FACTORIZATION OF ELLIPTIC BOUNDARY VALUE PROBLEMS BY INVARIANT EMBEDDING AND APPLICATION TO OVERDETERMINED PROBLEMS.

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Abstract

The purpose of this thesis is the factorization of elliptic boundary value problems defined in cylindrical domains, in a system of decoupled first order initial value problems. We begin with the Poisson equation with mixed boundary conditions, and use the method of invariant embedding: we embed our initial problem in a family of similar problems, defined in sub-domains of the initial domain, with a moving boundary, and an additional condition in the moving boundary. This factorization is inspired by the technique of invariant temporal embedding used in Control Theory when computing the optimal feedback, for, in fact, as we show, our initial problem may be defined as an optimal control problem. The factorization thus obtained may be regarded as a generalized block Gauss LU factorization. From this procedure emerges an operator that can be either the Dirichlet-to-Neumann or the Neumann-to-Dirichlet operator, depending on which boundary data is given on the moving boundary. In any case this operator verifies a Riccati equation that is studied directly by using an Yosida regularization. Then we extend the former results to more general strongly elliptic operators. We also obtain a $\mathcal{QR}$ type factorization of the initial problem, where $\mathcal{Q}$ is an orthogonal operator and $\mathcal{R}$ is an upper triangular operator. This is related to a least mean squares formulation of the boundary value problem.
In addition, we obtain the factorization of overdetermined boundary value problems, when we consider an additional Neumann boundary condition: if this data is not compatible with the initial data, then the problem has no solution. In order to solve it, we introduce a perturbation in the original problem and minimize the norm of this perturbation, under the hypothesis of existence of solution. We deduce the normal equations for the overdetermined problem and, as before, we apply the method of invariant embedding to factorize the normal equations in a system of decoupled first order initial value problems.

KEYWORDS: factorization, invariant embedding, Dirichlet-to-Neumann operator, Riccati equation, Yosida regularization, overdetermined problem.

Resumo

O objectivo desta tese é a fatorização de problemas elíticos com valores na fronteira definidos em domínios cilíndricos, num sistema desacoplado de problemas de primeira ordem de valores iniciais. Começamos com a equação de Poisson com condições de fronteira mistas, e usamos o método de imersão invariante: mergulhamos o problema inicial numa família de problemas semelhantes, definidos em subdomínios do domínio inicial, com uma condição adicional numa fronteira móvel. Esta fatorização inspira-se na técnica de imersão invariante temporal usada em Teoria do Controlo para calcular o feedback ótimo, pois, de facto, como demonstramos, o problema inicial pode definir-se como um problema de controlo ótimo. A fatorização assim obtida pode ser considerada como a generalização da fatorização LU por blocos obtida pelo método de Gauss, usada em Análise Numérica para resolver sistemas de equações lineares. Deste procedimento surge um operador, que no caso estudado, tanto pode ser o operador Dirichlet-Neumann, como o operador Neumann-Dirichlet, dependendo do tipo de dado atribuído na fronteira móvel. Em qualquer caso, o referido operador verifica uma equação de Riccati, equação esta que estudamos diretamente usando uma regularização de Yosida. Estendemos ainda os resultados anteriores a outros operadores fortemente elíticos. Também obtemos uma fatorização do tipo $\mathcal{QR}$, onde $\mathcal{Q}$ é um operador ortogonal e $\mathcal{R}$ um operador triangular superior, que resulta duma formulação em mínimos quadrados do problema de valores de fronteira.

Efetuamos ainda a fatorização de problemas elíticos sobredeterminados, quando se considera uma condição de fronteira de Neumann adicional: se esta condição não é compatível com os dados iniciais, então o problema não tem solução. Para o resolver, introduzimos uma perturbação no problema original e minimizamos a norma desta perturbação, sujeita à condição de existência de solução. Em seguida, deduzimos as equações normais para o problema sobredeterminado, e, tal como anteriormente, usamos o método de imersão invariante para obter uma fatorização das equações normais num sistema desacoplado de problemas de primeira ordem de valores iniciais.

PALAVRAS CHAVE: fatorização, imersão invariante, operador Dirichlet-Neumann, equação de Riccati, regularização de Yosida, problema sobredeterminado.

Contents

Introduction

The method of factorization by invariant embedding [2] is used [1] to derive analytical and numerical results in various fields such as, for instance, in atmospheric physics, wave propagation and transport theory. In Control Theory it is used [16, 4] to obtain the decoupling of systems arising from Optimal Control problems. This technique consists in embedding a given problem in a family of similar problems defined in sub-domains of the initial one with a moving boundary, and depending on a parameter governing the position of the moving boundary. In Control Theory the parameter used is the time variable, and in this thesis it is a spacial one. We begin by using this method to factorize second order elliptic boundary value problems defined in a cylindrical domain in a system of two decoupled first order initial value equations. A similar approach is followed by [9], and in [19] the same method is used to factorize elliptic problems in circular and star-shaped domains. We shall also apply the same method to obtain the factorization of a fourth order boundary value problem in a system of four decoupled first order initial value equations.
In the first Chapter we introduce our case study, problem $(\mathcal{P}_0)$, that we wish to factorize by the method of invariant embedding: the Poisson equation with mixed boundary conditions, defined in a cylindrical domain. We define its variational formulation and show that it is well-posed. We next consider a fourth order problem for the bi-Laplacian, defined in the same domain, and show its well-posedness. This result will be used in Chapter 2, to justify the well posedness of the particular fourth order problem that arises when we consider an overdetermined problem. Finally, in the last section, following [16], we present a brief explanation of the method of factorization by invariant embedding.
In Chapter 2, we begin by showing that the solution of $(\mathcal{P}_0)$ may be regarded as the optimal control of a particular control problem. Then we embed problem $(\mathcal{P}_0)$ in a family of similar problems defined in sub-domains of the initial domain, with an additional condition on a moving boundary: we obtain a system of two decoupled first order initial value problems, and a Riccati equation in an operator P that can be either the Dirichlet-to-Neumann or the Neumann-to-Dirichlet operator, depending on which boundary data is given on the moving boundary. In [9] it is shown that this decoupling may be regarded as a generalization to infinite dimension of the block Gauss LU factorization. Next, we deduce the normal equations for the overdetermined problem when we consider an additional Neumann boundary condition: we begin by introducing a perturbation in the optimal system equivalent to $(\mathcal{P}_0)$, and then minimize the norm of this perturbation. As before, we factorize the normal equations by using the method of invariant embedding: we obtain

the same Riccati equation as before, a Lyapounov equation in an operator Q, and four decoupled first order initial value problems. Next, we present solutions to the equations obtained, with the exception of the Riccati equation. This first part of the Chapter can be found in [13]. Finally, we derive a $\mathcal{QR}$ type factorization of problem $(\mathcal{P}_0)$, where $\mathcal{Q}$ is an orthogonal operator and $\mathcal{R}$ is an upper triangular operator.
In Chapter 3, we present a direct study of the Riccati equation by Yosida regularization. This study can be found in [5]. In [11] the same equation is studied in an Hilbert-Schmidt framework.
In Chapter 4, we deduce matrix formulae for the operators P and Q with the aid of an orthonormal basis of $H_0^1(\mathcal{O})$.
In Chapter 5, we obtain a simple factorized formula for the Dirichlet-to-Neumann operator, solution of the Riccati equation.
Finally, in Chapter 6, we use the method of invariant embedding to factorize more general elliptic problems.

Chapter 1

Preliminaries

In this chapter, we begin by introducing our case study, a particular elliptic problem, $(\mathcal{P}_0)$, defined in a cylindrical domain. Next, in section 1.2, we define the variational formulation for $(\mathcal{P}_0)$, and show that it is a well-posed problem. Then, in section 1.3, we extend the former results to more general elliptic problems that we show to be well-posed. In section 1.4, we introduce a fourth order problem for the bi-Laplacian, and show its well-posedness. Finally, in section 1.5, we present a brief explanation of the method of factorization by invariant embedding. This method will be used to factorize a second order elliptic boundary value problem in a system of decoupled first order initial value problems.

1.1 Position of the problem $(\mathcal{P}_0)$

Let Ω be the cylinder $\Omega =]0,1[\times\mathcal{O}$, $x' = (x,y) \in \mathbb{R}^n$, where x is the coordinate along the axis of the cylinder and $\mathcal{O}$, a bounded open set in $\mathbb{R}^{n-1}$, is the section of the cylinder. The lateral boundary of the cylinder is $\Sigma =]0,1[\times\partial\mathcal{O}$. For each $s \in]0,1[$, let $\Gamma_s = \{s\} \times \mathcal{O}$ and $\Sigma_s =]0,s[\times\partial\mathcal{O}$. Γ_0 and Γ_1 are the two faces of the cylinder. Given $f \in L^2(\Omega)$, $u_0 \in (H^{1/2}_{00}(\mathcal{O}))'$ and $u_1 \in H^{1/2}_{00}(\mathcal{O})$, we consider the following Poisson equation with mixed boundary conditions

$$(\mathcal{P}_0)\begin{cases} -\Delta u = -\dfrac{\partial^2 u}{\partial x^2} - \Delta_y u = f \quad \text{in } \Omega, \\ u|_{\Sigma} = 0, \\ -\dfrac{\partial u}{\partial x}|_{\Gamma_0} = u_0, \quad u|_{\Gamma_1} = u_1. \end{cases} \tag{1.1}$$

Remark 1.1 *The space $H^{1/2}_{00}(\mathcal{O})$ is defined in [17], Theorem 12.3, page 72, as the interpolated space of order $1/2$ between $H^1_0(\mathcal{O})$ and $L^2(\mathcal{O})$*

$$H^{1/2}_{00}(\mathcal{O}) = \left[H^1_0(\mathcal{O}), L^2(\mathcal{O})\right]_{\frac{1}{2}}$$

and, from Theorem 6.2, page 29 of [17], its dual is given by

$$(H^{1/2}_{00}(\mathcal{O}))' = \left[L^2(\mathcal{O}), H^{-1}(\mathcal{O})\right]_{\frac{1}{2}}.$$

Moreover,

$$H^1_0(\mathcal{O}) \subset H^{1/2}_{00}(\mathcal{O}) \subset L^2(\mathcal{O}),$$

each space being dense in the following one, so, by duality, we have

$$L^2(\mathcal{O}) \subset (H^{1/2}_{00}(\mathcal{O}))' \subset H^{-1}(\mathcal{O}),$$

each space dense in the following one.

Definition 1.2 *We define the following spaces*

$$X = L^2(0,1; H^1_0(\mathcal{O})) \cap H^1(0,1; L^2(\mathcal{O})),$$

$$X_0 = \{u \in X : u|_{\Gamma_1} = 0\},$$

$$Y = \{u \in X : \frac{\partial^2 u}{\partial x^2} \in L^2(0,1; H^{-1}(\mathcal{O}))\},$$

with the norms

$$\|u\|^2_X = \int_0^1 \|u(x)\|^2_{H^1_0(\mathcal{O})} dx + \int_0^1 \|u(x)\|^2_{L^2(\mathcal{O})} dx + \int_0^1 \left\|\frac{\partial u}{\partial x}(x)\right\|^2_{L^2(\mathcal{O})} dx, \tag{1.2}$$

$$\|u\|_{X_0}^2 = \int_0^1 \|u(x)\|_{H_0^1(\mathcal{O})}^2 dx + \int_0^1 \left\| \frac{\partial u}{\partial x}(x) \right\|_{L^2(\mathcal{O})}^2 dx, \tag{1.3}$$

$$\|u\|_Y^2 = \|u\|_X^2 + \int_0^1 \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{H^{-1}(\mathcal{O})}^2 dx. \tag{1.4}$$

Theorem 1.3 *The norms in X and in X_0 are equivalent to the norm in $H^1(\Omega)$. Moreover, X_0 is a closed subspace in X, and X is a Hilbert space for the norm of $H^1(\Omega)$.*

