- Equilibrium Outcomes of Repeated Two-Person Zero-Sum Games
- A Constructive Proof of the Nash Bargaining Solution
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Luís Carvalho

Nova School of Business and Economics
Universidade Nova de Lisboa

Supervisors:
Guido Maretto
Guilherme Carmona

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Equilibrium Outcomes of Repeated Two-Person Zero-Sum Games*

Guilherme Carmona Luís Carvalho

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Abstract

We consider discounted repeated two-person zero-sum games. We show that even when players have different discount factors (in which case the repeated game is not a zero-sum game), an outcome is subgame perfect if and only if all of its components are Nash equilibria of the stage game. This implies that in all subgame perfect equilibria, each player's payoff is equal to his minmax payoff. In conclusion, the competitive nature of two-player zero-sum games is not altered when the game is repeated.

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Keywords: Repeated games, two-person zero-sum games.

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1 Introduction

In a two-person zero-sum game, one player's gain is the other's loss. Therefore, this class of game is regarded as the prototype example of a strictly competitive game. A well known result (see, for example, Myerson (1997, Theorem 3.2)) shows that in all Nash equilibria of such games, every player receives his minmax payoff, i.e., the lowest payoff that he can guarantee to himself.

It is clear that the same conclusion applies to a discounted repeated two-person zero-sum game when both players have the same discount factor. Indeed, such a (repeated) game is itself a two-person zero-sum game. Furthermore, since all Nash equilibria yield the minmax payoff to both players, it follows easily that all subgame perfect equilibrium outcomes must consist of repetitions of (possibly different) Nash equilibria of the stage game. Thus, any departure from competitive, minmax behavior is impossible.

However, when players have different discount factors, the repeated game is no longer a zero-sum one. Thus, one may conjecture that the equilibrium set will expand, in particular, by allowing players to obtain higher payoffs than the minmax one. Indeed, intuitively, we could think that the player with the smaller discount factor is willing to bear losses in the future if she is compensated with some gains in the present; and that the player with the higher discount factor is willing to play accordingly since his losses today will be compensated with future gains. In short, intuition suggests that with different discount factors it might be possible to have subgame perfect equilibria where non stage-game equilibria are played in some periods.

Our main result shows that this intuition is misleading. In fact, for discounted repeated two-person zero-sum games with possibly different discount factors for both players, we show that the subgame perfect equilibrium outcomes consist of repetitions of Nash equilibria of the stage game and, consequently, that players receive their minmax payoff. Our result implies that the competitive character of two-player zero-sum games is not altered when the game is repeated, not even when the players discount the future at different rates.

2 Notation and Definitions

A *two-person zero-sum game* G is defined by $G = (A_1, A_2, u_1, u_2)$, where for all $i = 1, 2$, A_i is a finite set of player i's actions, $u_i : A_1 \times A_2 \to \mathbb{R}$ is player i's payoff function and players' payoff functions satisfy $u_1(a) + u_2(a) = 0$ for all $a \in A = A_1 \times A_2$. Let $S_i = \Delta(A_i)$, $S = S_1 \times S_2$ and $u_i : S \to \mathbb{R}$ be the usual mixed extension.

Let, for $i = 1, 2$, $v_i = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$, and $NE = \{s \in S : u_i(s) \geq u_i(\tilde{s}_i, s_{-i})$ for all $\tilde{s}_i \in S_i$ and $i = 1, 2\}$. The set $NE(G)$ is the set of Nash equilibria of G, and v_i is the minmax payoff for player i.

The *supergame of* G consists of an infinite sequence of repetitions of G taking place in periods $t = 1, 2, 3, \ldots$. At period t the players make simultaneous moves denoted by $s_i^t \in S_i$ and then each player learns his opponent's move.

For $k \geq 1$, a *k-stage history* is a k-length sequence $h_k = (s^1, \ldots, s^k)$, where, for all $1 \leq t \leq k$, $s^t \in S$; the space of all k-stage histories is H_k, i.e., $H_k = S^k$.[1] In the notation H_0 stands for the unique 0-stage history. The set of all histories is defined by $H = \bigcup_{n=0}^{\infty} H_n$.

It is assumed that at stage k each player knows h_k, that is, each player knows the actions that were played in all previous stages. A *strategy for player i*, $i = 1, 2$, is a function $f_i : H \to S_i$ mapping histories into actions. The set of player i's strategies is denoted by F_i, and $F = F_1 \times F_2$ is the joint strategy space. Every strategy $f = (f_1, f_2) \in F$ induces an outcome $\pi(f)$ as follows: $\pi^1(f) = f(H_0)$ and $\pi^k(f) = f(\pi^1(f), \ldots, \pi^{k-1}(f))$ for $k \in \mathbb{N}$. Let $\Pi = S \times S \times \cdots = S^{\infty}$.

Given an individual strategy $f_i \in F_i$ and a history $h = (s^1, \ldots, s^k) \in H$ we denote the *strategy induced by f_i at h* by $f_i|h$. This strategy is defined pointwise on H: for all $\bar{h} = (\bar{s}^1, \ldots, \bar{s}^{\bar{k}}) \in H$, then $(f_i|h)(\bar{h}) = f_i(s^1, \ldots, s^k, \bar{s}^1, \ldots, \bar{s}^{\bar{k}})$. We use $f|h$ to denote $(f_1|h, f_2|h)$ for every $f \in F$ and $h \in H$.

We assume that all players discount the future, although with a possibly different discount factor. Let $\delta_i \in (0, 1)$ denote the discount factor of player i, $i = 1, 2$. Thus

[1] As in Aumann (1964), we are assuming that players can observe the mixed strategies chosen. This assumption is not crucial to our work since, as Theorem 1 will show, equilibrium play is independent of the history.

the payoff in the supergame $G^\infty(\delta_1, \delta_2)$ of G is given by

$$U_i(f) = \sum_{t=1}^{\infty} \delta_i^{t-1} u_i(\pi^t(f)).$$

Also, for any $\pi \in \Pi$, $t \in \mathbb{N}$, and $i = 1, 2$, let $V_i^t(\pi) = \sum_{r=t}^{\infty} \delta_i^{r-t} u_i(\pi^r)$ be the continuation payoff of player i at date t if the outcome path π is played. For simplicity, we write $V_i(\pi)$ instead of $V_i^1(\pi)$.

A strategy vector $f \in F$ is a *Nash equilibrium* of $G^\infty(\delta_1, \delta_2)$ if $U_i(f) \geq U_i(\hat{f}_i, f_{-i})$ for all $i = 1, 2$ and all $\hat{f}_i \in F_i$. A strategy vector $f \in F$ is a *subgame perfect equilibrium* (SPE) of $G^\infty(\delta_1, \delta_2)$ if $f|h$ is a Nash equilibrium for all $h \in H$. An outcome path $\pi \in \Pi$ is a *subgame perfect outcome* if there exists a SPE f such that $\pi = \pi(f)$. We use $E\Pi(G, \delta_1, \delta_2)$ to denote the set of subgame perfect equilibrium outcomes of $G^\infty(\delta_1, \delta_2)$.

3 Equilibrium Outcomes

In this section we state and prove our main result. It shows that all equilibrium outcomes of the repeated game are repetitions of stage game Nash equilibria, implying that, although the supergame is not necessarily a zero-sum game, the strict competitive character of the stage game is maintained.

Theorem 1 *For all two-person zero-sum games G and all $\delta_1, \delta_2 \in (0,1)$, $E\Pi(G, \delta_1, \delta_2) = NE(G)^\infty$ and $u_i(\pi^k) = v_i$ for all $\pi \in E\Pi(G, \delta_1, \delta_2)$, $i = 1, 2$, and $k \in \mathbb{N}$.*

The conclusion of Theorem 1 is clear when $\delta_1 = \delta_2$ because in this case the supergame is itself a zero-sum game. Although this is no longer the case when players have different discount factors, the above case is still useful. In fact, our proof involves comparing the payoff of the most impatient player with the payoff he would obtain were he as patient as his opponent.

The comparison mentioned above requires the following result regarding power series. Before we state it, we recall the following notions (see, for instance, Rudin

(1964)). Given a sequence $\{a_k\}_{k=0}^{\infty}$ of real numbers and $x \in \mathbb{R}$, the series $\sum_{k=0}^{\infty} a_k x^k$ is called a power series. To the sequence $\{a_k\}_{k=0}^{\infty}$ corresponds $r \in \overline{\mathbb{R}}$ such that the series converges if $|x| < r$ and diverges if $|x| > r$; the (extended real) number is called the radius of convergence of the series. For all $x \in (-r, r)$ and $t \in \mathbb{N}$, define $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $f_t(x) = \sum_{k=0}^{\infty} a_{t+k} x^k$. Lemma 1 provides a useful formula for the derivative of f_t.

Lemma 1 *For all $x \in (-r, r)$ and $t \in \mathbb{N}$, $f_t'(x) = \sum_{k=1}^{\infty} f_{t+k}(x) x^{k-1}$.*

Proof. We have that $f_t'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$ and $f_t'(x)$ is absolutely convergent (see Rudin (1964, Theorem 8.1)). Let $A = \{(i,k) \in \mathbb{N}^2 : 1 \leq k < \infty, 1 \leq i \leq k\}$, $B = \{(i,k) \in \mathbb{N}^2 : i \leq k < \infty, 1 \leq i < \infty\}$ and note that $A = B$. Since $f_t'(x)$ is absolutely convergent, we obtain $f_t'(x) = \sum_{k=1}^{\infty} \sum_{i=1}^{k} a_{t+k} x^{k-1} = \sum_{A} a_{t+k} x^{k-1} = \sum_{B} a_{t+k} x^{k-1} = \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} a_{t+k} x^{k-1} = \sum_{i=1}^{\infty} x^{i-1} \sum_{k=0}^{\infty} a_{t+i+k} x^k = \sum_{i=1}^{\infty} f_{t+i}(x) x^{i-1}$ and the result follows. ■

We next turn to the proof of Theorem 1.

Proof of Theorem 1. Let $\delta_1, \delta_2 \in (0,1)$ and assume that $\delta_1 \leq \delta_2$. Define a game $\tilde{G}$ by $\tilde{G} = (A_1, A_2, \tilde{u}_1, u_2)$ where $\tilde{u}_1(a) = u_1(a) - v_1$. Hence, $\tilde{G}$ is a two-player, $(-v_1)$-sum game and $\tilde{v}_1 = 0$. Furthermore, $\tilde{V}_1^t(\pi) = V_1^t(\pi) - v_1/(1-\delta_1)$ for all outcomes π. Let M be such that $|u_i(s)| \leq M$ for all $i = 1, 2$.

The following claim establishes that the more patient a player is, the higher is his payoff of any SPE outcome.

Claim 1 *For all SPE outcomes π and all $t \in \mathbb{N}$, $\tilde{V}_1^t(\pi; \delta_1) \leq \tilde{V}_1^t(\pi; \delta_2)$.*

Proof of Claim 1. The conclusion is obvious if $\delta_1 = \delta_2$; hence, we may assume that $\delta_1 < \delta_2$. Denote $f_{t+k}(\delta) = \tilde{V}_1^{t+k}(\pi; \delta) = \sum_{j=0}^{\infty} \tilde{u}_1(\pi_{t+k+j}) \delta^j$ for all $k \in \mathbb{N}_0$. Since π is SPE, then $f_{t+k}(\delta_1) \geq \tilde{v}_1 = 0$ for all $k \in \mathbb{N}_0$.

We claim that $f_t^{(n)}(\delta_1) \geq 0$ for all $n, t \in \mathbb{N}_0$. By the above, we have that $f_t^{(0)}(\delta_1) \geq 0$ for all $t \in \mathbb{N}_0$. Suppose that $f_t^{(n-1)}(\delta_1) \geq 0$ for all $t \in \mathbb{N}_0$. Then, for all $t \in \mathbb{N}_0$, Lemma 1 implies that $f_t^{(n)}(\delta_1) = [f_t^{(n-1)}]'(\delta_1) = \sum_{k=1}^{\infty} f_{t+k}^{(n-1)}(\delta_1) \delta_1^{k-1} \geq 0$.

The result then follows using the Taylor series for f_t around δ_1 (see Rudin (1964, Theorem 8.4)) since

$$f_t(\delta_2) - f_t(\delta_1) = \sum_{k=1}^{\infty} \frac{f_t^{(k)}(\delta_1)}{k!}(\delta_2 - \delta_1)^k \geq 0.$$

■

If π is a SPE outcome of the supergame of $\tilde{G}$, then $\tilde{V}_i^t(\pi;\delta_i) \geq \tilde{v}_i/(1-\delta_i)$ for all $t \in \mathbb{N}$ and $i = 1, 2$. It then follows from Lemma 1 that

$$\frac{v_2}{1-\delta_2} = \frac{\tilde{v}_1}{1-\delta_1} + \frac{\tilde{v}_2}{1-\delta_2} \leq \tilde{V}_1^t(\pi;\delta_1) + \tilde{V}_2^t(\pi;\delta_2) \leq \tilde{V}_1^t(\pi;\delta_2) + \tilde{V}_2^t(\pi;\delta_2) = -\frac{v_1}{1-\delta_2} = \frac{v_2}{1-\delta_2}$$

Thus, $\tilde{V}_i^t(\pi) = \tilde{v}_i/(1-\delta_i)$, and so, $V_i^t(\pi) = v_i/(1-\delta_i)$ for all $t \in \mathbb{N}$ and $i = 1, 2$.

Let $\pi^{(0)}$ be a SPE outcome and let $(\pi^{(0)}, \pi^{(1)}, \pi^{(2)})$ be a SPE simple strategy supporting π^0 as the equilibrium outcome (see Abreu (1988, Proposition 5)). Then,

$$V_i^t(\pi^{(0)}) = u_i(\pi^{(0),t}) + \delta_i V_i^{t+1}(\pi^{(0)}) \geq \sup_{s_i \neq \pi_i^{(0),t}} u_i(s_i, \pi_{-i}^{(0),t}) + \delta_i V_i(\pi^{(i)}),$$

together with $V_i^{t+1}(\pi^{(0)}) = V_i(\pi^{(i)}) = v_i/(1-\delta_i)$ (since $\pi^{(i)}$ and $\{\pi^{(0),k}\}_{k=t+1}^{\infty}$ are SPE outcomes), implies that $u_i(\pi^{(0),t}) \geq \sup_{s_i \neq \pi_i^{(0),t}} u_i(s_i, \pi_{-i}^{(0),t})$ for all t and i. Hence, $\pi^{(0),t}$ is a Nash equilibrium of G for all t. ■

4 Conclusion

We have shown that equilibrium outcomes of repeated two-person, zero-sum games have the property that in every period a Nash equilibrium of the stage game is played. This result is interesting because it shows that the strict competitiveness embodied in two-person, zero-sum (normal-form) games extends to the repeated version even when players have different discount factors.

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A Constructive Proof of the Nash Bargaining Solution

Luís Carvalho

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Abstract

We consider the classical axiomatic Nash bargaining framework and propose a constructive proof of its solution. On the first part of this paper we prove Nash's solution is the result of a maximization problem; on the second part, through the properties of maximand's indifference curves we derive that it must be equal to xy.

Classification Numbers: C73.

Keywords: Nash Bargaining; Constructive Proof

1 Introduction

In the Nash Bargaining problem introduced in Nash (1950) two players decide unanimously on a utility allocation from a set $S \subset \mathbb{R}^2_+$ of possible alternatives, and an utility pair of $\boldsymbol{x}^* \in S$ is established as an agreement. If they do not cooperate and fail to reach an agreement the outcome is a vector of predetermined payoff $\boldsymbol{d} \in \mathbb{R}^2_+$, known as the disagreement point. The bargaining game is a pair $(S, \boldsymbol{d})$, in which S is the set of utilities available when a good is being bargained and $\boldsymbol{d}$ is the disagreement point. A bargaining solution $c(S, \boldsymbol{d})$ is a map that to each bargaining problem $(S, \boldsymbol{d})$ defines an agreement $c(S, \boldsymbol{d}) \in S \cup \{\boldsymbol{d}\}$.

Nash Bargaining solution is characterized by fulfilling some principles, the Nash axioms, is not a descriptive concept, as there is no strategic interaction among players, but a normative one, the choice function of the bargaining game must, in certain sense, be well behaved, it is as if there was a "fair arbitrator", Mariotti (1999), choosing what the final allocation should be if the principles were respected. The axioms *Pareto optimality, Symmetry, Affine transformation*, and *Independence of Irrelevant Alternatives* and in particular the relation that they create between the agreement of different bargaining games, are sufficient for the bargaining solution $c(\cdot)$ to be the result of a maximization process. That is, the requisite that a choice $c(\cdot)$ respects Nash axioms gives enough consistency and structure for the bargaining solution to be the result of the maximization of a function $f(\boldsymbol{x}, \boldsymbol{d})$ for $\boldsymbol{x} \in S$. This maximand function $f(\boldsymbol{x}, \boldsymbol{d})$ can be interpreted as a social function defined on the utility pair of the players $\boldsymbol{x} \in \mathbb{R}^2_+$. In particular Nash proved that the choice function must be $c(S) = \arg\max_{\boldsymbol{x} \in S} f(\boldsymbol{x}, \boldsymbol{d})$, the social function is $f(\boldsymbol{x}, \boldsymbol{d}) = (x - d_x)(y - d_y)$.

Nash The constructive proof this paper will derive is divided in two main parts. The first proves that the choice function defined on the sets $S \subset \mathbb{R}^2_+$ is the result of a maximization of a social function defined on the points $\boldsymbol{x} \in \mathbb{R}^2_+$. To establish this we will use a result from Peters and Wakker (1991) that states when a solution can be determined by a maximization process:

Corollary 5.7 1. *Let $c(\cdot)$ be a Pareto optimal, continuous choice function then the* *following two conditions are equivalent:*

1. *$c(\cdot)$ satisfies Independence of Irrelevant Alternatives*

2. *$c(S, \boldsymbol{d})$ maximizes a real valued function f on $S \in \mathbb{S}$.*

So, in order to prove that $c(\cdot)$ is the result of the maximization of a real valued function $f(\cdot)$, it is necessary to prove that the choice function is continuous. Deriving that $c(S, \boldsymbol{d}) = \arg\max_{\boldsymbol{x} \in S} f(\boldsymbol{x}, \boldsymbol{d})$.

The second part of the paper proves that the social function being maximized is $f(\boldsymbol{x}, \boldsymbol{d}) = (x - d_x)(y - d_y)$. Naturally, any strictly increasing transformation of $f(\cdot)$ can also be used as the social function. For this reason what is important, when identifying the social function, is to look their indifference curves, because they must be constant over all the alternative formulations.

In the next section we present notation and definitions we will use throughout the text, in section 3 we prove Nash Bargaining solution is a maximization process, in section 4 that the maximand of this process is $u(x, y)$ and then we conclude.

2 Notation and Definitions

A vector in $\mathbb{R}^2_+$ will be denoted by a bold letter usually $\boldsymbol{x}$ and its coordinates are represented like $\boldsymbol{x} = (x, y)$. The set of compact and convex sets of $\mathbb{R}^2_+$ is $\mathbb{S}$. For a set $S \in \mathbb{S}$, the maximum value of the first coordinate of S is $S^1 = \max\{x : \exists y \in \mathbb{R}, (x, y) \in S\}$, and the second coordinate maximum S^2 is defined in the same way. $\mathbb{S}^+$ is the set of the compact and convex subsets of $\mathbb{R}^2_+$ with $S^1 S^2 > 0$. For any $S \in \mathbb{S}^+$ there is a function $g_S : [0, S^1] \to [0, S^2]$ that defines the maximum value of the second coordinate when the first is x, hence for any $(x, y) \in S$, $y \leq g_S(x)$. Due to the convexity of S this function must be concave, next claim is proven on the appendix.

Claim 1. *There is a concave function $g_s(x) : [0, S^1] \to [0, S^2]$ such that $(x, g_S(x)) \in S$ and if $(x, y) \in S$ then $y \leq g_S(x)$.*

The bargaining problem is defined for pairs $(S, \boldsymbol{d})$, in which S is convex and compact, and it exist a $\boldsymbol{x} \in S$ such that $\boldsymbol{x} \gg \boldsymbol{d}$. The Nash bargaining solution is a correspondence that to each $(S, \boldsymbol{d})$ gives $c(S, \boldsymbol{d}) \subseteq S \cup \boldsymbol{d}$. We will normalize the disagreement point and work with $\boldsymbol{d} = \boldsymbol{0}$. This can be done without loss of generality because it is assumed that affine transformations of utility do not change the preference representation. Therefore, a bargaining game, from now on will be defined just on a utility set S. The hypothesis that exists a $\boldsymbol{x} \in S$ such that $\boldsymbol{x} \gg \boldsymbol{d}$ becomes $\boldsymbol{x} \gg \boldsymbol{0}$, so $S^1 S^2 > 0$ and the bargaining game will then be defined on $\mathbb{S}^+$. The bargaining solution is a correspondence $c : \mathbb{S}^+ \twoheadrightarrow \mathbb{R}^2_+$ with $c(S) \subseteq S$.

A set is *comprehensive* if for any $\boldsymbol{x} \in S$, any $\boldsymbol{x'} \leq \boldsymbol{x}$, $\boldsymbol{x'} \in S$, if for any $\boldsymbol{x} \in S$ the rectangle made by the vertices $\{(0,0); (x,0); (0,y); (x,y)\}$ is contained in the set. The *comprehensive hull* of a set $S \in \mathbb{S}$ is $comp(S) = \{\boldsymbol{x'} : \boldsymbol{x'} \leq \boldsymbol{x}, \text{ for any } \boldsymbol{x} \in S\}$. In the first part of the proof, in which we prove that the bargaining solution is a maximization process, we will work exclusively with comprehensive sets.

The *convex hull* of $S \in \mathbb{S}$ is the smallest convex set that contains S:

$$ch(S) = \{\boldsymbol{x'} : \boldsymbol{x'} = \lambda_1 \boldsymbol{x_1} + \lambda_2 \boldsymbol{x_2} \text{ with } \lambda_1 + \lambda_2 = 1, \lambda_1, \lambda_2 \geq 0, \forall \boldsymbol{x_1}, \boldsymbol{x_2} \in S\}$$

A set is *symmetric* if $(x, y) \in S$ implies $(y, x) \in S$. An *affine transformation* of $\boldsymbol{x} = (x, y) \in \mathbb{R}^2_+$, for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2_+$ and $\boldsymbol{\beta} = (\beta_1, \beta_2) \in \mathbb{R}^2_+$, is $\boldsymbol{\beta} + \boldsymbol{\alpha}\boldsymbol{x} = \Big(\beta_1 + \alpha_1 x, \beta_2 + \alpha_2 x\Big)$. An affine transformation of a set S is $\boldsymbol{\beta} + \boldsymbol{\alpha} S = \Big\{\boldsymbol{\beta} + \boldsymbol{\alpha}\boldsymbol{x} : \boldsymbol{x} \in S\Big\}$.

One transformation we need to use intensively in section 3 is to affine transform, with $\boldsymbol{\beta} = \boldsymbol{0}$, the set S in a way that $\boldsymbol{x} \in S$ is transformed into $\tilde{\boldsymbol{x}} \in \mathbb{R}^2_+$. This type of transformation will be denoted by $S_{(\boldsymbol{x}, \tilde{\boldsymbol{x}})}$ and the proportion factor $\boldsymbol{\alpha}$ is $\boldsymbol{\alpha} = \frac{\tilde{\boldsymbol{x}}}{\boldsymbol{x}}$, then $S_{(\boldsymbol{x}, \tilde{\boldsymbol{x}})} = \boldsymbol{\alpha} S = \frac{\tilde{\boldsymbol{x}}}{\boldsymbol{x}} S = \left(\frac{\tilde{x}}{x}, \frac{\tilde{y}}{y}\right) S$.

The *metrics* we use in this paper are such that the distance between two points $d(\boldsymbol{x}, \boldsymbol{x'}) = \max\{|x - x'|, |y - y'|\}$; the distance from a set to point is $d(\boldsymbol{x}, S') = \inf_{\boldsymbol{x'} \in S'} d(\boldsymbol{x}, \boldsymbol{x'})$; and the Hausdorff distance between two sets is

$$d(S, S') = \max\left\{\sup_{\boldsymbol{x} \in S} d(\boldsymbol{x}, S'), \sup_{\boldsymbol{x'} \in S} d(\boldsymbol{x'}, S)\right\}$$

.

The Nash axioms are:

Pareto Optimality(PO) $\forall S \in \mathbb{S}^+, \nexists \boldsymbol{x} \in S \setminus c(S) : \boldsymbol{x} \geq c(S)$

Independence of Irrelevant Alternatives(IIA) $\forall S, S' \in \mathbb{S}^+, S' \subseteq S$ with $c(S) \in S'$ then $c(S) = c(S')$

Symmetry (Sym) $\forall S \in \mathbb{S}^+$ symmetric then $c(S)_1 = c(S)_2$

Affine Transformations(AT) $\forall S \in \mathbb{S}, \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^2_+, c\Big(\boldsymbol{\beta} + \boldsymbol{\alpha} S\Big) = \boldsymbol{\beta} + \boldsymbol{\alpha} c(S)$

The interpretation of the first axiom, *PO*, is that a point should not be chosen if there is an option better for one player without damaging the other. *IIA* states that if the set of utilities is shrunk to $S' \subseteq S$ but the original solution $c(S)$ is still available in S', $c(S) \in S'$, then $c(S)$ should be the choice of the new bargaining $c(S') = c(S)$. All axioms implicitly assume that the choice of the bargaining is unique, although this doesn't have to be the case. We will prove that in fact the choice is unique. Until then we will use a different version of this axiom that allows for the multiplicity of choices [1]

IIAm $\forall S, S' \in \mathbb{S}, S' \subseteq S$ if $c(S, \boldsymbol{d}) \cap S' \neq \emptyset$ then $c(S', \boldsymbol{d}) = c(S, \boldsymbol{d}) \cap S'$

The *Sym* axiom defines the bargaining power of each player, and states that both players have equal strength, when facing a symmetric set the bargaining solution should establish an equal division for both. The *AT* axiom says that the change in the bargaining solution is equal to the change in the bargaining set, an affine transformation of players' utility set changes the agreement in exactly the same way.

3 Maximization of a Social Function

The proof of choice function continuity will be done by *reductio ad absurdum*, assuming that $c(S_k) = \boldsymbol{x_k}$ converges to $\boldsymbol{x'}$ a point different from $c(S) = \boldsymbol{x^*}$. Therefore there is a sequence of converging sets $S_k \to S$, $S_k, S \in \mathbb{S}^+$, such that $c(S_k) \nrightarrow c(S)$, contradicting the continuity of the bargaining solution.

[1] All the other axioms can be immediately adapted for the possibility of a multiplicity of agreements

In this case there is a sequence of convergent comprehensive sets $comp(S_k) \to comp(S)$ such that $c\Big(comp(S_k)\Big) \nrightarrow c\Big(comp(S)\Big)$, because $c\Big(comp(X)\Big) = c(X)$ by PO and $IIAm$. Hence, continuity of $c(\cdot)$ can be studied through comprehensive sets and in this section, even if it is not clearly mentioned, the sets are always comprehensive. Notice that in case S is a comprehensive set the function $g_S(\cdot)$ is nonincreasing.

What will be shown is that if the solution $c(\cdot)$ was not continuous, $c(S_k) \nrightarrow c(S)$, there would be a set S' with the bargaining solution $c(S')$ belonging to the interior of some S_k, being therefore worse than $c(S_k)$; and $c(S_k)$ belonging to the interior of S' and so worse than $c(S')$. Thus creating a contradiction, because $c(S_k)$ can not be worse than $c(S')$ that is worse than $c(S_k)$, as it will be prove. The set S', that will show this contradiction, will be an affine transformation of S, one that changes the point $\boldsymbol{x^*} = c(S)$ to a point in the interior of S, $\boldsymbol{\tilde{x}}$, so $S' = S_{(\boldsymbol{x},\boldsymbol{\tilde{x}})}$. The next lemma will prove that such a point $\boldsymbol{\tilde{x}}$, and therefore a set S' in the stated conditions, exists.

Lemma 1. *$\forall \boldsymbol{x}, \boldsymbol{x'} \in S$ with $\boldsymbol{x} \gg \boldsymbol{0}$ and $g_S(x) \neq g_S(x')$, then $\exists \boldsymbol{\tilde{x}} \in int(S)$ such that $\boldsymbol{x'} \in int\big(S_{(\boldsymbol{x},\boldsymbol{\tilde{x}})}\big)$.*

To show that $\boldsymbol{x'} \in \text{int}\big(S_{(\boldsymbol{x},\boldsymbol{\tilde{x}})}\big)$ we first need to understand when does a point that belongs to S also belong to $S_{(\boldsymbol{x},\boldsymbol{\tilde{x}})}$. We know that $\boldsymbol{x}$ belongs to the set S if its first coordinate is smaller than the maximum $x \leq S^1$ and the second coordinate smaller than the maximum at x, $y \leq g_S(x)$, next claim state sufficient conditions for this to happen in the affine transformed set $S_{(\boldsymbol{x},\boldsymbol{\tilde{x}})}$.

Claim 2. *If $\boldsymbol{x'} \in S$, $x' \leq \frac{\tilde{x}}{x} S^1$ and $g_S(x') \leq g_{S_{(\boldsymbol{x},\boldsymbol{\tilde{x}})}}(x')$ then $\boldsymbol{x'} \in S_{(\boldsymbol{x},\boldsymbol{\tilde{x}})}$.*

Proof. For any given set $\Sigma \in \mathbb{S}^+$ and $\boldsymbol{\alpha} \in \mathbb{R}^2_+$, if $\boldsymbol{x} = (x, y) \in \Sigma$, we know that $x \leq \Sigma^1$, and $\alpha_1 x \leq \alpha_1 \Sigma^1$, consequently $(\boldsymbol{\alpha}\Sigma)^1 = \alpha_1 \Sigma^1$. $S_{(\boldsymbol{x},\boldsymbol{\tilde{x}})} = \frac{\boldsymbol{\tilde{x}}}{\boldsymbol{x}} S$ then $S^1_{(\boldsymbol{x},\boldsymbol{\tilde{x}})} = \frac{\tilde{x}}{x} S^1$, as, by hypothesis $x' \leq \frac{\tilde{x}}{x} S^1$, then $x' \leq S^1_{(\boldsymbol{x},\boldsymbol{\tilde{x}})}$.

By definition of $g_S(\cdot)$, if $\boldsymbol{x'} \in S$ we must have $y' \leq g_S(x')$, by hypothesis $g_S(x') \leq g_{S_{(\boldsymbol{x},\boldsymbol{\tilde{x}})}}(x')$, then $y' \leq g_{S_{(\boldsymbol{x},\boldsymbol{\tilde{x}})}}(x')$. By reason of $x' \leq S^1_{(\boldsymbol{x},\boldsymbol{\tilde{x}})}$ and $y' \leq g_{S_{(\boldsymbol{x},\boldsymbol{\tilde{x}})}}(x')$, $\boldsymbol{x'}$ belongs to $S_{(\boldsymbol{x},\boldsymbol{\tilde{x}})}$. ■

The next claim states that if the affine transformation changes $\boldsymbol{x}$ to $\tilde{\boldsymbol{x}}$ then, as long as $\tilde{x}$ is between x and x', the first coordinate of $\boldsymbol{x'}$ will always fulfill the condition established in the previous claim.

Claim 3. *For $\boldsymbol{x}, \boldsymbol{x'} \in S$, with $x' \neq x$ for any $\tilde{x} \in \big(\min\{x, x'\}, \max\{x, x'\}\big)$ then $x' < \frac{\tilde{x}}{x} S^1$.*

Proof. If $x' = \min\{x', x\}$, $x' < \tilde{x} < x \leq S^1$, $S^1/x \geq 1$ and $x' < \tilde{x} \leq \tilde{x}(S^1/x)$; if $x' = \max\{x, x'\}$, $x < \tilde{x} < x'$, then $\frac{\tilde{x}}{x} > 1$, and $x' \leq S^1 < \frac{\tilde{x}}{x} S^1$ ∎

The following is an intermediate result that will help us prove for the second coordinate what the previous claim did for the first. If we affine transform the point $\boldsymbol{x}$ into one point of the frontier of S, $\tilde{\boldsymbol{x}} \in S$ with $\tilde{y} = g_S(\tilde{x})$, the maximum value of $x' \in [0, S^1_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}]$ on the new set , $g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x')$, will be bigger than $\frac{g_S(\tilde{x})}{g_S(x)} g_S(x \frac{x'}{\tilde{x}})$.

Claim 4. *$\forall \boldsymbol{x}, \tilde{\boldsymbol{x}} \in S$ with $\boldsymbol{x}, \tilde{\boldsymbol{x}} \gg \boldsymbol{0}$ and $\tilde{y} = g_S(\tilde{x})$, if $x' \in [0, S^1_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}]$ then $g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x') \geq \frac{g_S(\tilde{x})}{g_S(x)} g_S(\bar{x})$ in which $\bar{x} = x \frac{x'}{\tilde{x}}$.*

Proof. In general we know that for any $\boldsymbol{x'} \in \boldsymbol{\alpha} S$, $\exists \boldsymbol{x} \in S$ such that $\boldsymbol{\alpha x} = (\alpha_1 x, \alpha_2 y) = (x', y') = \boldsymbol{x'}$. As $y \leq g_S(x)$ and $x = \frac{x'}{\alpha_1}$, $y \leq g_S(\frac{x'}{\alpha_1})$, so $y' = \alpha_2 y \leq \alpha_2 g_s(x) = \alpha_2 g_S(\frac{x'}{\alpha_1})$. This inequality is valid for any y', inclusively for $y' = g_{\boldsymbol{\alpha} S}(x')$ and we can conclude that $g_{\boldsymbol{\alpha} S}(x') \leq \alpha_2 g_S(\frac{x'}{\alpha_1})$.

However we can find a point in S for which the inequality becomes an equality, thus proving that $g_{\boldsymbol{\alpha} S}(x') = \alpha_2 g_S(\frac{x'}{\alpha_1})$. For $(x, g_s(x)) \in S$, $\boldsymbol{\alpha}\big(x, g_s(x)\big) = \Big(\alpha_1 x, \alpha_2 g_S(x)\Big) = \Big(x', \alpha_2 g_S(\frac{x'}{\alpha_1})\Big) \in \alpha S$, and exists a $\boldsymbol{x'} \in \alpha S$ with $y' = \alpha_2 g_S(\frac{x'}{\alpha_1})$. Therefore

$$g_{\boldsymbol{\alpha} S}(x') = \alpha_2 g_S\big(\frac{x'}{\alpha_1}\big) \tag{1}$$

So $g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x') = g_{\frac{\tilde{\boldsymbol{x}}}{\boldsymbol{x}} S}(x')$, using equation (1) with $\boldsymbol{\alpha} = \frac{\tilde{\boldsymbol{x}}}{\boldsymbol{x}} = (\frac{\tilde{x}}{x}, \frac{\tilde{y}}{y})$, $g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x') = \frac{\tilde{y}}{y} g_S(\frac{x}{\tilde{x}} x') = \frac{\tilde{y}}{y} g_S(\bar{x})$. With $\tilde{y} = g_S(\tilde{x})$ and because $y \leq g_S(x)$

$$g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x') \geq \frac{g_S(\tilde{x})}{g_S(x)} g_S(\bar{x}) \tag{2}$$

$\blacksquare$

Now we will deduce a result for the second coordinate to fulfill the condition of claim (2).

Claim 5. *$\forall \boldsymbol{x}, \boldsymbol{x'} \in S$ with $\boldsymbol{x} \gg \mathbf{0}$ and $g_S(x) \neq g_S(x')$, for $\tilde{\boldsymbol{x}} \in S$ such that $\tilde{x} \in \Big(\min\{x, x'\}, \max\{x, x'\}\Big)$ and $\tilde{y} = g_S(\tilde{x})$ then $g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x') > g_S(x')$.*

Proof. As $\tilde{x} \in \Big(\min\{x, x'\}, \max\{x, x'\}\Big)$ claim (3) is applicable and $x' \leq \frac{\tilde{x}}{x} S^1 = S^1_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}$. Therefore conditions to apply claim (4) are satisfied and $g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x') \geq \frac{g_S(\tilde{x})}{g_S(x)} g_S(\bar{x})$ taking logarithms and considering $w(\cdot) = \log g_S(\cdot)$ we get

$$\log\Big(g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x')\Big) \geq w(\tilde{x}) - w(x) + w(\bar{x})$$

As $\min\{x, x'\} < \tilde{x} < \max\{x, x'\}$ there is a $0 < \theta < 1$ such that $\tilde{x} = x^{\theta} {x'}^{1-\theta}$ and $\bar{x} = \frac{x}{\tilde{x}} x' = x^{1-\theta} {x'}^{\theta}$. Using a particular case of Jensen's inequality we know that $x^{\theta} {x'}^{1-\theta} \leq \theta x + (1-\theta)x'$ and as the function $w(\cdot)$ is non increasing $w(\tilde{x}) = w(x^{\theta}{x'}^{1-\theta}) \geq w\Big(\theta x + (1-\theta)x'\Big) > \theta w(x) + (1-\theta) w(x')$, the last inequality is derived from strict concavity of $w_S(\cdot)$, (the logarithm is a strictly concave function and $g_S(\cdot)$ is a concave function) and $g_S(x) \neq g_S(x')$. Applying this reason to the entire equation

$$\begin{aligned}
&\log\big(g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x')\big) \geq w(\tilde{x}) - w(x) + w(\bar{x}) = w\Big(x^{\theta}{x'}^{1-\theta}\Big) - w(x) + w\Big(x^{1-\theta}{x'}^{\theta}\Big) \\
&\overbrace{\geq}^{\text{non-increasing}} \quad w\Big(\theta x + (1-\theta)x'\Big) - w(x) + w\Big((1-\theta)x + \theta x'\Big) \overbrace{>}^{\text{concavity}} \\
&\Big[\theta w(x) + (1-\theta) w(x')\Big] - w(x) + \Big[(1-\theta) w(x) + \theta w(x')\Big] = w(x')
\end{aligned}$$

Taking exponentials in this inequality we prove that $g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x') > g_S(x')$. $\blacksquare$

The next result involves almost no derivation, however it is essential for later use, and for that reason, it has a lemma of its own.

