Going twice to the polls?

Optimality of two-stage voting procedures

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Abstract

Two-stage voting systems are commonly used, not only in political elections but in many other types of contests such as the Academy of Motion Pictures Arts and Sciences (AMPAS) awards. These methods are nonetheless more costly than single-stage ones. In this paper we will compare the performance of different one-stage and two-stage voting systems. In particular, we will analyse the impact of the introduction of a second stage in the ability of electing the *Condorcet winner* and rejecting the *Condorcet loser*. Through simulation, we will conclude

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that under two-stage systems with only two nominees in the second stage, the likelihood of respecting the Condorcet criteria increases significantly. However, with three nominees in the second stage, results are ambiguous, depending on the degree of homogeneity of preferences.

1 Introduction

Selecting the best alternative from a given set would seem a fairly simple problem but the social choice literature shows that this is a very complex issue. Different methods may lead to different outcomes and, more importantly, all of them fail to simultaneously respect a given set of desirable conditions, both in terms of selecting a social preference and a social choice. This is what the well-known Arrow (1963) and Gibbard-Satterthwaite (1975) theorems respectively show.

Even though all systems fail to behave optimally, some may violate more conditions than others or simply violate a given condition more often (in a frequency analysis). Since the literature is far from an agreement on the best voting system, we observe a multiplicity of methods being used when it comes to selecting an alternative from a given set. For instance, in the main political election systems around world more than twenty different methods are used. Furthermore, these methods differ from those typically used in other contexts, such as sports.

Plurality voting[1] is one of the most well known voting systems. Under this method voters cast a single vote and the alternative with the highest number of votes wins. More than 75 countries use it in elections. *Approva*l voting is a simple system with attractive conditions but it is not that commonly used. Under this method, voters are allowed to vote for more than one candidate, and the candidate with the highest number of votes wins. It is currently considered by many social choice researchers as a good substitute for *plurality*, solving some of the problems that arise under this method. *Instant Runoff* Voting[2] is widely used in political elections, such as the election for party leader in the UK, the Irish presidential elections and the Australian elections for the House

[1] Also known as FPTP (first-past-the-post method used in USA presidential elections and in more than seventy five political elections around the world)

[2] Also referred to as *Alternative Voting* (UK), *Preferential Ballot* (Canada) or *Ranked Choice Voting* (US)

of Representatives. It is also used in some US states as well as in New Zealand. In addition, this method is used in the AMPAS awards when selecting the nominees and this year, for the first time, it was used as the voting system to select the best picture out of ten nominees. This method requires voters to rank their candidates and the winner is selected after a sequential process of elimination of weaker candidates. In each round, the candidate with the least number of first-place positions in individual rankings is eliminated from every voter's ranking. It is, therefore, a method where the whole ranking of alternatives matters.

According to Fishburn and Brams (1978), one-stage *approval* voting is "the most sincere and the most strategyproof" method[3]. This means that, under one-stage *approval* voting, agents will have fewer incentives to misreport their preferences. Moreover, *approval voting* ensures that whenever a *Condorcet winner* exists, under admissible strategies[4], it is the one chosen. In this specific sense, it is superior to many other widely used voting systems such as *plurality*. However, Fishburn and Brams do not include two-stage systems in their analysis.

Nurmi (1983) analyses a comprehensive list of five binary methods, three one-stage methods and five multistage non-binary systems. The main debated criteria are used to analyse these methods and the aim is a "synthesis of the assessments of procedures with respect to various criteria". In the same fashion, Richelson (1975, 1978, 1981) analyses a smaller group of systems with respect to a broader set of criteria. Both these analyses aim to synthesize and not to conclude about the best method to be used. Some methods respect more conditions than others, however we cannot infer which conditions are more socially desirable. Moreover, under such a binary analysis we are not able to analyse the frequency at which a given condition is violated by a specific method. For

[3] Compared to *plurality* and negative voting
[4] Dichotomous preferences

these reasons no definite conclusion on the best method can arise.

However, besides Nurmi (1983), little attention has been given to the optimality of two-stage systems. This is highly unexpected as these methods are more and more used in every type of election. More than seventy countries use the runoff system in their major political elections. This is the case of French and Portuguese presidential elections.

Nonetheless, two-stage systems cost more. Having two rounds, where voters need to cast a vote twice, means, *ceteribus paribus*, additional costs: more resources must be used to organize the extra ballot. Political instabilities may also arise inbetween the two rounds which inevitably adds to the economic cost of the election. Furthermore, it should be considered that, if voters are required to vote too often, a two-stage system might lead to voter fatigue resulting in a reduced voter turnout (lower percentage of people casting a vote/ going to the polls). Thus, the question to be answered is why are these systems used? What makes them an attractive alternative?

It is worth noticing that, in some cases, having two-stages is a necessary condition. This is the case of the AMPAS where a smaller list of alternatives must be created in order to become public and where they aim for different people voting at different stages of the election (i.e., only members of the directing branch vote for the nominees for directing but the whole membership votes for the winner). Therefore, a second ballot is mandatory. In these cases the question that remains is which method should be used? Or is the system currently in use optimal?

Summing up, two main questions need to be answered:

1. When it is not necessary to elect in two steps, which would be the reasons for using two-stage methods?

2. When there is a need for a two-stage election process, which method should be used? In other words, in which aspects is a specific system better than the others?

Note that these two questions are connected. In order to answer the first question, we need to study the impact of changing from a one-ballot voting scheme to a two-stage system. We will take three major one-stage voting systems with already known characteristics and add a second step to the process. We will analyse the differences in performances when a second stage is introduced where plurality is the system used to select the winner. Then, using this information, we are able to answer the second question as well.

In order to answer the set of questions, we will organize the paper in the following way. First, in section 2, we will present a brief summary of the conditions that the three one-stage methods (*plurality*, *approval* and *instant runoff*) violate or meet. In section 3, we will proceed with a binary analysis on the impact of changing from a one-stage system to a system where we add an extra stage where *plurality* is the system used. First, we will study the case where two nominees are selected to the second stage and afterwards we will add an extra nominee to the picture. Lastly in this section, we will compare all the two-stage systems and try to then infer if any of them seems to be more socially desirable. As mentioned before, a binary analysis is not enough to infer on the best election method. For that reson a deeper analysis should be performed. In section 4, we will focus on the Condorcet criteria and we will try to quantify, in terms of frequency, which of the methods seem to respect this criterion more often. To do this, we will proceed through simulation creating preferences, both under *impartial culture* and *single-peakedness* assumption, and consequently analyse the outcome of the different systems. In section 5, conclusions about the optimality of different two-stage methods will be drawn.

2 One-stage systems

In this chapter we will compare three different one-stage stage voting systems in their ability to respect a given set of desirable conditions.

After carefully considering all the conditions analysed in literature, we decided to focus our analysis on those more frequently used, excluding those that are respected by all the voting systems that we look into[5].

2.1 The Conditions

Let Υ be the nonempty finite set of all possible alternatives and let P_i be the individual linear strict preference relation over Υ. Then, consider function S that represents the social choice function. This function takes a given subset γ of Υ and chooses the best social set of alternatives in it (which can be a singleton) given the preferences of all agents, P. This set is therefore the social choice. Summing up, the function assigns a subset $S(\gamma, P)$ of Υ to each (γ, P) situation.

Every social choice relies on preferences that agents reveal. Having a social choice consistent with those individual preferences is highly desirable. The following criteria impose this consistency of social choice functions with individual preferences from different perspectives.

Criterion 1 *Pareto optimality (PO): Take the subset γ of Υ and consider x and y contained in γ. A social choice function, S, is Pareto optimal if and only if , when we observe that every single agent prefers x to y, then y cannot be chosen. Formally, $x \ P_i \ y, \forall i \Rightarrow y \notin S(\gamma, P)$.*

[5] Such as *neutrality* and *anonymity*

Criterion 2 *Condorcet winner (CW): A Condorcet winner is an alternative that wins every pairwise contest against all the other alternatives. In other words, x is the Condorcet winner of γ under P if $\sharp \{i : xP_iy\} > \sharp \{i : yP_ix\}$ for all $y \in \gamma$, $y \neq x$. Every social choice function S that guarantees the choice of the Condorcet winner, whenever one exists, respects the Condorcet winner criterion. So S will satisfy the Condorcet winner criterion if, when $x \in \gamma$ is the Condorcet winner of γ under P, we have $x \in S(\gamma, P)$.*

Criterion 3 *Condorcet Loser (CL): A Condorcet loser is an alternative that loses every pairwise comparison with all the alternatives available. In other words, x is the Condorcet loser of γ under P if $\sharp \{i : yP_ix\} > \sharp \{i : xP_iy\}$ for all $y \in \gamma$, $y \neq x$. Every social choice function S that guarantees that the Condorcet loser is never chosen, whenever one exists, respects the Condorcet loser criterion. So S will satisfy the Condorcet loser criterion if, when $x \in \gamma$ is the Condorcet loser of γ under P, we have $x \notin S(\gamma, P)$.*

The last three criteria are important, in the sense that they make the social choice more in line with individual preferences. Any violation of these criteria means that a strong inconsistency between the social choice function and the implicit individual preferences exists.