Proof. In fact

$$\|u\|_X^2 = \int_0^1 \int_{\mathcal{O}} |\nabla_y u|^2 dxdy + \int_0^1 \int_{\mathcal{O}} |u|^2 dxdy + \int_0^1 \int_{\mathcal{O}} \left| \frac{\partial u}{\partial x} \right|^2 dxdy = \|u\|_{H^1(\Omega)}^2, \quad \forall u \in X,$$

so we may conclude that the norms in X and in $H^1(\Omega)$ are equivalent. On the other hand, we have

$$\|u\|_{X_0}^2 = \int_0^1 \int_{\mathcal{O}} |\nabla_y u|^2 dxdy + \int_0^1 \int_{\mathcal{O}} \left| \frac{\partial u}{\partial x} \right|^2 dxdy = \int_\Omega |\nabla u|^2 dxdy \leq \|u\|_{H^1(\Omega)}^2, \quad \forall u \in X_0,$$

and

$$\|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 \leq (c+1)\|\nabla u\|_{L^2(\Omega)}^2 = (c+1)\|u\|_{X_0}^2, \quad \forall u \in X_0,$$

where c is the Poincaré constant, so the norm in X_0 is also equivalent to the norm in $H^1(\Omega)$. Next, we notice that $L^2(0,1;H_0^1(\mathcal{O}))$ is a Hilbert space for the norm

$$\|u\|_{L^2(0,1;H_0^1(\mathcal{O}))}^2 = \int_0^1 \|u(x)\|_{H_0^1(\mathcal{O})}^2 dx$$

and $H^1(0,1;L^2(\mathcal{O}))$ is a Hilbert space for the norm

$$\|u\|_{H^1(0,1;L^2(\mathcal{O}))}^2 = \int_0^1 \|u(x)\|_{L^2(\mathcal{O})}^2 dx + \int_0^1 \left\| \frac{\partial u}{\partial x}(x) \right\|_{L^2(\mathcal{O})}^2 dx,$$

consequently, X is a Hilbert space for the norm (1.2), or, which is equivalent, for the norm of $H^1(\Omega)$. Finally, we show that X_0 is closed in X. Consider a sequence $(u_n) \subset X_0$ such that $u_n \to u$ in X. Then $u \in X$ and $u_n \to u$ in $H^1(\Omega)$. The continuity of the trace mapping

$$u \mapsto u|_{\partial\Omega}$$

from $H^1(\Omega)$ into $H^{\frac{1}{2}}(\partial\Omega)$ implies, in particular, that

$$u_n|_{\Gamma_1} \to u|_{\Gamma_1}$$

so we derive that $u|_{\Gamma_1} = 0$ and, consequently, $u \in X_0$. ■

Remark 1.4 *We notice that, from Theorem 1.3, X_0 is a closed subspace of a Hilbert space and consequently is itself a Hilbert Space.*

Theorem 1.5 *Y is a Hilbert space.*

Proof. In fact, we can show that Y is complete for the norm

$$\|u\|_Y^2 = \|u\|_X^2 + \int_0^1 \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{H^{-1}(\mathcal{O})}^2,$$

taking into account the continuity of the derivation with respect to x in the sense of $\mathcal{D}'\left(]0,1[; H_0^1(\mathcal{O})\right)$. ■

Definition 1.6 *We define the following spaces*

$$W_1(0,1) = \{u \in L^2(0,1; H_0^1(\mathcal{O})) : \frac{\partial u}{\partial x} \in L^2(0,1; L^2(\mathcal{O}))\}$$

and

$$W_2(0,1) = \{u \in L^2(0,1; L^2(\mathcal{O})) : \frac{\partial u}{\partial x} \in L^2(0,1; H^{-1}(\mathcal{O}))\}.$$

Remark 1.7 *$W_1(0,1)$ is a Hilbert space for the norm*

$$\|u\|_{W_1(0,1)}^2 = \|u\|_{L^2(0,1;H_0^1(\mathcal{O}))}^2 + \left\| \frac{\partial u}{\partial x} \right\|_{L^2(0,1;L^2(\mathcal{O}))}^2$$

and the same is true for $W_2(0,1)$, with the norm

$$\|u\|_{W_2(0,1)}^2 = \|u\|_{L^2(0,1;L^2(\mathcal{O}))}^2 + \left\| \frac{\partial u}{\partial x} \right\|_{L^2(0,1;H^{-1}(\mathcal{O}))}^2.$$

Theorem 1.8 *For each $u \in Y$, we have*

$$\left(u, \frac{\partial u}{\partial x}\right) \in C\left([0,1]; H_{00}^{1/2}(\mathcal{O}) \times (H_{00}^{1/2}(\mathcal{O}))'\right),$$

and the mapping

$$u \mapsto \left(u, \frac{\partial u}{\partial x}\right)$$

is linear and continuous from Y *into* $C\left([0,1]; H_{00}^{1/2}(\mathcal{O}) \times (H_{00}^{1/2}(\mathcal{O}))'\right)$. *Besides, for each* $s \in [0,1]$, *the trace mapping*

$$u \mapsto \left(u|_{\Gamma_s}, \frac{\partial u}{\partial x}|_{\Gamma_s}\right)$$

is a linear and continuous mapping from Y *onto* $H_{00}^{1/2}(\mathcal{O}) \times (H_{00}^{1/2}(\mathcal{O}))'$.

Proof. Consider $u \in Y$. Then $u \in W_1(0,1)$, and, by Theorem 3.1, page 19 of [17], we may conclude that

$$u \in C\left([0,1]; H_{00}^{1/2}(\mathcal{O})\right),$$

and the identity mapping, I, is continuous from $W_1(0,1)$ into $C\left([0,1]; H_{00}^{1/2}(\mathcal{O})\right)$, which means that there exists a constant c_1 such that

$$\|u\|^2_{C\left([0,1]; H_{00}^{1/2}(\mathcal{O})\right)} \leq c_1 \|u\|^2_{W_1(0,1)}, \forall u \in W_1(0,1).$$

But

$$\|u\|^2_{W_1(0,1)} \leq \|u\|^2_Y, \forall u \in Y$$

so, the identity is continuous from Y into $C\left([0,1]; H_{00}^{1/2}(\mathcal{O})\right)$. Moreover, $\dfrac{\partial u}{\partial x} \in W_2(0,1)$ and, by the same Theorem, we conclude that

$$\frac{\partial u}{\partial x} \in C\left([0,1]; H_{00}^{1/2}(\mathcal{O})'\right).$$

Again by Theorem 3.1, page 19 of [17], we know that I is continuous from $W_2(0,1)$ into $C\left([0,1]; H_{00}^{1/2}(\mathcal{O})'\right)$. Therefore there exists a constant c_2 such that

$$\|v\|^2_{C\left([0,1]; H_{00}^{1/2}(\mathcal{O})'\right)} \leq c_2 \|v\|^2_{W_2(0,1)}, \forall v \in W_2(0,1),$$

so

$$\left\|\frac{\partial u}{\partial x}\right\|^2_{C\left([0,1]; H_{00}^{1/2}(\mathcal{O})'\right)} \leq c_2 \left\|\frac{\partial u}{\partial x}\right\|^2_{W_2(0,1)}, \forall u \in Y.$$

But

$$\left\| \frac{\partial u}{\partial x} \right\|^2_{W_2(0,1)} \leq \|u\|^2_Y, \forall u \in Y,$$

so the mapping

$$u \to \frac{\partial u}{\partial x}$$

is also continuous from Y into $C\left([0,1]; H^{1/2}_{00}(\mathcal{O})'\right)$. Finally, the last part of the theorem is a direct consequence of Theorem 3.2, page 21 of [17]. ■

1.2 Variational formulation of problem $(\mathcal{P}_0)$

Theorem 1.9 *The variational formulation of problem* $(\mathcal{P}_0)$ *is*

$$\int_\Omega \nabla u.\nabla v dxdy = \langle u_0, v|_{\Gamma_0} \rangle_{H^{1/2}_{00}(\mathcal{O})' \times H^{1/2}_{00}(\mathcal{O})} + \int_\Omega f v dxdy, \forall v \in X_0, \tag{1.5}$$

where $u \in X$ *verifies the constraint* $u|_{\Gamma_1} = u_1$.

Proof. Multiplying both sides of $-\Delta u = f$ by $v \in X_0$ and integrating in Ω, we obtain

$$\int_\Omega (-\Delta u) v dxdy = \int_\Omega f v dxdy. \tag{1.6}$$

Integrating by parts the left hand side of (1.6), we derive

$$\int_\Omega \nabla u.\nabla v dxdy - \int_{\partial\Omega} \frac{\partial u}{\partial n} v d\sigma = \int_\Omega f v dxdy, \tag{1.7}$$

and, taking into account that

$$v \in X_0 \Rightarrow v|_\Sigma = v|_{\Gamma_1} = 0$$

and $-\dfrac{\partial u}{\partial x}|_{\Gamma_0} = u_0$, we conclude that

$$\int_\Omega \nabla u.\nabla v dxdy = \int_{\Gamma_0} u_0 v|_{\Gamma_0} d\sigma + \int_\Omega f v dxdy.$$

But $v \in W_1(0,1)$, therefore, by Theorem 1.8, we derive that $v|_{\Gamma_0} \in H^{1/2}_{00}(\mathcal{O})$. We finally obtain

$$\int_\Omega \nabla u.\nabla v dxdy = \langle u_0, v|_{\Gamma_0} \rangle_{H^{1/2}_{00}(\mathcal{O})' \times H^{1/2}_{00}(\mathcal{O})} + \int_\Omega f v dxdy, \forall v \in X_0. \quad ■$$

Next we show the well posedness of problem (P_0) when $u_1 = 0$.

Theorem 1.10 *There exists a unique* $u \in X_0$*, such that*

$$\int_\Omega \nabla u.\nabla v dxdy = \langle u_0, v|_{\Gamma_0}\rangle_{H_{00}^{1/2}(\mathcal{O})' \times H_{00}^{1/2}(\mathcal{O})} + \int_\Omega f v dxdy, \forall v \in X_0. \tag{1.8}$$

Proof. Consider

$$a(u,v) = \int_\Omega \nabla u.\nabla v dxdy, \quad u, v \in X_0, \tag{1.9}$$

and

$$(f,v) = \langle u_0, v|_{\Gamma_0}\rangle_{H_{00}^{1/2}(\mathcal{O})' \times H_{00}^{1/2}(\mathcal{O})} + \int_\Omega f v dxdy, \quad v \in X_0. \tag{1.10}$$

By Hölder inequality, we have

$$|a(u,v)| \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

and, from Theorem 1.3, we conclude that $a(u,v)$ is a continuous bilinear form in X_0. Besides

$$\|u\|^2_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \leq (c+1)\|\nabla u\|^2_{L^2(\Omega)} = (c+1)a(u,u), \forall u \in X_0,$$

where c is the Poincaré constant, so we derive that

$$a(u,u) \geq \frac{1}{c+1}\|u\|^2_{H^1(\Omega)}, \forall u \in X_0,$$

which, again by Theorem 1.3, shows that $a(u,v)$ is coercive in X_0. On the other hand, by Hölder inequality

$$|\int_\Omega f v dxdy| \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}| \leq \|f\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}, \forall v \in X_0.$$

Moreover, from Theorem 1.8, we know that the mapping

$$v \mapsto v|_{\Gamma_0}$$

is continuous from $W_1(0,1)$ into $H_{00}^{1/2}(\mathcal{O})$, and, due to the fact that

$$\|v\|^2_{W_1(0,1)} \leq \|v\|^2_{X_0}, \forall v \in X_0,$$

we derive that the linear form

$$v \mapsto \langle u_0, v|_{\Gamma_0}\rangle_{H_{00}^{1/2}(\mathcal{O})' \times H_{00}^{1/2}(\mathcal{O})}$$

is continuous in X_0, for in fact, it is the composition of continuous mappings. Thus, (f, v) is a continuous linear form in X_0. Finally, by Lax-Milgram Theorem, there exists a unique solution $u \in X_0$ of (1.8). ■

We finally prove that problem (P_0) is well posed in the general case:

Theorem 1.11 *Problem (P_0) admits a unique solution $u \in H^1(\Omega)$.*

Proof. From Theorem 1.8, given $u_1 \in H_{00}^{1/2}(\mathcal{O})$ there exists $u^* \in Y$ such that $u^*|_{\Gamma_1} = u_1$. Let $\tilde{u} \in X_0$ be the unique solution of

$$\begin{cases} -\Delta\tilde{u} = -\dfrac{\partial^2\tilde{u}}{\partial x^2} - \Delta_y\tilde{u} = f + \Delta u^* \quad \text{in } \Omega, \\ \tilde{u}|_\Sigma = 0, \\ -\dfrac{\partial\tilde{u}}{\partial x}|_{\Gamma_0} = u_0 + \dfrac{\partial u^*}{\partial x}|_{\Gamma_0}, \quad \tilde{u}|_{\Gamma_1} = 0. \end{cases} \tag{1.11}$$

Then it is easy to show that $u = \tilde{u} + u^* \in X$ is a solution of (P_0). Finally we show that the solution of (P_0) is unique. In fact, if $\bar{u}, \hat{u} \in X$ are solutions of (P_0), then $u = \bar{u} - \hat{u} \in X_0$ verifies

$$\begin{cases} -\Delta u = 0 \quad \text{in } \Omega, \\ u|_\Sigma = 0, \\ -\dfrac{\partial u}{\partial x}|_{\Gamma_0} = 0, \quad u|_{\Gamma_1} = 0. \end{cases} \tag{1.12}$$

so, by Theorem 1.10, we conclude that $u = 0$. ■

1.3 Other elliptic problems

Let Ω be the cylinder defined in section 1.1. For the sake of simplicity we will denote an element of Ω by $x' = (x_1, y)$, with $x_1 \in]0, 1[$ and $y = (x_2, ..., x_n) \in \mathcal{O}$. Given $f \in L^2(\Omega)$, $u_0 \in (H_{00}^{1/2}(\mathcal{O}))'$ and $u_1 \in H_{00}^{1/2}(\mathcal{O})$, we consider the following equation with mixed boundary conditions

$$\begin{cases} Lu = f \quad \text{in } \Omega, \\ u|_\Sigma = 0, \\ (-\sum_{j=1}^{n} a_{1,j} \frac{\partial u}{\partial x_j})|_{\Gamma_0} = u_0, \quad u|_{\Gamma_1} = u_1. \end{cases} \tag{1.13}$$

where L is the operator defined by:

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{i,j}\frac{\partial}{\partial x_j}) \tag{1.14}$$

The $a_{i,j}$ are real and continuously differentiable functions in $\bar{\Omega}$ and L is a strongly elliptic operator in Ω, i.e., there exists a constant $c_0 > 0$ such that:

$$\sum_{i,j=1}^{n} a_{i,j}\xi_i\xi_j \geq c_0\|\xi\|^2, \quad \forall \xi \in \mathbb{R}^n, \tag{1.15}$$

Theorem 1.12 *The variational formulation of problem (1.13) is*

$$\sum_{i,j=1}^{n} \int_\Omega a_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx_1dy = \langle u_0, v|_{\Gamma_0}\rangle_{H_{00}^{1/2}(\mathcal{O})'\times H_{00}^{1/2}(\mathcal{O})} + \int_\Omega fvdx_1dy, \forall v \in X_0, \tag{1.16}$$

where $u \in X$ verifies the constraint $u|_{\Gamma_1} = u_1$.