Lemma 2. *$\forall \boldsymbol{x}, \boldsymbol{x'} \in S$ with $\boldsymbol{x} \gg \mathbf{0}$ and $g_S(x) \neq g_S(x')$ for $\tilde{\boldsymbol{x}} \in S$ such that $\tilde{x} \in \big(\min\{x, x'\}, \max\{x, x'\}\big)$ and $\tilde{y} = g_S(\tilde{x})$ then $\Big(x', g_S(x')\Big) \in S_{(\boldsymbol{x}, \tilde{\boldsymbol{x}})}$.*

Proof. By claim (3) $x' \leq \frac{\tilde{x}}{x} S^1$, claim (5) insures that $g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x') > g_S(x')$, and with these conditions we can apply claim (2) and derive that $\big(x', g_S(x')\big) \in S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}$. ■

By now we prepared the sufficient results to prove lemma (1) which will be the main instrument for proving the continuity of the bargaining solution $c(\cdot)$.

Proof. By claim (2) it is sufficient to find $\tilde{s} \in \operatorname{int}(S)$ such that $x' < \frac{\tilde{x}}{x} S^1$ and $g_S(x') < g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x')$. By claim (3) as long as $\tilde{x} \in \big(\min\{x, x'\}, \max\{x, x'\}\big)$, the first inequality is respected. With $\tilde{y} = g_S(\tilde{x})$, by claim (5), $g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x') > g_S(x') \geq y'$. We saw on claim (4) that $g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x') = \frac{\tilde{y}}{y} g_S(x' \frac{x}{\tilde{x}})$, and so is continuous on $\tilde{y}$, if instead of $\tilde{y} = g_S(\tilde{x})$ we choose a value of $\tilde{y}$ sufficiently close to $g_S(\tilde{x})$ the inequality is preserved and the second condition of claim (2) is satisfied. Concluding $\tilde{x} \in \big(\min\{x, x'\}, \max\{x, x'\}\big)$ and $\tilde{y} < g_S(\tilde{x})$, hence $\tilde{\boldsymbol{x}} = (\tilde{x}, \tilde{y}) \in \operatorname{int}(S)$. ■

Up until now we used only the *PO* and the *AT* axioms, (jointly with the convexity restriction of the bargaining set). The next two theorems will bring *Sym* and *IIAm* into play. In theorem (2) the first of these axioms is used to prove that the bargaining choice of any set has both players receiving strictly positive payoff, that is for any $S \in \mathbb{S}^+$, $c(S)_1 c(S)_2 > 0$. Theorem (1) proves, through the use of lemma (1), that the *IIAm* is equivalent to Nash's *IIA*, by saying that the set of choices for any set only has one element, $|c(S)| = 1$. For this reason after the proof we will use Nash's original axiom.

Theorem 1. *: $\forall S \in \mathbb{S}$, $|c(S)| = 1$.*

Proof. If $S = \{(0,0)\}$ the solution must be unique. If $S^1 = 0$ then by Pareto optimality $c(S) = (0, S^2)$ and $|c(S)| = 1$. If $S^i > 0$ for both $i = 1,2$ and the choices of the bargaining function are more than one, $|c(S)| \geq 2$, take $\boldsymbol{x}, \boldsymbol{x'} \in c(S)$. Suppose $x = x'$ then by Pareto optimality $y = g_s(x) = g_S(x') = y'$ and $\boldsymbol{x} = \boldsymbol{x'}$ contradicting the hypothesis of $|c(S)| \geq 2$. For this reason we must have $x \neq x'$ and $y \neq y'$. Due to $S^1 > 0$, $S^2 > 0$ and $g_S(x) = y \neq y' = g_S(x')$, we can apply lemma (1), $\exists \tilde{\boldsymbol{x}} \in \text{int}(S)$ such that $\boldsymbol{x'} \in \text{int}\Big(S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}\Big)$. We know $c\Big(S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}\Big) = c\Big(\frac{\tilde{\boldsymbol{x}}}{\boldsymbol{x}}S\Big) = \frac{\tilde{\boldsymbol{x}}}{\boldsymbol{x}}c(S) \ni \frac{\tilde{\boldsymbol{x}}}{\boldsymbol{x}}\boldsymbol{x} = \tilde{\boldsymbol{x}}$. As $\tilde{\boldsymbol{x}} \in S$ then $\tilde{\boldsymbol{x}} \in S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})} \cap S \subseteq S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}$, and, by *IIAm* axiom, we get $\tilde{\boldsymbol{x}} \in c\Big(S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})} \cap S\Big)$. Lemma (1) guarantees that $\boldsymbol{x'} \in \text{int}\Big(S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}\Big)$ then $\boldsymbol{x'} \in S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})} \cap S$ and by *IIAm*, $\boldsymbol{x'} \in c\Big(S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})} \cap S\Big)$. But we have an interior point, and therefore pareto dominated, as one of the choices of $S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}$, contradicting this way *PO* axiom. Thus we can not have $|c(S)| \geq 2$. ■

Theorem 2. *: If $S^i > 0$ then $c(S)_i > 0$ with $i = 1, 2$.*

Proof. If $C(S)_1 = 0$ due to the non increasing frontier of the set S, $g_S(\cdot)$ is non increasing, $g_S\big(0\big) = S^2$, and by *PO* $c(S)_2 = g_S\big(c(S)_1\big) = S^2$, so $C(S) = (0, S^2)$. Choosing $\boldsymbol{\alpha} \in \mathbb{R}^2_+$ such that $\alpha_1 = 1$ and $\alpha_2 = S^1/S^2$, then $c(\boldsymbol{\alpha}S) = (0, S^1)$. Consider $\Delta = ch\big\{(0,0),(0,S^1),(S^1,0)\big\}$, $\big\{(0,0),(0,S^1),(S^1,0)\big\} \subset \boldsymbol{\alpha}S$ and $\boldsymbol{\alpha}S$ is convex, then $\Delta \subseteq \boldsymbol{\alpha}S$ and by *IIA*, $c(\Delta) = (0, S^1)$. Δ is symmetric and by *symmetry* axiom $c_1(\Delta) = c_2(\Delta)$, we get a contradiction. ■

After this small digression we go back to proving the continuity of the bargaining solution. The next very simple claim shows that if a sequence of sets $\big\{S_k\big\}_{k=1}^{\infty}$ converge to S then a convergent sequence of points with $\boldsymbol{x_k} \in S_k$ must converge to a point in S, the limit of S_k.

Claim 6. *: If $S_k \to S$ and $S_k \ni \boldsymbol{x_k} \to \boldsymbol{x}$ then $\boldsymbol{x} \in S$*

Proof. if $\boldsymbol{x} \notin S$, define $d(\boldsymbol{x}, S) = \epsilon$. As $\boldsymbol{x_k} \to \boldsymbol{x}$, $\exists K \in \mathbb{N}$ where $\boldsymbol{x_k} \in B_{\epsilon/2}(\boldsymbol{x}), \forall k > K$. By triangle inequality

$$d(\boldsymbol{x}, S) \leq d(\boldsymbol{x_k}, \boldsymbol{x}) + d(\boldsymbol{x_k}, S) \Leftrightarrow d(\boldsymbol{x_k}, S) \geq d(x, S) - d(\boldsymbol{x_k}, \boldsymbol{x}) \Leftrightarrow d(\boldsymbol{x_k}, S) \geq \epsilon/2$$

Then $d(S_k, S) \geq \epsilon/2, \forall k > K$, meaning that $S_k \nrightarrow S$, a contradiction. ■

An immediate and simple implication of the *IIA* axiom is that two sets with different bargaining choices cannot simultaneously contain the other's solution. The next claim, which will be used in the next theorem, proves it.

Claim 7. *: $S, S' \in \mathbb{S}^+$, $c(S) \neq c(S')$ and $c(S) \in S'$, then $c(S') \notin S$.*

Proof. $c(S) \in S'$, so $c(S) \in S \cap S'$, then by IIA $c(S' \cap S) = c(S)$. If $c(S') \in S$ then by IIA $c(S' \cap S) = c(S')$, we get a contradiction once $c(S') \neq c(S)$. ∎

Theorem 3. *The function $c(\cdot)$ is continuous on $\mathbb{S}^+$.*

Proof. Assume $c(\cdot)$ is not continuous, then exists a sequence of sets $S_k \in \mathbb{S}^+$ convergent to $S \in \mathbb{S}^+$, $S_k \to S$, but the bargaining choice $c(S_k) = \boldsymbol{x_k}$ does not converge to $\boldsymbol{x^*} = c(S)$, $\boldsymbol{x_k} \nrightarrow \boldsymbol{x^*}$. Let's start by also assuming that $\{\boldsymbol{x_k}\}_{k=1}^{\infty}$ is convergent to the point $\boldsymbol{x'} \in \mathbb{R}^2_+$. As $S \in \mathbb{S}^+$, $S^1 > 0$ and $S^2 > 0$, theorem (2) insures $\boldsymbol{x^*} \gg \mathbf{0}$. With $\boldsymbol{\alpha} = (1, \frac{x*}{y*})$, $\boldsymbol{\alpha} S_k \to \boldsymbol{\alpha} S$, and $c(\boldsymbol{\alpha} S_k) = \boldsymbol{\alpha} c(S_k) \to \boldsymbol{\alpha x'} \neq \boldsymbol{\alpha x^*} = \boldsymbol{\alpha} c(S)$, with $\boldsymbol{\alpha} c(S) = (x^*, x^*)$. Therefore, if the sets $S_k \to S$ and $c(S_k) \nrightarrow c(S)$, then exists sets $S'_k \to S'$ with $c(S'_k) \nrightarrow c(S')$ with the additional property $c(S')_1 = c(S')_2$. So we will assume that $\boldsymbol{x^*} = (x^*, x^*)$.

Case 1: $x^* = x'$ or $x^* = y'$

We will show that exists a point in $\boldsymbol{\bar{x}_k} \in S_k$, and in the line $x = y$, so $\boldsymbol{\bar{x}_k} = (x_k, x_k)$, such that for large k $\boldsymbol{\bar{x}_k}$ it is better than the choice $c(S_k)$. To prove that such point exist we will start by claiming that there is a sequence of points $(d_k, d_k) \in S_k$ which converges to the solution of set S, $\boldsymbol{x^*} = c(S)$.

Claim 8. *If $c(S) = (x^*, x^*)$, $S_k \to S$, then $d_k = \max\{d : (d, d) \in S_k\} \to x^*$*

Proof. Define $S_{k|1} = \{s : (s, s) \in S_k\}$ and $S_{|1} = \{s : (s, s) \in S\}$. As $d(S_k, S) \to 0$, for any $\boldsymbol{\tilde{s}} \in S_{|1}$, $d(\boldsymbol{\tilde{s}}, S_k) \to 0$, that is, for any $\epsilon > 0$, $\exists \boldsymbol{\tilde{s}_k} \in S_k$ with $d(\boldsymbol{\tilde{s}}, \boldsymbol{\tilde{s}_k}) < \epsilon$. Take $\boldsymbol{\bar{s}_k} = (1, 1) \min\{\tilde{s}_{k,1}, \tilde{s}_{k,2}\}$, due to the comprehensive nature of the sets S_k, if $\boldsymbol{\tilde{s}_k} \in S_k$ then as $\boldsymbol{\bar{s}_k} \leq \boldsymbol{\tilde{s}_k}$, $\boldsymbol{\bar{s}_k} \in S_k$. The distance between $\boldsymbol{\bar{s}_k}$ and $\boldsymbol{\tilde{s}}$ at each coordinate is the same, as they both belong to the line $x = y$ and $d(\boldsymbol{\bar{s}_k}, \boldsymbol{\tilde{s}}) =$

$\max\{|\bar{s}_{k,1} - \tilde{s}_1|, |\bar{s}_{k,2} - \tilde{s}_2|\} = |\bar{s}_{k,1} - \tilde{s}_1| = |\min\{\tilde{s}_{k,1}, \tilde{s}_{k,2}\} - \tilde{s}_1|$. Thus we conclude that $d(\bar{\boldsymbol{s}}_{\boldsymbol{k}}, \tilde{\boldsymbol{s}}) \leq \max\{|\tilde{s}_{k,1} - \tilde{s}_1|, |\tilde{s}_{k,2} - \tilde{s}_1|\} = d(\tilde{\boldsymbol{s}}_{\boldsymbol{k}}, \tilde{\boldsymbol{s}}) < \epsilon$. The same calculation could be done for any point in $\tilde{\boldsymbol{s}}_{\boldsymbol{k}} \in S_{k|1}$, we then find a point $\tilde{\boldsymbol{s}} \in S$ with $d\big(\tilde{\boldsymbol{s}}_{\boldsymbol{k}}, \tilde{\boldsymbol{s}}\big) < \epsilon$ and for $\bar{\boldsymbol{s}} = (1,1)\min\{\tilde{s}_1, \tilde{s}_2\}$, $d\big(\tilde{\boldsymbol{s}}_{\boldsymbol{k}}, \bar{\boldsymbol{s}}\big) < \epsilon$ for k big enough, proving that $S_{k|1} \to S_{|1}$. Notice that due to pareto optimality $x^* = \max\{s : (s,s) \in S\} = \max\{s : s \in S_{|1}\}$ and $d_k = \max\{s : s \in S_{k|1}\}$. As the maximum function is continuous $d_k = \max\{s : s \in S_{k|1}\} \to \max\{s : s \in S_{|1}\} = x^*$. ■

The points we are looking for, points $\bar{x}_k$ which are better than $c(S_k)$, will be created by defining the mean of the coordinates of $c(S_k) = \boldsymbol{x_k} = (x_k, y_k)$, so $\bar{x}_k = \frac{x_k + y_k}{2}$. The next result shows that the point on the line $x = y$ with coordinates equal to $\bar{x}_k$, $\bar{\boldsymbol{x}}_{\boldsymbol{k}} = \bar{x}_k(1,1)$ does, for large k, also belong to set S_k.

Claim 9. $\exists K \in \mathbb{N} : k > K, \bar{\boldsymbol{x}}_{\boldsymbol{k}} \in S_k$

Proof. By hypothesis $\{\boldsymbol{x_k}\}_{k=1}^{\infty}$ is a convergent sequence, by claim (6) $\lim \boldsymbol{x_k} = \boldsymbol{x'} \in S$. Without loss of generality assume $x^* = x'$ (in the present case, either this is true or $x^* = y'$) as $\boldsymbol{x^*} \neq \boldsymbol{x'}$, by pareto optimality $y' < y^* = x^*$. Defining, consistently, $\bar{x}' = \frac{x'+y'}{2}$ we have that $\bar{x}_k \to \bar{x}' = \frac{x'+y'}{2} < \frac{x^*+y^*}{2} = x^*$. By claim (8) $d_k \to x^*$, so $\exists k \in \mathbb{N}$ such that $\bar{x}_k < d_k$. By definition of d_k, $\boldsymbol{d_k} = (d_k, d_k) \in S_k$ also $\boldsymbol{0} \in S_k$, and $\exists \alpha \in (0,1)$ such that $\bar{\boldsymbol{x}}_{\boldsymbol{k}} = \alpha \boldsymbol{d_k} + (1-\alpha)\boldsymbol{0}$, hence $\bar{\boldsymbol{x}}_{\boldsymbol{k}} \in S_k$ due to S_k being convex. ■

We know $\bar{\boldsymbol{x}}_{\boldsymbol{k}} \in S_k$ and that $c(S_k) = \boldsymbol{x_k}$, we will now find a set A_k with $c(A_k) = \bar{\boldsymbol{x}}_{\boldsymbol{k}}$ and $\boldsymbol{x_k} \in A_k$, contradicting in this way claim (7). The set $A_k = ch\,\{(0,0); (x_k, y_k), (y_k, x_k)\}$ is symmetric, and by axiom *Sym*, $c(A_k)$ must be such that $c(A_k)_1 = c(A_k)_2$, by *PO* axiom $c(A_k) = \bar{\boldsymbol{x}}_{\boldsymbol{k}}$. By the previous claim (9) $c(A_k) = \bar{\boldsymbol{x}}_{\boldsymbol{k}} \in S_k$ for large k, and by construction of A_k, $c(S_k) = \boldsymbol{x_k} \in A_k$, but $\bar{\boldsymbol{x}}_{\boldsymbol{k}} \neq \boldsymbol{x_k}$, (remember $\bar{x}_k = \bar{y}_k$ but $x_k \neq y_k$ for large k because $x' \neq y'$), we get a contradiction.

Case 2 Now we will consider the case in which $x^* \neq x'$ and $x^* \neq y'$. If we prove that it exists $\tilde{\boldsymbol{x}} \in S$, such that for at least one $k \in \mathbb{N}$, $\boldsymbol{x_k} \in S_{(\boldsymbol{x^*}, \tilde{\boldsymbol{x}})}$ and $\tilde{\boldsymbol{x}} \in S_k$ we get a contradiction, because $c\big(S_{(\boldsymbol{x^*}, \tilde{\boldsymbol{x}})}\big) = \tilde{\boldsymbol{x}}$ and $c(S_k) = \boldsymbol{x_k}$ contradicting claim (7), as long as $\boldsymbol{x_k} \neq \tilde{\boldsymbol{x}}$. As $S_i > 0$ and lemma (1) is applicable, $\exists \tilde{\boldsymbol{x}} \in \text{int}(S)$ such that $\boldsymbol{x'} \in \text{int}\Big(S_{(\boldsymbol{x}^*,\tilde{\boldsymbol{x}})}\Big)$, therefore, as $\boldsymbol{x_k} \to \boldsymbol{x'}$, $\boldsymbol{x_k} \in \text{int}\Big(S_{(\boldsymbol{x}^*,\tilde{\boldsymbol{x}})}\Big)$ for large k. Because $\tilde{\boldsymbol{x}} \in \text{int}(S)$ and $S_k \to S$, then $\tilde{\boldsymbol{x}} \in S_k$ for large k. The contradiction is obtained, and therefore $c(\cdot)$ must be continuous.

We started by assuming that $\{\boldsymbol{x_k}\}_{k=1}^{\infty}$ was convergent, if it is not convergent, as $S_k \to S$, for big values of k, $S_k \subset R = \Big\{\boldsymbol{s} : \boldsymbol{0} \leq \boldsymbol{s} \leq \big(S^1 + 1, S^2 + 1\big)\Big\}$. The rectangle R is compact and for the sequence $\{\boldsymbol{x_k}\}_{k=1}^{\infty} \subset R$ not to converge is because it has (at least) two subsequences converging to different values. However, as we just saw, any converging subsequence must converge to $\boldsymbol{x}^*$, hence it is impossible to have two subsequences converging to a value that is not $\boldsymbol{x}^*$. ■

Corollary 1. *If $c(\cdot)$ is symmetric, pareto optimal, IIA and ILT then $c(S)$ maximizes a real valued function f on $S \in \mathbb{S}$*

Proof. As $c(\cdot)$ is symmetric, pareto optimal, IIA and ILT then by theorem (3) $c(\cdot)$ is continuous in $\mathbb{S}$, and by corollary 5.7 in Peters and Wakker(91) $c(S)$ maximizes a real valued function. ■

4 The Social Function is $u(x, y) = xy$

Thus far we discovered that the choice function $c(\cdot)$ is the result of the maximization process of a social function $f(\cdot)$. In this chapter we will deduce the shape of this function, but as it is well known $f(\cdot)$ is not unique, any positive monotonic transformation of it can be used as a social function. To unveil the shape of one of those functions, we will initially concentrate on properties of the curves that represent the lower bound of the upper contour set, $y_k(\cdot)$. Later we will prove that these curves are the indifference curves of a particular social function $h(\cdot)$, and it is supported on this function that the Nash bargaining solution will be derived. But first of all, we will prove that any function $f(\cdot)$ representing the bargaining solution $c(S) = \arg\max_{\boldsymbol{x} \in S} f(\boldsymbol{x})$ must be strictly quasiconcave, this is a basic stepping stone on the following derivation. In this section we will use intensively lines with negative slope as the bargaining set, so, before advancing, some results and definitions need to be introduced.

Definition 1. *Let $\mathbb{L}^-$ be the set of negatively sloped lines in $\mathbb{R}^2_+$*

$$\mathbb{L}^- = \left\{ S \in \mathbb{S}^+ : \exists a > 0, b \leq 0, \forall (s_1, s_2) \in S, s_2 = a + bs_1 \right\} \tag{3}$$

The following set of results for lines in $\mathbb{L}^-$ are proved in the appendix. $a[L]$ is the constant coefficient, $b[L]$ is the slope of L; L^i is the maximum value assumed by the i^th argument.

Proposition 1. *For $L, L' \in \mathbb{L}^-$, and for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \gg \mathbf{0}$*

1. *$a[L] = L^2$ and $b[L] = -\frac{L^2}{L^1}$*
2. *$\boldsymbol{\alpha} L \in \mathbb{L}^-$ and $a[\boldsymbol{\alpha} L] = \alpha_2 a[L]$ and $b[\boldsymbol{\alpha} L] = \frac{\alpha_2}{\alpha_1} b[L]$*
3. *$\exists \boldsymbol{\alpha} \in \mathbb{R}^2_+$ such that $\boldsymbol{\alpha} L = L'$ with $\alpha_2 = \frac{a[L']}{a[L]}$ and $\alpha_1 = \frac{a[L']}{a[L]} \frac{b[L]}{b[L']}$*
4. *$\forall \boldsymbol{x} \in \mathbb{R}^2_+$, $\exists L \in \mathbb{L}^-$ such that $c(L) = \boldsymbol{x}$*

4.1 $f(x, y)$ is strictly quasiconcave

Theorem 4. *If $c(S) = \arg\max_{\boldsymbol{x} \in S} f(\boldsymbol{x})$ then $f(x, y)$ is strictly quasiconcave.*

Proof. A direct implication of the PO axiom is that the function $f(x, y)$ must be strictly increasing in both arguments. For any $L \in \mathbb{L}^-$, with $g_L(x) : [0, L^1] \to [0, L^2]$ such that $(x, g_L(x)) \in L$, define $t(x) = f\big(x, g_L(x)\big)$. We will prove that $\exists x_2 \in [0, L^1]$ such that $t(\cdot)$ is strictly increasing for $x \in [0, x_2]$ and strictly decreasing if $x \in [x_2, L^1]$. Choose $\boldsymbol{x_2} = c(S) = \big(x_2, g_L(x_2)\big)$, if, for any $x_0 < x_1 < x_2$ with $\boldsymbol{x_i} = (x_i, g_L(x_i))$, $\boldsymbol{x_0} \in L_{(\boldsymbol{x_2}, \boldsymbol{x_1})}$ then $t(\cdot)$ is strictly increasing for $x < x_2$. Because $\boldsymbol{x_1} = c\big(L_{(\boldsymbol{x_2}, \boldsymbol{x_1})}\big) = \arg\max_{\boldsymbol{x} \in L_{(\boldsymbol{x_2}, \boldsymbol{x_1})}} f(\boldsymbol{x})$, then $f(\boldsymbol{x_1}) > f(\boldsymbol{x}), \forall \boldsymbol{x} \in L_{(\boldsymbol{x_2}, \boldsymbol{x_1})}$. So we only need to prove that $\boldsymbol{x_0} \in L_{(\boldsymbol{x_2}, \boldsymbol{x_1})}$. In lemma (2) we proved that when $\tilde{x} \in \big(\min\{x, x'\}, \max\{x, x'\}\big)$, $\tilde{y} = g_S(\tilde{x})$ and $g_S(x) \neq g_S(x')$ then $\big(x', g_S(x')\big) \in S_{(\boldsymbol{x}, \tilde{\boldsymbol{x}})}$.

Replacing $x, x', \tilde{x}$ by x_2, x_0, x_1, then $x_1 \in (x_0, x_2)$ and we obtain the desired result that $\boldsymbol{x_0} \in S_{(\boldsymbol{x_2},\boldsymbol{x_1})}$. To prove that $f(\cdot)$ is decreasing when $x > x_2$ we use the same reasoning, this time with $x_1 \in (x_2, x_0)$, and prove again that $\boldsymbol{x_0} \in S_{(\boldsymbol{x_2},\boldsymbol{x_1})}$.

For the function $f(\cdot)$ to be strictly quasiconcave, for any $\boldsymbol{x_0}, \boldsymbol{x_1}$ and any $\alpha \in (0,1)$, $\boldsymbol{x_\alpha} = \alpha\boldsymbol{x_0} + (1-\alpha)\boldsymbol{x_1}$ we have that $f(\boldsymbol{x_\alpha}) > \max\{f(\boldsymbol{x_0}), f(\boldsymbol{x_1})\}$. Let $L[\boldsymbol{x_0}, \boldsymbol{x_1}] \in \mathbb{S}^+$ stand for the line that passes through $\boldsymbol{x_0}$ and $\boldsymbol{x_1}$. Let x_2 be the point at which $t(x_2) > t(x)$, $\forall x \in [0, L^1]$. When:

- $x_0 < x_1 \leq x_2$, as previously seen, $t(x)$ is increasing between x_0 and x_1 and, as $x_0 < x_\alpha < x_1$, $f(\boldsymbol{x_\alpha}) = t(x_\alpha) > t(x_0) \geq \min\{f(\boldsymbol{x_0}), f(\boldsymbol{x_1})\}$.

- $x_2 \leq x_0 < x_1$ then $x_0 < x_\alpha < x_1$, as $t(\cdot)$ is decreasing for $x > x_2$, $t(x_\alpha) > t(x_1)$, $f(\boldsymbol{x_\alpha}) > f(\boldsymbol{x_1}) \geq \min\{f(\boldsymbol{x_0}), f(\boldsymbol{x_1})\}$.

- $x_0 \leq x_2 \leq x_1$, and $x_0 < x_\alpha \leq x_2$, $t(x_\alpha) > t(x_0)$ and $f(\boldsymbol{x_\alpha}) > f(\boldsymbol{x_0}) \geq \min\{f(\boldsymbol{x_0}), f(\boldsymbol{x_1})\}$.

- $x_0 \leq x_2 \leq x_1$ and $x_2 \leq x_\alpha < x_1$, $t(x_\alpha) > t(x_1)$ and $f(\boldsymbol{x_\alpha}) > f(\boldsymbol{x_1}) \geq \min\{f(\boldsymbol{x_0}), f(\boldsymbol{x_1})\}$

We conclude that $f(\boldsymbol{x_\alpha}) = t(x_\alpha) > \min\{t(x_1), t(x_2)\} = \min\{f(\boldsymbol{x_1}), f(\boldsymbol{x_2})\}$ and for any possibility $f(\boldsymbol{x_\alpha}) = f(\alpha\boldsymbol{x_1} + (1-\alpha)\boldsymbol{x_2}) > \min\{f(\boldsymbol{x_1}), f(\boldsymbol{x_2})\}$, the function is strictly quasiconcave. ■

4.2 The Properties of $y_k(x)$

The curves $y_k(\cdot)$ are built on the following way: for a fixed $x \in \mathbb{R}^+$ and for a possible value of $f(\cdot)$, k, we find the set of y's such that $f(x,y) \geq k$, and from this set we choose the infimum. Firstly, we need to prove that $y_k(\cdot)$ is well defined, that is, that for the relevant values of k, the sets $\{y \in \mathbb{R}_+ : f(x,y) \geq k\}$ are non empty for all x. The next theorem will show it. In the proof of this result we will use an operator $T(L)$ which transforms a line $L \in \mathbb{L}^-$ into another line $T(L)$. The new line $T(L)$ is such that it passes through $c(L)$, this way we guarantee that the choice of the new line $c\big(T(L)\big)$ is better than $c(L)$, and also, due to theorem (4) better than any point on the line $T(L)$ between $c(L)$ and $c\big(T(L)\big)$.

What $T(L)$ will do, for a given line L in $\mathbb{L}^-$, is to pick the points $c(L) = (x^*, y^*)$, the bargaining solution, and the point $(\frac{x^*+x^m}{2}, 0)$, in which $x^m = L^1$, and with these two points define a new line $L' = T(L)$ that passes through them. L' is such that it's parameters $a = a[L']$ and $b = b[L']$ solve the following equations:

$$\begin{cases} y* &= a + bx^* \\ 0 &= a + b\frac{x^*+x^m}{2} \end{cases} \Leftrightarrow \begin{cases} b &= -2\frac{y^*}{x^m - x^*} \\ a &= -b\frac{x^*+x^m}{2} \end{cases} \tag{4}$$

In theorem (2) we derived that $c(L') \gg \mathbf{0}$ so $x^m > x^*$, hence $b < 0$ and $a > 0$, meaning that $L' \in \mathbb{L}^-$. Lets define the operation of transforming a line L in the new line L' formally.

Definition 2. *Let* $T : \mathbb{L}^- \to \mathbb{L}^-$ *with* $\boldsymbol{x^*} = c(L) \in T(L)$ *and* $(\frac{x*+x^m}{2}, 0) \in T(L)$.

We can apply the operator $T(\cdot)$ to a line that is already the result of an application of the operator, and have $T\big(T(L)\big) = T^2(L)$. We can proceed like this n times and getting the line $T^n(L)$. The next result establishes that the application of the $T(\cdot)$ operator n times is like multiplying the initial line L by a constant $\boldsymbol{\alpha} \in \mathbb{R}^2_+$ n times.

Lemma 3. $\forall n \in \mathbb{N}$ *and* $\forall L \in \mathbb{L}^-$, $T^n(L) = \boldsymbol{\alpha}^n L$ *with* $\alpha_1 < 1$

Proof. We will prove this result by induction. First, for $n = 1$, we will derive the value of $\boldsymbol{\alpha}$ such that $L' = T(L) = \boldsymbol{\alpha} L$. The maximum in the first coordinate of L' is, due to the negative slope of L', when $y = 0$, and this is by definition (2) of $T(\cdot)$, $x'^m = \frac{x^*+x^m}{2}$. By lemma (2) $y^* > 0$ then $x^* < x^m$, and $x'^m = \frac{x^*+x^m}{2} < x^m$. Also as $\boldsymbol{\alpha}$ is such that $\boldsymbol{\alpha} L = L'$ has, by (3) of proposition (1), $\alpha_1 = \frac{a[L']}{a[L]}\frac{b[L]}{b[L']}$ using number (1) of the same lemma we get that

$$\alpha_1 = \frac{L^{2'}}{L^2}\left[\frac{-\frac{L^2}{L^1}}{-\frac{L'^2}{L'^1}}\right] = \frac{L'^1}{L^1} = \frac{x'^m}{x^m} = \frac{1 + x^*/x^m}{2} < 1$$

.

Any point (x, y) belonging to the initial line L must, due to the equality (1) of proposition (1), satisfy the equation $y = y^m - \frac{y^m}{x^m}x$, then $y^* = y^m - \frac{y^m}{x^m}x^*$. Replacing this on equation (4), we obtain the formula $b = b[L'] = -2\frac{y^m - \frac{y^m}{x^m}x^*}{x^m - x^*} = -2y^m \frac{x^m - x^*}{x^m(x^m - x^*)} = -2\frac{y^m}{x^m} = 2b[L]$, the new line has twice the slope of the initial one. We know the slope of any line is the ratio of the maximums, this ratio of line $L' = T(L)$ is $\frac{y'^m}{x'^m}$ and so $\frac{y'^m}{x'^m} = 2\frac{y^m}{x^m}$. Using this equality on equation (2) of proposition (1) we derive $\alpha_2 = \frac{a[L']}{a[L]} = \frac{L'^2}{L^2} = \frac{y'^m}{y^m} = 2\frac{x'^m}{x^m} = 2\alpha_1$, $\boldsymbol{\alpha} = (\alpha_1, 2\alpha_1)$. The claim is true for $n = 1$.

Suppose the claim is true to $n-1$ and $T^{n-1}(L) = \boldsymbol{\alpha}^{n-1}L$. $T^n(L) = T\big(T^{n-1}(L)\big)$ so $T^n(L)$ passes by $c\big(T^{n-1}(L)\big) = c\big(\boldsymbol{\alpha}^{n-1}L\big) = \boldsymbol{\alpha}^{n-1}c(L) = \boldsymbol{\alpha}^{n-1}(x^*, y^*)$. If the maximum of the first coordinate in L is x^m the maximum in $\boldsymbol{\alpha}L$ is $\alpha_1 x^m$, consequently the maximum in $T^{n-1}(L)$ is $\alpha_1^{n-1}x^m$, and $T^n(L)$ passes also by $\left(\frac{\alpha_1^{n-1}x^m + \alpha_1^{n-1}x^*}{2}, 0\right) = \alpha_1^{n-1}\big(\frac{x^m + x^*}{2}, 0\big)$. As $T^n(L)$ and $T^{n-1}(L)$ are in $\mathbb{L}^-$ by proposition (1), $\exists \boldsymbol{\beta} \in \mathbb{R}^2_+$ such that $T^n(L) = \boldsymbol{\beta}T^{n-1}(L)$, we will find the value of such $\boldsymbol{\beta}$.

The coefficients, a_n and b_n, of the line that passes by $\alpha_1^{n-1}\big(\frac{x^m + x^*}{2}, 0\big)$ and $\boldsymbol{\alpha}^{n-1}(x^*, y^*)$ solve the following system of equations

$$\begin{cases} \alpha_2^{n-1}y^* & = a_n + b_n\alpha_1^{n-1}x^* \\ 0 & = a_n + b_n\alpha_1^{n-1}\frac{x^* + x^m}{2} \end{cases} \Leftrightarrow \begin{cases} b_n = -2\big(\frac{\alpha_2}{\alpha_1}\big)^{n-1}\frac{y^*}{x^m - x^*} \\ a_n = -b_n\alpha_1^{n-1}\frac{x^* + x^m}{2} \end{cases}$$

We already saw that $\frac{y*}{x^m - x*} = \frac{y^m}{x^m}$ and $b_n = -2^n\frac{y^m}{x^m}$, because $\alpha_2 = 2\alpha_1$. Due to $b_n = \frac{\beta_2}{\beta_1}b_{n-1}$, and to $b_{n-1} = -\frac{T^{n-1}(L)^2}{T^{n-1}(L)^1} = -2^{n-1}\frac{y^m}{x^m}$ we get $2^n\frac{y^m}{x^m} = \frac{\beta_2}{\beta_1}2^{n-1}\frac{y^m}{x^m}$, deriving $\frac{\beta_2}{\beta_1} = 2$. β_1 is the ratio of the maximums in the first component of $T^{n-1}(L)$ and $T^n(L)$, $\beta_1 = \frac{\alpha_1^{n-1}\frac{x^m + x^*}{2}}{\alpha_1^{n-1}x^m} = \alpha_1$. $\beta_1 = \alpha_1$ then $\beta_2 = 2\beta_1 = 2\alpha_1 = \alpha_2$. Concluding $T^n(L) = \boldsymbol{\beta}T^{n-1}(L) = \boldsymbol{\alpha}T^{n-1}(L) = \boldsymbol{\alpha}^n L$. ■

Lemma 4. *$\forall \boldsymbol{x_0} = (x_0, y_0) \in \mathbb{R}^2_+$ and $\forall 0 < x_1 < x_0$, $\exists y_1 : f(\boldsymbol{x_1}) > f(\boldsymbol{x_0})$.*

Proof. Let us first define $c\big(T^n(L)\big) = \boldsymbol{z^n} = (z_1^n, z_2^n)$. Due to the way $T(\cdot)$ was built, in particular to $\boldsymbol{z^{n-1}} = c(T^{n-1}(L)) \in T^n(L)$ and $\boldsymbol{z^n} \neq \boldsymbol{z^{n-1}}$ we know that $f(\boldsymbol{z^n}) > f(\boldsymbol{z^{n-1}})$, and we can conclude that $f(\boldsymbol{z^n}) > f(\boldsymbol{z^0})$. Choosing L in a way that $(x_0, y_0) \in L$ we have that $f(\boldsymbol{z^0}) > f(x_0, y_0)$ and $f(\boldsymbol{z^n}) > f(x_0, y_0)$. $z_1^n = \alpha_1^n z_1^0$, as $\alpha_1 < 1$, it is possible to find an n such that $z_1^n = \alpha_1^n z_1^0 \leq x_1 < \alpha_1^{n-1} z_1^0 = z_1^{n-1}$. Chose y_1 in a way that $\boldsymbol{x}_1 = (x_1, y_1) \in T^n(L)$. Due to theorem (4) we know that the function $f(\cdot)$ is decreasing along the line $T^n(L)$ for $x > z_1^n$, as $z_1^n \leq x_1 < z_1^{n-1}$ then $f(\boldsymbol{x}_1) > f(\boldsymbol{z^{n-1}})$ because both $\boldsymbol{x}_1$ and $\boldsymbol{z^{n-1}}$ belong to $T^n(L)$. Finally $f(\boldsymbol{x}_1) > f(\boldsymbol{z^{n-1}}) \geq f(\boldsymbol{x_0})$. ■

Theorem 5. *$\forall \boldsymbol{x} \in \mathbb{R}^2_+$, $\forall w \in \mathbb{R}^+$ the set $\{z : f(w, z) \geq f(\boldsymbol{x})\}$ is non empty.*

Proof. $\boldsymbol{x} = (x, y)$ if $w < x$ by lemma (4) $\exists z$ such that $f(w, z) > f(\boldsymbol{x})$ and the set $\{z : f(w, z) \geq f(\boldsymbol{x})\}$ is non empty. If $w > x$ then, due to pareto optimality, the function $f(\cdot)$ increasing in both arguments $f(w, y) > f(\boldsymbol{x})$. ■

Now we can define the function $y_k(\cdot)$ for any $k \in ch\big\{f(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^2_+\big\}$, knowing it is well defined for any x, once an immediate implication of the theorem (5) is that the set $Y_x^k = \{y \in \mathbb{R}_+ : f(x, y) \geq k\}$ is non empty.

Definition 3. *Let $U = ch\big\{f(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^2_+\big\}$, for $k \in U$ then*

$$y_k(x) = \inf\{y \in \mathbb{R}_+ : f(x, y) \geq k\} \tag{5}$$

The two next claims prove that $y_k(x)$ is strictly decreasing and strictly convex.

Claim 10. *The function $y_k(x)$ is decreasing and strictly decreasing if $y_k(x) > 0$.*

Proof. When $x > x'$, $f(x, y) \geq f(x', y)$ and $Y_x^k \supseteq Y_{x'}^k$, hence $\inf Y_x^k \leq \inf Y_{x'}^k$, meaning $y_k(x) \leq y_k(x')$, and $y_k(x)$ is decreasing.