In addition, we would desire changes in individual preferences to consistently change social preferences and consequently the social choice. This is what the following two criteria try to guarantee.

Criterion 4 *Monotonicity (MON): Take P, γ and $x \in \gamma$. Then, obtain P' from P by raising x in one or more preference orderings (and everything else remains unchanged). S is monotonic whenever $x \in S(\gamma, P) \Rightarrow x \in S(\gamma, P')$. So if x was already considered as one of the optimal* *social choices, then it should remain so when we raise its position in at least one agent's preference ordering.*

Criterion 5 *Independence of Irrelevant Alternatives (IIA)*[6]*:*

S satisfies independence of irrelevant alternatives if for all $(x, y) \subseteq \Upsilon$ and for all P and P':

For all i where $x\ P_i\ y$, now we have $x\ P_i'\ y$;

For all i where $y\ P_i\ x$, now we have $y\ P_i'\ x$;

Then we must have x socially preferred to y under P', if x was socially preferred to y under P, or y socially preferred to x under P' if it was also so under P. Formally $xP_sy \Rightarrow xP_s^{'}y$ and $yP_sx \Rightarrow yP_s^{'}x$ [7]*.*

Note that what is discussed here is more than the social choice: we discussed the social preference ordering. We will assume different ways of defininig criteria for social orderings according to each of the voting systems. However, we must be aware that many other different criteria could be assumed.

The next criterion can be seen as more of a consistency check. Guaranteeing that voters behave consistently while facing different subsets of a larger set is what the *weak axiom of revealed preferences* tries to ensure.

Criterion 6 *Weak Axiom of Revealed Preferences (WARP): take a subset m of γ, $m \subseteq \gamma$ and*

[6] *Also known as Sen's property alpha or as Chernoff's condition*

[7] *P_s stands for social strict preference relation, which can also be represented in a binary preference fashion as $\succ_s$*

consider P. A social choice function, S, respects WARP whenever: $m \cap S(\gamma, P) \neq \varnothing \Rightarrow m \cap S(\gamma, P) = S(m, P)$.

So, we take a subset m of γ and the set of socially optimal alternatives from γ, $S(\gamma, P)$. If m contains any of the previous optimal alternatives then the new set of optimal alternatives should consist of those that were previously optimal and are still available, $m \cap S(\gamma, P)$. As stated by Nurmi (1983) "methods satisfying *WARP* choose alternatives that are winners in all subsets of alternatives they belong to". Furthermore, "if two alternatives are chosen from a given subset of alternatives either both or neither of them will be chosen from any set that contains the subset " (Plott 1976).

2.2 Plurality, Approval and Instant Runoff

We will now proceed to the analysis of one-stage voting systems, defining and comparing three different systems in their ability to respect the above criteria. It is truly important to understand how these one-stage methods behave so that we can analyse the impact, in the performance of these systems, of the introduction of a second stage.

In our analysis, we will not be interested in studying abstention or blank votes and for that reason we will ignore these cases. This means that, under any voting system, agents will cast at least one vote (and obviously cannot vote for all the candidates).

2.2.1 Plurality

Under plurality, each voter casts one vote. The winning alternative (or set of alternatives) is the one that gets the highest number of votes. Formally, let $V(y)$ be the number of votes for the alternative y, for all $y \in \gamma$. Then the *plurality* rule is defined in the following way: $S(\gamma, P) = \{x \in \gamma : V(x) \geq V(y) \text{ for all } y \in \gamma\}$.

Plurality can also be used to derive a social ordering where the alternatives will be socially ranked according to the number of votes they get.

2.2.2 Approval

Suppose there are m alternatives. Under approval, each voter can give each alternative either one or no vote (approve or disapprove) with the restriction that no more than $m-1$ alternatives can be approved. In addition, we will impose the condition that voters must approve at least one of them so that abstention and blank votes are completely left out. The alternative that gets the highest number of approvals wins. Again, if there is a tie, more than one alternative can win the election. Formally, let $A(y)$ be the number of approvals that alternative y collects during the ballot, for all $y \in \gamma$. The *approval* rule goes as follows, $S(\gamma, P) = \{x \in \gamma : \ A(x) \geq A(y), \text{for all } y \in \gamma\}$.

Voters will approve all the alternatives that give them at least certain utility level, which we will denote as the threshold level of utility. This threshold level of utility may vary given the size of the set $m \in \gamma$. Intuitively, we would expect that the larger the size of the subset, the more alternatives should be approved and the smaller the set, the more demanding agents will become and so fewer alternatives should be approved.

We will take advantage of this possibly variable threshold level, in order to avoid situations in which voters who would vote for some candidates in a set would rather abstain in a subset of this set. To reconcile simplicity with the no-abstention assumption we will assume that the threshold utility level may only vary if the subset has no alternative that would be approved under γ or if the subset is only composed of previously approved alternatives. We will also assume that this threshold level varies the minimum possible. In the first case, only the alternative ranked first for the agent will be approved while in the second case, the agent will approve all except the last. It will be, as if the agent is forced to approve a candidate, in the first scenario, or to drop one *approval*, in the second scenario. Note that, under these assumptions and with only two candidates, *plurality* and *approval* are equivalent.

In a social ordering context, we will assume that alternatives will be socially ranked according to the number of approvals they get. The social choice is the most preferred alternative while the alternative with the smallest number of approvals is the least socially desirable one.

2.2.3 Instant Runoff

Under this voting system, agents are required to rank candidates from most to least preferred. Under sincere voting[8], this ranking corresponds to each voters' preference order. The alternative that is ranked first the fewest times is eliminated[9]. After elimination, results are readjusted so that, in each voter's ranking, the eliminated alternative is substituted by the one following it in the original one. This process is repeated, until no further elimination can be done. Note that,

[8] Voting uniquely to express preferences with no strategic intention; no preference misreporting.

[9] In a tie situation, different rules can be used. A possible criterion would be to eliminate the alternative with more last place attribution. A rule consistent to what we have defined for other voting systems would be to eliminate all the tied candidates so that no tie breaking criterion is used. This is the rule behind our formal definition of the *instant runoff* voting system.

although this voting system may involve a lot of steps, agents are only required to vote once.

Formally, define: $I_1(\gamma, P) = \{x \in \gamma : \ V(x) \leq V(y), \text{for all } y \in \gamma\}$. Taking only into account the set $\gamma \setminus I_1$ preferences are readjusted, and we get $I_2(\gamma \setminus I_1, P)$. This process is repeated until we get $I_{i+1}(\gamma \setminus I_1 \setminus I_2 \setminus \ldots \setminus I_i) = \{\oslash\}$,and I_i is the set of winners.

In what concerns the social ranking of alternatives we will follow an *order of elimination criterion*. In that sense, the alternative that is eliminated first will be the one we consider the less socially preferred, while the last one (the social choice) is, obviously, the most preferred one.

We will now give an example of how these three systems work.

Example 1 *Let there be ten voters and four alternatives* $\{x, y, v, z\}$ *and let* 1 *and* 0 *respectively stand for approval and non-approval:*

4 voters	*3 voters*	*2 voter*	*1 voter*
$x(1)$	$v(1)$	$y(1)$	$z(1)$
$y(1)$	$y(1)$	$v(0)$	$v(0)$
$v(0)$	$z(0)$	$z(0)$	$y(0)$
$z(0)$	$x(0)$	$x(0)$	$x(0)$

Table 1

- *Plurality winner: alternative x with four votes.*
- *Approval winner: alternative y with nine approvals.*
- *Instant runoff winner: alternative v. The process goes in the following way: z, with only one*

first place attribution, is the first alternative eliminated. We get:

4 voters	*3 voters*	*2 voters*	*1 voter*
x	v	y	v
y	y	v	y
v	x	x	x

Table 2

Now, y is eliminated because it is now the one alternative with the fewest first place attributions. Hence we get the final table:

4 voters	*3 voters*	*2 voters*	*1 voter*
x	v	v	v
v	x	x	x

Table 3

Hence, v is the winner under instant runoff.