Proof. Multiplying both sides of $Lu = f$ by $v \in X_0$ and integrating in Ω, we obtain

$$-\sum_{i,j=1}^{n} \int_\Omega \frac{\partial}{\partial x_i}(a_{i,j}\frac{\partial u}{\partial x_j})vdx_1dy = \int_\Omega fvdx_1dy. \tag{1.17}$$

Integrating by parts the left hand side of (1.17), we derive

$$\sum_{i,j=1}^{n} \int_\Omega a_{i,j}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}dx_1dy - \sum_{i,j=1}^{n} \int_{\partial\Omega} a_{i,j}\frac{\partial u}{\partial x_j}v\,\vec{n}.\vec{e}_{x_i}d\sigma = \int_\Omega fvdx_1dy. \tag{1.18}$$

Taking into account that

$$v \in X_0 \Rightarrow v|_\Sigma = v|_{\Gamma_1} = 0,$$

$(-\sum_{j=1}^{n} a_{1,j}\frac{\partial u}{\partial x_j})|_{\Gamma_0} = u_0$, and that $v \in W_1(0,1)$, which implies, by Theorem 1.8, that $v|_{\Gamma_0} \in H_{00}^{1/2}(\mathcal{O})$, we conclude that

$$\sum_{i,j=1}^{n} \int_\Omega a_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx_1dy = \langle u_0, v|_{\Gamma_0}\rangle_{H_{00}^{1/2}(\mathcal{O})'\times H_{00}^{1/2}(\mathcal{O})} + \int_\Omega fvdx_1dy, \forall v \in X_0. \qquad \blacksquare$$

Next we show the well-posedness of problem (1.13) when $u_1 = 0$.

Theorem 1.13 *There exists a unique* $u \in X_0$ *such that:*

$$\sum_{i,j=1}^{n} \int_\Omega a_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx_1 dy = \langle u_0, v|_{\Gamma_0} \rangle_{H_{00}^{1/2}(\mathcal{O})' \times H_{00}^{1/2}(\mathcal{O})} + \int_\Omega f v dx_1 dy, \forall v \in X_0. \quad (1.19)$$

Proof. Let

$$a(u,v) = \sum_{i,j=1}^{n} \int_\Omega a_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx_1 dy \quad (1.20)$$

and

$$(f,v) = \langle u_0, v|_{\Gamma_0} \rangle_{H_{00}^{1/2}(\mathcal{O})' \times H_{00}^{1/2}(\mathcal{O})} + \int_\Omega f v dx_1 dy, \forall v \in X_0. \quad (1.21)$$

By the Hölder inequality, we have:

$$\begin{aligned} |a(u,v)| &\leq M \sum_{i,j=1}^{n} \int_\Omega |\frac{\partial u}{\partial x_i}||\frac{\partial v}{\partial x_j}| dx_1 dy \leq \\ &\leq M \left(\int_\Omega |\nabla u|^2 dx_1 dy \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla v|^2 dx_1 dy \right)^{\frac{1}{2}} \leq M \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \end{aligned} \quad (1.22)$$

so, by Theorem 1.3, we conclude that $a(u,v)$ is a continuous bilinear form in X_0. Besides, using the fact that L is a strongly elliptic operator, we have:

$$\begin{aligned} a(u,u) &= \sum_{i,j=1}^{n} \int_\Omega a_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx_1 dy \geq c_0 \sum_{i=1}^{n} \int_\Omega |\frac{\partial u}{\partial x_i}|^2 dx_1 dy = \\ &= c_0 \int_\Omega \nabla u\|^2 dx_1 dy. \end{aligned} \quad (1.23)$$

On the other hand, we have:

$$\int_\Omega \|u\|^2 dx_1 dy + \int_\Omega \|\nabla u\|^2 dx_1 dy \leq (c+1) \int_\Omega \|\nabla u\|^2 dx_1 dy \leq \frac{c+1}{c_0} a(u,u) \quad (1.24)$$

where c is the Poincaré constant, and so we conclude that:

$$a(u,u) \geq \frac{c_0}{c+1} (\int_\Omega \|u\|^2 dx_1 dy + \int_\Omega \|\nabla u\|^2 dx_1 dy) = \frac{c_0}{c+1} \|u\|^2_{H^1(\Omega)}, \forall u \in X_0. \quad (1.25)$$

By Theorem 1.3, we may derive that $a(u,v)$ is coercive in X_0. Moreover, by Hölder inequality

$$| \int_\Omega f v dx_1 dy | \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} | \leq \|f\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}, \forall v \in X_0.$$

From Theorem 1.8, we know that the mapping

$$v \mapsto v|_{\Gamma_0}$$

is continuous from $W_1(0,1)$ into $H_{00}^{1/2}(\mathcal{O})$, and, due to the fact that

$$\|v\|^2_{W_1(0,1)} \leq \|v\|^2_{X_0}, \forall v \in X_0,$$

we derive that the linear form

$$v \mapsto \langle u_0, v|_{\Gamma_0} \rangle_{H_{00}^{1/2}(\mathcal{O})' \times H_{00}^{1/2}(\mathcal{O})}$$

is continuous in X_0, for in fact, it is the composition of continuous mappings. Thus, (f, v) is a continuous linear form in X_0. Finally, by the Lax-Milgram Theorem, there exists a unique solution $u \in X_0$ of (1.13), when $u_1 = 0$. ■

Finally we show the well-posedness of problem (1.13) in the general case:

Theorem 1.14 *Problem (1.13) admits a unique solution $u \in H^1(\Omega)$.*

Proof. From Theorem 1.8, given $u_1 \in H_{00}^{1/2}(\mathcal{O})$ there exists $u^* \in Y$ such that $u^*|_{\Gamma_1} = u_1$. Let $\tilde{u} \in X_0$ be the unique solution of

$$\begin{cases} L\tilde{u} = f - Lu^* \quad \text{in } \Omega, \\ \tilde{u}|_{\Sigma} = 0, \\ (-\sum_{j=1}^{n} a_{1,j} \dfrac{\partial \tilde{u}}{\partial x_j})|_{\Gamma_0} = u_0 + (\sum_{j=1}^{n} a_{1,j} \dfrac{\partial u^*}{\partial x_j})|_{\Gamma_0}, \quad \tilde{u}|_{\Gamma_1} = 0. \end{cases} \tag{1.26}$$

Then it is easy to show that $u = \tilde{u} + u^* \in X$ is a solution of (1.13). Finally we show that the solution of (P_0) is unique. In fact, if $\bar{u}, \hat{u} \in X$ are solutions of (1.13), then $u = \bar{u} - \hat{u} \in X_0$ verifies

$$\begin{cases} Lu = 0 \quad \text{in } \Omega, \\ u|_{\Sigma} = 0, \\ (-\sum_{j=1}^{n} a_{1,j} \dfrac{\partial u}{\partial x_j})|_{\Gamma_0} = 0, \quad u|_{\Gamma_1} = 0. \end{cases} \tag{1.27}$$

so, by Theorem 1.13, we conclude that $u = 0$. ■

1.4 A fourth order problem

In this section we are going to justify the well-posedness of the following problem for the bi-laplacian:

$$\begin{cases} \Delta^2 u = F, \quad \text{in } \Omega, \\ u|_\Sigma = 0, \quad \Delta u|_\Sigma = 0, \\ -\dfrac{\partial u}{\partial x}|_{\Gamma_0} = u_0, \quad \dfrac{\partial \Delta u}{\partial x}|_{\Gamma_0} = u_1, \\ u|_{\Gamma_1} = u_2, \ \Delta u|_{\Gamma_1} = u_3, \end{cases} \tag{1.28}$$

where $F \in L^2(\Omega)$, $u_0 \in H_{00}^{1/2}(\mathcal{O})$, $u_1 \in \left(H^{3/2}(\mathcal{O}) \cap H_{00}^{1/2}(\mathcal{O})\right)'$, $u_2 \in H^{3/2}(\mathcal{O}) \cap H_{00}^{1/2}(\mathcal{O})$ and $u_3 \in \left(H_{00}^{1/2}(\mathcal{O})\right)'$.

Definition 1.15 *We define the following spaces*

$$\mathbb{X} = L^2\left(0,1; H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})\right) \cap H^1(0,1; H_0^1(\mathcal{O})) \cap H^2(0,1; L^2(\mathcal{O})),$$

$$\mathbb{X}_0 = \{u \in \mathbb{X} : -\frac{\partial u}{\partial x}|_{\Gamma_0} = 0, u|_{\Gamma_1} = 0\},$$

It is easy to show that the norm in $\mathbb{X}$ is equivalent to the norm in $H^2(\Omega)$.

Theorem 1.16 *The variational formulation of problem (1.28) is*

$$\begin{aligned} &\textstyle\int_\Omega \Delta u \Delta v dxdy = \int_\Omega F v dxdy - \\ &- \left\langle \tfrac{\partial v}{\partial x}|_{\Gamma_1}, u_3 \right\rangle_{H_{00}^{1/2}(\mathcal{O})' \times H_{00}^{1/2}(\mathcal{O})} + \langle u_1, v|_{\Gamma_0} \rangle_{H_{00}^{1/2}(\mathcal{O})' \times H_{00}^{1/2}(\mathcal{O})}, \ \forall v \in \mathbb{X}_0. \end{aligned} \tag{1.29}$$

where $u \in \mathbb{X}$ verifies the aditional constraints: $-\dfrac{\partial u}{\partial x}|_{\Gamma_0} = u_0$, $u|_{\Gamma_1} = u_2$.

Proof. Multiplying both sides of $\Delta^2 u = F$ by $v \in \mathbb{X}_0$ and integrating in Ω, we obtain

$$\int_\Omega (\Delta^2 u) v dxdy = \int_\Omega F v dxdy. \tag{1.30}$$

Applying the Green formula to the left hand side of (1.30), we derive

$$-\int_\Omega \nabla(\Delta u).\nabla v dxdy + \int_{\partial\Omega} \frac{\partial \Delta u}{\partial n} v d\sigma = \int_\Omega F v dxdy, \tag{1.31}$$

and, taking into account that $v|_\Sigma = 0$ and $v|_{\Gamma_1} = 0$, and $\dfrac{\partial \Delta u}{\partial x}(0) = u_1$, we obtain:

$$-\int_\Omega \nabla(\Delta u).\nabla v dxdy - \int_{\Gamma_0} u_1 v d\sigma = \int_\Omega F v dxdy. \tag{1.32}$$

Integrating again by parts, and noting that $\dfrac{\partial v}{\partial x}|_{\Gamma_0} = 0$, $\Delta u|_\Sigma = 0$ and $\Delta u|_{\Gamma_1} = u_3$, we deduce that:

$$\int_\Omega \Delta u \Delta v dxdy - \int_{\Gamma_0} u_1 v d\sigma - \int_{\Gamma_1} \frac{\partial v}{\partial x} u_3 d\sigma = \int_\Omega F v dxdy. \tag{1.33}$$

From the regularity assumptions made at the beginning, we deduce that:

$$\begin{aligned} &\int_\Omega \Delta u \Delta v dxdy = \int_\Omega F v dxdy + \\ &+\left\langle \frac{\partial v}{\partial x}|_{\Gamma_1}, u_3 \right\rangle_{H_{00}^{1/2}(\mathcal{O})' \times H_{00}^{1/2}(\mathcal{O})} + \langle u_1, v|_{\Gamma_0}\rangle_{H_{00}^{1/2}(\mathcal{O})' \times H_{00}^{1/2}(\mathcal{O})}, \ \forall v \in \mathbb{X}_0. \end{aligned} \tag{1.34}$$

■

We next prove the well-posedness of problem (1.28) when $u_0 = u_2 = 0$.

