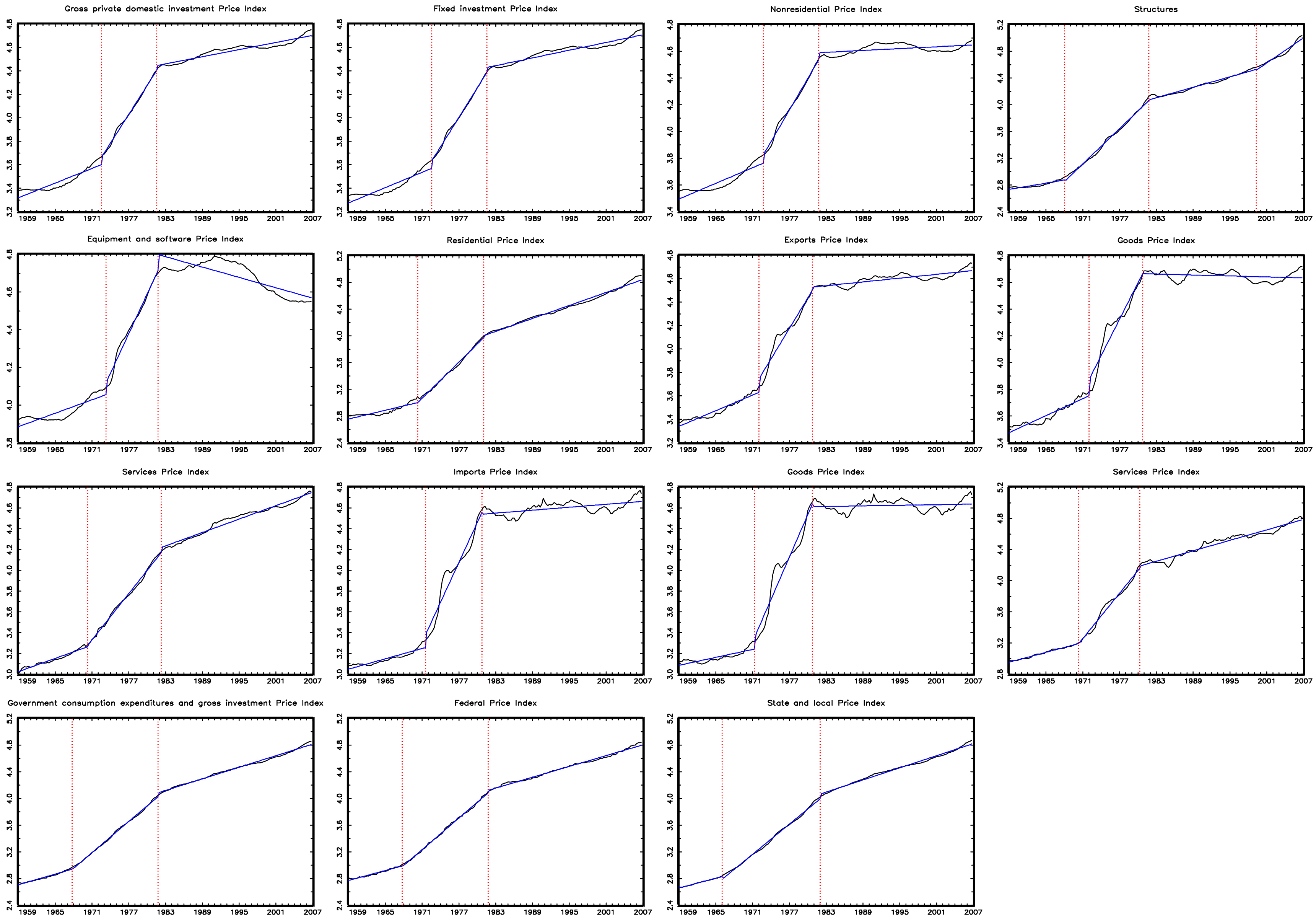

Fig. 1.8: Various U.S. macroeconomic time series and estimated Model B break dates (continued)

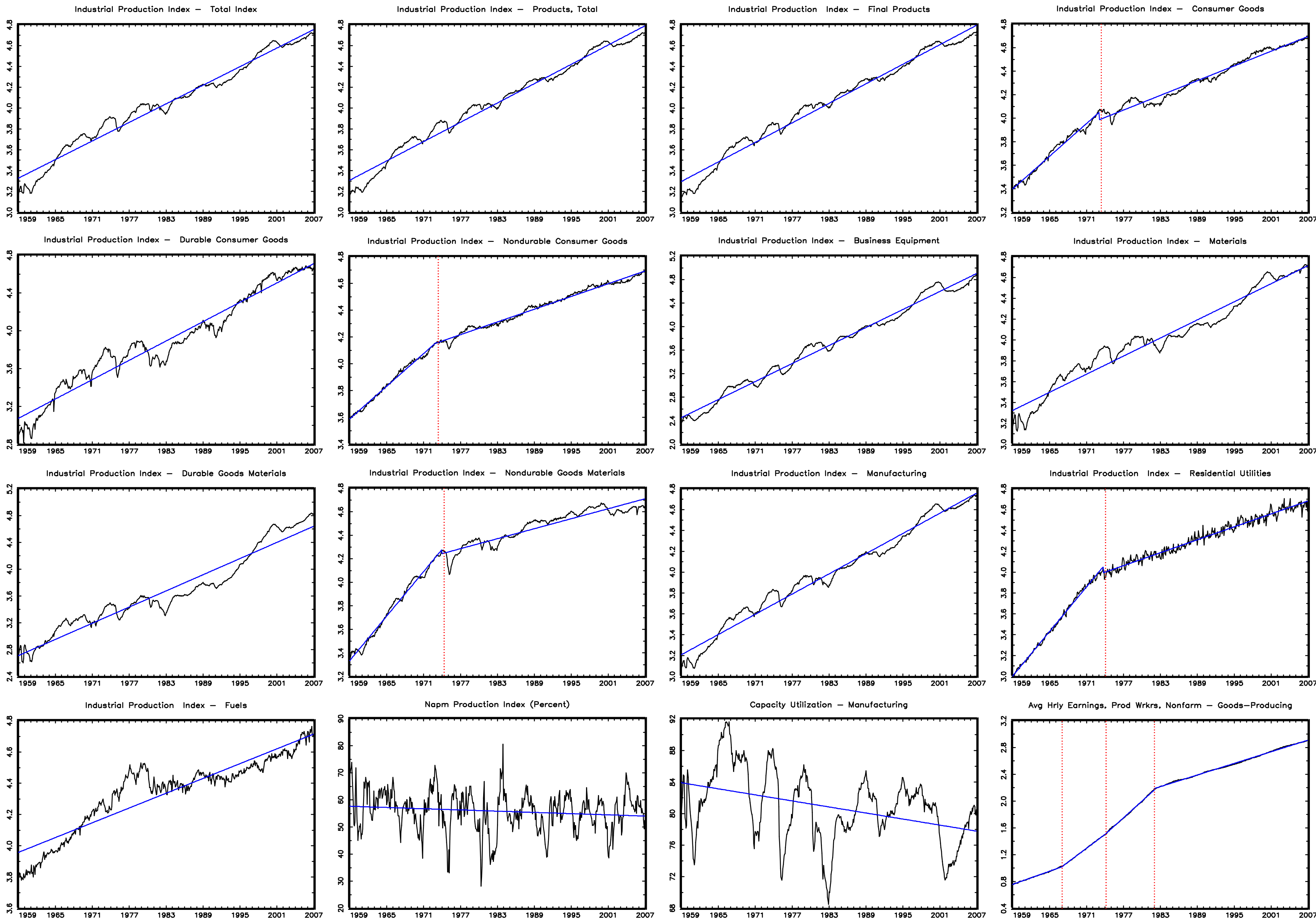

Fig. 1.9: Various U.S. macroeconomic time series and estimated Model B break dates (continued)

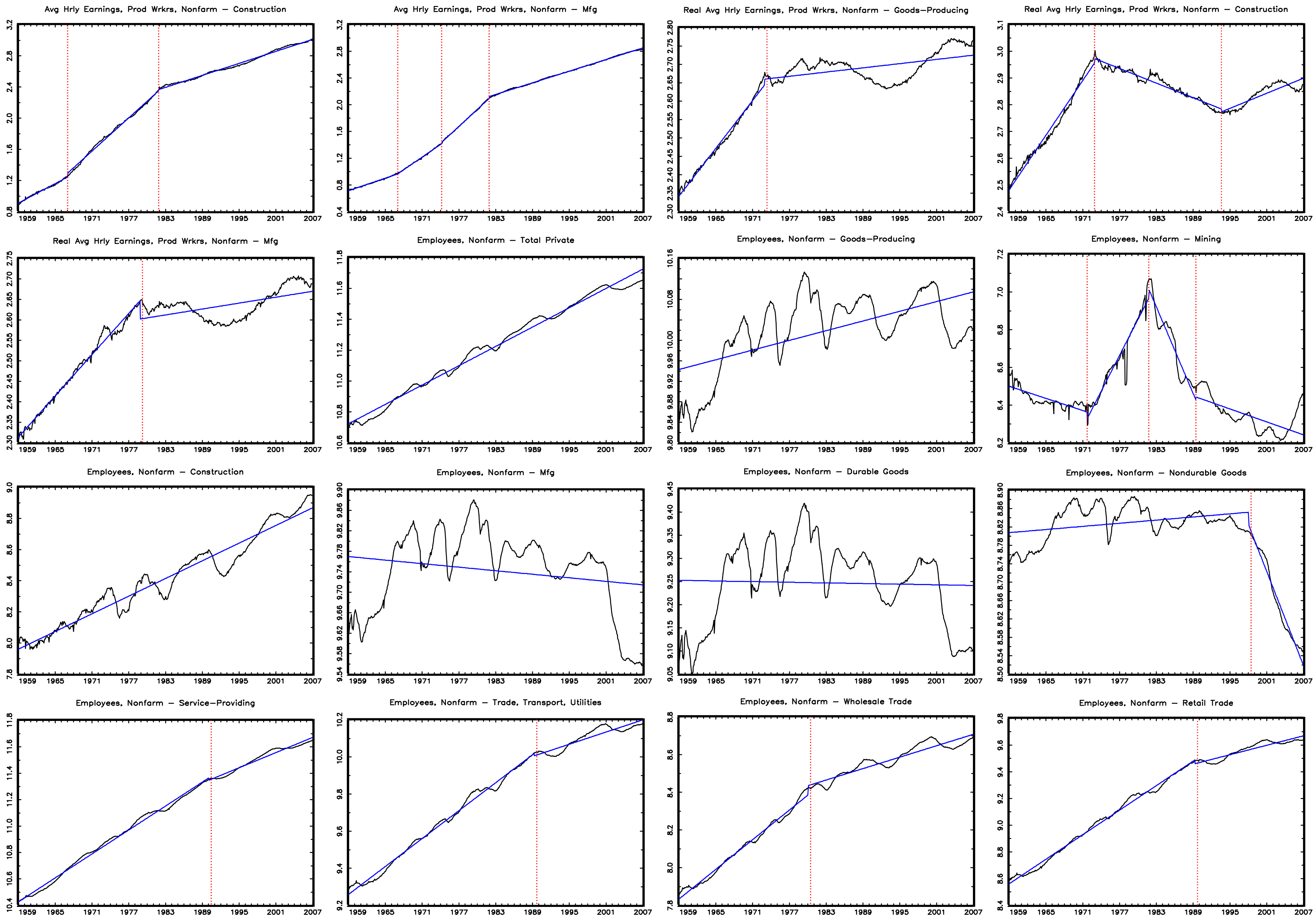

Fig. 1.10: Various U.S. macroeconomic time series and estimated Model B break dates (continued)

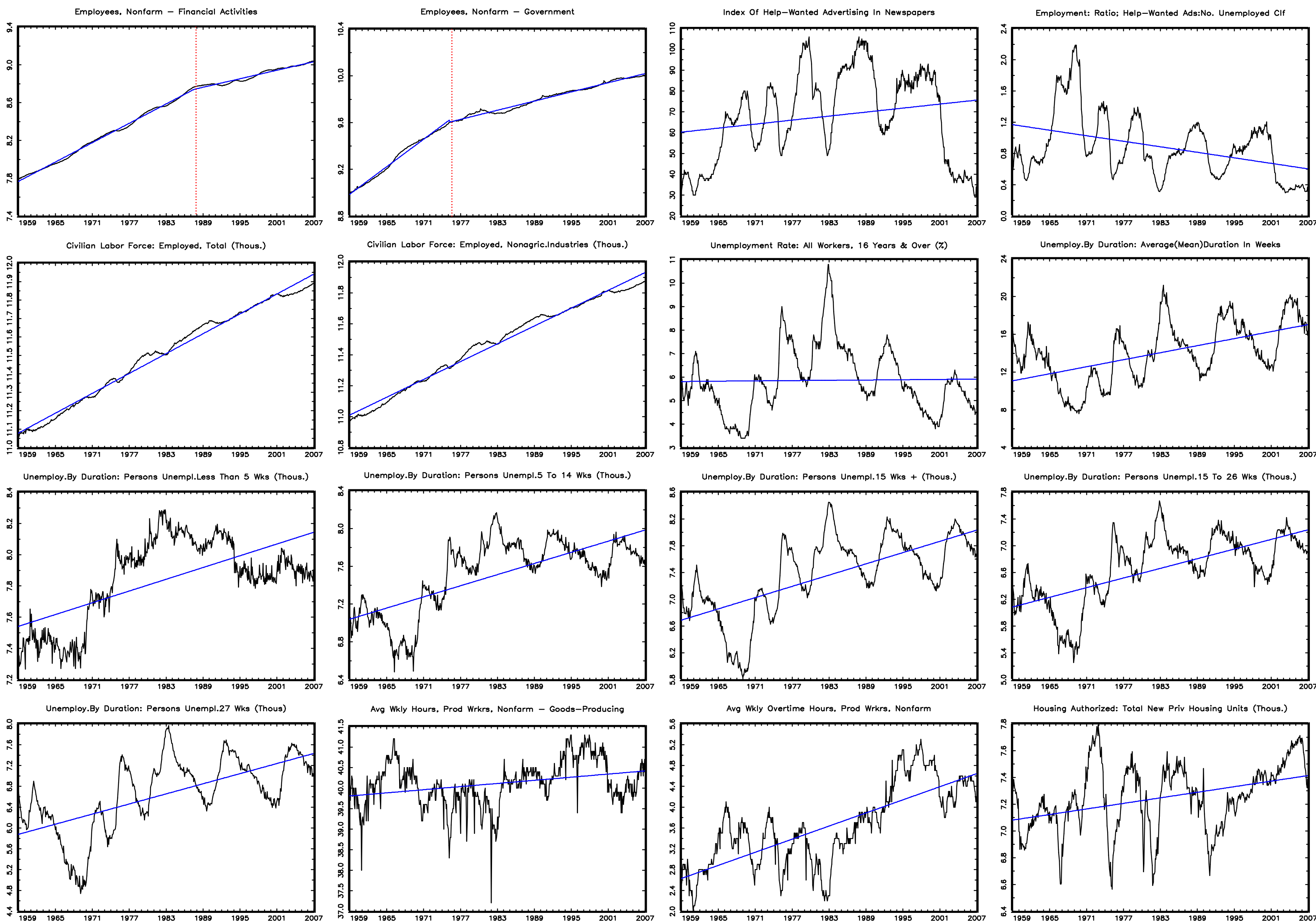

Fig. 1.11: Various U.S. macroeconomic time series and estimated Model B break dates (continued)

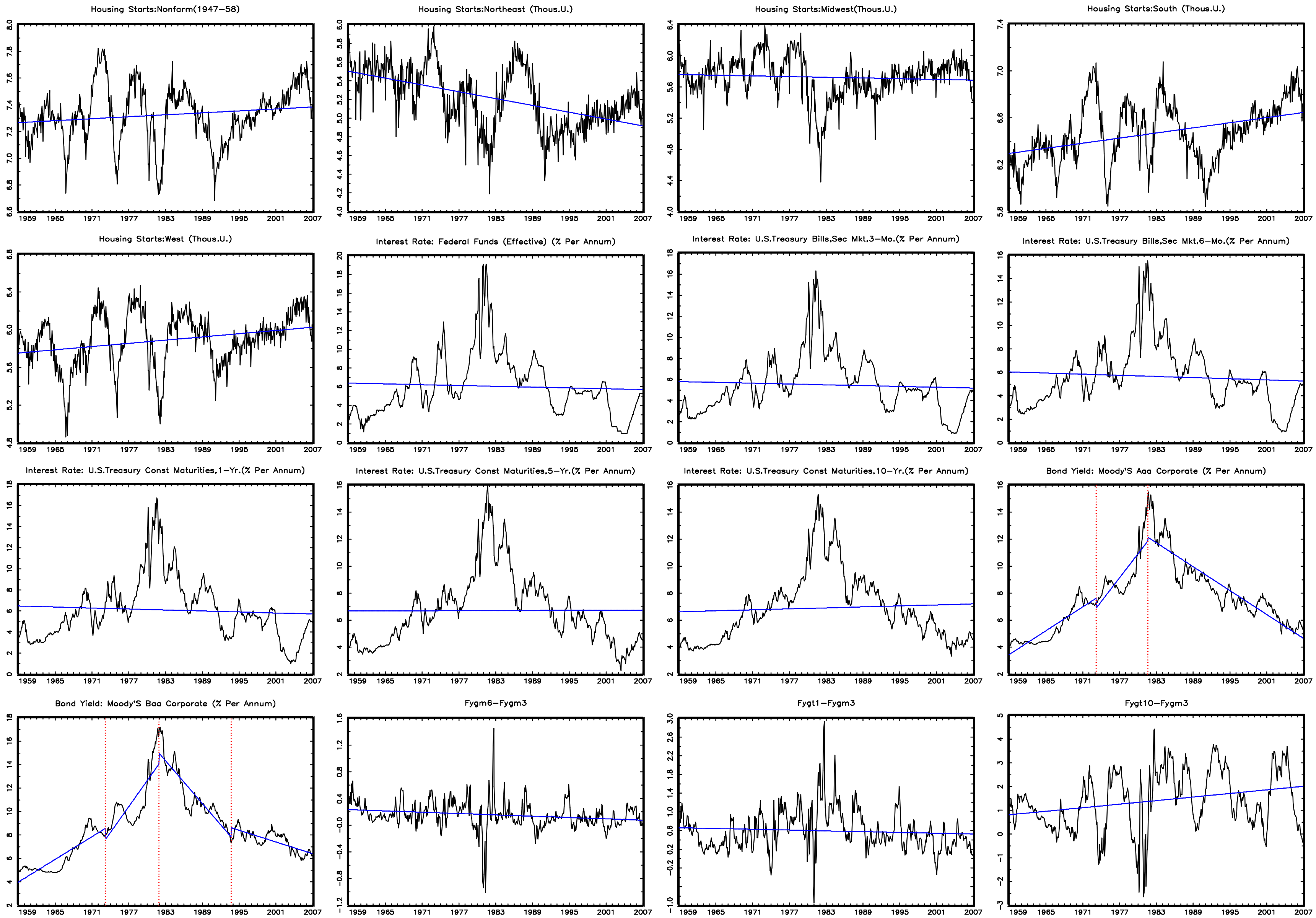

Fig. 1.12: Various U.S. macroeconomic time series and estimated Model B break dates (continued)

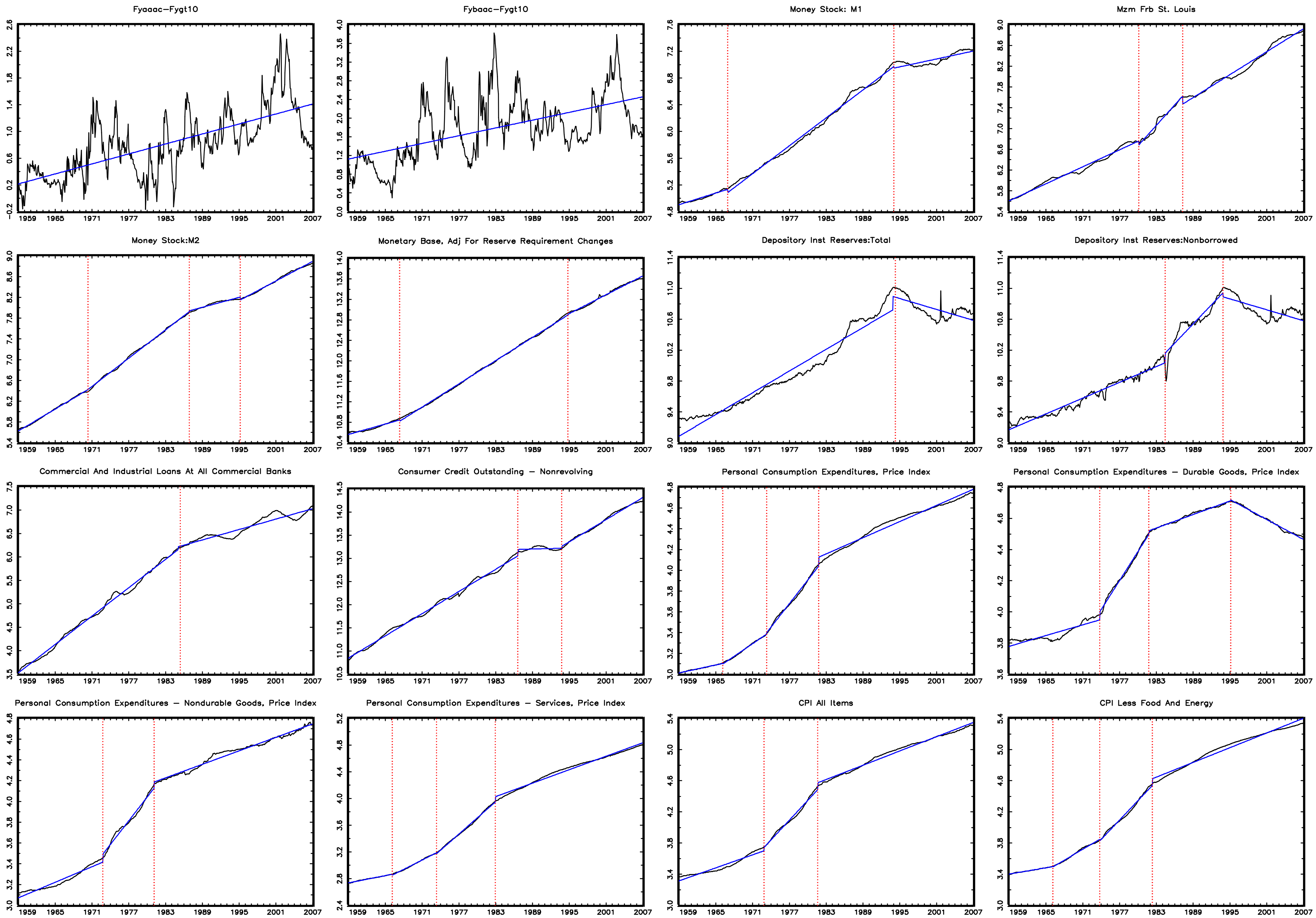

Fig. 1.13: Various U.S. macroeconomic time series and estimated Model B break dates (continued)

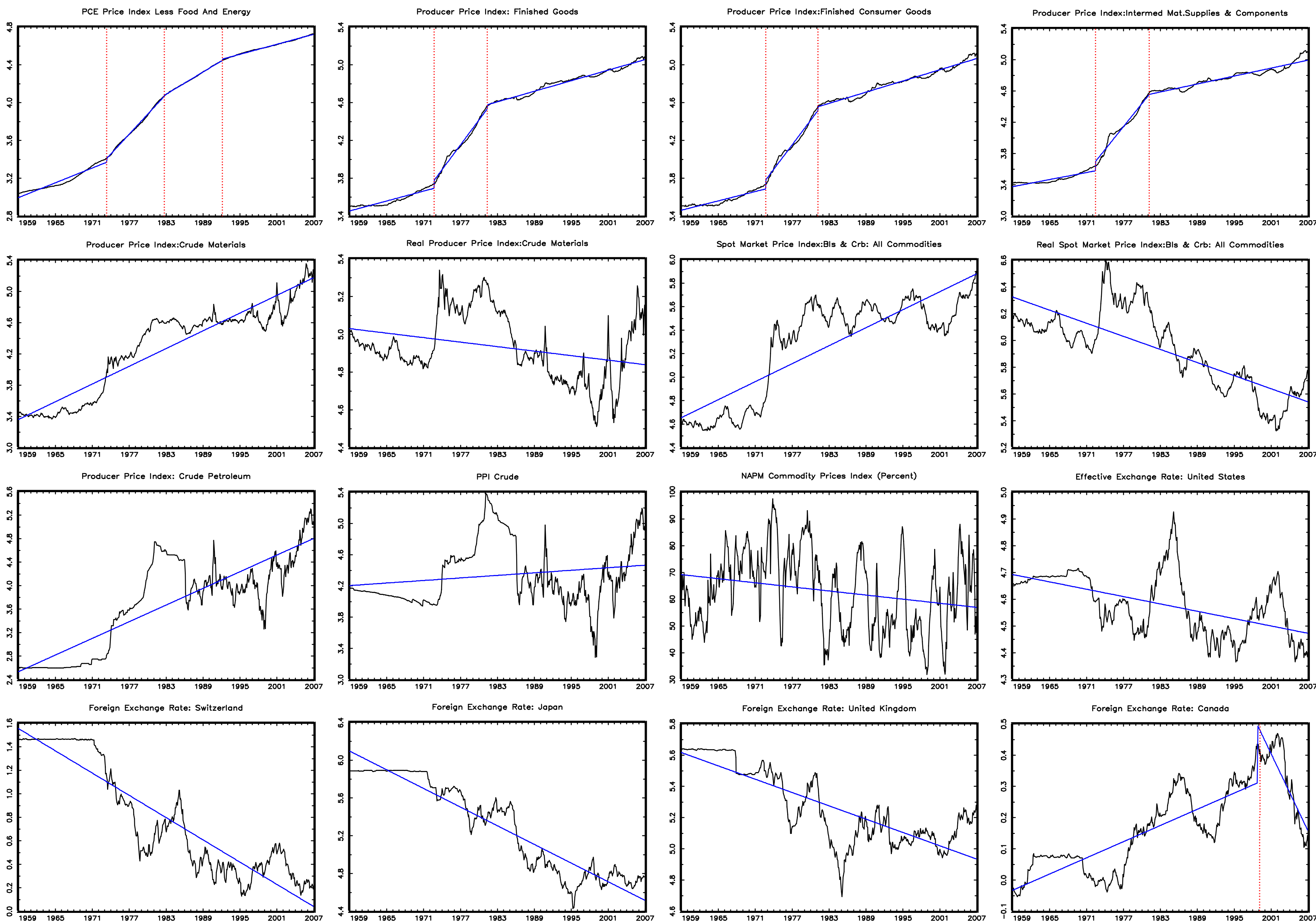

Fig. 1.14: Various U.S. macroeconomic time series and estimated Model B break dates (continued)

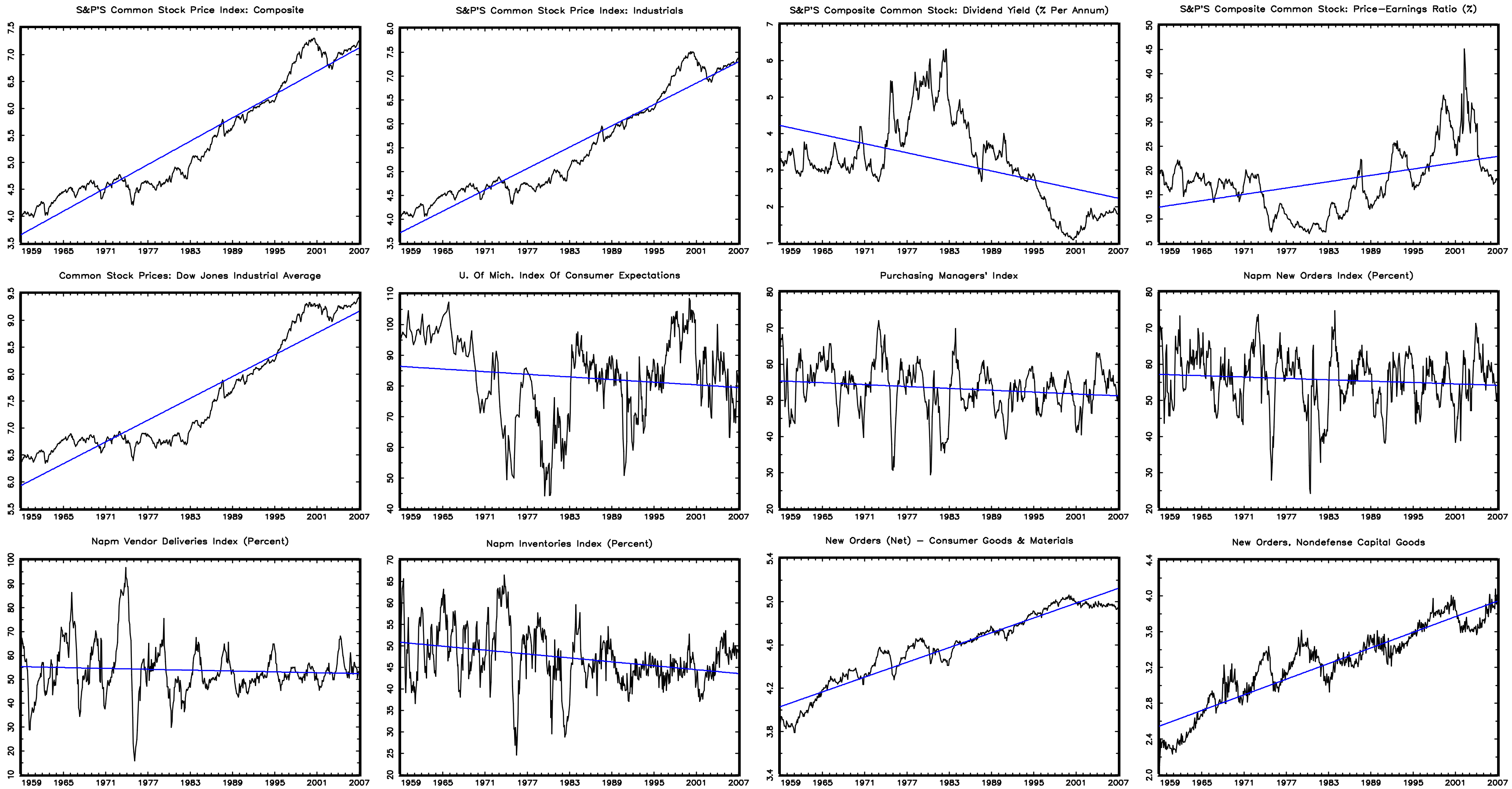

Fig. 1.15: Various U.S. macroeconomic time series and estimated Model B break dates (continued)