These three systems respect different optimality conditions. However, there is one thing they all have in common, which is the fact that all of them can induce strategic behaviour (although, as stated in Fisburn and Brams (1978), *approval* seems to do this less often). In addition, they all fail to ensure that the *Condorcet winner* is elected, whenever one exists, (Nurmi 1983). However, under "admissible strategies", *approval* voting is compatible with choosing the *Condorcet winner* (Fishburn and Brams 1981).

Fishburn amd Brams (1978) show that *plurality* fails to respect the *weak axiom of revealed preferences.* Moreover, under *plurality*, we can never assure that the *Condorcet loser* is not among the social choice. Since under our assumptions *plurality* may be constructed from *approval*, where voters only approve their preferred candidate, *approval* voting also fails to respect the *weak axiom of revealed preferences* and the *Condorcet loser criterion.*

In addition, *approval* is the only system of the three that violates *Pareto optimality.* As showed by Nurmi (1983), *plurality* obviously respects *Pareto optimality.* It is straightfoward that *instant runoff*, being a sequential elimination process based on *plurality* rule, would also never elect y when $x\ P_i\ y$ for all i. However, under one-stage *approval*, this can no longer be ensured in case of a tie. Consider the example:

Example 2 *Take* $m = \{x, y, z\}$ *and the following individual preferences:*

Voter 1	*Voter 2*	*Voter 3*
$x(1)$	$z(1)$	$x(1)$
$y(1)$	$x(0)$	$y(1)$
$z(0)$	$y(0)$	$z(0)$

Table 4

Under approval we would get $\{x, y\}$ *as the set of winners. Then we have* $x\ P_i\ y$ *for all* i *and at the same time* $y \in S(m, P)$. *Therefore, approval violates Pareto optimality.*

Instant runoff voting is the only method that respects *Condorcet loser criterion* but it is also the only one violating *monotonicity.* Under *instant runoff*, candidates are sequentially eliminated.

Therefore, any change in individual preferences is capable of changing the order of elimination of alternatives. This does not happen under *plurality* and *approval* as both are completly static methods. For the same reasons, the *weak axiom of revealed preferences* is not verified by *instant runoff voting.*

All the three one-ballot systems fail to respect the *independence of irrelevant alternatives* condition. To illustrate this, let us take a look of the following examples:

Example 3 $\gamma = \{x, y, z\}$ *and we have nine voters with the following preferences, P:*

4 voters	*2 voters*	*3 voters*
$x(1)$	$y(1)$	$z(1)$
$y(0)$	$x(0)$	$y(0)$
$z(0)$	$z(0)$	$x(0)$

Table 5

Under plurality (approval) the socially preferred candidate is x followed by z and the last one would be y: $x\ P_s\ z\ P_s\ y$

Now suppose that, for the last group of voters, we maintain x and y 's relative position and take z to the end of the ranking which is equivalent to removing z from the set. So we changed the position of an irrelevant alternative in what concerns x and y. If S was to respect IIA, we expected x to still be socially preferred to y.

4 voters	*2 voters*	*3 voters*
$x(1)$	$y(1)$	$y(1)$
$y(0)$	$x(0)$	$x(0)$
$z(0)$	$z(0)$	$z(0)$

Table 6

Note that now y is socially preferred to x which is in turn preferred to z. So, the social relative position of x and y has changed while individual ranking between x and y was kept unchanged. Therefore, plurality and, consequentially, approval violate independence of irrelevant alternatives.

Now let us take the case of *instant runoff* .

Example 4 *Take* $\gamma = \{x, y, v, z\}$,

4 voters	*3 voters*	*2 voters*	*1 voter*	*2 voters*
x	z	y	v	v
y	y	v	y	z
v	v	x	z	y
z	x	z	x	x

Table 7

Under instant runoff, the first elimination is y*, followed by* z *and then* x*. Hence,* v *is the winner. So, the social preference is as follows:* $v\ P_s\ x\ P_s\ z\ P_s\ y$.

Take the first group of voters and change x *and* y*'s positions without changing the others.*

4 voters	*3 voters*	*2 voters*	*1 voter*	*2 voters*
y	z	y	v	v
x	y	v	y	z
v	v	x	z	y
z	x	z	x	x

Table 8

Now, x *is the first one to be eliminated, followed by* z/v *and then by* v/z *(there is a tie between the two alternatives however, in this case, the order of elimination is irrelevant for the outcome of* *the election). In this case, y is the winner.*

Clearly, the order of elimination matters because if we take a glance at the social ranking, this is what we observe: $y\ P_s\ v\ P_s\ z\ P_s\ x$.

Even though y and v's individual relative position did not change, we observe that in the social ranking their position differ. The same can be concluded about x and z and also about y and z. Therefore, the instant runoff voting procedure also violates independence of irrelevant alternatives.

Summing up,

How do one-stage systems perform in what concerns the discussed criteria?

	Plurality	*Approval*	*Instant Runoff*
PO	1	0	1
CW	0	0	0
CL	0	0	1
MON	1	1	0
IIA	0	0	0
WARP	0	0	0

Table 9

If we take a look at these results we might be led to conclude that approval performs worse than plurality. In fact, in a binary prespective, and taking only these conditions into account, approval seems to violate more conditions that plurality does. In any case, we should take into account that Fishburn and Brams (1978) argue that *approval* voting respects strategyproofness more than *plurality* - and therefore plurality may actually violate the criteria it seems to meet. And although strategyproofness is not within the scope of our analysis, this binary analysis is still insufficient in order to compare the performances of different methods: we cannot infer about the frequency of violation of each of the conditions. In fact, we may have both systems violating a specific condition but then one of them violating this condition more often.

3 Introduction of a second stage

The first question we set ourselves to answer: do systems perform better when a second stage is introduced?

In a first step, we will consider a second stage, where only two alternatives compete under *plurality* rule. Therefore, in the first stage, two nominees are chosen according to a specific voting rule and then a second stage is needed to elect the winner through *plurality.* The reason why we want to analyse the two-nominees case is because it is the one that is most extensively used in political elections.

For the two-nominee scenario, we will not analyse the *two-stage instant runoff* as in this case the second ballot would become trivial. In other words, using *instant runoff* until one alternative is selected or interrupting the process when we get the two nominees and then selecting the winner by *plurality* always leads to the same result. Hence, there would be no difference between the one-stage and two-stage case.

However we will analyse the *two-stage instant runoff* case when we allow for a third nominee in the second stage. Under this three-nominee scenario, we will first analyse the impact of shifting from a one-stage system to a two-stage system. Afterwards, we will proceed analysing the three three-nominee two-stage systems, trying to draw some preliminary conclusions on which methods should be better and why.

Note that, in all runoff methods, a tie situation may occur in the first stage and in this type of cases a tie breaking criterion would be needed as runoff systems define a specific number of alternatives going to the second stage. We could specify for instance a lexicographic criterion[10]. Consider, however, that in large elections such as the presidential election the likelihood of a tie situation is extremely low. In fact, in many cases the election rules specify no tie-breaking criterion. For that reason and for all two-stage voting systems, we will not specify any criterion to solve ties in the first stage.

3.1 Two-nominee Case

3.1.1 Plurality Runoff

Under *two-nominee plurality runoff*, voters are required to vote twice. In the first stage, two nominees are chosen under *plurality*. Then, these two nominees will compete alone in a second stage. The nominee that gets more votes in the second stage wins. If both alternatives get the maximum number of votes, they will both be considered winners of the election.

It is straightforward that, under sincere voting, a second stage becomes trivial when one of the candidates gets the majority of votes in the first stage. This is the reason why, in reality, a second

[10] This way we could avoid randomness without losing *anonymity* of individuals. Nonetheless, *neutrality* is lost.

stage is only needed if the majority is not reached by any of the candidates in the first round. Yet, note that in case a second stage exists results remain unchanged. However, for simplicity, in our simulations we will always proceed to a second stage even when unnecessary.

3.1.2 Approval Runoff

This method goes as follows. In the first stage agents vote by approving or disapproving candidates. The two candidates with the highest number of approvals go to a second stage where the winner is selected through *plurality*. Analogously to *plurality runoff*, we would expect *approval runoff* to be the simple *approval* voting repeated twice. However, under our assumptions and with only two candidates, *plurality* and *approval* are equivalent. Hence, with a second phase with only two nominees, using *plurality* rule or *approval* is the same. Moreover, in the three-nominee scenario will also choose to use plurality in the second stage. This allows for comparisons with other two-stage methods that use plurality in the second stage[11]. Therefore, from now on, we will refer to *approval-plurality* as *approval runoff*.

3.1.3 Performance

The purpose of this analysis is to understand if there is any evidence of a better performance of methods, when a second stage is added to the picture. If we look carefully at *plurality runoff* and *approval* runoff, we can see that some of the criteria respected were lost but others now become respected. We will ignore tie situations in the nominees' selection. Let us then proceed with this analysis.