Assume $y_k(x) = y_k(x')$, then $\forall \tilde{x} \in (x', x)$, $y_k(x') \geq y_k(\tilde{x}) \geq y_k(x) = y_k(x')$, and $y_k(\tilde{x}) = y_k(x)$. Also $\forall \epsilon > 0$, $f\big(\tilde{x}, y_k(\tilde{x}) + \epsilon\big) > k$, if $f\big(\tilde{x}, y_k(\tilde{x}) + \epsilon\big) \leq k$, as $f(\cdot)$ is strictly increasing for any $\epsilon' < \epsilon$, $f\big(\tilde{x}, y_k(\tilde{x}) + \epsilon'\big) < f\big(\tilde{x}, y_k(\tilde{x}) + \epsilon\big) \leq k$, meaning that

$f(\tilde{x}, y) \geq k$ only if $y \geq y_k(\tilde{x}) + \epsilon$, then $y_k(\tilde{x}) = \inf\{y \in \mathbb{R}_+ : f(\tilde{x}, y) \geq k\} \geq y_k(\tilde{x}) + \epsilon$, and we get a contradiction $y_k(\tilde{x}) \geq y_k(\tilde{x}) + \epsilon$. By a similar argument $f\big(\tilde{x}, y_k(\tilde{x}) - \epsilon\big) < k$, otherwise $y_k(\tilde{x}) \leq y_k(\tilde{x}) - \epsilon$.

Pick a point $\tilde{\boldsymbol{x}} \in \mathbb{R}^2_+$ such that $\tilde{x} \in (x, x')$ and $\tilde{y} = y_k(\tilde{x})$, let the line $L \in \mathbb{L}^-$ be such that $c(L) = \tilde{\boldsymbol{x}}$ and chose $\bar{\boldsymbol{x}} = (\bar{x}, \bar{y})$ with $x < \bar{x} < \tilde{x}$. As $\bar{x} < \tilde{x}$ and $b[L] < 0$ we have that $\bar{y} = a[L] + b[L]\bar{x} > a[L] + b[L]\tilde{x} = \tilde{y}$, define $\frac{\tilde{y}}{\bar{y}} < \alpha_2 < 1$. With $\boldsymbol{\alpha} = (1, \alpha_2)$ the choice $c\big(\boldsymbol{\alpha}L\big) = \boldsymbol{\alpha}c(L) = \big(\tilde{x}, \alpha_2 y_k(\tilde{x})\big)$, and as $\alpha_2 y_k(\tilde{x}) < y_k(\tilde{x})$ we have, for some $\epsilon = y_k(\tilde{x}) - \alpha_2 y_k(\tilde{x}) > 0$, $f\Big(c\big(\boldsymbol{\alpha}L\big)\Big) = f\big(\boldsymbol{\alpha}c(L)\big) = f\big(\tilde{x}, \alpha_2 y_k(\tilde{x})\big) = f\big(\tilde{x}, y_k(\tilde{x}) - \epsilon\big) < k$. However $\boldsymbol{\alpha}(\bar{x}, \bar{y}) = (\bar{x}, \alpha_2\bar{y})$ and $\alpha_2\bar{y} > \tilde{y} = y_k(\tilde{x})$, $y_k(\cdot)$ is constant in (x, x') $y_k(\tilde{x}) = y_k(\bar{x})$, hence $\alpha_2\bar{y} > y_k(\bar{x})$. And with $\epsilon = \alpha_2\bar{y} - y_k(\bar{x}) > 0$, $f(\bar{x}, \alpha_2\bar{y}) = f(\bar{x}, y_k(\bar{x}) + \epsilon) > k$. So $f(\boldsymbol{\alpha}\bar{\boldsymbol{x}}) = f(\bar{x}, \alpha_2\bar{y}) > f\Big(c\big(\boldsymbol{\alpha}L\big)\Big)$, we got a contradiction because $f\Big(c\big(\boldsymbol{\alpha}L\big)\Big)$ should be the biggest in $\boldsymbol{\alpha}L$, $y_k(\cdot)$ must be strictly decreasing. ■

Claim 11. *The function $y_k(x)$ is strictly convex.*

Proof. Pick two points $\boldsymbol{x}, \boldsymbol{x'}$ in $\mathbb{R}^2_+$, with both points on the same indifference curve, $\boldsymbol{x} = \big(x, y_k(x)\big)$ and $\boldsymbol{x'} = \big(x', y_k(x')\big)$, for any $\alpha \in [0,1]$ define $x^\alpha = \alpha x + (1-\alpha)x'$. Take two converging sequences $\{y^n_x\}^\infty_{n=1}$ and $\{y^n_{x'}\}^\infty_{n=1}$ with $y^n_x \to y_k(x)$, $y^n_{x'} \to y_k(x')$ and $y^n_x > y_k(x)$, $y^n_{x'} > y_k(x')$. As we saw in the previous proof $f(x, y^n_x) > k$ and $f(x, y^n_{x'}) > k$, with $y^\alpha_n = \alpha y^n_x + (1-\alpha)y^n_{x'}$, by function $f(\cdot)$ quasiconcavity, lemma (4), we know that $f(x^\alpha, y^\alpha_n) \geq \min\{f\big(x, y^n_x\big), f\big(x', y^n_{x'}\big)\} > k$. Therefore $y^\alpha_n \in Y^k_{x^\alpha}$, and $y_k(x^\alpha) = \inf Y^k_{x^\alpha} \leq y^\alpha_n$, and $y_k(x^\alpha) \leq \lim_n y^\alpha_n = \lim_n \alpha y^n_x + (1-\alpha)y^n_{x'} = \alpha y_k(x) + (1-\alpha)y_k(x')$. This result is verified for any x, x' and x^α, consequently the function $y_k(\cdot)$ is convex.

If $y_k(\cdot)$ is not strictly convex, exists x and x' such that $y_k(x^\alpha) = \alpha y_k(x) + (1-\alpha)y_k(x')$, as $y_k(\cdot)$ is convex we know Avriel, Diewert, Schaible, and Zang (2010, p.17) that the function $R(x', x) = \frac{y_k(x') - y_k(x)}{x' - x}$ is nondecreasing in x for a fixed x', (and nondecreasing in x' for a fixed x). Then $R(x^\alpha, x) = \frac{y_k(x^\alpha) - y_k(x)}{x^\alpha - x} = \frac{(1-\alpha)\big(y_k(x') - y_k(x)\big)}{(1-\alpha)(x' - x)} = R(x', x)$, as the function $R(x', x)$ nondecreases for $x^\alpha \leq x^\beta \leq x'$, $R(x^\alpha, x) \leq R(x^\beta, x) \leq$

$R(x',x)$, then $R(x',x) = R(x^\beta,x)$, which is equivalent to $y_k(x^\beta) = \beta y_k(x) + (1-\beta)y_k(x')$. By a similar argument to the one used in claim (10), we know $f\big(x^\beta, y_k(x^\beta)-\epsilon\big) < k < f\big(x^\beta, y_k(x^\beta)+\epsilon\big)$, for any $\epsilon > 0$.

Pick a point $\tilde{\boldsymbol{x}}$ over the line $y_k(\cdot)$ such that: $\tilde{x} \in (x,x')$ and $\tilde{y} = y_k(\tilde{x})$; pick the line $\tilde{L}$ such that $c(\tilde{L}) = \tilde{\boldsymbol{x}}$ and $b[L] \neq b = \frac{y_k(x)-y_k(x')}{x-x'}$. A point $\tilde{\boldsymbol{x}}$ and a line $\tilde{L}$ with those conditions always exists. Consider the line $L = \big\{(x,y) : y = 2-x\big\}$, it is symmetric and $c(L) = (1,1)$, so $\tilde{\boldsymbol{x}} = c(\tilde{\boldsymbol{x}}L)$. If the initial point chosen $\tilde{\boldsymbol{x}}$ is such that the slope $b(\tilde{\boldsymbol{x}}L) = b$, pick $\bar{\boldsymbol{x}}$, another point over the indifference curve $y_k(\cdot)$, so $\bar{y} = y_k(\bar{x})$. The line $\bar{\boldsymbol{x}}L$, that has $\bar{\boldsymbol{x}} = c\big(\bar{\boldsymbol{x}}L\big)$ can be rewritten as $\frac{\bar{\boldsymbol{x}}}{\tilde{\boldsymbol{x}}}\tilde{\boldsymbol{x}}L$, with $\boldsymbol{\alpha} = \frac{\bar{\boldsymbol{x}}}{\tilde{\boldsymbol{x}}} = \big(\frac{\bar{x}}{\tilde{x}}, \frac{\bar{y}}{\tilde{y}}\big)$ and using point (2) of proposition (1), $b[\bar{\boldsymbol{x}}L] = b[\boldsymbol{\alpha}\tilde{\boldsymbol{x}}L] = \frac{\alpha_2}{\alpha_1} b[\tilde{\boldsymbol{x}}L] = \frac{\alpha_2}{\alpha_1} b$. Calculating $\frac{\alpha_2}{\alpha_1} = \frac{\bar{y}/\tilde{y}}{\bar{x}/\tilde{x}} = \frac{\bar{y}/\bar{x}}{\tilde{y}/\tilde{x}} = \frac{a/\bar{x}+b}{a/\tilde{x}+b} \neq 1$, and $b[\bar{\boldsymbol{x}}L] \neq b$, because $\tilde{y} = y_k(\tilde{x}) = a + b\tilde{x}$ and $\bar{y} = y_k(\bar{x}) = a + b\bar{x}$.

Having chosen the point $\tilde{\boldsymbol{x}}$ and the line $\tilde{L}$ and knowing that $y_k(\cdot)$ is a line for any value between x and x', we can write that $y_k(\bar{x}) = a + b\bar{x}$ for any $x < \bar{x} < x'$. The point $\tilde{\boldsymbol{x}} = (\tilde{x}, \tilde{y})$ is on both lines, $\tilde{L}$ and $y_k(\cdot)$, so $y_k(\tilde{x}) = a + b\tilde{y} = a[\tilde{L}] + b[\tilde{L}]\tilde{x}$. Suppose that $b[\tilde{L}] > b$, pick a point $\bar{x}$ such that $x < \tilde{x} < \bar{x} < x'$ and let $\bar{\boldsymbol{x}} = (\bar{x}, \bar{y}) \in \tilde{L}$, as the point is on the line $\tilde{L}$, $\bar{y} = a[\tilde{L}] + b[\tilde{L}]\bar{x}$. Calculate $\bar{y} - y_k(\bar{x})$

$$\begin{aligned}
\bar{y} - y_k(\bar{x}) =& \Big[a[\tilde{L}] + b[\tilde{L}]\bar{x}\Big] - \Big[a + b\bar{x}\Big] \\
=& \Big[a[\tilde{L}] + b[\tilde{L}]\tilde{x} + b[\tilde{L}](\bar{x}-\tilde{x})\Big] - \Big[a + b\tilde{x} + b(\bar{x}-\tilde{x})\Big] \\
=& b[\tilde{L}](\bar{x}-\tilde{x}) - b(\bar{x}-\tilde{x}) = (b[\tilde{L}] - b)(\bar{x}-\tilde{x})
\end{aligned}$$

As $b[\tilde{L}] > b$ and $\bar{x} > \tilde{x}$, $\bar{y} - y_k(\bar{x}) > 0$. Chose α_2 with $\frac{y_k(\bar{x})}{\bar{y}} < \alpha_2 < 1$ and $\boldsymbol{\alpha} = (1, \alpha_2)$, we know $c(\boldsymbol{\alpha}\tilde{L}) = (\tilde{x}, \alpha_2\tilde{y})$ and $\alpha_2\tilde{y} < \tilde{y} = y_k(\tilde{x})$, so $f\Big(c\big(\boldsymbol{\alpha}\tilde{L}\big)\Big) < k$. However, $\boldsymbol{\alpha}\bar{\boldsymbol{x}} \in \boldsymbol{\alpha}L$ and $\alpha_2\bar{y} > y_k(\bar{x})$, therefore $f(\boldsymbol{\alpha}\bar{\boldsymbol{x}}) > k$ and we get a contradiction, because $f\Big(c\big(\boldsymbol{\alpha}\tilde{L}\big)\Big) < k < f(\boldsymbol{\alpha}\bar{\boldsymbol{x}})$, $c\big(\boldsymbol{\alpha}\tilde{L}\big)$ is not the best. $y_k(\cdot)$ must be strictly convex. If $b[\tilde{L}] < b$ the proof follows the same lines but choosing a point $\bar{\boldsymbol{x}}$ with $x < \bar{x} < \tilde{x} < x'$. ∎

Convexity of $y_k(\cdot)$ implies that lateral derivatives of $y_k(\cdot)$ exist everywhere Avriel, Diewert, Schaible, and Zang (2010, p. 20). The next theorem will prove that $y_k(\cdot)$ is differentiable, and from there goes to show that it is at the tangency between L and $y_k(\cdot)$ that the maximum is attained. This theorem starts to unveil that the curves of $y_k(\cdot)$ are the indifference curves of the social function.

Theorem 6. *The point $\boldsymbol{x} = (x, y) \in \mathbb{R}^2_+$ is the choice of a line $L = \{(x, y) : y = a + bx\}$, $c(L) = \boldsymbol{x}$, if and only if for $k = f(\boldsymbol{x})$, $y'_k(x) = b$*

This theorem assumes one strong fact not yet proven, namely the differentiability of $y_k(x)$, so we will need to show that this is true. Before that, however, we need to prove a milder version of the necessary condition of the theorem, that the point $\tilde{\boldsymbol{x}}$ on the line L with $y'_k(\tilde{x}^-) \leq b \leq y'_k(\tilde{x}^+)$ must be the choice on the set L, $\tilde{\boldsymbol{x}} = c(L)$.

Claim 12. *If the line $L = \{(x, y) : y = a + bx\}$ is such that $\tilde{\boldsymbol{x}} = \big(\tilde{x}, y_k(\tilde{x})\big) \in L$ and $y'_k(\tilde{x}^-) \leq b \leq y'_k(\tilde{x}^+)$ then $c(L) = \tilde{\boldsymbol{x}}$.*

Proof. As $\tilde{\boldsymbol{x}} \in L$ then $y_k(\tilde{x}) = a + b\tilde{x}$. We know $R_k(x, \tilde{x}) = \frac{y_k(x) - y_k(\tilde{x})}{x - \tilde{x}}$ is increasing in x, so $y'_k(\tilde{x}^-) = \lim_{\epsilon \downarrow 0} R_k(\tilde{x} - \epsilon, \tilde{x}) > R_k(x, \tilde{x})$ for $x < \tilde{x}$. Then $R_k(x, \tilde{x}) < y'(\tilde{x}^-) \leq b$, and after some easy calculation we get that $b(x - \tilde{x}) < y_k(x) - y_k(\tilde{x})$, using that $y_k(\tilde{x}) = a + b\tilde{x}$, $a + bx < y_k(x)$ for any $x < \tilde{x}$. So the function $g_k(x) = y_k(x) - (a + bx)$ is positive whenever $x < \tilde{x}$. The function $g_k(\cdot)$ is, due to $y_k(\cdot)$ convexity, also convex. And $\bar{R}_k(x, \tilde{x}) = \frac{g_k(x) - g_k(\tilde{x})}{x - \tilde{x}}$ is increasing in x. For $x_0 < x1 < \tilde{x}$, $\bar{R}_k(x_0, \tilde{x}) < \bar{R}_k(x_1, \tilde{x})$ due to $g_k(\tilde{x}) = 0$ we get $\frac{g_k(x_0)}{x_0 - \tilde{x}} < \frac{g_k(x_1)}{x_1 - \tilde{x}}$ and as $x_0 - \tilde{x} < 0$, $g_k(x_0) > g_k(x_1)\frac{x_0 - \tilde{x}}{x_1 - \tilde{x}}$, as $\frac{\tilde{x} - x_0}{\tilde{x} - x_1} > 1$ we get $g_k(x_0) > g_k(x_1)$. The function $g_k(x)$ is decreasing for $x < \tilde{x}$. Applying the same calculations for $x > \tilde{x}$ we conclude $g_k(x)$ is increasing in this case. Therefore, for any given $\epsilon > 0$, we can chose the maximum of $g_k(x)$ in a closed neighbourhood of $\tilde{x}$, $cl\big(N_\epsilon(\tilde{x})\big)$, and we know from the increasing-decreasing nature of $g_k(x)$ that the maximum will be on one of the most distant point from $\tilde{x}$, we define $a_\epsilon = \sup_{x \in cl\left(N_\epsilon(\tilde{x})\right)} g_k(x) = \max\{g_k(\tilde{x} - \epsilon), g_k(\tilde{x} + \epsilon)\}$. Let $A_\epsilon = \{x : g_k(x) \leq a_\epsilon\}$ be the set of points x for which the line L is at distance a_ϵ or less from $y_k(x)$. Again by the nature of $g_k(x)$, those points must belong to the ϵ neighbourhood of $\tilde{x}$, $A_\epsilon \subseteq cl\big(N_\epsilon(\tilde{x})\big)$. Define

$$L_{a_\epsilon} = \{(x,y) : y = a + bx + a_\epsilon\}$$

For the points $(x,y) \in L_{a_\epsilon}$: if $x \in A_\epsilon$ then $g_k(x) \leq a_\epsilon \Leftrightarrow y_k(x) \leq a + bx + a_\epsilon = y$ and $f(x,y) \geq k$; if $x \notin A_\epsilon$, $g_k(x) > a_\epsilon \Leftrightarrow y_k(x) > a + bx + a_\epsilon = y$ and $f(x,y) < k$. Hence with $c(L_{a_\epsilon}) = \arg\max_{\boldsymbol{x} \in L_{a_\epsilon}} f(\boldsymbol{x})$, $c(L_{a_\epsilon})_1 \in A_\epsilon \subseteq cl\big(N_\epsilon(\tilde{x})\big)$. $g_k(x)$ is continuous then $a_\epsilon \downarrow 0$ when $\epsilon \downarrow 0$, and $L_{a_\epsilon} \to L$, by theorem (3) $c(L_{a_\epsilon}) \to c(L)$. As $c(L^\epsilon)_1 \in cl\big(N_\epsilon(\tilde{x})\big) \to \{\tilde{x}\}$, $c(L) = \big(\tilde{x}, y_k(\tilde{x})\big) = \tilde{\boldsymbol{x}}$. ∎

The previous claim is sufficient to establish that functions $y_k(\cdot)$ are differentiable everywhere.

Claim 13. *$\forall k$, $y_k(\cdot)$ is differentiable.*

Proof. If $y_k(\cdot)$ is not differentiable $\exists x$ such that $y_k'(x^-) < y_k'(x^+)$, pick two lines L and L' to which $\big(x, y_k(x)\big)$ belongs, and $y_k'(x^-) < b[L] < b[L'] < y_k'(x^+)$. By claim (12) the optimal point is $c(L) = c(L') = \big(x, y_k(x)\big)$, but the $\boldsymbol{\alpha} \in \mathbb{R}^2_+$ such that $\boldsymbol{\alpha}L = L'$ must be different from $(1,1)$, otherwise $L = L'$, therefore $c(L') = C(\boldsymbol{\alpha}L) = \boldsymbol{\alpha}c(L) \neq c(L)$, a contradiction. It cannot exist x with $y_k'(x^-) < y_k'(x^+)$. ∎

We can now prove the theorem (6)

Proof. By claim (13) the curves $y_k(\cdot)$ are differentiable, using this result on claim (12) we get that if $\boldsymbol{x} = (x,y) \in L$, with $y = y_k(x)$ and $b[L] = y_k'(x)$ then $c(L) = \boldsymbol{x}$. To prove sufficiency, assume that it is possible for $(x,y) = c(L)$ with $y_k(x) = y$ but $b[L] = b \neq y_k'(x)$. Let's assume $y_k'(x) > b$, then $\lim_{\epsilon \to 0} \frac{y_k(x+\epsilon) - y_k(x)}{\epsilon} > b$, meaning that there is an $\bar{\epsilon} > 0$ such that for any $|\epsilon'| < \bar{\epsilon}$, $\frac{y_k(x+\epsilon') - y_k(x)}{\epsilon'} > b$, for the point x we know $y_k(x) = a + bx$ and for $\epsilon' < 0$, we get $y_k(x+\epsilon') < y_k(x) + b\epsilon' = a + b(x+\epsilon')$, in which, naturally, as $a = a[L]$ and $b = b[L]$, $\big(x+\epsilon', a + b(x+\epsilon')\big) \in L$, if $x + \epsilon' > 0$. But by theorem (2) $c(L) \gg \mathbf{0}$ and for small ϵ', $x + \epsilon' > 0$, due to $a + b(x+\epsilon') > y_k(x+\epsilon')$, $f\big(x+\epsilon', y_k(x+\epsilon')\big) > k = f(\boldsymbol{x})$, a contradiction, $\boldsymbol{x}$ can not be $c(L)$. ∎

Claim 14. *$\forall b < 0$, $\forall k \in U$, $\exists x \in \mathbb{R}^+$ such that $y'_k(x) = b$.*

Proof. If condition is not satisfied then either $b \in ch\Big\{y'_k(x) : x \in \mathbb{R}^+\Big\}$ or $b \notin ch\Big\{y'_k(x) : x \in \mathbb{R}^+\Big\}$. If it is the first case then $y_k(\cdot)$ must be discontinuous at one point $\bar{x}$, and $\lim_{x\to\bar{x}^-} y'_k(x) > \lim_{x\to\bar{x}^+} y'_k(x)$, by an argument equal to the one used in claim (13) we prove such a discontinuity creates a contradiction with AT axiom. If $b \notin ch\Big\{y'_k(x) : x \in \mathbb{R}^+\Big\}$, lets assume that $b \leq \inf\Big\{y'_k(x) : x \in \mathbb{R}^+\Big\}$, as the $y'_k(\cdot)$ is increasing (a convex function derivative is increasing) then $b \leq \lim_{x\to 0^+} y'_k(x)$. For $x > 0$, $b < y'_k(x)$, and we know $R(x, x') = \frac{y_k(x)-y_k(x')}{x-x'}$ is increasing in x' for a fixed x, so $\lim_{\epsilon\to 0} R(x, x+\epsilon) > b$, then $R(x, x') > b$, for $x' > x$. So $\frac{y_k(x)-y_k(x')}{x-x'} > b$, and noticing that $x - x' < 0$, $y_k(x) - y_k(x') < b(x - x')$ as this inequality is valid for any positive x, we can take limits, $\lim_{x\to 0} y_k(x) - y_k(x') < -bx'$. Then $\lim_{x\to 0} y_k(x) < y_k(x') - bx'$, and the function $y_k(\cdot)$ is bounded near the origin, let $\theta = \lim_{x\to 0} y_k(x)$.

Let the line $L_\epsilon = \Big\{(x, y) : y = \theta_\epsilon + bx\Big\}$ with $\theta_\epsilon = y_k(\epsilon) - b\epsilon$. If $(x, y) \in L_\epsilon$ and $x < \epsilon$, as $R(\epsilon, x)$ is increasing and $y'_k(x) > b$ then $R(\epsilon, x) > y'_k(x) > b$. So $\frac{y_k(\epsilon)-y_k(x)}{\epsilon-x} > b$ and $y_k(\epsilon) - y_k(x) > b(\epsilon - x)$, replacing $y_k(\epsilon) = \theta_\epsilon + b\epsilon$, $\theta_\epsilon + bx > y_k(x)$, and $f(x, y) = f(x, \theta_\epsilon + bx) > k$. If $(x, y) \in L_\epsilon$ and $x > \epsilon$ then $f(x, y) = f(x, \theta_\epsilon + bx) < k$. Therefore $c(L_\epsilon)_1 \leq \epsilon$. And $L = \Big\{(x, y) : y = \theta_+ bx\Big\}$ is the limit of L_ϵ, and by theorem (3) $c(L_\epsilon) \to c(L)$ and $c(L)_1 = 0$, contradicting theorem (2). Therefore it must exist a $x \in \mathbb{R}$ with $y'_k(x) = b$.

If $b \geq \sup\Big\{y'_k(x) : x \in \mathbb{R}^+\Big\}$ the reasoning is similar. We know $R(x, x')$ is increasing in x and $R(x, x') < R(x' + \epsilon, x') < y'_k(x') < b < 0$, for any $x < x'$. Then $\frac{y_k(x)-y_k(x')}{x-x'} < b$, and $y_k(x') < y_k(x) + b(x' - x)$. As $b < 0$ then the last inequality means that for a big value of x', with a fixed x, $y_k(x') < 0$, a contradiction as $y_k(\cdot)$ is positive. ■

4.3 The Function $h(x)$

Any increasing transformation of the social function also represents the same preferences, meaning that we can be dealing with different kinds of functions. I will use one particular function to represent the choice $c(S)$. This function $h(\boldsymbol{x})$ is based on the values of $f(\boldsymbol{x})$ when $x = 1$. For a given $f(\boldsymbol{x})$, such that $c(S) = \arg\max_{\boldsymbol{x}\in S} f(\boldsymbol{x})$, for any point $\boldsymbol{x}$ we pick $\gamma(\boldsymbol{x}) = \inf\{z : f(1, z) \geq f(\boldsymbol{x})\}$. That is, along the vertical line $x = 1$, we are picking the smallest y among those that have a f-value bigger then $f(\boldsymbol{x})$. That $\gamma(\boldsymbol{x})$ is well defined is an immediate consequence of lemma (4), and we can now define $h(\boldsymbol{x}) = f\big(1, \gamma(\boldsymbol{x})\big)$. We will show that $h(\cdot)$ represents the same choice as the function $f(\cdot)$ for any set $S \in \mathbb{S}$. For that we need first to establish the function $h(\cdot)$ is strictly quasiconcanve.

Theorem 7. *$h(x, y)$ is strictly quasiconcave.*

Claim 15. *If $f(\boldsymbol{x}) > f(\boldsymbol{x'})$ then $h(\boldsymbol{x}) \geq h(\boldsymbol{x'})$; and if $f(\boldsymbol{x}) = f(\boldsymbol{x'})$ then $h(\boldsymbol{x}) = h(\boldsymbol{x'})$*

Proof. If $f(\boldsymbol{x}) > f(\boldsymbol{x'})$ and $z \in \mathbb{R}^+$ is such that $f(1, z) \geq f(\boldsymbol{x})$ then $f(1, z) \geq f(\boldsymbol{x})$ meaning that $\{z : f(1, z) \geq f(\boldsymbol{x})\} \subseteq \{z : f(1, z) \geq f(\boldsymbol{x'})\}$ and $\gamma(\boldsymbol{x}) = \inf\{z : f(1, z) \geq f(\boldsymbol{x})\} \geq \inf\big\{z : f(1, z) \geq f(\boldsymbol{x'})\big\} = \gamma(\boldsymbol{x'})$. From this we conclude $h\big(\boldsymbol{x}\big) = f\big(1, \gamma(\boldsymbol{x})\big) \geq f\big(1, \gamma(\boldsymbol{x'})\big) = h\big(\boldsymbol{x'}\big)$. If $f(\boldsymbol{x}) = f(\boldsymbol{x'})$, the equality of the sets $\{z : f(1, z) \geq f(\boldsymbol{x})\} = \{z : f(1, z) \geq f(\boldsymbol{x'})\}$ establishes $\gamma(\boldsymbol{x}) = \gamma(\boldsymbol{x'})$ and naturally $h\big(\boldsymbol{x}\big) = h\big(\boldsymbol{x'}\big)$. ■

Claim 16. *If $h(\boldsymbol{x^\alpha}) = \min\{h(\boldsymbol{x}), h(\boldsymbol{x'})\}$ then $h(\boldsymbol{x^\beta}) = \min\{h(\boldsymbol{x}), h(\boldsymbol{x'})\}$ either for all β with $0 \leq \beta \leq \alpha$, or for all β, with $\alpha \leq \beta \leq 1$*

Proof. If $\boldsymbol{x} \ll \boldsymbol{x'}$ and $h(\boldsymbol{x^\alpha}) = \min\{h(\boldsymbol{x}), h(\boldsymbol{x'})\} = h(\boldsymbol{x})$ then $h(\boldsymbol{x^\alpha}) = h(\boldsymbol{x})$, and due to the $f(\cdot)$ being strictly increasing in both arguments, for $0 \leq \beta \leq \alpha$, $x \ll \boldsymbol{x^\beta} \ll \boldsymbol{x^\alpha}$ and $f(\boldsymbol{x}) < f(\boldsymbol{x^\beta}) < f(\boldsymbol{x^\alpha})$, by claim (15) $h(\boldsymbol{x}) \leq h(\boldsymbol{x^\beta}) \leq h(\boldsymbol{x^\alpha}) = h(\boldsymbol{x})$.

If $\boldsymbol{x} \not\ll \boldsymbol{x'}$ and $\boldsymbol{x'} \not\ll \boldsymbol{x}$ the line passing through both points $L[\boldsymbol{x}, \boldsymbol{x'}]$ has negative slope: $(\tilde{x}, \tilde{y}) \in L[\boldsymbol{x}, \boldsymbol{x'}]$, $\tilde{y} = y + \frac{y'-y}{x'-x}(\tilde{x} - x)$, therefore the slope of the line $b\Big[L[\boldsymbol{x}, \boldsymbol{x'}]\Big] = \frac{y'-y}{x'-x}$ is negative. According to theorem (4), there is a $x^* \in (x, x')$ such that for points in $L[\boldsymbol{x}, \boldsymbol{x'}]$, $f\big(\tilde{x}, \tilde{y}\big)$ is increasing for values of $\tilde{x} < x*$ and decreasing for $\tilde{x} > x*$. If $x^\alpha \in [x, x*]$ then$f(\boldsymbol{x}) < f(\boldsymbol{x^\alpha})$ and $\forall 0 \leq \beta \leq \alpha$, $f(\boldsymbol{x}) < f(\boldsymbol{x^\beta}) < f(\boldsymbol{x^\alpha})$ and

$\min\{h(\boldsymbol{x}), h(\boldsymbol{x'})\} \leq h(\boldsymbol{x}) \leq h(\boldsymbol{x^\beta}) \leq h(\boldsymbol{x^\alpha}) = \min\{h(\boldsymbol{x}), h(\boldsymbol{x'})\}$, hence $h(\boldsymbol{x^\beta}) = h(\boldsymbol{x^\alpha}) = \min\{h(\boldsymbol{x}), h(\boldsymbol{x'})\}$.

If $x^\alpha \in [x^*, x']$ then$\forall \beta$, $\alpha \leq \beta \leq 1$, $x^* \leq x^\alpha < x^\beta < x'$, by theorem (4), $f(\boldsymbol{x^\alpha}) > f(\boldsymbol{x^\beta}) > f(\boldsymbol{x'})$, and $\min\{h(\boldsymbol{x}), h(\boldsymbol{x'})\} = h(\boldsymbol{x^\alpha}) \geq h(\boldsymbol{x^\beta}) \geq h(\boldsymbol{x}) \geq \min\{h(\boldsymbol{x}), h(\boldsymbol{x'})\}$, hence $h(\boldsymbol{x^\beta}) = h(\boldsymbol{x^\alpha}) = \min\{h(\boldsymbol{x}), h(\boldsymbol{x'})\}$. ■

Claim 17. *If* $\exists \boldsymbol{x}, \boldsymbol{x'} \in \mathbb{R}^2_+$ *such that* $h(\boldsymbol{x^\beta}) = h(\boldsymbol{x})$, $\forall \beta \in [\theta_1, \theta_2]$ *then,* $\exists \gamma \in \mathbb{R}^+$ *with* $f(1, \gamma - \epsilon) < f(\boldsymbol{x^\beta}) < f(1, \gamma + \epsilon)$,$\forall \epsilon > 0$ *and* $\forall \beta \in [\theta_1, \theta_2]$

Proof. As $h(\boldsymbol{x^\beta}) = f\big(1, \gamma(\boldsymbol{x^\beta})\big)$ and $f(1, z)$ is strictly increasing in z, when $h(\boldsymbol{x^\beta}) = h(\boldsymbol{x})$ then $\gamma(\boldsymbol{x^\beta}) = \gamma(\boldsymbol{x}) = \gamma$. Therefore $\gamma = \inf\big\{z : f(1, z) \geq f(\boldsymbol{x^\beta})\big\}$, so for any $\epsilon > 0$, $\exists \gamma' < \gamma + \epsilon$ such that $f(1, \gamma') \geq f(\boldsymbol{x^\beta})$ and as the function is increasing $f(1, \gamma + \epsilon) > f(1, \gamma') \geq f(\boldsymbol{x^\beta})$ for any $\beta \in [\theta_1, \theta_2]$. Any $\gamma' < \gamma$ is not in the set $\big\{z : f(1, z) \geq f(\boldsymbol{x^\beta})\big\}$ meaning that for any $\epsilon > 0$, $f(1, \gamma - \epsilon) < f(1, \gamma) \leq f(\boldsymbol{x^\beta})$. ■

Lemma 5. *For any* $k_0 < k_1$ *such that* $\exists x \in \mathbb{R}^+$ *with* $y_{k_1}(x) = y_{k_0}(x)$ *then* $\forall x' \in \mathbb{R}^+, y_{k_1}(x') = y_{k_0}(x')$

Proof. Suppose a x' with $y_{k_1}(x') \neq y_{k_0}(x')$ exists, and without loss of generality assume $x' > x$. Pick the maximum from the set $A = \{\tilde{x} \in \mathbb{R}^+ : y_{k_1}(\tilde{x}) = y_{k_0}(\tilde{x})$ and $x \leq \tilde{x} \leq x'\}$. The maximum exists because: the set is nonempty, $x \in A$; it is limited, because $x \leq \tilde{x} \leq x'$; and it is closed, as a result of $y_{k_1}(\cdot)$ and $y_{k_0}(\cdot)$ being differentiable by claim (13), and hence continuous functions. The existence of a maximum is guaranteed by Weirstrass extreme value theorem, and for simplicity let's assume the maximum is x. The function $g(z) := y_{k_1}(z) - y_{k_0}(z)$ is differentiable, it is the difference of two differentiable functions, and using the mean value theorem we know that $\exists c \in (x, x')$ such that $\frac{g(x') - g(x)}{x' - x} = g'(c)$. Noticing that $g(x) = y_{k_1}(x) - y_{k_0}(x) = 0$, and, as $k_1 > k_0$, $y_{k_1}(x') > y_{k_0}(x')$, we conclude that $b = \frac{g(x') - g(x)}{x' - x} > 0$. Therefore $\exists c \in (x, x')$ such that $y'_{k_1}(c) = b + y'_{k_0}(c)$ and $y'_{k_0}(c) \leq y'_{k_1}(c)$. Due to convexity $y'_{k_1}(\tilde{x})$ is increasing, then the point d such that $y'_{k_1}(d) = y'_{k_0}(c)$, must be $d < c$.

We now will pick two parallel lines L, L' with a slope equal to $y'_{k_0}(c) = y'_{k_1}(d)$, L passing by $(c, y_{k_0}(c))$ and L' passing by $(d, y_{k_1}(d)$. By claim (12) that $c(L) = \big(c, y_{k_0}(c)\big)$ and $c(L') = \big(d, y_{k_1}(d)\big)$. We will then see that the relation $L' = \boldsymbol{\alpha} L$ creates a contradiction with $d < c$. The line $L = \big\{(x, y) : y = y_{k_0}(c) + y'_{k_0}(c)(x - c), \forall x, y \geq 0\big\}$ is tangent to $y_{k_0}(\cdot)$ at the point c, $c(L) = c(L) = \big(c, y_{k_0}(c)\big)$. The line $L' = \big\{(x, y) : y = y_{k_1}(d) + y'_{k_0}(c)(x - d), \forall x, y \geq 0\big\}$ is tangent to $y_{k_1}(\cdot)$ at the point d and has $c(L') = \big(d, y_{k_1}(d)\big)$. It is possible to find $\boldsymbol{\alpha} \in \mathbb{R}^2_+$ such that $L' = \boldsymbol{\alpha} L$, and we know by proposition (1) that $b[\alpha L] = \frac{\alpha_1}{\alpha_2} b[L]$. $b[L] = b[L']$ so $\alpha_1 = \alpha_2$. By the same lemma $a[L'] = a[\boldsymbol{\alpha} L] = \alpha_1 a[L]$, and $\alpha_1 = \frac{a[L']}{a[L]}$. From L and L' definitions we know that $a[L] = y_{k_0}(c) - y'_{k_0}(c)c$ and that $a[L'] = y_{k_1}(d) - y'_{k_1}(d)d = y_{k_1}(d) - y'_{k_0}(c)d$.

We will study the the sign of $a[L'] - a[L]$, knowing $d < c$

$$\begin{aligned}
a[L'] - a[L] =& \big[y_{k_1}(d) - y'_{k_0}(c)d\big] - \big[y_{k_0}(c) - y'_{k_0}(c)c\big] \\
=& y_{k_1}(d) - y_{k_0}(c) + y'_{k_0}(c)(c - d) \\
>& y_{k_0}(d) - y_{k_0}(c) + y'_{k_0}(c)(c - d) \\
=& (c - d)\big[y'_{k_0}(c) - \frac{y_{k_0}(c) - y_{k_0}(d)}{c - d}\big]
\end{aligned}$$

For a strictly convex function $R(\tilde{x}, \tilde{x}') = \frac{y_{k_0}(\tilde{x}) - y_{k_0}(\tilde{x}')}{\tilde{x} - \tilde{x}'}$ is strictly increasing in $\tilde{x}'$ for a fixed $\tilde{x}$ so as $d < c$, $R(c, c+\epsilon) > R(c, d)$ and $y'_{k_0}(c) = \lim_{\epsilon \downarrow 0} R(c, c+\epsilon) > R(c, d)$. And so we conclude that $a[L'] - a[L] > 0$ for the case $d < c$, and $\alpha_1 = \frac{a[L']}{a[L]} > 1$.

With $c(L') = c(\boldsymbol{\alpha} L) = \boldsymbol{\alpha} c(L)$, then $d = c(L')_1 = \alpha_1 c(L)_1 = \alpha_1 c > c$ but this contradicts the previous conclusion that $d < c$, so we can't have $y_{k_1}(x') \neq y_{k_0}(x')$. ∎

We have now gathered sufficient results to prove theorem (7).

Proof. That $h(\cdot)$ is quasiconcave is straightforward, $\boldsymbol{x}^{\boldsymbol{\alpha}} = \alpha \boldsymbol{x} + (1 - \alpha)\boldsymbol{x'}$, by $f(\cdot)$ quasiconcacvity $f(\boldsymbol{x}^{\alpha}) \geq \min\{f(\boldsymbol{x}), f(\boldsymbol{x'})\}$ and by claim (15) we derive $h(\boldsymbol{x}^{\boldsymbol{\alpha}}) \geq \min\{h(\boldsymbol{x}), h(\boldsymbol{x'})\}$.