Theorem 1.17 *There exists a unique $u \in \mathbb{X}_0$ such that:*

$$\begin{aligned} &\int_\Omega \Delta u \Delta v dxdy = \int_\Omega F v dxdy + \\ &+\left\langle \frac{\partial v}{\partial x}|_{\Gamma_1}, u_3 \right\rangle_{H_{00}^{1/2}(\mathcal{O})' \times H_{00}^{1/2}(\mathcal{O})} + \langle u_1, v|_{\Gamma_0}\rangle_{H_{00}^{1/2}(\mathcal{O})' \times H_{00}^{1/2}(\mathcal{O})}, \ \forall v \in \mathbb{X}_0. \end{aligned} \tag{1.35}$$

Proof. We define the following norm in $\mathbb{X}_0$:

$$|||u|||^2_{\mathbb{X}_0} = \int_\Omega |\Delta u|^2 dxdy. \tag{1.36}$$

Let

$$\begin{aligned} &a(u,v) = \int_\Omega \Delta u \Delta v dxdy, \\ &(f,v) = \int_\Omega F v dxdy + \left\langle \frac{\partial v}{\partial x}|_{\Gamma_1}, u_3 \right\rangle_{H_{00}^{1/2}(\mathcal{O})' \times H_{00}^{1/2}(\mathcal{O})} + \langle u_1, v|_{\Gamma_0}\rangle_{H_{00}^{1/2}(\mathcal{O})' \times H_{00}^{1/2}(\mathcal{O})}. \end{aligned} \tag{1.37}$$

We have:

$$|a(u,v)| \leq |||u|||_{\mathbb{X}_0} |||v|||_{\mathbb{X}_0} \tag{1.38}$$

and consequently $a(u,v)$ is a coercive and continuous bilinear form in $\mathbb{X}_0$. On the other hand, from the regularity assumptions made at the beginning of the section, it can be easily shown that (f,v) is a continuous linear form in $\mathbb{X}_0$. The conclusion now follows directly from the Lax-Milgram Theorem.

1.5 The method of factorization by invariant embedding

Following J.L. Lions ([16]), in this section we present the method of factorization by invariant embedding. Let V, H be Hilbert spaces such that V is dense in H. Assuming that H' (the dual space of H) is identified with H, we have $V \subset H \subset V'$, where each space is dense in the following one. Considering the Hilbert space of controls $\mathcal{U}$ and the Hilbert space of observations $\mathcal{H}$, we are given a continuous and coercive bilinear form on V, that is

$$\varphi, \psi \to a(t; \varphi, \psi)$$

for each $t \in (0, T)$, such that we may write

$$a(t; \varphi, \psi) = (A(t)\varphi, \psi)\,, A(t)\varphi \in V',$$

and verifying $A(t) \in \mathcal{L}(L^2(0, T; V); L^2(0, T; V'))$. Let us denote by $\Lambda_{\mathcal{U}}$ the canonical isomorphism of $\mathcal{U}$ onto $\mathcal{U}'$. Given

$$B \in \mathcal{L}(\mathcal{U}; L^2(0, T; V')), f \in L^2(0, T; V'), y_0 \in H,$$

we consider the control problem

$$\begin{cases} \dfrac{\partial}{\partial t} y(v) + A(t)y(v) = f + Bv, \\ y(v)|_{t=0} = y_0, \\ y(v) \in L^2(0, T; V). \end{cases} \tag{1.39}$$

The function $y(v)$ is the state of the system. The observation is given by

$$z(v) = Cy(v), C \in \mathcal{L}(L^2(0, T; V), \mathcal{H})$$

and N is such that

$$N \in \mathcal{L}(\mathcal{U}; \mathcal{U}), (Nu, u)_{\mathcal{U}} \geq c\|u\|^2_{\mathcal{U}}, c > 0.$$

The cost function is

$$J(v) = \|Cy(v) - z_d\|^2_{\mathcal{H}} + (Nv, v)_{\mathcal{U}}.$$

Let Λ represent the isomorphism between $\mathcal{H}$ and $\mathcal{H}'$. The adjoint state $p(v)$ is then introduced by

$$\begin{cases} -\dfrac{\partial}{\partial t}p(v) + A^*(t)p(v) = C^*\Lambda(Cy(v) - z_d) \text{ in }]0,T[, \\ p(T;v) = 0 \\ p(v) \in L^2(0,T;V). \end{cases} \tag{1.40}$$

We have the folowing result:

Theorem 1.18 *The optimal control verifies (1.39), (1.40) and*

$$\Lambda_{\mathcal{U}}^{-1} B^* p(u) + N(u) = 0, \forall u \in \mathcal{U}. \tag{1.41}$$

Now let us assume that $\mathcal{U} = L^2(0,T;E)$ and $\mathcal{H} = L^2(0,T;F)$, E and F being separable Hilbert spaces. Moreover, let $B(t) \in \mathcal{L}(E;V')$, $C(t) \in \mathcal{L}(V;F)$, $\forall t \in]0,T[$ such that $t \to (B(t)e,\psi)$ and $t \to (C(t)\varphi, f')$ are measurable $\forall e \in E, \psi \in V, \varphi \in V, f' \in F'$, and $\|B(t)\|_{\mathcal{L}(E;V')} \leq c, \|C(t)\|_{\mathcal{L}(V;F)} \leq c$. These formulae give us $B \in \mathcal{L}(\mathcal{U}; L^2(0,T;V'))$ and $C \in \mathcal{L}((L^2(0,T;V));\mathcal{H})$. Furthermore let us assume that $N(t) \in \mathcal{L}(E;E), (N(t)e,e_1)$ measurable, $\|N(t)\|_{\mathcal{L}(E;E)} \leq c$ and $(N(t)e,e)_E \geq v\|e\|_E^2, \forall e \in E$. Representing by Λ_E (resp. Λ_F) the canonical isomorphism of E (resp. F) into its dual space, then $\Lambda_{\mathcal{U}} u(t) = \Lambda_E u(t)$ almost everywhere and $\Lambda f(t) = \Lambda_F f(t)$ almost everywhere. Finally we set $D_1(t) = B(t)N(t)^{-1}\Lambda_E^{-1}B(t)^*$ and $D_2(t) = C(t)^*\Lambda_F C(t)$. We have $D_1(t) \in \mathcal{L}(V;V')$, $D_2(t) \in \mathcal{L}(V;V')$, $t \to (D_1(t)\varphi,\psi)$ measurable $\forall \varphi$, $\psi \in V$, $\|D_i(t)\|_{\mathcal{L}(V;V')} \leq c$, $i = 1,2$, and $D_1(t)^* = D_1(t)$, $D_2(t)^* = D_2(t)$. Then equations (1.39) and (1.40) may be written as:

$$\begin{cases} \dfrac{\partial y}{\partial t} + A(t)y + D_1(t)p = f, \quad t \text{ in }]0,T[, \\ -\dfrac{\partial p}{\partial t} + A^*(t)p - D_2(t)y = g, \quad t \text{ in }]0,T[, \end{cases} \tag{1.42}$$

with $g(t) = -C^*(t)\Lambda_F z_d(t)$, and $y(0) = y_0$, $p(T) = 0$. The solution $\{y,p\}$ of (1.42) may then be written in a unique manner as:

$$p(t) = P(t)y(t) + r(t), \forall t \in]0,T[, \tag{1.43}$$

where $y, p \in C([0,T]; H)$, and $P(t) \in \mathcal{L}(\mathcal{H}, \mathcal{H})$. $P(t)$ and $r(t)$ are defined as follows:

1. We solve

$$\begin{cases} \dfrac{d\beta}{dt} + A(t)\beta + D_1(t)\gamma = 0 \quad \text{ in }]s, T[, \\ -\dfrac{d\gamma}{dt} + A^*(t)\gamma - D_2(t)\beta = 0 \quad \text{ in }]s, T[, \\ \beta(s) = h, \gamma(T) = 0; \end{cases} \tag{1.44}$$

and set:

$$P(s)h = \gamma(s); \tag{1.45}$$

2. We solve

$$\begin{cases} \dfrac{d\eta}{dt} + A(t)\eta + D_1(t)\xi = f \text{ in }]s, T[, \\ -\dfrac{d\xi}{dt} + A^*(t)\xi - D_2(t)\eta = g \text{ in }]s, T[, \\ \eta(s) = 0, \xi(T) = 0; \end{cases} \tag{1.46}$$

and define:

$$r(s) = \xi(s). \tag{1.47}$$

Formal differentiation of (1.43) and substitution in (1.42) leads to:

$$\begin{cases} -\dfrac{\partial P}{\partial t} + PA + A^*P + PD_1P = D_2 \text{ in }]0, T[, \\ -\dfrac{\partial r}{\partial t} + A^*r + PD_1r = Pf + g \text{ in }]0, T[. \end{cases} \tag{1.48}$$

Furthermore, from (1.43) at $t = T$ we have $p(T) = P(T)y(T) + r(T) = 0$, so $P(T) = 0$ and $r(T) = 0$.

Chapter 2

Factorization of $(\mathcal{P}_0)$

In this chapter we begin by showing that problem $(\mathcal{P}_0)$ may be defined as an optimal control problem. In section 2.2, we apply the method of invariant embedding to obtain the factorization of problem $(\mathcal{P}_0)$ in a system of decoupled first order initial value problems. In addition, we obtain an operator Riccati equation where the unknown, P, is the Dirichlet-to-Neumann operator defined on a section of the domain. Then, in section 2.3, we apply the same method in the reverse sense and we obtain another factorization of $(\mathcal{P}_0)$. Next, in section 2.4, we consider an additional Neumann boundary condition at point 1, and deduce the normal equations for this overdetermined problem. In section 2.5, we apply the method of invariant embedding to factorize the fourth order problem obtained in the previous section, into a system of decoupled first order initial value problems. We obtain the same operator Riccati equation as before, and a Lyapunov equation in an operator Q that arises from the invariant embedding. Next, we study some properties of the operators P and Q. Then, by using evolution operators, we study the equations obtained in the factorization. Finally, in section 2.8, we deduce a $\mathcal{QR}$ type factorization of problem $(\mathcal{P}_0)$.

2.1 Associated control problem

In this section, for the sake of simplicity, we consider $u_0 = 0$. We define an optimal control problem that we will show to be equivalent to $(\mathcal{P}_0)$. The control variable is v and the state u verifies equation (2.1) below. Let $\mathcal{U} = L^2(\mathcal{O})$ be the space of controls . For each $v \in \mathcal{U}$, we represent by $u(v)$ the solution of the problem:

$$\begin{cases} \dfrac{\partial u}{\partial x} = v \quad \text{in } \Omega, \\ u(1) = u_1. \end{cases} \tag{2.1}$$

We consider the following set of admissible controls:

$$\mathcal{U}_{ad} = \{v \in \mathcal{U} : u(v) \in X_{u_1}\}$$

where

$$X_{u_1} = \{h \in L^2(0,1; H_0^1(\mathcal{O})) \cap H^1(0,1; L^2(\mathcal{O})) : h(1) = u_1\}.$$

The cost function is

$$J(v) = \|u(v) - u_d\|^2_{L^2(0,1;H_0^1(\mathcal{O}))} + \|v\|^2_{L^2(\Omega)} = \int_0^1 \|\nabla_y u(v) - \nabla_y u_d\|^2_{L^2(\mathcal{O})} dx +$$
$$+ \int_0^1 \int_{\mathcal{O}} v^2 dx dy, \quad v \in \mathcal{U}_{ad}.$$

The desired state u_d is defined in each section by the solution of

$$\begin{cases} -\Delta_y \varphi(x) = f(x) \quad \text{in } \mathcal{O}, \\ \varphi|_{\partial\mathcal{O}} = 0, \end{cases} \tag{2.2}$$

where $\varphi \in L^2(0,1; H_0^1(\mathcal{O}))$. Consequently, we have

$$u_d = (-\Delta_y)^{-1} f \in L^2(0,1; H_0^1(\mathcal{O})).$$

Now we look for $z \in \mathcal{U}_{ad}$, such that

$$J(z) = \inf_{v \in \mathcal{U}_{ad}} J(v).$$

Taking into account that $\mathcal{U}_{ad}$ is not a closed subset in $L^2(\Omega)$, we cannot apply the usual techniques to solve the problem, even it is not clear under that form that this problem has a solution. Nevertheless we can rewrite it as an equivalent minimization problem with respect to the state

$$\mathcal{U}_{ad} = \left\{ \frac{\partial h}{\partial x} : h \in X_{u_1} \right\}$$

and, consequently

$$J(z) = \inf_{v \in \mathcal{U}_{ad}} J(v) = \inf_{h \in X_{u_1}} \bar{J}(h) = \bar{J}(u)$$

where $\frac{\partial u}{\partial x} = z$, and

$$\bar{J}(h) = \|h - u_d\|^2_{L^2(0,1;H^1_0(\mathcal{O}))} + \|\frac{\partial h}{\partial x}\|^2_{L^2(\Omega)} = \int_0^1 \|\nabla_y h - \nabla_y u_d\|^2_{L^2(\mathcal{O})} dx +$$
$$+ \int_0^1 \int_{\mathcal{O}} \left| \frac{\partial h}{\partial x} \right|^2 dxdy.$$