[11] Moreover it makes simulations more easy to perform (section 4)

Claim 1 *Take x and y of a given set γ, where $x \; P_i \; y$, for all i. Then x is the winner in a pairwise comparison between x and y. Moreover, when more alternatives are added to the picture and using plurality as the voting system two scenarios with two different implications can occur:*

1. $V(x) > 0 \Rightarrow V(x) > V(y)$ *or,*

2. $V(x) = 0 \Rightarrow V(x) = V(y)$

It is easy to understand that if no agent ranks x first ($V(x) = 0$), then no agent will rank y as well, because all of them strictly prefer x to y. In addition, it is straighforward that, in all other scenarios x would always get a higher number of votes that y does.

Proposition 1 *Two-nominee plurality runoff is Pareto optimal*

Proof. Suppose not. Assume that we have $x \; P_i \; y$, for all i and at the same time $S(\gamma, P) = y$. If y wins the election then it had to win a pairwise comparison in the second stage. For that reason we can be sure that x was not among the nominees (Claim 1). This means that when we compared all the alternatives in the first stage y had at least as many votes as x. Through Claim 1 we know that it can only be the case of $V(x) = V(y) = 0$. This means that no agent ranks x or y first. Two things can be concluded so far:

1. A tie breaking criteria had to be used in the first stage and y was the selected alternative.

2. The second nominee has to be the only alternative of the set with $V(.) > 0$ and, for that reason, ranks first in all individual preferences (otherwise y would not be among the nominees).

Then y could not win a pairwise comparison with this other nominee which contradicts $S(\gamma, P) = y$. ■

Proposition 2 *Two-nominee approval runoff is inconsistent with Pareto optimality.*

We could be led to think that the fact that *Pareto optimality* is respected by *plurality*, together with fact that, in the last stage, *plurality* is the system used, would be sufficient to guarantee that if $x \ P_i \ y$ for all i, then y could not win. However, this is not so. In particular when we consider specific tie situations in the first stage.

Example 5 *Take the following situation:*

voter 1	*voter 2*	*voter 3*
$x(1)$	$v(1)$	$x(1)$
$y(1)$	$x(0)$	$y(1)$
$z(0)$	$z(0)$	$v(1)$
$v(0)$	$y(0)$	$z(0)$

Table 10

Let $A(.)$ *be the function that assigns the number of approvals for each alternative. We have* $A(z) = 0$, $A(x) = A(y) = A(v) = 2$. *Therefore, there is a tie. Now suppose that, under a given tie-breaking criterion, we get* v *and* y *as the alternatives selected to the second stage. In that case,* y *will be the winner.*

Note that, nonetheless, *approval runoff* would only break *Pareto optimality* in this specific tie situation which is unlikely in large elections.

Proposition 3 *Two-nominee plurality runoff and approval runoff are inconsistent with the monotonicity criterion.*

Proof. *Suppose the following situation,*

6 voters	*5 voters*	*4 voters*	*2 voters*
$x(1)$	$z(1)$	$y(1)$	$y(1)$
$y(0)$	$x(0)$	$z(0)$	$z(0)$
$z(0)$	$y(0)$	$x(0)$	$x(0)$

Table 11

Clearly, x and y, with six votes each, will be the alternatives passing on to the second stage. In a pairwise comparison between x and y this is what we observe,

6 voters	*5 voters*	*4 voters*	*2 voters*
$x(1)$	$x(1)$	$y(1)$	$y(1)$
$y(0)$	$y(0)$	$x(0)$	$x(0)$

Table 12

Therefore, x is the winner under both two-nominee plurality runoff and approval runoff.

Now suppose that we raise the relative position of x for the last group of voters. Note that *no assumption was made on how the threshold utility level in the approval decision would vary in these situations. Take the case where voters decide to only approve one alternative (equivalent to plurality). According to the monotonicity criterion, x should still be the optimal social choice under plurality runoff and approval runoff. However, any change in relative positions may now change the alternatives that proceed to the second stage and for that reason, it may influence the result. Let us take a more detailed look into the consequences of this change.*

6 voters	*5 voters*	*4 voters*	*2 voters*
$x(1)$	$z(1)$	$y(1)$	$x(1)$
$y(0)$	$x(0)$	$z(0)$	$y(0)$
$z(0)$	$y(0)$	$x(0)$	$z(0)$

Table 13

Now, x and z are the two nominees, and so these will be the alternatives competing in the second stage.

6 voters	*5 voters*	*4 voters*	*2 voters*
$x(1)$	$z(1)$	$z(1)$	$x(1)$
$y(0)$	$x(0)$	$x(0)$	$z(0)$

Table 14

Therefore, z beats x and wins the election. ■

Obviously, we could guarantee that approval runoff ensures *monotonicity* under a specific assumption on how the threshold level of utility varies. More specifically, if we assume that voters will approve the exact same alternatives than they approved before together with the raised alternative, whenever reasonable (e.g. the alternative was raised to the top or right next to an approved alternative). This would be enough to guarantee that the exact same alternatives would go to the second stage and, hence, the social choice would be the same. In these cases, no alternative would be hurt if it were better ranked in a specific individual preference ordering.

However, in general, under a two-stage system, a candidate can be clearly made worse off by being "liked more" in a group of voters. Equivalently, we observe that an alternative can be helped if it is in a lower position in a given individual ranking of preferences. In our example, x is the winner under some initial preferences and loses its victory when a group of voters actually likes x even more than they did before. In contrast, z is the new winner and is in a worse position in a specific individual ranking and in the exact same position in all others. This happens because y is now at a lower position, in such a way that it has no longer sufficient votes to move on to the second stage. Consequently, x has now to compete with z instead which has a stronger relative position than x in the majority of voters' preferences.

The need of pre-selecting candidates makes any change of relative position of alternatives very much relevant even if it might seem irrelevant at first glance. This is why *monotonicity* is now violated but was not before, under the one-stage version of the runoff systems.

For the very same reasons, we can say that *independence of irrelevant alternatives* is still unrespected by the two two-nominee runoff systems. The reasoning is straightforward and therefore we will state it without proof.

Proposition 4 *Independence of irrelevant alternatives is still not respected by two-nominee plurality runoff and approval runoff.*

Other conditions that were previously violated still are. This is the case of the *weak axiom of revealed preferences* and *Condorcet winner criterion.*

Proposition 5 *Two nominee plurality runoff and approval runoff violate the weak axiom of revealed preferences.*

Proof. Consider the following case,

$\gamma = \{v, x, y, z\}$

3 voters	2 voters	3 voters
$x(1)$	$y(1)$	$v(1)$
$v(0)$	$z(0)$	$y(0)$
$y(0)$	$x(0)$	$z(0)$
$z(0)$	$v(0)$	$x(0)$

Table 15

x and v are the two nominees. Hence, in the second stage we obtain,

3 voters	2 voters	3 voters
$x(1)$	$x(1)$	$v(1)$
$v(0)$	$v(0)$	$x(0)$

Table 16

The selected alternative is therefore x.

Now take a subset m of γ , $m = \{x, y, z\}$

3 voters	2 voters	3 voters
$x(1)$	$y(1)$	$y(1)$
$y(0)$	$z(0)$	$z(0)$
$z(0)$	$x(0)$	$x(0)$

Table 17

If *plurality* and *approval runoff* were to respect the weak axiom of revealed preferences, x should still be the winner, as $m \cap S(\gamma, P) \neq \emptyset$, hence $S(m, P) = m \cap S(\gamma, P) = x$. However, this is not the case. The fact that v is no longer available, changes the alternatives going to the second stage. Now, x and y compete in a second round and, in this case, x is no longer the winner. y wins and therefore, $S(m, P) \neq m \cap S(\gamma, P)$. Both *plurality runoff* and *approval runoff* violate WARP. ■

Proposition 6 *Condorcet winner criterion is still not respected by the two-nominee approval runoff and plurality runoff*

Proof. Consider the following situation,

4 voters	3 voters	2 voters
$x(1)$	$y(1)$	$z(1)$
$z(0)$	$z(0)$	$y(0)$
$y(0)$	$x(0)$	$x(0)$

Table18

The two nominees are x and y. However the *Condorcet winner* is z as it beats both x and y in pairwise comparisons. Clearly, the *Condorcet winner* criterion is violated by both *plurality runoff* and consequently by *approval runoff.* ■

So far, we have verified that *monotonicity* is lost and that the *weak axiom of revealed preferences*, *independence of irrelevant alternatives* and the *Condorcet winner* criterion are still violated when a second stage is added.