If $h(\cdot)$ is quasiconcave but not strict quaisconvave function then $\exists \boldsymbol{x}, \boldsymbol{x'} \in \mathbb{R}^2_+$ and $\alpha \in (0, 1)$ such that $\boldsymbol{x}^{\boldsymbol{\alpha}} = \min\{h(\boldsymbol{x}), h(\boldsymbol{x'})\}$, and by claim (16), $h(\boldsymbol{x}^{\boldsymbol{\beta}}) =$

$\min\{h(\boldsymbol{x}), h(\boldsymbol{x'})\}$ either for all $0 \leq \beta \leq \alpha$ or for $\alpha \leq \beta \leq 1$. Let's assume that it is the first case $0 \leq \beta \leq \alpha$. Function $f(\cdot)$ is strictly quasiconcave and we know it is increasing-decreasing along any line, therefore it has a maximum and a minimum along a limited line, $m = \min_{\beta \in [0,\alpha]} f(\boldsymbol{x^\beta})$ and $M = \max_{\beta \in [0,\alpha]} f(\boldsymbol{x^\beta})$. By claim (17), $\exists \gamma > 0$ such that for any $\epsilon > 0$, $f(1, \gamma - \epsilon) < m < M < f(1, \gamma + \epsilon)$, hence $\gamma = \inf\{y : f(1, y) \geq m\} = \inf\{y : f(1, y) \geq M\}$, which is the same as $y_m(1) = y_M(1)$ and by lemma (5) we know $y_m(x) = y_M(x)$ for any x. Due to $m \leq f(x^\beta, y^\beta) \leq M$, $y_M(x^\beta) = \inf\big\{y : f(x^\beta, y) \geq M\big\} \geq y^\beta$ and $y_m(x^\beta) = \inf\big\{y : f(x^\beta, y) \geq m\big\} \leq y^\beta$. Due to $y_m(x^\beta) = y_M(x^\beta)$ we get that $y^\beta \leq y_M(x^\beta) = y_m(x^\beta) \leq y^\beta$, and therefore $y_m(x^\beta) = y^\beta$. For any $\beta \in [0, \alpha]$, $y_m\big(\beta x + (1-\beta)x'\big) = \beta y + (1-\beta)y'$ and the function $y_m(\cdot)$ is not strictly convex contradicting claim (11). $h(\cdot)$ must be a strictly quasiconvave function. ■

The conditions are now gathered to show that the function $h(\cdot)$ also represents the bargaining solution $c(\cdot)$.

Theorem 8. $\boldsymbol{x}^* = \arg\max_{\boldsymbol{x} \in S} h(\boldsymbol{x})$ *if and only if* $\boldsymbol{x}^* = \arg\max_{\boldsymbol{x} \in S} f(\boldsymbol{x})$*,* $\forall S \in \mathbb{S}$*.*

Proof. As the function $h(\cdot)$ is striclty quasiconcave if $h(\boldsymbol{x}^*) = h(\boldsymbol{x'})$, $h(\boldsymbol{x^\alpha}) > h(\boldsymbol{x}^*)$, for this reason the maximizer of $h(\cdot)$ in S, when S is a convex set, is unique. If $h(\boldsymbol{x}^*) > h(\boldsymbol{x}), \forall \boldsymbol{x} \in S$ by negation of the first result in claim (15) we get $f(\boldsymbol{x}^*) \geq f(\boldsymbol{x})$, also by negation of the second result of the same claim, $h(\boldsymbol{x}^*) \neq h(\boldsymbol{x})$ implies $f(\boldsymbol{x}^*) \neq f(\boldsymbol{x})$, so $f(\boldsymbol{x}^*) > f(\boldsymbol{x})$. We may conclude that if $\boldsymbol{x}^* = \arg\max_{\boldsymbol{x} \in S} h(\boldsymbol{x})$ then $\boldsymbol{x}^* = \arg\max_{\boldsymbol{x} \in S} f(\boldsymbol{x})$

Again by claim (15) when $f(\boldsymbol{x}^*) > f(\boldsymbol{x'})$, $h(\boldsymbol{x}^*) \geq h(\boldsymbol{x'})$, if $h(\boldsymbol{x}^*) = h(\boldsymbol{x'})$, strict quasiconcavity of $h(\cdot)$ implies $h(\boldsymbol{x^\alpha}) > h(\boldsymbol{x}^*)$ and from what was seen previously this also means $f(\boldsymbol{x^\alpha}) > f(\boldsymbol{x}^*)$, a contradiction once $f(\boldsymbol{x}^*)$ is the maximum, therefore $h(\boldsymbol{x}^*) \geq h(\boldsymbol{x'})$ and $h(\boldsymbol{x}^*) \neq h(\boldsymbol{x'})$. We may conclude that $\boldsymbol{x}^* = \arg\max_{\boldsymbol{x} \in S} h(\boldsymbol{x})$ if $\boldsymbol{x}^* = \arg\max_{\boldsymbol{x} \in S} f(\boldsymbol{x})$. ■

The proof that the social function being maximized is in fact $u(x, y) = xy$ relies on the fact that $y_k(\cdot)$ are the indifference curves of the function $h(\cdot)$. The next lemma proves it.

Lemma 6. *If* $h(x,y)=k$ *then* $y=y_k(x)$.

Proof. Suppose $h(x,y)=k$ and $y<y_k(x)$, for any $y<\tilde{y}<y_k(x)$, $f(x,\tilde{y})<k$, if $f(x,\tilde{y})\geq k$, then $\tilde{y}\in\{y':f(x,y')\geq k\}$ and $y_k(x)=\inf\{y:f(x,y)\geq k\}\leq\tilde{y}$, a contradiction. So $k=h(x,y)=f\big(1,\gamma(x,y)\big)>f(x,\tilde{y})$, and $\gamma(x,y)\in\{z:f(1,z)\geq f(x,\tilde{y})\}$ and naturally $\gamma(x,y)\geq\inf\{z:f(1,z)\geq f(x,\tilde{y})\}=\gamma(x,\tilde{y})$. This implies $h(x,y)=f(1,\gamma(x,y))\geq f(1,\gamma(x,\tilde{y}))=h(x,\tilde{y})$, but as $y<\tilde{y}$ this result contradicts $h(\cdot)$ being strictly increasing in both factors. If $h(x,y)=k$ and $y>y_k(x)$ the prrof is done in the same way. ■

Theorem 9. *The function* $u(x,y)=xy$ *represents the bargaining solution* $c(\cdot)$

Proof. First we will prove that $\forall k$, $\frac{xy'_k(x)}{y(k)}=-1$. Let L be a line with $c(L)=\boldsymbol{x^*}$, then for a certain k, $h(\boldsymbol{x^*})=k$, by lemma (6) $y^*=y_k(x^*)$, and by claim (12), it must be that $y'_k(x^*)=b[L]$. Chose $\alpha_1>0$ and calculate $y_k(\alpha_1 x)$ then derive α_2 from $\alpha_2 y_k(x^*)=y_k(\alpha_1 x*)$. This way $\boldsymbol{x^*}=(x^*,y_k(x^*))$ and $\boldsymbol{\alpha x^*}=(\alpha_1x^*,\alpha_2y_k(x^*))=(\alpha_1x^*,y_k(\alpha_1x*))$ belong to the same indifference curve and $h(\boldsymbol{\alpha x^*})=k$. Using the result of claim (12) again, for $\boldsymbol{\alpha x^*}=c(\boldsymbol{\alpha}L)$ it must be that $y'_k(\alpha_1x^*)=b[\boldsymbol{\alpha}L]=\frac{\alpha_2}{\alpha_1}y'_k(x^*)$, where the last equality comes from proposition (1). Simplifying both equations,

$$\begin{cases} \alpha_2 y_k(x^*)=y_k(\alpha_1 x*) \\ y'_k(\alpha_1x^*)=\frac{\alpha_2}{\alpha_1}y'_k(x^*) \end{cases} \Leftrightarrow \begin{cases} \alpha_2=\frac{y_k(\alpha_1x^*)}{y_k(x^*)} \\ \frac{y'_k(\alpha_1x^*)}{y'_k(x^*)}=\frac{y_k(\alpha_1x^*)}{\alpha_1y_k(x^*)} \end{cases}$$

these equations are valid for any $\alpha_1>0$ so with $\alpha_1=\frac{x'}{x^*}$ replaced in the second, we get $x'\frac{y'_k(x')}{y_k(x')}=x^*\frac{y'_k(x^*)}{y_k(x^*)}$, and we conclude that $x\frac{y'_k(x)}{y_k(x)}$ is equal to all values of x and must be constant. When $b[L]=-1$ or $L_1^m=L_2^m$ the line L is symmetric and $c(L)_1=c(L)_2$, $y_k(x^*)=x^*$ and at this point, using again the claim (12), $y'_k(x^*)=b[L]=-1$, so $x^*\frac{y'_k(x^*)}{y_k(x^*)}=-1$ and due to $x\frac{y'_k(x)}{y_k(x)}$ being constant, $x\frac{y'_k(x)}{y_k(x)}=-1$.

Now we will prove that the function $u(x,y) = xy$ represents the bargaining solution $c(\cdot)$. Start by defining the function $u(\boldsymbol{x}) = k_1$ if $h(1,k_1) = h(\boldsymbol{x})$. Clearly the function $u(\cdot)$ represents the same ordering as the function $h(\cdot)$. If $\boldsymbol{x}^* = \arg\max_{x\in S} h(\boldsymbol{x}) \Leftrightarrow h(\boldsymbol{x}^*) > h(\boldsymbol{x}), \forall x \in S \Leftrightarrow h\big(1,\gamma(\boldsymbol{x}^*)\big) > h\big(1,\gamma(\boldsymbol{x})\big), \forall x \in S$ then by definition of the function $u(\cdot)$,$u(\boldsymbol{x}^*) = \gamma(\boldsymbol{x}^*) > \gamma(\boldsymbol{x}) = u(\boldsymbol{x}), \forall x \in S \Leftrightarrow \boldsymbol{x}^* = \arg\max_{x\in S} u(\boldsymbol{x})$. The indifference curves are the same under $u(\cdot)$ and $h(\cdot)$. Consider the indifference curve $H_k = \big\{\boldsymbol{x} : h(\boldsymbol{x}) = k\big\}$, it then exists a z_k such that $h(1,z_k) = k$, that is $(1,z_k) \in H_k$. So if $\boldsymbol{x'} \in H_k \Leftrightarrow \boldsymbol{x'} \in \big\{\boldsymbol{x} : \gamma(\boldsymbol{x}) = z_k\big\} \Leftrightarrow \boldsymbol{x'} \in \big\{\boldsymbol{x} : u(\boldsymbol{x}) = z_k\big\}$. And the indifference curves are the same.

Solving the differential equation of the previous claim $x\frac{y_k'(x)}{y_k(x)} = -1$ we get that $y_k(x) = \frac{1}{x}a_k$. For $x = 1$, $y_k(1) = k$ because $u(1,k) = k$, thus $xy_k(x) = k$. So $u(x,y) = k$ when $xy = k$, then $u(x,y) = xy$. ■

5 Conclusion

In this paper we developed a new method to find Nash's solution to the bargaining problem. Peters and Wakker (1991) provides the conditions for the result that the Nash bargaining solution to be the result of a maximization process. Then, from the properties of this maximand's indifference curves the Nash solution is found. The mathematical arguments used in this paper are mainly of real analysis origin and are not directly adaptable to different bargaining structures, such as for example those defined in Peters and Vermeulen (2012), Conley and Wilkie (1996) or to Kalai and Smorodinsky (1975). However, axiomatic bargaining does exhibit algebraic properties which can be explored in future research to overcome this limitation. Namely, we can regard the AT axiom as a morphism, and with the right definition of the multiplication operation on the bargaining sets, each bargaining model can then be interpreted algebraically. The study of the different axiomatic bargainings under this algebraic and more general framework will likely extend the understanding we detain of them.

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Appendices

A

Claim 1. *There is a concave function $g_s(x) : [0, S^1] \to [0, S^2]$ such that $(x, g_S(x)) \in S$ and if $(x, y) \in S$ then $y \leq g_S(x)$.*

Proof. Consider $g_S(x) = \max_{(x,y)\in S} y$. As S is compact and the function $f(y) = y$ is continuous, $g_S(x)$ is well defined for all x. For any two points in $x, x' \in [0, S^1]$, define $\boldsymbol{x} = \big(x, g_S(x)\big)$ and $\boldsymbol{x'} = \big(x', g_S(x')\big)$, clearly, by definition of $g_S(\cdot)$, $\boldsymbol{x}, \boldsymbol{x'} \in S$. Due to convexity of S for any $\alpha \in [0, 1]$, $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{x'} = \Big(\alpha x + (1-\alpha)x', \alpha g_S(x) + (1-\alpha)g_S(x')\Big) \in S$, as $g_S(\tilde{x}) \geq \tilde{y}, \forall y$ with $(\tilde{x}, \tilde{y}) \in S$ then $g_S\Big(\alpha x + (1-\alpha)x'\Big) \geq \alpha g_S(x) + (1-\alpha)g_S(x')$, the function is concave. ∎

Proposition 1. *For $L, L' \in \mathbb{L}^-$, for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \gg \boldsymbol{0}$*

1. *$a[L] = L_2^m$ and $b[L] = -\frac{L_2^m}{L_1^m}$*
2. *$\boldsymbol{\alpha}L \in \mathbb{L}^-$ and $a[\boldsymbol{\alpha}L] = \alpha_2 a[L]$ and $b[\boldsymbol{\alpha}L] = \frac{\alpha_2}{\alpha_1} b[L]$*
3. *$\exists\boldsymbol{\alpha} \in \mathbb{R}_+^2$ such that $\boldsymbol{\alpha}L = L'$ with $\alpha_2 = \frac{a[L']}{a[L]}$ and $\alpha_1 = \frac{a[L']}{a[L]}\frac{b[L]}{b[L']}$*
4. *$\forall\boldsymbol{x} \in \mathbb{R}_+^2$, $\exists L \in \mathbb{L}^-$ such that $c(L) = \boldsymbol{x}$*

Proof. (1)If the set $L = \big\{(x, y) : y = a + bx\big\} \in L^-$, then $a[L] = a > 0$ and $b[L] = b < 0$. $y(x) = a + bx < a = y(0)$, the maximum value of the second argument is $L_m^2 = a$. Inverting the equation $y(x) = a + bx$ as $x(y) = \frac{1}{b}y - \frac{a}{b}$ as $\frac{1}{b} < 0$ $x(y) \leq -\frac{a}{b} = x(0)$ and the maximum value of the first argument is $L_m^1 = -\frac{a}{b}$, and $b = -\frac{L_m^2}{L_m^1}$.

(2) If $S = \{(s_1, s_2) : s_2 = a + bs_1, \text{ with } s_1 \geq 0, s_2 \geq 0\}$,

$$\begin{aligned}\alpha S =&\{(\alpha_1 s_1, \alpha_2 s_2) : s_2 = a + bs_1, \text{ with } s_1 \geq 0, s_2 \geq 0\}\\ =&\{(\alpha_1 s_1, \alpha_2 s_2) : \alpha 2 s_2 = \alpha_2 a + \frac{\alpha_2}{\alpha_1} b\alpha_1 s_1, \text{ with } \alpha_1 s_1 \geq 0, \alpha_2 s_2 \geq 0\}\end{aligned}$$

$$=\{(\tilde{s}_1, \tilde{s}_2) : \tilde{s}_2 = \tilde{a} + \tilde{b}\tilde{s}_1, \tilde{s}_1 \geq 0, \tilde{s}_2 \geq 0\}$$

In the last equality we used that $\tilde{a} = \alpha_2 a$ and $\tilde{b} = \frac{\alpha_2}{\alpha_1} b$, as $a > 0$, $b \leq 0$, $\alpha_1 > 0$ and $\alpha_2 > 0$, then $\tilde{a} > 0$ and $\tilde{b} \leq 0$. Therefore $\alpha S \in \mathbb{L}^-$, $a[\boldsymbol{\alpha} L] = \alpha_2 a[L]$ and $b[\boldsymbol{\alpha} L] = \frac{\alpha_2}{\alpha_1} b[L]$

(3) For two lines to be equal, both coefficients of the lines must be equal, if $\boldsymbol{\alpha} L = L'$ then $a[\boldsymbol{\alpha} L] = a[L']$ and $b[\boldsymbol{\alpha} L] = b[L']$ using the result of point (2) we get

$$\begin{cases} a[\boldsymbol{\alpha} L] & = \alpha_2 a[L] = a[L'] \\ b[\boldsymbol{\alpha} L] & = \frac{\alpha_2}{\alpha_1} b[L] = b[L'] \end{cases} \Leftrightarrow \begin{cases} \alpha_2 & = \frac{a[L']}{a[L]} \\ \alpha_1 & = \alpha_2 \frac{b[L]}{b[L']} \end{cases}$$

(4) Consider symmetric the line $L = \{(x, y) : x + y = 2\}$, the choice must be along the line $x = y$, $c(L) = (1, 1)$, for any $\boldsymbol{x} \in \mathbb{R}^2_+$, take $\boldsymbol{x} L$ and $c(\boldsymbol{x} L) = \boldsymbol{x} c(L) = \boldsymbol{x}$.

■

Equilibria and Outcomes in Multiplayer Bargaining

Luís Carvalho

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Abstract

Multiplayer bargaining is a game in which all possible divisions are equilibrium outcomes. This paper presents the classical subgame perfect equilibria strategies and analyses their weak robustness, namely the use of weakly dominated strategies. The paper then develops a refined equilibrium concept, based on trembling hand perfection, in order to overcome such weakness. Concluding that none of the classical equilibrium strategies survives the imposition of the extra robustness and, albeit using more complex strategies, the equilibrium outcomes don't change.

Journal of Economic Literature Classification Numbers: C72, C78

Keywords: Multiplayer Bargaining; Equilibrium Refinements; Perfect Equilibria

1 Introduction

In n-players bargaining there is an infinite divisible good to be shared among them. The division is obtained by the following procedure: at each moment a player proposes a division, the other $n-1$ players vote in favor or against it. If all agree the division is made accordingly; if at least one player votes against it, the game goes on to another round, with another player proposing a different division and a new suffrage taking place. The game ends when a proposal is accepted by all. At each round, the good in question loses value by δ.

The classical and better known result on multiplayer bargaining is that all divisions are *Subgame Perfect Nash Equilibria (SPNE)* outcomes of the game, meaning that all divisions can be agreed in equilibria. Crucial to obtain this result is the existence of a credible and painful threat for deviators of the "right" track. Herrero (1985) proposes an ingenious mechanism of doing this, creating a strategy in which at least one player is unsatisfied with a deviation proposal. For this strategy she used a state variable, if the proponent does not propose as implied by the state, the state changes to a new one in which the player worst off in this division receives everything. Players do not want to deviate because in the punishment state they will receive nothing. For this strategy to be an equilibrium the discount value cannot be very small, namely with 3 players $\delta > 1/2$. Haller (1986) noted that an equilibrium for all divisions could be extended to $\delta \leq 1/2$. This strategy also uses a state variable and punishment threats that attribute everything to one player only, the main difference is in the repliers' actions, with players accepting only if the proposition is equal to the state - any difference, even if all repliers are awarded , is rejected. The belief players have that the proposition will be rejected renders them indifferent between accepting and rejecting the offer, and they thus opt for refusing it. Other equilibrium strategies can be developed, namely one with *agreement at time T* and other with *no agreement* at all. On both of these the major force is also the belief that others will reject different proposals, without it players would act differently from what is defined. Of notice is that all these equilibria do not depend on the replies, and that it is unorthodox for players not to accept better proposals unless they are punished by doing so. This is a major shortcoming of this equilibrium: players, without being punished by acting differently, are choosing to play a dominated strategy.

This is an evident weakness of the equilibria concept used. In *Haller*'s strategy players in specific history states accept zero offerings because they do not expect to receive more in the future if they reject them. They are powerless to change the outcome, it is a resigned acceptance. In *Herrero*'s strategy players propose divisions in which they receive zero. Again this is a hopeless proposition and only happens thanks to the belief that others will also follow a resigned action course, as players believe others will reject, they believe their own actions do not have any effect. The need of unanimity gives total power to all players in terms of rejecting a proposal, and other players' actions will have no impact. This case, of the players' actions having no effect on the outcome of the game, may result in the best and more accurate strategies not being played and originates non sensible equilibria. Players only choose their best available actions in singleton information sets, if, for example, players knew what others had voted before them, making their information set at the moment of voting a singleton, then players knew that if they accepted a good proposition then others could also do it. This conviction would make them vote in favor of the good division. This type of structure in games and the possible appearance of non sensible equilibria is very well known, and has been studied and solved by the use of refined equilibria notions.

In this work we develop two different equilibrium concepts to analyse the game, based on Selten's (1975) perfect equilibria, and introduce the possibilities of small mistakes by the players. *Perfect Equilibrium(PE)* imposes that all players in all histories commit a minor mistake, and therefore imposes mistakes in an infinite game with a continuum of actions, to our knowledge a concept with such characteristics was not defined previously in the literature. Although it involves some different options it owes much to the work of Simon and Stinchcombe (1995) and Carbonell-Nicolau (2011), that developed existence results of the *Perfect Equilibria* strategies for normal form games with a continuum action space with continuous and discontinuous utilities, respectively. The use of trembles involves some distortion of the game and should be used with parsimony. As the reason for the *SPNE* not to work is the non singleton information set at the reply, and in order to introduce the minimum distortions possible, we also create an equilibrium refinement that only imposes trembles on the replies, *Perfect Equilibrium in Replies (PER)*. When referring simultaneously to both these concepts we will talk of them as *Trembling Hand*.

When a perturbed game is played, if the strategy does not punish replies, as is the case in the strategies already described, players will always accept propositions that give them more than what they would receive in future if the proposition were to be refused, (although this may seems obvious it is not what happens in the *Haller* equilibrium, in which better propositions are rejected in face of the expected rejection of the other replier). Thus, they accept this proposition even if the chance of others accepting it is very small. This property of the *Trembling Hand* equilibrium strategies which are simultaneously independent of replies is the pivotal point to show that *Haller*'s strategy is not *Trembling Hand*. This strategy support the equilibrium by punishing a deviator with attributing him zero, and he has no possibility of receiving more unless someone deviates along the way. But if any player can make a mistake, for example accepting a different proposition, the deviator will never accept zero, he will keep looking for an opponent to make a mistake. The deviator will always refuse a zero proposition and the strategies are not equilibrium strategies.

However, if in *Herrero* we modify the punishment divisions slightly, in a way in which all players receive a positive quantity, even if very small, and maintain the rest of the equilibrium structure, then this *Modified Herrero* strategy is a *PER*. And with a further modification it is *PE*. These strategies serve as a good example because one is *PER* and not *PE*, the other, with the additional modification, is *PE* and not *PER*, which shows that the *PE* set is not stricter than *PER*; it also illustrates the implications of *PE* definition. With a less restrictive notion the initial modified strategy would be a *PE*, however the example also clarifies the rationale for the restrictive definition of *PE*.

For $\delta \leq 1/2$ there is no easy equilibrium solution that works for all points in the simplex, there are points in which all players receive a strictly positive fraction of the good that are not *Trembling Hand* outcomes for reply independent strategies. So, for these divisions to be a *Trembling Hand* equilibrium outcome, we need a strategy that also punishes the replies. The strategy we will build, strategy ρ, has two punishment states per player, instead of one as hitherto: one to punish deviations and another one to punish deviations from deviations. The trick used is that the second punishment state avoids deviations from the first punishment state and the first punishment state avoids deviations from the second. Thus, a sensible choice of these punishment divisions is enough to insure that all strictly positive divisions are outcomes of a *Trembling Hand* strategy. As in *Haller*'s strategy, in ρ players will only accept the proposition correspondent to the state, but in this strategy they are punished if they don't. The remarkable in this ρ strategy is the undominance of it, it is the only best reply, and for this reason respects the conditions necessary to be even a stronger equilibria notion. We can generalize this strategy, the same way as we have done with *Haller*'s, to allow an agreement date latter than the initial moment. For the sake of completeness, another strategy that permits divisions with players receiving zero will also be presented. This strategy is naturally weakly dominated, but on the approximation games it is not. There is a mechanism of awarding the well behaved players when another one deviates, the chance of receiving this award serves of an incentive for players to accept receiving or proposing for themselves zero. They are hoping that a player deviates and they receive the premium for the compliance.

We will now proceed to introduce notation and the classical equilibrium strategies of *Haller* and *Herrero* in section 2. In section 3 the new equilibrium concepts are defined and a proof that the classical equilibrium strategies are not *Trembling Hand* is given. In the section 4 some new strategies, that are *Trembling Hand*, are defined. Finally in section 5 a conclusion is provided.

2 Notation and Classical Equilibria

2.1 Game and Notation

Although the majority of the results to be presented are easily generalized for more players in this paper we will focus on the game with only 3 players. The set of players is $I = \{1, 2, 3\}$. At the moment $t \in \mathbb{N}$ a proposal is one point of the unitary simplex $p^t = (p_1^t, p_2^t, p_3^t) \in \Delta$, with $\Delta = \left\{(x_1, x_2, x_3) : \sum_{i=1}^{3} x_i \leq 1, x_i \geq 0\right\}$ and p_i^t the part attributed to player i. The proponent at t is the player $i(t)$; with $i(t)$ the function that determines the proponent, it has a cycle of period 3, $i(t) : T \mapsto I$ and $i(t) = \{i \in I : \exists m \in \mathbb{N}_0, t = i+3m\}$. $t(i) : I \rightrightarrows T$ is the correspondence that defines the moments in which player i proposes, these moments are $t(i) = \{t \in T : \exists m \in \mathbb{N}_0, t = i + 3m\}$.

Player's $j \neq i(t)$ response to the proposal is an action taken on $\{0, 1\}$; with a_j^t, the action of j at t, being 0 if j rejects the proposition received, and 1 if the player accepts it. So $a_i^t \in \{0, 1\}$ if $i(t) \neq i$ or $a_i^t \in \Delta$ if $i(t) = i$. For the sake of simplicity define the set of actions available for i at t by

$$A_i^t = \begin{cases} \{0, 1\} \text{ if } i \neq i(t) \\ \Delta \text{ if } i = i(t) \end{cases}$$

The vector of all actions taken at moment t is $a^t = (a_1^t, a_2^t, a_3^t)$ and the space of all actions at t is $A_t = A_1^t \times A_2^t \times A_3^t = \{0, 1\}^2 \times \Delta = \bar{\Delta}$.

For $t \geq 1$, a stage history can be either a history after or before the proposition is done, and a distinction between these two cases is necessary. We therefore define at moment t in history h the stage history is $h^{t,1}$ for the proposal and $h^{t,2}$ for the responses, $h^t = (h^{t,1}, h^{t,2})$. A $(t-1, 2)$-*history* in which $t-1$ propositions and voting have taken place is denoted by $h^{|t-1,2} = (a^1, \ldots, a^{t-1})$; and a $(t, 1)$-*history*, when a proposition has already been done at t but no replies have been received yet, $h^{|t,1} = (a^1, \ldots, a^{t-1}, a_{i(t)}^t)$, in which, for all $1 \leq k \leq t-1$, $a^k \in \bar{\Delta}$ and $a_{i(t)}^t \in \Delta$; the space of $(t, 2)$-stage histories is $H^{t,2} = \prod_{k=1}^{t} \bar{\Delta} = \bar{\Delta}^t$, and the space of all $(t, 1)$-histories is $H^{t,1} = H^{(t-1),2} \times \Delta = \bar{\Delta}^{t-1} \times \Delta$. $H^{0,2}$ stands for $\emptyset$ the unique 0-stage history. The set of all histories is $H = \bigcup_{t=1}^{\infty}(H^{t,1} \cup H^{t,2})$.

The length of a history, $\tau(h)$ is a function from the set of histories into the stage moment $\tau : H \mapsto \mathbb{N}_0 \times \{1,2\}$, so $\tau(h) = (t,k)$ $t \in \mathbb{N}_0$ being the moment of the history, and $k \in \{1,2\}$ whether the voting has already been made $k = 2$ or not $k = 1$. $t(h)$ is the moment of history h, so $\tau(h) = (t(h), k)$ and $i(h) = i(t(h))$ the proponent at h. For a history h with $t(h) > t$, $h^{|t,k}$ is the history h until stage (t,k). h^+ and h^- are respectively the history h plus one more stage or without the last stage, and it will be used only when the marginal actions are obvious from the context. It is assumed that at stage (t,k) each player knows $h^{|t,k}$, that is, each player knows the actions that were played in all previous stages. $(h, \bar{h})$ is the history h followed by $\bar{h}$.

A *pure strategy for player* i is a function $s_i : H \to \{0,1\} \cup \Delta$, with $s_i(h) \in A_i^{t(h^+)}$ mapping histories into actions. The set of player i pure strategies is denoted by S_i, and $S = S_1 \times S_2 \times S_3$ is the joint pure strategy space. Every pure strategy $s = (s_1, s_2, s_3) \in S$ induces a path after the history h, $\varpi_s(h)$. At h the action will be $s(h)$, then if an agrement has not been reached $s\big(h, s(h)\big)$ are the actions played, so we can define the future after h when s is the strategy as $\varpi_s(h) = \{h, s(h), s\big(h, s(h)\big), s\Big(s\big(h, s(h)\big)\Big), \dots\}$. A strategy s induces, as well, a division $d(s)$ and a moment in which the agreed division occurs $t(s)$. The moment $t(s) = t$ is when $\min_{i \in -i(h)} s_i^t(h_{t,2}^s) = 1$[1], and division is $d(s) = h^{t(s),1}$. If there is no agreement, by convention, $e(s) = +\infty$ and $d(s) = \bar{0}$. The utility for a given strategy is $\Pi_i^t(s|h) = v_i\Big(t(s|h), d_i(s|h)\Big)$, is increasing with the share received $d_i(s)$ and decreasing with the time until agreement $t(s)$, $\Pi_i^t(s|h) = \delta^{t(s|h)-t} d(s|h)$, payment function can also be written (in a similar fashion to the definition of payment when mixed actions are used) as $\Pi_i^t(s|h) = \sum_{\bar{h} \in \varpi_s(h)} \delta^{t(\bar{h})-t(h)} \pi(h, \bar{h})$, in which $\pi(\tilde{h})$ is the value of the division agreed at the last moment of $\tilde{h}$, and therefore is the product of the last moment actions $\pi(\tilde{h}) = \tilde{h}^{t,1} \tilde{h}_j^{t,2} \tilde{h}_k^{t,2}$, $k, j \notin -i(\tilde{h})$.

Herrero (1985) was the first[2] to prove that all points in Δ are equilibria outcomes when, $\delta > 1/2$. Later Haller noted that if the repliers' strategies were stricter the

[1] The usual notation will be followed for a player $i \in I$, $-i = I \setminus \{i\}$

[2] Although he never published his results, Shaked is also attributed with the creation of such strategies, see, for example, Sutton (1986) or Osborne and Rubinstein (1990)

equilibria could extend to any δ. Due to the dynamic character of the game the equilibrium concept used is the Subgame Perfect Nash Equilibrium that we hereby define.

Definition 1. *$s \in S$ is a **SPNE** if $\Pi_i^t(s|h) \geq \Pi_i^t(s_i', s_{-i}|h)$ $\forall h \in H, \forall i \in I$ and $\forall s_i' \in S_i$*

The utility function in the bargaining game can be written, as noted before, in the form $\Pi_i^t(s) = \sum_{\tau=1}^{\infty} \delta^\tau a_\tau$ with a_τ the payments at $t+\tau$, that is either zero or the value of the agreed division at $t+\tau$, and is bounded by 1. It is relatively straightforward to see that if two strategies share the same future path for a long period their actualized payment will be similar, therefore the utility function is continuous at infinity and the one shot deviation principle is valid. To prove that a given strategy is an *SPNE* we need only to look for alternative strategies that are different on one information set. For this purpose we define the one shot deviation strategy.

Definition 2. *The set of **One Shot Deviation(OSD)** strategies from s_i at h is $OSD(s_i, h) = \{\gamma_i \in S_i : \gamma_i(h) \neq s_i(h)$ and $\gamma_i(h') = s_i'(h'), \forall h' \in H \setminus h\}$*

2.2 Haller Equilibrium Strategy

In this subsection we will present the equilibrium defined by Haller (1986), a proof that such strategy is a *SPNE* will be presented for completeness. [3] This strategy uses a state function $r(h) : H \to E$ that, for any history h, tracks if any player has deviated from the planed, and induces a punishment for that player. There is a bond between the state and the division to be proposed under the strategy, for this reason we use the same symbol for a state and the division associated with it. $E = \{e^0, e^1, e^2, e^3\}$ is set of states, e^0 is any point in Δ, e^i is the division in which player i receives 1, $e_k^i = \begin{cases} 1 \text{ , if } k = i \\ 0 \text{ , if } k \neq i \end{cases}$. At $h \in H^{t,2}$, if the player $i = i(t)$ did not propose $r(h)$ the state changes to $e^{i(t+1)}$, in which the player i receives nothing. The state at the initial

[3] In the proof we are only looking for better pure strategies, if no pure strategy is better then no mixed strategy can be better either.

moment $h = \emptyset$ is $r(h) = e^0$. Transition takes place immediately after the proposal and before the replies so for $\tau(h) = (t, 2)$, $r(h) = r(h^-)$. For $\tau(h) = (t, 1)$,

$$r(h) = \begin{cases} r(h^-) & \text{if } h^{t,1} = r(h^-) \\ e^{i(t+1)} & \text{if } h^{t,1} \neq r(h^-) \end{cases}$$

Now we will present the equilibrium strategy.

Definition 3. ***Haller Equilibrium Strategy*** *In Haller's equilibrium strategy for h such that* $\tau(h) = (t-1, 2)$, $s_{i(t)}(h) = r(h)$, *so the proposition will always be equal to the state. For* $\tau(h) = (t, 2)$ *replier's* $j \neq i(h)$ *strategy is*

$$s_j(h) = \begin{cases} 1 & se\ h^{t,1} = r(h^-) \\ 0 & se\ h^{t,1} \neq r(h^-) \end{cases}$$

Repliers accept the proposition if it is equal to the state and reject it if it is different, note that for replier j the share offered to him is as important as the share offered to others, what matters is that the proposition is equal to $r(h^-)$ so the share of all players is relevant.

Table 1: Haller's Strategy

	State	e^j
Player i	Proposal	e^j
	Accept p	$p = e^j$

Theorem 1. *Haller's strategy is an* SPNE*and any* $e^0 \in \Delta$ *is an*SPNE*equilibrium outcome.*

Proof. s is Haller's strategy with $r(\emptyset) = e^0$, for any but fixed $e^0 \in \Delta$. We will prove that there is no history h after which one player i can change his strategy to $s'_i \in OSD(s_i, h)$ and improve his payment. Let us start by noting that due to $r(h) = r(h^-)$ for $\tau(h) = (t, 2)$, $h^{t,2}$ has no influence in the state, whatever are the responses the state does not change.

For $\boldsymbol{i = i(t)}$, $\boldsymbol{\tau(h) = (t-1,2)}$ If all players play according to the strategy s, i proposes $r(h)$ and all others accept, $\Pi_i^t(s|h) = r_i(h)$. If $s'_i \in OSD(s_i, h)$ then $p = s'_i(h) \neq s_i(h) = r(h)$, i made a different proposal, repliers j, k only accept if the proposal is $p = r(h^+) = e^{i(t+1)}$, the state after the deviated proposition. So if there is an immediate agreement i's payoff is $\Pi_i^t(s'_i, s_{-i}|h) = e_i^{i(t+1)} = 0$, if there is not $\Pi_i^t(s'_i, s_{-i}|h) = \delta\Pi_i^{t+1}(s'_i, s_{-i}|h^+) = \delta\Pi_i^{t+1}(s|h^+) = \delta e_i^{i(t+1)} = 0$. Clearly $\Pi_i^t(s'_i, s_{-i}|h) \leq \Pi_i^t(s|h)$ for any $OSD(s_i, h)$, the proponent $i(t)$ has no advantage in altering his strategy.

For $\boldsymbol{j \neq i(t)}$ and $\boldsymbol{\tau(h) = (t,1)}$ we have two possibilities for the player to act unaccording to s, either to accept a proposal different from $r(h)$ or to reject the proposal of $r(h)$. When the proposal is equal to the state $\boldsymbol{h^{t,1} = r(h)}$, if all players act by s the proposition is accepted and $\Pi_j^t(s|h) = r_j(h)$. If $s'_j \in OSD(s_j, h)$, j refuses the proposition, $s'_j(h) = 0$, we can define the stage history $h^{t,2} = \big(s'_j(h), s_k(h)\big) = (0,1)$ and $h^+ = (h, h^{t,2})$. The state does not change, as the state is independent of the replies, so $r(h^+) = r(h)$. j's refusal delays the agreement one period, because after h^+ all players follow s and the agreement is $r(h^+) = r(h)$. $\Pi_j^t(s'_j, s_{-j}|h) = \delta\Pi_j^{t+1}(s'_j, s_{-j}|h^+) = \delta\Pi_j^{t+1}(s|h^+) = \delta r_j(h^+) = \delta r_j(h) \leq r_j(h)$, and we conclude that $\Pi^t(s'_j, s_{-j}|h) \leq \Pi_j^t(s|h)$. When the proposal is not equal to the state $\boldsymbol{h^{t,1} \neq r(h)}$, that mean the proponent $i(h)$ has deviated from the strategy and the state is $r(h) = e^{i(t+1)}$. If $-i(h)$ follow s the proposal is refused, the state is $r(h^+) = r(h) = e^{i(t+1)}$, where $h^+ = (h, (0,0))$, and $\Pi_j^t(s|h) = \delta\Pi_j^{t+1}(s|h^+) = \delta e_j^{i(t+1)}$. If j follows $s'_j \in OSD(s_j, h)$ accepting the proposition, $s'_j(h) = 1$. The proposal will still be declined by the other player and there will be no change in state caused by j response, and $r(\bar{h}^+) = e^{i(t+1)}$, with $\bar{h}^+ = (h, (1,0))$. $\Pi_j^t(s'_j, s_{-j}|h) = \delta\Pi_j^{t+1}(s|\bar{h}^+) = \delta e_j^{i(t+1)} = \delta\Pi_j^{t+1}(s|h^+) = \Pi_j^t(s|h)$. Player j does not improve by changing strategy. $\square$

2.3 Herrero's Strategy

Being less general than Haller's strategy Herrero proposed an equilibrium strategy that is less fragile. In this case the players' acceptance is not reduced to one division only, they *apparently* consider only their own share, and the acceptance rule has a threshold. The punishment scheme is activated if a player does not propose what he was supposed to. A state function defining the state at history h and which division should be proposed (again there is an identification between state and proposal) $r(h) : H \to E$, is updated after each proposal but before the replies, so $r(h) = r(h^-)$ when $\tau(h) = (t, 2)$. The states are again $E = \{e^0, e^1, e^2, e^3\}$, with e^i the division in which player i receives the totality, the initial state is $r(\emptyset) = e^0$.