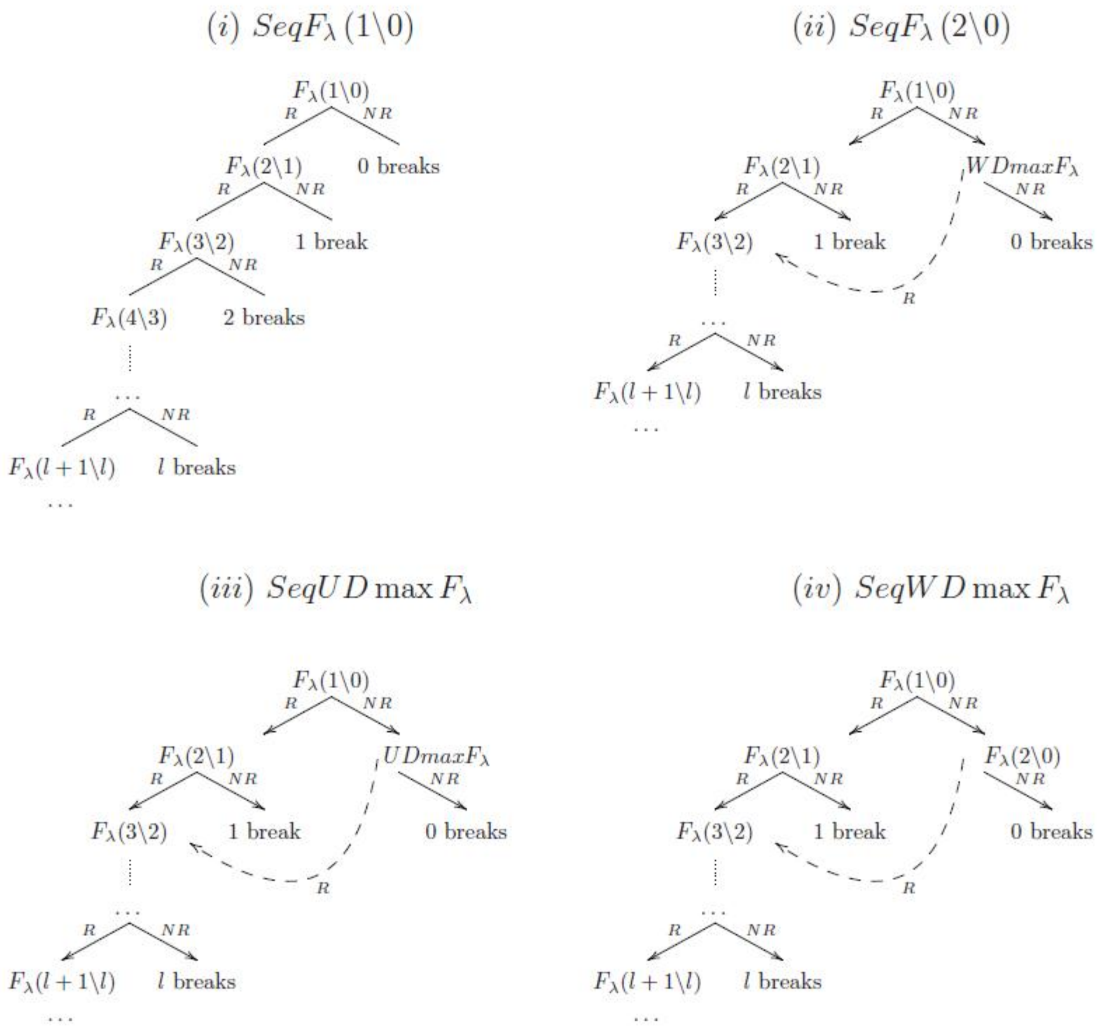

Fig. 1.16: Sequential Tests procedure

2. TREND BREAKS IN MULTIVARIATE TIME SERIES

With Luis C. Nunes[1]

2.1 Introduction

Structural changes in economics can occur for a variety of reasons, such as changes in economic policy, changes in the structure of the economy, or an invention that changes a specific industry. As a result, this concept has widespread use in economics. In econometrics it is usually modelled as changes in the population regression function over the course of the sample. If such changes, or "breaks" occur, then a regression model that neglects those changes can provide a misleading basis for inference and forecasting. Correctly detecting and identifying a structural change can also have profound effect on policy evaluation and recommendation. As a result, structural changes have always been an important concern in econometric modeling.

The statistics and econometrics literature both contain a vast amount of work on issues related to structural changes with unknown break dates, most of it specifically designed for the case of a single change (see Perron, 2006, for an extensive survey).

Because a myriad of political and economic factors may alter the data generating process, multiple changes may be a more accurate characterization of economic time series. Hence, the problem of multiple structural changes has received more attention recently, mostly in the context of a single regression. Bai and Perron (1998, 2003) provided a comprehensive treatment of various issues: consistency of estimates of the break dates,

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tests for structural changes, confidence intervals for the break dates, methods to select the number of breaks, and efficient algorithms to compute the estimates.

However, they preclude the presence of trending or nonstationary regressors while, in fact, formal testing of whether a time series contains structural breaks or not depend on whether the stochastic part of the process is stationary or not. Most tests trying to assess whether structural change is present will reject the null hypothesis of no structural change when the process has a unit root component but constant model parameters. Moreover, doing a structural change test using first-difference data or growth rates to correct for possible I(1) shocks leads to tests with very poor finite sample properties when the series has an I(0) noise component.

A possible solution would be to apply stationary or unit root tests in a first step but these also suffer from similar problems since their properties are in turn affected by the stability of the deterministic components. The leading cases are when data have changes in the mean or slope of a linear trend: unmodelled trend breaks can bias unit root tests towards the non-rejection of the unit root hypothesis when the errors are I(1) (see Perron, 1989), while including unnecessary broken trends greatly reduces power to reject the unit root null under I(0) errors (see Marsh, 2005, for example). A circular testing problem therefore arises between tests on the parameters of the trend function and unit root/stationarity tests. This creates particular difficulties in applied work, since both are of definite practical importance in economic applications.

Hence, the problem of testing for structural changes in a linear model with errors that are either I(0) or I(1) is of substantial interest when testing for breaks in the mean or slope of a linear trend. Only recently have some solutions to this dilemma been proposed in the literature. These resort to statistical tests of the null hypothesis of a constant linear trend against the alternative of a one break at some unknown date that do not require a priori knowledge of whether the noise component is I(0) or I(1). Perron and Yabu (2009) proposed a Feasible Quasi Generalized Least Squares approach to estimate the slope of the trend function. By truncating the estimate of the sum of the autoregressive coefficients of the disturbance term to take the value of one whenever the estimate is in a neighborhood of one, they have shown that the limiting distribution of the t-statistic becomes Normal regardless of the persistence of the error term. Sayginsoy and Vogelsang (2004) proposed a Mean Wald and a Sup-Wald statistic scaled by a factor based on unit root tests to smooth the discontinuities in the asymptotic distributions of the test statistics as the errors go from I(0) to I(1). The scaling factor approach is based on Vogelsang (1998) who proposed test statistics for general linear hypothesis regarding the parameters of the trend function which do not require knowledge as to whether the innovations are I(0) or I(1). Harvey et al. (2009) employed a weighted average of the appropriate regression t-statistics used to test the existence of a broken trend when the errors are I(0) and I(1). Nunes and Sobreira (2010) built on the framework proposed by Harvey et al. (2009) to provide tests of the null hypothesis of no trend breaks against the alternative of one or more breaks in the trend slope which do not require knowledge of the form of serial correlation in the data and are robust as to whether the underlying shocks are stationary or have a unit-root.

The objective of this paper is to extend their work into a multivariate framework and study the problem of testing for multiple structural changes in the trend function of a multivariate time series which do not require knowledge of the form of serial correlation in the data and are robust as to whether the data is stationary, non stationary, cointegrated or not cointegrated. This problem is of practical importance for several reasons. Many macroeconomic time series are characterized by a clear tendency to grow over time, that is, as having a deterministic time trend component. This implies that many interesting economic applications involve statistical inference on the parameters of the trend function. Examples can be found in the continuous time macroeconomic modelling (see Bergstrom et al., 1992, Nowman, 1998), in international trade with the Prebish-Singer hypothesis testing (see Bunzel and Vogelsang, 2005), in the empirical debate regarding regional convergence in per capita income (see Sayginsoy and Vogelsang, 2004), or in environmental economics on the future consequences of global warming (see Vogelsang and Franses, 2005). Also, one is often interested in testing whether the rate of growth of some macroeconomic variables, such as Real GDP, exhibits a structural change. With data in logarithmic form, the coefficient on the trend component represents this average growth rate.

However, empirical work, in general, relies on inference about the trend function in a single regression framework rather than multivariate systems. Many factors which are generally deemed important for the presence of a structural change can result in the growth rates breaking contemporaneously across series. This suggests that gains in precision might be achieved by a multivariate treatment. While it may be difficult to identify a break point with a single series, it should be, intuitively, much easier to locate the common break point using a number of series together. For example, Bai et al. (1998) have shown that dating the slowdown in the postwar U.S. was somewhat difficult due to a very imprecise univariate estimate of the break date for U.S. output. However, dynamic economic theories suggest that a discrete productivity slowdown,an oil price shock or a tax policy change will be reflected in lower growth rates not only of output, but of series that are cointegrated with output, in particular, consumption and investment (see King et al., 1988, for example). When modelling these variables as a trivariate system Bai et al. (1998) found a statistically significant common slowdown in the growth rate around the first quarter of 1969.

In spite of the substantial payoffs for using multivariate rather than univariate techniques, work on structural change issues arising in the context of a system of multivariate equations is relatively scarce. Bai et al. (1998) considered asymptotically valid inference for the estimate of a single break date in multivariate time series allowing stationary or integrated regressors as well as trends. They show that the width of the confidence interval decreases in an important way when series having a common break are treated as a group and estimation is carried using a quasi-maximum likelihood (QML) procedure. Also, Bai (2000) considers the consistency, rate of convergence and limiting distribution of estimated break dates in a segmented stationary VAR model estimated again by QML when the breaks can occur in the parameters of the conditional mean, the covariance matrix of the error term or both. Qu and Perron (2007) considered a more general framework and their theoretical analysis shows how substantial efficiency gains can be obtained by casting the analysis in a system of regressions. In addition, the result of Bai et al. (1998), that when a break is common across equations the precision increases in proportion to the number of equations, is extended to the multiple break case. More importantly, the precision of the estimate of a particular break date in one equation can increase when the system includes other equations even if the parameters of the latter are invariant across regimes. All that is needed is that the correlation between the errors be non-zero. However, Bai et al. (1998) was only designed for the single break case and deals with I(0) and I(1) dynamic models in a separate fashion. Bai (2000) and Qu and Perron (2007) do not permit models with integrated or trending regressors. Hence, techniques for inference about multiple break dates in the trend function in multivariate systems are currently unavailable in spite of the substantial gains from analyzing multiple equations.

It is important to realize that to do appropriate inference about structural breaks in multivariate equations it is necessary to have a priori knowledge about the stationarity and the cointegrating relations among the variables and, in general, this information is not available. The limiting distribution of test statistics depends on number of common stochastic trends (see, for example, Sims et al., 1990, Park and Phillips, 1988, 1989) so that methods of inference that are robust to different possibilities are needed.

A possible solution would be to apply popular Likelihood ratio (LR) tests for the cointegrating rank proposed by Johansen (1991) in a first step as this represents the natural extension of unit root tests into the multivariate framework. However, the limit distribution under the null of cointegrating rank depends on nuisance parameters related to the deterministic components, in particular, if there are breaking trends or not. In recognition of this fact, Johansen et al. (2000) show how the traditional cointegration analysis can be used to identify some types of structural breaks with known break points in the deterministic components. Within their framework they show how to identify (and test for) shifts in the trends, but not in the levels. Lütkepohl and Saikkonen (2000a,b) and Lütkepohl et al. (2004, 2008) in a sequence of papers proposed an alternative two-step approach where in the first step the deterministic part of the DGP is estimated by a generalized least squares (GLS) procedure and then removed from the series. Thereafter an

LR type test for the cointegrating rank is applied. In Monte Carlo simulations Lütkepohl et al. (2008) (LST, henceforth) proved that the test proposed has considerably better small sample properties than the Johansen et al. (2000) test. However, they only consider testing the cointegration rank with trend breaks assuming that the number of break points and the break dates are known. Misspecification of the number of change points and break dates may have a profound effect on the finite sample properties of LR tests for the Cointegrating Rank from Johansen et al. (2000) and LST. Hence, the circular testing problem described above between tests on the parameters of the trend function and unit root/stationarity tests also arises with cointegration/commmon stochastic trends tests.

This paper provides tests of the null hypothesis of no trend breaks against the alternative of one or more breaks in the trend slope in multivariate time series. Our proposed tests do not require knowledge of the form of serial correlation in the data; in particular, no prior knowledge is needed as to whether the the multivariate system is stationary, nonstationary, cointegrated or not cointegrated, thereby breaking the circular testing problem discussed above between structural change and cointegration testing.

The general setup is a VAR process with a linear trend term which may have level shifts and breaks in the trend slope at unknown points in time. If a break is believed to have occurred in the deterministic part of the process only and does not affect the stochastic part, it seems natural to strictly separate the deterministics from the stochastic part in setting up the model. Therefore, as in Lütkepohl et al. (2008), the deterministic part is added to a zero mean purely stochastic process in our setup. The details about the model and assumptions can be found in Section 2.2. In section 2.3 we describe the procedure used to estimate the deterministic components in the stationary and nonstationary directions. These estimators form the basis for statistical inference about the slope of the deterministic trend.

We construct test statistics that are weighted averages of the appropriate Wald statistics to test the existence of multiple trend breaks when the disturbance term is stationary, nonstationary, cointegrated or non cointegrated. The weighting function we employ is based on tests for common stochastic trends from Nyblom and Harvey (2000) and Busetti

(2002) applied to the levels and first differenced data. In section 2.4.1, we start by considering the case where the true break fractions are known and prove that the proposed statistics converge in distribution to a chi-square distribution under the null. Next, in section 2.4.2 we consider the case where the trend break fractions are unknown and need to be estimated. We transform our statistic in the same spirit as Bai et al. (1998) and Qu and Perron (2007) and find those break dates that globally maximize the value of the Wald test over the set of admissible partitions under a trimming restriction. Then we evaluate the Wald statistic on those estimated break points. Here, the weight function is obtained through the minimization of the tests for common stochastic trend over all permissible change points and we prove that its large sample behavior is similar to the known break case regardless of the number of break fractions estimated and the number of structural breaks in the trend function. Finally, in Section 2.5 we propose a sequential test procedure that can be used to estimate the number of trend breaks and that are also robust to stationarity, nonstationarity and cointegration on the multivariate system. In both the known and unknown break dates settings, our proposed tests are made robust to short memory serial correlation in the shocks via the use of lagged dependent variables as regressors. Some Monte Carlo experiments and an empirical application are provided in Sections 2.6 and 2.7 to highlight the practical usefulness of our proposed tests. Section 2.8 offers some concluding remarks.

2.2 *The Model and Assumptions*

Consider the following n-dimensional time series $y_t = (y_{1,t}, \ldots, y_{n,t})$ that is known to be generated by a process with a first-order linear trend and m possible local disjoint broken trends ($m+1$ regimes):

$$y_t = \mu_0 + \mu_1 t + \sum_{j=1}^{m} \delta_j DU_t^j + \sum_{j=1}^{m} \gamma_j DT_t^j + u_t, \quad t = 1, \ldots, T \tag{2.1}$$

where δ_j and γ_j for $j = 0, \ldots, m$ are unknown vectors of coefficients and $DU_t^j := 1(t > T_j^*)$ and $DT_t^j := 1(t > T_j^*)(t - T_j^*)$ are dummy variables capturing, respectively, the eventual j^{th} change in the intercept and in the slope coefficients occurring at date $T_j^* := \lfloor \tau_j^* T \rfloor$ with associated break fraction $\tau_j^* \in (0,1)$ and $0 < \tau_1^* < \ldots < \tau_m^* < 1$. In the above model, we are interested in testing if there are common trend breaks in y_t and in estimating the number and dates of breaks in the multivariate time series process, independently of whether u_t is stationary, non stationary, cointegrated or not cointegrated. Therefore, we would like to test the null hypothesis $H_0 : \gamma_1 = \gamma_2 = \ldots = \gamma_m = 0$ against the two sided alternative: $H_1 : \gamma_1 \neq 0 \vee \gamma_2 \neq 0 \vee \ldots \vee \gamma_m \neq 0$ in equation (2.1). We make the following assumption on the stochastic part of the Model, u_t.

Assumption 2. *u_t is an unobservable error process which we assume that follows a p^{th} order zero mean VAR process:*

$$u_t = A_1 u_{t-1} + \ldots + A_p u_{t-p} + \varepsilon_t \tag{2.2}$$

where the initial values $u_t, t \leq 0$ are assumed to be equal to zero.

where the A_i are $(n \times n)$ coefficient matrices, for $i = 1, \ldots, p$. The disturbance term ε_t is assumed to satisfy the following assumption:

Assumption 3. *Let ε_t be a zero mean Gaussian white noise so that $\varepsilon_t \overset{iid}{\sim} \mathcal{N}_n(\mathbf{0}, \mathbf{\Omega})$ with the covariance matrix Ω definite positive.*

To exclude the possibility of having explosive and seasonal roots, we use following assumption regarding the characteristic polynomial of u_t:

Assumption 4. *Let $A(z) = I_n - \sum_{i=1}^{p} A_i z^i$ be the characteristic polynomial of u_t. Then $A(z)$ satisfies the condition that if $|A(z)| = 0$, then either $|z| = 1$ or $|z| > 1$.*

If we subtract u_{t-1} on both sides of (2.2) and rearrange terms, we can write u_t in error correction (EC) form:

$$\Delta u_t = \Pi u_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta u_{t-i} + \varepsilon_t \tag{2.3}$$

where $\Pi = -(I_n - A_1 - \ldots - A_p)$, $\Gamma_i = -(A_{i+1} - \ldots - A_p)$ and Δ is the usual first-difference operator. The process u_t is assumed to be at most $I(1)$ and may be cointegrated or not, which implies that matrix Π can be full rank, be equal to the null matrix or have reduced cointegrating rank $0 \leq r \leq n$. If u_t is cointegrated with $0 < r < n$ then the matrix Π can be written as $\Pi = \alpha\beta'$ where α and β are $(n \times r)$ matrices of full column rank. Since we want to rule out the possibity of finding processes that are integrated of order higher than one, we impose the following condition:

Assumption 5. *We assume that* $|\alpha'_{\perp}\Psi\beta_{\perp}| \neq 0$, *with* $\Psi = I_n - \sum_{i=1}^{p-1} \Gamma_i$.

Here as well as below, if B is an $(n \times H)$ matrix of full column rank $n > H$ we let $B_{\perp}$ stand for an orthogonal complement, that is, $B_{\perp}$ is an $(n \times (n - H))$ matrix of full column rank and such that $B'B_{\perp} = 0$. The orthogonal complement of a nonsingular square matrix is zero and the orthogonal complement of zero is an identity matrix of suitable dimension. Given these assumptions, according to the Granger Representation Theorem, the solution of (2.3) has the representation:

$$u_t = C \sum_{i=1}^{t} \varepsilon_i + \xi_t \tag{2.4}$$

where, apart from the specification of the initial values, ξ_t is a zero mean stationary process and $C = \beta_{\perp}(\alpha'_{\perp}\Psi\beta_{\perp})^{-1}\alpha'_{\perp}$. We use the following identity for the characteristic polynomial of u_t:

$$A(L) = I_n - \sum_{i=1}^{p} A_i L^i = I_n \Delta - \Pi L - \sum_{i=1}^{p-1} \Gamma_i \Delta L^i \tag{2.5}$$

It will be useful later on to note the relation between the two representations of $A\left(L\right)$:

$$
\begin{aligned}
A_1 &= I_n + \alpha\beta' + \Gamma_1 \\
A_i &= \Gamma_i - \Gamma_{i-1}, i = 2, \ldots, p-1 \\
A_p &= -\Gamma_{p-1}
\end{aligned}
\tag{2.6}
$$

Multiplying (2.1) by $A(L)$, we obtain the error correction (EC) form of y_t:

$$
\begin{aligned}
\Delta y_t =& \mu + \alpha \left(\beta' y_{t-1} - \beta' \mu_1 (t-1) - \sum_{j=1}^{m} \beta' \gamma_j DT_{t-1}^j \right) + \sum_{i=1}^{p-1} \Gamma_i \Delta y_{t-i} \\
& + \sum_{j=1}^{m} \sum_{i=0}^{p-1} \upsilon_{i,j} D_{t-i}^j + \sum_{j=1}^{m} \eta_j DU_{t-1}^j + \varepsilon_t, t = p+1, \ldots,
\end{aligned}
\tag{2.7}
$$

where $\mu = -\Pi\mu_0 + \Psi\mu_1$, $\eta_j = \Psi\gamma_j - \Pi\delta_j$ and

$$
\upsilon_{i,j} = \begin{cases} \delta_j + \Pi\delta_j + \Gamma_1\gamma_j + \ldots + +\Gamma_{p-1}\gamma_j & , \ i = 0, \\ -\Gamma_i\delta_j + \Gamma_{i+1}\gamma_j + \ldots + \Gamma_{p-1}\gamma_j & , \ i = 1, \ldots, p-2, \\ -\Gamma_{p-1}\delta_j & , \ i = p-1. \end{cases}
$$

Also, notice that D_t^j is an impulse dummy which takes the value 1 at $t = T_j^* + 1$ and 0 elsewhere. To write the EC form more compactly define $\Phi = (\mu, \upsilon_{0,1}, \ldots, \upsilon_{p-1,m}, \eta_1, \ldots, \eta_m)$, $X_{DU,t} = \left(1, D_t^1, \ldots, D_{t-p+1}^m, DU_{t-1}^1, \ldots, DU_{t-1}^m\right)'$, $\phi = (\beta'\mu_1, \beta'\gamma_1, \ldots, \beta'\gamma_m)'$, $X_{DT,t} = \left(t, DT_t^1, \ldots, DT_t^m\right)'$, $\Gamma = (\Gamma_1, \ldots, \Gamma_{p-1})$ and $\Delta y_{p+1,t} = (\Delta y_{t-1}, \ldots, \Delta y_{t-p+1})'$. Then equation (2.7) can be rewritten as:

$$
\Delta y_t = \Phi X_{DU,t} + \alpha \left(\beta' y_{t-1} - \phi' X_{DT,t-1}\right) + \Gamma \Delta y_{p+1,t} + \varepsilon_t, t = p+1, \ldots, \tag{2.8}
$$

As in TSL we shall use the VECM for the observed series y_t (equations (2.7) and (2.8)) to obtain first stage estimators for the parameters of the error process x_t, that is, for α, β, Γ_i $(i = 1, \ldots, p-1)$ and Ω. A conventional reduced rank (RR) or estimated generalized least squares (EGLS) regression of Δy_t on $(y_{t-1}, t-1, DT^1_{t-1}, \ldots, DT^m_{t-1})$ corrected for $(1, \Delta y_{t-1}, \ldots, \Delta y_{t-p+1}, D^1_t, \ldots, D^1_{t-p+1}, \ldots, D^m_t, \ldots, D^m_{t-p+1}, DU^1_{t-1}, \ldots, DU^m_{t-1})$ may be used. We adopt the latter method firstly proposed by Ahn and Reinsel (1990) and Saikonnen (1992) since it has some theoretical and practical advantages relative to Johansen's reduced rank maximum likelihood estimation (see Brüggemann and Lütkepohl, 2005, Herwartz and Lütkepohl, 2011, for more details).

2.3 *The estimation method*

Our method uses the VECM in (2.7) and applies feasible GLS to the model (2.1) as it was proposed in TSL to estimate the parameters of the deterministic part in the direction of β and $\beta_\perp$, respectively. Then, we construct a sequence of Wald statistics for a broken trend appropriate for all possible number of stochastic trends we may have in y_t. To see how the estimation method works consider first the case where the process u_t in (2.3) is known to be non stationary and cointegrated with known cointegrating rank $0 < r < n$ and cointegration vectors, β, so that $|A(1)| = 0$ and $\Pi = \alpha\beta'$.

Then the EC form is given in (2.7) whose parameters can be estimated with an estimated generalized least squares (EGLS) regression.

To see how the EGLS works, we concentrate out the short-run adjustment and deterministic components outside the cointegrating relations and consider the concentrated model corresponding to (2.7):

$$R_{\Delta y,t} = \alpha\left(\beta' R_{y,t-1} - \phi' R_{DT,t-1}\right) + e_t$$

where $R_{z,t}$ denotes the residuals from regressing z on $(X_{DU,t}, \Delta y_{p+1,t})'$ for $z = \Delta y, y, DT$. Suppose that α and Ω are known and β is normalized such that:

$$\beta = \begin{bmatrix} I_r \\ \beta_{(k)} \end{bmatrix}$$

Then, the only unknown elements are $\beta_{(k)}$ and ϕ which can be estimated with the application of OLS to the following multivariate linear regression:

$$R_{\Delta y,t} - \alpha R^{(r)}_{y,t-1} = \alpha \left(\beta'_{(k)} R^{(k)}_{y,t-1} - \phi' R_{DT,t-1} \right) + e_t \tag{2.9}$$

where $R^{(r)}_{y,t-1}$ and $R^{(k)}_{y,t-1}$ are defined, respectively, as the first r and the last $k = n - r$ elements of $R_{y,t-1}$. This procedure becomes feasible GLS with consistent first stage estimators of α and Ω. Given the previous normalization, it is readily seen that the first r columns of $\widetilde{\Pi}$ are equal to $\widetilde{\alpha}$ and the usual covariance matrix estimator from the unrestricted LS estimator can be shown to be a consistent estimator of Ω. Hence, for the proposed $\widetilde{\alpha}$ and $\widetilde{\Omega}$, the FGLS estimator of $\beta^{\phi\prime}_{(k)}$ is given by:

$$\widetilde{\beta}^{\widetilde{\phi}\prime}_{(k)} = \left(\widetilde{\alpha}' \widetilde{\Omega}^{-1} \widetilde{\alpha} \right)^{-1} \widetilde{\alpha}' \widetilde{\Omega}^{-1} \left[\sum_{t=1}^{T} \left(R_{\Delta y,t} - \widetilde{\alpha} R^{(r)}_{y,t-1} \right) R^{(k)\prime}_{y,DT,t-1} \right] \left[\sum_{t=1}^{T} R^{(k)}_{y,DT,t-1} R^{(k)\prime}_{y,DT,t-1} \right]^{-1} \tag{2.10}$$

where $\beta^{\phi\prime}_{(k)} = \left(\beta'_{(k)}, \phi' \right)$ and $R^{(k)}_{y,DT,t-1} = \left(R^{(k)}_{y,t-1}, R_{DT,t-1} \right)'$. Now since $\beta'\gamma$ is obtained from the last m columns of matrix ϕ' we can use the corresponding submatrix of $\widetilde{\phi}'$ as an estimator of $\beta'\gamma$.

Now we discuss the estimation of γ in the direction of $\beta'_{\perp}$. One possible simple and fast method applies LS directly to the regression model:

$$\widetilde{\beta}'_{\perp} \Delta y_t = \sum_{j=1}^{m} \widetilde{\beta}'_{\perp} \delta_j D^j_t + \widetilde{\beta}'_{\perp} \mu_1 + \sum_{j=1}^{m} \widetilde{\beta}'_{\perp} \gamma_j DU^j_t + \widehat{\beta}'_{\perp} \Delta u_t, \quad t = 1, \ldots, T \tag{2.11}$$

Since η_t is stationary, the resulting estimator $\widetilde{\beta}'_{\perp} \widetilde{\gamma}$ is consistent for $\beta'_{\perp}\gamma$ and it can be shown with similar arguments from Lütkepohl and Saikkonen (2000b) that it achieves asymptotic normality. However since the error term of (2.11) ignores all possible short-run dynamics we consider additionally an alternative method that fits a finite autoregressive model to u_t and then applies FGLS. Specifically, if equation (2.1) is multiplied from the left by $A(L)$ we have that:

$$A(L)\, y_t = G_{0t}\mu_0 + H_{0t}\mu_1 + \sum_{j=1}^{m} G_{jt}\delta_j + \sum_{j=1}^{m} H_{jt}\gamma_j + \varepsilon_t \tag{2.12}$$

where $y_t = 0$ for $t \leq 0$, $G_{0t} = A(L)a_t$, $H_{0t} = A(L)b_t$, $G_{jt} = A(L)DU_t^j$ and $H_{jt} = A(L)DT_t^j$ for $j = 1, \dots, m$ with

$$a_t = \begin{cases} 0 & \text{, for } t \leq 0, \\ 1 & \text{, for } t \geq 1, \end{cases} \qquad b_t = \begin{cases} 0 & \text{, for } t \leq 0, \\ t & \text{, for } t \geq 1, \end{cases}$$

Moreover, if we define:

$$Q = \left[\Omega^{-1}\alpha \left(\alpha'\Omega^{-1}\alpha\right)^{-\frac{1}{2}} \quad \alpha_\perp \left(\alpha_\perp'\Omega\alpha_\perp\right)^{-\frac{1}{2}} \right] \tag{2.13}$$

It is straightforward to see that:

$$QQ' = \Omega^{-1}\alpha \left(\alpha'\Omega^{-1}\alpha\right)^{-1} \alpha'\Omega^{-1} + \alpha_\perp \left(\alpha_\perp'\Omega\alpha_\perp\right)^{-1} \alpha_\perp' = \Omega^{-1}$$

Now, if we pre-multiply equation (2.12) by Q' the resulting error vector from the transformed multivariate regression vector will have a spherical covariance matrix. Hence, as in GLS estimation with a known covariance structure, we have a transformation which renders a regression model with an error term with standard properties. Since the parameters from the matrix Q and the characteristic polynomial $A(L)$ are not known in practice suitable estimators for α, β, Γ_i $(i = 1, \dots, p-1)$ and Ω are needed. We substitute these estimators for the corresponding theoretical parameters according to (2.6) to obtain estimators of the A_i coefficient matrices, denoted by $\widetilde{A}_i$, $i = 1, \ldots, p$.