Proposition 7 *Both two-nominee plurality runoff and approval runoff respect the Condorcet loser criterion.*

Proof. A *Condorcet loser* is the alternative that loses all pairwise comparisons. Hence, if x is the Condorcet loser of the set γ and $V(.)$ is the function that counts the number of votes that each alternative has, we know that $V(x) < V(y)$, where y is the alternative pairwise compared to x, and this is true for all $y \in \gamma$.

A necessary condition for a social choice, in any of these two runoff systems, is to win the second round of elections. Taking x and an unspecified $y \in \gamma$ as the two nominees in the second stage, for x to win it will need $V(x) \geq V(y)$. So, either x is not a *Condorcet loser* or it does not win the second round. In the instance of both happening we would therefore reach a contradiction. ■

Hence, even if it happens to be among the nominees, the very definition of *Condorcet loser* ensures that it cannot win the last stage of the election.

3.2 Three-nominee case

3.2.1 Plurality and Approval Runoff

When another nominee is added to the picture, the conditions that were violated before will remain so, as the reason why they were violated before had nothing to do with the number of nominees but due to the voting system procedure itself. The fact that *independence of irrelevant alternatives, monotonicity* and the *weak axiom of revealed preferences*, are violated, by both *two-nominee plurality* and *approval runoff*, is a consequence of the existence of a second stage and not of the number of nominees in the second stage. Changes in the individual order of preferences, or in the set of alternatives may change the alternatives that make it to the second stage and hence, the winner might be different. This could happen whether we have two, three or more nominees. Moreover, the fact that *Pareto optimality* is violated under approval runoff has only to do with the method used in the first stage (approval) which is exactly the same when we introduce another nominee. This is why we state without proof that these three criteria are still violated when we add an extra nominee to the procedure.

Two-nominee plurality runoff and *approval runoff* were only able to respect two and one condition respectively. One of these conditions, *Pareto optimality* (only respected by *plurality runoff*), is respected because of the voting system itself and the fact that it is respected has again no relation with the number of nominees in the second stage. Note, however, that the reason why both the runoff voting systems were consistent with *Condorcet loser* criterion before was not due to the scheme used in the second stage. In fact, *plurality* voting is typically inconsistent with *Condorcet loser* condition. What guarantees that the *Condorcet loser* is not among the elected alternatives, is the fact that we have two alternatives being compared in the second stage, and *Condorcet losers* are denominated so because they lose all pairwise comparisons. Hence, by adding one more nominee to the second stage, *Condorcet loser* condition is now violated by both *plurality* and *approval runoff*.

3.2.2 Two-stage instant runoff

In a similar fashion to the *approval runoff*, the *two-stage instant runoff* will not be the *instant runoff* repeated twice. As a matter of fact, this would lead to a trivial and so unnecessary extra stage. Results would differ and due to that the only plausible reason to use such scheme would be the necessity of a second stage. But even in the Oscar case, where the phase with nominees is necessary, *plurality* is the system chosen for the second stage after a first stage where *instant runoff* is used[12]. We will denote the voting system where *instant runoff* is used in the first stage and *plurality* in the second to select the best alternative from a set of three nominees as the *two-stage instant runoff*.

Again, all previously violated criteria will remain so. Furthermore, some previously respected

[12] With the exception of the 2010 Best Picture Academy Award, that now asks the members of the academy to rank the ten nominees according to their preferences. In this case, *instant runoff* is repeated twice.

criteria might be lost. The ability to reject the *Condorcet loser* is the only condition that is lost when an extra stage is added to the picture and the reason is again straightforward. The two-stage *instant runoff* is still *Pareto optimal.* If $xP_i\ y$, for all i, even though we cannot guarantee that y is not among the nominees, we know that if it is, so is x and then the fact that we use *plurality* in the second stage ensures that y will never be chosen.

We can summarize the results in the following tables:

	Plurality	*Plurality Runoff (two-nominee)*	*Plurality Runoff (three-nominee)*
PO	1	1	1
CW	0	0	0
CL	0	1	0
MON	1	0	0
IIA	0	0	0
WARP	0	0	0

Table 19

	Approval	*Approval Runoff (two-nominee)*	*Approval Runoff (three-nominee)*
PO	0	0	0
CW	0	0	0
CL	0	1	0
MON	1	0	0
IIA	0	0	0
WARP	0	0	0

Table 20

	Instant Runoff	*two-stage Instant Runoff (three-nominee)*
PO	1	1
CW	0	0
CL	1	0
MON	0	0
IIA	0	0
WARP	0	0

Table 21

It is worth remarking that we have assumed throught that agents will not vote strategically.

This assumption will be discussed in more detail in the next section.

Apart from that, if we take a look at the tables, we could get the wrong impression and trust that, by introducing a second stage, nothing gets significantly better. This, however, would be at odds with the worldwide use of two-stage systems (even when two stages are not strictly necessary) which are more costly and time-consuming. Also, we might be led to conclude that no significant difference exists between the analysed voting systems.

Note however that this is a binary analysis that provides little information. Some conditions may be less or more violated than they were before and so, stating that a given criterion is still violated is not enough for a thorough analysis. Having that in mind, a frequency analysis with preference simulation will be our next step, focusing on the two Condorcet criteria.

4 Frequency analysis

In this section, we will analyse the effects of the introduction of a second stage but from a different perspective.

The *Condorcet winne*r is the alternative that beats all the others in pairwise contests. As stated by Saari and Newenhizen (1985), "the *Condorcet winner* appears to capture the true choice of the voters". However, none of our voting systems can guarantee that the *Condorcet winner* is elected whenever it exists.

In turn, the election of the *Condorcet loser* could be seen as even more problematic. The fact that an alternative that loses all pairwise comparisons is elected as the winner of a given set means that the system leads to a poor reflection of individual preferences. Hence, respecting the

Condorcet loser criterion would be clearly desirable. Unfortunately, not all voting systems are capable of behaving this way. For that reason, it is also important to understand whether we can see an improvement in this context by introducing a second stage.

Through simulation of preferences, it is possible to analyse the scale to which a given condition is violated under different voting systems. In our case, we will be able to tell if, by introducing a second stage, we are able to elect the *Condorcet winner* more often than we were with a single-stage system. The same approach will allow us to check if the *Condorcet loser* is selected fewer times, when a second stage is introduced. Moreover, we will be able to compare the different two-stage systems in terms of frequency of violation of the Condorcet criteria.

4.1 Simulation method

Simulation is needed to get individual preferences. A lot of assumptions can be made about these preferences. We will start by simulating the model using the most common assumption about preferences. Afterwards, we will try to confirm the results we got using different assumption about preferences. This will allow us to get more reliable results.

4.1.1 Impartial Culture

The *impartial culture* assumption (IC) is widely used due to its simplicity. Under IC, each individual has equal probability of having each of the possible preference profiles. If we have n alternatives then, the probability of a given individual having a specific profile is 1/n!. This seems a pretty obvious and simple assumption in general however, in many situations, it is unrealistic. This is the case of presidential elections. Typically, the fact that someone prefers a given candidate might influence my choice and at the same time my favourite candidate is correlated to my second favourite candidate, and so on. Take the following example.

x=extreme right wing candidate

y=centre candidate

z=extreme left wing candidate

Suppose that x is my favourite candidate. Typically, z will be my least preferred candidate as its platform ideals supposedly differ more from mine than y's ideals do. So, preferring y after x is much more likely than preferring z after x and therefore this leaves y as the worst alternative. We may then observe that preference profiles are not equally likely.

Anyway, in many situations, this assumption makes sense and due to its lack of complexity we shall proceed based on it. But we will also consider an alternative approach in order to check whether the results are robust to changes in assumptions.

4.1.2 Single-peakedness

Under *impartial culture,* preferences are completely heterogeneous which might not be as realistic as desirable. To simulate more homogeneous preferences, we will work under the assumption of single-peaked preferences. Under this assumption, we know that voters' preferences reflect the fact that agents are worse off, the further away they are from their ideal choice.

Example 6 *Suppose the case of five alternatives, organized left to right in one dimension:* $\{a, b, c, d, e\}$. *Suppose that* c *is the ideal choice for a given voter. Then, when this voter compares alternatives to* *the left of c it strictly preferes the one that its closer to the ideal choice. The same is true while comparing alternatives to the right of c.*

Under single-peakedness, preferences ordered like $\{c, e, d, b, a\}$ *are not allowed because e should be less preferred than d due to the fact that e is further from the ideal point c. Note, however, that we are not able to tell which of the alternatives, b and d, are "more similar" to c hence, we can only compare alternatives to the right of the preferred one or to the left between themselves. Preferences ordered like* $\{c, b, a, d, e\}$ *are allowed as we may, in fact, have a preferred to d even if a is further from ideal choice c. This makes sense as for a multiplicity of reasons voters may for instance have a preference for the alternatives to the left of the ideal choice over the right alternatives.*

So, after simulating the model under *impartial culture*, we will try to understand how things work if we assume *single-peakedness* of preferences.