Define $k(p, t)$ as the replier worst off in proposition p made at t (of smaller index if there is more than one), $k(p, t) = \min\left\{j \in I \setminus i(t) : p_j = \min_{k \in I \setminus i(t)} p_k\right\}$. The state is defined in the following way for $\tau(h) = (t, 1)$

$$r(h) = \begin{cases} r(h^-) & \textit{if } h^{t,1} = r(h^-) \\ e^k & \textit{if } h^{t,1} \neq r(h^-) \end{cases}$$

Briefly, if the player made the expected proposal, $h^{t,1} = r(h^-)$, there is no state change; if he did not, then the strategy enters in a punishment scheme of $i(h)$ that gives everything to player $k = k(h^{t,1}, t)$. Herrero's strategy is resumed on the following table and formally defined subsequently.

Table 2: Herrero's startegy

	State	e^j
Player i	Proposal	e^j
	Reply	$p_i \geq \delta e_i^j$

Definition 4. ***Herrero's Strategy*** *The proponent always proposes* $r(h)$*,* $s_{i(h)}(h) = r(h)$*, the strategy for repliers* $j \neq i(h)$ *is*

$$s_j(h) = \begin{cases} 1 \textit{ if } h_j^{t,1} \geq \delta r(h)_j \\ 0 \textit{ if } h_j^{t,1} < \delta r(h)_j \end{cases}$$

Theorem 2. *For* $\delta > 1/2$ *Herrero's strategy is*SPNE*for any* $e^o \in \Delta$*.*

Proof. We will use the one shot deviation principle once more. Let's start by seeing that at $h \in H^{t-1,2}$ the player $\boldsymbol{i = i(t)}$ gains nothing by acting differently from s; when all players act accordingly, i utility is $\Pi_i^t(s|h) = r(h)_i$. If i uses $s_i' \in OSD(s, h)$

and makes a different proposition, $p \neq r(h)$, there is immediately a change of state to $r(h^+) = e^k$ with $k = k(p,t) \neq i$. If $h^{++} = (h,p,r)$ where r is the reply to $h^{t,1}$, $r \in \{0,1\}^2$, if $\min r_j = 0$, at least one player refused the proposition and $\Pi_i^t(s_i', s_{-i}|h) = \delta\Pi_i^{t+1}(s|h^{++}) = \delta r(h^{++})_i = \delta e_i^k = 0 \leq \Pi_i^t(s_i, s_{-i}|h)$. Then the only way i can improve is when all players accept. After proposition $p \neq r(h)$, state becomes e^k, with k the player receiving the minimum, according to s for k to accept $p_k = \min\{p_j, p_k\} \geq \delta$, then $p_j \geq \delta$ the total amount given to the repliers for both of them to accept the proposal must be at least 2δ, as the total cannot be bigger than a unity we conclude that $\delta \leq 1/2$, contradicting the initial hypothesis. So both repliers can not accept the out of equilibrium proposition simultaneously. For $\boldsymbol{j \neq i(t)}$ and $\boldsymbol{\tau(h) = (t,1)}$ the payment for player j under s depends on the actions of the other replier k as well, if $h_\iota^{t,1} \geq \delta r(h)_\iota$, for $\iota = j,k$ all repliers will accept, $\min_{\iota \in -i(h)} s_\iota(h) = 1$, payment is immediate and equal to $h_j^{t,1} = \Pi_j^t(s|h)$; if any of the repliers reject (due to his share being smaller than the established by the state), $\min_{\iota \in -i(h)} s_\iota(h) = 0$ the agreement is delayed one period but the state is not changed, as the state do not depend on the replies, $h^+ = \Big(h, \big(s_j(h), s_k(h)\big)\Big) \in H^{t,2}$ and $r(h^+) = r(h)$. In this case $\Pi_j^t(s|h) = \delta\Pi_j^{t+1}(s|h^+) = \delta r(h^+)_j = \delta r(h)_j$. And we can conclude that $\Pi_j^t(s|h) \geq \delta r(h)_j$ independently of the replies. At this moment there are two ways in which the players can act contrarily to the strategy s: to accept a proposal that should be refused or to reject one that should be accepted. In neither one does the player improve. If $\boldsymbol{s_j(h) = 1}$, player j chooses $s_j' \in OSD(s_j, h)$, then $s_j'(h) = 0$ his payment is $\Pi_j^t(s_j', s_{-j}|h) = \delta\Pi_j^{t+1}(s_j', s_{-j}|h^+)$, with $h^+ = \Big(h, \big(s_j'(h), s_k(h)\big)\Big)$, as $r(h^+) = r(h)$, the state do not depend on the replies, $\Pi_j^{t+1}(s_j', s_{-j}|h^+) = \Pi_j^{t+1}(s_j, s_{-j}|h^+) = r(h^+)_j = r(h)_j$. j's rejection leads to $\Pi_j^t(s_j', s_{-j}|h) = \delta r(h)_j$, $\Pi_j^t(s_i', s_j|h) \leq \Pi_j^t(s|h)$. When $\boldsymbol{s_j(h) = 0}$ then a strategy $s_j' \in OSD(s_j, h)$ has $s_j'(h) = 1$. If player k accepts, $s_k(h) = 1$, the agreement is immediate and the payment of j is $h_j^{t,1}$. It is smaller than $\delta r(h)_j$ because according to s_j a proposal should only be rejected, $s_j(h) = 0$, if $h^{t,1} < \delta r(h)_j$. If $s_k(h) = 0$ the agreement is postponed and j's payment is $\delta\Pi_j^{t+1}(s|h^+)$. We can therefore define the payment of j as

$$
\begin{aligned}
\Pi_j^t(s'_j, s_{-j}|h) =& s_k(h)h_j^{t,1} + (1 - s_k(h))\delta\Pi_j^{t+1}(s'_j, s_{-j}|h^+) \\
=& s_k(h)h_j^{t,1} + (1 - s_k(h))\delta\Pi_j^{t+1}(\boldsymbol{s_j}, s_{-j}|h^+) \\
=& s_k(h)h_j^{t,1} + (1 - s_k(h))\delta r(h^+)_j \\
\leq& s_k(h)\delta r(h)_j + (1 - s_k(h))\,\delta r(\boldsymbol{h})_j = \delta r(h)_j \leq \Pi_j^t(s|h)
\end{aligned}
$$

□

It is of note that under Haller's and Herrero's strategies all divisions are equilibrium outcomes, even Pareto dominated divisions, however agreement is always reached at $t = 1$. It is possible to use the fact that player's payment is zero to obtain an*SPNE*for which the agreement is reached later, $t(s) > 1$, a minor adaptation of Haller's strategies is enough.[4] The two following theorems are proved in appendix A.

Theorem 3. $\forall e^0 \in \Delta, \forall T \in \mathbb{N}$, *exists a strategy* s SPNE*with* $e(s) = T$ *and* $d(s) = e^0$

Another atypical equilibrium outcome is when an agreement is never obtained. This case happens when at least one player at each round refuse the received proposal, no agreement is then established at a finite moment and the game is played indefinitely. The next theorem will prove the existence of such kind of equilibrium strategies.

Theorem 4. *There is an* SPNE*strategy* $s \in S$ *in which no division is agreed upon and* $e(\sigma) = \infty$.

3 Trembling Hand Equilibria

3.1 Trembling Hand Equilibria

In *Haller's* strategy repliers, without being punished by acting differently, reject propositions that leave them better off, they are choosing weakly dominated strate-

[4] The adaptation could be made in Herrero's strategy, the principle would be the same, if a player deviates before the agreement date T, the punishment path of Herrero's strategy is triggered.

gies. At the moment of an answer, when player j rejects the proposition, whatever k does, the proposal will still be rejected, the agreement moment will be delayed, and j's action is, for the time being, useless, then he can either accept or reject, that his payment doesn't change.

This is typical of voting systems, a similar problem is, for example, presented in Acemoglu, Egorov, and Sonin (2009), in which three individuals are choosing by majority rule between a or b, and each player strictly prefers option a to b. The non-intuitive possibility that all three individuals vote for option b is a Nash equilibrium. When any two players vote for b, it is a weak best response for the third one to do so as well. It is the belief that all other players will vote for the worst option that makes him vote for it as well. The same happens in multiplayer bargaining, when a replier believes the other is rejecting the proposal, he is indifferent between accepting and rejecting it. If both players think the same way there may be a rejection of a good proposal for both. This problem is an amply known weakness of SPNE, and was in the origin of the sequential and perfect equilibrium concepts, for example. Van Damme (1991, p.9) identifies the problem with the fact that not all information sets are singletons,

> (...)for a subgame perfect equilibrium to be sensible, it is necessary that this equilibrium prescribes at each information set which is singleton a choice which maximizes the expected payoff after that information set. Note that the restriction to singleton information sets is necessary to ensure that the the expected payoff after the information set is well defined. *This restriction, however, has the consequence that not all subgame perfect equilibria which satisfy this additional condition are sensible.*

So, if all information sets are singleton, the *SPNE* is sensible, if they are not then there might be a problem in some equilibria strategies. If the information set is non singleton a choice of an action that is not the best may happen, the use of the concept is, in this case, questionable. Haller's strategy clearly demonstrates that a refined equilibrium concept should be used in the multibargaining game.

For the purpose of this paper we propose two concepts in the vein of Perfect Equilibria of Selten (1975), different from *SPNE*, that try to overcome the described problem by adding small randomness to player's actions. This way all player's actions are decisive in every moment, all their actions and choices have an impact on the future payments. The concepts used are very similar in their philosophy, but the first only imposes trembles on replies, while the second imposes trembles also at the proposition moment. We adopt an equilibrium notion in which players only make mistakes in replies, because it is at these moments that the information sets are non singleton. The proponent's information set is a singleton, he always knows what the repliers have just done and all the previous history. His strategy must thus maximize the payment after all histories, as proposals always impact the outcome, and *SPNE* is a sensible equilibrium for these cases. In this way, in order to avoid unnecessary complications and due to the requirement of trembling inducing distortions to the game, we opted for introducing the minimum distortions necessary by using the concept of *Perfect Equilibrium in Replies (PER)*. However there is a limitation in using this concept, we are imposing mistakes in a moment where players only have two possibilities but do not impose it when the players have a continuum of possibilities. For the sake of completeness, we will also develop our analysis for the case in which trembles happens at every moments of the game, and we will call this *Perfect Equilibrium (PE)*. The use of two different concepts also shows that the core of results obtained is not dependent on the particular notion used. Before defining these new concepts it is indispensable to define what a mixed strategy is.

3.2 Mixed Strategy

A mixed strategy for this game will be defined in terms of behavioral mixed strategies, meaning that to each h the player will chose a probability distribution over the possibilities A_h available at the time.[5] According to Aumann (1961), to choose a mixed distribution at each h is equivalent to choosing a mixed strategy over all simple strategies. This result is Khun's theorem adaptation for the case of infinite extensive

[5]With the natural definition $A_h = \{0, 1\}$ if $\tau(h) = (t, 1)$ and $A_h = \Delta$ if $\tau(h) = (t-1, 2)$.

games with continuum space of actions. Denoting $\mathfrak{F}(X,\sigma_X)$ the set of probabilities measures over the set X with σ-algebra σ_X. At moment h, with A_h the actions available to the players, a *behavioral strategy at* h for each i is to pick a probability measure $\sigma_i(h) \in \mathfrak{F}(A^h, \mathfrak{B}(A_h))$[6]. A *behavioral mixed strategy for player* i, σ_i is a behavioral mixed strategy for every history $\sigma_i(h)$, $\forall h \in H$, the set of all possible behavioral mixed strategy is Σ_i. A *behavioral mixed strategy* is $\sigma = (\sigma_1, \sigma_2, \sigma_3)$.

To define the payment function it is important to know not only the agreement distribution over Δ, *i.e.* to know what is the probability measure on $\mathfrak{B}(\bar{\Delta})$, but also the moment that agreement is done. For that purpose we will define one probability measure based on the *behavioral mixed strategy*, σ. ${}_k\bar{\sigma}_h$ defines the probability over the future histories of dimension k after h, it is therefore defined on the sigma-algebra $\mathfrak{B}\big(\bar{\Delta}^k\big)$.

${}_k\bar{\sigma}_h$ will be defined iteratively. We start by the probability measure of the histories ending on the period next to h. For that, for each $h \in H^{t,2}$, define ${}_1\sigma_h(O) = \sigma_h(O)$, with $O \in \mathfrak{B}(\bar{\Delta})$. If at h the proposal was accepted and $h^{t,2} = (1,1)$ then no path was followed and in that case ${}_1\sigma_h(O) = 0$ for any $O \in \mathfrak{B}(\bar{\Delta})$. Define the probability measure over future histories of size 2 like

$$ {}_2\bar{\sigma}_h(O) = \int_{\bar{h}\in\bar{\Delta}} \sigma_{(h,\bar{h})}(O_{|\bar{h}})\partial\big({}_1\bar{\sigma}_h\big) $$

In which $O_{|\bar{h}}$ is the projection of $O \in \mathfrak{B}(\bar{\Delta})^2$ on the last coordinate $O_{|\bar{h}} = \big\{\tilde{h} \in \bar{\Delta} : \big(\bar{h},\tilde{h}\big) \in O\big\}$, and clearly a measurable set on $\mathfrak{B}(\bar{\Delta})$. Using the same idea it is possible to define, recursively, ${}_{k+1}\bar{\sigma}_h$ the probability measure among the histories with duration $k+1$ superior to h when σ is the played strategy, for $O \in \mathfrak{B}(\bar{\Delta}^{k+1})$

$$ {}_{k+1}\bar{\sigma}_h(O) = \int_{\bar{h}\in\bar{\Delta}^k} \sigma_{(h,\bar{h})}(O_{|\bar{h}})\partial\big({}_k\bar{\sigma}_h\big) $$

.

For $\bar{h} \in \bar{\Delta}^k$ means that $\bar{h} = (\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_k)$, where $\bar{h}_j \in \bar{\Delta}$, *i.e.* $\bar{h}_{j,1} \in \Delta$ and $\bar{h}_{j,2}, \bar{h}_{j,3} \in \{0,1\}$. Define for $h \in H^{\tilde{t},2}$, the immediate payment at time $\tilde{t}$, $\pi(h) =$

[6] For $A_h = \Delta$ we will use the Borelian σ-algebra

$h_1^{\tilde{t}} h_2^{\tilde{t}} h_3^{\tilde{t}}$, if both repliers accept $\pi(h) = h_1^{\tilde{t}}$, if either rejects $\pi(h) = \bar{0}$. $\pi(h)$ is clearly continuous in h. The payment at $t = t(h)$, under the mixed strategy σ can be defined as

$$\Pi_i^t(\sigma|h) = \sum_k \delta^k \int_{\bar{h}\in\bar{\Delta}^k} \pi(h,\bar{h})\partial\big({}_k\bar{\sigma}_h\big)$$

The expected payment is a discounted sum of a stream of expected values received at each moment when σ is played, at h player i expects to receive $\int_{\bar{h}\in\bar{\Delta}^k} \pi(h,\bar{h})\partial\big({}_k\sigma_h\big)$ in the moment $t(h)+k$.

The next result will show the continuity of the function $\Pi_i^t(\cdot|h)$, so if $\sigma^n \to \sigma$ then $\Pi_i^t(\sigma^n|h) \to \Pi_i^t(\sigma|h)$ for any h. The convergence concept we will use in the strategies space is the strong convergence of measures, so $\sigma^n \to \sigma$ if $\sigma_h^n(O) \to \sigma_h(O)$ for all $O \in \mathfrak{B}(A_h)$, we mean the setwise convergence, with the metric $d(\mu,\nu) = \sup_{h\in H}\sup_{A\in\mathfrak{B}(A_h)} |\mu(A) - \nu(A)|$. The next theorem proves the payment function continuity.

Theorem 5. $\Pi_i^t(\sigma|h)$ *is continuous for all* $h \in H$.

Proof. If $\sigma_h^n \to \sigma_h$, as for each n, ${}_1\sigma_h^n(O) = \sigma_h^n(O)$ then ${}_1\bar{\sigma}_h^n(O) \to {}_1\bar{\sigma}_h(O)$. By induction and using Fatou's lemma with varying measure Royden (1968, p. 231),

$$\begin{aligned}\liminf_n {}_{k+1}\bar{\sigma}_h^n(O) &= \liminf_n \int_{\bar{h}\in\bar{\Delta}^k} \sigma_{(h,\bar{h})}^n(O_{|\bar{h}})\partial\big({}_k\bar{\sigma}_h^n\big) \geq \int_{\bar{h}\in\bar{\Delta}^k} \liminf_n \sigma_{(h,\bar{h})}^n(O_{|\bar{h}})\partial\big({}_k\bar{\sigma}_h\big)\\ &= \int_{\bar{h}\in\bar{\Delta}^k} \sigma_{(h,\bar{h})}(O_{|\bar{h}})\partial\big({}_k\bar{\sigma}_h\big) = {}_{k+1}\bar{\sigma}_h(O)\end{aligned}$$

When $\liminf_n {}_{k+1}\bar{\sigma}_h^n(O) \geq {}_{k+1}\bar{\sigma}_h(O)$ is valid for all open sets then ${}_{k+1}\bar{\sigma}_h^n$ weakly converges to ${}_{k+1}\sigma_h$, and by Portmaentau lemma we know this implies $\int_{\bar{h}\in\bar{\Delta}^k} \pi(h,\bar{h})\partial\big({}_{k+1}\bar{\sigma}_h^n\big) \to \int_{\bar{h}\in\bar{\Delta}^k} \pi(h,\bar{h})\partial\big({}_{k+1}\bar{\sigma}_h\big)$, as $\pi(h,\bar{h})$ is continuous in $\bar{\Delta}^k$. And it follows that $\Pi_i^t(\sigma|h)$ is continuous. □

We can define a best repply to σ at h for player i as

$$BR_i(\sigma|h) = \left\{\sigma'_h \in \mathfrak{F}(A^h, \mathfrak{B}(A_h)) : \Pi_i^t(\sigma'_h, \sigma_{-h}|h) \geq \sup_{\mu_h \in \mathfrak{F}(A^h, \mathfrak{B}(A_h))} \Pi_i^t(\mu_h, \sigma_{-h}|h)\right\}$$

3.3 Trembling Hand Equilibria

The questions raised by Osborne and Rubinstein (1990, p.250), justify the use of an agent strategic form of the game in both *Trembling Hand* definitions. The *PER* is almost a direct translation of Selten's Perfect Equilibrium for the multiplayer bargaining game. We use approximation games in which both actions at the moment of replies are played with at least ϵ probability, and for a strategy to be *PER* it must be an accumulation point of the *SPNE*of the approximation games, with $\epsilon \downarrow 0$. The reason to assume that all actions at all replies must be played with the same probability is due to the symmetric character of the game. To allow one player to accept with a different probability than another one is to destroy this character and to implicitly change an important characteristic of the game. For this reason, even in the *PE*, we will always assume equal restrictions at equal moments, that is, at the replies the restrictions are equal no matter the moment or the player, and at the propositions as well.

Definition 5. *Let* $\Sigma_i^\epsilon = \left\{\sigma_i \in \Sigma_i : \sigma_i(k|h) \geq \epsilon, \forall h \in H^1, k \in \{0,1\}\right\}$, σ *is a* ***Perfect Equilibrium in Replies*** *if it is an accumulation point of a sequence of* $\left\{\sigma^\epsilon\right\}_{\{\epsilon\downarrow 0\}}$, *with* σ^ϵ *a best reply at all histories* h *in the set* Σ_i^ϵ, *that is*

$$\Pi_i^t(\sigma^\epsilon|h) \geq \Pi_i^t\left(\sigma'^\epsilon_{h,i}, \sigma^\epsilon_{-(h,i)}|h\right), \forall \sigma'^\epsilon_{h,i} \in \Sigma_i^\epsilon \cap OSD_i(\sigma^\epsilon, h) \tag{1}$$

In contrast to *PER*, that imposes trembles in a finite set, the *Perfect Equilibrium* notion imposes it also in the uncountable set Δ, and thus is more difficult to define. To our knowledge there is no theory or good examples where to draw from for this concept. The main difficulty is the extensive structure of the bargaining game together with a continuum of actions (at the propositions). Simon and Stinchcombe (1995) developed a concept of Perfect Equilibria for normal form games with a continuum of actions, Carbonell-Nicolau (2011) creates an alternative but equivalent characterization of Perfect Equilibria in the context of games with discontinuous utilities. These will serve as a basis to define the equilibrium notion on extensive games. The Selten (1975) PE on a finite action demands that all points must be chosen with a strict positive probability, on the approximation games; Simon and Stinchcombe (1995) transposed this imposition in the continuous action case to all open sets which must be played with positive probability (on the approximation games). So for all $h \in H^2$, if O is an open subset of Δ then $\sigma^{\epsilon}_{i(h)}(O|h) > 0$. Again if we only used this type of restriction at each moment we would be destroying the game symmetry. It can happen that if some actions are chosen with a certain probability (in the trembles) a strategy is an equilibrium, but if a kind of uniform restriction was set to all actions (in the trembles), *i.e.* a blindness imposition on the trembles, then this strategy might not hold. Later we will present a case where the specific shape of this criteria makes a difference. In multiplayer bargaining the need a stricter criteria is clear by the symmetric nature of the game, if there is not a stronger restriction on the type of allowed mistakes this symmetric nature can be lost, and this changes and distorts the structure of the game entirely. For this reason the criteria we will use is $\sigma^{\epsilon}_{i(h)}(O|h) \geq \epsilon\lambda(O)$, with $\lambda(\cdot)$ proportional to Lebesgue measure in order for $\lambda(\Delta) = 1$, this way we insure a certain blindness, and all mistakes are equally (un)probable. σ^{ϵ} should also be a rest reply at all moments of history and converge (strongly) to the equilibrium strategy. In this game there is no obvious reason to assume strong convergence, however due to the strategic nature of games the approximation strategies should play each action or set of actions with almost the same probability as the equilibrium strategy, to assume a more weak convergence notion might lead to equilibrium strategies in the initial game which were not played in the approximation games.

Definition 6. *$\overline{\Sigma}^{\epsilon}_h = \{\sigma_h \in \Sigma_h : \sigma_h(O) \geq \epsilon\lambda(O), \forall O \subseteq A_h$ open set$\}$. σ is a **Perfect Equilibria** if it is an accumulation point of a sequence of $\{\sigma^{\epsilon}\}_{\{\epsilon\downarrow 0\}}$ with σ^{ϵ} a best reply at all histories h in the set $\bar{\Sigma}^{\epsilon}_i$, that is*

$$\Pi^t_i(\sigma^{\epsilon}|h) \geq \Pi^t_i({\sigma'}^{\epsilon}_{h,i}, \sigma^{\epsilon}_{-(h,i)}|h), \forall {\sigma'}^{\epsilon}_{h,i} \in \overline{\Sigma}^{\epsilon}_h \cap OSD_i(\sigma^{\epsilon}, h)$$

3.4 Trembling Hand Equilibria and Classical Strategies

One property common to all equilibria presented in section (2) is that replies do not play a role in the future of the game. In case of the rejection of a proposal, who rejected the proposal is not relevant to the future path of the game. In this type of strategies, defined as Reply Independent, when one of the trembling hand concepts is in use, as there are no future consequences of accepting or rejecting proposals, and there is always the possibility that the other player accepts, those that leave the players better off should be accepted. The next result will prove this, but first we formally define a *Reply Independent* strategy, as a strategy where the same action is taken for two histories with the same propositions (but possibly with different replies).

Definition 7. *The strategy σ is* ***Reply Independent*** *if for any h and $\tilde{h}$ with $\tau(h) = \tau(\tilde{h})$ and $h^{t,1} = \tilde{h}^{t,1}$, $\forall t \leq t(h)$, then $\sigma(h) = \sigma(\tilde{h})$.*

$\Sigma_p \subset \Sigma$ is the set of all Reply Independent strategies.

If a strategy is Reply Independent, $\sigma \in \Sigma_p$, when a proposal is rejected the payment is always the same no matter what the concrete reply vector $r \in R$ is, with $R = \{(0,0),(0,1),(1,0)\}$ the set of responses where a proposition is rejected. So $\Pi_i^{t+1}\big(\sigma|h,r\big) = \Pi_i^{t+1}\big(\sigma|h,r'\big)$, $\forall r,r' \in R$. We can then define, for a Reply Independent strategy, the future payment after a proposal being refused $p_i^\sigma(h) = \Pi_i^{t+1}\big(\sigma|h,r\big)$, $\forall r \in R$ with $\tau(h) = (t,1)$. We are now in conditions to show that if a strategy is trembling hand equilibrium and reply independent, then good proposals are always accepted.

Theorem 6. *If a simple, reply independent with immediate agreement at each history strategy σ, is* Trembling Hand *then $\sigma_j(1|h) = 1$ if $h_j^{t,1} > p_j^\sigma(h)$ and $\sigma_j(1|h) = 0$ for $h_i^{t,1} < p_j^\sigma(h)$.*

Proof. As the strategy is simple it exists a $d \in \Delta$ such that $\sigma_{i(h)}\big(d\big|h\big) = 1$ and $\sigma_j\big(1\big|h,d\big) = 1$ for $j \neq i(h)$. Therefore, given the definition of a *PE*, the approximation strategy σ^ϵ, for the proponent $i(h)$, must be $\sigma_{i(h)}^\epsilon\big(O\big|h\big) = \epsilon\lambda(O) + (1-\epsilon)\chi_d(O)$, otherwise the distance between the strategies would be bigger than ϵ[7]. For the same reason, the replier, receiving the expected proposition d, chooses to $\sigma_j^\epsilon\big(1\big|h,d\big)=1-\epsilon$. When a proposition $\tilde{d}$ different from d is received the payoff in each of the possibilities is

$$\begin{aligned}
\Pi_j^t\big(1,\sigma_{-\{j,h\}}^\epsilon\big|h,\tilde{d}\big) =&\sigma_k^\epsilon\big(1\big|h,\tilde{d}\big)\tilde{d}_j+\delta\sigma_k^\epsilon\big(0\big|h,\tilde{d}\big)\Pi_j^{t+1}\big(\sigma^\epsilon\big|h,\tilde{d},1,0\big)\\
\Pi_j^t\big(0,\sigma_{-\{j,h\}}^\epsilon\big|h,\tilde{d}\big) =&\delta\sigma_k^\epsilon\big(1\big|h,\tilde{d}\big)\Pi_j^{t+1}\big(\sigma^\epsilon\big|h,\tilde{d},0,1\big)+\delta\sigma_k^\epsilon\big(0\big|h,\tilde{d}\big)\Pi_j^{t+1}\big(\sigma^\epsilon\big|h,\tilde{d},0,0\big)
\end{aligned}$$

Simplifying the notation $\tilde{h}^r=\big(h,\tilde{d},r\big)$ and $\Pi_\epsilon^r=\Pi_j^{t+1}\big(\sigma^\epsilon\big|h,\tilde{d},r\big)$. Player j accepts the proposition $\tilde{d}$, with $\tilde{d}_j>\delta p_j^\sigma(\tilde{h})$, if

$$\Pi_j^t\Big(1,\sigma_{-\{j,h\}}^\epsilon\Big|h,\tilde{d}\Big)>\Pi_j^t\Big(0,\sigma_{-\{j,h\}}^\epsilon\big|h,\tilde{d}\Big)\Leftrightarrow\frac{\sigma_k^\epsilon\big(1\big|h,\tilde{d}\big)}{1-\sigma_k^\epsilon\big(1\big|h,\tilde{d}\big)}>\delta\frac{\Pi_\epsilon^{00}-\Pi_\epsilon^{10}}{\tilde{d}_j-\delta\Pi_\epsilon^{01}}$$

The following claim will be used to calculate $\Pi_\epsilon^{00}-\Pi_\epsilon^{10}$.

Claim 1. *For $h_1,h_2\in H^{t,1}$ with the same proposition's history, $h_1^{k,1}=h_2^{k,1}$ for $k\le t$, then $\Big|\Pi_j^t\big(\sigma^\epsilon\big|h_1\big)-\Pi_j^t\big(\sigma^\epsilon\big|h_2\big)\Big|\le 7\epsilon$.*

Proof. If $h_1,h_2\in H^{t,1}$ then a proposition has already been done at t and the payoff can be divided into the several components one for each possible reply pair

$$\begin{aligned}
\Big|\Pi_j^t\big(\sigma^\epsilon\big|h_1\big)-\Pi_j^t\big(\sigma^\epsilon\big|h_2\big)\Big| &\le\delta\Big|\sum_{r\in R}\sigma^\epsilon(r|h_1)\Pi_j^{t+1}\big(\sigma^\epsilon\big|h_1,r\big)-\sigma^\epsilon(r|h_2)\Pi_j^{t+1}\big(\sigma^\epsilon\big|h_2,r\big)\Big|\\
&\quad+h_j^{t,1}\Big|\sigma^\epsilon(1,1|h_1)-\sigma^\epsilon(1,1|h_2)\Big|\le\\
&\le\delta\sum_{r\in R}\sigma^\epsilon(r|h_1)\Big|\Pi_j^{t+1}\big(\sigma^\epsilon\big|h_1,r\big)-\Pi_j^{t+1}\big(\sigma^\epsilon\big|h_2,r\big)\Big|+\Big|\sigma^\epsilon(r|h_1)-\sigma^\epsilon(r|h_2)\Big|\Pi_j^{t+1}\big(\sigma^\epsilon\big|h_2,r\big)\\
&\quad+h_j^{t,1}\Big|\sigma^\epsilon(1,1|h_1)-\sigma^\epsilon(1,1|h_2)\Big|\le
\end{aligned}$$

[7] With $\chi_a(B)$ the indicator function

$$\chi_a(B)=\begin{cases}1 & \text{if } a\in B\\ 0 & \text{if } a\notin B\end{cases}$$

$$\leq \delta \sum_{r\in R} \sigma^{\epsilon}(r|h_1)\big(1-(1-\epsilon)^3\big) + 2\epsilon \Pi_j^{t+1}\big(\sigma^{\epsilon}\big|h_2, r\big) + 2\epsilon h_j^{t,1} \leq 7\epsilon$$

The first inequality is the result of the triangle inequality; the second from the equality $\sigma^{\epsilon}(r|h_2)\Pi_j^{t+1}\big(\sigma^{\epsilon}\big|h_2,r\big) = \sigma^{\epsilon}(r|h_1)\Pi_j^{t+1}\big(\sigma^{\epsilon}\big|h_2,r\big) + \Big(\sigma^{\epsilon}(r|h_2) - \sigma^{\epsilon}(r|h_1)\Big)\Pi_j^{t+1}\big(\sigma^{\epsilon}\big|h_2,r\big)$; the third inequality from the fact that after the proposition, $|\sigma^{\epsilon}(r|h) - \sigma(r|h)| \leq \epsilon$, then, due to reply independence, $\sigma(r|h_1) = \sigma(r|h_2)$, and by $\sigma^{\epsilon}\big(p,1,1\big|h_1,r\big) = \sigma^{\epsilon}\big(p,1,1\big|h_2,r\big) = (1-\epsilon)^3$ and therefore $\Big|\Pi_j^t\big(\sigma^{\epsilon}\big|h_1\big) - \Pi_j^t\big(\sigma^{\epsilon}\big|h_2\big)\Big| \leq 1-(1-\epsilon)^3$. □

Calculating $\Pi_{\epsilon}^{00} - \Pi_{\epsilon}^{10}$,

$$\Pi_{\epsilon}^{00} = \int_{p\in\Delta\backslash d} \Pi_j^{t+1}\big(\sigma^{\epsilon}\big|\tilde{h}^{00}, p\big)\partial\Big(\sigma^{\epsilon}(\tilde{h}^{00})\Big) + (1-\epsilon)^3 d$$
$$\Pi_{\epsilon}^{10} = \int_{p\in\Delta\backslash d} \Pi_j^{t+1}\big(\sigma^{\epsilon}\big|\tilde{h}^{10}, p\big)\partial\Big(\sigma^{\epsilon}(\tilde{h}^{10})\Big) + (1-\epsilon)^3 d$$

As seen before, due to the reply independence of σ, the proposition is the same after $\tilde{h}^{00}$ and $\tilde{h}^{10}$, σ^{ϵ} is also equal after $\tilde{h}^{00}$ and $\tilde{h}^{10}$ and $\Pi_{\epsilon}^{00} - \Pi_{\epsilon}^{10} = \int_{p\in\Delta\backslash d} \Pi_j^{t+1}\big(\sigma^{\epsilon}\big|\tilde{h}^{00},p\big) - \Pi_j^{t+1}\big(\sigma^{\epsilon}\big|\tilde{h}^{10},p\big)\partial\Big(\sigma_i^{\epsilon}(\cdot|\tilde{h}^r)\Big)$. Using the result of claim(1), $\Big|\Pi_{\epsilon}^{00} - \Pi_{\epsilon}^{10}\Big| \leq \int_{p\in\Delta\backslash d} 7\epsilon\partial\Big(\sigma_i^{\epsilon}(\cdot|\tilde{h}^r)\Big) = 7\epsilon^2$. We know that $\sigma_k^{\epsilon}\big(1\big|h,\tilde{d}\big) \geq \epsilon$ and for small ϵ, $\frac{\epsilon}{1-\epsilon} > 7\epsilon^2$, then it must be that

$$\frac{\sigma_k^{\epsilon}\big(1\big|h,\tilde{d}\big)}{1-\sigma_k^{\epsilon}\big(1\big|h,\tilde{d}\big)} > \frac{\Pi_{\epsilon}^{00} - \Pi_{\epsilon}^{10}}{\tilde{d}_j - \delta\Pi_{\epsilon}^{01}}$$

If the proposition is better than the future payment $\tilde{d}_j > \delta p_j^{\sigma}(\tilde{h})$ then in the approximating strategy player j always accepts the proposal $\sigma_j^{\epsilon}(1|h,\tilde{d}) = 1-\epsilon$.

The same reasoning can be applied for a strategy σ to be *PER*, the future propositions, even in σ^{ϵ}, are the same whatever the actions of the repliers, and so a better proposition will always be accepted.

□

An immediate consequence of the previous result is that Haller's strategy (and it's derivatives) are not *Trembling Hand* equilibria, since repliers only accept a unique proposal and for that reason it cannot sustain the hypothesis of small errors. Without penalizing the answers it was relatively clear that this would happen.

Corollary 1. *Haller's strategy is not Trembling Hand equilibrium.*

Herrero's strategy is different, it respects the previous result, but it still maintains a shortcoming, not all the played strategies are non-dominated, for instance when a player accepts a division that attributes him zero he is playing a weakly dominated strategy. The next theorem shows that *Herrero*'s is not a *Trembling Hand* equilibrium, the proof is done in the appendix for the *Perfect Equilibrium* concept, but it could be done for *PER* following the exact same lines, step by step, for this reason we do not explicitly prove it herein.

Theorem 7. Herrero*'s strategy is not a Trembling Hand equilibrium*

4 New Equilibra Strategies

4.1 Herrero Modified

On this section we will construct two strategies based on *Herrero*, HM_1 and HM_2, the first is *PER* and the second *PE* and in both almost all divisions can be established as equilibrium outcomes. They serve as an example of two important properties of the *Trembling Hand*. Counterintuitevely the set of strategies that are *PE* is not a subset of those that are *PER*. In fact HM_1 is *PER* without being *PE*, and HM_2 is *PE* without being *PER*. And secondly they show, specially HM_2, how crucial the details of *PE* are definition and how they might affect which strategies are equilibria.

These strategies are equal except for the reply to a very specific proposal, in HM_1 that proposal is accepted, in HM_2 it is rejected. The main change to the original *Herrero* strategy is the states and the punishment divisions which they establish, instead of the vectors e^i, both *HM* use $\bar{e}^1, \bar{e}^2, \bar{e}^3$

$$\bar{e}^i_j = \begin{cases} 1 - 2\eta & \text{if } j \neq i \\ \eta & \text{if } j = i \end{cases}$$

With $0 < \eta < \delta/2 - 1/4$. This way we ensure to all players a positive payment, although small, and this positive payoff creates an incentive for the players not to reject the proposition, because if they do it the payment they can get devalues with time.

The function that tracks the state (and therefore the proposition which should be settled on) is as in *Herrero*'s strategy, it changes only after the proposition. When the proposition does not coincide with the state, there is a change to a state that rewards the worst off player in the "out of equilibrium" proposition. Formally state transition is defined in the same way as *Herrero*'s but with the states changed from e^i to $\bar{e}^i$. HM_1 is equal to Herrero's strategy but with the new states instead of e^i. Both strategies are presented in the next two tables.

Table 3: HM1

	State	$\bar{e}^j$
Player i	Proposal	$\bar{e}^j$
	Accept	$p_i \geq \delta \bar{e}^j_i$

The moment in which HM_2 is different from HM_1 is when the share proposed to the replier that will not propose on the next round coincides to what he would receive there, that is the share proposed to him is $\delta \bar{e}^j_i$. Only in this case are the strategies different with that replier accepting the proposal in HM_1 and rejecting it in HM_2.