Then we define $\widetilde{A}(L) = I_n - \sum_{i=1}^{p} \widetilde{A}_i L^i$, $\widetilde{G}_{0t} = \widetilde{A}(L)\, a_t$, $\widetilde{H}_{0t} = \widetilde{A}(L)\, b_t$, $\widetilde{G}_{jt} = \widetilde{A}(L)\, DU_t^j$ and $\widetilde{H}_{jt} = \widetilde{A}(L)\, DT_t^j$ for $j = 1, \ldots, m$. Finally, the estimator $\widetilde{Q}$ may be obtained if we replace in (2.13) by their respective estimators. Now we can use these estimators to get the feasible form of equation (2.12):

$$\widetilde{Q}'\widetilde{A}(L)\, y_t = \widetilde{Q}'\widetilde{G}_{0t}\mu_0 + \widetilde{Q}'\widetilde{H}_{0t}\mu_1 + \sum_{j=0}^{m} \widetilde{Q}'\widetilde{G}_{jt}\delta_j + \sum_{j=0}^{m} \widetilde{Q}'\widetilde{H}_{jt}\gamma_j + \varsigma_t \tag{2.14}$$

Now, we can use equation (2.14) to obtain the estimator for $\beta_{\perp}'\gamma$.

We use conventional FGLS for the extreme cases $r = 0$ and $r = n$. Suppose now one knows that the process u_t is non stationary and not cointegrated, with $|A(1)| = 0$ and $\Pi = 0$, so that $r = 0$. Then, we construct the feasible characteristic polynomial $\widetilde{A}(L)$ with a first stage estimation of the regression model:

$$\Delta y_t = \mu + \sum_{i=1}^{p-1} \Gamma_i \Delta y_{t-i} + \sum_{j=1}^{m} \sum_{i=1}^{p-1} \upsilon_{i,j} D_{t-i}^j + \sum_{j=1}^{m} \eta_j DU_{t-1}^j + \varepsilon_t, t = p+1, \ldots, \tag{2.15}$$

which is a the particular case of equation (2.7) restricted by $\Pi = \alpha\beta' = 0$. Then the estimators of the coefficient matrices $\widetilde{A}_i$ are recovered analogously using (2.6) with the aforementioned restriction. Using $\widetilde{Q} = \widetilde{\Omega}^{-\frac{1}{2}}$, the estimators $\widetilde{\delta}_j$ and $\widetilde{\gamma}_j$ $(j = 0, \ldots, m)$ are obtained using (2.14). Notice that, in this case, the regressors $\widetilde{G}_{jt}$ $(j = 0, \ldots, m)$ are equal to zero except on a fixed number of p time indices and so behave as an impulse dummy. Furthermore, $\widetilde{H}_{jt}$ $(j = 0, \ldots, m)$ are similar to the constant term and level shift dummies (see expressions (2.31), (2.32), (2.33) in the Mathematical Appendix).

Finally, suppose one knows that the process u_t is stationary, with $|A(1)| \neq 0$, so that $r = n$. The parameter matrices $\widetilde{A}_i$ are obtained estimating the following equation:

$$y_t = \mu_* + \sum_{i=1}^{p} A_i y_{t-i} + \sum_{j=1}^{m} \delta_{j*} DU_t^j + \sum_{j=1}^{m} \gamma_{j*} DT_t^j + \sum_{j=1}^{m} \sum_{i=1}^{p-1} \upsilon_{i,j*} D_{t-i}^j + \varepsilon_t, t = p+1, \ldots, \quad (2.16)$$

where the it is easy to see the relation between μ_*, δ_{j*}, γ_{j*} and $\upsilon_{i,j*}$ and the parameters in (2.7). Setting $\widetilde{Q} = \widetilde{\Omega}^{-\frac{1}{2}}$, the estimators $\widetilde{\mu}_0$, $\widetilde{\mu}_1$, $\widetilde{\delta}_j$ and $\widetilde{\gamma}_j$ $(j = 1, \ldots, m)$ are computed as before with (2.14).

As a matter of notation we use bold letters to denote the *vec* operator applied to a matrix. For example, we have $\widetilde{\boldsymbol{\beta}}'_\perp \widetilde{\boldsymbol{\gamma}}\,(\boldsymbol{\tau}) = vec\left(\widetilde{\beta}'_\perp \widetilde{\gamma}\,(\tau)\right)$ and $\widetilde{\boldsymbol{\beta}}' \widetilde{\boldsymbol{\gamma}}\,(\boldsymbol{\tau}) = vec\left(\widetilde{\beta}' \widetilde{\gamma}\,(\tau)\right)$. The next theorem establishes the asymptotic distribution of estimators of the magnitude of the break assuming that we have specified correctly the number of common stochastic trends, i.e, $k = k^*, \qquad k^* = 0, \ldots, n$:

Theorem 6. *If assumptions 2-5 and u_t has $k^* = n - r^*$ common stochastic trends then, under $H_0 : \gamma_1 = \ldots = \gamma_m = 0$, the asymptotic distribution of the estimators is given by:*

(a)

$$T^{1/2} \widetilde{\boldsymbol{\beta}}'_\perp \widetilde{\boldsymbol{\gamma}}\,(\boldsymbol{\tau}) \Rightarrow \left\{ \left[\int_0^1 RU\,(s,\tau)\, RU\,(s,\tau)'\, ds \right]^{-1} \otimes I_{k^*} \right\} \left\{ \int_0^1 RU\,(s,\tau) \otimes dB_{k^*}^{\beta_\perp}(s) \right\}$$

for $k^* = 1, \ldots, n$.

(b)

$$T^{3/2} \widetilde{\boldsymbol{\beta}}' \widetilde{\boldsymbol{\gamma}}\,(\boldsymbol{\tau}) \Rightarrow \left\{ \left[\int_0^1 RT\,(s,\tau)\, RT\,(s,\tau)'\, ds \right]^{-1} \otimes I_{r^*} \right\} \left\{ \int_0^1 RT\,(s,\tau) \otimes dB_{r^*}^{\alpha}(s) \right\}$$

for $k^* = 0, \ldots, n-1$, where $dB_{k^*}^{\beta_\perp}(s) = \beta'_\perp C \Omega^{\frac{1}{2}} B_n(s)$ and $dB_{r^*}^{\alpha}(s) = \left(\alpha' \Omega^{-1} \alpha\right)^{-1} \alpha' \Omega^{-\frac{1}{2}} B_n(s)$ and $B_n(s)$ is a n-dimensional standard Brownian motion. Here we also have that $RU\,(s,\tau) = (RU\,(s,\tau_1), \ldots, RU\,(s,\tau_m))'$ and $RT\,(s,\tau) = (RT\,(s,\tau_1), \ldots, RT\,(s,\tau_m))'$, where $RU\,(s,\tau_j)$ is the continuous time residual from a projection of $1\,(s > \tau_j)$ onto the space spanned by $\{1\}$ and $RT\,(s,\tau_j)$ is the continuous time residual from a projection of $1\,(s > \tau_j)\,(s - \tau_j)$ onto the space spanned by $\{1, 1\,(s > \tau_1), \ldots, 1\,(s > \tau_m)\}$.

2.4 Testing for Common Breaks in Trend

2.4.1 Known Break Fractions

We start by considering the case where the number of breaks m is fixed and the vector of true break fractions $\tau^* = (\tau_1^*, \tau_2^*, \ldots, \tau_m^*)'$ and hence all the eventual dates when the slope changes occur are known. We consider a Wald type test of no common breaks in the trend ($m = 0$) versus the alternative hypothesis that there are m breaks in (2.1) robust as to whether the stochastic part of the process described by u_t disturbance term is stationary, nonstationary, cointegrated or not cointegrated. The idea underlying the proposed test is to construct a weighted average of Wald statistics appropriate to test the existence of multiple broken trends for each possible case. For a fixed r, let k be the number of common stochastic trends in y_t, i.e., $k = n-r$. For each k we decompose γ from (2.14) in stationary and non stationary directions using $P_{\beta_\perp} + P_\beta = \beta\,(\beta'\beta)^{-1}\beta + \beta_\perp\,(\beta_\perp'\beta_\perp)^{-1}\beta_\perp = I_n$. If H_0 holds so that $\gamma_j = 0$ $(j = 1, \ldots, m)$ we have $\beta'\gamma_j = 0$ and $\beta_\perp'\gamma_j = 0$. On the other hand, under the alternative, it must be that $\beta_\perp'\gamma_j \neq 0$ or $\beta'\gamma_j \neq 0$. Therefore the idea is to test jointly the restrictions $\beta'\gamma_j = 0$ and $\beta_\perp'\gamma_j = 0$ by the Wald principle. Since the estimators $\widetilde{\beta}_\perp'\widetilde{\gamma}$ and $\widetilde{\beta}'\widetilde{\gamma}$ are asymptotically independent as proved in Theorem 7, the test can be written as the sum of the appropriate Wald statistics for testing $\beta'\gamma_j = 0$ and $\beta_\perp'\gamma_j = 0$. Therefore, the $W_k(\tau^*)$ statistic is defined as:

$$
\begin{aligned}
W_k(\tau^*) &= W^k_{\boldsymbol{\beta}_\perp'\boldsymbol{\gamma}}(\tau^*) + W^k_{\boldsymbol{\beta}'\boldsymbol{\gamma}}(\tau^*) \\
&= \left(\widetilde{\boldsymbol{\beta}}_\perp'\widetilde{\boldsymbol{\gamma}}\right)' \left[\widetilde{Avar}(\widetilde{\boldsymbol{\beta}_\perp'\boldsymbol{\gamma}})\right]^{-1} \left(\widetilde{\boldsymbol{\beta}}_\perp'\widetilde{\boldsymbol{\gamma}}\right) + \left(\widetilde{\boldsymbol{\beta}}'\widetilde{\boldsymbol{\gamma}}\right)' \left[\widetilde{Avar}(\widetilde{\boldsymbol{\beta}'\boldsymbol{\gamma}})\right]^{-1} \left(\widetilde{\boldsymbol{\beta}}'\widetilde{\boldsymbol{\gamma}}\right)
\end{aligned}
\qquad (2.17)
$$

$$
\widetilde{Avar}(\widetilde{\boldsymbol{\beta}'\boldsymbol{\gamma}}) = \left[\sum_{t=1}^{T} RT_t\,(\tau)\, RT_t\,(\tau)'\right]^{-1} \otimes (\widetilde{\alpha}'\widetilde{\Omega}^{-1}\widetilde{\alpha})^{-1}
$$

$$
\widetilde{Avar}(\widetilde{\boldsymbol{\beta}_\perp'\boldsymbol{\gamma}}) = \left[\sum_{t=1}^{T} RU_t\,(\tau)\, RU_t\,(\tau)'\right]^{-1} \otimes \left(\widetilde{\beta}_\perp'\widetilde{C}\widetilde{\Omega}\widetilde{C}'\widetilde{\beta}_\perp\right)
$$

with $RT_t(\tau) = (RT_t(\tau_1), \ldots, RT_t(\tau_m))$ and $RU_t(\tau) = (RU_t(\tau_1), \ldots, RU_t(\tau_m))$, where

$RT_t(\tau_j)$ are the residuals from a regression of DT_t^j on $\left(1, t, DU_t^1, \ldots, DU_t^m\right)$ and $RU_t(\tau_j)$ are the residuals from a regression of DU_t^j on $\left(1, D_t^1, \ldots, D_t^m\right)$. Notice that if $k = 0$ then y_t is stationary and so $\beta = I_n$, $\beta_\perp = 0$ which implies that $W_k\left(\tau^*\right) = W_{\beta'\gamma}^k\left(\tau^*\right)$. On the other hand, if $k = n$ then y_t is non stationary and not cointegrated and we conclude that $W_k\left(\tau^*\right) = W_{\beta'_\perp\gamma}^k\left(\tau^*\right)$.

We now establish the asymptotic distribution of $W_k(\tau^*)$ statistics for $k = 0, \ldots, n$:

Theorem 7. *If Assumptions 2-5 hold and u_t has $k^* = n - r^*$ common stochastic trends, i.e, $rank(\Pi) = r^*$ then, under H_0:*

$$W_{k^*}(\tau^*) \xrightarrow{d} \chi^2_{nm}$$

where χ^2_{nm} denotes the chi-square distribution with Nm degrees of freedom.

In view of the above results, and given that number of common stochastic trends (or the number of cointegrating relations) in u_t is not known in practice or, in other words, the $rank(\Pi)$ is unknown, it is a fairly natural step to consider constructing a procedure that employs some auxiliary routine which ensures that, asymptotically at least, the statistic $W_k(\tau^*)$ is selected when u_t has k common stochastic trends or $rank(\Pi) = n - k$, thereby ensuring that the asymptotically optimal test is selected in the limit. To that end we extend the approach of Harvey et al. (2009) and Nunes and Sobreira (2010) and construct data-dependent weighted averages of the sequence of $W_k(\tau^*)$ statistics for $k = 0, \ldots, n$ in the following way:

$$W_\lambda(\tau^*) = \sum_{k=0}^{n} \left[\lambda_k\left(\tau^*\right) - \lambda_{k-1}\left(\tau^*\right)\right] W_k(\tau^*) \tag{2.18}$$

where $\lambda_k\left(\tau^*\right) = \lambda(\xi_{y,k}(\tau^*), \xi_{\Delta y,k}(\tau^*))$ if $k = 0, \ldots, n-1$ and equal to zero if $k = -1$ and to unity if $k = n$. Here, $\xi_{y,k}(\tau^*)$ and $\xi_{\Delta y,k}(\tau^*)$ are auxiliary statistics chosen such that, as the sample size diverges to positive infinity, the difference between weights functions $\lambda_k(.,.) - \lambda_{k-1}(.,.)$ converges to unity when u_t has, in fact, k common stochastic trends ($n - k$ cointegrating relations) and to zero when u_t does not have k common stochastic trends, such that $W_\lambda(\tau^*)$ will collapse to $W_k(\tau^*)$ when u_t has k common stochastic trends.

Because the auxiliary routine needs to be ambivalent between H_0 and H_1, the $\xi_{y,k}(\tau^*)$ and $\xi_{\Delta y,k}(\tau^*)$ statistics must also be invariant with respect to the parameters from the model defined in (2.1)and (2.3). We therefore need to choose appropriate auxiliary statistics, $\xi_{y,k}(\tau^*)$ and $\xi_{\Delta y,k}(\tau^*)$, and weight function, $\lambda(.,.)$. For the former we shall adopt the multivariate KPSS common trends test statistics calculated from the ordered eigenvalues of residuals obtained from regressions of (2.1) and first differenced form of (2.1). Specifically, let $\Lambda_{y,1} \geq \ldots \geq \Lambda_{y,n}$ and $\Lambda_{\Delta y,1} \geq \ldots \geq \Lambda_{\Delta y,n}$ be the ordered eigenvalues of $\Sigma_y^{-1} C_y$ and $\Sigma_{\Delta y}^{-1} C_{\Delta y}$, obtained from $|C_y - \Lambda_{y,j}\Sigma_y| = 0$ and $|C_{\Delta y} - \Lambda_{\Delta y,j}\Sigma_{\Delta y}| = 0$ for $j = 1, \ldots, n$, respectively, where $C_y = T^{-2}\sum_{t=1}^{T}\left[\sum_{i=1}^{t}\widetilde{u}_i\right]\left[\sum_{i=1}^{t}\widetilde{u}_i\right]'$, $\Sigma_y = T^{-1}\sum_{i=1}^{t-1} J\left(i/l\right) \sum_{t=i+1}^{T}\widetilde{u}_t\widetilde{u}_{t-i}'$, $C_{\Delta y} = T^{-2}\sum_{t=1}^{T}\left[\sum_{i=1}^{t}\widetilde{v}_i\right]\left[\sum_{s=1}^{t}\widetilde{v}_i\right]'$ and $\Sigma_{\Delta y} = T^{-1}\sum_{i=1}^{t-1} J\left(i/l\right) \sum_{t=i+1}^{T}\widetilde{v}_t\widetilde{v}_{t-i}'$. $\widetilde{u}_t$ and $\widetilde{v}_t$ are the residuals from the regression of y_t on $\left\{1, t, DU_t^1, \ldots, DU_t^m, DT_t^1, \ldots, DT_t^m\right\}$ and of Δy_t on $\left\{1, D_t^1, \ldots, D_t^m, DU_t^1, \ldots, DU_t^m\right\}$, respectively. In what follows we shall make use of the Bartlett kernel for $J\left(.\right)$, with the data dependent formula proposed by Andrews (1991) for the bandwidth $l = [4(T/100)^{1/4}]$. The $\xi_{y,k}(\tau^*)$ and $\xi_{\Delta y,k}(\tau^*)$ tests are defined as the sum of the $n-k$ smallest eigenvalues, that is,

$$\xi_{y,k}(\tau^*) = \Lambda_{y,k+1} + \ldots + \Lambda_{y,n} \tag{2.19}$$

and

$$\xi_{\Delta y,k}(\tau^*) = \Lambda_{\Delta y,k+1} + \ldots + \Lambda_{\Delta y,n} \tag{2.20}$$

The relevant large sample properties of these two test statistics are given in the following lemma:

Lemma 3. *If Assumptions 2-5 hold, then:*

(a)

$$\xi_{y,k}(\tau^*) = \begin{cases} O_p(1), \text{ if } k \geq k^* \\ O_p(T/l), \text{ if } k < k^* \end{cases}$$

(b)

$$\xi_{\Delta y,k}(\tau^*) = \begin{cases} O_p(l/T), \text{ if } k \geq k^* \\ O_p(1), \text{ if } k < k^* \end{cases}$$

The results in Lemma 3 suggest a weight function of the form:

$$\lambda_k\left(\tau^*\right) = \lambda(\xi_{y,k}(\tau^*), \xi_{\Delta y,k}(\tau^*)) := \exp[-\{g\xi_{y,k}(\tau^*)\xi_{\Delta y,k}(\tau^*)\}^6] \tag{2.21}$$

where g is a positive constant, since this will converge to unity if $k \geq k^*$ and to zero if $k < k^*$. The following corollary summarizes the asymptotic properties of the weight function $\lambda(\xi_{y,k}(\tau^*), \xi_{\Delta y,k}(\tau^*))$ and of $W_\lambda(\tau^*)$ test statistic.

Corollary 9. *If u_t has $0 \leq k^* \leq n$ common stochastic trends then:*

$$\lambda_k\left(\tau^*\right) = \lambda(\xi_{y,k}(\tau^*), \xi_{\Delta y,k}(\tau^*)) \xrightarrow{p} \begin{cases} 1, \text{ if } k \geq k^* \\ 0, \text{ if } k < k^* \end{cases} \tag{2.22}$$

Also we have that $W_\lambda(\tau^*) = W_{k^*}(\tau^*) + o_p\left(1\right) \xrightarrow{d} \chi^2_{nm}$ for any $0 \leq k^* \leq n$.

Remark 12. The results from Corollary 9 show that, regardless of the number of stochastic trends, $W_\lambda(\tau^*)$ is asymptotically equivalent to $W_{k^*}(\tau^*)$, i.e, the Wald statistic appropriate for testing broken trends if one knew the number of stochastic trends in our multivariate time-series. This occurs with the aid of the weight function (2.21) which ensures that the asymptotically optimal Wald test is selected in the limit. Furthermore, under H_0 W_λ statistic converges to a chi-square distribution with Nm degrees of freedom and so this test is easily implemented using the critical values from this distribution.

Remark 13. Since the difference $\lambda_k - \lambda_{k-1}$ converges in probability to unity if $k = k^*$ and to zero when $k \neq k^*$ at an exponential rate in T, i.e., at a faster rate than any finite polynomial rate, each individual term $\{\lambda_k - \lambda_{k-1}\} W_k, k \neq k^*$ is asymptotically negligible even if W_k diverges in probability at a polynomial rate.

2.4.2 Unknown Break Fractions

In this section, we are interested in testing for common broken trends in equation (2.1) in cases where the change points $\tau = (\tau_1, \ldots, \tau_m)$ are unknown. This testing problem does not fit into the standard testing framework since the unknown parameter τ is only present under the alternative and not under the null. We follow the approach of Andrews (1993), Andrews and Ploberger (1994) and extended by Bai et al. (1998), Bai (2000) and Qu and Perron (2007) to the multivariate setting. Our testing procedure is based on the supremum of the sequence of $W_k(\tau)$ statistics for testing H_0 for $\tau_j = \tau_j^L, \ldots, \tau_j^U$, $(j = 1, \ldots, m)$:

$$W_k = \sup_{\tau \in \Lambda_\epsilon^m} W_k(\tau) \tag{2.23}$$

where $\Lambda_\epsilon^m = \{(\tau_1, \ldots, \tau_m) : \tau_1 \geq \epsilon, \tau_m \leq 1 - \epsilon, |\tau_{j+1} - \tau_j| \geq \epsilon\}$ and it is assumed that $\tau^* \in \Lambda_\epsilon^m$.To solve the problem of unknown number of common stochastic trends we follow the same strategy as the known breaks fraction case and write the analogue of test statistic W_λ:

$$W_\lambda = \sum_{k=0}^{n} \left[\lambda_k(\widehat{\tau}, \widetilde{\tau}) - \lambda_{k-1}(\widehat{\tau}, \widetilde{\tau})\right] W_k \tag{2.24}$$

Here, the sequence of multivariate KPSS statistics are now replaced by $\xi_{y,k}(\widehat{\tau}) = \inf_{\tau \in \Lambda_\epsilon^m} \xi_{y,k}(\tau)$ and $\xi_{\Delta y,k}(\widetilde{\tau}) = \inf_{\tau \in \Lambda_\epsilon^m} \xi_{\Delta y,k}(\tau)$. To derive the asymptotic behavior of W_λ, we must study the large sample behavior of the weight function and the W_k statistics. The next theorem establishes the asymptotic distribution of individual W_k assuming that we made right guess on the number of common stochastic trends in the multivariate system or, more succinctly, if $k = k^*$:

Theorem 8. *If assumptions 2-5 and u_t has $k^* = n - r^*$ common stochastic trends then, under $H_0 : \gamma_1 = \ldots = \gamma_m = 0$, then the asymptotic distributions of the Wald Tests are the following:*

(a)

$$W^{k^*}_{\beta'_{\perp}\gamma}(\tau) \Rightarrow \left\{ \int_0^1 RU\left(s,\tau\right) \otimes dB_{k^*}\left(s\right) \right\}' \left\{ \left[\int_0^1 RU\left(s,\tau\right) RU\left(s,\tau\right)' ds \right]^{-1} \otimes I_{k^*} \right\}$$
$$\left\{ \int_0^1 RU\left(s,\tau\right) \otimes dB_{k^*}\left(s\right) \right\} \equiv J^m_{k^*}\left(\tau\right)$$

for $k^* = 1, \ldots, n$

(b)

$$W^{k^*}_{\beta'\gamma}(\tau) \Rightarrow \left\{ \int_0^1 RT\left(s,\tau\right) \otimes dB_{r^*}\left(s\right) \right\}' \left\{ \left[\int_0^1 RT\left(s,\tau\right) RT\left(s,\tau\right)' ds \right]^{-1} \otimes I_{r^*} \right\}$$
$$\left\{ \int_0^1 RT\left(s,\tau\right) \otimes dB_{r^*}\left(s\right) \right\} \equiv J^m_{r^*}\left(\tau\right)$$

for $k^* = 0, \ldots, n-1$, where $\left\{B_{r^*}\left(s\right)', B_{k^*}\left(s\right)'\right\}$ is a n-dimensional vector of independent standard Brownian Motion processes.

(c)

$$W_{k^*} \Rightarrow \sup_{\tau \in \Lambda^m_\epsilon} J^m_{k^*}\left(\tau\right) + J^m_{r^*}\left(\tau\right)$$

The fixed τ representations of the asymptotic distribution of $\widetilde{\beta}'_{\perp}\widetilde{\gamma}$, $\widetilde{\beta}'\widetilde{\gamma}$, $W_{\beta'_{\perp}\gamma}(\tau)$, $W_{\beta'\gamma}(\tau)$ are shown in Theorem 7. Since the sup function is continuous, the stated result in part *(c)* of Theorem 8 follows directly with the application of the Continuous Mapping Theorem (CMT). Next, we obtain the large sample behavior of the auxiliary statistics $\xi_{y,k}(\widehat{\tau})$ and $\xi_{\Delta y,k}(\widetilde{\tau})$ when the stochastic part of the Model has $0 \leq k \leq n$ stochastic trends. The continuous mapping theorem applied to the inf function and fixed τ representations of the asymptotic distributions of the $\xi_{y,k}\left(\tau\right)$ and $\xi_{\Delta y,k}\left(\tau\right)$ presented in Busetti (2002) and Nyblom and Harvey (2000) (that are trivial to extend to our multiple breaks setting) allow us to show that the rates of convergence of these statistics are the same as in the known break fraction case:

Lemma 4. *Let assumptions 2-5 hold. If the magnitude of the shifts $\delta_{s,j}$ and $\gamma_{s,j}$ decreases to zero at faster rates than $T^{-1/2}$ for $\gamma_{1,j}, \ldots, \gamma_{k^*,j}, \delta_{k^*+1,j}, \ldots, \delta_{n,j}$ and $T^{-3/2}$ for $\gamma_{k^*+1,j}, \ldots, \gamma_{n,j}$ then we have that*

(a)

$$\xi_{y,k}(\widehat{\tau}) = \begin{cases} O_p(1), \ \textit{if } k \geq k^* \\ O_p(T/l), \ \textit{if } k < k^* \end{cases}$$

(b)

$$\xi_{\Delta y,k}(\widetilde{\tau}) = \begin{cases} O_p(l/T), \ \textit{if } k \geq k^* \\ O_p(1), \ \textit{if } k < k^* \end{cases}$$

Since the multivariate KPSS statistics converge in probability at the same rate as in Section 2.4.1, one readily obtains that the weight function $\lambda\left(\xi_{y,k}(\widehat{\tau}), \xi_{\Delta y,k}(\widetilde{\tau})\right) \xrightarrow{p} 0$ if $k < k^*$ and $\lambda\left(\xi_{y,k}(\widehat{\tau}), \xi_{\Delta y,k}(\widetilde{\tau})\right) \xrightarrow{p} 1$ if $k \geq k^*$ as in Corollary 9. Consequently, from this fact and Theorem 8 we are in position to establish the asymptotic distribution of the weighted Wald statistic, W_λ:

Corollary 10. *Let assumptions 2-5 and $H_0 : \gamma_1 = \ldots = \gamma_m = 0$ hold. If u_t has $0 \leq k^* \leq n$ common stochastic trends then:*

$$\lambda(\xi_{y,k}(\widehat{\tau}), \xi_{\Delta y,k}(\widetilde{\tau})) \xrightarrow{p} \begin{cases} 1, \ \textit{if } k \geq k^* \\ 0, \ \textit{if } k < k^* \end{cases} \tag{2.25}$$

Also we have that $W_\lambda = W_{k^*} + o_p(1) \Rightarrow \sup_{\tau \in \Lambda_\epsilon^m} J_{k^*}^m(\tau) + J_{r^*}^m(\tau)$ for any $0 \leq k^* \leq n$.

Notice that contrary to the known break fraction case, the asymptotic distribution of W_{k^*} is different if u_t has k^* common stochastic trends for $k^* = 0, \ldots, n$ and no longer converges to a chi-square distribution with mN degrees of freedom. In this case using the same reasoning as Vogelsang (1998), we could choose a constant such that, for a given significance level ψ under H_0, the critical values to be used in the testing procedure become the same irrespective of the number of stochastic trends, k^*, present in y_t. However, after simulating the critical values of $\sup_{\tau\in\Lambda_\epsilon^m} J_{k^*}^m(\tau) + J_{r^*}^m(\tau)$ across $0 \leq k^* \leq n$ we found these to be very similar and, hence, from a practical point of view this fact won't make difference on the finite sample behavior of the test. Critical values for the asymptotic distributions of the W_λ statistic were obtained via simulations. The vector standard Brownian Motion $B(s)$ is approximated with partial sums $\sum_{i=1}^{[Ts]} \epsilon_i$ where ϵ_i is i.i.d. $\mathcal{N}(0, I_n)$ for $T = 1000$ and 5000 replications. In table 2.1 we present critical values for the 1 trend break case ($m = 1$) up to 8 dependent variables ($n = 2, \ldots, 8$).