4.2 The model

In this section, our concern is to understand how the ability of electing the *Condorcet winner* changes according to the voting system. We know that none of the methods used is able to guarantee the election of the pairwise comparison winner when it exists. However, we are now interested in analysing which method respects this condition more often. In particular, we want to verify if the introduction of a second stage generates an improvement in this context. The same should be verified about the *Condorcet loser*.

To replicate a voting situation we need voters, with their respective preferences, and a voting system to select the winner. In theory, besides preferences, we would still need the actual vote however, we are assuming sincere voting and therefore, votes reflect true preferences of each agent. This may seem a poor assumption as agents tend to maximize the utility of their vote, the so-called strategic vote. However, Blais, Laslier, Sauger and Van der Straeten (2009) show that sincere voting works better for a very complex voting system while strategic voting works better for simple methods. Furthermore, they suggest that the probability of sincere voting might also depend on the number of candidates. In addition, Laslier (2009) proves that for a large electorate, in the absence of ties, the best response under *approval* voting is sincere.

The fact that, in our model, we will rely on a voting situation where most complex voting systems are used (runoff systems) and the fact that we will have seven candidates competing (while most studies use three) makes the sincere voting assumption more realistic than it would seem at first glance. Under some voting systems, such as *instant runoff*, the fifty voters would find it difficult to fully understand each of the seven candidates' probability of winning (pivot probabilities) and so, a complete rational behaviour would be difficult in reality. Understanding the optimal strategy would require a complicated reasoning and consequently, agents would hardly find a clear utility maximizing voting strategy. Computational complexity would most likely lead to sincere voting.

In the model, we replicate a voting situation with fifty voters expressing their true preferences for seven candidates where ties are not allowed and so only one of the alternatives can win. Ties are broken randomly.

We only allow for one winner mainly due to practical reasons when comparing outcomes of the different voting systems. Furthermore, this is what typically happens in reality where the majority of situations asks for a single winner. Therefore, we observe the need of tie breaking rules in contests and in elections. A random choice of the winner in a case of tying seems to be a plausible assumption in this case. In these cases, we will therefore assume a random choice of the winner.

In *approval* voting, the *approval* criterion that we have chosen was a random one. The computer would randomly select voters' threshold line. A potential improvement of the model would be, for instance, to understand how in reality an *approval* decision is made and to try to replicate it in the model. When Laslier and Van der Straeten (2008) collected the French data for the presidential elections, they concluded that, under a set of sixteen alternatives, voters would approve, on average 3,15 of them. They also found that the distribution around this point was smooth and that one-name ballots were rare. This is the kind of information that can be used to replicate real situations and, therefore, to reach more reliable results.

We will compare eight different voting systems' outcomes with what would be the Condorcet method winner, the so-called *Condorcet winner* (if it exists), and the *Condorcet loser*. Keep in mind that our aim is to see if the introduction of a second stage makes the *Condorcet winner* and the *Condorcet loser* criteria more often respected.

In order to do so, we will simulate a hundred thousand voting situations. These situations only differ in voters' preferences which are a hundred thousand times drawn randomly and independently from a given set of preferences. This set of preferences is different depending on the assumptions we make about them. Under *single-peakedness*, for instance, we have the set of all possible single-peaked preferences while under *impartial culture* all preference orderings may appear. In each of the individual situations, composed of fifty voters and seven candidates, we will have a set of ten different outcomes, each one corresponding to a given method (eight different voting systems plus the *Condorcet winner* and *Condorcet loser*). This will be repeated a hundred thousand times and so, in the end, we will have a hundred thousand by ten matrix of results.

The ideia in this simulation was to deviate from the typical three candidate approach. Because we are working with two-stage systems we will be making a pre-selection of alternatives before the winner is actually chosen. Then, due to the reason that we want our results to apply not only to political elections, e.g. presidential elections, but also to a multiplicity of situations such as the Oscar case we will be working in two distinct situations: two-nominee two-stage systems and three-nominee two-stage systems. The fact that we are making pre-extractions of alternatives makes it already clear that three candidates is not enough. Then, the fact that we will need in some situations to work with three nominees makes it clear that even five nominees might be insufficient. However, having more candidates makes our analysis more and more complex. In particular, in the case of three nominees and *impartial culture* we have a set of 6 possible agents' preference profiles while in the case of seven candidates for instance this number rises to 5040. Moreover, constantly working with information of 3*50*10000 dimension is quite different from working with a 7*50*10000 dimension information . For that reason we were not able to analyse the case of more than seven candidates, due to computational restrictions. Trying to reconcile an acceptable number of candidates with an acceptable number of voters together with an acceptable number of simulations and analysing at the same time eight voting systems and two criteria is a complex problem and restrictions will bind.

The results will be analysed as described previously and a consistent frequency analysis will be drawn.

It is important to underline that, in spite of computational complexity, the number of candidates and voters is completely adjustable. In fact, we would suggest as a further improvement of this study to change these variables and see how the results would differ accordingly. Another possible change for a more complete study would be to choose a different criterion and repeat the procedure.

Some conditions could be easily programmed and so the same kind of study could be done with no significant extra effort. In addition, this model could be useful in cases where we have data on agents' preferences as it would allow us to predict which candidate would be the winner under sincere voting and afterwards, compare it with the actual winner. Furthermore, we could verify how the voting system choice would determine the election winner.

Before looking at the outcome of the simulation, let us first take a look at some important theoretical results. Note that these results hold independently of the assumption on preferences.

Claim 2 *In the absence os ties, whenever we select candidates for a second stage, under any of the methods, the winner of the equivalent one-stage method will be among the set of nominees .*

The reasoning of this claim is straightforward. Note that, for instance, under *plurality runoff*, we use *plurality* to elect the nominees. The same can be said about *approval runoff* and *two-stage instant runoff*. Hence, if a given alternative was the winner under *plurality* it is obviously among the two or three more voted alternatives. Therefore, this alternative will be among the set of nominees.

Claim 3 *In every voting situation, where the Condorcet winner is chosen under a one-stage method and no ties have occured, it will also be chosen under its two-stage equivalent when only two nominees are pre-selected.*

Proof. If the *Condorcet winner* was chosen, under a one-stage system, we know that it will for sure be among the two nominees (Claim 2). Then, by the definition of a *Condorcet winner*, we know that it will be elected through *plurality* when only two alternatives are compared. ■

For these reasons, we already get the intuition that when a second stage with two nominees is introduced, things very hardly get worse. This fact is reflected in our simulation's results under both the assumptions made on preferences.

It is important to stress the fact that all the results obtained were completely robust as if we were to simulate the voting situations 10 times instead of 10000 there would be no significant change.

4.2.1 Results under IC

After the voting situations were simulated, we were able to verify how often *Condorcet winner*s were elected, under each of the voting systems in question.

Let us take a take a look at the results:

Condorcet winners in the simulation under IC

Total number of CW	44694
Relative number of CW	$44,694\%$

Table 22

Voting systems and the *Condorcet winner* criterion under IC

Voting System	Absolute Frequency of CW election	Relative Frequency of CW election
Plurality	24113	$53,951\%$
Plurality Runoff (two-nominee)	33495	$74,943\%$
Plurality Runoff (three-nominee)	31801	$71,153\%$
Approval	27499	$61,527\%$
Approval Runoff (two-nominee)	37266	$83,380\%$
Approval Runoff (three-nominee)	35101	$78,536\%$
Instant Runoff	39378	$88,106\%$
Two-stage instant runoff	33009	$73,856\%$

Table 23

For *approval* and *plurality*, introducing a second stage increases the probability of electing the *Condorcet winner*, when one exists. This improvement is particularly strong in the two-nominee case. This leads us to the intuitive conclusion that, many times, when Condorcet winners are not elected, they are, in fact, among the two or three most-voted alternatives.

In the case of *instant runoff*, the introduction of a second stage is not beneficial. Note however, that *instant runoff* was already a multi-stage process but where voters only needed to vote once. In the seven candidates case, *instant runoff* is in fact, a six-stage process of elimination of weaker candidates. What we actually do in the two-stage *instant runoff* is to accelerate the process, by skipping one of the steps. The first stage is an *instant runoff* procedure interrupted after its fourth stage and then, we add an extra step, where *plurality* is used to select to winner from three nominees. The *two-stage instant runoff* is therefore a five-stage process, yet, voters are required to vote twice. Consequently, it has, in practice, fewer stages than *instant runoff* has. The result is as expected.