Table 4: HM2

	State	$\bar{e}^j$
Player i	Proposal	$\bar{e}^j$
	Accept	$p_i > \delta \bar{e}^j_i$ and $i \neq i(h^+)$
		$p_i \geq \delta \bar{e}^j_i$ and $i = i(h^+)$

Theorem 8. *For $\delta > 1/2$, HM_1 is* PER *and it is not* PE*, HM_2 is* PE *and it is not* PER

Proof. We will start seeing that HM_1 is *PER* and can not be *PE*, this happens because in the approximating games the only best reply is when a proposition of

$\delta\bar{e}_i^j$ is received, to accept it, and by this reason HM_2 can never be the limit of an approximating strategy. Take s for the HM_1 and s^ϵ for the approximating strategy. At $h \in H^{t-1,2}$ for the proponent $\boldsymbol{i = i(t)}$ we will prove that $\Pi_i^t(s|h) > \Pi_i^t(s_i', s_{-i}|h)$ for all $s_i' \in OSD(s_i, h)$, and due to continuity of payment function $\Pi_i^t(s_i, s_{-i}^\epsilon|h) > \Pi_i^t(s_i', s_{-i}^\epsilon|h)$. If s is played after $\tau(h) = (t-1, 2)$, i receives $\Pi_i^t(s|h) = r(h)_i \geq \eta$. If it makes a proposition p different from $r(h)$, the state changes before voting to $r\big(h, p\big) = \bar{e}^k$ with $k = k(p, t) \neq i$. If the proposal $h^{t,1}$ is rejected the future payment is $\delta\bar{e}_i^k = \delta\eta < \eta$. If it is accepted $k = k(p, t)$ accepts only if $h_k^{t,1} \geq \delta(1 - 2\eta)$; as $h_j^{t,1} \geq h_k^{t,1}$, then $h_j^{t,1} \geq \delta(1-2\eta)$, and $h_i^{t,1} = 1 - \sum_{j \neq i} h_j^{t,1} = 1 - 2\delta(1-2\eta)$. But, due to $\eta < \delta/2 - 1/4$, $1 - 2\delta(1-2\eta) < \eta$ and for the proposal to be accepted by all the repliers the proponent must receive less than $\eta \leq \Pi_i^t(s|h) = r(h)_i$. So, no matter whether the proposal is accepted or rejected, the proponent always gets worst by playing a different strategy. For s_i^ϵ to be a best reply it must coincide with s_i as PER does not impose any restriction on the proposition distribution $s_i^\epsilon\big(O\big|h\big) = s_i\big(O\big|h\big) = \chi_{r(h)}(O)$.

For a replier $\boldsymbol{j \neq i(t)}$. The strategy s is simple, reply independent and the agreement is established at the first moment therefore as proved in theorem (6), $s_j^\epsilon(1|h) = 1 - \epsilon$ if $h_j^{t,1} > \delta r(h)_j$ and $s_j^\epsilon(1|h) = \epsilon$ if if $h_j^{t,1} < \delta r(h)_j$. The only case that rest to be analysed in when $h_j^{t,1} = \delta r(h)_j$. Define the set of possible histories when the strategy σ^ϵ is played after h

$$H_{s^\epsilon}(h) = \{\bar{h} \in H : s^\epsilon(\bar{h}|h) > 0\}$$

For $\bar{h} \in H_{s^\epsilon}$ and $\tau(\bar{h}) = (t+1, 2)$, we know that the state of $\bar{h}$ is $r(h)$ and that the proposition at $\bar{h}^{t+1,1}$ was $r(h)$. This because in the proposition stage s^ϵ and s define the same action, $s_{i(\bar{h})}(\bar{h}^-) = r(\bar{h}^-) = r(h)$ and $s_{i(\bar{h})}^\epsilon\big(r(h)|\bar{h}\big) = 1$. This way $\bar{h}^{t+1,1} = r(h)$. The proposition coincides with the state and there is no state change, $r(\bar{h}) = r(h)$. Using the same reasoning for any t' if $\tau(h') = (t', 2)$ and $h' \in H_{s^\epsilon}(h)$, we conclude that $r(h') = r(h)$, and the proposition after h is always the same $r(h)$ if strategy s^ϵ is played. If proposal $h^{t,1}$ is rejected the continuation payment can be written as

$$\Pi_j^{t+1}\big(s^\epsilon\big|h,r\big)=\delta\sum_k p(a_{t+k}=1)\delta^k r(h)_j<\delta r(h)_j$$

In which $p(a_{t+k}=1)$ is the probability that an agreement is reached $t+k$ periods after h, when s^ϵ is played. Then, when the player accepts, his payment is $h_j^{t,1}s_k^\epsilon(1|h)+s_k^\epsilon(0|h)\Pi_j^{t+1}\big(s^\epsilon\big|h\big)$. If he rejects the payment is $\Pi_j^{t+1}\big(s^\epsilon\big|h\big)$. So when $h_j^{t,1}\geq\delta r(h)_j>\Pi_j^t\big(s^\epsilon\big|h\big)$ the best for j it to accept with maximum allowed probability, and $s_j^\epsilon(1|h)=1-\epsilon$. The convergent strategy is

$$s_{i(h)}^\epsilon(x|h)=\begin{cases}1 & \textit{if } x=r(h)\\ 0 & \textit{if } x\neq r(h)\end{cases}\qquad\qquad s_j^\epsilon(x|h)=\begin{cases}1-\epsilon & \textit{if } h^{t,1}\geq r(h)_j\\ \epsilon & \textit{if } h^{t,1}<r(h)_j\end{cases}$$

The same derivation is valid for the strategy HM_2 to be a *PER*. By the dominance principle the proposition must be equal to the state; because of strategy's reply independence, whenever the proposition is bigger than future earnings to accept it is the best reply and when it is smaller the best is to reject it; and in case of equality as the future proponents will always play $r(h)$ the replier should accept the proposition to avoid devaluation. Therefore the best reply approximation strategy must be equal to s^ϵ, but these strategies converge to HM_1, and we conclude that HM_2 can not be *PER*.

To prove that HM_1 was a *PER* we also proved that HM_2 was not *PER*, because if a best reply approximation strategy to HM_2 existed, it had to be equal to the approximation strategy to HM_1, and this approximation strategy, obviously, cannot converge to HM_2. The same happens with the proof that HM_2 is a *PE*, it will also prove that HM_1 is not a *PE* because the approximation strategy should be the same. The proposition in HM_2 is the same as in HM_1 the player always proposes a division equal to the state, and if he does not he is strongly penalized afterwards, $\Pi_i^t(\sigma|h)>\Pi_i^t(\sigma_i',\sigma_{-i}|h)$, therefore for an approximation strategy σ^ϵ, $\Pi_i^t(\sigma_i,\sigma_{-i}^\epsilon|h)>\Pi_i^t(\sigma_i',\sigma_{-i}^\epsilon|h)$. So, for an approximation strategy to σ to be a best reply in the approximation game, it should be equal to σ with maximum probability, so $\sigma_{i(h)}^\epsilon\big(r(h)\big|h\big)=1-\epsilon$ and to respect the other condition to be PE we must have that for any $O \in \mathfrak{B}(\Delta)$

$$\sigma^{\epsilon}_{i(h)}\big(O\big|h\big) = (1-\epsilon)\chi_{r(h)}(O) + \epsilon\lambda(O)$$

HM_2 is a simple, reply independent with immediate agreement strategy, the derivation made in theorem(6) is applicable and at history h if the proposition $h_j^{t,1}$ is such that $h_j^{t,1} > \delta r(h)_j$ the best reply in the approximation strategy is to accept with probability $1-\epsilon$ and if $h_j^{t,1} < \delta r(h)_j$ the best is to reject with maximum probability. Then for $j \neq i(h)$

$$\sigma^{\epsilon}_j(1\big|h) = \begin{cases} 1-\epsilon & \text{if } h_j^{t,1} > \delta r(h) \\ \epsilon & \text{if } h_j^{t,1} < \delta r(h) \end{cases}$$

Before looking at what happens when $h_j^{t,1} = \delta r(h)_j$ two notes are important. First, all the conclusions were equally valid if instead we were trying to prove that HM_1 is PE. Second, what we determine of the structure of σ^{ϵ} is enough to conclude that if two histories $h, h' \in H^{t-1,2}$ have the same state, players' payoff, under the strategy σ^{ϵ}, is the same, because the only moment where the actions can be different is when one of the players received a proposition $p_j = \delta r(h)_j$, but the measure of this case is null. The next claim proves this.

Claim 2. *If $h, h' \in H^{t-1,2}$ then $\Pi^t_j\big(\sigma^{\epsilon}\big|h\big) = \Pi^t_j\big(\sigma^{\epsilon}\big|h'\big)$ with $r(h) = r(h')$*

Proof. Define $\bar{\Delta}_* = \big\{(p,r) \in \Delta \times \{0,1\}^2 : p_i = \delta\bar{e}^j_i, i,j = 1,2,3\big\}$, and ${}_1\sigma^{\epsilon}\big(\bar{\Delta}_*\big|h\big) = \sigma^{\epsilon}\big(\bar{\Delta}_*\big|h\big) \leq \sum_{i,j}\sigma^{\epsilon}\Big(\big\{p \in \Delta : p_i = \delta\bar{e}^j_i\big\}\big|h\Big) = \sum_{i,j}\epsilon\lambda\Big(\big\{p \in \Delta : p_i = \delta\bar{e}^j_i\big\}\big|h\Big) = 0$. If $\bar{\Delta}_+ = \bar{\Delta} \setminus \bar{\Delta}_*$ then $\sigma^{\epsilon}\big(\bar{\Delta}_+\big|h\big) = 1$. By induction

$$\begin{aligned} {}_{k+1}\bar{\sigma}^{\epsilon}\big(\bar{\Delta}^{k+1}_+\big|h\big) &= \int_{\bar{h}\in\bar{\Delta}^k} \sigma^{\epsilon}_{h,\bar{h}}\Big(\bar{\Delta}^{k+1}_{+|\bar{h}}\Big)\partial\big({}_k\bar{\sigma}^{\epsilon}{}_h\big) = \int_{\bar{h}\in\bar{\Delta}^k_+} \sigma^{\epsilon}_{h,\bar{h}}\Big(\bar{\Delta}_+\Big)\partial\big({}_k\bar{\sigma}^{\epsilon}{}_h\big) \\ &= \int_{\bar{h}\in\bar{\Delta}^k_+} \sigma^{\epsilon}_{h,\bar{h}}\big(\bar{\Delta}\big)\partial\big({}_k\bar{\sigma}^{\epsilon}{}_h\big) = {}_{k+1}\bar{\sigma}^{\epsilon}\big(\bar{\Delta}^{k+1}\big|h\big) \end{aligned}$$

Where the first equality holds because ${}_k\bar{\sigma}^{\epsilon}\big(\bar{\Delta}^k_+\setminus\bar{\Delta}^k_+\big|h\big) = 0$ by induction hypothesis and because if $\bar{h} \in \bar{\Delta}^k_+$ then $\bar{\Delta}_{+|\bar{h}} = \Delta_+$. Therefore

$$\begin{aligned}\Pi_j^t\big(\sigma^\epsilon\big|h\big) =&\sum_{k=0}^{\infty}\delta^k\int_{\bar{h}\in\bar{\Delta}^k}\pi(h,\bar{h})\partial\big({}_k\bar{\sigma^\epsilon}_h\big)=\sum_{k=0}^{\infty}\delta^k\int_{\bar{h}\in\bar{\Delta_+}^k}\pi(h,\bar{h})\partial\big({}_k\bar{\sigma^\epsilon}_h\big)\\ =&\sum_{k=0}^{\infty}\delta^k\int_{\bar{h}\in\bar{\Delta_+}^k}\pi(h,\bar{h})\partial\big({}_k\bar{\sigma^\epsilon}_{h'}\big)=\sum_{k=0}^{\infty}\delta^k\int_{\bar{h}\in\bar{\Delta}^k}\pi(h,\bar{h})\partial\big({}_k\bar{\sigma^\epsilon}_{h'}\big)\\ =&\Pi_j^t\big(\sigma^\epsilon\big|h'\big)\end{aligned}$$

□

There are two possibilities of $h_j^{t,1}=\delta\bar{e}_j^i$, either $i=j$ or $i\neq j$. If it is the first case, in the proposition $h^{t,1}$, j was the worst player, he has received the smaller share so it must be that $h_j^{t,1}\leq\frac{1}{2}$, but η is such that $\delta(1-2\eta)>\frac{1}{2}$, an implication of $\eta<\delta/2-1/4$, so $h_j^{t,1}<\delta\bar{e}_j^j$. This way we need to consider only when the proposition $h^{t,1}$ attributes to player j the share $h_j^{t,1}=\delta\eta$, and attributes less to the other replier k. To determine the best reply of j we must compare $\Pi_j^t(\sigma_j^a,\sigma_{-j}^\epsilon|h)$ with $\Pi_j^t(\sigma_j^r,\sigma_{-\{j,h\}}^\epsilon|h)$, knowing that the other replier k rejects the proposition with probability $1-\epsilon$.

$$\begin{cases}\Pi_j^t(\sigma_j^r,\sigma_{-\{j,h\}}^\epsilon|h) & =(1-\epsilon)\delta\Pi_j^{t+1}\big(\sigma^\epsilon\big|h,0,0\big)+\delta\epsilon\Pi_j^{t+1}\big(\sigma^\epsilon|h,0,1\big)\\ \Pi_j^t(\sigma_j^a,\sigma_{-\{j,h\}}^\epsilon|h) & =(1-\epsilon)\delta\Pi_j^{t+1}\big(\sigma^\epsilon\big|h,1,0\big)+\epsilon\delta\eta\end{cases}$$

Due to the state being the same in the histories $(h,1,0)$ and $(h,0,0)$ we know that $\Pi_i^{t+1}(\sigma^\epsilon|h,1,0)=\Pi^{t+1}\big(\sigma^\epsilon|h,0,0\big)$, by claim (2), the difference in payment between the two actions resumes itself to $\Pi_j^t(\sigma_j^a,\sigma_{-i}^\epsilon|h)-\Pi_j^t(\sigma_j^r,\sigma_{-j}^\epsilon|h)=\delta\epsilon\Big(\eta-\Pi_j^{t+1}\big(\sigma^\epsilon|h,0,1\big)\Big)$.

First, we will see in detail what is $\Pi_j^{t+1}\big(\sigma^\epsilon|h,0,1\big)$. It has 3 "types" of payment: when there are no mistakes, the proponent proposes $r(h)$ and both repliers accept, this happens with probability $(1-\epsilon)^3$, and contribution to j's payoff is $(1-\epsilon)^3\eta$; when the proponent does the right proposal but at least one of the repliers do a mistake and the expected payoff from this histories is $(1-\epsilon)\delta\sum_{r\in R}p_r\Pi_j^{t+1}\big(\sigma^\epsilon\big|h,r(h),r\big)$, with p_r the probability of the reply be r, if $r=(k,j)\in\{(0,0),(0,1),(1,0)\}$ then $p_r=(1-\epsilon)^{k+j}\epsilon^{2-k-j}$; and lastly, when the proponent makes the wrong proposal. For the moment, let us call this expected value ϵE_j. As before $\Pi_j^{t+2}\big(\sigma^\epsilon|\tilde{h},r(h),r\big)$ does not depend on the specific reply r and we can relabel it as Π_j^{t+2}, with $\tilde{h} = (h, 0, 1)$. Summing all up we get

$$\begin{aligned}\eta - \Pi_j^{t+1}\big(\sigma^\epsilon|h,0,1\big) &= \eta - \Big[(1-\epsilon)^3\eta + (1-\epsilon)\delta\sum_{r\in R} p_r\Pi_j^{t+2} + \epsilon E_j\Big] \\ &= \eta(3\epsilon - 3\epsilon^2 - \epsilon^3) - (1-\epsilon)\big(1-(1-\epsilon)^2\big)\delta\Pi_j^{t+2} - \epsilon E_j \\ &= \epsilon\Big[\eta(3 - 3\epsilon - \epsilon^2) - (2 - 3\epsilon + \epsilon^2)\delta\Pi_j^{t+2} - E_j\Big]\end{aligned}$$

The signal of $\Pi_j^t(1, \sigma^\epsilon_{-\{j,h\}}|h) - \Pi_j^t(0, \sigma^\epsilon_{-\{j,h\}}|h)$ for small values of ϵ is equal to $3\eta - 2\delta\eta - e$, we assumed that $E_j \to e$ and used that Π_j^{t+2} in the expression converges to η.It is the signal of $3\eta - 2\delta\eta - e$ that determines the best action for player j. This signal depends crucially on the value of e and to understand it we need to look to the structure of E_j. When a proposition is not done according to the state, there is a state transition to $\bar{e}^w$. Define Δ_w the set of points of Δ that, when proposed, the state change to $\bar{e}^w$. Divide each of these sets in three: $\Delta_w^{1,0}$ in which, when σ is being played, player j accepts the propositions but player k rejects; $\Delta_w^{0,1}$ player j rejects but player k accepts; and, $\Delta_w^{0,0}$ in which both players reject[8][9][10]

$$\begin{aligned}E_j &= \sum_{w\in -i(t+1)}\sum_{r\in R}\int_{p\in\Delta_w^r} \delta\sigma^\epsilon(r|\tilde{h},p)\Pi_j^{t+2}\big(\sigma^\epsilon|\tilde{h},p,r\big)\partial\sigma^\epsilon(\tilde{h}) + O(\epsilon) \\ &= \sum_{w\in -i(t+1)}\sum_{r\in R}\int_{p\in\Delta_w^r} \delta(1-\epsilon)^2\Pi_j^{t+2}\big(\sigma^\epsilon|\tilde{h},p,r\big)\partial\sigma^\epsilon(\tilde{h}) + O(\epsilon) \\ &= \sum_{w\in -i(t+1)}\int_{p\in\Delta_w} \delta(1-\epsilon)^2\Pi_j^{t+2}\big(\sigma^\epsilon\big|\bar{e}^w\big)\partial\sigma^\epsilon(\tilde{h}) + O(\epsilon) \\ &= \delta\Pi_j^{t+2}\big(\sigma^\epsilon\big|\bar{e}^{w_1}\big)\sigma^\epsilon\big(\Delta_{w_1}\big|\tilde{h}\big)(1-\epsilon)^2 + \delta\Pi_j^{t+2}\big(\sigma^\epsilon\big|\bar{e}^{w_1}\big)\sigma^\epsilon(\Delta_{w_2}|\tilde{h})(1-\epsilon)^2 + O(\epsilon)\end{aligned}$$

[8]Notice that at least one of the players rejects the proposition namely if the state changes to e^k the player k always reject under σ, and by that $\Delta_k^{0,1}$ is empty. It is only defined to ease the formulas.

[9]We are using the big O notation, symbolizing that $f(\epsilon) \in O(\epsilon)$ when $\frac{f(\epsilon)}{\epsilon} \le k$ for small values of ϵ for some k

[10]There is a slight abuse on notation here because $\sigma^\epsilon(\Delta|h) = \epsilon$ and we are using it as if $\sigma^\epsilon(\Delta|h) = 1$, E was multiplied by ϵ on the payment function.

The first equality results from the division of the set Δ into several parts and from knowing that at each set Δ_w^r the probability of a reply different from r is at most $\epsilon(1-\epsilon)$; the second from the probability of r being $(1-\epsilon)^2$; the third is due to the union of the sets Δ_w^r for a fixed player w and by claim(2) the payoff is equal for all histories that have the same state, we define $\Pi_j^{t+2}\big(\sigma^\epsilon\big|\bar{e}^w\big)$ as the payment in the state $\bar{e}^w$.

Taking limits of the expression for E_j: if the player j is the proponent at $t+1$ then he will always receive the same in both states $\Pi_j^{t+2}\big(\sigma^\epsilon\big|\bar{e}^{w_1}\big) = \Pi_j^{t+2}\big(\sigma^\epsilon\big|\bar{e}^{w_2}\big) = \eta$, and $e = \delta\eta$. And we can calculate that $\eta - \Pi_j^{t+1}\big(\sigma^\epsilon|h,0,1\big)$ converges to $3-2\delta\eta-\delta\eta > 0$. For small values of ϵ, if j is the next proponent $j = i(t+1)$, he should accept the proposition $h^{t,1}$ when $h_j^{t,1} = \delta\bar{e}_j^i$ with maximum probability; if j is not the next proponent, as $\sigma^\epsilon(\Delta_{w_1}|\tilde{h}) = \sigma^\epsilon(\Delta_{w_2}|\tilde{h}) = 1/2$ the limit of E is $e = \delta\frac{1}{2}(1-\eta)$, and $3\eta - 2\delta\eta - e = 3\eta - 2\delta\eta - \delta\frac{1}{2}(1-\eta) = 1/2\big(6\eta - 3\delta\eta - \delta\big)$, and it is better for j to accept the proposition if $\eta > \frac{\delta}{6-3\delta}$. But this inequality is incompatible with $\eta < \delta/2 - 1/4$, these conditions cannot be fulfilled simultaneously. Therefore for small values of ϵ the replier should reject the proposition $h_j^{t,1} = \delta\eta$.

Summing up the conclusions, if $h_j^{t,1} = \delta\eta$ and $r(h) \neq \bar{e}^j$ then

$$\sigma_j^\epsilon(1|h) = \begin{cases} 1-\epsilon & \text{if } j = i(t+1) \\ \epsilon & \text{if } j \neq i(t+1) \end{cases} \tag{2}$$

Clearly σ^ϵ is convergent to HM_2, so it is a *PE*. For HM_1 to be a *PE* σ^ϵ must be the approximation strategy, and therefore HM_1 is not a *PE*. □

The subtleties of the *PE* definition are evident, if we adopted a less stricter notion and didn't impose the condition that $\sigma^\epsilon(O|h) \geq \epsilon\lambda(O)$ but one similar to that of Simon and Stinchcombe (1995) then HM_1 would also be *PE*. For example if the trembles in the proposition had a very small $\sigma^\epsilon(\Delta_j|h)$ we could find a different convergent sequence, but that would mean the proponent was usingthe distribution function on the trembles strategically.

4.2 A Strictly Undominated Equilibrium Strategy for all δ

We saw that if the strategies do not penalize the replies players will accept better proposals. One clear conclusion is that whenever the proposal is greater than δ, the repliers will accept it. If this discount factor is small, $\delta < 1/2$, it is possible to propose a division that both repliers accept. Given that $\delta < 1/2$, pick an ϵ and set $p_j^t = \delta + \epsilon$ for $j \neq i(h)$, $p_j^t > \delta \geq \delta \Pi_j^{t+1}(\sigma|(h,r))$, and due to theorem (6) all j accept the proposition. Any division d with $d_i(t) < 1 - 2\delta$ is not a *Trembling Hand*.

In this section we will build a strategy that is *Trembling Hand* equilibrium for all δ, and for almost all divisions ($d(\sigma) \gg \bar{0}$). This strategy is strictly non-dominated, so the actions taken at each information set are the unique best reply. To establish it and according to theorem (6) it is necessary either to use mixed strategies, that the strategy does not establish the agreement immediately or to penalize the replies. In this case we opt to penalize the out of equilibrium replies, for this we will use two punishment "states" (by player). The set of all states for this strategy is $E = \{e^0, e^1, e^2, e^3, e_a^1, e_a^2, e_a^3\}$. The idea of two states per player is to allow to punish a proposer i when he deviates from e^k with e_a^i, and punish with e^i when he deviates from e_a^i. Again to each state corresponds one particular division, therefore $e \in E$ is a division vector, with the share given to player k being k^{th}-coordinate of e. To the corresponding state the division vector is

$$e_k^i = \begin{cases} \epsilon_1 & \textbf{if } k \neq i \mod 3+1 \\ 1-2\epsilon_1 & \text{if } k = i+1 \end{cases} \qquad e_{a,k}^i = \begin{cases} \epsilon_1 & \text{if } k \in -\{i, i \mod 3+1\} \\ \epsilon_2 & \text{if } k = i \\ 1-\epsilon_1-\epsilon_2 & \text{if } k = i \mod 3+1 \end{cases}$$

ϵ's are chosen in a way that: $\delta\epsilon_1 < \epsilon_2 < \epsilon_1$; and, $\epsilon_1 < \min_{i \in I} e_i^0$. Notice that due to $\epsilon_1 < \min_{i \in I} e_i^0$ and $\sum_{i \in I} e_i^0 \leq 1$, $3\epsilon_1 \leq 1$ and naturally $2\epsilon_1 < 1$ guaranteeing that $e^i \in \Delta$.

The strategy, ρ, defines that players should make a proposition equal to the state and reject all the propositions which are different (in this sense it is like Haller's strategy, but this time robust to minor randomness, because it penalizes out of the path answers). State changes happen after the voting, so the transition takes place in histories that belong to $H^{t,2}$. The state effectively changes to a different state only when the proposal or one of the responses is different from what is defined by ρ; when both are different it is the reply that defines the new state; if both players reply in an unexpected way it is the player with the smaller index who is punished. The following table sums up the transition.

State tansition		
	Different Proposal	Different Reply
state	player $i(h)$	player k
e^0	$e^{i(h)}$	e^k_a
e^j	$e^{i(h)}_a$	e^k_a
e^j_a	$e^{i(h)}$	e^k_a

In a rigorous manner, the function $r(h)$ that determines the state, $r(.) : H \to E$, at the initial state is $r(\emptyset) = e^0$; as there is no state change from $(t,1)$ to $(t,2)$ no matter what the proposition at $(t,1)$ was, for histories h with $\tau(h) = (t,1)$, $r(h) = r(h^-)$; for a history h ending after a voting stage $\tau(h) = (t,2)$ the state is

$$r(h) = \begin{cases} \begin{cases} e^{i(h)} & \text{if } r(h^{|t-1,2}) \in \{e^0, e^1_a, e^2_a, e^3_a\} \\ e^{i(h)}_a & \text{if } r(h^{|t-1,2}) \in \{e^1, e^2, e^3\} \end{cases} & h^{t,1} \neq \ r(h^{|t-1,2}) \text{ or } h^{t,2} = \bar{0} \\ e^k_a \text{ if } h^{t,2} \neq \rho_j(h^{|t,1}) \end{cases}$$

The first and second branches define the new state when a player makes a proposition different from the state $h^{t,1} \neq r(h^{|t-1,2})$ and all repliers act accordingly voting against the proposal, $h^{t,2} = \bar{0}$; the new state is $e^{i(h)}$ or $e^{i(h)}_a$ depending on the initial one. The third branch defines the state when a replier is incongruent with the strategy. k is the player of smaller index who plays differently from what was expected, $k = \min\{j \in -i(h) : h^{t,2}_j \neq \rho_j(h^{|t,1})\}$.

The strategy for the proponent $i(h)$ is $\rho_{i(h)}(x|h) = \chi_{r(h)}(x)$. The player $i(h)$ plays x with probability 1 if $x = r(h)$ and with probability 0 if it is not. For the replier

$j \neq i(h)$, ρ_j defines that if a proposal is the same as the state it should always be accepted, if it is different it should be rejected, formally

$$\rho_j(x|h) = \begin{cases} \chi_{\{1\}}(x) & \text{if } h^{t,1} = r(h^-) \\ \chi_{\{0\}}(x) & \text{if } h^{t,1} \neq r(h^-) \end{cases}$$

for $x \in \{0, 1\}$, meaning that j always accepts if the proposal is equal to the state and always rejects otherwise.

To prove ρ is an equilibrium for any δ we will use the next two claims, both proved in the appendix. The first proves that a strategy is a *Trembling Hand* equilibrium, without the need to look for the specific ϵ strategy that converges to it, if dominant. The second claim proves that if a strategy is strictly better than all *OSD* strategies, then it is strictly better than all strategies.

Claim 3. *When* $\inf\left\{\Pi_i^t(\sigma|h) - \Pi_i^t(\sigma_i', \sigma_{-i}|h) : \forall h \in H, \forall \sigma_i' \in OSD(\sigma_i, h)\right\} > 0$ *then* σ *is an* Trembling Hand *equilibrium.*

The classical *OSD* result state that $\Pi_i^t(s|h) \geq \Pi_i^t(s_i', s_{-i}|h)$ for all $s_i' \in S_i$ is equivalent to $\Pi_i^t(s|h) \geq \Pi_i^t(\bar{s}_i, s_{-i}|h)$ for all $\bar{s}_i \in OSD(s_i, h)$ and all h. Given the previous claim, a similar result but with a strict inequality is useful, this way we only need to prove the strict inequality for *OSD* strategies. The following simple claim proves this result for our game.

Claim 4. *If* $\Pi_i^t(s|h) > \Pi_i^t(s_i', s_{-i}|h)$, *for all* $s_i' \in OSD(s_i, h)$ *and for all* $h \in H$, *then* $\Pi_i^t(s|h) > \Pi_i^t(\bar{s}_i, s_{-i}|h)$, *for all* $\bar{s}_i \in \Sigma_i \setminus s_i$ *and for all* $h \in H$

Theorem 9. $\Pi_i^t(\rho|h) > \Pi_i^t(\rho_i', \rho_{-i}|h)$, $\forall i \in I, h \in H$ *and* $\rho_i' \in \Sigma_i$.

Proof. If ρ is strictly better than all $OSD(\rho_i, h)$ for all i and h, then by claim(4) we get the intended result. In the state $e \in E$ after history $\tau(h) = (t-1, 2)$, when the proposition at t has not been made yet, if all players follow the strategy ρ, $i(h)$ proposes e, repliers accept, agreement is immediate and the player's payment is $\Pi_i^t(\rho|h) = e_i$. If $\tau(h) = (t, 1)$ and the proposal has been made, payment depends on whether the proposition coincided with the state or not. If it did, agreement is immediate; if it did not, agreement is postponed one period, and the proposition is $r\big(h, \bar{0}\big)$. Remember state transition only happens after voting, so $e = r(h) = r(h^-)$

$$\Pi_i^t(\rho|h) = \chi_{\{e\}}(h^{t,1})e_i + \big(1 - \chi_{\{e\}}(h^{t,1})\big)\, \delta r(h, \bar{0})_i$$

The vector of zeros $\bar{0}$ appears because, according to the strategy ρ, all players reject a proposal not equal to state. So if a proposal is equal to state $\chi_{\{e\}}(h^{t,1}) = 1$ and $\Pi_i^t(\rho|h) = e_i$; if not, $\chi_{\{e\}}(h^{t,1}) = 0$ repliers vote against it and payment is $\delta r(h, \bar{0})_i$. Now we will see that using *OSD* strategies leaves the players worst off than getting along with ρ.

When $\boldsymbol{r(h) = e^k}$, $\boldsymbol{k \neq 0}$ and $\boldsymbol{\tau(h) = (t-1, 2)}$, with $i = i(t)$, $\rho_i' \in OSD(\rho_i, h)$, player i makes a proposition $h^{t,1}$ different from e^k, all players $j \in -i$ reject the proposition $\rho_j(0|h^+) = 1$ so $h^{t,2} = \bar{0}$. With $h^{++} = (h, h^{t,1}, h^{t,2})$ $r(h^{++}) = e_a^i$, then $\Pi_i^t(\rho_i', \rho_{-i}|h) = \delta\Pi_i^{t+1}(\rho_i', \rho_{-i}|h^{++}) = \delta\Pi_i^{t+1}(\rho_i, \rho_{-i}|h^{++}) = \delta r(h^{++}) = \delta e_{a,i}^i = \delta\epsilon_2$. We already derived $\Pi_i^t(\rho|h) = e_i^k \geq \epsilon_1$ as $\delta\epsilon_2 < \epsilon_1$ we conclude that player i is strictly worse.

For $\boldsymbol{\tau(h) = (t, 1)}$, two cases for a deviating strategy are possible: $h^{t,1} = r(h^-)$, the proposal was according to the state or it was not, $h^{t,1} \neq r(h^-)$. If it was not and a player j accepts the proposal, $\rho_j' \in OSD(\rho_j, h)$ considering $1 = \rho_j'(1|h) \neq \rho_j(1|h) = 0$. In h the other replier rejects and the proposal is not accepted, then $\Pi_j^t\big(\rho_j', \rho_{-j}|h\big) = \delta\Pi_j^{t+1}(\rho|h^+)$ with $h^+ = (h, h^{t,2})$ and $h_k^{t,2} = \chi_j(k)$, only j accepts the proposal. $r(h^+) = e_a^j$ and $\Pi_j^t(\rho_j', \rho_{-j}|h) = \delta e_{a,j}^j = \delta\epsilon_2$. If j had rejected the state had changed as well, because $h^{t,1} \neq r(h^-)$, but this time to $r(h, \bar{0}) = e_a^{i(h)}$, $\Pi_j^t(\rho|h) = \delta\Pi_j^{t+1}(\rho|h, \bar{0}) = \delta r(h, \bar{0})_j = \delta e_{a,j}^{i(h)} \geq \delta\epsilon_1$ remembering $\epsilon_2 < \epsilon_1$, it is proved that j's payment worsens.

If the proposition was equal to the state $\boldsymbol{h^{t,1} = r(h^-)}$, for j not following ρ means the player rejects $h^{t,1}$. If it behaved as ρ_j the agreement was immediate and $\Pi_j^t(\rho|h) = e_j^k \geq \epsilon_1$. But if the player j refused and he was the only one to do it, proposal was rejected and $r(h^+) = e_a^j$. This way $\Pi_j^t(\rho_j', \rho_{-j}|h) = \delta e_{a,j}^j = \delta\epsilon_2$; $\epsilon_1 > \delta\epsilon_2$, player j worsens if he does not follow the strategy.

When $\boldsymbol{r(h) = e_a^k}$, $\tau(h) = (t-1, 2)$. Suppose proposer i follows $\rho'_i \in OSD(\rho_i, h)$, proposing something different from $r(h)$, by ρ definition all repliers will refuse it, the state will change to $r(h^{++}) = e^i$ and future payment is $\Pi_i^t(\rho'_i, \rho_{-i}|h) = \delta e_i^i = \delta\epsilon_1 < \epsilon_2 \leq e_{a,i}^k = \Pi_i^t(\rho|h)$, player i gets strictly worse by playing ρ'_i.

$\tau(h) = (t, 1)$ **and** $j \neq i(h)$, two possibilities for the strategy to be OSD of ρ_j in pure strategies, when $h^{t,1} = r(h^-)$ and $\rho'_j(0|h) = 1$ or when $h^{t,1} \neq r(h^-)$ and $\rho'_j(1|h) = 1$. In the first case j rejects the $h^{t,1}$, the agreement is only established in the next moment, the state changes to $r(h^+) = e_a^j$, then $\Pi_j^t(\rho'_j, \rho_{-j}|h) = \delta e_{a,j}^j = \delta\epsilon_2$; if j followed ρ_j, $\Pi_j^t(\rho|h) = e_{a,j}^k \geq \epsilon_2 > \delta\epsilon_2 = \Pi_j^t(\rho'_j, \rho_{-j}|h)$. In the other case, when j accepts improperly $h^{t,1} \neq r(h^-)$, the other replier rejected it and delayed the agreement to next period, the state changed to e_a^j penalizing j for the proposal's acceptance, $\Pi_j^t(\rho'_j, \rho_{-j}|h) = \delta e_{a,j}^j = \delta\epsilon_2$. If j had rejected the proposition the state would change to $e^{i(h)}$ and the payment $\Pi_j^t(\rho|h) = \delta e_j^{i(h)} \geq \delta\epsilon_1$, and $\Pi_j^t(\rho'_j, \rho_{-j}|h) < \Pi_j^t(\rho|h)$.

$\boldsymbol{r(h) = e^0}$, if $i = i(h)$ opts for a $OSD(\rho_i, h)$, the proposition is not e^0, repliers refuse it, the state changes to e^i, the agreement is delayed to $t+1$, and i is harmed $\Pi_i^t(\rho'_i, \rho_{-i}|h) = \delta\Pi_i^t(\rho|h^+) = \delta\epsilon_1 < \epsilon_1 < \min_{k\in I} e_k^0 \leq e_i^0 = \Pi_i^t(\rho|h)$.

For the repliers there are, once more, two hypothesis for OSD in pure strategies, when $h^{t,1} = r(h^-)$ with $\rho'_j(0|h) = 1$, or when $h^{t,1} \neq r(h^-)$ and $\rho'_j(1|h) = 1$. In the first case the proposition is refused by the other repliers and j is penalized on the deal agreed in the next period $\Pi_j^t(\rho'_j, \rho_{-j}|h) = \delta e_{a,j}^j = \delta\epsilon_2 < e_j^0 = \Pi_j^t(\rho|h)$, j is worse off. On the second case j accepts a proposal that is refused, the agreement obtained in the next moment is e_a^j where j is clearly worse than in $e^{i(h)}$, $\Pi_j^t(\rho'_j, \rho_{-j}|h) = \delta\Pi_j^t(\rho|h^+) = \delta\epsilon_2 < \delta\epsilon_1 \leq \delta e_j^{i(h)} = \Pi_j^t(\rho|h)$. □

4.3 All divisions are a PE outcome

So far we provided strategies that establish as agreement outcomes only elements of the set $\left\{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^n x_i \leq 1, x_i > 0\right\}$. The next strategy using an out off equilibrium incentive mechanism for players that follow it establishes that all possible divisions in Δ are *Trembling Hand* equilibrium outcomes.

For that consider the set of states $E = \bigcup_{i \neq j} \{e^i, e^{i,j}, \bar{e}^{i,j}\}$, where $e^i \in \Delta$ are as previously defined and the new states $e^{ij} \in \Delta$ are: $e^{ij}_i = \gamma_1$, $e^{ij}_j = \gamma_2$, and $e^{ij}_k = 0$, for $k \notin \{i,j\}$, for example $e^{31} = (\gamma_2, 0, \gamma_1)$; $\bar{e}^{ij}_i = \gamma_3$, $\bar{e}^{ij}_j = \gamma_4$, and $\bar{e}^{ij}_k = 0$, for $k \notin \{i,j\}$, so, for example, $\bar{e}^{23} = \big(0, \gamma_3, \gamma_4\big)$. For each history h there is a state $r(h) \in E$. The strategy for $h \in H^{t-1,2}$ is for the proponent to always propose a division equal to the state $s_{i(h)}(h) = r(h)$; for $h \in H^{t,1}$ the player $j \in -i(h)$ accepts if the proposal was equal to the state and rejects otherwise.

$$s_j(h) = \begin{cases} 1, \text{ if } h^{t,1} = r(h^-) \\ 0, \text{ if } h^{t,1} \neq r(h^-) \end{cases}$$

To define the state transition we need to use a function from history to the set of subsets of player $g(h) : H^2 \to 2^I$, that tracks which players moved as defined in s at the last moment $h^t = (h^{t,1}, h^{t,2})$.

$$g(h) = \Big\{ i \in I : \big(i \neq i(h) \text{ and } s_i(h^{|t,1}) = h^{t,2}_i \big) \text{ or } \big(i = i(h) \text{ and } s_i(h^{|t-1,2}) = h^{t,1} \big) \Big\}$$

When all players follow s the agreement is immediate, the proponent plays $r(h)$ and both repliers accept it, so if h was not an ending history, some of the players did not play according to the strategy and either the proponent or at least one replier deviated. Therefore there is an impossibility of $g(h) = I$ in a non ending history h. That is, a history with $h^{t,2} \neq (1,1)$ must have $g(h) \neq I$.