2.5 *A test of l versus $l+1$ common broken trends*

As in Qu and Perron (2007) and Kejriwal and Perron (2010) we extend our methodology to a test of the null hypothesis of l common broken trends against the alternative of $l+1$ breaks. This test allow us to build a sequential procedure that can be used to determine the number of trend breaks in our system of equations. The test is implemented as follows. First we obtain the estimates of the break dates $\left(\widetilde{T}_1, \ldots, \widetilde{T}_l\right)$ as maximizers of the log-likelihood function under the hypothesis of l breaks in the trend for the model in levels which is equivalent to have:

$$\left(\widetilde{T}_1, \ldots, \widetilde{T}_l\right) = \arg \inf_{\tau\in\Lambda_\epsilon^l} \log \left|\Sigma_y\left(T_1, \ldots, T_l\right)\right|$$

with $\Sigma_y\left(T_1, \ldots, T_l\right) = T^{-1} \sum_{t=1}^{T} \widetilde{u}_t \widetilde{u}_t'$ where the residuals $\widetilde{u}_t$ are obtained from the estimated equation (2.1) with the dummy variables evaluated at dates $(T_1, \ldots, T_l)$. Next, we proceed by testing for the presence of an additional break in each of the $(l+1)$ segments obtained with the estimated partition $\left(\widetilde{T}_1, \ldots, \widetilde{T}_l\right)$. In particular, for each segment $s = 1, \ldots, l+1$ we estimate the VECM by EGLS and the model in levels by FGLS in the direction of $\beta_\perp'$ as described in equations (2.9) and (2.14),respectively. The regression equations are then given, respectively, by:

$$R_{\Delta y,t} - \alpha^{(s)} R_{y,t-1}^{(r)} = \alpha^{(s)} \left(\beta_{(k)}^{(s)\prime} R_{y,t-1}^{(k)} - \phi^{(s)\prime} R_{DT,t-1}\right) + e_t$$

and

$$\widetilde{Q}'\widetilde{A}(L)\,y_t = \widetilde{Q}'\widetilde{G}_{0t}\mu_0^{(s)} + \widetilde{Q}'\widetilde{H}_{0t}\mu_1^{(s)} + \widetilde{Q}'\widetilde{G}_{1t}\delta^{(s)} + \widetilde{Q}'\widetilde{H}_{1t}\gamma^{(s)} + \varsigma_t$$

for $t = \widetilde{T}_{s-1} + 1 \ldots \widetilde{T}_s$ with $\widetilde{T}_0 = 0$ and $\widetilde{T}_{l+1} = T$ for every $k = 0, \ldots, n$. Here $R_{\Delta y,t}$, $R_{y,t}^{(r)}$, $R_{y,t}^{(k)}$, $R_{DT,t}$, $\widetilde{G}_{j,t}$ and $\widetilde{H}_{j,t}$, $j = 0, 1$ are as defined in Section 2.3 but with t replaced $(t - \widetilde{T}_{s-1})$. All the estimators are obtained using the subsample from observation $t = \widetilde{T}_{s-1} + 1$ to $\widetilde{T}_s$.

The test now amounts to testing the null hypothesis of no break in the slope of the trend function $H_0 : \gamma^{(s)} = 0$ against the alternative of a single break $H_1 : \gamma^{(s)} \neq 0$ in each segment $s = 1, \ldots, l+1$ with an unknown break date. We conclude in favour of the $l+1$ changes if the overall maximum value of the $(l+1)$ Wald statistics is sufficiently high. The Wald test statistic for a fixed break date ζ is then given by:

$$W_k^{(s)}(\widetilde{T}_{l-1}, \zeta, \widetilde{T}_l) = \left(\widetilde{\boldsymbol{\beta}}^{(s)\prime}\widetilde{\boldsymbol{\gamma}}^{(s)}\right)' \left[\widetilde{Avar}\left(\widetilde{\boldsymbol{\beta}^{(s)\prime}\boldsymbol{\gamma}^{(s)}}\right)\right]^{-1} \left(\widetilde{\boldsymbol{\beta}}^{(s)\prime}\widetilde{\boldsymbol{\gamma}}^{(s)}\right) \tag{2.26}$$

$$+ \left(\widetilde{\boldsymbol{\beta}}_{\perp}^{(s)\prime}\widetilde{\boldsymbol{\gamma}}^{(s)}\right)' \left[\widetilde{Avar}\left(\widetilde{\boldsymbol{\beta}_{\perp}^{(s)\prime}\boldsymbol{\gamma}^{(s)}}\right)\right]^{-1} \left(\widetilde{\boldsymbol{\beta}}_{\perp}^{(s)\prime}\widetilde{\boldsymbol{\gamma}}^{(s)}\right) \tag{2.27}$$

$$\widetilde{Avar}(\widetilde{\boldsymbol{\beta}^{(s)\prime}\boldsymbol{\gamma}^{(s)}}) = \left[\sum_{t=\widetilde{T}_{l-1}+1}^{\widetilde{T}_l} RT_t^{(s)}(\zeta)\, RT_t^{(s)}(\zeta)'\right]^{-1} \otimes (\widetilde{\alpha}^{(s)\prime}\widetilde{\Omega}^{(s)-1}\widetilde{\alpha}^{(s)})^{-1}$$

$$\widetilde{Avar}(\widetilde{\boldsymbol{\beta}_{\perp}^{(s)\prime}\boldsymbol{\gamma}^{(s)}}) = \left[\sum_{t=\widetilde{T}_{l-1}+1}^{\widetilde{T}_l} RU_t^{(s)}(\zeta)\, RU_t^{(s)}(\zeta)'\right]^{-1} \otimes \left(\widetilde{\beta}_{\perp}^{(s)\prime}\widetilde{C}^{(s)}\widetilde{\Omega}^{(s)}\widetilde{C}^{(s)\prime}\widetilde{\beta}_{\perp}^{(s)\prime}\right)$$

The sequential test is then defined as the maximum of the $W^{(s)}(\widetilde{T}_{l-1}, \zeta, \widetilde{T}_l)$ over all $s = 1, \ldots, l+1$:

$$W_k(l+1|l) = \max_{1 \le s \le l+1} \sup_{\zeta \in \Lambda_{s,\epsilon}} W_k^{(s)}(\widetilde{T}_{s-1}, \zeta, \widetilde{T}_s)$$

where the possible eligible break dates are contained in the following set:

$$\Lambda_{s,\epsilon} = \left\{\zeta : \widetilde{T}_{s-1} + \left(\widetilde{T}_s - \widetilde{T}_{s-1}\right)\epsilon \le \zeta \le \widetilde{T}_s - \left(\widetilde{T}_s - \widetilde{T}_{s-1}\right)\epsilon\right\}$$

The next theorem establishes the asymptotic distribution of $W^{k^*}(l+1|l)$, that is, the sequential Wald statistic that would be appropriate to use if we knew the true number of common stochastic trends ($k = k^*$):

Theorem 9. *If assumptions 2-5 and u_t has $k^* = n - r^*$ common stochastic trends then, under the null that there are $m = l$ breaks, we have $\lim_{T\to\infty} P\left(W_{k^*}\left(l+1|l\right) \leq x\right) = G_{k^*,\epsilon}\left(x\right)^{l+1}$ where $G_{k^*\epsilon}\left(x\right)$ is the distribution function of $\sup_{\tau\in\Lambda_\epsilon^m} J_{k^*}^m\left(\tau\right) + J_{r^*}^m\left(\tau\right)$ for $m = 1$.*

Since in general the number of common stochastic trends, k^* is not known, the practical implementation of this test is rather limited. Hence, as in sections 2.4.1 and 2.4.2 we built data dependent weighted averages of the $W_k\left(l+1|l\right)$ statistics for $k = 0, \ldots, n$ that ensure that the appropriate sequential test statistic is selected at least asymptotically. Therefore, the $W_\lambda\left(l+1|l\right)$ statistic is given by:

$$W_\lambda\left(l+1|l\right) = \sum_{k=0}^{n}\left[\lambda_k\left(\widehat{\tau}^{l+1}, \widetilde{\tau}^{l+1}\right) - \lambda_{k-1}\left(\widehat{\tau}^{l+1}, \widetilde{\tau}^{l+1}\right)\right] W_k\left(l+1|l\right)$$

where $\widehat{\tau}^{l+1} = (\widehat{\tau}_1, \ldots, \widehat{\tau}_{l+1}) = \arg\inf_{\tau\in\Lambda_\epsilon^{l+1}} \xi_{y,k}(\tau)$ and $\widetilde{\tau}^{l+1} = (\widetilde{\tau}_1, \ldots, \widetilde{\tau}_{l+1}) = \arg\inf_{\tau\in\Lambda_\epsilon^{l+1}} \xi_{\Delta y,k}(\tau)$. Since under H_0 we have $m = l$ it is readily seen that the asymptotic behaviour of the multivariate KPSS statistics $\xi_{y,k}(\widehat{\tau}^{l+1})$ and $\xi_{\Delta y,k}(\widetilde{\tau}^{l+1})$ and the weight functions $\lambda_k\left(\widehat{\tau}^{l+1}, \widetilde{\tau}^{l+1}\right)$ for $k = 0, \ldots, n$ is precisely the same as described in Lemma 4 and Corollary 10. Hence, we may state the following corollary that describes the asymptotic behaviour of $W_\lambda\left(l+1|l\right)$:

Corollary 11. *If assumptions 2-5 hold and u_t has k^* common stochastic trends then, under the null that there are $m = l$ breaks:*

$$\lambda(\xi_{y,k}(\widehat{\tau}^{l+1}), \xi_{\Delta y,k}(\widetilde{\tau}^{l+1})) \xrightarrow{p} \begin{cases} 1, \text{ if } k \geq k^* \\ 0, \text{ if } k < k^* \end{cases} \tag{2.28}$$

and we have $\lim_{T\to\infty} P\left(W_\lambda\left(l+1|l\right) \leq x\right) = G_{k^*,\epsilon}\left(x\right)^{l+1}$ where $G_{k^*,\epsilon}\left(x\right)$ is the distribution function of $\sup_{\tau\in\Lambda_\epsilon^m} J_{k^*}^m\left(\tau\right) + J_{r^*}^m\left(\tau\right)$ for $m = 1$.

The results in Corollary 11 show that critical values for the sequential tests can be computed from the quantiles of the asymptotic distribution of the $W_*^{k^*}$ statistic for the case of just one break ($m = 1$). In table 2.2 we present critical values $W_\lambda (l + 1|l)$ test for $l = 0, \dots, 4$ up to 8 dependent variables ($n = 2, \dots, 8$). The $W_\lambda (l + 1|l)$ can, then, be used to estimate the number of common broken deterministic trends without making any assumption about the error process being $I(0)$ or $I(1)$ cointegrated or not cointegrated. The procedure starts with $l = 0$, by using $W_\lambda (1|0)$ to test for the presence of one break. If the null hypothesis is rejected, we set $l = 1$ and perform the $W_\lambda (2|1)$. The procedure is repeated in a similar fashion until the $W_\lambda (l + 1|l)$ cannot reject the null hypothesis of l breaks. The estimated number of breaks is then obtained as the number of rejections. This sequential procedure can be made consistent with the same arguments as in Hosoya (1989) by adopting a significance level for the test $W_\lambda (l + 1|l)$ that decreases to zero, at a suitable rate, as the sample size increases.

2.6 *Finite Sample Simulations*

In this section we provide results of several Monte Carlo simulations. All the results were computed over 5000 replications using the rndn pseudo random number generator in Gauss. The trimming parameter ϵ is set equal to 0.15. Asymptotic critical values were obtained with discrete approximations ($T = 1000$) of the asymptotic distributions. We report the critical values, with different significant levels denoted by ψ, for the test of the null of no break against the alternative of a broken trend, with the break date unknown on Table 2.1. To apply these tests we need to choose constant g from the weight function. After studying the finite sample behaviour of the sequence of weight functions, $\{\lambda_k (\widehat{\tau}, \widetilde{\tau})\}_{k=0}^{n}$ with several Monte Carlo simulations we found that the rule $g_{k,n,m} = (500 + 750 (m - 1)) \dfrac{(n + k)}{(n - k)^2}$ was the best overall and presents decent finite sample size and power over the range of experiments considered. More specifically, to analyze the power and size properties we use 5000 simulations with different number of observations ranging from $T = 100$ to $T = 1000$ derived from the following general DGPs:

$$y_t = \mu_0 + \mu_1 t + \sum_{j=1}^{m} \delta_j DU_t^j + \sum_{j=1}^{m} \gamma_j DT_t^j + u_t$$

Throughout, we set μ_0 and μ_1 to zero ($\mu_0 = \mu_1 = 0$) since the test results are invariant to the true parameter values of the intercept and trend terms. The disturbance term, u_t, is based on Toda (1994, 1995) where he has shown that the following process can be seen as a canonical form for investigating the properties of LR type cointegration tests:

$$u_t = \begin{bmatrix} \rho \mathbf{I}_{n-k^*} & \mathbf{0}_{k^*} \\ \mathbf{0}_{n-k^*} & \mathbf{I}_{k^*} \end{bmatrix} u_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}_n\left(\mathbf{0}_n, \mathbf{I}_n\right) \tag{2.29}$$

Since both the Wald test and the multivariate KPSS statistic are invariant to affine linear transformations, that is, $y_t \mapsto Py_t + a$ this class of DGPs also represent a canonical form to study the properties of W_λ test statistic.

Table 2.3 reports size ($\delta_j = \gamma_j = 0$) for $j = 1, \ldots, m$ of $W_\lambda(\tau^*)$ test statistic for the known break fraction case with $\tau^* = 0.5$. We used the values 0, 0.4 and 0.8 for the autoregressive parameter, ρ, of the $I(0)$ time series. Naturally, for the k^* $I(1)$ time series the autoregressive parameter is set to 1 as it can be seen in (2.29). We set the dimension of the y_t vector as $n = 3$ and considered all possible number of common stochastic trends $k^* = 0, \ldots, n$. Hence, for $k^* = n$, u_t is a pure multivariate random walk whereas if $k^* = 0$, u_t is a multivariate I(0) process where each element of the vector y_t is and AR(1) with autoregressive parameter ρ. In the case of pure $I(0)$ shocks, we observe that the W_λ statistic tends to be somewhat undersized, specially, for a small number of observations ($T = 100$). However, as we increase the sample size the empirical rejection frequencies get closer to the 5% significance level. Conversely, if u_t is a pure non cointegrated $I(1)$ process the test is slightly oversized. The same oversizing pattern occurs if the shocks are $I(1)$ and cointegrated (except for $T = 100$) and this effect is specially pronounced as we increase the persistence of the $I(0)$ components of y_t. However, we see these distortions as finite sample effects because as we increase the sample size the the oversizing magnitude decreases and approaches the 5% rejection frequency.

Table 2.4 reports rejection frequencies of W_λ test statistic for the unknown break fraction case. We simulated bivariate ($n = 2$) and trivariate ($n = 3$) processes generated according to (2.29). We used values of 0 and 0.5 for the autoregressive parameter of

$n - k^*$ I(0) time series and we considered sample sizes of $T = 100$ and $T = 200$. We set $\gamma = \delta = 0$ and $\gamma = \frac{\delta}{2} = \{0.2, 0.4\}$ to analyze, respectively, finite sample size and power of the test. In general, the behaviour of the empirical size is qualitatively similar to the known break case with underrejections in cases where the shocks are I(0) and a sudden shift from an undersized to an oversized test as the sample increased from 100 to 200 observations for $k^* = 1, \ldots, n-1$. However, it is important to realize that in most cases the degree of size distortions is higher than in the known break fraction case. We provide some results regarding power and without showing more detailed results we observe a general rule: the test is much more effective in detecting the existence of breaks in the trend if they occurred in the stationary time series than if they occurred in the I(1) region. This should come as no surprise as the order of probability of the estimator of the magnitude of the break γ is much higher in the $I(0)$ region than in the $I(1)$ region as it can be readily seen in Theorem 6.

2.7 *Empirical Application*

The construction of accurate and cross-country comparable top income share estimates attracted a considerable amount of attention in the economic inequality literature during the last decade. The share of total income concentrated on the richest people is now considered to be a reasonable proxy for income inequality due to the observed strong and positive correlation between the top income share and other measures of income inequality as, for example, relative property. Furthermore, changes in distribution in top income share may also have very important socio-economic implications: for example, a higher concentration may increase the influence of the richest class on political outcomes and on important decisions from major industry sectors. It may also generate "expenditure cascades" of the middle class as the median income people change the positional goods bundle that they consider "adequate" to keep their social stating. Hence the important consequences of changing top income shares justify an extensive research agenda both to understand the main causes and quantifying the socioeconomic impact of these movements.

In a recent paper Roine and Waldenström (2011) added a valuable contribution to this literature. They conducted an extensive study about the number and timing of structural breaks in the trend of top income shares across eighteen countries. The authors identified and distinguished break dates common to all countries, common to groups of countries and specific to each individual country. The empirical analysis used Bai and Perron (1998) method to detect country specific breaks in the trend and Qu and Perron (2007) methodology (henceforth PQ) to detect structural changes occurring simultaneously on all countries and across groups. These 2 methods preclude trending and unit root regressors. Now if we compare our test with PQ critical values we find very similar values and so it seems that, in practice, we can detect accurately changes in trend with PQ algorithm. However, if we do not take into account (non-)stationarity properties of the driving shocks then this may lead to structural breaks tests to have very poor size and power properties as discussed in previous sections.

Hence, in this section we use exactly the dataset from Roine and Waldenström (2011) (data are generously provided on Waldenstrom's website). Table 1 from that paper enumerates the sources used to collect data for different countries. These sources essentially obtain income shares through national level personal income tax returns. For each country, data are topically drawn from income tax tabulations that report for a large number of different income groups the corresponding number of tax payers, total income and tax liability. Then, the standard practice is to assume that data follow a Pareto distribution and use interpolation techniques to produce top income series. A discussion of different methodologies by which this can be done can be found in Atkinson and Piketty (2007). We analyze natural logarithm of top percentile income shares.

This dataset is used to identify breaks in the trend function that are common for groups and for all countries under our proposed econometric sequential testing methodology. Our objective is to complement their empirical analysis both confirming and strengthening some of their important findings and highlighting possible important differences on our results that may be attributed to common cross section stochastic trends observed in the data.

The main results of PQ and our sequential W_λ test statistic applied to country groups top 1% income share are given in Table 2.5. The break dates estimated under both methodologies are superimposed on the plots of Figure 2.1. According to PQ test, there is statistical evidence at the 5% level to find 2 trend breaks for the continental group and 3 trend breaks for the asian group. However, the sequential W_λ statistic cannot confirm these structural breaks for both country groups. The results illustrate the importance of taking into account the number of stochastic trends if they exist. These idea is reinforced when we look at differences on the results from other country groups: The results from the PQ test show strong evidence for 3 trend breaks in anglo-saxon and nordic group but we only find evidence for 1 trend break according to the W_λ statistic. We also report the values of PQ and W_λ statistics for testing the null of 0 against 1 break in trend. Here we observe that the value of the PQ statistic is substantially higher than the W_λ statistic except for the nordic group. Thus caution should be taken in interpreting certain events as causing common structural breaks in trend of top income shares. In fact, according to both methodologies the second World War and the first oil price shock can be regarded as exogenous shocks, respectively, for nordic and anglo-saxon groups. However, in contrast to PQ, the W_λ test statistic reveals that other estimated trend breaks in all country groups should be regarded as shocks or combination of shocks from the errors of the underlying data-generating process of top income shares.

2.8 *Conclusion*

In this paper we presented tests for the presence of multiple structural change in the trend slope of a multivariate time series which do not require knowledge of the form of serial correlation in the data and are valid regardless of the vector of shocks being $I(0)$, $I(1)$, cointegrated or not cointegrated. We have considered a Disjoint Broken Trends Model. We have extended the test procedure proposed by Harvey et al. (2009) to multivariate setting and constructed a weighted average of a sequence of Wald statistics appropriate for testing the existence of breaks in trend if one knows the number of stochastic trends in the data. We start by considering the case in which the empirical researcher is sure about the break dates if there is any structural change in the trend function and proved that the proposed test has a chi-square limiting distribution, regardless of the number of stochastic trends. Next, we propose tests for known number of trend breaks but unknown break dates under the alternative. Here, the estimated break dates are global maximizers of the sequence of Wald statistics evaluated at all admissible partitions. We also proposed a sequential procedure that may be used to estimate the number of breaks along the lines of Qu and Perron (2007). We analyzed Monte Carlo evidence to study finite sample properties of the proposed tests.

Proof of Theorem 6. (a) The proof of this part of the theorem is similar to the proof of Lemma 3.1 from TSL. First, notice that all relevant quantities are invariant to normalizations of $\widetilde{\alpha}$ and $\widetilde{\beta}$, so we can assume some kind of normalization and use the following results (see Ahn and Reinsel, 1990, Saikkonen, 1992, Paruolo, 2002, for example):

$$\begin{array}{lll} \widetilde{\alpha} = \alpha + O_p\left(T^{-\frac{1}{2}}\right) & \widetilde{\beta} = \beta + O_p\left(T^{-1}\right) & \widetilde{\beta}_\perp = \beta_\perp + O_p\left(T^{-1}\right) \\ \widetilde{\Gamma}_i = \Gamma_i + O_p\left(T^{-\frac{1}{2}}\right) & \widetilde{\Omega} = \Omega + O_p\left(T^{-\frac{1}{2}}\right) & \end{array} \tag{2.30}$$

From the definitions of $\widetilde{G}_{jt}$ and $\widetilde{H}_{jt}$, for $j = 0, \ldots, m$, we have that:

$$\widetilde{G}_{0t} = \begin{cases} I_n, \text{ if } t = 1 \\ I_n - \sum_{j=1}^{t-1} \widetilde{A}_j, \text{ if } t = 2, \ldots, p \\ -\widetilde{\alpha}\widetilde{\beta}', \text{ if } t = p+1, \ldots, T \end{cases} \qquad \widetilde{H}_{0t} = \begin{cases} I_n, \text{ if } t = 1 \\ tI_n - \sum_{j=1}^{t-1} (t-j)\widetilde{A}_j, \text{ if } t = 2, \ldots, p \\ \widetilde{\Psi} - (t-1)\widetilde{\alpha}\widetilde{\beta}', \text{ if } t = p+1, \ldots, T \end{cases} \tag{2.31}$$

$$\widetilde{G}_{jt} = \begin{cases} 0, \text{ if } t < T_j^* \\ I_n, \text{ if } t = T_j^* \\ I_n - \sum_{J=1}^{t-T_j^*} \widetilde{A}_j, \text{ if } t = T_j^* + 1, \ldots, T_j^* + p - 1 \\ -\widetilde{a}\widetilde{\beta}', \text{ if } t = T_j^* + p, \ldots, T \end{cases} \tag{2.32}$$

$$\widetilde{H}_{jt} = \begin{cases} 0, \text{ if } t < T_j^* \\ I_n, \text{ if } t = T_j^* \\ I_n - \sum_{J=1}^{t-T_j^*} \left(t - T_j^* + 1 - j\right) \widetilde{A}_j, \text{ if } t = T_j^* + 1, \dots, T_j^* + p - 1 \\ \widetilde{\Psi} - \left(t - T_j^*\right) \widetilde{\alpha}\widetilde{\beta}', \text{ if } t = T_j^* + p, \dots, T \end{cases} \tag{2.33}$$

for $j = 1, \dots, m$. The idea is to consider the asymptotic properties of the estimators in the direction of $\beta_\perp$. For this purpose, consider the parameter vectors $\underline{B}_1 = \left\{\widetilde{\beta}'\mu_0, \widetilde{\beta}'\delta_1, \dots, \widetilde{\beta}'\delta_m, \widetilde{\beta}'\mu_1, \widetilde{\beta}'\gamma_1, \dots, \widetilde{\beta}'\gamma_m\right\}'$, $\underline{B}_2 = \left\{\widetilde{\beta}'_\perp\mu_1, \widetilde{\beta}'_\perp\gamma_1, \dots, \widetilde{\beta}'_\perp\gamma_m\right\}'$ and $\underline{B}_3 = \left\{\widetilde{\beta}'_\perp\mu_0, \widetilde{\beta}'_\perp\delta_1, \dots, \widetilde{\beta}'_\perp\delta_m\right\}'$. Now, to express equation (2.14) in terms of $\underline{B}_1$, $\underline{B}_2$ and $\underline{B}_3$. We transform the matrices $\widetilde{G}_{jt}$ and $\widetilde{H}_{jt}$ $(j = 0, \dots, m)$ accordingly and we define:
$\widetilde{F}_{1t} = \widetilde{Q}'\left[\widetilde{G}_{0t}\overline{\overline{\beta}} : \dots : \widetilde{G}_{mt}\overline{\overline{\beta}} : \widetilde{H}_{0t}\overline{\overline{\beta}} : \dots : \widetilde{H}_{mt}\overline{\overline{\beta}}\right]$, $\widetilde{F}_{2t} = \widetilde{Q}'\left[\widetilde{H}_{0t}\overline{\overline{\beta}}_\perp : \dots : \widetilde{H}_{mt}\overline{\overline{\beta}}_\perp\right]$ and $\widetilde{F}_{3t} = \widetilde{Q}'\left[\widetilde{G}_{0t}\overline{\overline{\beta}}_\perp : \dots : \widetilde{G}_{mt}\overline{\overline{\beta}}_\perp\right]$, where $\overline{\overline{\beta}} = \widetilde{\beta}\left[\widetilde{\beta}'\widetilde{\beta}\right]^{-1}$ and $\overline{\overline{\beta}} = \widetilde{\beta}_\perp\left[\widetilde{\beta}'_\perp\widetilde{\beta}_\perp\right]^{-1}$. Then (2.14) can be rewritten as:

$$\widetilde{Q}'\widetilde{A}(L)\, y_t = \widetilde{F}_{1t}\underline{B}_1 + \widetilde{F}_{2t}\underline{B}_2 + \widetilde{F}_{3t}\underline{B}_3 + \varsigma_t \tag{2.34}$$

Now notice that equation (2.34) differs from equation (A.1) in TSL only in the number of structural breaks that we may allow in the deterministic component. Since the intercept and slope dummies behave in the same way, respectively, as the constant and linear trend the rates of convergence of the LS estimators $\widetilde{\underline{B}}_1$, $\widetilde{\underline{B}}_2$ and $\widetilde{\underline{B}}_3$ will be the same as in TSL. Hence, taking into account that $\widetilde{F}_{3t}$ takes nonzero values only for a fixed number of time indices t we conclude that the appropriately standardized moment matrix is asymptotically block diagonal between $\widetilde{F}_{3t}$ and $\left[\widetilde{F}_{1t} : \widetilde{F}_{2t}\right]$ and $\widetilde{\underline{B}}_1 = \underline{B}_1 + O_p(1)$. Also, the aforementioned arguments allow us to drop $\widetilde{F}_{3t}$ on the right hand side of (2.34) and conclude that the asymptotic properties of estimators from equation (2.34) are the same from the following equation:

$$\widetilde{y}_t = \left\{c'_{1t} \otimes \left(-\widetilde{Q}'\widetilde{\alpha}\right)\right\}\underline{B}_1 + \left\{c'_{2t} \otimes \widetilde{Q}'\widetilde{\Psi}\widetilde{\beta}_\perp\left(\widetilde{\beta}'_\perp\widetilde{\beta}_\perp\right)^{-1}\right\}\underline{B}_2 + \varsigma_t \tag{2.35}$$

where $\widetilde{y}_t = \widetilde{Q}'\widetilde{A}(L)\,y_t$, $c_{1t} = \left\{1, DU_t^1, \ldots, DU_t^m, t, DT_t^1, \ldots, DT_t^m\right\}'$, $c_{2t} = \left\{1, DU_t^1, \ldots, DU_t^m\right\}'$ and $\eta_t = \widetilde{Q}'\widetilde{A}(L)\,y_t$. Then we have that:

$$\begin{bmatrix} \widetilde{\underline{B}}_2 - \underline{B}_2 \\ \widetilde{\underline{B}}_3 - \underline{B}_3 \end{bmatrix} = \begin{bmatrix} C_{11} \otimes \widetilde{A}_{11} & C_{12} \otimes \widetilde{A}_{12} \\ C_{21} \otimes \widetilde{A}_{21} & C_{22} \otimes \widetilde{A}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \sum_{t=1}^{T} c_{1t} \otimes \left(-\widetilde{\alpha}'\widetilde{\Omega}^{-1}\widetilde{A}(L)\,u_t\right) \\ \sum_{t=1}^{T} c_{2t} \otimes \left(\widetilde{\beta}_\perp'\widetilde{\beta}_\perp\right)^{-1}\widetilde{\beta}_\perp'\widetilde{\Psi}'\widetilde{\Omega}^{-1}\widetilde{A}(L)\,u_t \end{bmatrix}$$

where

$$C_{ij} = \sum_{t=1}^{T} c_{it}c_{jt}'$$

$$\widetilde{A}_{11} = \widetilde{\alpha}'\widetilde{\Omega}^{-1}\widetilde{\alpha}$$

$$\widetilde{A}_{12} = \widetilde{A}_{21} = -\widetilde{\alpha}'\widetilde{\Omega}^{-1}\widetilde{\Psi}\widetilde{\beta}_\perp\left(\widetilde{\beta}_\perp'\widetilde{\beta}_\perp\right)^{-1}$$

$$\widetilde{A}_{22} = \left(\widetilde{\beta}_\perp'\widetilde{\beta}_\perp\right)^{-1}\widetilde{\beta}_\perp'\widetilde{\Psi}'Q\widetilde{\Omega}^{-1}\widetilde{\Psi}\widetilde{\beta}_\perp\left(\widetilde{\beta}_\perp'\widetilde{\beta}_\perp\right)^{-1} = \widetilde{A}_{21}\widetilde{A}_{11}^{-1}\widetilde{A}_{12} + \widetilde{D}^{-1}$$

with

$$\widetilde{D} = \widetilde{\beta}_\perp'\widetilde{C}\widetilde{\Omega}\widetilde{C}'\widetilde{\beta}_\perp$$

Let's turn now to the asymptotic distribution of $\widetilde{\beta}_\perp'\widetilde{\gamma} - \beta_\perp'\gamma$. To simplify the notation let $RU_t := \left\{RU_t^1, \ldots, RU_t^m\right\}$ where RU_t^j are the OLS residuals from the regression of DU_t^j on $\left\{1, D_t^1, \ldots, D_t^m\right\}$. Then, using the FWLT we can write the asymptotic bias of $\widetilde{\boldsymbol{\beta}}_\perp'\widetilde{\boldsymbol{\gamma}}$ as:

$$T^{\frac{1}{2}}\left(\widetilde{\boldsymbol{\beta}}'_{\perp}\widetilde{\boldsymbol{\gamma}}-\boldsymbol{\beta}'_{\perp}\boldsymbol{\gamma}\right)=\left(\left[T^{-1}\sum_{t=1}^{T}RU_tRU_t'\right]^{-1}\otimes\underbrace{B\left(A_{21}A_{11}^{-1}\alpha'+\left(\beta_{\perp}'\beta_{\perp}\right)^{-1}\beta_{\perp}'\Psi'\right)}_{=\beta_{\perp}'C\Omega}\Omega^{-1}\right)$$

$$\left(T^{-\frac{1}{2}}\sum_{t=1}^{T}RU_t\otimes\widetilde{A}\left(L\right)u_t\right)+o_p\left(1\right)=$$

$$=\left(\left[T^{-1}\sum_{t=1}^{T}RU_tRU_t'\right]^{-1}\otimes I_{k^*}\right)T^{-\frac{1}{2}}\sum_{t=1}^{T}RU_t\otimes\beta_{\perp}'C\varepsilon_t+o_p\left(1\right)$$