So, given this interpretation of the *two-stage instant runoff*, we may conclude that under *impartial culture* more stages increase the probability of election of a *Condorcet winner.* This is the reason why, *instant runoff*, is the most efficient of the eight voting systems in what respects the *Condorcet winner* criterion. By looking at the data, we conclude that, under *instant runoff*, we are able to elect more fifteen thousand and two hundred sixty five Condorcet winners that were not previously selected through *plurality.* This is a large number and comes as a result of an efficient elimination of weaker alternatives.

In what respects the two-stage voting procedures, *approval runoff* voting seems to be the most efficient of all. *Plurality runoff* comes off as the least appealing in what concerns the *Condorcet winner* election. Although differences between the three two-stage systems are not as strong as the differences between one-stage and two-stage systems, we observe that, if we look to the relative frequency as an efficiency rate, *three-nominee approval runoff* is 4,48 percentage points more efficient than *two-stage instant runoff* and 7,383 percentage points more efficient than *three-nominee plurality runoff.*

We may acknowledge that each version of *approval* voting beats *plurality* in its equivalent counterpart and therefore this supports the idea that *approval* is a good substitute for *plurality.*

The very same analysis was done for the *Condorcet loser* criterion. Here, our goal is to verify if the introduction of a second stage reduces the probability of electing a *Condorcet loser.* In fact, having an additional stage decreases significantly the likelihood of selecting a *Condorcet loser* as the social choice. In the extreme cases, where there are only two nominees in the second stage, the likelihood of this is reduced to zero.

The results are as following:

Condorcet losers in the simulation under IC

Total Number of CL	44659
Relative number of CL	$44,659\%$

Table 24

Voting systems and the *Condorcet loser* criterion under IC

	Absolute frequency of CL election	Relative frequency of CL election
Plurality	418	$0,93523\%$
Plurality Runoff (2 nominees)	0	0
Plurality Runoff (3 nominees)	80	$0,17914\%$
Approval	89	$0,19929\%$
Approval Runoff (2 nominees)	0	0
Approval Runoff (3 nominees)	17	$0,03807\%$
Instant Runoff	0	0
Two-stage instant runoff	78	$0,174657\%$

Table 25

If we take a look at the results, the first thing to be noticed is that the violation of the *Condorcet loser* criterion is a problem with a significantly smaller dimension than the violation of the *Condorcet winner* criterion. The frequency at which each voting system selects the *Condorcet loser* as the election winner is never superior to 1%, while the frequency of violation of the *Condorcet winner* criterion reaches 46% in *plurality*'s case.

Although, overall, numbers are almost insignificant in the *Condorcet loser* criterion's case, we observe that, under *plurality* and *approval*, when we add an extra stage, even if we use three nominees, we are able to reduce the probability of an undesirable election of the *Condorcet loser* by more than 80%. Obviously, and for reasons stated previously, when we have only two nominees, we are able to guarantee that no Condorcet loser is ever selected as the social choice. In these cases, the efficiency rate, which can be seen as the relative frequency of non-election of the *Condorcet loser* when one exists, is 100%. Note that, this is also the case of *instant runoff,* although in theory it is a one-stage voting system. A *two-stage instant runoff*, having three nominees competing in the last stage, can only perform worse than *instant runoff*. In fact, *two-stage instant runoff* only has an efficiency rate of approximately 99,83 %, as with this method, the *Condorcet loser* is elected 78 times.

Again, excluding *instant runoff* and focusing on three-nominee runoffs, *approval runoff* is the most efficient in what respects the *Condorcet loser* criterion. Furthermore, all types of *approval* voting outperform their *plurality* counterparts.

Exactly in the same fashion as in the *Condorcet winner* analysis, *instant runoff* is the most efficient one-stage method and the reason for this is that it is, in fact, a six-stage voting system even though voters only need to cast a vote once. In what concerns three-nominee two-stage voting systems, *approval runoff* voting excels, electing the *Condorcet loser* 79 % less often than *plurality runoff* and 78 % less often than *two-stage instant runoff*.

We can once again conclude that, under *impartial culture*, introducing a second stage improves voting systems ability to respect the *Condorcet loser* criterion.

4.2.2 Results under single-peakedness

The results under *single-peakedness* are quite different and the reason is clear. Under *single-peakedness*, the center alternatives are the most likely to win. Take the case of seven candidates organized left to right in one dimension, $\{a, b, c, d, e, f, g\}$. Note that there is only one possible preference profile ranking a as the top alternative, which is $\{a, b, c, d, e, f, g\}$, while there are twenty possible preference profiles ranking d at the top. In fact, the set of all possible single-peaked preferences, with seven candidates, is composed of sixty-four different preference orderings.

The distribution of single-peaked preference orderings according to the top-ranked alternative

Number of preferences ranking a first	1
Number of preferences ranking b first	6
Number of preferences ranking c first	15
Number of preferences ranking d first	20
Number of preferences ranking e first	15
Number of preferences ranking f first	6
Number of preferences ranking g first	1

Table 26

Therefore, a random draw from the whole set of single-peaked preferences will make it more likely to have d as the *Condorcet winner*. Comparing a and d, we know that agents preferring candidates to the right of d will prefer d to a. Moreover, some agents preferring candidates to the left might also prefer d to a. Candidate d could therefore easily get a majority against a. The fact that, in our sample, the number of preferences where d is the top candidate is much larger than the ones where b is the top candidate also makes it easier for d to beat b. Finally, we would expect d to beat c as well, even though we might expect this to be less likely than the previous cases (and note that for e,f and g the reasoning is symmetric). In fact, 98% of the times where a *Condorcet winner* exists in our simulation, it is indeed d. In the remaining 2% of cases, the winner is either c or e.

Moreover, under homogenous preferences, we expect the existence of both a *Condorcet winner* and *Condorcet loser* to be easier to obtain. Actually, in our simulation, we concluded that $98,263\%$ of the times we had a *Condorcet winner*. This is almost 120% more *Condorcet winners* than we had under *impartial culture*. As expected, the probability of the existence of a *Condorcet loser* also increased substantially. Under *impartial culture* we found a *Condorcet loser* approximately 45% of the times while under *single-peakedness* this number rises to 89%

In what concerns the *Condorcet loser* criterion and under single-peakedness, an important conclusions is that very seldom would a method choose a or g as winners. In fact, none of the methods, each using one hundred thousand simulations, ever elects the *Condorcet loser*. However, this was not the case under *impartial culture* because preferences were much more heterogeneous and so, at first glance, it would not be so easy to detect a *Condorcet loser* in a specific situation with fifty voters and seven candidates. Here, however, we would barely need calculations to see that a *Condorcet loser* would be either a or g.

Claim 4 *Under single peakedness, every preference profile ranks either a or g as the least preferred alternative.*

The reasoning of this claim is straightfoward and so we state it without proof.

Proposition 8 *Under single-peakedness, whenever a Condorcet loser exists, it will be one of the more extreme alternatives.*

Proof. Take for instance, without loss of generality, the case of a given set of seven alternatives, organized left to right as follows: $\{a, b, c, d, e, f, g\}$. Here, whenever a *Condorcet loser* exists, we can be sure that it is either a or g.

Through claim 4, we know that one of three things can happen:

1. a is the alternative that ranks last in the majority of voters' preferences

2. g is the alternative that ranks last in the majority of votes' preferences

3. a and g rank last with the exact same frequency

If we consider case 1, we can immediatly conclude that if we compare a to any other alternative, we would have a lose that pairwise contest because a is ranked last in more than 50% of the preference orderings. This way, no candidate other than a could be the *Condorcet loser*.

The reasoning is similar for scenario 2 but with g as the *Condorcet loser* instead.

Under Scenario 3, we know that if we compare a to g, the two alternatives will tie and therefore neither a nor g can be the *Condorcet loser*, which by definition requires the alternative to strictly lose every pairwise comparison. Aditionally, note that no other alternative can in fact be the *Condorcet loser*. Both a and g are ranked in a lower position than all the other alternatives in at least 50% of the preferences' profiles.

We can therefore conclude that, under single-peakedness, when a Condorcet loser exists, it is either the extreme right alternative (in this case g) or the extreme left alternative (a). ■

This is the reason why, in our simulation, we always observe either a, g or no *Condorcet loser* at all. Moreover, it becomes almost impossible for one of our methods to elect the *Condorcet loser*.

Let us check specific results.