There is an order for the players at each moment of time determined by the next moment that the players propose. Define at each t and for each player i, $t_i = \min\{\tilde{t} : \tilde{t} > t \text{ and } \tilde{t} \in t(i)\}$, and we say i proposes before j at t, $i \prec_t j$, if $t_i < t_j$. Take $\bar{g}(h)$ to be the ordered pair with the same elements of $g(h)$ ordered by $\prec_{t(h)}$. One example, if $g(h) = \{1,3\}$ and $t(h) = 4$ the next proponent is player 2, then player 3 followed by 1, so $3 \prec_4 1$ and $\bar{g}(h) = (3,1)$.

Transition occurs only after the voting stage, so if $\tau(h) = (t,2)$, $r(h) = r(h^-)$.

For $h = (t, 2)$[11]

$$r(h) = \begin{cases} r(h^{|t,1}), & \text{if } g(h) = \emptyset \\ e^{\bar{g}(h)}, & \text{if } i(h) \notin g(h) \\ \bar{e}^{\bar{g}(h)}, & \text{if } i(h) \in g(h) \end{cases}$$

Players that did not follow the strategy are punished by receiving zero in the next state. A player is willing to accept (or propose) 0 based on the possibility of the other player making a mistake, and in that case, the well behaved player receives a premium. The proof that this strategy is a *Trembling Hand* equilibrium is fastidious and cumbersome and left for the appendix.

Theorem 10. *The strategy s is a* PE[12]*.*

5 Conclusion

The present work serves the purpose of refining the equilibrium theory of the multi-player bargaining. After introducing common *SPNE* equilibrium strategies the following is an attempt of creating a sound equilibrium refinement of repeated with continuum of action games. It proves that none of the classical equilibria resists minor refinements, but that using more complex strategies it is possible to sustain any division as an equilibrium outcome.

[11] For notation convenience on the definition of $r(h)$ let $\bar{e}^i = e^i$

[12] The strategy defined is *PER* as well, and all points in Δ are also *PER* outcomes. In fact a simpler strategy can sustain any division as a *PER*, one with less states, in particular with half the states of s, with $\gamma_1 = \gamma_3 = 2/3$ and $\gamma_2 = \gamma_4 = 1/3$

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Appendices

A

A.1 Equilibria with $e(s) \neq 1$

Theorem 3. $\forall e^0 \in \Delta, T \in \mathbb{N}$, *exists* $s \in S$ SPNE*with* $e(s) = T$ *and* $d(s) = e^0$

Proof. For any $e^0 \in \Delta$ and $T \in \mathbb{N}$, $r(h)$ defines the state for any history, there is one more state than in Haller's strategy, the set of states is $E = \left\{\delta^{T+1}e^0, e^0, e^1, e^2, e^3\right\}$. For the initial history the state is $r(\emptyset) = \delta^{T+1}e^0$; when $t > 0$ and $h \in H^{t,2}$ is $r(h) = r(h^-)$; and for $h \in H^{t,1}$ is

$$r(h) = \begin{cases} r(h^-) & \text{if } h^{t,1} = r(h^-) \text{ and } t \neq T-1 \\ e^0 & \text{if } h^{t,1} = r(h^-) \text{ and } t = T-1 \\ e^{i(t+1)} & \text{if } h^{t,1} \neq r(h) \end{cases}$$

The strategy is like in Haller's to propose a division equal to the state $s_{i(h)}(h) = r(h)$ with $\tau(h) = (t-1, 2)$, and when $\tau(h) = (t, 1)$ repliers $j \neq i(h)$ follow

$$s_j(h) = \begin{cases} 1 & \text{if } h^{t,1} = r(h) \text{ and } t \geq T \\ 0 & \text{if } h^{t,1} \neq r^{t-1}(h) \text{ or } t < T \end{cases}$$

We need to prove two distinct points, first that $e(s) = T$ and $d(s) = e^0$; second that s is SPNE. The first result is relatively straightforward lets define ϖ_s as the history path when the strategy s was played since the beginning. If all players act accordingly to s, $s_j(h) = 0$ for all repliers $j \neq i(h)$ and history h with $t(h) < T$, then the time of agreement must be $t(s) \geq T$. At the stage $(T-1, 1)$, $r(\varpi_s^{|T-1,1}) = \delta^{T+1}e^0$, proposition is done, and according to the transition state function the state changes to e^0 and repliers both reject the proposition. At time T proposition will be e^0, considering $i = i(T)$, $s_i(\varpi_s^{|T-1,2}) = e^0 = r(\varpi_s^{|T-1,1})$, repliers will accept $s_j(\psi^{|T,1}) = 1$. The agreement is established at $t(s) = T$ and the division reached $d(s) = e^0$.

s **is SPNE.** If there was some deviation on propositions the strategy enters in a punishment scheme, meaning that $r(h) \in \left\{e^1, e^2, e^3\right\}$ and at these states it just replicate Haller's strategy, which we already proved is an SPNE. If a history h has $t(h) \geq T$ the strategy is just equal to Haller's so it respects $SPNE$condition. To prove s is an $SPNE$it is only missing that s is the best option for histories with $t(h) < T$ and in which the players did not deviate, $h = \psi^{\tau(h)}$. If the proponent $i = i(h)$ does not deviate is payment is $\Pi_j^t(s|h) = \delta^{T-t}e_i^0$ if he proposes something different, both repliers reject the proposal and the game enters in a punishment of i $\Pi_j^t(s_i', s_{-i}|h) = \delta\Pi_j^{t+1}(s|h^+) = e_i^k = 0$. The proponent does not improve. If $j \neq i(h)$, $\tau(h) = (t,1)$ and $h^{t,1} = r(h)$ the agreement will be reached in $T-t$ periods, the payment of following s is $\Pi_j^t(s|h) = \delta^{T-t}e_i^0$. When j plays $s_j' \in OSD(s_j, h)$, $s_j'(h) = 1$ player j contradicts s accepting the proposal, but replier k still rejects it and so, with $h^+ = \big(h, (0,1)\big)$, $\Pi_j^t(s_j', s_{-j}|h) = \delta\Pi_j^{t+1}(s|h^+) = \delta\left[\delta^{T-t-1}e^0\right] = \Pi_j^t(s|h)$. Player j receives the same, not improving by changing strategy. $\square$

A.2 Equilibria with $e(s) = \infty$

Theorem 4. *There is a strategy $s \in S$ SPNEin which no division is agreed upon and $e(\sigma) = \infty$.*

Proof. This strategy involves players that are not interested in bargaining, they want all or nothing, this way they always propose everything to themselves and reject everything that is less than it. So, the strategy is the following, for any proponent $i = i(h)$, $s_i(h) = e^i$. For any replier $j \neq i$, $s_j(h) = \begin{cases} 1 & \text{if } h^{t,1} = e_j \\ 0 & \text{if } h_j^{t,1} \neq e_j \end{cases}$. It is clear that no agreement can be reached in finite time, the replier j only accept e^j and the other replier, k, e^k, therefore they will never accept the same proposal, so $\forall h \in H$, $\Pi_i^t(s|h) = 0$ and by convention $t(s) = \infty$. We still need to prove that s is an SPNE. When $\boldsymbol{\tau(h) = (t-1, 2)}$, whatever the proposal $s_i' \in OSD(s_i, h)$ of $i = i(h)$ it will be always rejected by one of the repliers, and $i(h)$ payment does not increase by using it. So, whatever the proposition $\bar{h}^{t,1} = s_i'(h)$, $\Pi_i^t(s_i', s_{-i}|h) = \delta\Pi_i^{t+1}(s|\bar{h}) = 0$.

When $\boldsymbol{\tau(h) = (t,1)}$, the replier $j \neq i(h)$ cannot improve his payment. If under s he rejected the proposal, $s_j(h) = 0$ on the alternative strategy $s'_j \in OSD(s_j, h)$ he must accept it, $1 = s'_j(h)$. The payment of j is: $h_j^{t,1}$ if k accepted the proposition; and is $\delta\Pi_j^{t+1}(s|h^+) = 0$, if k rejected it, with $h^+ = (h,(1,0))$,. So j's payment is $\Pi_j^t(s'_j, s_{-j}|h) = s_k(h)h_j^{t,1} + (1 - s_k(h))\,\delta\Pi_j^{t+1}(s|h^+) = s_k(h)h_j^{t,1}$. But $s_k(h) = 1$, only when $h^{t,1} = e^k$, meaning that $h_j^{t,1} = e_j^k = 0$, and $\Pi_j^t(s'_j, s_{-j}|h) = 0$. Replier cannot improve by accepting when before he was rejecting. If under s the replier accepted the proposal, that meant the proposal was $h^{t,1} = e^j$, but the other replier rejected it, so $\Pi_j^t(s|h) = \delta\Pi_j^{t+1}(s|h^+) = 0$. If $s'_j(h) = 0$ then nothing really changes the game goes to the next round and players will again try to get everything to themselves, so $\Pi_j^t(s'_j, s_{-j}|h) = \delta\Pi_j^{t+1}(s|h^+) = 0$ the change of reply does not improve replier's payoff. □

A.3 Herrero's strategy is not a *PE*

Theorem 7. Herrero*'s strategy is not a* PE*.*

Proof. Let σ be the Herrero's strategy, assume it is *PE* and σ^ϵ is the approximation sequence converging to it, $\sigma^\epsilon \xrightarrow{\epsilon\downarrow 0} \sigma$, with σ^ϵ having the properties of definition (6).

Claim 5. *When i is the proponent at h, $\exists\bar{\epsilon} > 0$, $\forall\epsilon < \bar{\epsilon}$, $\Pi_i^t(\sigma^\epsilon|h) > \epsilon$.*

Proof. Consider $\gamma^i \in \Delta$ is a vector with $\gamma_j^i = \gamma_0/2$ for $j \neq i$ and $\gamma_i^i = 1 - \gamma_0$, with $\gamma_0 > 0$. Define the strategy $\sigma'_i \in \Sigma_i^\epsilon$ in which i proposes with probability $1 - \epsilon$ the vector γ^i, and rejects any offer made with probability $1 - \epsilon$. So $\sigma'^{\epsilon}_i(x|h) = 1 - \epsilon$ when: $i(h) \neq i$ and $x = 0$; or when $i(h) = i$ and $x = \gamma^i$.

At the history h, if i proposes γ^i the state will be $r(h^+) = e^k$, with $h^+ = (h, \gamma^i)$, for $j \neq k$, $\gamma_0/2 > 0 = \Pi_j^t(\sigma|h)$; and for the replier k we have $\gamma_0/2 < \delta = \Pi_k^t(\sigma|h^+)$ then by theorem (6), for small ϵ

$$\sigma_j^\epsilon(1|h^+) = 1 - \epsilon \tag{3}$$

$$\sigma_k^\epsilon(1|h^+) = \epsilon \tag{4}$$

Defining $a_h \in \{0,1\}$ as $a_h = 0$ if an agreement is not reached in the stage that starts at h and $a_h = 1$ if an agreement is obtained. We can calculate $p(a_h = 0)$ as the probability of no agreement at $h \in H^{t,1}$ when $\left(\sigma'^{\epsilon}_i, \sigma^{\epsilon}_{-i}\right)$ is being played and i is the proponent $i(h) = i$.

$$
\begin{aligned}
p\big(a_h = 0\big) =& \int_{\bar{h}\in\Delta} \sigma^{\epsilon}_{-i}(R\big|h,\bar{h})\partial\big(\sigma'^{\epsilon}_i\big) \geq \int_{\bar{h}\in\Delta} \sigma^{\epsilon}_{-i}\big(\{0,1\}\times\{0\}\big|h,\bar{h}\big)\partial\big(\sigma'^{\epsilon}_i\big) \\
=& \int_{\bar{h}\in\Delta} \sigma^{\epsilon}_j\big(0\big|h,\bar{h}\big)\partial\big(\sigma'^{\epsilon}_i\big) \geq \sigma'^{\epsilon}_i\big(\gamma^i\big|h\big)\sigma^{\epsilon}_j\big(0\big|h,\gamma^i\big) \\
=&(1-\epsilon)^2
\end{aligned}
$$

Notice that the lower bound $p\big(a_h = 0\big) \geq (1-\epsilon)^2$ is independent of the history and for any t such that $i(t) = i$ we get that the probability of no agreement at t has the same lower bound, we can then define a_t in coherence with a_h and calculate $p\big(a_t = 0\big) \geq (1-\epsilon)^2$ if $i(t) = i$. The probability of agreement $p\big(a_h = 1\big) \geq \sigma'^{\epsilon}_i\big(\gamma^i\big|h\big)\sigma^{\epsilon}_j\big(1\big|h,\gamma^i\big)\sigma^{\epsilon}_k\big(1\big|h,\gamma^i\big) = (1-\epsilon)^2\epsilon$, and $p\big(a_t = 1\big) \geq (1-\epsilon)^2\epsilon$, with $i(t) \neq i$.

If the proponent is $i(h) = j \neq i$ then the probability of no agreement in the moment after h, must be $p\big(a_h = 0\big) \geq 1-\epsilon$ because in σ'^{ϵ}_i player i refuse any proposition with probability $1-\epsilon$.

$$
\begin{aligned}
p\big(a_h = 0\big) =& \int_{\bar{h}\in\Delta} \sigma'^{\epsilon}_{-j}(R\big|h,\bar{h})\partial\big(\sigma^{\epsilon}_j\big) \geq \int_{\bar{h}\in\Delta} \sigma'^{\epsilon}_i(0\big|h,\bar{h})\partial\big(\sigma^{\epsilon}_j\big) \\
=& \int_{\bar{h}\in\Delta} \Big(1-\epsilon\Big)\partial\big(\sigma^{\epsilon}_j\big) = 1-\epsilon
\end{aligned}
$$

In this case, it happens as when $i(h) = i$, the calculations to find the lower bound do not depend on the specific history and therefore $p(a_t = 0) \geq 1-\epsilon$. Define q_t as the probability of not obtaining an agreement on the round that starts at t, *i.e.* q_t is the probability a disagreement is obtained in $t, t+1, t+2$, $q_t = p(a_t = 0)p(a_{t+1} = 0)p(a_{t+2} = 0) \geq (1-\epsilon)^2(1-\epsilon)(1-\epsilon) = \eta_1$.

Denoting P_{t+k} by the probability an agreement of γ is reached at $t+k$. If at

t, $i(t) = i$, then the probability the agreement is immediate is $P_t = p(a_t = 1) \geq \sigma_i'^{\epsilon}(\gamma|h)\sigma_j^{\epsilon}(1|h,\gamma)\sigma_k^{\epsilon}(1|h,\gamma) = (1-\epsilon)^2\epsilon = \eta_0$; for an agreement to be delayed until $t+3$, then no agreement can happen in $t, t+1$ or $t+2$, so $P_{t+3} = q_t p(a_{t+3} = 1) \geq \eta_1\eta_0$; for P_{t+3k} we must have no agreement in any round starting at $t+3\tau$, $0 \leq \tau < k$, therefore $P_{t+3k} = \prod_{\tau=0}^{k-1} q_{t+\tau} p(a_{t+3k} = 1) \geq \eta_1^k\eta_0$. Now we can calculate a lower bound for the payment of player i under the strategy $\sigma_i'^{\epsilon}$

$$\begin{aligned}
\Pi_i^t(\sigma'^{\epsilon}|h) =& \sum_{k=0}^{+\infty} \delta^k \int_{\bar{h}\in\bar{\Delta}^k} \pi(h,\bar{h})_i \partial\big({}_k\sigma'^{\epsilon}_h\big) \geq \sum_{k\in t(i)} \delta^k \int_{\bar{h}\in\bar{\Delta}^k} \pi(h,\bar{h})_i \partial\big({}_k\sigma'^{\epsilon}_h\big) \\
=& \sum_{k=0}^{+\infty} \delta^{3k} \int_{\bar{h}\in\bar{\Delta}^{3k}} \pi(h,\bar{h})_i \partial\big({}_{3k}\sigma'^{\epsilon}_h\big) \geq \sum_{k=0}^{+\infty} \delta^{3k}(1-\gamma)P_{t+3k} \\
\geq& \sum_{k=0}^{+\infty} \delta^{3k}(1-\gamma)\eta_1^k\eta_0 = \frac{(1-\gamma)\eta_0}{1-\delta^3\eta_1} = \frac{(1-\gamma)(1-\epsilon)^2}{1-\delta^3(1-\epsilon)^4}\epsilon
\end{aligned}$$

As $\frac{(1-\gamma)(1-\epsilon)^2}{1-\delta^3(1-\epsilon)^4} \to \frac{1-\gamma}{1-\delta^3}$. If $\gamma < \delta^3$, $\Pi_i^t(\sigma_i', \sigma_{-i}^{\epsilon}|h) > \epsilon$ for small values of ϵ.

□

For σ to be an PE it must be an accumulation point of a sequence of approximating games, consider the ϵ approximating game and the σ^{ϵ} equilibrium. We will look for a particular history in which no convergent σ^{ϵ} to σ is simultaneously the best reply. For that consider the history h with $\tau = (t,1)$, in which player 1 was the proponent, $i(h) = 1$, and the proposition done was equal to the state $h^{t,1} = r(h^-) = e^3$; in this specific history player 3, by theorem (6), accepts with maximum probability, $1-\epsilon$.

For player 2 consider the strategies $\sigma_2^a, \sigma_2^r \in OSD(\sigma_2^{\epsilon}, h)$ in which $\sigma_2^a(1|h) = 1$, $\sigma_2^r(0|h) = 1$. For σ to be a PE, for small ϵ, $\Pi_2^t(\sigma_2^a, \sigma_{-2}^{\epsilon}|h) \geq \Pi_2^t(\sigma_2^r, \sigma_{-2}^{\epsilon}|h)$ To simplify the following formulas we write $\Pi_2^{t+1}\Big(\sigma^{\epsilon}\Big|h, r\Big) = \Pi_{\epsilon}^r$

If player 2 accepts the proposition his payment is

$$\begin{aligned}
\Pi_2^t(\sigma_i^a, \sigma_{-i}^{\epsilon}|h) =& \delta\sigma_3^{\epsilon}(0|h)\Pi_2^{t+1}\Big(\sigma^{\epsilon}\Big|h,(1,0)\Big) + \delta\sigma_3^{\epsilon}(1|h)e_2^3 \\
=& \delta\epsilon\Pi_2^{t+1}\Big(\sigma^{\epsilon}\Big|h,(1,0)\Big) = \delta\epsilon\Pi_{\epsilon}^{1,0}
\end{aligned}$$

If 2 rejects his payment is

$$
\begin{aligned}
\Pi_2^t(\sigma_i^r, \sigma_{-i}^\epsilon|h)) =&\delta\sigma_3^\epsilon(0|h)\Pi_2^{t+1}\Big(\sigma^\epsilon\Big|h,(0,0)\Big) + \delta\sigma_3^\epsilon(1|h)\Pi_2^{t+1}\Big(\sigma^\epsilon\Big|h,(0,1)\Big)\\
=&\delta\epsilon\Pi_2^{t+1}\Big(\sigma^\epsilon\Big|h,(0,0)\Big) + \delta(1-\epsilon)\Pi_2^{t+1}\Big(\sigma^\epsilon\Big|h,(0,1)\Big)\\
=&\delta\epsilon\Pi_\epsilon^{0,0} + \delta(1-\epsilon)\Pi_\epsilon^{0,1}
\end{aligned}
$$

Rewriting the necessary inequality for σ to be PE,

$$
\begin{aligned}
\Pi_2^t\big(\sigma_2^a, \sigma_{-2}^\epsilon\big|h\big) \geq \Pi_2^t\big(\sigma_2^r, \sigma_{-2}^\epsilon\big|h\big) \Leftrightarrow \epsilon\Pi_\epsilon^{1,0} \geq \epsilon\Pi_\epsilon^{0,0} + (1-\epsilon)\Pi_\epsilon^{0,1}\\
\Leftrightarrow\epsilon\left(\Pi_\epsilon^{1,0} - \Pi_\epsilon^{0,0}\right) \geq (1-\epsilon)\Pi_\epsilon^{0,1} \Leftrightarrow \frac{\Pi_\epsilon^{1,0} - \Pi_\epsilon^{0,0}}{\Pi_\epsilon^{0,1}} \geq \frac{1-\epsilon}{\epsilon} \qquad (5)
\end{aligned}
$$

Because of:

- $\Pi_\epsilon^{1,0} = \Pi_2^{t+1}\Big(\sigma^\epsilon\Big|h,(e_3,1,0)\Big) < 1$;
- $\Pi_\epsilon^{0,0} = \Pi_2^{t+1}\Big(\sigma^\epsilon\Big|h,(e_3,0,0)\Big) > \epsilon$, because σ_i^ϵ must be a best reply to σ^ϵ at all histories h, then $\Pi_2^{t+1}\Big(\sigma^\epsilon\Big|h,(e_3,0,0)\Big) \geq \Pi_2^{t+1}\Big(\sigma'^\epsilon_2, \sigma_{-2}^\epsilon\Big|h,(e_3,0,0)\Big) > \epsilon$ by claim (5);
- $\Pi_\epsilon^{0,1} = \Pi_2^{t+1}\Big(\sigma^\epsilon\Big|h,(e_3,0,1)\Big) > \epsilon$ by the same reason as previous point.

the inequality (5) can't be verified, σ is not a *PE*. □

A.4 Dominating Strategy is Trembling Hand

Claim 3. *When* $\inf\Big\{\Pi_i^t(\sigma|h) - \Pi_i^t(\sigma_i', \sigma_{-i}|h) : \forall h \in H, \forall\sigma_i' \in OSD(\sigma_i,h)\Big\} > 0$ *then* σ *is* PE *and a* PER.

Proof. This proof is tailored for the *PE*, but it's adaptation for the existence of *PER* is immediate and direct. First we will prove that the strategy σ to be in the theorem conditions need to be simple, that is $\forall h \in H$, $\exists\{a\} \in A_h$ such that $\sigma_i(a|h) = 1$

Lemma 1. *If* $\Pi_i^t(\sigma|h) > \Pi_i^t(\sigma_i', \sigma_{-i}|h)$, $\forall\sigma_i' \in OSD(\sigma_i,h)$ *then* $\sigma_i(.|h)$ *is a simple strategy.*

Proof. If $\sigma_i(.|h)$ is not a simple action, there exists a $A \subseteq A_h$, such that $\sigma_i(A|h) > 0$ and $\sigma_i(A^c|h) > 0$, where $A^c = A_h \setminus A$. Define $\sigma_i'(O|h) = \frac{\sigma_i(O \cap A|h)}{\sigma_i(A|h)}$ and $\sigma_i''(O|h) = \frac{\sigma_i(O \cap A^c|h)}{\sigma_i(A^c|h)}$.

$$\begin{aligned}
\Pi_i^t(\sigma|h) = &\int_{a\in A_h} \Pi_i^t(a, \sigma_{-h}|h) \partial \sigma_i(.|h) \\
=&\sigma(A) \int_{a\in A} \Pi_i^t(a, \sigma_{-h}|h) \partial \left(\frac{\sigma_i(.|h)}{\sigma(A)} \right) + \sigma(A^c) \int_{a\in A^c} \Pi_i^t(a, \sigma_{-h}|h) \partial \left(\frac{\sigma_i(.|h)}{\sigma(A^c)} \right) \\
=&\sigma(A) \int_{a\in A} \Pi_i^t(a, \sigma_{-h}|h) \partial \sigma_i'(.|h) + \sigma(A^c) \int_{a\in A^c} \Pi_i^t(a, \sigma_{-h}|h) \partial \sigma_i''(.|h) \\
=&\sigma(A) \int_{a\in A_h} \Pi_i^t(a, \sigma_{-h}|h) \partial \sigma_i'(.|h) + \sigma(A^c) \int_{a\in A_h} \Pi_i^t(a, \sigma_{-h}|h) \partial \sigma_i''(.|h) \\
=&\sigma(A) \Pi_i^t(\sigma_i', \sigma_{-i}|h) + \sigma(A^c) \Pi_i^t(\sigma_i'', \sigma_{-i}|h)
\end{aligned}$$

By hypothesis $\Pi_i^t(\sigma|h) > \Pi_i^t(\sigma_i', \sigma_{-i}|h)$ and $\Pi_i^t(\sigma|h) > \Pi_i^t(\sigma_i'', \sigma_{-i}|h)$ but as $\Pi_i^t(\sigma|h) = \sigma_i(A|h)\Pi_i^t(\sigma'|h) + \sigma_i(A^c|h)\Pi_i^t(\sigma''|h)$ we would conclude that $\Pi_i^t(\sigma|h) > \Pi_i^t(\sigma|h)$. So $\sigma_i(.|h)$ must be simple. □

For each $\epsilon > 0$ define $\sigma_i^\epsilon(O|h) = (1-\epsilon)\sigma_i(O|h) + \epsilon\lambda_i(O|h)$, if $h \in H^{t,2}$, with $\lambda(\cdot)$ the measure proportional to Lebesgue measure; and $\sigma_i^\epsilon(O|h) = (1-\epsilon)\sigma_i(O|h) + \epsilon\lambda_i(O|h)$ in which $\lambda_i(O|h) = \frac{|O|}{2}$, $O \subseteq \{0, 1\}$. It is clear that $\sigma^\epsilon(O|h) \to \sigma_i(O|h)$ and to be a PE we just need to insure it is the best reply. Next result proves that there is an absolute convergence of $\Pi_i^t(\sigma^\epsilon|h)$ to $\Pi_i^t(\sigma|h)$ in h.

Lemma 2. *$\forall \xi > 0$, $\exists \bar{\epsilon} > 0$, that $\forall h \in H$, and $\forall \epsilon < \bar{\epsilon}$, $|\Pi_i^t(\sigma^\epsilon|h) - \Pi_i^t(\sigma|h)| < \xi$*

Proof. By the previous lemma (1) σ is simple, accordingly it is possibly to define the path after h when σ is played $\varpi_\sigma(h) = \{h, h_\sigma^0, h_\sigma^1, \ldots\}$, ($\varpi_\sigma(h)$ can be finite). Without loss of generality we suppose player 1 is proposing, $h_\sigma^0 = (h_{\sigma,1}^0, h_{\sigma,2}^0, h_{\sigma,3}^0)$, and

$$\sigma^\epsilon(h_\sigma^0|h) = \sigma_1^\epsilon(h_{\sigma,1}^0|h)\sigma_2^\epsilon(h_{\sigma,2}^0|h, h_{\sigma,1}^0)\sigma_3^\epsilon(h_{\sigma,3}^0|h, h_{\sigma,1}^0)$$

Due to $\sigma_i^\epsilon \geq (1-\epsilon)\sigma_i$,

$$\sigma^\epsilon(h_\sigma^0|h) \geq \left[(1-\epsilon)\sigma_1(h_{\sigma,1}^0|h)\right]\left[(1-\epsilon)\sigma_2(h_{\sigma,2}^0|h, h_{\sigma,2}^0)\right]\left[(1-\epsilon)\sigma_3(h_{\sigma,3}^0|h, h_{\sigma,1}^0)\right] = (1-\epsilon)^3$$

we are only considering when the players follow σ, which happens for each player with probability $1-\epsilon$. Then ${}_1\sigma^{\epsilon}(h^0_\sigma|h) \geq (1-\epsilon)^3$, and

$$\begin{aligned}\int_{\bar{h}\in\bar{\Delta}} \pi(h,\bar{h})\partial\big({}_1\bar{\sigma}^{\epsilon}_h\big) &= \pi(h,h^0_\sigma)(1-\epsilon)^3 + \int_{\bar{h}\in\bar{\Delta}} \pi(h,\bar{h})\partial\big({}_1\tilde{\sigma}^{\epsilon}_h\big) \\ &= \pi(h,h^0_\sigma)(1-\epsilon)^3 + R_1\end{aligned}$$

Where ${}_1\tilde{\sigma}^{\epsilon}_h(O) = {}_1\sigma^{\epsilon}_h(O) - (1-\epsilon)^3\chi_{h^0_\sigma}(O)$, for any $O \in \mathfrak{B}(\bar{\Delta})$, and $R_1 = \int_{\bar{h}\in\bar{\Delta}} \pi(h,\bar{h})\partial\big({}_1\tilde{\sigma}^{\epsilon}_h\big)$. The next moment if σ is played after h is the triple of actions h^1_σ, the probability of observing (h^0_σ, h^1_σ), when σ^ϵ is played is ${}_2\sigma^{\epsilon}(h^0_\sigma, h^1_\sigma|h) \geq {}_1\sigma^{\epsilon}(h^0_\sigma|h)\sigma^{\epsilon}(h^1_\sigma|h, h^0_\sigma)$, by the same reasoning as before $\sigma^{\epsilon}(h^1_\sigma|h,h^0_\sigma) \geq (1-\epsilon)^3$, and ${}_2\sigma^{\epsilon}(h^1_\sigma|h) \geq (1-\epsilon)^6$. Hence,

$$\begin{aligned}\int_{\bar{h}\in\bar{\Delta}^2} \pi(h,\bar{h})\partial\big({}_2\bar{\sigma}^{\epsilon}_h\big) &= \pi(h,h^0_\sigma,h^1_\sigma)\,{}_2\bar{\sigma}^{\epsilon}(h^0_\sigma,h^1_\sigma|h) + \int_{\bar{h}\in\bar{\Delta}^2} \pi(h,\bar{h})\partial\big({}_2\tilde{\sigma}^{\epsilon}_h\big) \\ &= \pi(h,h^0_\sigma,h^1_\sigma)(1-\epsilon)^6 + R_2\end{aligned}$$

With the natural definition for $R_2 = \int_{\bar{h}\in\bar{\Delta}^2} \pi(h,\bar{h})\partial\big({}_2\tilde{\sigma}^{\epsilon}_h\big)$ and ${}_2\tilde{\sigma}^{\epsilon}_h(O) = {}_2\sigma^{\epsilon}_h(O) - (1-\epsilon)^6\chi_{(h^0_\sigma,h^1_\sigma)}(O)$ for any $O \in \mathfrak{B}(\bar{\Delta}^2)$.Abusing slightly on the notation, defining $h^{|k}_\sigma = (h, h^0_\sigma, \dots h^k_\sigma)$ and developing the previous calculations for all the moments, the payment when σ^ϵ is played is $\Pi^t_i(\sigma^\epsilon|h) = \sum_k \delta^k \int_{\bar{h}\in\bar{\Delta}^k} \pi(h,\bar{h})\partial\big({}_k\sigma^{\epsilon}_h\big) = \sum_k \delta^k \Big[\pi(h^{|k}_\sigma)(1-\epsilon)^{3k} + R_k\Big]$. With $R_k = \int_{\bar{h}\in\bar{\Delta}^k} \pi(h,\bar{h})\partial\big({}_k\tilde{\sigma}^{\epsilon}_h\big) \leq \int_{\bar{h}\in\bar{\Delta}^k} 1\partial\big({}_k\tilde{\sigma}^{\epsilon}_h\big) = {}_k\bar{\sigma}^{\epsilon}_h(\tilde{\Delta}^k) = 1-(1-\epsilon)^{3k}$. We may write $\Pi^t_i(\sigma|h) = \sum_k \delta^k\pi(h^{|k}_\sigma)$ consequently

$$\begin{aligned}\Big|\Pi^t_i(\sigma|h) - \Pi^t_i(\sigma^\epsilon|h)\Big| &\leq \Big|\sum_k \delta^k \pi(h^{|k}_\sigma)\Big(1-(1-\epsilon)^{3k}\Big)\Big| + \sum_k \delta^k R_k \\ &\leq \sum_k \delta^k\Big(1-(1-\epsilon)^{3k}\Big) + \sum_k \delta^k\Big(1-(1-\epsilon)^{3k}\Big) \\ &= 2\sum_k \Big(\delta^k - \big(\delta(1-\epsilon)^3\big)^k\Big) = 2\frac{1}{1-\delta} - \frac{1}{1-\delta(1-\epsilon)^3} \\ &= 2\delta\frac{1-(1-\epsilon)^3}{(1-\delta)(1-\delta(1-\epsilon)^3)} \leq 2\delta\frac{3\epsilon-3\epsilon^2+\epsilon^3}{(1-\delta)^2} = f(\epsilon)\end{aligned}$$

Any ${\sigma'}_i^\epsilon \in OSD(\sigma_i^\epsilon, h)$ can be built as $\sigma'^\epsilon \in \Sigma^\epsilon$ can be written as ${\sigma'}_i^\epsilon(O|h) = (1-\epsilon){\sigma'}_i(O|h)+\epsilon\lambda_i(O|h)$, with ${\sigma'}_i \in OSD(\sigma_i, h)$ and $\lambda_i(O|h)$ defined as before. It can be proved in the same way as the previous claim that $\left|\Pi_i^t({\sigma'}_i^\epsilon, \sigma_{-i}^\epsilon|h) - \Pi_i^t({\sigma'}_i, \sigma_{-i}|h)\right| < f(\epsilon)$. □

Then with $a = \inf\left\{\Pi_i^t(\sigma|h) - \Pi_i^t(\sigma_i', \sigma_{-i}|h) : \forall h \in H, \forall \sigma_i' \in OSD(\sigma_i, h)\right\}$, if $\xi = \frac{a}{2}$, then it is possible to find ϵ such that, $\left|\Pi_i^t(\sigma|h) - \Pi_i^t(\sigma^\epsilon|h)\right| < \frac{a}{2}$ and $\left|\Pi_i^t(\sigma_i', \sigma_{-i}|h) - \Pi_i^t({\sigma'}_i^\epsilon, \sigma_{-i}^\epsilon|h)\right| < \frac{a}{2}$. Therefore $\Pi_i^t(\sigma^\epsilon|h) > \Pi_i^t(\sigma_i|h) - \frac{a}{2} \geq \Pi_i^t(\sigma_i', \sigma_{-i}|h) + \frac{a}{2} > \Pi_i^t({\sigma'}_i^\epsilon, \sigma_{-i}^\epsilon|h)$, and σ^ϵ is in fact the best reply for any h in Σ^ϵ. □

A.5 OSD Dominating Strategy is Dominating

Claim 4. *If $\Pi_i^t(s|h) > \Pi_i^t(s_i', s_{-i}|h)$, for all $s_i' \in OSD(s_i, h)$ and for all $h \in H$, then $\Pi_i^t(s|h) > \Pi_i^t(\bar{s}_i, s_{-i}|h)$, for all $\bar{s}_i \in \Sigma_i \setminus s_i$ and for all $h \in H$*

Proof. By lemma (1) the strategy s is a simple strategy. We will now see that s_i also strongly dominate all $\bar{s}_i \in S_i$ when s is being played. As s is a simple strategy we can define the sequence of future histories after h, $H(s|h)$, and has only one sequence. For the strategy $\bar{s}$ assume it is also simple and that the future play ends on finite time, therefore $(h, \bar{h}_1, \ldots, \bar{h}_{\overline{T}}) = H(\bar{s}|h)$, is the only path of the strategy $\bar{s}$ after h. The payment only depend on this path and so we can easily define that $\Pi_i^t(\bar{s}|h) = \delta^T\pi(h, \bar{h}_1, \ldots, \bar{h}_{\overline{T}})_i$; and, as $(h, h_1, \ldots, h_T) = H(s|h)$, $\Pi_i^t(s|h) = \delta^T\pi(h, h_1, \ldots, h_T)_i$.

Now supported on the strategy $\bar{s}$ we will construct $\overline{T}$ strategies that are equal to $\bar{s}$ at one stage of history and equal to s everywhere else. These new strategies will be either equal to s or $OSD(s, \cdot)$. For that purpose define $\tilde{h}_0 = h$, for $1 \leq k \leq \overline{T}$, $\tilde{h}_k = (h, \bar{h}_1, \ldots, \bar{h}_k)$ and the strategy $\tilde{s}^k$ for player i as

$$\tilde{s}_i^k(\tilde{h}) = \begin{cases} s_i(\tilde{h}) \textbf{ if } \tilde{h} \neq \tilde{h}_k \\ \bar{s}_i(\tilde{h}) \textbf{ if } \tilde{h} = \tilde{h}_k \end{cases}$$

So $\tilde{s}^k$ is in fact either a $OSD(s_i, \tilde{h}_k)$ or is equal to s_i everywhere, if $\bar{s}_i(\tilde{h}^k) = s_i(\tilde{h}^k)$. For the other players $j \neq i$ $\tilde{s}_j = \bar{s}_j = s_j$. Using these new strategies $\tilde{s}^k(\cdot)$ we can rewrite $\Pi_i^t(s|h) - \Pi_i^t(\bar{s}|h) = \Pi_i^t(s|\tilde{h}_0) - \Pi_i^t(\tilde{s}^0|\tilde{h}_0) + \Pi_i^t(\tilde{s}^0|\tilde{h}_0) - \Pi_i^t(\bar{s}|h)$, and repeating this procedure for $\Pi_i^t(\tilde{s}^1|\tilde{h}_1)$ we get

$$\begin{aligned}\Pi_i^t(s|h) - \Pi_i^t(\bar{s}|\tilde{h}_0) =& \left[\Pi_i^t(s|h) - \Pi_i^t(\tilde{s}^0|\tilde{h}_0)\right] + \left[\Pi_i^t(\tilde{s}^0|\tilde{h}_0) - \delta\Pi_i^{t+1}(\tilde{s}^1|\tilde{h}_1)\right] + \left[\delta\Pi_i^{t+1}(\tilde{s}^1|\tilde{h}_1) - \Pi_i^t(\bar{s}|h)\right]\\ =& \left[\Pi_i^t(s|h) - \Pi_i^t(\tilde{s}^0|\tilde{h}_0)\right] + \delta\left[\Pi_i^{t+1}(s|\tilde{h}_1) - \Pi_i^{t+1}(\tilde{s}^1|\tilde{h}_1)\right] + \left[\delta\Pi_i^t(\tilde{s}^1|\tilde{h}_1) - \Pi_i^t(\bar{s}|h)\right]\end{aligned}$$

For the equality we used that $\tilde{s}^0(\tilde{h}_0) = \bar{s}(\tilde{h}) = \bar{s}(\tilde{h}_0) = \bar{h}_1$, so if $\overline{T} > 1$, $\bar{h}_1$ is a non ending history, and $\Pi_i^t(\tilde{s}^0|\tilde{h}_0) = \delta\Pi_i^{t+1}(\tilde{s}^0|\tilde{h}_0, \bar{h}_1) = \delta\Pi_i^{t+1}(s|\tilde{h}_1)$, by definition $\tilde{s}^0$ is equal to s for all histories different from $\tilde{h}_0$. Repeating $\overline{T}$ times we get

$$\begin{aligned}\Pi_i^t(s|h) - \Pi_i^t(\bar{s}|h) =& \sum_{k=0}^{\overline{T}-1} \delta^k \left(\Pi_i^{t+k}(s|\tilde{h}_k) - \Pi_i^{t+k}(\tilde{s}^k|\tilde{h}_k)\right) + \delta^{\overline{T}-1}\Pi_i^{t+\overline{T}-1}\left(\tilde{s}^{\overline{T}-1}|\tilde{h}_{\overline{T}-1}\right) - \Pi_i^t(\bar{s}|h)\\ =& \sum_{k=0}^{\overline{T}-1} \delta^k \left(\Pi_i^{t+k}(s|\tilde{h}_k) - \Pi_i^{t+k}(\tilde{s}^k|\tilde{h}_k)\right)\end{aligned}$$

The equality results from $\tilde{s}^{\overline{T}-1}\left(\tilde{h}_{\overline{T}-1}\right) = \bar{s}(\tilde{h}_{\overline{T}-1}) = \bar{h}_T$, as this is an ending history, $\Pi_i^{t+\overline{T}-1}(\tilde{s}^{\overline{T}-1}\big|\tilde{h}_{\overline{T}-1}) = \delta\pi\left(\tilde{h}_{\overline{T}-1}, \bar{h}_T\right)$, so $\delta^{\overline{T}-1}\Pi_i^{t+\overline{T}-1}\left(\tilde{s}^{\overline{T}-1}|\tilde{h}_{\overline{T}-1}\right) = \Pi_i^t(\bar{s}|h)$. As s_i and $\bar{s}_i$ are different, at least one of $\tilde{s}_i^k$ is different from s_i and $\tilde{s}_i^k \in OSD(s_i, \tilde{h}_k)$, by hypothesis $\Pi_i^{t+k}(s|\tilde{h}_k) > \Pi_i^{t+k}(\tilde{s}^k|\tilde{h}_k)$, for the others we know $\Pi_i^{t+k}(s|\tilde{h}_k) = \Pi_i^{t+k}(\tilde{s}^k|\tilde{h}_k)$ and we conclude $\Pi_i^t(s|h) - \Pi_i^t(\bar{s}|h) > 0$.