Now entirely standard results allow us to establish the following weak convergence result:

$$T^{\frac{1}{2}}\left(\widetilde{\boldsymbol{\beta}}'_{\perp}\widetilde{\boldsymbol{\gamma}}-\boldsymbol{\beta}'_{\perp}\boldsymbol{\gamma}\right)\Rightarrow\left[\int_0^1 RURU'\otimes I_{k^*}\right]^{-1}\left[\int_0^1 RU\otimes dB_{k^*}^{\beta_{\perp}}\right]$$

(b) We first prove that that the result of the theorem holds for the GLS estimator, i. e., assuming the unrealistic assumption that α, Ω are known. Let $e_t=R_{\Delta y,t}-\alpha\left(\beta' R_{y,t-1}-\phi' R_{DT,t-1}\right)$. Then, if we replace $R_{\Delta y,t}-\alpha R_{y,t-1}^{(r)}$ in the expression of the GLS estimator (see (2.3) with $\widetilde{\alpha}$ and $\widetilde{\Omega}$ replaced by α and Ω) by $\alpha\left(\beta'_{(k)}R_{y,t-1}^{(k)}-\phi' R_{DT,t-1}\right)+e_t$ and rearrange terms we obtain:

$$\left(\widetilde{\beta}'_{(k)},\widetilde{\phi}'\right)-\left(\beta'_{(k)},\phi'\right)=\left(\alpha'\Omega^{-1}\alpha\right)^{-1}\alpha'\Omega^{-1}\left[\sum_{t=1}^{T}e_tR_{y,DT,t-1}^{(k)\prime}\right]\left[\sum_{t=1}^{T}R_{y,DT,t-1}^{(k)}R_{y,DT,t-1}^{(k)\prime}\right]^{-1}$$

Since $R_{y,t-1}^{(k)}$ possibly exhibits a segmented deterministic trend with up to $m+1$ regimes it is convenient to use $R_{DT,t-1}$, the last component of the vector $R_{y,DT,t-1}^{(k)}$, to detrend the levels of this process taking into account the trend breaks. In order to do so and to control the different asymptotic rates of convergence of the estimators, we define the adjustment matrix $Q_T=\begin{bmatrix}TB_{22} & T^{\frac{3}{2}}P_{R_{DT}}\\ 0 & T^{\frac{3}{2}}I_{m+1}\end{bmatrix}$ such that $\begin{bmatrix}TB_{22} & T^{\frac{3}{2}}P_{R_{DT}}\\ 0 & T^{\frac{3}{2}}I_{m+1}\end{bmatrix}^{-1}=\begin{bmatrix}\frac{1}{T}B_{22}^{-1} & -T^{\frac{3}{2}}B_{22}^{-1}P_{R_{DT}}\\ 0 & T^{-\frac{3}{2}}I_{m+1}\end{bmatrix}$ where B_{22} is the lower right hand $((k)\times(k))$ block of B^{-1} with $B=\begin{bmatrix}\beta'\\ \alpha'_{\perp}\end{bmatrix}$ and $P_{R_{DT}}$ is the projection matrix of R_{DT}. For ease of exposition we use the following notation. For any two processes K_t and L_t we define the residuals:

$$(K_t|L_t) = K_t - \sum_{i=1}^{T} K_i L_i' \left(\sum_{i=1}^{T} L_i L_i' \right)^{-1} L_t$$

Then we have that:

$$R_{y,DT,t-1}^{(k)} = Q_T \begin{pmatrix} T^{-1} B_{22}^{-1} R_{y,t-1}^{(k)} | R_{DT,t-1} \\ T^{-\frac{3}{2}} R_{DT,t-1}' \end{pmatrix} \tag{2.36}$$

Using (2.36) for the moment matrix $\sum_{t=1}^{T} R_{y,DT,t-1}^{(k)} R_{y,DT,t-1}^{(k)\prime}$, we have that:

$$\left[\sum_{t=1}^{T} R_{y,DT,t-1}^{(k)} R_{y,DT,t-1}^{(k)\prime} \right]^{-1} = Q_T'^{-1} \begin{bmatrix} M_T^{11} & 0 \\ 0 & M_T^{22} \end{bmatrix} Q_T^{-1} + o_p(1) \tag{2.37}$$

where $M_T^{11} = \left(T^{-2} \sum_{t=1}^{T} \beta_\perp' \left(u_{t-1} | X_{DT,t-1}, X_{DU,t-1} \right) \left(u_{t-1} | X_{DT,t-1}, X_{DU,t-1} \right)' \beta_\perp \right)^{-1}$ and $M_T^{22} = \left(T^{-3} \sum_{t=1}^{T} \left(X_{DT,t-1} | X_{DU,t-1} \right) \left(X_{DT,t-1} | X_{DU,t-1} \right)' \right)^{-1}$. and for the cross products between the components e_t the regressors of $R_{y,DT,t-1}^{(k)\prime}$, we see that:

$$\sum_{t=1}^{T} e_t R_{y,DT,t-1}^{(k)\prime} = \left(T^{-1} \sum_{t=1}^{T} \varepsilon_t \left(u_{t-1} | X_{DT,t-1}, X_{DU,t-1} \right)' \beta_\perp \right)$$
$$\left(T^{-\frac{3}{2}} \sum_{t=1}^{T} \varepsilon_t \left(X_{DT,t-1} | X_{DU,t-1} \right)' \right) Q_T' + o_p(1) \tag{2.38}$$

Hence, combining (2.37) and (2.38) we find the asymptotic dominant terms and asymptotic distribution of the appropriately standardized estimators $\left(\widetilde{\beta}_{(k)}^{GLS\prime}, \widetilde{\phi}^{GLS\prime} \right)$.

Now we prove that $\left(\widetilde{\beta}_{(k)}^{FGLS\prime}, \widetilde{\phi}^{FGLS\prime} \right)$ is asymptotically close to $\left(\widetilde{\beta}_{(k)}^{GLS\prime}, \widetilde{\phi}^{GLS\prime} \right)$ by proving that $\left(\left(\widetilde{\beta}_{(k)}^{FGLS\prime}, \widetilde{\phi}^{FGLS\prime} \right) - \left(\widetilde{\beta}_{(k)}^{GLS\prime}, \widetilde{\phi}^{GLS\prime} \right) \right) Q_T = o_p(1)$.

Let $\nu_t = R_{\Delta y,t} - \widetilde{\alpha} \left(R_{y,t-1}^{(r)} - \widetilde{\phi}' R_{DT,t-1} \right) - \widetilde{\alpha} \left(\beta_{(k)}^{\phi\prime} R_{y,DT,t-1}^{(k)} \right)$. If we replace $R_{\Delta y,t} -$

$\widetilde{\alpha}R_{y,t-1}^{(r)}$ by $\widetilde{\alpha}\left(\beta_{(k)}^{\phi\prime}R_{y,DT,t-1}^{(k)}\right)+e_t$ and rearrange terms, we obtain:

$$\left(\widetilde{\beta}_{(k)}^{FGLS\prime},\widetilde{\phi}^{FGLS\prime}\right)-\left(\widetilde{\beta}_{(k)}^{\prime},\widetilde{\phi}\right)=\left(\widetilde{\alpha}^{\prime}\widetilde{\Omega}^{-1}\widetilde{\alpha}\right)^{-1}\widetilde{\alpha}^{\prime}\widetilde{\Omega}^{-1}\left[\sum_{t=1}^{T}\nu_t R_{y,DT,t-1}^{(k)\prime}\right]\left[\sum_{t=1}^{T}R_{y,DT,t-1}^{(k)}R_{y,DT,t-1}^{(k)\prime}\right]^{-1}$$

Now:

$$\left(\widetilde{\beta}_{FGLS(k)}^{\widetilde{\phi}\prime}-\widetilde{\beta}_{GLS(k)}^{\widetilde{\phi}\prime}\right)Q_T=(I)+(II) \tag{2.39}$$

where

$$(I)=\left(\left(\widetilde{\alpha}^{\prime}\widetilde{\Omega}^{-1}\widetilde{\alpha}\right)^{-1}\widetilde{\alpha}^{\prime}\widetilde{\Omega}^{-1}-\left(\alpha^{\prime}\Omega^{-1}\alpha\right)^{-1}\alpha^{\prime}\Omega^{-1}\right)\left[\sum_{t=1}^{T}e_t R_{y,DT,t-1}^{(k)\prime}\right]\left[\sum_{t=1}^{T}R_{y,DT,t-1}^{(k)}R_{y,DT,t-1}^{(k)\prime}\right]^{-1}Q_T$$

and

$$(II)=\left(\widetilde{\alpha}^{\prime}\widetilde{\Omega}^{-1}\widetilde{\alpha}\right)^{-1}\widetilde{\alpha}^{\prime}\widetilde{\Omega}^{-1}\left(\sum_{t=1}^{T}\left(\nu_t-e_t\right)R_{y,DT,t-1}^{(k)\prime}\right)\left(\sum_{t=1}^{T}R_{y,DT,t-1}^{(k)}R_{y,DT,t-1}^{(k)\prime}\right)Q_T$$

Since $\nu_t-e_t=(\widetilde{\alpha}-\alpha)\left(\beta^{\prime}R_{y,t-1}-\phi R_{DT,t-1}\right)$ by virtue of the consistency of $\widetilde{\alpha}$ and $\widetilde{\Omega}$ and the fact that $\beta^{\prime}R_{y,t-1}-\phi R_{DT,t-1}$ is and $I(0)$ process "cleaned" from the deterministic components we see that the right hand side in (2.39) is $o_p(1)$. From (2.37) and (2.38) and $\left(\widetilde{\beta}_{FGLS(k)}^{\widetilde{\phi}\prime}-\widetilde{\beta}_{GLS(k)}^{\widetilde{\phi}\prime}\right)Q_T=o_p(1)$, we find that:

$$\left(\widetilde{\phi}^{FGLS\prime}-\phi^{\prime}\right)Q_T\xrightarrow{d}\int_0^1 B_{r^*}^{\alpha}\left(X_{DT}|X_{DU}\right)^{\prime}\left(\int_0^1\left(X_{DT}|X_{DU}\right)\left(X_{DT}|X_{DU}\right)^{\prime}\right)^{-1}$$

and so applying the FWL Theorem we have that:

$$T^{-\frac{3}{2}}\widetilde{\boldsymbol{\beta}}^{\prime}\widetilde{\boldsymbol{\gamma}}\xrightarrow{d}\left[\int_0^1 RTRT^{\prime}\otimes I_r\right]^{-1}\left[\int_0^1 RT\otimes dB_{r^*}^{\alpha}\right]$$

□

Proof of Theorems 7 and 8. Using arguments similar to the proof of Lemma A.2 from Lütkepohl and Saikkonen (2000a), we find that:

$$T\widetilde{Avar}\left(\widetilde{\boldsymbol{\beta}'_{\perp}\boldsymbol{\gamma}}\right) \Rightarrow \left(\int_0^1 RURU\right)^{-1} \otimes \left(\beta'_{\perp} C\Omega C' \beta_{\perp}\right) \tag{2.40}$$

and

$$T^3\widetilde{Avar}\left(\widetilde{\boldsymbol{\beta}'\boldsymbol{\gamma}}\right) \Rightarrow \left(\int_0^1 RTRT\right)^{-1} \otimes \left(\alpha'\Omega^{-1}\alpha\right)^{-1} \tag{2.41}$$

Therefore, one readily obtains:

$$W^k_{\boldsymbol{\beta}'_{\perp}\boldsymbol{\gamma}} = \left(\widetilde{\boldsymbol{\beta}}'_{\perp}\widetilde{\boldsymbol{\gamma}}\right)' \left[\widetilde{Avar}\left(\widetilde{\boldsymbol{\beta}'_{\perp}\boldsymbol{\gamma}}\right)\right]^{-1} \left(\widetilde{\boldsymbol{\beta}}'_{\perp}\widetilde{\boldsymbol{\gamma}}\right) \Rightarrow \chi^2_{mk}$$

and

$$W^k_{\beta'_{\perp}\gamma} = \left(\widetilde{\boldsymbol{\beta}}'\widetilde{\boldsymbol{\gamma}}\right)' \left[\widetilde{Avar}\left(\widetilde{\boldsymbol{\beta}'\boldsymbol{\gamma}}\right)\right]^{-1} \left(\widetilde{\boldsymbol{\beta}}'\widetilde{\boldsymbol{\gamma}}\right) \Rightarrow \chi^2_{mr}$$

The result from Theorem 7 now follows from the asymptotic unconditional independence of $T^{\frac{1}{2}}\widetilde{\beta}'_{\perp}\widetilde{\gamma}$ and $T^{\frac{3}{2}}\widetilde{\beta}'\widetilde{\gamma}$ and the fact that the sum of independent chi-square random variables is also chi-square distributed. Hence, it follows that:

$$W^k_{\gamma} = W^k_{\boldsymbol{\beta}'_{\perp}\boldsymbol{\gamma}} + W^k_{\boldsymbol{\beta}'\boldsymbol{\gamma}} \Rightarrow \chi^2_{mn}$$

Now we define the following the following standard Brownian motions:

$$\begin{pmatrix} B_{r^*}(s) \\ B_{k^*}(s) \end{pmatrix} = \begin{pmatrix} \left(\alpha'\Omega^{-1}\alpha\right)^{-\frac{1}{2}} \alpha'\Omega^{-\frac{1}{2}} B_n(s) \\ \left(\beta'_{\perp} C\Omega C'\beta_{\perp}\right)^{-\frac{1}{2}} \beta'_{\perp} C\Omega^{\frac{1}{2}} B_n(s) \end{pmatrix} = BM \begin{pmatrix} I_{r^*} & 0 \\ 0 & I_{k^*} \end{pmatrix}$$

The result from Theorem 8 now follows from the definition of $B_{r^*}(s)$ and $B_{k^*}(s)$, Theorem 6,(2.40) and (2.41) and the CMT. $\square$

Proof of Lemma 3. Result (a) follows from Proposition 3 of Busetti (2002) for the cases $k < k^*$ and $k = k^*$. With similar arguments from Theorem B.4 of Nyblom and Harvey (2000) we can see $\Lambda_{y,k} = O_p(T/l)$ for $k = 1, \ldots, k^*$ and $\Lambda_{y,k} = O_p(1)$ if $k = k^* + 1, \ldots, n$. Hence we conclude that $\xi_{y,k} = O_p(1)$ when $k > k^*$. Result (b) follows from Corollary B.6 of Nyblom and Harvey (2000) for the case $k < k^*$. With results from Leybourne et al. (2007) we proceed in the same way as Theorem B.4 from Nyblom and Harvey (2000) to prove that $\Lambda_{\Delta y,k} = O_p(1)$ for $k = 1, \ldots, k^*$ and $\Lambda_{\Delta y,k} = O_p(l/T)$ if $k = k^* + 1, \ldots, n$. Consequently, $\xi_{y,k} = O_p(l/T)$ when $k > k^*$. □

Proof of Lemma 4. To avoid inessential algebraic complexities we prove these results for $m = 1$ and $\mu_0 = \mu_1 = \delta = 0$. These assumptions have no effect on the orders of probability. Throughout the proof, we will make use of the following partition of $\Sigma_y(\tau)$, $\Sigma_{\Delta y}(\tau)$, $C_y(\tau)$ and $C_{\Delta y}(\tau)$:

$$\Sigma_y(\tau) = \begin{bmatrix} \Sigma_{11,y}(\tau) & \Sigma_{12,y}(\tau) \\ \Sigma_{21,y}(\tau) & \Sigma_{22,y}(\tau) \end{bmatrix} \Sigma_{\Delta y}(\tau) = \begin{bmatrix} \Sigma_{11,\Delta y}(\tau) & \Sigma_{12,\Delta y}(\tau) \\ \Sigma_{21,\Delta y}(\tau) & \Sigma_{22,\Delta y}(\tau) \end{bmatrix}$$

$$C_y(\tau) = \begin{bmatrix} C_{11,y}(\tau) & C_{12,y}(\tau) \\ C_{21,y}(\tau) & C_{22,y}(\tau) \end{bmatrix} C_{\Delta y}(\tau) = \begin{bmatrix} C_{11,\Delta y}(\tau) & C_{12,\Delta y}(\tau) \\ C_{21,\Delta y}(\tau) & C_{22,\Delta y}(\tau) \end{bmatrix}$$

where $\Sigma_{11,y}(\tau)$, $\Sigma_{11,\Delta y}(\tau)$, $C_y(\tau)$, $C_{11,\Delta y}(\tau)$ are $k^* \times k^*$ matrices.

(a) We analyze first the asymptotic properties of the OLS variance matrix estimator for the model in levels: Σ_y with $l = 0$. Using the same arguments as in Harvey et al. (2009) we find that the dominant term of the difference $\Sigma_y(\tau) - \Sigma_y(\tau^*)$ is given by:

$$\frac{(dT)^2}{36} (\tau - 1)^3 (4\tau^* - \tau - 3)\, \gamma\gamma'$$

where $d = \tau - \tau^*$. Now since $\gamma = o\left(T^{-\frac{1}{2}}\right)$ for $s = 1, \ldots, , k^*$ and $\gamma = o\left(T^{-\frac{3}{2}}\right)$ for $s = k^* + 1, \ldots, n$ it follows that $\|\Sigma_{22,y}(\tau) - \Sigma_{22,y}(\tau^*)\| = o_p\left(T^{-1}\right)$, $\|\Sigma_{12,y}(\tau) - \Sigma_{12,y}(\tau^*)\| = o_p(1)$, $\|\Sigma_{21,y}(\tau) - \Sigma_{21,y}(\tau^*)\| = o_p(1)$ and $\|\Sigma_{11,y}(\tau) - \Sigma_{11,y}(\tau^*)\| = o_p(T)$. Now we relax the restriction on l and use the last 2 results to obtain the order of probability for the difference of the long run covariance matrix estimator:

$$
\begin{aligned}
\left\|\Sigma_y(\tau)-\Sigma_y(\tau^*)\right\| \leq & \left\|\frac{1}{T}\sum_{t=1}^{T}e_t(\tau)\,e_t(\tau)'-\frac{1}{T}\sum_{t=1}^{T}e_t(\tau^*)\,e_t(\tau^*)'\right\| \\
& +\left\|2\underbrace{\sum_{i=1}^{l}\left(1-\frac{i}{l+1}\right)}_{O(l)}\left(\frac{1}{T}\sum_{t=1}^{T}e_t(\tau)\,e_{t-i}(\tau)'-\frac{1}{T}\sum_{t=1}^{T}e_t(\tau^*)\,e_{t-i}(\tau^*)'\right)\right\|
\end{aligned}
\tag{2.42}
$$

Hence, we conclude that $\|\Sigma_{22,y}(\tau)-\Sigma_{22,y}(\tau^*)\| = o_p\left(lT^{-1}\right)$, $\|\Sigma_{12,y}(\tau)-\Sigma_{12,y}(\tau^*)\| = o_p(l)$, $\|\Sigma_{21,y}(\tau)-\Sigma_{21,y}(\tau^*)\| = o_p(l)$ and $\|\Sigma_{11,y}(\tau)-\Sigma_{11,y}(\tau^*)\| = o_p(lT)$. We now turn to the order of probability of C_y. We start to establish the asymptotic behaviour of the partial sum of the vector of residuals. We can rewrite $\sum_{t=1}^{i}\widehat{u}_t(\tau)$ as:

$$
\begin{aligned}
\sum_{t=1}^{i}\widehat{u}_t(\tau) &= \sum_{t=1}^{i}\left(u_t-\gamma f_t(\widehat{\tau},\tau^*)\right)+\sum_{t=1}^{i}DT_t(\tau)\left(\gamma\frac{\sum_{t=1}^{T}f_t(\tau,\tau^*)\,DT_t(\tau)}{\sum_{t=1}^{T}DT_t(\tau)^2}-\frac{\sum_{t=1}^{T}DT_t(\tau)\,u_t}{\sum_{t=1}^{T}DT_t(\tau)^2}\right) \\
&= (I)+(II)
\end{aligned}
\tag{2.43}
$$

where $f_t(\tau,\tau^*) = 1(T\tau^* < t \leq T\tau)[t-T\tau^*] + 1(t > T\tau)\,T(\tau-\tau^*)$. Now for (I) in (2.43), we have that:

$$
\left\|\sum_{t=1}^{i}\left(u_t-\gamma f_t(\tau,\tau^*)\right)\right\| \leq \left\|\sum_{t=1}^{i}u_t\right\| + \left\|\gamma\frac{1}{2}(Td)^2\right\| + \left\|\gamma T^2 d\,(i-\tau^*)\right\|
$$

Hence, from the shrinking shifts assumption, we have that $T^{-\frac{3}{2}}\sum_{t=1}^{i}\left(u_{st}-\gamma_s f_t(\widehat{\tau},\tau^*)\right) = T^{-\frac{3}{2}}\sum_{t=1}^{i}u_{st}+o_p(1)$ for $s=1,\ldots,k^*$ and $T^{-\frac{1}{2}}\sum_{t=1}^{i}\left(u_{st}-\gamma_s f_t(\widehat{\tau},\tau^*)\right) = T^{-\frac{1}{2}}\sum_{t=1}^{i}u_{st}+o_p(1)$ for $s=k^*+1,\ldots,n$. The same line of proof can be used to show that the dominant term of (II) in (2.43) is $-\sum_{t=1}^{i}DT_t(\tau)\frac{\sum_{t=1}^{T}DT_t(\tau)\,u_t}{\sum_{t=1}^{T}DT_t(\tau)^2}$. The described asymptotic properties for (I) and (II) implies that $C_{11,y}=O_p\left(T^2\right)$ $C_{12,y}=O_p(T)$, $C_{21,y}=O_p(T)$ and $C_{22,y}=O_p(1)$. Now the proof follows similar lines from Busetti (2002). The eigenvalues of

$\Sigma_y(\tau)^{-1} C_y(\tau)$ are the solution of the characteristic polynomial:

$$\begin{aligned}
|C_y - \Lambda_{y,j}\Sigma_y| &= |C_{11,y} - \Lambda_{y,j}\Sigma_{11,y}| \\
&\quad \times \left| C_{22,y} - \Lambda_{y,j}\Sigma_{22,y} - (C_{21,y} - \Lambda_{y,j}\Sigma_{21,y}) (C_{11,y} - \Lambda_{y,j}\Sigma_{11,y})^{-1} (C_{12,y} - \Lambda_{y,j}\Sigma_{12,y}) \right| \\
&= 0
\end{aligned}$$

Therefore, $\Lambda_{y,s} = O_p\left(\frac{T}{l}\right)$ for $s = 1, \ldots, k^*$ and $\Lambda_{y,s} = O_p(1)$ for $s = k^* + 1, \ldots, n$ and $\left(\frac{l}{T}\right) \xi_{y,k}(\tau) = O_p(1)$ if $k < k^*$ and $\xi_{y,k}(\tau) = O_p(1)$ if $k \geq k^*$.

(b) With the same line of proof from Harvey et al. (2009) we observe that the dominant term of the difference on the OLS variance matrix estimators for the model in differences, $\Sigma_{\Delta y}(\tau) - \Sigma_{\Delta y}(\tau^*)$ for $l = 0$, is given by:

$$\frac{d(\tau^* - 1)}{(2\tau^* - \tau - 1)} \gamma\gamma'$$

Given that $\gamma = o\left(T^{-\frac{1}{2}}\right)$ for $s = 1, \ldots,, k^*$ and $\gamma = o\left(T^{-\frac{3}{2}}\right)$ for $s = k^* + 1, \ldots, n$ it follows that $\|\Sigma_{11,\Delta y}(\tau) - \Sigma_{11,\Delta y}(\tau^*)\| = o_p\left(T^{-\frac{1}{2}}\right)$. Now if we relax the restriction on l it follows as in 2.42 that $\|\Sigma_y(\tau) - \Sigma_y(\tau^*)\| = O_p\left(lT^{-\frac{1}{2}}\right)$. As regards to $C_{\Delta y}$, we again analyze first the asymptotic properties of the partial sum of the vector of residuals from the model in differences. We can rewrite $\sum_{t=1}^{i} \widehat{v}_t$ as:

$$\begin{aligned}
\sum_{t=1}^{i} \widehat{v}_t(\tau) = &\left(\gamma \sum_{t=1}^{i} f_{DU_t}(\tau^*, \tau) + \sum_{t=1}^{i} v_t \right) \\
&+ \sum_{t=1}^{i} DU_t(\tau) \left(\frac{\sum_{t=1}^{T} DU_t(\tau) v_t}{\sum_{t=1}^{T} DU_t(\tau)^2} - \gamma \frac{\sum_{t=1}^{T} DU_t(\tau) f_{DU_t}(\tau^*, \tau)}{\sum_{t=1}^{T} DU_t(\tau)^2} \right) = (I) + (II)
\end{aligned}$$

where $f_{DU_t}(\tau^*, \tau) = 1(T\tau^* < t \leq T\tau)$. Now for (I) we have that $\|(I)\| \leq \left\| \sum_{t=1}^{i} v_t \right\| + \|\gamma T d\|$ which implies that $\gamma_s \sum_{t=1}^{i} f_{DU_t}(\tau^*, \tau) + \sum_{t=1}^{i} v_{s,t} = T^{-\frac{1}{2}} \sum_{t=1}^{i} v_{s,t} + o_p(1)$ for $s = 1, \ldots, k^*$ and $\gamma_s \sum_{t=1}^{i} f_{DU_t}(\tau^*, \tau) + \sum_{t=1}^{i} v_{s,t} = \sum_{t=1}^{i} v_{s,t} + o_p(1)$ for $s = k^* + 1, \ldots, n$.

With the same arguments it is possible to show that the dominant term of (II) is

$-\sum_{t=1}^{i} DU_t(\tau) \frac{\sum_{t=1}^{T} DU_t(\tau) v_t}{\sum_{t=1}^{T} DU_t(\tau)^2}$. Hence it follows that $C_{11,y} = O_p(1)$ $C_{12,y} = O_p\left(T^{-\frac{1}{2}}\right)$, $C_{21,y} = O_p\left(T^{-\frac{1}{2}}\right)$ and $C_{22,y} = O_p\left(T^{-1}\right)$. The eigenvalues of $\Sigma_{\Delta y}(\tau)^{-1} C_{\Delta y}(\tau)$ are the solution of the characteristic polynomial:

$$
\begin{aligned}
|C_{\Delta y} - \Lambda_{\Delta y,j}\Sigma_{\Delta y}| = {} & |C_{11,\Delta y} - \Lambda_{\Delta y,j}\Sigma_{11,\Delta y}| \\
& \times |C_{22,\Delta y} - \Lambda_{\Delta y,j}\Sigma_{22,\Delta y} - \\
& (C_{21,\Delta y} - \Lambda_{\Delta y,j}\Sigma_{21,\Delta y})(C_{11,\Delta y} - \Lambda_{\Delta y,j}\Sigma_{11,\Delta y})^{-1}(C_{12,\Delta y} - \Lambda_{\Delta y,j}\Sigma_{12,\Delta y})| \\
& = 0
\end{aligned}
$$

Therefore, $\Lambda_{\Delta y,s} = O_p(1)$ for $s = 1, \ldots, k^*$ and $\Lambda_{\Delta y,s} = O_p\left(\frac{l}{T}\right)$ for $s = k^* + 1, \ldots, n$ which determines that $\xi_{\Delta y,k}(\tau) = O_p(1)$ if $k < k^*$ and $\left(\frac{T}{l}\right) \xi_{\Delta y,k}(\tau) = O_p(1)$ if $k \geq k^*$. □

Proof of Theorem 9. The proof follows the same lines of the proof of Proposition 7 from Bai and Perron (1998) and Theorem 1 from Kejriwal and Perron (2010) and is, therefore, omitted for the sake of brevity. □

Tab. 2.1: Asymptotic critical values of the W_*^λ test statistic for the the 1 trend break case, $m = 1$.

	Number of Dependent variables, n						
ψ	2	3	4	5	6	7	8
0.10	10.19	12.72	14.24	16.77	18.13	20.70	21.85
0.05	12.17	14.30	16.65	18.49	20.86	22.24	23.92
0.01	16.68	18.90	20.83	22.65	24.71	26.28	27.25

Tab. 2.2: Asymptotic critical values for the sequential test $W_\lambda(l+1|l)$.

		Number of Dependent Variables, n						
l	ψ	2	3	4	5	6	7	8
	0.1	10.19	12.72	14.24	16.77	18.13	20.70	21.85
0	0.05	12.17	14.30	16.65	18.49	20.86	22.24	23.92
	0.01	16.68	18.90	20.83	22.65	24.71	26.28	27.25
	0.1	12.15	14.21	16.63	18.41	20.67	22.21	23.88
1	0.05	14.69	16.13	18.19	20.55	22.27	24.20	25.36
	0.01	17.57	19.82	22.56	23.40	25.29	28.51	29.23
	0.1	13.55	15.54	17.37	19.81	21.76	22.90	24.84
2	0.05	15.42	17.69	18.96	21.21	23.23	24.87	26.30
	0.01	18.80	21.32	22.84	23.98	25.83	29.81	29.76
	0.1	14.63	16.10	18.17	20.45	22.23	24.20	25.34
3	0.05	16.16	18.57	20.35	22.01	24.13	25.70	26.78
	0.01	19.16	22.44	23.52	24.44	26.79	30.15	30.28
	0.1	15.01	16.80	18.66	21.06	22.82	24.77	25.74
4	0.05	16.59	18.86	20.77	22.56	24.70	26.27	27.25
	0.01	19.24	23.15	24.15	24.67	27.51	30.34	30.54

Tab. 2.3: Empirical size of $W_\lambda(\tau^*)$ test for $\tau^* = 0.5$, 5% nominal level .