Condorcet winners in the simulation under *single-peakedness*

Total number of CW	98263
Relative number of CW	$98,263\%$

Table 27

Voting systems and the *Condorcet winner* criterion under *single-peakedness*

Voting System	Absolute Frequency of CW election	Relative Frequency of CW election
Plurality	66033	$67,200\%$
Plurality Runoff (2 nominees)	87934	$89,488\%$
Plurality Runoff (3 nominees)	34770	$35,385\%$
Approval	73004	$74,294\%$
Approval Runoff (2 nominees)	91821	$93,444\%$
Approval Runoff (3 nominees)	34254	$34,860\%$
Instant Runoff	86695	$88,228\%$
Two-stage instant runoff	31625	$32,184\%$

Table 28

As expected (claim 3), introducing a second stage, with two nominees to be elected, only makes things better. If we look at the results, *plurality runoff* with two nominees elects the *Condorcet* *winner* around 33% more times than *plurality*. This number reduces to 25% when we compare *two-nominee approval runoff* with one-stage *approval*. Consequently, we find evidence that the introduction a second stage, where the winner is chosen from a set of two nominees, improves the chances of electing the *Condorcet winner*.

Suprisingly enough, this improvement is no longer observed when we add a nominee to the second stage. Let us then try to understand why three-nominee two-stage procedures perform this inefficiently under *single-peakedness* while under *impartial culture* they performed with much more success and accuracy than any one-stage method.

Under single-peaked preferences and in particular in our simulation, taking all the methods into consideration, more than 90% of the times the three nominees are the center candidates. For instance, if we have $\{a, b, c, d, e, f, g\}$ as the set of alternatives, the center candidates will be $\{c, d, e\}$. After getting these three nominees, *plurality* will be the system used to select the winner under all the two-stage systems. Note that c will obtain all the votes to the left of d, e will obtain all the votes to the right and d will only obtain votes of agents who rank d as the top candidate. So, it is enough to have less than one third of the agents ranking d as the top alternative to know that c or e will win the election, even if we have d as the *Condorcet winner*. For that reason, we have three-nominee two-stage systems performing much worse than their one-stage counterparts.

Condorcet losers in the simulation under *single-peakedness*

Total Number of CL	89 044
Relative number of CL	$89,044$ %

Table 29

As stated before, under *single-peakedness*, no method, in a total of a hundred thousand simulations, ever elects the *Condorcet loser*. The main reason for this is the fact that under *single-peakedness* we have much more homogeneous preferences than before. We now have 99% more Condorcet losers than under *impartial culture* which is demonstrates the homogeneity of preferences. Under single-peakedness, we would seldom find a *Condorcet loser* among the two or three second-stage nominees and, for that reason, it is much easier to reject a *Condorcet loser* than it was under *impartial culture*. Actually, we can say that, in practice, it becomes almost impossible to violate the *Condorcet loser* criteria although theoretically it is not (under one-stage and three-nominee two-stage methods). The following example shows that it is indeed possible to violate the *Condorcet loser* criterion, even under single peakedness.

Example 7 *Take the case of seven alternatives* $\{a, b, c, d, e, f, g\}$ *and* 26 *voters with the following preferences over the set of candidates:*

11 *voters*	6 *voters*	5 *voters*	4 *voters*
$a(1)$	$g(1)$	$c(1)$	$f(1)$
$b(0)$	$f(0)$	$d(0)$	$g(0)$
$c(0)$	$e(0)$	$e(0)$	$e(0)$
$d(0)$	$d(0)$	$f(0)$	$d(0)$
$e(0)$	$c(0)$	$g(0)$	$c(0)$
$f(0)$	$b(0)$	$b(0)$	$b(0)$
$g(0)$	$a(0)$	$a(0)$	$a(0)$

Table 30

Note that a loses all pairwise comparisons with any alternative other that itself and therefore, by definition, it is the Condorcet loser of this set of preferences.

However, at the same time, we elect it as the social choice, $S(\gamma, P)$, under one-stage plurality and approval as well under two-stage instant runoff and three nominee two-stage plurality and approval.

5 Conclusions

Two-stage systems are more and more commonly used. We see them in the main political elections all over the world as well as many other important contests. The Academy of Motion Picture Arts and Sciences (AMPAS), for instance, uses a two-stage system in the process os selecting the Oscar winner, for each category.

However, *ceteris paribus*, adding a second stage is costly. These systems require voters to go to the polls twice and, in the case of political elections, political instability may rise in-between stages. Therefore, it is imperative to understand why people use these systems. Trying to understand how two-stage systems perform better than one-stage systems, is the main focus of this paper.

We have analysed the impact of moving from a one-stage to a two-stage procedure in three different methods: *plurality*, *approval* and *instant runoff* voting system. We have studied both cases of two-nominee and three-nominee selection. The performance of the different methods was evaluated according to their ability to respect six different criteria which are currently among the most debated ones in the literature.

At first, in a first binary approach, nothing seems to get significantly better, when one adds to the picture a second stage where *plurali*ty is the system used. For this reason, we decided to focus on the Condorcet criteria and proceed with a frequency analysis in order to understand if these criteria are respected more or less frequently, when a second stage is introduced. The ability to select the *Condorcet winne*r, by many considered the true winner, and rejecting the *Condorcet loser* is somewhat desirable. For this reason, the two Condorcet criteria were the ones chosen to be analysed in a frequency context.

Through simulation, under specific conditions, such as the number of candidates and number of voters, we were able to verify that, in the cases of *plurality and approval*, the *Condorcet winner* criterion is less frequently violated if a second stage is introduced, where *plurality* is the system used to select the winner from a set of two nominees. This fact was verified under two distinct assumptions on preferences: *impartial culture* and *single-peakedness*.

However, when we have three nominees competing together in the second stage, we can no longer state that an extra stage increases the probability of electing the *Condorcet winner*, when one exists. It will now depend on the degree of homogeneity of preferences. Under *impartial culture*, for instance, we observe that although the three-nominee runoffs do not perform as efficiently as the two-nominee ones, they perform much more successfully than one-stage systems. Hence, having an extra stage is undoubtedly beneficial in what concerns voting systems' capability of electing the so called *Condorcet winner*. Unfortunately, this is no longer true when we face more homogeneous preferences such as the case of single-peaked ones. Under this strong assumption on preferences, having an extra stage where three candidates will compete under *plurality*, has a negative effect on the method's ability to respect the *Condorcet winner* criterion. We observed that one-stage systems are capable of electing at least one hundred percent more *Condorcet winner*s than their three-nominee two-stage equivalents. These numbers are significant and should be taken into account while discussing the optimality of two-stage systems.

Instant runoff is the most efficient one-stage system. This is because it is in fact a multi-stage procedure with only two nominees in the last stage, except that voters only need to cast their vote once. In fact, under *impartial culture*, it performs better than all the two-stage systems as well. In addition, under any assumption on preferences, *instant runoff* performs better than its three-nominee two-stage equivalent, the *two-stage instant runoff*. Among the two-nominee two-stage systems, *approval runoff* is the one that stands out in its capability of electing the *Condorcet winner*.

In what concerns the *Condorcet loser* criterion, introducing a second stage is beneficial under any voting system. Under two-nominee runoffs, Condorcet losers, by definition, can never be elected. Moreover, under three nominees, the probability of electing a *Condorcet loser* decreases significantly, independently of the assumptions made on preferences. *Approval runoff* is again the two-stage system that more efficiently satisfies this Condorcet criterion. Elections that by design require a second stage should carefully consider the hypothesis of adopting *approval runoff* as their voting method to select the winner.

Note that, under any assumption on preferences, all *approval* methods beat their *plurality* counterparts in the ability to respect the Condorcet criteria. This may leads us to think that if we were to use approval in the second stage, results would most likely be improved. In particular, under single-peakedness, it could actually be the case that three-nominee runoffs would again perform better than their one-stage counterparts, in what concers the *Condorcet winner* criterion.

Although very strong assumptions were made on preferences, it is important to understand, that under some circumstances, having a second stage, where *plurality* is the system used, might not be beneficial in what concerns the *Condorcet winner* criterion. Its positive input depends really on the number of nominees and the type of preferences agents have. The more homeogeneous preferences are, the less appealing the three-nominee two-stage systems will be. The fact that these systems are used among many different contexts, and in very important elections makes this kind of analysis particularly useful.

However, this analysis is not enough to infer about the overall optimality of two-stage systems. In spite of the computational complexity, using approval in the second stage would allow for insightful conclusions on the subject. The same can be said of repeating the simulation for a different number of candidates and voters. Additional criteria could also be analysed so that we may fully understand the main advantages of using a more expensive system, such as the two-stage system. It would also be benefitial for the study to analyse how the incentives for strategic behaviour change, when a second stage is introduced.

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