We remark that X_{u_1} is a closed convex subset in the Hilbert space

$$X = L^2(0,1; H^1_0(\mathcal{O})) \cap H^1(0,1; L^2(\mathcal{O}))$$

and, for $u_d = 0$, $(\bar{J}(h))^{\frac{1}{2}}$ is a norm equivalent to the norm in X. Then by Theorem 1.3, chapter I, of [16], there exists a unique $u \in X_{u_1}$, such that:

$$\bar{J}(u) = \inf_{h \in X_{u_1}} \bar{J}(h)$$

which is uniquely determined by the condition

$$\bar{J}'(u)(h - u) \geq 0, \forall h \in X_{u_1}.$$

But X_0 is a subspace, and so the last condition is equivalent to

$$\bar{J}'(u)(h) = 0, \forall h \in X_0. \tag{2.3}$$

Now we have

$$\bar{J}'(u)(h) = 0 \Leftrightarrow \lim_{\theta \to 0^+} \frac{1}{\theta}[\bar{J}(u + \theta h) - \bar{J}(u)] = 0 \Leftrightarrow \int_0^1 \int_{\mathcal{O}} \nabla_y (u - u_d) . \nabla_y h dxdy +$$
$$+ \int_0^1 \int_{\mathcal{O}} \frac{\partial u}{\partial x} \frac{\partial h}{\partial x} dxdy = 0, \forall h \in X_0$$

which implies that

$$\int_0^1 \langle -\Delta_y(u-u_d), h\rangle_{H^{-1}(\mathcal{O})\times H_0^1(\mathcal{O})}\, dx + \int_0^1 \int_{\mathcal{O}} \frac{\partial u}{\partial x}\frac{\partial h}{\partial x} dxdy = 0, \forall h \in X_0.$$

Then, taking into account that $u_d = (-\Delta_y)^{-1} f$, we obtain

$$\int_0^1 \langle -\Delta_y(u) - f, h\rangle_{H^{-1}(\mathcal{O})\times H_0^1(\mathcal{O})}\, dx + \int_0^1 \int_{\mathcal{O}} \frac{\partial u}{\partial x}\frac{\partial h}{\partial x} dxdy = 0, \forall h \in X_0.$$

If we consider $h \in \mathcal{D}(\Omega)$, then

$$\left\langle -\Delta_y u - \frac{\partial^2 u}{\partial x^2} - f, h \right\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = 0, \forall h \in \mathcal{D}(\Omega)$$

so, we may conclude that $-\Delta u = f$ in the sense of distributions. But $f \in L^2(\Omega)$, and so we deduce that $u \in Y$, where

$$Y = \left\{ v \in X_{u_1} : \Delta v \in L^2(\Omega) \right\}.$$

We now introduce the adjoint state :

$$\begin{cases} \dfrac{\partial p}{\partial x} = -\Delta_y u - f \quad \text{in } \Omega, \\ p(0) = 0. \end{cases}$$

We know that $-\Delta_y u - f \in L^2(0,1; H^{-1}(\mathcal{O}))$. For each $h \in X_0$

$$\int_0^1 \langle -\Delta_y u - f, h\rangle_{H^{-1}(\mathcal{O})\times H_0^1(\mathcal{O})}\, dx = \int_0^1 \left\langle \frac{\partial p}{\partial x}, h \right\rangle_{H^{-1}(\mathcal{O})\times H_0^1(\mathcal{O})} dx =$$
$$= -\int_0^1 \int_{\mathcal{O}} p \frac{\partial h}{\partial x} dydx$$

and so $p \in L^2(\Omega)$. Using the optimality condition (2.3), we obtain:

$$\int_0^1 \int_{\mathcal{O}} \left(-p + \frac{\partial u}{\partial x} \right) \frac{\partial h}{\partial x} dydx = 0, \forall h \in X_0,$$

which implies that

$$-p + \frac{\partial u}{\partial x} = 0 \tag{2.4}$$

Furthermore, from (2.4) we have that:

$$0 = \int_0^1 \int_{\mathcal{O}} (-p + \frac{\partial u}{\partial x}) \frac{\partial h}{\partial x} dydx = -\int_0^1 \int_{\mathcal{O}} \frac{\partial}{\partial x}(-p + \frac{\partial u}{\partial x}) hdydx + \\ + \int_{\mathcal{O}} (-p + \frac{\partial u}{\partial x})(1)h(1)dy - \int_{\mathcal{O}} (-p + \frac{\partial u}{\partial x})(0)h(0)dy, \quad \forall h \in X_0, \tag{2.5}$$

and, taking into account that $h(1) = p(0) = 0$, we deduce that $\frac{\partial u}{\partial x}(0) = 0$. We have thus shown that problem

$$(\mathcal{P}_{1,u_1}) \begin{cases} \dfrac{\partial u}{\partial x} = p \quad \text{in } \Omega, \quad u\,(1) = u_1, \\ \dfrac{\partial p}{\partial x} = -\Delta_y u - f \quad \text{in } \Omega, \quad p\,(0) = 0, \end{cases} \tag{2.6}$$

admits a unique solution $\{u, p\} \in L^2(\Omega) \times L^2(\Omega)$, where u is the solution of $(\mathcal{P}_0)$.

We can represent the optimality system (2.6) in matrix form as follows:

$$\mathcal{A} \begin{pmatrix} p \\ \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ \\ f \end{pmatrix}, \quad u(1) = u_1, \; p(0) = 0, \tag{2.7}$$

with

$$\mathcal{A} = \begin{pmatrix} -I & \frac{\partial}{\partial x} \\ \\ -\frac{\partial}{\partial x} & -\Delta_y \end{pmatrix}.$$

Remark 2.1 *We notice that problem $(\mathcal{P}_0)$ is equivalent to the optimality system written in matrix form (2.7).*

2.2 Direct invariant embedding

Following R. Bellman [2], we embed problem $(\mathcal{P}_{1,u_1})$ in the family of similar problems defined on $\Omega_s =]0, s[\times \mathcal{O}, 0 < s \leq 1$:

$$(\mathcal{P}_{s,h}) \begin{cases} \dfrac{\partial \varphi}{\partial x} - \psi = 0 \quad \text{in } \Omega_s, \quad \varphi(s) = h, \\ \varphi|_{\Sigma} = 0, \\ -\dfrac{\partial \psi}{\partial x} - \Delta_y \varphi = f \quad \text{in } \Omega_s, \quad \psi(0) = -u_0, \end{cases} \tag{2.8}$$

where h is given in $H_{00}^{1/2}(\mathcal{O})$. When $s=1$ and $h=u_1$ we obtain problem $(\mathcal{P}_{1,u_1})$. Due to the linearity of the problem, the solution $\{\varphi_{s,h},\psi_{s,h}\}$ of $(\mathcal{P}_{s,h})$ verifies

$$\psi_{s,h}(s)=P(s)h+r(s), \tag{2.9}$$

where $P(s)$ and $r(s)$ are defined as follows:

1) We solve

$$\begin{cases} \dfrac{\partial\beta}{\partial x}-\gamma=0 \ \text{ in } \Omega_s, \quad \beta\,(s)=h, \\ \beta|_{\Sigma}=0, \\ -\dfrac{\partial\gamma}{\partial x}-\Delta_y\beta=0 \ \text{ in } \Omega_s, \quad \gamma\,(0)=0, \end{cases} \tag{2.10}$$

and define $P(s)$ as:

$$P(s)h=\gamma(s).$$

We remark that $P(s)$ is the Dirichlet-to-Neumann operator on Γ_s relative to the domain Ω_s.

2) We solve

$$\begin{cases} \dfrac{\partial\eta}{\partial x}-\xi=0 \ \text{ in } \Omega_s, \quad \eta\,(s)=0, \\ \eta|_{\Sigma}=0, \\ -\dfrac{\partial\xi}{\partial x}-\Delta_y\eta=f \ \text{ in } \Omega_s, \quad \xi\,(0)=-u_0. \end{cases} \tag{2.11}$$

The remainder $r(s)$ is defined by:

$$r(s)=\xi(s).$$

Furthermore, the solution $\{u,p\}$ of $(\mathcal{P}_{1,u_1})$ restricted to $]0,s[$ satisfies $(\mathcal{P}_{s,u|_{\Gamma_s}})$, for $s\in]0,1[$, and so one has the relation

$$p(x)=P(x)u(x)+r(x), \forall x\in]0,1[. \tag{2.12}$$

From (2.12) and the boundary conditions at $x=0$, we easily deduce that

$$P(0)=0, \ \ r(0)=-u_0.$$

Formally, taking the derivative with respect to x on both sides of equation (2.12), we obtain:

$$\frac{\partial p}{\partial x}(x) = \frac{dP}{dx}(x)u(x) + P(x)\frac{\partial u}{\partial x}(x) + \frac{dr}{dx}(x)$$

and, substituting from (2.6) and (2.12) we conclude that:

$$\begin{aligned}&-\Delta_y u - f = \frac{dP}{dx}(x)u(x) + P(x)(P(x)u(x) + r(x)) + \frac{dr}{dx} \Leftrightarrow \\ &(\frac{dP}{dx} + P^2 + \Delta_y)u + \frac{dr}{dx} + Pr + f = 0.\end{aligned} \tag{2.13}$$

Then, taking into account that $u(x) = h$ is arbitrary, we obtain the following decoupled system:

$$\frac{dP}{dx} + P^2 + \Delta_y = 0, \quad P(0) = 0, \tag{2.14}$$

$$\frac{\partial r}{\partial x} + Pr = -f, \quad r(0) = -u_0, \tag{2.15}$$

$$\frac{\partial u}{\partial x} - Pu = r, \quad u(1) = u_1, \tag{2.16}$$

where P and r are integrated from 0 to 1, and finally u is integrated backwards from 1 to 0. We remark that P is an operator on functions defined on $\mathcal{O}$ verifying a Riccati equation.

We have factorized problem $(\mathcal{P}_0)$ as:

$$\text{“} - \Delta\text{”} = -\left(\frac{d}{dx} + P\right)\left(\frac{d}{dx} - P\right).$$

This decoupling of the optimality system (2.6) may be seen as a generalized block LU factorization. In fact, for this particular problem, we may write

$$\mathcal{A} = \begin{pmatrix} I & 0 \\ -P & -\frac{d}{dx} - P \end{pmatrix} \begin{pmatrix} -I & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & -P \\ 0 & \frac{d}{dx} - P \end{pmatrix}.$$

We will see in section 2.6 that P is self adjoint. So, the first and third matrices are adjoint of one another and are, respectively, lower triangular and upper triangular.

2.3 Backwards invariant embedding

In this section we are going to obtain another factorization of $(\mathcal{P}_0)$. For that purpose, we embed problem $(\mathcal{P}_0)$ in the family of similar problems defined in $\Omega'_s =]s,1[\times\mathcal{O}$ with additional boundary conditions in $\Gamma_s = \{s\}\times\mathcal{O}$:

$$(\hat{\mathcal{P}}_{s,h}) \begin{cases} -\Delta u = f \quad \text{in } \Omega'_s, \\ u|_\Sigma = 0, \\ -\dfrac{\partial u}{\partial x}|_{\Gamma_s} = h, \\ u|_{\Gamma_1} = u_1 \end{cases} \tag{2.17}$$

We know that, for each $h \in (H^{1/2}_{00}(\mathcal{O}))'$ and each $s \in [0,1[$, (2.17) is a well posed problem. Besides, when $s = 0$ and $h = u_0$, we obtain problem $(\mathcal{P}_0)$. Then the solution $\hat{u}_{s,h}$ of (2.17) verifies:

$$\hat{u}_{s,h}|_{\Gamma_s} = \hat{P}(s)h + \hat{r}(s), \tag{2.18}$$

where $\hat{P}$ and $\hat{r}$ are defined in the following way:

1. We solve

$$\begin{cases} -\Delta\gamma = 0 \quad \text{in } \Omega'_s, \\ \gamma|_\Sigma = 0, \\ -\dfrac{\partial\gamma}{\partial x}|_{\Gamma_s} = h, \\ \gamma|_{\Gamma_1} = 0. \end{cases} \tag{2.19}$$

Then $\hat{P}(s)h = \gamma|_{\Gamma_s}$, for $s \in [0,1[$, and $\hat{P}(1) = 0$.