ρ	T	k^*			
		0	1	2	3
	100	0.005	0.020	0.033	0.078
	200	0.034	0.065	0.076	0.072
0	300	0.042	0.066	0.075	0.063
	400	0.039	0.071	0.067	0.057
	500	0.043	0.068	0.066	0.051
	1000	0.049	0.058	0.056	0.054
	100	0.007	0.024	0.039	0.082
	200	0.039	0.081	0.081	0.074
0.4	300	0.043	0.083	0.082	0.065
	400	0.047	0.080	0.078	0.055
	500	0.048	0.077	0.069	0.060
	1000	0.050	0.060	0.058	0.050
	100	0.024	0.046	0.053	0.073
	200	0.074	0.164	0.132	0.071
0.8	300	0.070	0.149	0.129	0.065
	400	0.071	0.127	0.104	0.061
	500	0.071	0.120	0.093	0.052
	1000	0.054	0.083	0.067	0.049

Tab. 2.4: Empirical size and power of W_λ test, 5% nominal level .

$n = 2$

γ	ρ	T	k^*		
			0	1	2
0			0.008	0.046	0.057
0.2		100	0.989	0.844	0.07
0.4			0.99	0.862	0.086
0	0		0.019	0.07	0.03
0.2		200	1	0.993	0.041
0.4			1	0.994	0.051
0			0.01	0.076	0.057
0.2		100	0.945	0.73	0.067
0.4			0.962	0.801	0.09
0	0.5		0.022	0.101	0.033
0.2		200	1	0.989	0.044
0.4			1	0.993	0.051

$n = 3$

γ	ρ	T	k^*			
			0	1	2	3
0			0.003	0.014	0.017	0.021
0.2		100	0.914	0.732	0.307	0.029
0.4			0.919	0.774	0.363	0.035
0	0		0.018	0.088	0.134	0.029
0.2		200	1	1	0.962	0.034
0.4			1	1	0.965	0.046
0			0.008	0.032	0.026	0.022
0.2		100	0.749	0.525	0.161	0.027
0.4			0.796	0.642	0.248	0.036
0	0.5		0.033	0.166	0.156	0.029
0.2		200	1	0.999	0.929	0.035
0.4			1	1	0.936	0.053

Tab. 2.5: Group countries common trend breaks in the top 1% Income Share

	Number of breaks (Sequential)		Break dates		Test statistic $m = 1$	
Country group	W_λ	PQ	W_λ	PQ	W_λ	PQ
Anglo-saxon	1	3	1979	1937, 1953, 1982	19.7**	172.3***
Continental Europe	0	2	-	1943, 1976	7.3	153.5***
Nordic	1	3	1939	1939, 1961, 1991	40.3***	25.7***
Asia	0	3	-	1945, 1959, 1983	7.9	192.3***

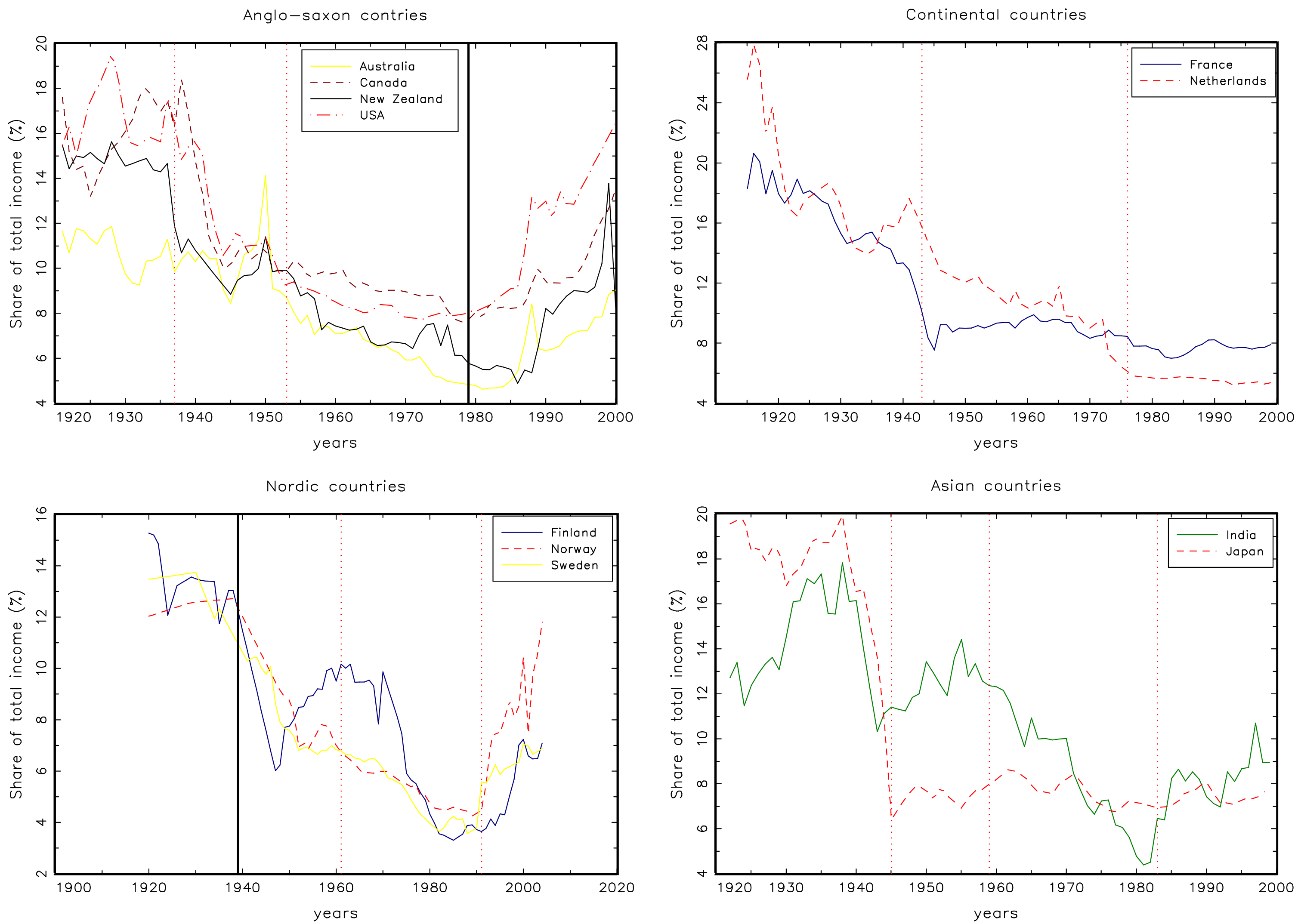

Fig. 2.1: Top 1% Income shares for country groups with break dates detected with PQ (dotted) and W_λ (solid) methodologies

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3. NEOCLASSICAL, SEMI-ENDOGENOUS OR ENDOGENOUS GROWTH THEORY? EVIDENCE BASED ON NEW STRUCTURAL CHANGE TESTS

With Luis C. Nunes and Paulo M. M. Rodrigues[1]

3.1 Introduction

Determining the nature of the trend (*i.e.* whether it is deterministic or stochastic) and whether structural breaks are present in per capita output has been of considerable interest in the literature. These two important and interrelated topics have very important macroeconomic and econometric implications. First, as firstly put forward by Nelson and Plosser (1982), if per capita output has a unit root (stochastic trend) then real disturbances are likely to be the most important source of macroeconomic fluctuations as opposed to disturbances with only a transitory impact, in agreement with the Real Business Cycle Theory. However, if the trend in per capita output is deterministic then it is expected to have small and infrequent real shocks and so disturbances with only a transitory impact such as monetary shocks are the ones that explain a large fraction of business cycle fluctuations. Second, the interpretation and usefulness of simple linear regression models in which output is involved depends on the nature of the trend as OLS may produce spurious results in the presence of a stochastic trend as shown by Granger and Newbold (1974) and later demonstrated analytically by Phillips (1986).

The seminal work of Nelson and Plosser (1982) contrasted the null hypothesis of a unit root against the alternative of trend stationarity for 14 U.S. long historical time series and did not reject the unit root hypothesis for U.S. real per capita GNP.

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A vast discussion in the literature followed this work and tried to confirm or corroborate Nelson and Plosser's conclusions through new and improved unit root tests, but with no apparent consensus. An important consideration on unit root testing put forward by Perron (1989) is that with unmodeled structural breaks in the deterministic trend one can hardly reject the unit root hypothesis even if the series is trend stationary (albeit with breaks). Perron (1989) argued by simple visual inspection that the 1929 crash was responsible for a trend break. Using 2.5 percent significance level, he rejected the unit root hypothesis in real per capita GNP contradicting Nelson and Plosser's results. However, Perron's (1989) exogeneity assumption, corresponding to the Great Depression, was subject to strong criticism (see Christiano, 1992) and, consequently, to a considerable number of new unit-root test procedures which estimate the break point endogenously under the alternative hypothesis (see,*inter alia*, Zivot and Andrews, 1992, Perron, 1997, Vogelsang and Perron, 1998, Perron and Rodriguez, 2003). For example, Zivot and Andrews (1992) clearly does not find statistical evidence against the unit root hypothesis in per capita output as opposed to Perron (1989), but in Perron (1997) statistical evidence is much more ambiguous. Recently, this line of work has also attracted significant criticism because these procedures do not allow for a structural break to occur under the null hypothesis, only under the alternative and hence are not invariant to the magnitude of the shift in level and/or slope of the trend function (see Kim and Perron, 2009, Carrion-i Silvestre et al., 2009, Harris et al., 2009). Kim and Perron (2009) devised testing procedures which allow for one trend break under both the null and alternative hypotheses and rejected the unit root hypothesis for per capita output supporting Perron (1989).

Additionally to the debate on unit root nonstationarity versus stationarity with breaks, the issue of structural change in the deterministic component of per capita output also deserves careful assessment in its own right. If one does not appropriately specify the trend function then the model will provide inconsistent estimates and poor forecasting performance. Moreover, if one writes a simple linear regression model of log real per capita output on a time trend, the trend coefficient will represent the average growth rate, a quantity of substantial interest and that we will give special attention in this paper.

One of the important topics that highlights the importance of studying the stability of the output growth rate is the competition between neoclassical, semi-endogenous and endogenous growth theories for the model that best describes what we observe in the data. Jones (1995a, 2002, 2005) contrasted the observed substantial and permanent rise of investment in human capital and R&D with the remarkable stability of U.S. per capita output. If we take these models seriously, then we should have observed permanent positive shifts on the rate of economic growth, according to the endogenous growth literature, or, at least, short run increases and long run "level effects" according to the neoclassical and semi-endogenous growth models. However, the growth rate of U.S. per capita output has been remarkably stable since the end of the 19th century. Moreover, Jones (1995b) documents that several variables that should lead to permanent changes in the long run growth rate or, at least, have "level effects" exhibited large, persistent movements, generally in the "growth-increasing" direction in OECD economies, at least, since the World War II. Based on the documented increase of these variables, Papell and Prodan (2005) classified several countries according to three mutually exclusive hypotheses, each compatible with a certain class of economic growth models:

(a) The "Summer-Weil-Jones" or "constant trend" hypothesis, originally suggested by David Weil and Lawrence Summers and subsequently considered in Jones (1995b), which argues that a simple time trend with slope equal to the average growth rate should describe very accurately the log of per capita output. Some temporary departures from this line are allowed, corresponding to large exogenous shocks on the economy and subsequent recovery, but the linear trend should return to its original path. Jones (2002) developed a model to reconcile the conflicting evidence between the rising investment in human capital and R&D and the stability of the U.S. growth rate and provided explanations to this phenomenon: either the permanent effects associated with all these factors have been offseting leaving the growth rate constant or the sequence of transitional dynamics has been generating higher average growth rates than the steady-state value.

(b) 'The "Jones-Solow" or "level shift" hypothesis, which favors the neoclassical (Solow, 1956) and the Jones' (Jones, 1995a, 2005) semi-endogenous growth theories. It defends that, after policy changes such as rise in the human capital or R & D investment, output growth may change in the short run but should return to its original value in the long run. However, these changes should lead to long-run increases in the level of per capita GDP.

(c) The "Romer" or "slope shift" hypothesis postulated by Romer (1986) suggests that policy changes should alter the growth rate of per capita output permanently .

The objective of this paper, considering the hypotheses previously indicated (i.e. the "constant trend", the "level shift" and the "slope shift" hypotheses), is to analyze which economic growth theory seems to better characterize the growth path of per capita output of a large set of countries. The literature closely related to this paper which also addresses this issue is Ben-David and Papell (1995) who pre-tested the unit root hypothesis with the Zivot and Andrews (1992) approach and then used the Vogelsang (1997) test, with critical values corresponding to the resultant order of integration, to search for evidence for one break in the trend function. Papell and Prodan (2005) and Papell and Prodan (2011) pre-tested for the existence of a unit root with the ADF test discussed in Papell and Prodan (2007) that allows for two endogenous break points but with the second break restricted to have only a slope shift. After filtering out the non stationary countries, they used a modification of the sequential procedure by Bai (1999), as suggested by Prodan (2008), to estimate the number of breaks. Finally, for countries with more than 1 break they formally tested the constant trend and level shift hypotheses with a standard F statistic. However, these approaches have several limitations: first, the unit root pre-testing procedure imposes, but does not estimate the number of breaks in the trend function. Second, the unit root test is based on search procedures under the alternative hypothesis and does not render pivotal asymptotic distributions in the presence of trend breaks under the null hypothesis as previously indicated. Third, it is well known that this sequence of pre-testing procedures can generate substantial size and power distortions (even asymptotically) specially if the first step statistics have poor finite sample properties.

Recent developments have provided different solutions to the problem of testing for the presence of structural breaks without unit root pre-testing. For instance, Perron and Yabu (2009), Kejriwal and Perron (2010), Harvey et al. (2009) and Nunes and Sobreira (2010) (hereafter NS) introduced statistical procedures to test for and estimate structural breaks in the trend function that are robust as to whether the noise component is I(0) or I(1) so that no unit root pre-testing is needed. Kejriwal and Lopez (2012) took advantage of these recent econometric developments to test three hypotheses labeled with the same names as ours but they actually used different definitions for each hypothesis. For the "constant trend" hypothesis they do not allow a country to return to its original level of per capita GDP and GDP growth after the transitional period following a large shock. For the "level shift" hypothesis they do not allow a country to return to its steady state value of GDP growth after the transitional period following a large shock.

To categorize countries according to the "constant trend", "level shift" and "growth shift" hypotheses we need first to identify when large and exogenous shocks occurred for each country. We use the framework in NS as it allows for direct estimation of the number and timing of breaks in the slope of the deterministic trend function. If no breaks are found then we interpret that result as evidence favoring the "constant trend" hypothesis and consequently suggesting the neoclassical growth theory. If only one break is detected, that favors the "slope shift" hypothesis and the endogenous growth models are favored in this case. Finally, if two breaks are found then three situations may occur: i) it may happen that, after the last break, both the level and growth rate of per capita output return to its long run hypothetical value if there were no trend breaks. This situation enters in the "constant trend" setup; ii) it is possible that, after the last break, the growth rate but not the level of per capita output returns to its original path. This enters in the "level shift" hypothesis case. Finally, we may observe that neither the level nor the growth rate of per capita GDP have returned to their original paths. This favors the "slope shift" hypothesis. To test these hypotheses we need to test additional restrictions on the coefficients of the linear regression model conditional on the regimes estimated with the aforementioned framework of NS.

Hence, this paper provides a further contribution to the econometric literature: How can we test additional restrictions on the trend function given the estimated break dates obtained in a first step? It turns out that given the fast rate of convergence of the estimators of the break dates we prove that a standard F-test will converge asymptotically to the usual chi-square distribution with the number of degrees of freedom corresponding to the number of restrictions. This result turns out to be very useful not only for the particular problem treated in this paper but for any study in which general linear restrictions of the trend function across regimes need to be tested after estimating the number and timing of the breaks in a first step.

We apply our procedure to long historical per capita GDP series for an extensive set of countries. Statistical evidence supports the "constant trend" hypothesis for nine countries: Austria, Germany, Switzerland, Canada, United States, Chile, Sweden, Australia and New Zealand. Only six countries seem to be compatible with the "level shift" hypothesis: France, Netherlands, Brazil, Denmark, Japan and Italy. Finally, we found evidence to conclude that eight countries satisfy the "growth shift" hypothesis: Belgium, Uruguay, Finland, Norway, United Kingdom, Sri Lanka, Portugal and Spain.

This paper is organized as follows. The introductory note in Section 3.2 briefly discusses the motivation to employ this two-step procedure to classify countries according to the three economic growth hypotheses. Section 3.2.1 presents the general econometric setup and underlying assumptions to analyze possible changes and patterns in the steady state growth rate. Section 3.2.2 presents testing procedures for our first step analysis that estimates the number of breaks in the steady state growth rate and respective break dates. Section 3.2.3 presents the statistic to test general linear restrictions on the coefficients, conditional on the results from the first step, discusses its asymptotic properties and shows how it can be used for the objective of the paper. Section 3.3 presents and discusses empirical results and provides a definite categorization of the countries analyzed. Section 3.4 provides some brief concluding remarks.

3.2 Assumptions and Methodology

In this section we discuss the empirical approach used to classify countries according to the "constant trend", the "level shift" and the "slope shift" hypotheses.

In section 3.2.1 we present a general econometric model for long-term per capita output and describe in detail its underlying assumptions. In section 3.2.2 we describe the procedure proposed by NS that tests for the existence, the number and the timing of trend breaks. This is the first step in our approach and, contrary to Papell and Prodan (2005) and Papell and Prodan (2011), we do not need to pre-test the unit root hypothesis since these tests are robust as to whether the underlying errors are I(0) or I(1).

If statistical evidence indicates that all countries have zero or only one break in the per capita GDP growth rate then our empirical analysis would stop, since no evidence for the existence of trend breaks favors the "constant trend" hypothesis. On the other hand, if there is evidence for the presence of one break in trend then this favors the "slope shift" hypothesis, as a changing steady-state growth rate is compatible with the Romer-type endogenous growth models. Finally, if our testing procedure detects the presence of two or more breaks in the trend function, then either the neoclassical, semi-endogenous or the endogenous growth theory may hold. A first possibility is that, after the first large shock (which typically coincides with the World Wars or The Great Depression), the output growth rate deviated from its steady state value but, after enough time has passed, transition dynamics return the economy to its steady state growth path. This reasoning is in line with the "constant trend" hypothesis which defends that not only the steady-state growth rate but also the trend function as a whole should be equal except in the transition period. A second possibility, compatible with the neoclassical or semi-endogenous "level shift" hypothesis, occurs when only the steady state growth rates remain the same before the first break and after transitional dynamics. As a final possibility, we may observe that, after the recovery from the shock, the economy enters a new and different steady state growth path in contradiction with the neoclassical growth theory but perfectly compatible with endogenous growth models. These three different and mutually exclusive behaviors of the long-run trend and growth rates are associated with specific linear restrictions on the breaking trend model described in section 3.2.3. Additionally, that section outlines the approach used to test general linear restrictions and establishes the large sample properties of our proposed statistics.

3.2.1 Econometric Model and Assumptions

The most general setup to model the behavior of long-term per capita output is the disjoint broken linear trend model as discussed in NS and Kejriwal and Perron (2010). We will consider using their framework now to test for additional restrictions on the trend breaks coefficients. Hence, the log real per capita GDP, denoted by y_t ($t = 1, \ldots, T$), is a univariate time series process that is assumed to be generated by the following equation that includes a constant, a linear trend and m structural breaks in the trend function which may occur at dates $\{T_1^*, \ldots, T_m^*\}$:

$$y_t = \alpha + \beta t + \sum_{j=1}^{m} \delta_j DU_t\left(\tau_j^*\right) + \sum_{j=1}^{m} \gamma_j DT_t\left(\tau_j^*\right) + u_t \qquad t = 1, ..., T, \tag{3.1}$$

where $DU_t\left(\tau_j^*\right) := \mathbb{1}\left(t > T_j^*\right)$ and $DT_t\left(\tau_j^*\right) := \mathbb{1}\left(t > T_j^*\right)\left(t - T_j^*\right)$ capture the eventual j^{th} break, in the level and slope, respectively, occurring at date $T_j^* := \lfloor \tau_j^* T \rfloor$ for $j = 1, ..., m$. Notice that the first differenced form of equation (3.1) is given by:

$$\Delta y_t = \beta + \sum_{j=1}^{m} \delta_j D_t\left(\tau_j^*\right) + \sum_{j=1}^{m} \gamma_j DU_t\left(\tau_j^*\right) + v_t \qquad t = 2, ..., T, \tag{3.2}$$

where $D_t\left(\tau_j^*\right) = \mathbb{1}\left(t = T_j^* + 1\right)$. From both equations (3.1) and (3.2), it is readily seen that the slope coefficient is the long-run, or steady state, growth rate. Hence, in this unrestricted version of the model we allow for different steady state growth rates across regimes. Until the occurrence of the first structural break at T_1^*, the slope coefficient is equal to β. After T_1^* the long-run growth rate changes from β to $\beta + \gamma_1$ and the level shifts by δ_1. At break point T_2^* the steady-state growth rate changes from $\beta + \gamma_1$ to $\beta + \gamma_1 + \gamma_2$ and the level shifts by δ_2. Generally, in period T_j^* the slope coefficient changes from $\beta + \sum_{i=1}^{j-1} \gamma_i$ to $\beta + \sum_{i=1}^{j} \gamma_i$ while the level shifts by δ_j. Note that whenever $\delta_j \neq 0$, the trend function becomes discontinuous at the break date T_j^*.

The disturbance term u_t is assumed to have an AR(1) representation,

$$u_t = \rho u_{t-1} + \varepsilon_t, \qquad t = 2, ..., T, \qquad u_1 = \varepsilon_1, \tag{3.3}$$

where ε_t in (3.3) satisfies the following assumption (see Sayginsoy and Vogelsang, 2004, pp. 2-3, for more details):

Assumption 6. *The stochastic process ε_t is such that:*

$$\varepsilon_t = C(L)\eta_t, \qquad C(L) = \sum_{i=0}^{\infty} c_i L^i$$

with $C(1)^2 > 0$ and $\sum_{i=0}^{\infty} i|c_i| < \infty$, and where η_t is a martingale difference sequence with unit conditional variance and $\sup_t E\left(\eta_t^4\right) < \infty$.

Notice that the conditions stated in Assumption 6 are quite general. In particular, we allow for the presence of substantial serial correlation in the errors of the AR(1) representation of u_t. The autoregressive coefficient, ρ, is allowed to be either smaller or equal to 1 in absolute value so that real per capita output can either be I(0) or I(1), respectively.

Our goal is to classify countries according to the “constant trend”, the “level shift” and the “slope shift” hypotheses. We approach this problem in two steps: first, we test for the existence of slope breaks in the trend function and estimate both the number and the timing of the change points. This is done unrestrictedly using the methods suggested by NS which are briefly discussed in the next section. Second, conditional on the estimated number of breaks, break dates and coefficients, we build a statistical framework to test general linear restrictions on the coefficients of the linear disjoint broken trend model in section 3.2.3. This amounts to testing the null hypothesis $H_0 : R\Phi = r$ against the two-sided alternative hypothesis $H_0 : R\Phi \neq r$ where R is a q by $2\,(m+1)$ matrix with rank q and r is a q-dimensional vector of constants. These procedures are all made robust to whether u_t is I(0) or I(1) so that $|\rho| \leq 1$. Then we show how the three aforementioned hypotheses can be formulated as linear restrictions on the parameters of the breaking trend model.

3.2.2 Detection and estimation of the number of breaks

In this section we present the methodology used to estimate the number of breaks in the slope of the trend function of per capita output and the respective break dates. We use the extension of Harvey et al. (2009) to the multiple structural breaks setting developed by NS.

Initially, NS analyzes a *sup* F type test of no slope breaks against the alternative hypothesis that there are m slope breaks. The test involves estimating equations (3.1) and (3.2) by OLS for all candidate break fractions $\tau^m = (\tau_1, \ldots, \tau_m)$. The *sup* F statistics are obtained from,

$$\mathcal{F}_0^*(m|0) := \sup_{\tau^m \in \Lambda_m} \mathcal{F}_0(\tau^m) \tag{3.4}$$

and

$$\mathcal{F}_1^*(m|0) := \sup_{\tau^m \in \Lambda_m} \mathcal{F}_1(\tau^m) \tag{3.5}$$

where $\mathcal{F}_0(\tau^m)$ and $\mathcal{F}_1(\tau^m)$ denote, respectively, standard F statistics for testing $\gamma_1 = \ldots = \gamma_m$ from the estimated equations (3.1) and (3.2). To account for general forms of serial correlation in the data, $\mathcal{F}_0(\tau^m)$ and $\mathcal{F}_1(\tau^m)$ were "standardized" by a Bartlett long run variance estimate obtained from the residuals of the estimated equations (3.1) and (3.2). Λ_m specifies the dates allowed for the search of the structural breaks and is given by $\Lambda_m = \{(\tau_1, ..., \tau_m) : |\tau_{i+1} - \tau_i| \geq \eta, \tau_1 \geq \eta, \tau_m \leq 1 - \eta\}$. Basically, this set rules out dates that are close to each other and/or close to the beginning/end of the sample to guarantee invertibility of the moments matrix and enough neighborhood observations to identify the true break points (see Andrews and Ploberger, 1994, Bai and Perron, 1998, for more details). Finally, the break point estimators are the global maximizers of the objective functions: $\widehat{\tau}^m := \arg\sup_{\tau^m \in \Lambda_m} \mathcal{F}_0(\tau^m)$ and $\widetilde{\tau}^m := \arg\sup_{\tau^m \in \Lambda_m} \mathcal{F}_1(\tau^m)$.

Now, since $\mathcal{F}_0^*(m|0)$ and $\mathcal{F}_1^*(m|0)$ converge to a non degenerate asymptotic distribution if and only if the data are, respectively, I(0) and I(1) (see Theorem 5 from NS), these test statistics were weighted by a weight function which is asymptotically binary and ensures that, in the limit, $\mathcal{F}_0^*(m|0)$ is selected if u_t is I(0) and $\mathcal{F}_1^*(m|0)$ is chosen when u_t is I(1). Hence, this weighted $\mathcal{F}$ statistic, $\mathcal{F}_\lambda^*(m|0)$, is given by:

$$\mathcal{F}_\lambda^*(m|0) := \lambda(\widehat{\tau}^m, \widetilde{\tau}^m)\, \mathcal{F}_0^*(m|0) + b_\xi^m \left[1 - \lambda(\widehat{\tau}^m, \widetilde{\tau}^m)\right] \mathcal{F}_1^*(m|0) \tag{3.6}$$

where b_ξ^m is a constant that ensures that for a given significance level ξ and null hypothesis of no trend breaks the critical values of the asymptotic distribution of $\mathcal{F}_\lambda^*$ is the same in both I(0) and I(1) cases. NS studied different forms of the weight function suggested by Harvey et al. (2009) and concluded that the one with the best finite sample properties for the multiple trend breaks case was given by:

$$\lambda(\widehat{\tau}^m, \widetilde{\tau}^m) := \exp\left[-\left\{g_m S_0(\widehat{\tau}^m) S_1(\widetilde{\tau}^m)\right\}^6\right] \tag{3.7}$$

where $g_m = 500 + 750 \times (m-1)$ and $S_0(\widehat{\tau}^m)$ and $S_1(\widetilde{\tau}^m)$ denote the KPSS statistics based on the residuals from the estimated equations (3.1) and (3.2) with associated break fractions $\widehat{\tau}^m$ and $\widetilde{\tau}^m$. The $\mathcal{F}_\lambda^*(m|0)$ statistic can then be used to test the null of no slope breaks against the alternative hypothesis that there are m slope breaks without making any assumptions about the errors being I(0) or I(1) since $\lambda(\widehat{\tau}^m, \widetilde{\tau}^m) \xrightarrow{p} 1$ if u_t is I(0) and $\lambda(\widehat{\tau}^m, \widetilde{\tau}^m) \xrightarrow{p} 0$ if u_t is I(1) and b_ξ ensures comparability with the same critical value in both cases. The final estimator for the vector of break fractions is obtained from $\lambda(\widehat{\tau}^m, \widetilde{\tau}^m)\, \widehat{\tau}^m + \left[1 - \lambda(\widehat{\tau}^m, \widetilde{\tau}^m)\right] \widetilde{\tau}^m$.

A problem arises in this setup as we have to specify the number of breaks under the alternative hypothesis and we may not have that information. Following Bai and Perron (1998), NS considered the class of double maximum tests of the null of no trend break against the alternative hypothesis of an unknown number of breaks in the trend slope up to some maximum M whose robust version can generally be written as:

$$D\max \mathcal{F}_\lambda^* := \left\{\lambda\left(\widehat{\tau}^M, \widetilde{\tau}^M\right) \times D\max \mathcal{F}_0^*\right\} + b_\xi^M \left\{\left[1 - \lambda\left(\widehat{\tau}^M, \widetilde{\tau}^M\right)\right] \times D\max \mathcal{F}_1^*\right\} \tag{3.8}$$

where $D\max \digamma_d^* := \max_{1\leq m\leq M} a_{d,m} \digamma_d^* (m|0), d = 0, 1$, and b_ξ^M denotes a constant that can be chosen, as before, in a way that guarantees the same critical values for both I(0) and I(1) cases.

NS analyzed the two standard choices for the constants $a_{d,j}$: the $UD\max \digamma_\lambda^*$ test where $a_{d,1} = ... = a_{d,M} = 1$, and the $WD\max \digamma_\lambda^*$ test with $a_{d,1} = 1$ and for $m > 1$, $a_{d,m} = \dfrac{C_d(\xi,1)}{C_d(\xi,m)}$ where $C_d(\xi, m)$ is the asymptotic critical value of the test $\digamma_d^*$ for a significance level ξ and m breaks.