If $H(\bar{s}|h)$ is of infinite size, let the size of history h be $t = t(h)$ and τ be the size of the first history in $H(\bar{s}|h)$ in which the strategies s and $\bar{s}$ define different actions, so $\bar{s}(\tilde{h}_\tau) \neq s(\tilde{h}_\tau)$, and $\bar{s}(\tilde{h}_{\tau'}) = s(\tilde{h}_{\tau'})$, for any $0 \leq \tau' < \tau$. Set $\epsilon = \Pi_i^{t+\tau}(s|\tilde{h}_\tau) - \Pi_i^{t+\tau}(\tilde{s}^\tau\big|\tilde{h}_\tau)$, $\epsilon > 0$ due to $\tilde{s}_i^\tau \in OSD(s_i, \tilde{h}_\tau)$ and the hypothesis that s_i is strictly better than all *OSD* at the departure history. The first $t + \overline{T}$ moments of $H(\bar{s}|h)$ are $\tilde{h}_{\overline{T}}$, if $\overline{T}$ is such that $\delta^{\overline{T}-\tau} < \epsilon$. Developing the same calculations as before

$$\Pi_i^t(s|h) - \Pi_i^t(\bar{s}|h) = \sum_{k=0}^{\overline{T}-1} \delta^k \left(\Pi_i^{t+k}(s|\tilde{h}_k) - \Pi_i^{t+k}(\tilde{s}^k|\tilde{h}_k)\right) + \delta^{\overline{T}-1}\Pi_i^{t+\overline{T}-1}\left(\tilde{s}^{\overline{T}-1}|\tilde{h}_{\overline{T}-1}\right) - \Pi_i^t(\bar{s}|h)$$

$$\geq \delta^{\tau}\epsilon + \delta^{\overline{T}-1}\Pi_i^{t+\overline{T}-1}\left(\tilde{s}^{\overline{T}-1}|\tilde{h}_{\overline{T}-1}\right) - \Pi_i^t(\bar{s}|h) > 0$$

□

A.6 All divisions are a PE outcome

Theorem 10. *The strategy s is a* PE.

Remember that the strategy is

$$s_j(h) = \begin{cases} 1, \text{ if } h^{t,1} = r(h^-) \\ 0, \text{ if } h^{t,1} \neq r(h^-) \end{cases}$$

with $g(h)$ the set of player that did not follow the strategy

$$g(h) = \left\{ i \in I : \left(i \neq i(h) \text{ and } s_i(h^{|t,1}) \neq h_i^{t,2}\right) \text{ or } \left(i = i(h) \text{ and } s_i(h^{|t-1,2}) \neq h^{t,1}\right) \right\}$$

the state transition is

$$r(h) = \begin{cases} r(h^{|t,1}), & \text{if } g(h) = I \\ e^{\bar{g}(h)}, & \text{if } i(h) \notin g(h) \\ \bar{e}^{\bar{g}(h)}, & \text{if } i(h) \in g(h) \neq I \end{cases}$$

For s to be a *PE* there must exist a sequence of approximating strategies s^ϵ, with $s^\epsilon \xrightarrow{\epsilon\downarrow 0} s$. This strategy is a totally mixed strategy, and in replies both possibilities assume positive probability, but the action that doesn't coincide with s is played only with ϵ probability, so for $j \neq i(h)$ and $h \in H^{t,1}$

$$s_j^\epsilon(1|h) = \begin{cases} 1-\epsilon, & \text{if } h^{t,1} = r(h^{|t-1,2}) \\ \epsilon, & \text{if } h^{t,1} \neq r(h^{|t-1,2}) \end{cases}$$

For $h \in H^{t-1,2}$, we assume that in $s_{i(h)}^\epsilon(h)$, $i(h)$ plays $r(h)$ with probability $1-\epsilon$ and has a uniform distribution on Δ, $s_{i(h)}^\epsilon(O|h) = \epsilon\lambda(O)$. It is clear that $s^\epsilon \to s$, and for

s to be a PE we need prove that s^ϵ is a best reply.

Before calculating the payment there are some facts about s^ϵ that facilitate the job. Note that the strategy, as function of h, only depends on the state of history $e = r(h)$, so it could be defined $s^\epsilon(a|h) = s^\epsilon(a|e)$ for $a \in A_h$. When s^ϵ is played after h the future states are determined by the initial state $r(h) = e$, the actions taken (dependent on $s^\epsilon(a|e)$) and by the proponent at h, $i(h)$. So for two different histories h and $\tilde{h}$ if they share the proponent $i(h) = i(\tilde{h})$ and the state $r(h) = r(\tilde{h})$ then the future play will have the same distribution, *i.e.* ${}_k s^\epsilon_h = {}_k s^\epsilon_{\tilde{h}}$ for all $k \in \mathbb{N}$. By this reason the future payment is the same at h and at $\tilde{h}$, $\Pi_i^{t(h)}(s^\epsilon|h) = \Pi_i^{t(\tilde{h})}\big(s^\epsilon|\tilde{h}\big)$. Therefore, we can define equivalent classes of histories where the future payment is the same if the strategy s^ϵ is played. For $e \in E$ and $i \in I$ define the classes $[e, k] = \big\{h \in H : r(h) = e \text{ and } i(h) = k\big\}$.

Without loss of generality we focus on player 1 and for notation simplicity define $\Pi_1^{t(h)}(s^\epsilon|h) = \Pi_e^k$, if $h \in [e, k]$. When all players follow s^ϵ, 1 is the proponent, p the proposal and e the state, 1's payment Π_e^1 is composed of several parcels presented in the following table.

Table 5: Π_e^1 parcels

		Player 1			
		$p = e$		$p \neq e$	
		Player 2			
		Accept	Reject	Accept	Reject
Player 3	Accept	e_1	$\Pi^2_{e^{31}}$	p_1	$\Pi^2_{\bar{e}^2}$
	Reject	$\Pi^2_{e^{21}}$	$\Pi^2_{e^1}$	$\Pi^2_{\bar{e}^3}$	$\Pi^2_{\bar{e}^{23}}$

The content on the table will be explained through the example of one cell. Suppose player 1 proposed e and player 2 accepted, as it should, but player 3 rejected, the proposition is rejected and agreement is delayed, the players that followed the strategy s were 1 and 2, $g(h) = (1, 2)$, as 1 was the proposer, next round proposer is 2 so $\bar{g}(h) = (2, 1)$. The proposer, player 1, played accordingly, the new state will be e^{21}, and 1's payment that comes from future agreement is $\delta\Pi^2_{e^{21}}$. All the possibilities are covered in the table. To obtain 1's expected payoff we multiply each possibility by the respective probability.

$$\begin{aligned}\Pi_e^1 =&(1-\epsilon)\Big[e_1 s_{-1}^{\epsilon}\big(1,1|h^{t,1}=e\big)+\delta s_{-1}^{\epsilon}\big(0,1|h^{t,1}=e\big)\Pi_{e^{31}}^2+\\ &\delta s_{-1}^{\epsilon}\big(1,0|h^{t,1}=e\big)\Pi_{e^{21}}^2+\delta s_{-1}^{\epsilon}\big(0,0|h^{t,1}=e\big)\Pi_{e^{1}}^2\Big]+E^1 \\ =&e_1(1-\epsilon)^3+\delta\epsilon(1-\epsilon)^2\Pi_{e^{31}}^2+\delta\epsilon(1-\epsilon)^2\Pi_{e^{21}}^2+\delta(1-\epsilon)\epsilon^2\Pi_{e^{1}}^2+E^1\end{aligned} \tag{6}$$

For two different states e and $\tilde{e}$ all but the first term on (6) are equal, so $\Pi_e^1-\Pi_{\tilde{e}}^1=(e_1-\tilde{e}_1)(1-\epsilon)^3$. This equality simplify extremely Π_e^k, for example, we use the fact that player 1 receives nothing in the states e^2, e^3 and $\bar{e}^{23}$, to state that $\Pi_{e^2}^1=\Pi_{e^3}^1=\Pi_{e^{23}}^1$.

The expected payment from a proposition p different from the state when 1 is the proponent can be: p_1 if accepted by all repliers, (this happens with probability ϵ^2, propositions that are different from state are accepted only with ϵ probability by each player); $\delta\Pi_{e^2}^2$ if player 3 rejected and 2 accepted; $\delta\Pi_{e^3}^2$ if player 2 rejected and 3 accepted; and $\delta\Pi_{\bar{e}^{23}}^2$ if both players rejected the proposition $p\neq e$, and the state changed to $\bar{e}^{23}$

$$\begin{aligned}E_1=&\int_{p\in\Delta\setminus r(h)} p_1\epsilon^2+\delta\epsilon(1-\epsilon)\Pi_{e^2}^2+\delta(1-\epsilon)\epsilon\Pi_{e^3}^2+\delta(1-\epsilon)^2\Pi_{\bar{e}^{23}}^2\partial\big(s_i^{\epsilon}(h)\big)\\ =&\int_{p\in\Delta\setminus r(h)} p_1\epsilon^2+\delta(1-\epsilon^2)\Pi_{e^2}^2\partial\big(s_i^{\epsilon}(h)\big)\\ =&\bar{p}^{s^{\epsilon}}\epsilon^3+\delta\epsilon(1-\epsilon^2)\Pi_{e^2}^2\end{aligned}$$

The payoff of player 1 when 2 and 3 are the proponents is:

$$\begin{aligned}\Pi_e^2=&e_1(1-\epsilon)^3+\delta\epsilon(1-\epsilon)^2\Pi_{e^{12}}^3+\delta\epsilon(1-\epsilon)^2\Pi_{e^{32}}^3+\delta\epsilon^2(1-\epsilon)\Pi_{e^2}^3+E^2\\ \Pi_e^3=&e_1(1-\epsilon)^3+\delta\epsilon(1-\epsilon)^2\Pi_{e^{13}}^1+\delta\epsilon(1-\epsilon)^2\Pi_{e^{23}}^1+\delta\epsilon^2(1-\epsilon)\Pi_{e^3}^1+E^3\end{aligned} \tag{7}$$

Developing the same fastidious calculous for the trembling on propositions when player 2 and 3 are the proponents, E_2 and E_3, that we did for E_1, replacing $\tilde{\Delta}=\Delta\setminus r(h)$ and remembering that $\bar{e}^{31}=(\gamma_4,0,\gamma_3)$ and $\bar{e}^{12}=(\gamma_3,\gamma_4,0)$ we get

$$
\begin{aligned}
E_2 =& \int_{p\in\tilde{\Delta}} p\epsilon^2 + \delta\epsilon(1-\epsilon)\Pi^3_{e^3} + \delta(1-\epsilon)\epsilon\Pi^3_{e^1} + \delta(1-\epsilon)^2\Pi^3_{\bar{e}^{31}}\partial s^\epsilon_2(h)\\
=&\bar{p}^{s^\epsilon}\epsilon^3 + \delta(1-\epsilon)\int_{p\in\tilde{\Delta}} \epsilon\Pi^3_{e^2} + \epsilon\Big[(1-\epsilon)^3 + \Pi^3_{e^2}\Big] + (1-\epsilon)\Big[\gamma_4(1-\epsilon)^3 + \Pi^3_{e^2}\Big]\partial s^\epsilon_2(h)\\
=&\bar{p}^{s^\epsilon}\epsilon^3 + \delta(1-\epsilon)\int_{p\in\tilde{\Delta}} \epsilon(1-\epsilon)^3 + \gamma_4(1-\epsilon)^4 + (1+\epsilon)\Pi^3_{e^2}\partial s^\epsilon_2(h)\\
=&\bar{p}^{s^\epsilon}\epsilon^3 + \epsilon\delta(1-\epsilon)^4\left[\epsilon + \gamma_4(1-\epsilon)\right] + \epsilon\delta(1-\epsilon^2)\Pi^3_{e^2}
\end{aligned}
$$

$$
\begin{aligned}
E_3 =& \int_{p\in\tilde{\Delta}} p\epsilon^2 + \delta\epsilon(1-\epsilon)\Pi^1_{e^1} + \delta(1-\epsilon)\epsilon\Pi^1_{e^2} + \delta(1-\epsilon)^2\Pi^1_{\bar{e}^{12}}\partial s^\epsilon_3(h)\\
=&\bar{p}^{s^\epsilon}\epsilon^3 + \delta(1-\epsilon)\int_{p\in\tilde{\Delta}} \epsilon\left[\Pi^1_{e^2} + (1-\epsilon)^3\right] + \epsilon\Pi^1_{e^2} + (1-\epsilon)\left[\Pi^1_{e^2} + \gamma_3(1-\epsilon)^3\right]\partial s^\epsilon_3(h)\\
=&\bar{p}^{s^\epsilon}\epsilon^3 + \delta(1-\epsilon)\int_{p\in\tilde{\Delta}} (1-\epsilon)^3\left[\epsilon + \gamma_3(1-\epsilon)\right] + (1+\epsilon)\Pi^1_{e^2}\partial s^\epsilon_3(h)\\
=&\bar{p}^{s^\epsilon}\epsilon^3 + \epsilon\delta(1-\epsilon)^4\left[\epsilon + \gamma_3(1-\epsilon)\right] + \epsilon\delta(1-\epsilon^2)\Pi^3_{e^2}
\end{aligned}
$$

We will focus on the state e^2, later we prove no more state needs to be analysed. Replacing e by e^2 and using relations like $\Pi^k_{e^{12}} = \Pi^k_{e^{13}}$, $\Pi^2_{e^{13}} = \Pi^2_{e^2} + \gamma_1(1-\epsilon)^3$ and $\Pi^2_{e^1} = \Pi^2_{e^2} + (1-\epsilon)^3$ in the equations (6) and (7).

$$
\begin{aligned}
\Pi^1_{e^2} =&\delta\epsilon(1-\epsilon)^2\Pi^2_{e^{31}} + \delta\epsilon(1-\epsilon)^2\Pi^2_{e^{21}} + \delta\epsilon^2(1-\epsilon)\Pi^2_{e^1} + E^1\\
=&\delta\epsilon(1-\epsilon)\left[2(1-\epsilon)\Pi^2_{e^{31}} + \epsilon\Pi^2_{e^1}\right] + E^1\\
=&\delta\epsilon(1-\epsilon)\left[2(1-\epsilon)\left(\Pi^2_{e^2} + (1-\epsilon)^3\gamma_2\right) + \epsilon\left(\Pi^2_{e^2} + (1-\epsilon)^3\right)\right] + E^1\\
=&\delta\epsilon(1-\epsilon)\left[\left(2(1-\epsilon) + \epsilon\right)\Pi^2_{e^2} + 2(1-\epsilon)^4\gamma_2 + \epsilon(1-\epsilon)^3\right] + E^1\\
=&\delta\epsilon(1-\epsilon)\left[(2-\epsilon)\Pi^2_{e^2} + (1-\epsilon)^3\left(2(1-\epsilon)\gamma_2 + \epsilon\right)\right] + \left[\bar{p}^{s^\epsilon}\epsilon^3 + \delta\epsilon(1-\epsilon^2)\Pi^2_{e^2}\right]\\
=&\delta\epsilon(1-\epsilon)\left[3\Pi^2_{e^2} + (1-\epsilon)^3\left(2(1-\epsilon)\gamma_2 + \epsilon\right)\right] + \bar{p}^{s^\epsilon}\epsilon^3\\
=&3\delta\epsilon(1-\epsilon)\Pi^2_{e^2} + \delta\epsilon(1-\epsilon)^4\left[2(1-\epsilon)\gamma_2 + \epsilon\right] + \bar{p}^{s^\epsilon}\epsilon^3
\end{aligned}
$$

$$
\begin{aligned}
\Pi^2_{e^2} &= \delta\epsilon(1-\epsilon)^2\Pi^3_{e^{12}} + \delta\epsilon(1-\epsilon)^2\Pi^3_{e^{32}} + \delta\epsilon^2(1-\epsilon)\Pi^1_{e^2} + E^2 \\
&= \delta\epsilon(1-\epsilon)\left[(1-\epsilon)\Pi^3_{e^{12}} + (1-\epsilon)\Pi^3_{e^{32}} + \epsilon\Pi^1_{e^2}\right] + E^2 \\
&= \delta\epsilon(1-\epsilon)\left[(1-\epsilon)\left(\Pi^3_{e^2} + \gamma_1(1-\epsilon)^3\right) + (1-\epsilon)\Pi^3_{e^2} + \epsilon\Pi^1_{e^2}\right] + E^2 \\
&= \delta\epsilon(1-\epsilon)\left[(2-\epsilon)\Pi^3_{e^2} + \gamma_1(1-\epsilon)^4\right] + E^2 \\
&= \delta\epsilon(1-\epsilon)\left[(2-\epsilon)\Pi^3_{e^2} + \gamma_1(1-\epsilon)^4\right] + \left[\bar{p}^{s^\epsilon}\epsilon^3 + \epsilon\delta(1-\epsilon)^4\left[\epsilon + \gamma_4(1-\epsilon)\right] + \delta\epsilon(1-\epsilon^2)\Pi^3_{e^2}\right] \\
&= 3\delta\epsilon(1-\epsilon)\Pi^3_{e^2} + \epsilon\delta(1-\epsilon)^4\left[\epsilon + (1-\epsilon)(\gamma_4+\gamma_1)\right] + \bar{p}^{s^\epsilon}\epsilon^3
\end{aligned}
$$

$$
\begin{aligned}
\Pi^3_{e^2} =& \delta\epsilon(1-\epsilon)^2\Pi^1_{e^{13}} + \delta\epsilon(1-\epsilon)^2\Pi^1_{e^{23}} + \delta\epsilon^2(1-\epsilon)\Pi^1_{e^3} + E^3 \\
=& \delta\epsilon(1-\epsilon)\left[(1-\epsilon)\left(\Pi^1_{e^2} + \gamma_1(1-\epsilon)^3\right) + \Pi^1_{e^2}\right] + E^3 \\
=& \delta\epsilon(1-\epsilon)\left[(2-\epsilon)\Pi^1_{e^2} + \gamma_1(1-\epsilon)^4\right] + \left[\bar{p}^{s^\epsilon}\epsilon^3 + \epsilon\delta(1-\epsilon)^4\left[\epsilon + \gamma_3(1-\epsilon)\right] + \epsilon\delta(1-\epsilon^2)\Pi^3_{e^2}\right] \\
=& 3\delta\epsilon(1-\epsilon)\Pi^3_{e^2} + \epsilon\delta(1-\epsilon)^4\left[\epsilon + (\gamma_3+\gamma_1)(1-\epsilon)\right] + \bar{p}^{s^\epsilon}\epsilon^3
\end{aligned}
$$

We get the following system of equations

$$
\begin{cases}
\Pi^1_{e^2} = & 3\delta\epsilon(1-\epsilon)\Pi^2_{e^2} + \delta\epsilon(1-\epsilon)^4\left[\epsilon + (1-\epsilon)2\gamma_2\right] + \bar{p}^{s^\epsilon}\epsilon^3 \\
\Pi^2_{e^2} = & 3\delta\epsilon(1-\epsilon)\Pi^3_{e^2} + \epsilon\delta(1-\epsilon)^4\left[\epsilon + (1-\epsilon)(\gamma_4+\gamma_1)\right] + \bar{p}^{s^\epsilon}\epsilon^3 \\
\Pi^3_{e^2} = & 3\delta\epsilon(1-\epsilon)\Pi^1_{e^2} + \epsilon\delta(1-\epsilon)^4\left[\epsilon + (1-\epsilon)(\gamma_3+\gamma_1)\right] + \bar{p}^{s^\epsilon}\epsilon^3
\end{cases}
$$

With $\xi_0 = 3\delta\epsilon(1-\epsilon)$, $\xi_1 = \epsilon\delta(1-\epsilon)^4$ and $\beta_1 = \epsilon+(1-\epsilon)2\gamma_2$, $\beta_2 = \epsilon+(1-\epsilon)(\gamma_4+\gamma_1)$, $\beta_3 = \epsilon + (1-\epsilon)(\gamma_3+\gamma_1)$ we can rewrite the system as

$$
\begin{cases}
\Pi^1_{e^2} = & \xi_1\beta_1 + \xi_0\Pi^2_{e^2} + \bar{p}^{s^\epsilon}\epsilon^3 \\
\Pi^2_{e^2} = & \xi_1\beta_2 + \xi_0\Pi^3_{e^2} + \bar{p}^{s^\epsilon}\epsilon^3 \\
\Pi^3_{e^2} = & \xi_1\beta_3 + \xi_0\Pi^3_{e^1} + \bar{p}^{s^\epsilon}\epsilon^3
\end{cases}
$$

Solving the system we get the values of $\Pi^k_{e^2}$, for $k = 1, 2, 3$

$$\begin{cases} \Pi^1_{e^2} = & \frac{\xi_1}{1-\xi_0^3}(\beta_1 + \xi_0\beta_2 + \xi_0^2\beta_3) + \bar{p} \\ \Pi^2_{e^2} = & \frac{\xi_1}{1-\xi_0^3}(\beta_2 + \xi_0\beta_3 + \xi_0^2\beta_1) + \bar{p} \\ \Pi^3_{e^2} = & \frac{\xi_1}{1-\xi_0^3}(\beta_3 + \xi_0\beta_1 + \xi_0^2\beta_2) + \bar{p} \end{cases}$$

And can now calculate the following limits for later use.

$$\begin{cases} \lim_{\epsilon\downarrow 0} \Pi^1_{e^2}/\epsilon = 2\delta\gamma_2 \\ \lim_{\epsilon\downarrow 0} \Pi^2_{e^2}/\epsilon = \delta(\gamma_4 + \gamma_1) \\ \lim_{\epsilon\downarrow 0} \Pi^3_{e^2}/\epsilon = \delta(\gamma_3 + \gamma_1) \end{cases} \tag{8}$$

To analyse the best response of player 1 in state e^2 when s^ϵ is being played we consider the strategies $s_1^a, s_1^r \in OSD(s_1^\epsilon, h)$ in which $s_1^a(1|h) = 1$, $s_1^r(0|h) = 1$. For s to be a PE, for small ϵ, $\Pi_1^t(s_1^a, s_{-1}^\epsilon|h) \geq \Pi_1^t(s_1^r, s_{-1}^\epsilon|h)$

When player 3 is the proponent and proposed e^2 with $r(h^-) = e^2$, the payment for player 1 in each of his actions is:

- $\Pi_1^t(s_1^a, s_{-1}^\epsilon|h) = 0.(1-\epsilon) + \delta\epsilon\Pi^1_{e^{13}} = \delta\epsilon\Pi^1_{e^{13}}$
- $\Pi_1^t(s_1^r, s_{-1}^\epsilon|h) = (1-\epsilon)\delta\Pi^1_{e^{23}} + \epsilon\delta\Pi^1_{e^3}$

And the difference between the two payoffs is

$$\begin{aligned} \Pi_1^t(s_1^a, s_{-1}^\epsilon|h) - \Pi_1^t(s_1^r, s_{-1}^\epsilon|h) =& \delta\epsilon\Pi^1_{e^{13}} - \left[(1-\epsilon)\delta\Pi^1_{e^{23}} + \epsilon\delta\Pi^1_{e^{31}}\right] \\ =& \delta\epsilon\left[\Pi^1_{e^3} + \gamma_1(1-\epsilon)^3\right] - (1-\epsilon)\delta\Pi^1_{e^{32}} - \epsilon\delta\Pi^1_{e^3} \\ =& \delta\epsilon\gamma_1(1-\epsilon)^3 - (1-\epsilon)\delta\Pi^1_{e^{32}} \\ =& (1-\epsilon)\delta\left(\gamma_1\epsilon(1-\epsilon)^2 - \Pi^1_{e^2}\right) \\ =& (1-\epsilon)\epsilon\delta\left(\gamma_1(1-\epsilon)^2 - \frac{\Pi^1_{e^2}}{\epsilon}\right) \end{aligned}$$

As $\frac{\Pi^1_{e2}}{\epsilon} \to 2\delta\gamma_2$, if $\gamma_1 > 2\delta\gamma_2$ the inequality $\Pi_1^t(s_1^a, s_{-1}^\epsilon|h) \geq \Pi_1^t(s_1^r, s_{-1}^\epsilon|h)$ is verified for small values of ϵ.

If in the state e^2 player 3 made a proposition $e \neq e^2$ player 1 payment in case of acceptance is $\Pi_1^t(s_1^a, s_{-1}^\epsilon|h) = e_1 s_2^\epsilon(1|h,e) + \delta s_2^\epsilon(0|h,e)\Pi_{e^2}^1 \leq \epsilon + \delta(1-\epsilon)\Pi_{e^2}^1$ or in case of rejecting $\Pi_1^t(s_1^r, s_{-1}^\epsilon|h) = \delta s_2^\epsilon(1|h,e)\Pi_{e^1}^1 + \delta s_2^\epsilon(0|h,e)\Pi_{\bar{e}^{12}}^1$. As $s_2^\epsilon(1|h,e) \to 0$, $\Pi_{e^2}^1 \to 0$ and $\Pi_{\bar{e}^{12}}^1 \to \gamma_3$, $\Pi_1^t(s_1^r, s_{-1}^\epsilon|h) - \Pi_1^t(s_1^a, s_{-1}^\epsilon|h) \to \delta\gamma_3 > 0$, for small ϵ the best option to player 1 is to reject the proposal.

If player 2 proposed e^2 with $r(h^-) = e^2$,the payment for player 1 in each of his actions is:

- $\Pi_1^t(s_1^a, s_{-1}^\epsilon|h) = 0.(1-\epsilon) + \delta\epsilon\Pi_{e^{12}}^3 = \delta\epsilon\Pi_{e^{13}}^1$
- $\Pi_1^t(s_1^r, s_{-1}^\epsilon|h) = (1-\epsilon)\delta\Pi_{e^{32}}^1 + \epsilon\delta\Pi_{e^2}^1$

And the difference between the two payoffs is

$$\begin{aligned}
\Pi_1^t(s_1^a, s_{-1}^\epsilon|h) - \Pi_1^t(s_1^r, s_{-1}^\epsilon|h) =& \delta\epsilon\Pi_{e^{12}}^3 - \left[(1-\epsilon)\delta\Pi_{e^{32}}^3 + \epsilon\delta\Pi_{e^2}^3\right] \\
=& \delta\epsilon\left[\Pi_{e^2}^3 + \gamma_1(1-\epsilon)^3\right] - (1-\epsilon)\delta\Pi_{e^2}^3 - \epsilon\delta\Pi_{e^2}^3 \\
=& \delta(1-\epsilon)\left(\epsilon\gamma_1(1-\epsilon)^2 - \Pi_{e^2}^3\right) \\
=& \delta\epsilon(1-\epsilon)\left(\gamma_1(1-\epsilon)^2 - \frac{\Pi_{e^2}^3}{\epsilon}\right)
\end{aligned}$$

As seen in (8) $\frac{\Pi_{e2}^3}{\epsilon} \to \delta(\gamma_3 + \gamma_1)$, and $\left[\Pi_1^t(s_1^a, s_{-1}^\epsilon|h) - \Pi_1^t(s_1^r, s_{-1}^\epsilon|h)\right]/\epsilon \to \gamma_1 - \delta(\gamma_3 + \gamma_1)$ and if $\gamma_1 > \frac{\delta}{1-\delta}\gamma_3$ the necessary inequality is verified.

In the case player 2 made a proposition different from the state, it can be proved that player 1 is better by rejecting the proposition, this is done in the same way as when player 3 proposed a different division. Nothing changes in the proof.

When player 1 is proposing, and state is e^2, consider the two $OSD(s_1^\epsilon, h)$, $s_1^{nd}(e^2|h) = 1$, the "non-deviating" strategy in which 1 always proposes e^2 after h, and the "deviating" strategy with player always proposing e, $s_1^d(e|h) = 1$, different from e^2. For s to be *Perfect Equilibria* $\Pi_1^t(s_1^{nd}, s_{-1}^\epsilon|h) \geq \Pi_1^t(s_1^d, s_{-1}^\epsilon|h)$ for small values of ϵ.

$$
\begin{aligned}
\Pi_1^t(s_1^d, s_{-1}^\epsilon|h) &\leq 1.\epsilon^2 + \delta\epsilon(1-\epsilon)\Pi_{e^3}^2 + \delta\epsilon(1-\epsilon)\Pi_{e^2}^2 + \delta(1-\epsilon)^2\Pi_{\bar{e}^{23}}^2 \\
&= \epsilon^2 + \delta(1-\epsilon^2)\Pi_{e^2}^2 \\
\Pi_1^t(s_1^{nd}, s_{-1}^\epsilon|h) &= \delta\epsilon(1-\epsilon)\Pi_{e^{31}}^2 + \delta\epsilon(1-\epsilon)\Pi_{e^{21}}^2 + \delta\epsilon^2\Pi_{e^1}^2 \\
&= 2\delta\epsilon(1-\epsilon)\Pi_{e^{31}}^2 + \delta\epsilon^2\Pi_{e^1}^2 \\
&= \delta(2-\epsilon)\epsilon\Pi_{e^2}^2 + 2\delta\epsilon(1-\epsilon)^4\gamma_2 + \delta\epsilon^2(1-\epsilon)^3 \\
&= \delta(2-\epsilon)\epsilon\Pi_{e^2}^2 + \delta\epsilon(1-\epsilon)^3\left[2\gamma_2(1-\epsilon)+\epsilon\right]
\end{aligned}
$$

$$
\begin{aligned}
\Pi_1^t(s_1^{nd}, s_{-1}^\epsilon|h) - \Pi_1^t(s_1^d, s_{-1}^\epsilon|h) &\geq \left[\delta(2-\epsilon)\epsilon - \delta(1-\epsilon^2)\right]\Pi_{e^2}^2 + \delta\epsilon(1-\epsilon)^3\left[2\gamma_2(1-\epsilon)+\epsilon\right] - \epsilon^2 \\
&= \delta(2\epsilon-1)\Pi_{e^2}^2 + \delta\epsilon(1-\epsilon)^3\left[2\gamma_2(1-\epsilon)+\epsilon\right] - \epsilon^2 \\
&= \delta\epsilon\left\{(2\epsilon-1)\frac{\Pi_{e^2}^2}{\epsilon} + (1-\epsilon)^3\left[2\gamma_2(1-\epsilon)+\epsilon\right] - \frac{\epsilon}{\delta}\right\}
\end{aligned}
$$

And the expression inside the curly brackets, using again (8), converges to $-\delta(\gamma_1+\gamma_4)+2\gamma_2$, and if $2\gamma_2 > \delta(\gamma_1+\gamma_4)$ the necessary inequality is assured.

The set of inequalities for s to be a PE are

$$
\left\{\begin{array}{l}
\gamma_1 \geq 2\delta\gamma_2 \\
\gamma_1 \geq \frac{\delta}{1-\delta}\gamma_3 \\
2\gamma_2 \geq \delta(\gamma_1+\gamma_4)
\end{array}\right.
\Leftrightarrow
\left\{\begin{array}{l}
\gamma_1 \geq \frac{2\delta}{1+2\delta} \\
\gamma_1 \geq \frac{\delta}{1-\delta}\gamma_3 \\
\gamma_1 \leq \frac{2-\delta}{2+\delta} + \frac{\delta}{2+\delta}\gamma_3
\end{array}\right.
\tag{9}
$$

We assumed $\gamma_1+\gamma_2 = \gamma_3+\gamma_4 = 1$. These equations are all compatible, if $\gamma_1 > \frac{2\delta}{1+2\delta}$ and $\gamma_3 < \min\left\{\frac{1-\delta}{\delta}, \frac{2-\delta}{1+2\delta}\frac{1-\delta}{\delta}\right\}$ and solution, for each δ exists.

We will now see that for the other states $e \in E$, player 1 never improve is payment by deviating from strategy s^ϵ. First when 1 is the proponent. Notice that for the proponent the expected payment of a deviation does not depend on the state, it is always equal no matter what the initial state was, $\Pi_1^t(s_1', s_{-1}^\epsilon|e) = \Pi_1^t(s_1', s_{-1}^\epsilon|e^2)$. Hence, if the proposition is equal to the state, as $\Pi_1^t(s^\epsilon|e) \geq \Pi_1^t(s^\epsilon|e^2)$, and if in e^2 deviating was not profitable in e it is not as well, $\Pi_1^t(s^\epsilon|e) \geq \Pi_1^t(s^\epsilon|e^2) \geq \Pi_1^t(s_1', s_{-1}^\epsilon|e^2) = \Pi_1^t(s_1', s_{-1}^\epsilon|e)$.

When 1 is the replier and the proposition is not equal to the state e, 1's expected payment by rejecting the proposal is the same as when rejecting a proposition not equal to the state and the state was e^2. So if $r(h^-) = e$ and $h^{t,1} \neq e$, and $r(\tilde{h}^-) = e^2$ and $\tilde{h}^{t,1} \neq e^2$. With $s_1^r \in OSD(s_1^\epsilon, h)$ and $\tilde{s}_1^r \in OSD(s_1^\epsilon, \tilde{h})$, are the OSD strategies that reject the deviating proposition at h and $\tilde{h}$, respectively, $\Pi_1^t(s_1^r, s_{-1}^\epsilon|h) = \Pi_1^t(\tilde{s}_1^r, s_{-1}^\epsilon|\tilde{h})$. The same is valid if the player accepts the deviating proposition, his payment is exactly the same in state e to what it was in state e^2. Defining $s_1^a \in OSD(s_1^\epsilon, h)$ and $\tilde{s}_1^a \in OSD(s_1^\epsilon, \tilde{h})$ as the OSD strategies that accept the deviating proposition at h and $\tilde{h}$, respectively, $\Pi_1^t(s_1^a, s_{-1}^\epsilon|h) = \Pi_1^t(\tilde{s}_1^a, s_{-1}^\epsilon|\tilde{h})$. Accordingly, if in $r(\tilde{h}) = e^2$ there was no advantage in accepting a deviating proposal, $\Pi_1^t(s_1^a, s_{-1}^\epsilon|\tilde{h}) \geq \Pi_1^t(s_1^r, s_{-1}^\epsilon|\tilde{h})$ in $r(h) = e$ there is no advantage also, because the payments are equal in both states, $\Pi_1^t(s_1^a, s_{-1}^\epsilon|h) \geq \Pi_1^t(s_1^r, s_{-1}^\epsilon|h)$.

The same reasoning can be applied to the histories in which the last proposition was equal to the state $r(h) = h^{t,1} = e$. The player's payoff by rejecting the proposition is equal to the payoff when he rejects $r(\tilde{h}) = \tilde{h}^{t,1} = e^2$. That is, the OSD strategies that reject the propositions, $s_1^r \in OSD(s_1^\epsilon, h)$ and $\tilde{s}_1^r \in OSD(s_1^\epsilon, \tilde{h})$, have the same payment $\Pi_1^t(s_1^r, s_{-1}^\epsilon|h) = \Pi_1^t(\tilde{s}_1^r, s_{-1}^\epsilon|\tilde{h})$. As $\Pi_1^t(s^\epsilon|h) - \Pi_1^t(s^\epsilon|\tilde{h}) = (e_1 - e_1^2)(1-\epsilon)^3$. Due to the state's definition, for any $e \in E$, $e_1 \geq e_1^2$, therefore $\Pi_1^t(s^\epsilon|h) \geq \Pi_1^t(s^\epsilon|\tilde{h})$, and we conclude that $\Pi_1^t(s^\epsilon|h) \geq \Pi_1^t(s^\epsilon|\tilde{h}) \geq \Pi_1^t(\tilde{s}_1^r, s_{-1}^\epsilon|\tilde{h}) = \Pi_1^t(s_1^r, s_{-1}^\epsilon|h)$. Not to deviate is the best for player 1 when the proposition coincide with the state. This way 1 has no advantage in choosing a different strategy for any of states in E.

Due to the symmetry of the strategies used in s^ϵ to exist a state in which any player i had something to gain by deviating then there must also exist a state where 1 would gain by playing the same deviating strategy. As there is not such case, there is no player and no state in which there is a profitable deviation, for this reason s^ϵ is a best reply to itself, and s is a *PE*.