2. We solve:

$$\begin{cases} -\Delta\beta = f \quad \text{in } \Omega'_s, \\ \beta|_\Sigma = 0, \\ -\dfrac{\partial\beta}{\partial x}|_{\Gamma_s} = 0, \\ \beta|_{\Gamma_1} = u_1, \end{cases} \tag{2.20}$$

and set: $\hat{r}(s) = \beta|_{\Gamma_s}$, and $\hat{r}(1) = u_1$. Moreover, the solution u of $(\mathcal{P}_0)$ restricted to $]s, 1[$ verifies $(\hat{\mathcal{P}}_{s,u|_{\Gamma_s}})$, for $s \in]0, 1[$, so we have the relation:

$$u(x, y) = (\hat{P}(x)(-\frac{\partial u}{\partial x}|_{\Gamma_x}))(y) + (\hat{r}(x))(y), \quad \forall x \in]0, 1[. \tag{2.21}$$

By formal derivation with respect to x of this formula, we obtain:

$$\frac{\partial u}{\partial x} = \frac{\partial \hat{P}}{\partial x}(-\frac{\partial u}{\partial x}) + \hat{P}(-\frac{\partial^2 u}{\partial x^2}) + \frac{d\hat{r}}{dx}. \tag{2.22}$$

Then substituting from (2.21) and (2.17) we derive:

$$\frac{\partial u}{\partial x} = \frac{\partial \hat{P}}{\partial x}(-\frac{\partial u}{\partial x}) + \hat{P}f + \hat{P}\Delta_y[\hat{P}(-\frac{\partial u}{\partial x}) + \hat{r}] + \frac{d\hat{r}}{dx}. \tag{2.23}$$

Finally, taking into account that $-\frac{\partial u}{\partial x}|_{\Gamma_s} = h$ is arbitrary, we deduce

$$\begin{cases} \dfrac{\partial \hat{P}}{\partial x} + \hat{P}\Delta_y \hat{P} + I = 0, \quad \hat{P}(1) = 0 \\ \dfrac{d\hat{r}}{dx} + \hat{P}\Delta_y \hat{r} = -\hat{P}f, \quad \hat{r}(1) = u_1 \\ \hat{P}\dfrac{\partial u}{\partial x} + u = \hat{r}, \quad -\dfrac{\partial u}{\partial x}|_{\Gamma_0} = u_0. \end{cases} \tag{2.24}$$

2.4 Normal equations for the overdetermined problem

From now on, we suppose $u_0 \in H^{1/2}_{00}(\mathcal{O})$, $u_1 \in H^{3/2}(\mathcal{O}) \cap H^{1/2}_{00}(\mathcal{O})$, $u_2 \in H^{1/2}_{00}(\mathcal{O})$, $f \in H^2(\Omega) \cap L^2(0, 1; H^1_0(\mathcal{O}))$ and $f|_\Sigma = 0$.

Assuming we have an extra information, given by a Neumann boundary condition at point 1, we consider the overdetermined system

$$\mathcal{A}\begin{pmatrix} p \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}, \; u(1) = u_1, \; p(0) = -u_0, \; \frac{\partial u}{\partial x}(1) = u_2. \tag{2.25}$$

If the data are not compatible with (2.7), this system should be satisfied in the least square sense.

We introduce a perturbation ,

$$\mathcal{A}\begin{pmatrix} p \\ \\ u \end{pmatrix} = \begin{pmatrix} \delta g \\ \\ f+\delta f \end{pmatrix},\ u(1)=u_1,\ p(0)=-u_0,\ \frac{\partial u}{\partial x}(1)=u_2. \tag{2.26}$$

We want to minimize the norm of the perturbation,

$$J(\delta f,\delta g)=\frac{1}{2}\int_0^1\left(\|\delta f\|^2_{L^2(\mathcal{O})}+\|\delta g\|^2_{L^2(\mathcal{O})}\right)dx, \tag{2.27}$$

subject to the constraint given by (2.26). This defines problem $(\mathcal{P}_1)$. Like in section 2.1, we know that the final optimality problem is well-posed, and we consider the corresponding Lagrangian.

Taking, for convenience, the Lagrange multiplier of the second equation of (2.26) as $\bar{u}-f$, where $\bar{u}$ verifies the additional boundary condition $\bar{u}|_\Sigma = f$, we have:

$$\begin{aligned} L(\delta f,\delta g,u,p,\bar{u},\bar{p}) &= J(\delta f,\delta g)+\int_0^1\left(\bar{p},\frac{\partial u}{\partial x}-p-\delta g\right)_{L^2(\mathcal{O})}dx+ \\ &+\int_0^1\left(\bar{u}-f,-\frac{\partial p}{\partial x}-\Delta_y u-f-\delta f\right)_{L^2(\mathcal{O})}dx+\left(\mu,\frac{\partial u}{\partial x}(1)-u_2\right)_{L^2(\mathcal{O})}. \end{aligned} \tag{2.28}$$

Taking into account that $\frac{\partial u}{\partial x}(1)=p(1)+\delta g(1)$, we obtain

$$\left(\frac{\partial L}{\partial u},\varphi\right)=\int_0^1(\bar{u}-f,-\Delta_y\varphi)_{L^2(\mathcal{O})}dx+\int_0^1\left(\bar{p},\frac{\partial\varphi}{\partial x}\right)_{L^2(\mathcal{O})}dx,\ \forall\varphi\in\mathcal{Y},$$

where

$$\mathcal{Y}=\left\{\varphi\in H^1(\Omega):\Delta\varphi\in L^2(\Omega),\ \varphi|_\Sigma=0,\ \frac{\partial\varphi}{\partial x}(0)=0,\ \varphi(1)=0\right\}$$

and, integrating by parts, we derive

$$\left(\frac{\partial L}{\partial u},\varphi\right)=\int_0^1(-\Delta_y(\bar{u}-f),\varphi)_{L^2(\mathcal{O})}dx-(\bar{p}(0),\varphi(0))+\int_0^1\left(-\frac{\partial\bar{p}}{\partial x},\varphi\right)_{L^2(\mathcal{O})}dx.$$

Now, if $\bar{p}(0)=0$, and because all the functions are null on Σ, we conclude that:

$$\frac{\partial L}{\partial u}=0\Leftrightarrow-\frac{\partial\bar{p}}{\partial x}-\Delta_y\bar{u}=-\Delta_y f.$$

On the other hand

$$\left(\frac{\partial L}{\partial p},\psi\right)=\int_0^1\left(\bar{u}-f,-\frac{\partial \psi}{\partial x}\right)_{L^2(\mathcal{O})}dx+\int_0^1(\bar{p},-\psi)_{L^2(\mathcal{O})}\,dx+$$

$$(\mu,\psi\,(1))_{L^2(\mathcal{O})}=\int_0^1\left(\frac{\partial\,(\bar{u}-f)}{\partial x},\psi\right)_{L^2(\mathcal{O})}dx+(\bar{u}\,(0)-f\,(0)\,,\psi\,(0))_{L^2(\mathcal{O})}-$$

$$-\,(\bar{u}\,(1)-f\,(1)\,,\psi\,(1))_{L^2(\mathcal{O})}+\int_0^1(-\bar{p},\psi)_{L^2(\mathcal{O})}\,dx+(\mu,\psi\,(1))_{L^2(\mathcal{O})}$$

and, if $\psi\,(0)=0$ and $\bar{u}\,(1)-f\,(1)=\mu$ arbitrary, then

$$\frac{\partial L}{\partial p}=0\Leftrightarrow\frac{\partial \bar{u}}{\partial x}-\bar{p}=\frac{\partial f}{\partial x}.$$

We have thus obtained:

$$\begin{cases}\dfrac{\partial \bar{u}}{\partial x}-\bar{p}=f_1:=\dfrac{\partial f}{\partial x}, \quad \bar{u}\,(1)\text{ arbitrary}, \quad \bar{u}=f\text{ on }\Sigma,\\ -\dfrac{\partial \bar{p}}{\partial x}-\Delta_y\bar{u}=f_2:=-\Delta_y f, \quad \bar{p}\,(0)=0.\end{cases}\tag{2.29}$$

We finally evaluate the optimal values for δf and δg. We have:

$$\left(\frac{\partial L}{\partial(\delta f)},\gamma\right)=\int_0^1(\delta f,\gamma)_{L^2(\mathcal{O})}dx+\int_0^1(\bar{u}-f,-\gamma)_{L^2(\mathcal{O})}dx,\ \ \forall\gamma\in L^2(\Omega)$$

and for all $\xi\in L^2(\Omega)$ such that $\frac{\partial\xi}{\partial x}\in L^2(\Omega)$,

$$\left(\frac{\partial L}{\partial(\delta g)},\xi\right)=\int_0^1(\delta g,\xi)_{L^2(\mathcal{O})}dx+\int_0^1(\bar{p},-\xi)_{L^2(\mathcal{O})}dx.$$

At the minimum, we must have

$$\frac{\partial L}{\partial(\delta f)}=0\Leftrightarrow\delta f=\bar{u}-f$$

and

$$\frac{\partial L}{\partial(\delta g)}=0\Leftrightarrow\delta g=\bar{p}.$$

In conclusion, we obtain

$$\mathcal{A}\begin{pmatrix}p\\ \\ u\end{pmatrix}=\begin{pmatrix}\delta g\\ \\ f+\delta f\end{pmatrix}=\begin{pmatrix}\bar{p}\\ \\ \bar{u}\end{pmatrix},\tag{2.30}$$

and the normal equation is given by

$$\mathcal{A}^2 \begin{pmatrix} p \\ \\ u \end{pmatrix} = \begin{pmatrix} f_1 \\ \\ f_2 \end{pmatrix}, \; p(0) = -u_0, \; u(1) = u_1, \; \frac{\partial u}{\partial x}(1) = u_2, \; -\frac{\partial u}{\partial x}(0) = u_0. \tag{2.31}$$

From (2.29), we have

$$-\Delta \bar{u} = -\frac{\partial^2 \bar{u}}{\partial x^2} - \Delta_y \bar{u} = \frac{\partial^2 f}{\partial x^2} - \Delta_y f = -\Delta f \tag{2.32}$$

and, from (2.29) and (2.30),

$$-\Delta f = -\Delta \bar{u} = -\Delta \left(-\frac{\partial p}{\partial x} - \Delta_y u \right) = -\Delta \left(\frac{\partial \bar{p}}{\partial x} - \frac{\partial^2 u}{\partial x^2} - \Delta_y u \right)$$
$$= -\Delta \left(-\Delta_y \bar{u} + \Delta_y f - \Delta u \right) = \Delta^2 u + \Delta_y \left(\Delta \bar{u} - \Delta f \right) = \Delta^2 u.$$

We now notice that

$$\frac{\partial^2 \bar{p}}{\partial x^2} = \frac{\partial}{\partial x} \left(\Delta_y f - \Delta_y \bar{u} \right) = \Delta_y \left(\frac{\partial f}{\partial x} - \frac{\partial \bar{u}}{\partial x} \right) = -\Delta_y \bar{p}$$

and, remarking that $\bar{p}(0) = 0$, we derive $-\Delta_y \bar{p}(0) = 0$ which implies that

$$\frac{\partial(\Delta u)}{\partial x}(0) = \frac{\partial^2 \bar{p}}{\partial x^2}(0) - \frac{\partial \bar{u}}{\partial x}(0) = -\Delta_y \bar{p}(0) - \bar{p}(0) - \frac{\partial f}{\partial x}(0) = -\frac{\partial f}{\partial x}(0).$$

Furthermore, taking into account that $-\Delta u = f$ in Ω and $f|_\Sigma = 0$, we conclude that $\Delta u|_\Sigma = 0$. Now we can write the normal equation as

$$(\mathcal{P}_2) \quad \begin{cases} \Delta^2 u = -\Delta f, \quad \text{in } \Omega, \\ u|_\Sigma = 0, \quad \Delta u|_\Sigma = 0, \\ -\dfrac{\partial u}{\partial x}(0) = u_0, \quad \dfrac{\partial \Delta u}{\partial x}(0) = -\dfrac{\partial f}{\partial x}(0), \\ u(1) = u_1, \; \dfrac{\partial u}{\partial x}(1) = u_2. \end{cases} \tag{2.33}$$

2.5 Factorization of the normal equation by invariant embedding

In order to factorize problem (2.33) we consider $f \in H^2(\Omega) \cap L^2(0,1; H_0^1(\mathcal{O}))$, $u_1 \in H^{3/2}(\mathcal{O}) \cap H_{00}^{1/2}(\mathcal{O})$, $u_0, u_2 \in H_{00}^{1/2}(\mathcal{O})$. We embed (2.33) in the family of problems $(\mathcal{P}_{s,h,k})$

defined in $\Omega_s =]0, s[\times \mathcal{O}$, for each $h \in H^{\frac{3}{2}}(\mathcal{O}) \cap H^{1/2}_{00}(\mathcal{O})$ and each $k \in (H^{\frac{1}{2}}_{00}(\mathcal{O}))'$. Afterwards we will show the relation between $(\mathcal{P}_{s,h,k})$ for $s = 1$ and problem (2.33).

$$(\mathcal{P}_{s,h,k}) \quad \begin{cases} \Delta^2 u = -\Delta f, \quad \text{in } \Omega_s, \\ u|_\Sigma = 0, \quad \Delta u|_\Sigma = 0, \\ -\dfrac{\partial u}{\partial x}(0) = u_0, \quad \dfrac{\partial \Delta u}{\partial x}(0) = -\dfrac{\partial f}{\partial x}(0), \\ u|_{\Gamma_s} = h, \ \Delta u|_{\Gamma_s} = k. \end{cases} \tag{2.34}$$

These problems may be decomposed in two second order boundary value problems, as follows:

1. First, we solve:

$$\begin{cases} -\Delta v = \Delta f, \quad \text{in } \Omega_s, \\ v|_\Sigma = 0, \\ -\dfrac{\partial v}{\partial x}(0) = \dfrac{\partial f}{\partial x}(0), \\ v|_{\Gamma_s} = k \end{cases} \tag{2.35}$$

From section 1.2, we know that this is a well-posed problem, and it admits a unique solution $v \in H^1(\Omega)$.