To consistently estimate both the true number and timing of breaks, NS proposed a sequential testing procedure in the same spirit as Bai and Perron (1998). The sequential test statistic for testing the null hypothesis of l breaks against the alternative of $l + 1$ breaks is constructed as a weighted average of the maximum value from $(l + 1)$ *sup* F type statistics associated with testing the null hypothesis $\gamma_{l+1} = 0$ versus the alternative $\gamma_{l+1} \neq 0$ in the model in levels:

$$y_t = \alpha+\beta t+\sum_{j=1}^{l}\delta_j DU_t(\widehat{\tau}_j)+\sum_{j=1}^{l}\gamma_j DT_t(\widehat{\tau}_j)+\delta_{l+1}DU_t(\zeta)+\gamma_{l+1}DT_t(\zeta)+u_t \qquad t = 1, ..., T, \tag{3.9}$$

and in first differences:

$$\Delta y_t = \beta+\sum_{j=1}^{l}\delta_j D_t(\widetilde{\tau}_j)+\sum_{j=1}^{l}\gamma_j DU_t(\widetilde{\tau}_j)+\delta_{l+1}D_t(\zeta)+\gamma_{l+1}DU_t(\zeta)+v_t \qquad t = 2, ..., T, \tag{3.10}$$

in each segment set by the estimated partitions $(\widehat{\tau}_1, \ldots, \widehat{\tau}_l)$ and $(\widetilde{\tau}_1, \ldots, \widetilde{\tau}_l)$. Formally, if we let $\digamma_0(\widehat{\tau}_1, .., \widehat{\tau}_{i-1}, \zeta, \widehat{\tau}_i, .., \widehat{\tau}_l)$ and $\digamma_1(\widetilde{\tau}_1, .., \widetilde{\tau}_{i-1}, \zeta, \widetilde{\tau}_i, .., \widetilde{\tau}_l)$ denote, respectively, the standard F-statistics for testing the null hypothesis $\gamma_{l+1} = 0$ versus the alternative $\gamma_{l+1} \neq 0$ from the estimated equations (3.9) and (3.10) then the sequential test statistic for the model in levels is given by:

$$\digamma_0^*(l+1|l) := \max_{1\leq i\leq l+1} \sup_{\zeta\in\Lambda_{0,i}} \digamma_0(\widehat{\tau}_1, .., \widehat{\tau}_{i-1}, \zeta, \widehat{\tau}_i, .., \widehat{\tau}_l)$$

and for the model in first differences, the sequential test statistic is,

$$\mathcal{F}_1^*(l+1|l) := \max_{1\leq i\leq l+1} \sup_{\zeta\in\Lambda_{1,i}} \mathcal{F}_1(\widetilde{\tau}_1, .., \widetilde{\tau}_{i-1}, \zeta, \widetilde{\tau}_i, .., \widetilde{\tau}_l)$$

where the possible eligible break fractions ζ are contained in the following sets in which η is the trimming parameter:

$$\Lambda_{0,i} = \{\zeta : \widehat{\tau}_{i-1} + (\widehat{\tau}_i - \widehat{\tau}_{i-1})\,\eta \leq \zeta \leq \widehat{\tau}_i - (\widehat{\tau}_i - \widehat{\tau}_{i-1})\,\eta\} \tag{3.11}$$

and

$$\Lambda_{1,i} = \{\zeta : \widetilde{\tau}_{i-1} + (\widetilde{\tau}_i - \widetilde{\tau}_{i-1})\,\eta \leq \zeta \leq \widetilde{\tau}_i - (\widetilde{\tau}_i - \widetilde{\tau}_{i-1})\,\eta\} \tag{3.12}$$

with $\widehat{\tau}_0 = 0$ and $\widehat{\tau}_{l+1} = 1$. Here, as before, Bartlett long run variance estimates are used for $\mathcal{F}_0^*(l+1|l)$ and $\mathcal{F}_1^*(l+1|l)$. For the exact same reasons outlined above for $\mathcal{F}_0^*(m|0)$ and $\mathcal{F}_1^*(m|0)$, the I(0)/I(1) dichotomy demands a weighted average of $\mathcal{F}_0^*(l+1|l)$ and $\mathcal{F}_1^*(l+1|l)$ so that the new weighted sequential $\mathcal{F}$ statistic can be used to estimate the number of breaks without making any assumption about the errors being I(0) or I(1). The weighted sequential $\mathcal{F}$ statistic, $\mathcal{F}_\lambda^*(l+1|l)$, is then given by:

$$\mathcal{F}_\lambda^*(l+1|l) := \lambda\left(\widehat{\tau}^{l+1}, \widetilde{\tau}^{l+1}\right)\mathcal{F}_0^*(l+1|l) + b_\xi^{l+1|l}\left[1 - \lambda\left(\widehat{\tau}^{l+1}, \widetilde{\tau}^{l+1}\right)\right]\mathcal{F}_1^*(l+1|l) \tag{3.13}$$

where, as before, $b_\xi^{l+1|l}$ is the constant that ensures that for a given significance level ξ and null hypothesis of l trend breaks the critical values of the asymptotic distribution of $\mathcal{F}_\lambda^*(l+1|l)$ are the same in both I(0) and I(1) cases.

The $\mathcal{F}_\lambda^*(l+1|l)$ can then be used to estimate the number of breaks in the trend slope without making any assumption about the errors being I(0) or I(1). The benchmark procedure starts with $l = 0$, by using the $\mathcal{F}_\lambda^*(1|0)$ to test for the presence of one break. If the null hypothesis is rejected, we set $l = 1$ and perform the $\mathcal{F}_\lambda^*(2|1)$ test. The procedure is repeated until the $\mathcal{F}_\lambda^*(l+1|l)$ test cannot reject the null hypothesis of l breaks.

In small samples, for some particular combinations of breaks in the trend slope, this sequential procedure may not perform well. For instance, in the presence of two breaks of opposite sign, the $F_{\lambda}^{*}(1|0)$ may have low power in identifying the two breaks, causing the sequential estimation procedure to stop too soon as can be observed in Table 4 from NS. To obviate this problem, NS suggested the use of the $F_{\lambda}^{*}(2|0)$ or a double maximum test $D\max F_{\lambda}^{*}$ whenever the $F_{\lambda}^{*}(1|0)$ does not reject the null hypothesis of no break. If $F_{\lambda}^{*}(2|0)$ or the double maximum test do not reject H_0 then we conclude that there are no trend breaks. Otherwise, we proceed to $F_{\lambda}^{*}(3|2)$. They called these sequential procedures $SeqF_{\lambda}^{*}(1|0)$, $SeqF_{\lambda}^{*}(2|0)$, $SeqUD\max F_{\lambda}^{*}$ and $SeqWD\max F_{\lambda}^{*}$. Figure 3.1 summarizes the necessary steps to implement each type of the sequential tests presented. Critical values and constants, $b_{\xi,m}$ and $b_{\xi}^{l+1|l}$, necessary for the implementation of each test are reported in Tables 1 and 2 of NS for a trimming parameter $\eta = 0.15$ which is going to be used throughout this paper as well.

3.2.3 Testing for general linear restrictions on the trend function across regimes

The sequential procedure discussed in Section 3.2.2 acts as a formal statistical pre-test for the presence of structural breaks in the per capita output growth rate. It also allows, in a first stage, to estimate the number of structural breaks and the timing in which these have occurred. Now after establishing the regimes set by the partitions $\widehat{\tau}^m$ and $\widetilde{\tau}^m$ we are in a position to construct a statistic to test for general linear restrictions on the coefficients of the linear disjoint broken trend model, conditional on the estimated number of breaks, break dates and coefficients. This statistical test will then be used to categorize countries according to the "constant trend", the "level shift" and the "slope shift" hypotheses previously discussed. For notational convenience, we suppress the index m from $\widehat{\tau}^m$ and $\widetilde{\tau}^m$. Hence, $\widehat{\tau}$ and $\widetilde{\tau}$ are the estimated break fractions if the true number of structural breaks in the trend function are set in (3.4) and (3.5).

We still do not require any *a priori* knowledge as to whether the noise component is I(0) or I(1). Consequently, since the asymptotic behavior of the test statistics based on levels and first differences depends on the I(0)/I(1) dichotomy, as in Section 3.2.2, we construct the test procedure as a weighted average of the tests appropriate for the case of I(0) and I(1) environments so that it becomes robust to both possibilities.

To expose explicitly how the method works, it is useful to express equations (3.1) and (3.2) in matrix notation. We start by stacking all the coefficients from (3.1) except α in a $2m+1$ vector, *i.e.*, $\Phi = (\delta_1, \ldots, \delta_m, \beta, \gamma_1, \ldots, \gamma_m)'$. We do not include α in Φ because this parameter is not identified in the first-differenced model (3.2). We also stack the regressors from the model in levels in the $2m+1$ vector $X_{DT,t}(\tau) = (DU_t(\tau_1), \ldots, DU_t(\tau_m), t, DT_t(\tau_1), \ldots, DT_t(\tau_m))'$. Hence, equation (3.1) can be written as,

$$y_t = \alpha + X_{DT,t}(\tau^*)'\Phi + u_t \qquad t = 1, ..., T. \tag{3.14}$$

Similarly, also the regressors from the model in first differences can be stacked in a $2m+1$ vector, as $X_{DU,t}(\tau) = (D_t(\tau_1), \ldots, D_t(\tau_m), 1, DU_t(\tau_1), \ldots, DU_t(\tau_m))'$ so that (3.2) can be rewritten as,

$$\Delta y_t = X_{DU,t}(\tau^*)'\Phi + \Delta u_t \qquad t = 2, ..., T. \tag{3.15}$$

Now suppose first that m and τ^* are known and u_t is known to be I(0). We want to build a statistical procedure to test general linear restrictions on the coefficient vector Φ. This amounts to testing the null hypothesis $H_0 : R\Phi = r$ against the two-sided alternative hypothesis $H_A : R\Phi \neq r$ where R is a q by $2m+1$ matrix with rank q and r is a q dimensional vector of constants. Then, the appropriate statistical inference method of testing H_0 against H_A rejects for H_0 large values of the F statistic computed from (3.14) by OLS. In other words, the statistic of interest, F_0^R is

$$F_0^R = \left(R\widehat{\Phi} - r\right)' \left[R\widehat{V}(\widehat{\Phi})R'\right]^{-1} \left(R\widehat{\Phi} - r\right)/q \tag{3.16}$$

where

$$\widehat{\Phi} = \left[\sum_{t=1}^{T} \left\{X_{DT,t}(\widehat{\tau}) - \overline{X}_{DT}(\widehat{\tau})\right\} \left\{X_{DT,t}(\widehat{\tau}) - \overline{X}_{DT}(\widehat{\tau})\right\}'\right]^{-1} \left[\sum_{t=1}^{T} \left\{X_{DT,t}(\widehat{\tau}) - \overline{X}_{DT}(\widehat{\tau})\right\} y_t\right] \tag{3.17}$$

and

$$\widehat{V}(\widehat{\Phi}) = \widehat{\omega}^2 \left[\sum_{t=1}^{T} \left\{X_{DT,t}(\widehat{\tau}) - \overline{X}_{DT}(\widehat{\tau})\right\} \left\{X_{DT,t}(\widehat{\tau}) - \overline{X}_{DT}(\widehat{\tau})\right\}'\right]^{-1} \tag{3.18}$$

where $\widehat{\omega}^2$ denotes the Bartlett long run variance estimator obtained from the residuals of the regression described by equation (3.1). Under these assumptions, it is well known that $q \cdot \mathcal{F}_0^R$ has a χ_q^2 asymptotic distribution.

On the other hand, suppose now that m and τ^* continue to be known but u_t is now I(1), that is $\rho = 1$ in (3.3). The appropriate statistical inference method of testing H_0 against H_A consists of estimating the coefficient vector Φ in equation (3.15) by OLS so that the noise component becomes I(0) and reject H_0 for large values of the $\mathcal{F}_1^R$ statistic defined as,

$$\mathcal{F}_1^R = \left(R\widetilde{\Phi} - r\right)' \left[R\widetilde{V}(\widetilde{\Phi})R'\right]^{-1} \left(R\widetilde{\Phi} - r\right)/q \tag{3.19}$$

$$\widetilde{\Phi} = \left[\sum_{t=2}^{T} X_{DU,t}(\widetilde{\tau})X_{DU,t}(\widetilde{\tau})'\right]^{-1} \left[\sum_{t=2}^{T} X_{DU,t}(\widetilde{\tau})\Delta y_t\right] \tag{3.20}$$

and

$$\widetilde{V}(\widetilde{\Phi}) = \widetilde{\omega}^2 \left[\sum_{t=2}^{T} X_{DU,t}(\widetilde{\tau})X_{DU,t}(\widetilde{\tau})'\right]^{-1} \tag{3.21}$$

where $\widetilde{\omega}^2$ is the Bartlett long run variance estimator obtained from the residuals of the regression described by equation (3.2). Under these assumptions and the normality of the errors, we have that $q \cdot \mathcal{F}_1^R$ also has a χ_q^2 asymptotic distribution. As discussed in Remarks 1 and 5 from Perron and Yabu (2009) the normality of the noise component is needed because the level shift dummies, $DU_t(\tau^*)$, become impulse dummies, $D_t(\tau^*)$, with a single outlier at $T^* + 1$ when we apply first differences to Model (3.1). Consequently, if the linear restrictions to be tested do not involve parameters $\delta_1, \ldots, \delta_m$ it is possible to rule out the normality assumption and still attain the chi-square asymptotic distribution.

In practice, the precise number of structural breaks and their dates are rarely known. The approach to overcome this limitation is to use first the sequential procedure described in Section 3.2.2 to obtain $\widehat{m}$ and $\widehat{\tau}$ and replace τ^* in (3.16) and (3.19) by $\widehat{\tau}$ and $\widetilde{\tau}$, respectively. The next theorem shows that the asymptotic distribution of $\mathcal{F}_0^R$ and $\mathcal{F}_1^R$ is the same regardless of whether we use the true or the estimated break fractions.

Theorem 10. *Let the time series process y_t be generated according to (3.1) and (3.3) with $\gamma_j \neq 0$, $j = 1, ..., m$ under $H_0 : R\Phi = r$ and let Assumption 6 hold. If:*

(a) u_t *is I(0), then* $q \cdot \mathcal{F}_0^R \xrightarrow{d} \chi_q^2$.

(b) u_t *is I(1), then* $q \cdot \mathcal{F}_1^R \xrightarrow{d} \chi_q^2$.

Now since the order of integration of u_t is not known in practice we need a weight function that converges to unity if u_t is I(0) and to zero if u_t is I(1) such that the weighted F-statistic collapses asymptotically to the F-statistic corresponding to the true order of integration. Since the KPSS tests applied to the levels and first differenced data are invariant with respect to the values of parameters $\alpha, \beta, \delta_1, \ldots, \delta_m, \gamma_1, \ldots, \gamma_m$ in (3.1) we conclude that the relevant large sample properties of the KPSS procedure applied to the levels and first differenced data are exactly the same as described in Lemma 1 from NS for the known break fraction case and in Lemma 2 from NS for the unknown break fraction case regardless of whether H_0 or H_A holds. Hence, we have that both under H_0 and H_A, $\lambda(\widehat{\tau}, \widetilde{\tau}) \xrightarrow{p} 1$ if u_t is I(0) and $\lambda(\widehat{\tau}, \widetilde{\tau}) \xrightarrow{p} 0$ if u_t is I(1). Moreover, it does so at an exponential rate which ensures that the appropriate F statistic is selected asymptotically even if the other F statistic diverges in probability at a polynomial rate. Based on these results, the proposed statistic to test general linear restrictions on the trend function across regimes is an analogue of the $\mathcal{F}_\lambda^*$ statistics in (3.6) and (3.13) and is given by,

$$\mathcal{F}_\lambda^R = \lambda(\widehat{\tau}, \widetilde{\tau}) \mathcal{F}_0^R + [1 - \lambda(\widehat{\tau}, \widetilde{\tau})] \mathcal{F}_1^R \tag{3.22}$$

From the arguments presented above, we are now in position to state the following corollary regarding the large sample behavior of the $\mathcal{F}_\lambda^R$ statistic:

Corollary 12. *Let the time series process* y_t *be generated according to (3.1) and (3.3) with* $\gamma_j \neq 0$, $j = 1, ..., m$ *under* $H_0 : R\Phi = r$ *and let Assumption 6 hold. If:*

(a) u_t *is I(0), then* $q \cdot \mathcal{F}_\lambda^R \xrightarrow{d} \chi_q^2$.

(b) u_t *is I(1), then* $q \cdot \mathcal{F}_\lambda^R \xrightarrow{d} \chi_q^2$.

From Corollary 12 we conclude that, regardless of whether u_t is I(0) or I(1), $q \cdot \mathcal{F}_\lambda^R$ achieves the chi-square distribution with q degrees of freedom and so the two-sided test of H_0 against H_A is straightforward to implement using critical values from a chi-squared distribution with degrees of freedom corresponding to the total number of restrictions being tested. The F_λ^R statistic is going to be a useful statistical tool to classify the countries according to the "linear trend", "level shift" and "growth shift" hypothesis.

Note that, as mentioned in the introductory note of Section 3.2, if we find evidence for the presence of two or more trend breaks this result is not sufficient to favor any of these three hypothesis. To support the "linear trend" hypothesis the deterministic trend following the last break has to be a linear projection of the trend function until the first break. This amounts to formally test the following two restrictions:

$$\gamma_1 + \cdots + \gamma_m = 0 \tag{3.23}$$

which imposes the slope of the trend function to be the same in the first and final regimes, and

$$\delta_1 + \ldots + \delta_m + \gamma_1 \left(T_m^* - T_1^*\right) + \ldots + \gamma_{m-1}\left(T_m^* - T_{m-1}^*\right) = 0 \tag{3.24}$$

that restricts the trend function from the last regime to be equal to the deterministic trend from the first regime. This set of restrictions can be casted in the format $R\Phi = r$ if R and r are defined as,

$$R = \begin{bmatrix} 0 & \ldots & 0 & 0 & 1 & \ldots & 1 & 1 \\ 1 & \ldots & 1 & 0 & \left(T_m^* - T_1^*\right) & \ldots & \left(T_m^* - T_{m-1}^*\right) & 0 \end{bmatrix}, \qquad r = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.25}$$

If we perform the F_λ^R test and fail to reject this set of restrictions then we conclude that the corresponding country satisfies the neoclassical 'linear trend" hypothesis. Rejection of the set of restrictions in (3.25) does not automatically imply the choice of the endogenous growth theory. In fact, both Jones' semi-endogenous and Solow's neoclassical growth models allow for changing growth rates. Jones (1995a, 2002, 2005) documents that, at least, since the World War II, several policy variables exhibited large, persistent movements, generally in the "growth-increasing" direction in several OECD countries. According to the semi-endogenous and neoclassical theories, per capita output should have deviated from the steady state level after these changes inducing temporary higher than steady state growth rates. However, transitional dynamics should force a gradual decline in the growth rate until it attains its steady-state value. After these shocks, we should observe the same original steady state growth rate but a higher long run per capita output level. Hence, the "level shift" hypothesis is tested formally with the first restriction in (3.25) that the slope coefficient before the first break and after the last break should be equal:

$$R = \begin{bmatrix} 0 & \dots & 0 & 0 & 1 & \dots & 1 \end{bmatrix}, \qquad r = 0 \tag{3.26}$$

The failure to reject restriction (3.26) with F_{λ}^{R} test is taken to imply that the "level shift" hypothesis holds for the analyzed country. Finally, if both sets of restrictions defined in (3.25) and in (3.26) are rejected this is interpreted as evidence against the "neoclassical" and "semi-endogenous" predictions and compatible with Romer endogenous growth theory or the "growth shift" hypothesis.

Since, in practice, we do not know $(T_1^*, \dots, T_m^*)$ we replace these values in (3.25) and (3.26) by its estimates obtained from the first step procedure.

3.3 Results of the Economic Growth hypotheses tests

After describing the econometric methodology to be used we are now in position to classify the countries according to the "linear trend", "level shift" and "growth shift" hypotheses.

We used data on per capita GDP from 1870 to 2008 for the following countries: Austria, Belgium, France, Germany, Netherlands, Switzerland, Canada, United States, Brazil, Chile, Uruguay, Sweden, Denmark, Finland, Norway, United Kingdom, Japan, Sri Lanka, Australia, New Zealand, Italy, Portugal and Spain. This dataset was obtained from Maddison (2009).

3.3.1 *Testing for Breaks in Steady State Growth*

Our analysis starts by identifying which shocks have affected significantly the real per capita GDP growth rate. Given that our dataset includes long historical time series for an extensive set of countries, by simple inspection of Economic History it is straightforward to write a large list of candidate economic events that could have had a strong impact on the output growth path of each country. A data dependent algorithm is therefore needed to select those shocks that in fact had a statistically significant effect on the steady state growth rate and to specify exactly when the consequent change in trend started.

Hence, the first step tests for the existence of (one or multiple) structural breaks in the trend function without assuming any *a priori* knowledge of the candidate break points. Table 3.1 reports results from application of $F_\lambda^*(m|0)$ for $m = 1, 2, 3$, the $UD\max F_\lambda^*$ and $WD\max F_\lambda^*$ tests with $M = 3$ to per capita GDP series for various countries at the 10%, 5% and 1% significance levels. When the null is rejected at 5% level, we present the estimated break dates in parentheses. All tests fail to reject the null of no trend break at all significance levels considered for Switzerland, Canada, United States, Chile, Sweden and Australia. The $F_\lambda^*(3|0)$ rejects the no break in trend hypothesis for New Zealand at 10% level but not at 5% level. Since all other tests fail to reject the null, we consider that there is not enough evidence to conclude that this country had any structural break in the slope of the trend function. Therefore, all these countries are in favor of the "constant trend" hypothesis.

In opposition, we reject the null of no trend break in all tests considered for Sri Lanka, Portugal, Spain (at all significance levels considered), Japan, Italy (at all significance levels considered except for $F_\lambda^*(1|0)$), Belgium, Netherlands, Finland and Norway (at 5% level or higher). Interestingly, for the United Kingdom the constant trend hypothesis is rejected when we apply the $F_\lambda^*(1|0)$ and $F_\lambda^*(2|0)$ tests even at 1% significance level but doesn't reject the null neither for $F_\lambda^*(3|0)$ nor for the $D\max F_\lambda^*$ tests for all significance levels considered. This may be explained by the loss of power due to allowing for more breaks than necessary as observed in Figures 1 to 3 from NS.

Since the implementation of $F_\lambda^*(m|0)$ tests require the specification of the number of trend breaks under the alternative hypothesis and the $D\max F^*_\lambda$ tests do not estimate the break dates if the null is rejected, additional statistical procedures are needed to determine the exact number and timing of trend breaks. Hence, it is of practical relevance to implement recursive methods as described in Section 3.2.2 to estimate the number of structural breaks. Table 3.2 reports number of breaks and respective break dates estimated from the implementation of the sequential procedures to GDP per capita series of the countries analyzed. Results for all the sequential procedures in Table 3.2 show statistical evidence of two trend breaks for Netherlands, Japan, United Kingdom and Italy and one break in slope for Belgium, Finland, Norway, Sri Lanka, Portugal and Spain. Hence, our results clearly support the "growth shift" hypothesis for the second enumerated group of countries but are not conclusive for those countries where two breaks have been found. For this group of countries, we need to apply restricted structural change tests to classify between the three economic growth hypotheses. The results of these tests are discussed in the next section.

We find ambiguous results for Uruguay as the decision to reject or not the null hypothesis depends on the test implemented: we reject the null with $F^*_\lambda(1|0)$, $WD\max F^*_\lambda$, $F^*_\lambda(3|0)$ tests but not with $F^*_\lambda(2|0)$ and the $UD\max F^*_\lambda$ tests at 5% significance level. To help solving this discrepancy we take advantage of results from the sequential procedures in Table 3.2. Here the results are unanimous and identify one trend break for Uruguay which is supportive of the "growth shift" hypothesis and provide no evidence for breaks in France in line with the "constant trend" hypothesis.

The results for the remaining countries may also seem startling at first sight: for Austria, Germany and Brazil, the application of F^*_λ class of statistics rejects the null against two and three trend breaks under the alternative but surprisingly fails to reject against one trend break at 5% significance level. The $D\max F^*_\lambda$ tests seem to confirm the results from $F^*_\lambda(2|0)$ and $F^*_\lambda(3|0)$ as they always reject the no breaking trend hypothesis at 5% level. France and Denmark again fail to reject the null against one trend break even at 10% level but the remaining tests show more ambiguous results: for Denmark the "constant trend" hypothesis is rejected if we use $F^*_\lambda(2|0)$ test but is only rejected at 10%

level according to $F_\lambda^*(3|0)$ and $D\max F_\lambda^*$ tests. For France the $F_\lambda^*(3|0)$ and $WD\max F_\lambda^*$ tests find evidence for trend breaks at 5% level but the $F_\lambda^*(2|0)$ and $UD\max F_\lambda^*$ tests only reject the null at 10% significance level.

This mixed evidence is also observed for the sequential procedures: the $SeqF_\lambda^*(1|0)$ procedure finds no evidence for trend breaks in total opposition to the two breaks evidenced by the $SeqF_\lambda^*(2|0)$ except for France where two breaks are only detected if we use $SeqWD\max F_\lambda^*$ method. The $SeqUD\max F_\lambda^*$ and $SeqWD\max F_\lambda^*$ procedures reinforce the no breaks conclusion of $SeqF_\lambda^*(1|0)$ for Denmark and the two breaks conclusion of $SeqF_\lambda^*(2|0)$ for Austria, Germany and Brazil. We pursue our analysis with two trend breaks for these five countries and actually the battery of tests discussed in the next section provide insights to this conflicting evidence.

3.3.2 Restricted Structural Breaks and Economic Growth hypotheses

After the first structural break, did GDP per capita growth rate deviated from its steady state value but transition dynamics returned the economy to its steady state growth path? Or even in a stronger sense did per capita output trend returned to the no break counterfactual trend path? Or, contrarily, after the structural break, no transition dynamics is observed and the economy continues on a new and different steady state growth path?

The statistical answers to these questions are discussed in this section. Table 3.3 reports results for restricted structural change tests applied to countries that have shown evidence for 2 trend breaks. The second and third columns present F-statistics and p-values associated with testing that steady state growth rates from the first and last regimes are equal. This amounts to testing the null hypothesis defined in (3.26) with $m = 2$. The F_λ^R fails to reject the null hypothesis at 5% significance level for all listed countries except United Kingdom. We conclude that evidence favors the "growth shift" hypothesis for the United Kingdom. These results also explain the disparate evidence as regards to the number of slope changes in the trend function for Austria, France, Germany, Brazil and Denmark: Prodan (2008) and NS document that it is very likely that the standard sequential procedure cannot reject the null of no breaks in the presence of structural breaks of opposite sign. These countries represent the most problematic case because not only the direction is opposite but statistical evidence shows that $\gamma_1 = \gamma_2$, i.e, the second structural break cancels the effect on the growth rate of the first structural break.

But can we say that not only the steady state growth rate but the all the trend function has been constant over time except during the period between the two estimated break dates? The fourth and fifth columns report F-statistics and p-values for testing that the trend function from the last regime is a linear projection of the trend from the first regime. Here the null hypothesis is given by (3.25) under the assumption that two breaks occurred at times $\left(\widehat{T}_1, \widehat{T}_2\right)$ if the model is estimated in levels or $\left(\widetilde{T}_1, \widetilde{T}_2\right)$ if the model is estimated in first differences. We fail to reject the null even at 20% significance level for Austria and Germany. This result clearly supports the "constant trend" hypothesis. We obtain rejections at 5% level for Netherlands and Denmark and even at 1% level for France, Brazil, Japan and Italy and so we conclude that the "weaker" "level shift" hypothesis holds for these countries.

Figures 3.2 to 3.6 plot the variable of interest, GDP per capita measured in logarithms, for the countries analyzed. We superimposed the estimated break dates and the fitted values of the unrestricted model. For those countries with two statistically significant structural breaks we also superimposed the fitted values of the model restricted by the "level shift" hypothesis and restricted by the "constant trend" hypothesis. From simple visual inspection, we think that the estimated break dates correspond reasonably well to the timings when the trend function behavior changes in an important way. Also, for countries that did not reject the restrictions, the fitted restricted model seems to adjust well to the observed movements of the data.

In summary, according to the previous econometric analysis we may divide countries considered according to the economic growth theory hypotheses in the following way:

Economic Growth Hypotheses	Countries that best fit each hypothesis
"Summer-Weil-Jones" or "constant trend"	Austria, Germany, Switzerland, Canada, United States, Chile, Sweden, Australia, New Zealand
"Jones-Solow" or "level shift"	France, Netherlands, Brazil, Denmark, Japan, Italy
"Romer" or "slope shift"	Belgium, Uruguay, Finland, Norway, United Kingdom, Sri Lanka, Portugal, Spain

3.4 Conclusion

In this paper we have proposed an econometric procedure to classify countries according to the economic growth hypothesis that best describes the behavior of its real GDP per capita. Our method is implemented in two steps: first, we select the number and timing of changes in the slope of the per capita output deterministic trend. However, this information may not be enough for proper classification because if we detect more than one trend break then different configurations of the slope changes may assign each country to different hypotheses. Hence, in the second step, given the estimated number and timing of the trend breaks, we build a statistical framework to test for general linear restrictions on the level and slope of the linear trend function.

In the same spirit as Harvey et al. (2009), both tests are made robust to the I(0)/I(1) dichotomy via the use of weighted averages of two conventional F statistics, one appropriate for an I(0) environment and the other when the data are I(1). Hence, our approach surpasses technical and methodological limitations from previous approaches to the same research question.

Since the economic growth hypotheses considered are formulated as linear restrictions on the parameters of the breaking trend model, we are now able to classify the countries according to the different hypotheses.