2. Then we solve:

$$\begin{cases} \Delta u = v, \quad \text{in } \Omega_s, \\ u|_\Sigma = 0, \\ -\dfrac{\partial u}{\partial x}(0) = u_0, \\ u|_{\Gamma_s} = h \end{cases} \tag{2.36}$$

Again, by section 1.2, this is a well-posed problem, and it admits a unique solution $u \in H^2(\Omega)$, and so we may evaluate its trace $\dfrac{\partial u}{\partial x}(s)$, which is an affine function of h and k:

$$\frac{\partial u}{\partial x}(s) = P_1(s)h + Q(s)k + \tilde{r}(s). \tag{2.37}$$

In order to define P_1 we consider the problem:

$$\begin{cases} \Delta^2 \gamma_1 = 0, \quad \text{in } \Omega_s, \\ \gamma_1|_\Sigma = 0, \quad \Delta\gamma_1|_\Sigma = 0, \\ \dfrac{\partial \gamma_1}{\partial x}(0) = 0, \quad \dfrac{\partial \Delta\gamma_1}{\partial x}(0) = 0, \\ \gamma_1|_{\Gamma_s} = h, \ \Delta\gamma_1|_{\Gamma_s} = 0. \end{cases} \tag{2.38}$$

This problem reduces to:

$$\begin{cases} \Delta \gamma_1 = 0, \quad \text{in } \Omega_s, \\ \gamma_1|_\Sigma = 0, \\ \dfrac{\partial \gamma_1}{\partial x}(0) = 0, \quad \gamma_1|_{\Gamma_s} = h. \end{cases} \tag{2.39}$$

We set $P_1(s)h = \dfrac{\partial \gamma_1}{\partial x}(s)$. This defines P_1 as an operator from $H^{3/2}(\mathcal{O}) \cap H^{1/2}_{00}(\mathcal{O})$ into $H^{1/2}_{00}(\mathcal{O})$. From (2.10) we may conclude that $P_1 = P$.

To define Q, we consider

$$\begin{cases} \Delta^2 \gamma_2 = 0, \quad \text{in } \Omega_s, \\ \gamma_2|_\Sigma = 0, \quad \Delta\gamma_2|_\Sigma = 0, \\ \dfrac{\partial \gamma_2}{\partial x}(0) = 0, \quad \dfrac{\partial \Delta\gamma_2}{\partial x}(0) = 0, \\ \gamma_2|_{\Gamma_s} = 0, \ \Delta\gamma_2|_{\Gamma_s} = k, \end{cases} \tag{2.40}$$

and set:

$$Q(s)k = \frac{\partial \gamma_2}{\partial x}(s).$$

Problem (2.40) can be decomposed in two second order boundary value problems as follows:

1. First we solve:

$$\begin{cases} \Delta v = 0, \quad \text{in } \Omega_s, \\ v|_\Sigma = 0, \\ -\dfrac{\partial v}{\partial x}(0) = 0, \\ v|_{\Gamma_s} = k \end{cases} \tag{2.41}$$

2. Next we solve:

$$\begin{cases} \Delta \gamma_2 = v, \quad \text{in } \Omega_s, \\ \gamma_2|_\Sigma = 0, \\ -\dfrac{\partial \gamma_2}{\partial x}(0) = 0, \\ \gamma_2|_{\Gamma_s} = 0 \end{cases} \tag{2.42}$$

Finally, we solve:

$$\begin{cases} \Delta^2 \beta = -\Delta f, \quad \text{in } \Omega_s, \\ \beta|_\Sigma = \Delta\beta|_\Sigma = 0, \\ -\dfrac{\partial \beta}{\partial x}(0) = u_0, \quad \dfrac{\partial \Delta\beta}{\partial x}(0) = -\dfrac{\partial f}{\partial x}(0), \\ \beta|_{\Gamma_s} = \Delta\beta|_{\Gamma_s} = 0 \end{cases} \tag{2.43}$$

and set:

$$\tilde{r}(s) = \frac{\partial \beta}{\partial x}(s).$$

Then, the solution of the normal equation restricted to $]0, s[$, verifies $(\mathcal{P}_{s,u|_{\Gamma_s},\Delta u|_{\Gamma_s}})$, for $s \in]0, 1[$. So, one has the relation

$$\frac{\partial u}{\partial x}|_{\Gamma_s} = P(s)u|_{\Gamma_s} + Q(s)\Delta u|_{\Gamma_s} + \tilde{r}(s). \tag{2.44}$$

From (2.44), it is easy to see that $Q(0) = 0$ and $\tilde{r}(0) = -u_0$. On the other hand, we may consider the following second order problem on Δu as a subproblem of problem (2.34)

$$\begin{cases} \Delta(\Delta u) = -\Delta f, \quad \text{in } \Omega_s, \\ \Delta u|_\Sigma = 0, \\ \dfrac{\partial \Delta u}{\partial x}(0) = -\dfrac{\partial f}{\partial x}(0), \\ \Delta u|_{\Gamma_1} = c. \end{cases} \tag{2.45}$$

where c is to be determined later, in order to be compatible with the other data. From (2.15) and (2.16), it admits the following factorization:

$$\begin{cases} \dfrac{\partial t}{\partial x} + Pt = -\Delta f, \quad t(0) = -\dfrac{\partial f}{\partial x}(0), \\ -\dfrac{\partial \Delta u}{\partial x} + P\Delta u = -t, \quad \Delta u(1) = c. \end{cases} \tag{2.46}$$

Formally, taking the derivative with respect to x on both sides of (2.44), we obtain:

$$\frac{\partial^2 u}{\partial x^2}(x) = \frac{dP}{dx}(x)u(x) + P(x)\frac{\partial u}{\partial x}(x) + \frac{dQ}{dx}(x)\Delta u(x) + Q(x)\frac{\partial \Delta u}{\partial x}(x) + \frac{d\tilde{r}}{dx}(x)$$

and, substituting from (2.44) and (2.46), we obtain:

$$\Delta u - \Delta_y u = \frac{dP}{dx}u + P(Pu + Q\Delta u + \tilde{r}) + \frac{dQ}{dx}\Delta u + Q(P\Delta u + t) + \frac{d\tilde{r}}{dx} \tag{2.47}$$

or which is equivalent

$$(\frac{dP}{dx} + P^2 + \Delta_y)u + (\frac{dQ}{dx} + PQ + QP - I)\Delta u + \frac{d\tilde{r}}{dx} + P\tilde{r} + Qt = 0. \tag{2.48}$$

Now, taking into account that $u|_{\Gamma_s} = h$ and $\Delta u|_{\Gamma_s} = k$ are arbitrary, we derive

$$\frac{dP}{dx} + P^2 + \Delta_y = 0, \quad P(0) = 0, \tag{2.49}$$

$$\frac{dQ}{dx} + PQ + QP = I, \quad Q(0) = 0, \tag{2.50}$$

$$\frac{\partial t}{\partial x} + Pt = -\Delta f, \quad t(0) = -\frac{\partial f}{\partial x}(0), \tag{2.51}$$

$$\frac{\partial \tilde{r}}{\partial x} + P\tilde{r} = -Qt, \quad \tilde{r}(0) = -u_0, \tag{2.52}$$

$$\frac{\partial \Delta u}{\partial x} - P\Delta u = t, \quad \Delta u(1) = c, \tag{2.53}$$

$$\frac{\partial u}{\partial x} - Pu = Q\Delta u + \tilde{r}, \quad u(1) = u_1. \tag{2.54}$$

It is easy to see, from the definition, that $Q(1)$ is a bijective operator from $(H_{00}^{\frac{1}{2}}(\mathcal{O}))'$ to $H_{00}^{\frac{1}{2}}(\mathcal{O})$, so we can define $(Q(1))^{-1}$. From (2.37) and the regularity assumptions made at beginning of this section, we can define:

$$c = (Q(1))^{-1}(u_2 - P(1)\, u_1 - \tilde{r}(1)). \tag{2.55}$$

Remark 2.2 *We remark the interest of the factorized form if the same problem has to be solved many times for various sets of data (u_1, u_2). Once the problem has been factorized, that is P and Q have been computed, and t and $\tilde{r}$ are known, the solution for a data set (u_1, u_2) is obtained by solving (2.55) and then the Cauchy initial value problems (2.53), (2.54) backwards in x.*

We have factorized problem $(\mathcal{P}_2)$. We may write

$$\begin{pmatrix} -\frac{d}{dx} - P & 0 \\ -Q & -\frac{d}{dx} - P \end{pmatrix} \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \frac{d}{dx} - P & -Q \\ 0 & \frac{d}{dx} - P \end{pmatrix} \begin{pmatrix} u \\ \Delta u \end{pmatrix} = \begin{pmatrix} -\Delta f \\ 0 \end{pmatrix}$$

2.6 Some properties of P and Q

In this section we study some properties of the operators P and Q appearing in the previous sections.

Definition 2.3 *Given $s \in]0, 1]$, we define the following spaces:*

$$X_s = L^2(0, s; H_0^1(\mathcal{O})) \cap H^1(0, s; L^2(\mathcal{O})),$$

$$Y_s = \{u \in X_s : \frac{\partial^2 u}{\partial x^2} \in L^2(0, s; H^{-1}(\mathcal{O}))\},$$

Lemma 2.4 *For each $s \in [0, 1]$, and each $h \in H_{00}^{1/2}(\mathcal{O})$, $(\mathcal{P}_{s,h})$ is a well-posed problem, and $P(s) \in \mathcal{L}(H_{00}^{1/2}(\mathcal{O}); (H_{00}^{1/2}(\mathcal{O}))')$. Moreover, for each $s \in [0, 1]$, $r(s) \in (H_{00}^{1/2}(\mathcal{O}))'$.*

Proof. The linearity of $P(s)$ is an imediate consequence of the linearity of problem $(\mathcal{P}_{s,h})$. On the other hand, similarly to section 1.2, it is easy to show that for each $s \in [0, 1]$ and each $h \in H_{00}^{1/2}(\mathcal{O})$, $(\mathcal{P}_{s,h})$ admits a unique solution $u_s \in Y_s$. Let $\gamma_s \in Y_s$ be the solution of (2.10). Then $\dfrac{\partial\gamma_s}{\partial x} \in L^2(\Omega_s)$, and its trace $\dfrac{\partial\gamma_s}{\partial x}|_{\Gamma_s} \in (H_{00}^{1/2}(\mathcal{O}))'$. On the other hand, let $\beta_s \in Y_s$ be the solution of (2.11). Then $r(s) = \dfrac{\partial\beta_s}{\partial x}|_{\Gamma_s} \in (H_{00}^{1/2}(\mathcal{O}))'$. ■

Lemma 2.5 *For each $s \in [0,1]$, $P(s)$ is a self-adjoint non negative operator and it is positive if $s \neq 0$.*

Proof. In fact, the property is obviously true when $s = 0$. On the other hand, let $s \in]0,1]$, $h_1, h_2 \in L^2(\mathcal{O})$, and $\{\beta_1, \gamma_1\}$,$\{\beta_2, \gamma_2\}$ the corresponding solutions of (2.10) and (2.11) to h_1 and h_2. From the definition of P we may conclude that $P(s)h_i = \dfrac{\partial\beta_i}{\partial x}|_{\Gamma_s}$, where β_i is the solution of:

$$\begin{cases} -\Delta\beta_i = 0 \quad \text{in } \Omega_s, \\ \beta_i|_\Sigma = 0, \\ -\dfrac{\partial\beta_i}{\partial x}|_{\Gamma_0} = 0, \quad \beta_i|_{\Gamma_s} = h_i. \end{cases}$$

We then have that:

$$0 = \int_{\Omega_s} (-\Delta\beta_1)\beta_2 dxdy = \int_{\Omega_s} \nabla\beta_1\nabla\beta_2 dxdy - \int_{\partial\Omega_s} \frac{\partial\beta_1}{\partial x}\beta_2 d\sigma$$

and, taking into account that $\beta_2|_\Sigma = 0$, $\dfrac{\partial\beta_1}{\partial x}|_{\Gamma_0} = 0$ and $\beta_2|_{\Gamma_s} = h_2$, we conclude that:

$$(P(s)h_1, h_2) = \int_{\Gamma_s} \frac{\partial\beta_1}{\partial x}(s)\beta_2(s)d\sigma = \int_{\Omega_s} \nabla\beta_1\nabla\beta_2 dxdy$$

which shows that $P(s)$ is a self-adjoint and positive operator. ■

Lemma 2.6 *For each $s \in [0,1]$, $Q(s) \in \mathcal{L}((H_{00}^{1/2}(\mathcal{O}))'; H_{00}^{1/2}(\mathcal{O})) \cap \mathcal{L}(L^2(\mathcal{O}); H_0^1(\mathcal{O}))$ is a self-adjoint, non negative operator in $L^2(\mathcal{O})$, and it is positive if $s \neq 0$.*

Proof. In fact, the result is obviously verified if $s = 0$. On the other hand, if $s \in]0,1]$, $k_i \in L^2(\mathcal{O})$, and γ_i are the solutions of the problems:

$$\begin{cases} \Delta^2\gamma_i = 0, \quad \text{in } \Omega_s, \\ \gamma_i|_\Sigma = 0, \quad \Delta\gamma_i|_\Sigma = 0, \\ \dfrac{\partial\gamma_i}{\partial x}(0) = 0, \quad \dfrac{\partial\Delta\gamma_i}{\partial x}(0) = 0, \\ \gamma_i|_{\Gamma_s} = 0, \ \Delta\gamma_i|_{\Gamma_s} = k_i, \quad i = 1, 2, \end{cases} \tag{2.56}$$

then, by Green's formula, noticing that $\gamma_1|_\Sigma = \gamma_1|_{\Gamma_s} = 0$ and $\frac{\partial \Delta\gamma_2}{\partial x}(0) = 0$, we have:

$$0 = \int_{\Omega_s} \gamma_1 \Delta^2 \gamma_2 dxdy = -\int_{\Omega_s} \nabla\gamma_1 \nabla(\Delta\gamma_2) dxdy$$

and, again by Green's formula, remarking that $\Delta\gamma_2|_\Sigma = 0$ and $\frac{\partial \gamma_1}{\partial x}(0) = 0$, we obtain

$$\langle Q(s)k_1, k_2\rangle = \int_{\Gamma_s} \frac{\partial \gamma_1}{\partial x}(s)\Delta\gamma_2(s) d\sigma = \int_{\Omega_s} \Delta\gamma_1 \Delta\gamma_2 dxdy$$

which shows that $Q(s)$ is a self-adjoint non negative operator in $L^2(\mathcal{O})$. On the other hand

$$\langle Q(s)k, k\rangle = 0 \Leftrightarrow \int_{\Omega_s} (\Delta\gamma)^2 dxdy = 0 \Rightarrow \Delta\gamma = 0 \text{ in } \Omega_s \Rightarrow k = \Delta\gamma|_{\Gamma_s} = 0$$

and so $Q(s)$ is positive for $s \in]0, 1]$. ■

2.7 Solving the equations

In this section we are going to present formal solutions of the equations (2.50) through (2.54), obtained by means of an evolution operator. As for equation (2.49), it is studied in chapter 3 by Yosida regularization.