We find evidence favoring the "constant trend" hypothesis for nine countries: Austria, Germany, Switzerland, Canada, United States, Chile, Sweden, Australia and New

Zealand. The results of our tests support the "level shift" hypothesis for six countries: France, Netherlands, Brazil, Denmark, Japan and Italy. Finally, there is a third group of eight countries where statistical evidence favors the "growth shift" hypothesis: Belgium, Uruguay, Finland, Norway, United Kingdom, Sri Lanka, Portugal and Spain.

To conclude we briefly discuss some issues that are on the research agenda: First, since the results from the restricted structural change tests are asymptotic by nature, there is certainly the need to evaluate the quality of the asymptotic approximation and the finite sample power of the tests via Monte Carlo simulations. Second, we have focused in this paper on pre-testing slope changes in the deterministic trend function allowing for simultaneous breaks in level. If the test does not detect a change in slope this automatically assigned the country to the "constant trend" hypothesis. For those countries with no evidence for a significant change in slope, it would also be useful to apply robust methods to detect level breaks while accommodating a deterministic linear trend developed by Harvey et al. (2010). The level shifts may or may not prevent the linear trend following the last level shift to be strictly a linear projection of the trend preceding the first level shift. In spite of the invariant steady state growth rates across regimes, it is debatable as to whether the first case corresponds to the "level shift" hypothesis and so it would be interesting to accommodate this extension in our analysis. Finally, since the econometric framework analyzed is quite general it would be interesting to implement the two step econometric procedure to a tourism dataset where it is very important to infer about how soon can the industry recover from previous significantly negative shocks.

Proof of Theorem 10. According to Perron and Zhu (2005) $\widehat{\tau} = \tau^* + O_p\left(T^{-1}\right)$ if u_t is I(0) and from Bai and Perron (1998) $\widetilde{\tau} = \tau^* + O_p\left(T^{-1}\right)$ if u_t is I(1). Though the proof from Perron and Zhu (2005) is for the single break case, their results continue to hold for the multiple breaks case as argued by Kejriwal and Perron (2010). Using these results on the asymptotic properties of $\widehat{\tau}$ and $\widetilde{\tau}$ it is possible to show that $\Upsilon_0(\widehat{\Phi}(\widehat{\tau}) - \widehat{\Phi}(\tau^*)) \xrightarrow{p} 0$ and $\Upsilon_0\left(\widehat{V}(\widehat{\Phi}(\widehat{\tau})) - \widehat{V}(\widehat{\Phi}(\tau^*))\right) \xrightarrow{p} 0$ if u_t is I(0). Similarly, for the model in differences we find that $\Upsilon_1(\widetilde{\Phi}(\widetilde{\tau}) - \widetilde{\Phi}(\tau^*)) \xrightarrow{p} 0$ and $\Upsilon_1\left(\widetilde{V}(\widetilde{\Phi}(\widetilde{\tau})) - \widetilde{V}(\widetilde{\Phi}(\tau^*))\right) \xrightarrow{p} 0$ if u_t is I(1). Here Υ_0 and Υ_1 are the appropriate normalization matrices of the corresponding OLS estimators. Hence $\mathcal{F}_0^R(\widehat{\tau}) - \mathcal{F}_0^R(\tau^*) \xrightarrow{p} 0$ if u_t is I(0) and $\mathcal{F}_1^R(\widetilde{\tau}) - \mathcal{F}_1^R(\tau^*) \xrightarrow{p} 0$ if u_t is I(1). The rest of the proof now follows from the fact that $q \cdot \mathcal{F}_0^R(\tau^*) \xrightarrow{d} \chi_q^2$ if u_t is I(0) and $q \cdot \mathcal{F}_1^R(\tau^*) \xrightarrow{d} \chi_q^2$ if u_t is I(1). $\square$

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Tab. 3.1: Empirical Application of F^*_λ and $D\max F^*_\lambda$ tests to real GDP per capita

Countries	$F^*_\lambda(m\|0)$			$UD\max F^*_\lambda$	$WD\max F^*_\lambda$
	$m=1$	$m=2$	$m=3$		
Austria	7.98* (1944)	10.90*** (1943,1964)	8.62** (1919,1943,1964)	10.35**	11.78**
Belgium	11.30** (1941)	9.09** (1941,1973)	8.83*** (1920,1941,1973)	11.44**	11.24**
France	7.24	8.17* (1922,1943)	9.61*** (1922,1943,1972)	8.87*	12.23**
Germany	5.63	12.40*** (1944,1965)	10.09*** (1922,1944,1965)	11.78**	13.41**
Netherlands	10.85** (1943)	10.83*** (1923,1944)	10.45*** (1922,1943,1969)	10.99**	13.31**
Switzerland	2.37	6.18	4.87	5.87	6.68
Canada	2.41	4.75	4.11	4.51	5.23
United States	2.13	2.49	2.52	2.36	3.21
Brazil	8.50* (1892)	10.50** (1940,1979)	10.66*** (1917,1940,1979)	9.97**	13.57**
Chile	5.29	5.53	5.59	5.35	7.11
Uruguay	10.40** (1922)	1.71	8.19** (1906,1953,1968)	8.51*	10.71**
Sweden	3.49	5.10	5.55	5.13	7.07
Denmark	6.70	9.39** (1939,1972)	7.00* (1909,1939,1972)	8.92*	10.15*
Finland	11.72** (1916)	9.33** (1916,1937)	7.38** (1916,1937,1972)	11.87**	11.37**
Norway	12.00** (1943)	9.84** (1942,1979)	7.96** (1904,1942,1979)	12.15**	11.64**
United Kingdom	48.63*** (1935)	30.56*** (1902,1924)	4.01	6.04	5.78
Japan	10.93** (1943)	44.50*** (1943,1972)	36.14*** (1914,1943,1972)	42.26***	48.11***
Sri Lanka	17.57*** (1974)	11.14*** (1898,1974)	10.05*** (1898,1946,1974)	17.79***	17.04***
Australia	6.25	5.42	3.92	6.32	6.05
New Zealand	4.74	6.97	6.70* (1909,1931,1965)	6.62	8.53
Italy	11.98** (1943)	18.43*** (1943,1968)	16.77*** (1914,1943,1968)	17.51***	21.35***
Portugal	16.01*** (1940)	21.18*** (1950,1972)	18.95*** (1920,1950,1972)	20.12***	24.12***
Spain	15.88*** (1948)	13.84*** (1948,1973)	13.67*** (1927,1948,1973)	16.08***	17.40***

Notes: *, ** and *** refers to rejection at the 10%, 5% and 1% significance level, respectively. Where rejections are obtained for the $F^*_\lambda(0|m)$ test at 5% significance level , the estimated break dates are reported in parentheses.

Tab. 3.2: Empirical Application of Sequential tests to real GDP per capita

Countries\Test	$SeqF_{\lambda}^{*}(1\|0)$	$SeqF_{\lambda}^{*}(2\|0)$	$SeqUD\max F_{\lambda}^{*}$	$SeqWD\max F_{\lambda}^{*}$
Austria	0	2 (1943,1964)	2 (1943,1964)	2 (1943,1964)
Belgium	1 (1941)	1 (1941)	1 (1941)	1 (1941)
France	0	0	0	2 (1922,1943)
Germany	0	2 (1944,1965)	2 (1944,1965)	2 (1944,1965)
Netherlands	2 (1923,1944)	2 (1923,1944)	2 (1923,1944)	2 (1923,1944)
Switzerland	0	0	0	0
Canada	0	0	0	0
United States	0	0	0	0
Brazil	0	2 (1940,1979)	2 (1940,1979)	2 (1940,1979)
Chile	0	0	0	0
Uruguay	1 (1922)	1 (1922)	1 (1922)	1 (1922)
Sweden	0	0	0	0
Denmark	0	2 (1939,1972)	0	0
Finland	1 (1916)	1 (1916)	1 (1916)	1 (1916)
Norway	1 (1943)	1 (1943)	1 (1943)	1 (1943)
United Kingdom	2 (1902,1924)	2 (1902,1924)	2 (1902,1924)	2 (1902,1924)
Japan	2 (1943,1972)	2 (1943,1972)	2 (1943,1972)	2 (1943,1972)
Sri Lanka	1 (1974)	1 (1974)	1 (1974)	1 (1974)
Australia	0	0	0	0
New Zealand	0	0	0	0
Italy	2 (1943,1968)	2 (1943,1968)	2 (1943,1968)	2 (1943,1968)
Portugal	1 (1940)	1 (1940)	1 (1940)	1 (1940)
Spain	1 (1948)	1 (1948)	1 (1948)	1 (1948)

Tab. 3.3: Restricted Structural Change Tests

Countries\Test	"Level shift" hypothesis		"Constant trend" hypothesis	
	F_λ^R statistic	p-value	F_λ^R statistic	p-value
Austria	1.76	0.42	1.41	0.24
France	3.06*	0.22	5.36***	0.00
Germany	0.53	0.77	1.45	0.23
Netherlands	2.92*	0.23	3.32**	0.04
Brazil	0.06	0.97	6.30***	0.00
Denmark	0.50	0.78	4.42**	0.01
United Kingdom	53.18***	0.00	40.42***	0.00
Japan	0.33	0.85	10.38***	0.00
Italy	1.78	0.41	9.80***	0.00

Tab. 3.4: Estimated growth rates , in percentage terms, for the "growth shift"\"level shift" hypothesis

Countries\Growth rates	Unrestricted Model (growth shift)			Restricted Model (level shift)		
	1st regime	2nd regime	3rd regime	1st regime	2nd regime	3rd regime
Austria	1.07	3.00	2.65	1.65	3.00	1.65
France	1.26	-1.36	3.46	2.48	-1.36	2.48
Germany	1.62	3.55	1.89	1.72	3.55	1.72
Netherlands	0.98	-3.07	3.52	2.36	-3.07	2.36
Brazil	0.80	3.47	0.76	0.79	3.47	0.79
Denmark	1.59	3.04	1.62	1.60	3.04	1.60
United Kingdom	1.09	0.49	1.84	1.63	0.49	1.63
Japan	1.84	4.98	1.97	1.88	4.98	1.88
Italy	0.97	5.45	1.88	1.29	5.45	1.29

Tab. 3.5: Estimated growth rates , in percentage terms, for the "growth shift"\"constant trend" hypothesis

Countries\Growth rates	Unrestricted Model (growth shift)			Restricted Model (constant trend)		
	1st regime	2nd regime	3rd regime	1st regime	2nd regime	3rd regime
Austria	1.07	3.00	2.65	1.86	1.93	1.86
France	1.26	-1.36	3.46	1.79	2.25	1.79
Germany	1.62	3.55	1.89	1.76	3.34	1.76
Netherlands	0.98	-3.07	3.52	1.59	1.00	1.59
Brazil	0.80	3.47	0.76	1.59	1.50	1.59
Denmark	1.59	3.04	1.62	1.82	2.37	1.82
United Kingdom	1.09	0.49	1.84	1.45	1.39	1.45
Japan	1.84	4.98	1.97	2.49	2.81	2.49
Italy	0.97	5.45	1.88	1.87	2.91	1.87

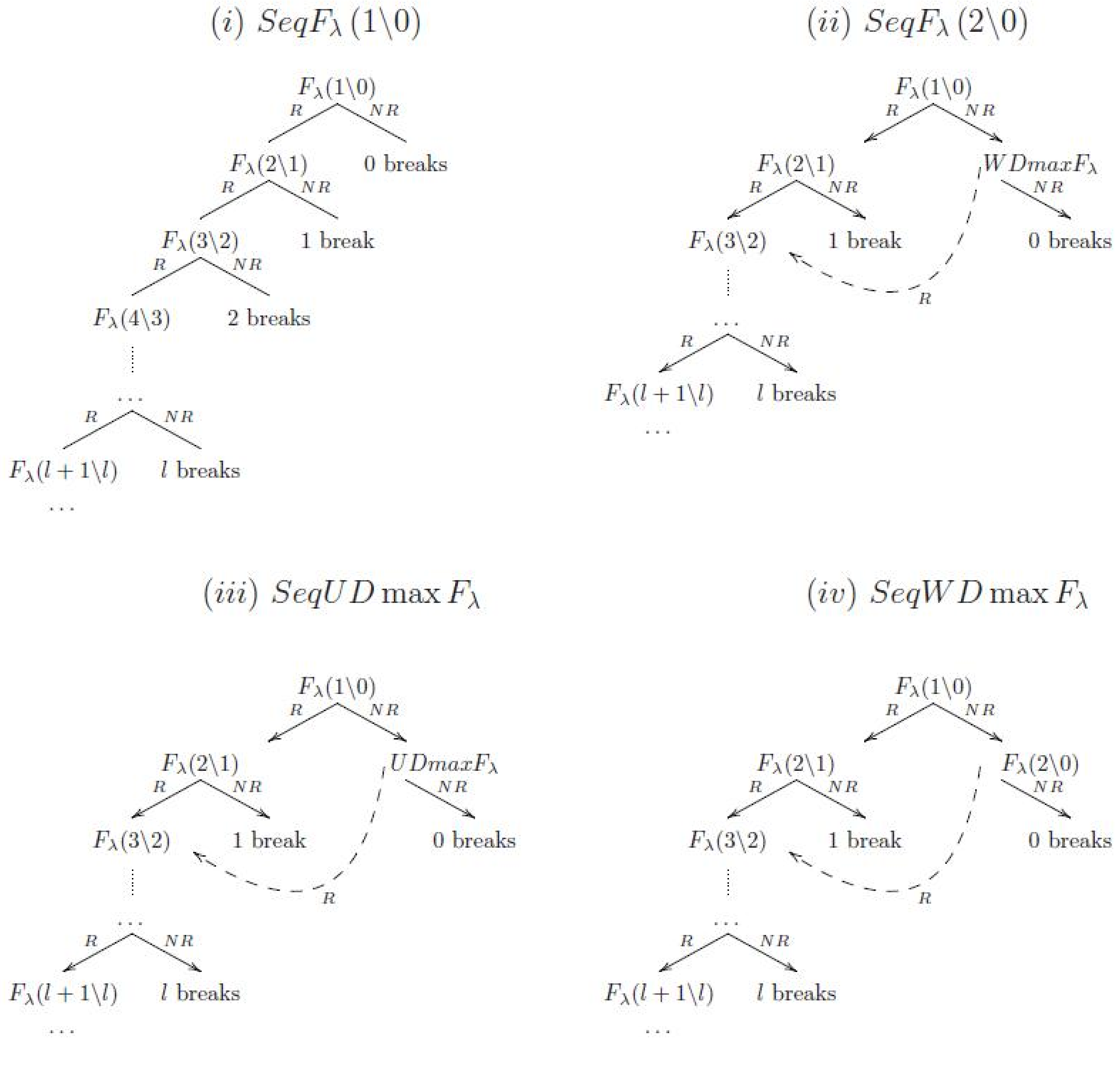

Fig. 3.1: Sequential Tests procedure

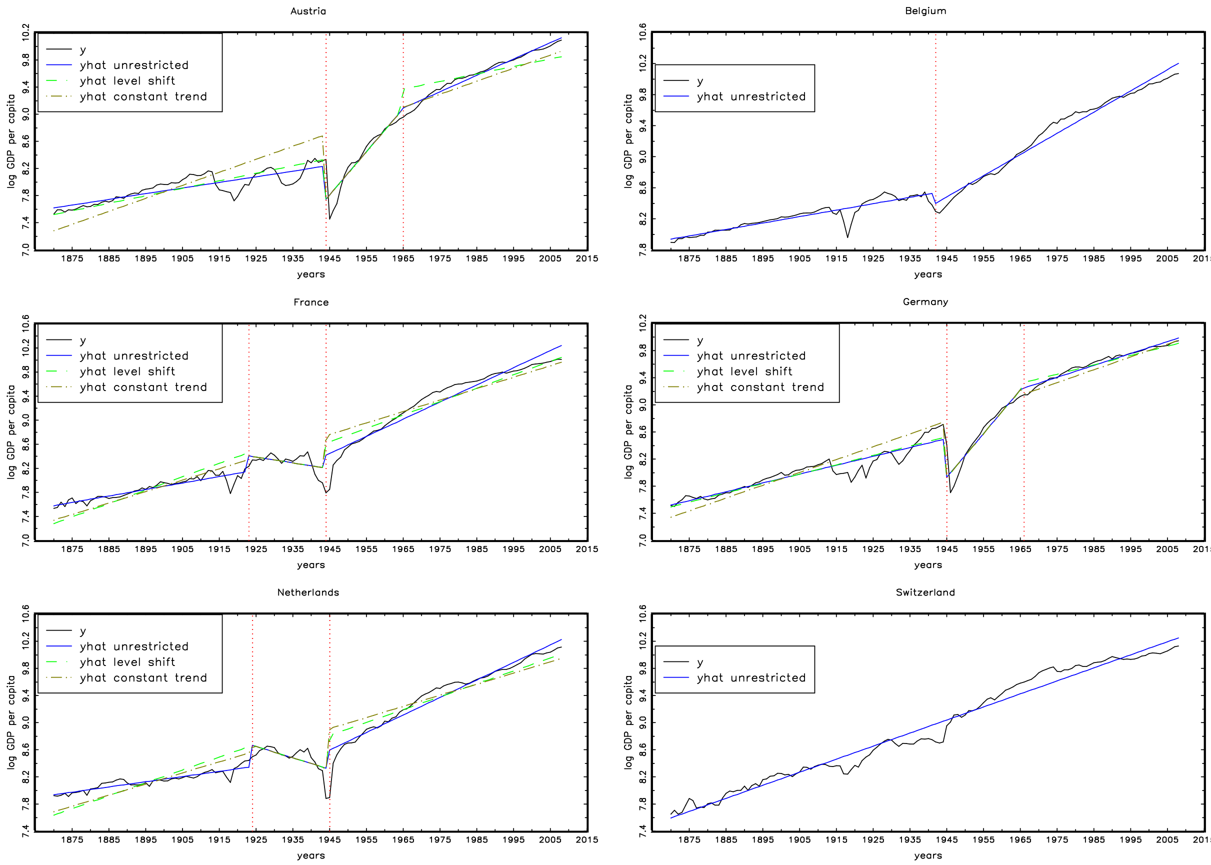

Fig. 3.2: Real GDP per capita - Western Europe

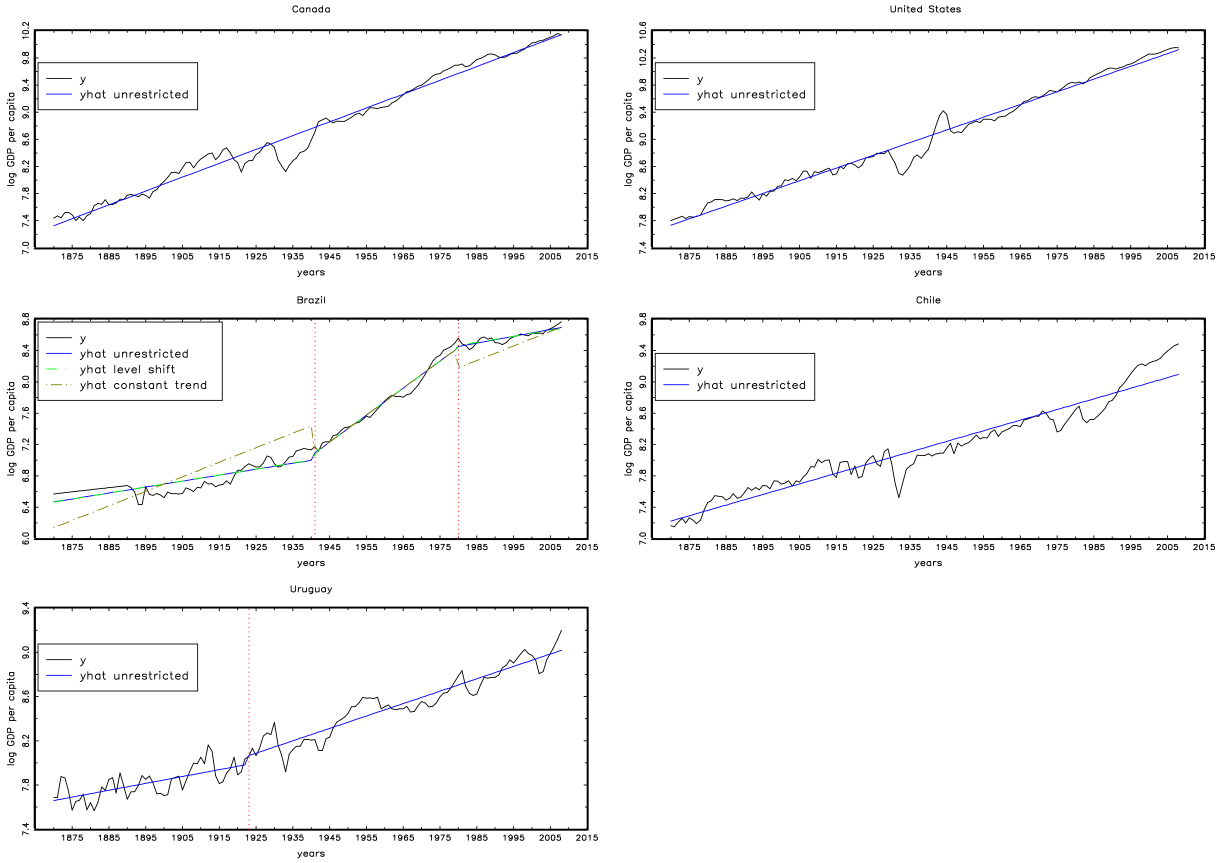

Fig. 3.3: Real GDP per capita - North/South America

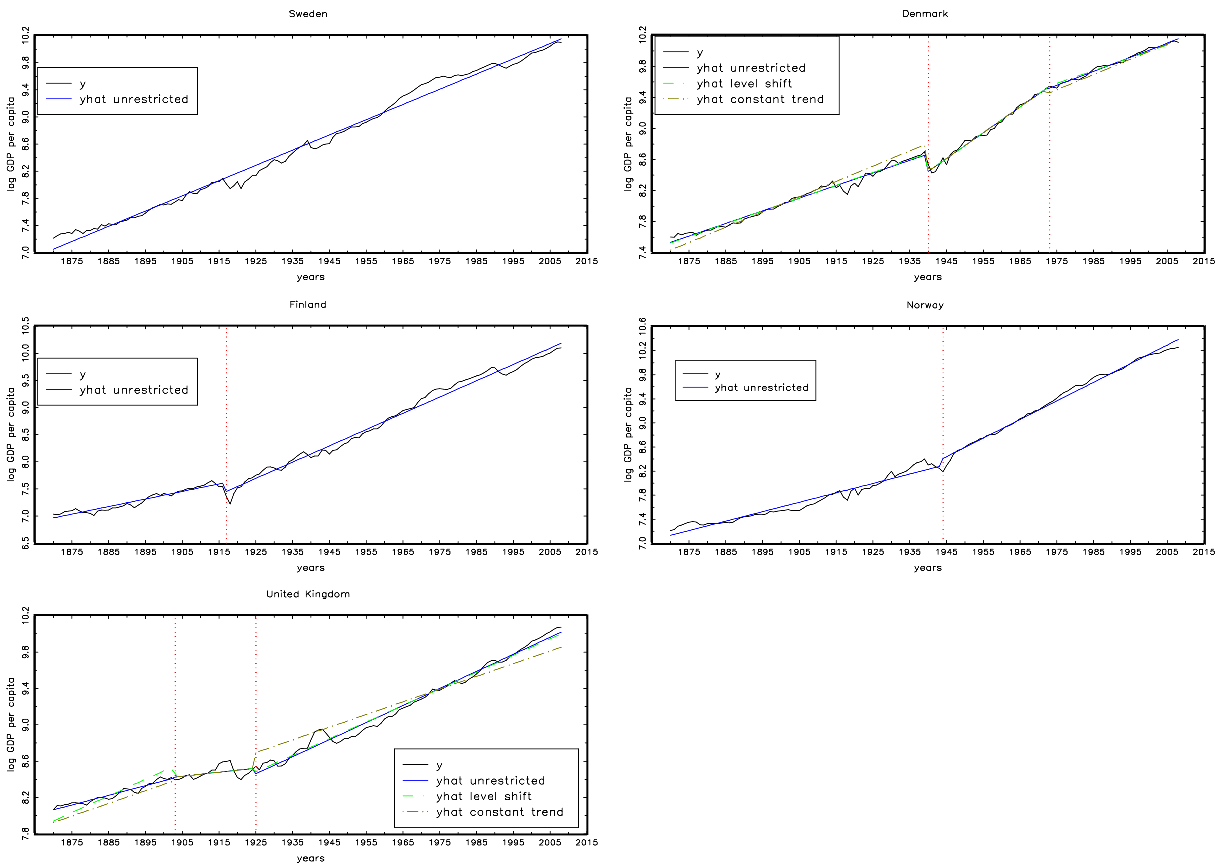

Fig. 3.4: Real GDP per capita - Northern Europe

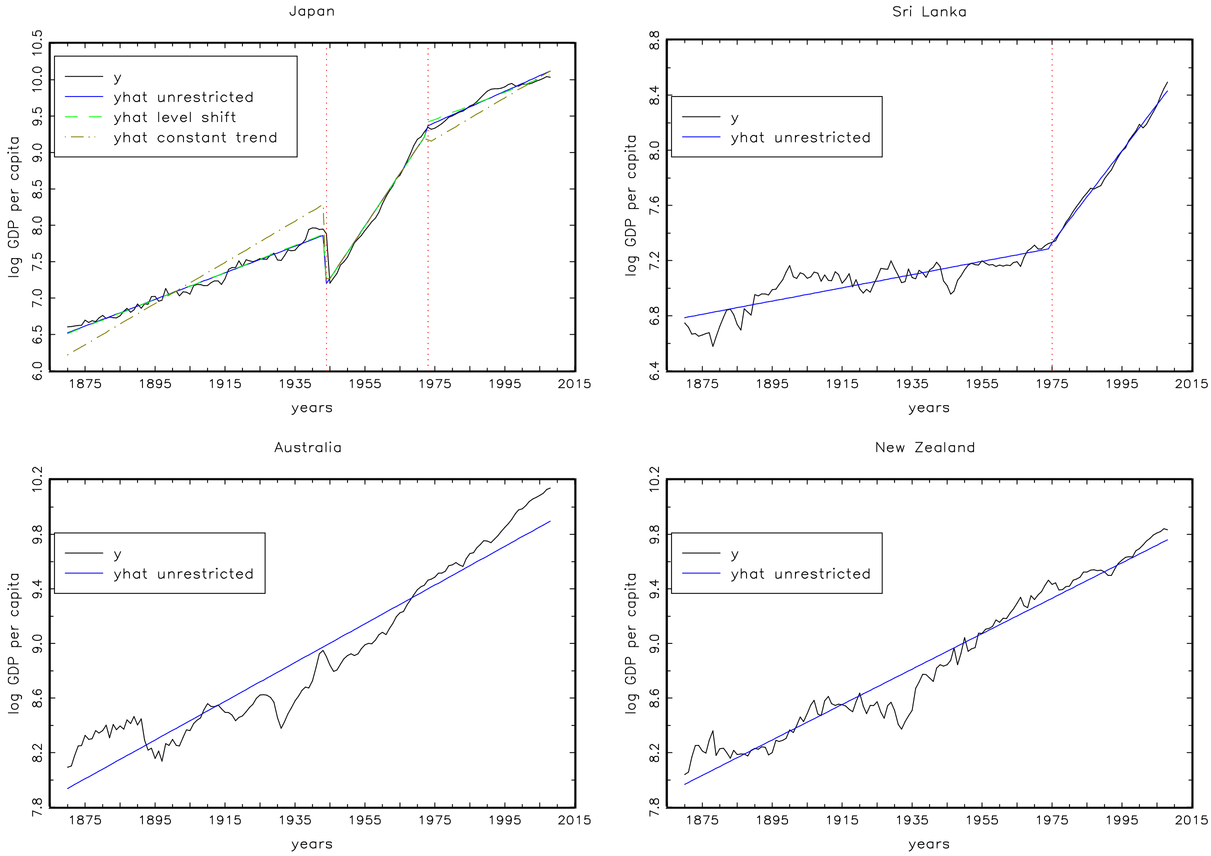

Fig. 3.5: Real GDP per capita - Asia and Oceania

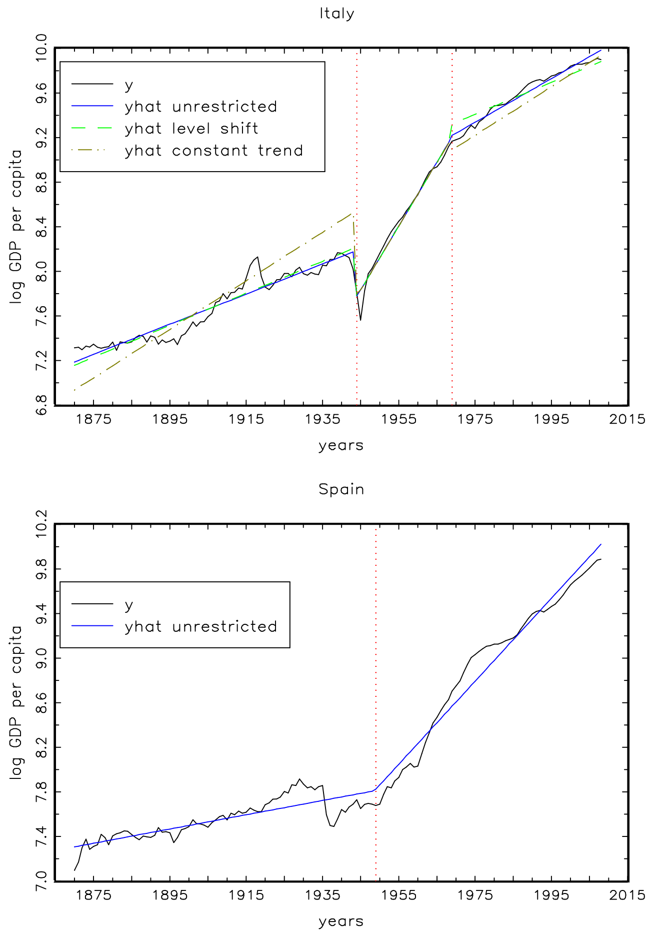

Fig. 3.6: Real GDP per capita - Southern Europe