Lemma 2.7 *For each $x \in [0, 1]$, $-P(x)$ is the infinitesimal generator of a strongly continuous semigroup of contractions in $L^2(\mathcal{O})$.*

Proof. In fact we know that, for each $x \in [0, 1]$, $P(x)$ is an unbounded and self-adjoint operator from $L^2(\mathcal{O})$ into $L^2(\mathcal{O})$ with domain $H_0^1(\mathcal{O})$. By [6], proposition II.16, page 28, $-P(x)$ is a closed operator. On the other hand

$$(-P(x)h, h) \leq 0, \forall h \in H_0^1(\mathcal{O})$$

so, $-P(x)$ is a dissipative operator. Finally, by [18], Corollary 4.4, page 15, $-P(x)$ is the infinitesimal generator of a strongly continuous semigroup of contractions in $L^2(\mathcal{O})$, $\{\exp(-tP(x)\}_{t\geq 0}$. ■

Definition 2.8 *An evolution operator in a Hilbert space $\mathcal{H}$ is a two parameter family of bounded linear operators in $\mathcal{H}$, $U(x,s)$, $0 \le s \le x \le 1$, verifying $U(x,x) = I$, $U(x,r)U(r,s) = U(x,s)$, $0 \le s \le r \le x \le 1$, and such that $(x,s) \longmapsto U(x,s)$ is strongly continuous for $0 \le s \le x \le 1$.*

Lemma 2.9 *There exists a unique evolution operator $U(x,s)$ in $L^2(\mathcal{O})$, solution of the equation:*

$$\frac{\partial}{\partial s}U(x,s)h = U(x,s)P(s)h, \quad \forall h \in H_0^1(\mathcal{O}), \quad \textit{a.e. in} \;\; 0 \le s \le x \le 1.$$

Proof. It is easy to see that the family $\{-P(x)\}_{x\in[0,1]}$ verifies the conditions of Theorem 3.1, with the slight modification of remark 3.2, of [18]. This implies that there exists a unique evolution operator $U(x,s)$ in $L^2(\mathcal{O})$, such that $\|U(x,s)\|_{\mathcal{L}(L^2(\mathcal{O}))} \le 1$ and

$$\frac{\partial}{\partial s}U(x,s)h = U(x,s)P(s)h, \quad \forall h \in H_0^1(\mathcal{O}), \quad \text{a.e. in} \;\; 0 \le s \le x \le 1. \qquad \blacksquare$$

Formally, from equation (2.50), we have:

$$\frac{\partial}{\partial s}(U(x,s)Q(s)U^*(x,s)) = U(x,s)U^*(x,s).$$

Integrating from 0 to x, and remarking that $Q(0) = 0$,

$$Q(x) = \int_0^x U(x,s)U^*(x,s)\,ds.$$

Definition 2.10 *We define a mild solution of the Lyapunov equation (2.50) by*

$$(Q(x)h, \bar{h}) = \int_0^x (U^*(x,s)h, U^*(x,s)\bar{h})\,ds, \qquad \forall h, \bar{h} \in H_0^1(\mathcal{O}).$$

By the preceeding remarks,

Lemma 2.11 *Equation (2.50) has a unique mild solution.*

Again formally, from equation (2.51), we have

$$\frac{\partial}{\partial s}(U(x,s)t(s)) = U(x,s)\frac{\partial t}{\partial s} + U(x,s)P(s)t = -U(x,s)\Delta f,$$

Definition 2.12 *We define a mild solution of (2.51) by*

$$t(x) = -U(x,0)\frac{\partial f}{\partial x}(0) - \int_0^x U(x,s)\Delta f \, ds.$$

For equations (2.52), (2.53) and (2.54) we proceed in a similar way, noting that for (2.53) and (2.54) the integral is taken between x and 1.

2.8 $\mathcal{QR}$ type factorization of problem $(\mathcal{P}_0)$

In Numerical Analysis, when solving a system $AX = B$, instead of obtaining the LU factorization of A, where L is a lower triangular matrix and U an upper triangular matrix, for the sake of stability we often prefer a QR factorization. We proceed in the following way: let $A^TA = R^TR$ be a factorization of the normal equation, with R an upper triangular matrix. Then $A = A^{-T}R^TR$, and defining $Q = A^{-T}R^T$, we have that $Q^TQ = I$, and the problem reduces to solving $QY = B$, $RX = Y$. Similarly to this procedure, in this chapter we are going to obtain a $\mathcal{QR}$ type factorization of problem $(\mathcal{P}_0)$, where $\mathcal{Q}$ is an orthogonal operator and $\mathcal{R}$ is an upper triangular operator. For this purpose we proceed in the following fashion: consider the problem $Au = f$, and let $A^*A = R^*R$ be a factorization of the normal equation, with $\mathcal{R}$ an upper triangular operator. Then $A = A^{-*}R^*R$, and defining $\mathcal{Q} = A^{-*}R^*$, the operator $\mathcal{Q}$ is orthogonal, $A = \mathcal{QR}$, and the problem reduces to solving $\mathcal{QR}u = f$.

We return to section 2.5 and equations (2.49) through (2.54): eliminating t from (2.52) and (2.53), we have:

$$(\frac{\partial}{\partial x} + P)Q^{-1}(\frac{\partial}{\partial x} + P)\tilde{r} = \Delta f. \tag{2.57}$$

From (2.53) and (2.54) we obtain:

$$\begin{aligned} &Q^{-1}(\frac{\partial}{\partial x} - P)u = \Delta u + Q^{-1}\tilde{r}, \\ &(\frac{\partial}{\partial x} - P)Q^{-1}(\frac{\partial}{\partial x} - P)u = t + (\frac{\partial}{\partial x} - P)Q^{-1}\tilde{r}. \end{aligned} \tag{2.58}$$

Substituting t from (2.52) in the last equation, we derive:

$$(\frac{\partial}{\partial x} - P)Q^{-1}(\frac{\partial}{\partial x} - P)u = (-Q^{-1}(\frac{\partial}{\partial x} + P) + (\frac{\partial}{\partial x} - P)Q^{-1})\tilde{r}, \tag{2.59}$$

and, consequenyly we deduce that:

$$Q(\frac{\partial}{\partial x} - P)Q^{-1}(\frac{\partial}{\partial x} - P)u = (-(\frac{\partial}{\partial x} + P)Q + Q(\frac{\partial}{\partial x} - P))Q^{-1}\tilde{r}. \tag{2.60}$$

Using (2.50), we have that:

$$-(\frac{\partial}{\partial x} + P)Q + Q(\frac{\partial}{\partial x} - P) = -I, \tag{2.61}$$

and so we obtain:

$$Q^2(\frac{\partial}{\partial x} - P)Q^{-1}(\frac{\partial}{\partial x} - P)u = -\tilde{r}. \tag{2.62}$$

Finally, by substituting r from last equation in (2.57), we have:

$$(\frac{\partial}{\partial x} + P)Q^{-1}(\frac{\partial}{\partial x} + P)Q^2(\frac{\partial}{\partial x} - P)Q^{-1}(\frac{\partial}{\partial x} - P)u = -\Delta f. \tag{2.63}$$

Letting $\mathcal{R} = Q(\frac{\partial}{\partial x} - P)Q^{-1}(\frac{\partial}{\partial x} - P)$, we have thus obtained the factorization of (2.33) (or, which is equivalent, of Δ^2 plus boundary conditions) as $\mathcal{R}^*\mathcal{R}$. Let $A = -\Delta$. Then: $A^*A = A^2 = \mathcal{R}^*\mathcal{R}$, and so $A = A^{-1}\mathcal{R}^*\mathcal{R}$. By defining $\mathcal{Q} = A^{-1}\mathcal{R}^*$, we have that $A = \mathcal{Q}\mathcal{R}$, and the problem of solving $Au = -\Delta u = f$ reduces to solving $\mathcal{Q}\mathcal{R}u = f$, in the following way:

1. Given f regular enough, we evaluate $Af = -\Delta f$, then we solve

$$\frac{\partial t}{\partial x} + Pt = -\Delta f, \quad t(0) = -\frac{\partial f}{\partial x}(0)$$

followed by

$$\frac{\partial \tilde{r}}{\partial x} + P\tilde{r} = -Qt, \quad \tilde{r}(0) = -u_0 \tag{2.64}$$

and evaluate g from $\tilde{r} = Qg$.

2. Next we have to solve $\mathcal{R}u = g$. Letting $z = \Delta u + Q^{-1}\tilde{r}$, and using equations (2.50), (2.52) and (2.53), we may easily deduce that:

$$\frac{\partial z}{\partial x} - Pz = -Q^{-1}g = -Q^{-2}\tilde{r}, \quad z(1) = c + Q^{-1}(1)\tilde{r}(1), \tag{2.65}$$

where c is given by (2.55). We now evaluate z from (2.65) and finally u from

$$\frac{\partial u}{\partial x} - Pu = Qz, \quad u(1) = u_1.$$

Next we show that $\mathcal{Q}$ is an orthogonal operator. From the definition of $\mathcal{Q}$, it is easy to see that given $f \in L^2(\Omega)$, then $g = \mathcal{Q}f$ is evaluated in the following way:

1. We first apply: $\mathcal{R}^* : f \to \mathcal{R}^* f = (\frac{\partial}{\partial x} + P)Q^{-1}(\frac{\partial}{\partial x} + P)Qf = h$.
2. We solve: $-\Delta g = h$.

We then have:

$$\mathcal{Q}\mathcal{Q}^*(f) = (-\Delta)^{-1}\mathcal{R}^*\mathcal{R}(-\Delta)^{-1}(f) = (-\Delta)^{-1}\mathcal{R}^*\mathcal{R}u = (-\Delta)^{-1}\Delta^2 u = -\Delta u = f. \quad (2.66)$$

On the other hand, we have that:

$$\mathcal{Q}^*\mathcal{Q} = \mathcal{R}(-\Delta)^{-1}(-\Delta)^{-1}\mathcal{R}^* = \mathcal{R}(\Delta^2)^{-1}\mathcal{R}^* = \mathcal{R}(\mathcal{R}^*\mathcal{R})^{-1}\mathcal{R}^* = I. \quad (2.67)$$

and so we conclude that $\mathcal{Q}$ is an orthogonal operator.

Chapter 3

Direct study of the Riccati equation by Yosida regularization

In this chapter we present a direct study of the Riccati equation

$$\frac{dP}{dx} + P^2 = -\Delta_y, \quad P(0) = 0, \tag{3.1}$$

satisfied by the Dirichlet-Neumann operator P defined on a section of the domain. The additional difficulty of this problem is due to the unboundedness of the right-hand side and of the solution of the equation. This causes a problem to define powers of P_0. The Yosida regularization is used to overcome it. In [?] its well-posedness was proved by adapting the Galerkin method used by J.L. Lions in [16]. In [10] a direct study of the operator equation was made in a Hilbert-Schmidt operator framework inspired from [21]. Due to the unboundedness of the operator, the fixed point argument used in [4] and [20] does not work any more. In [4] (p. 405) the case of the unbounded observation is studied, but the assumptions relate this unboundedness to the one of the generator of the evolution semi-group (which is here 0) and are not satisfied in our case.

3.1 Yosida regularization

Let A be the unbounded operator $-\Delta_y$ in the Hilbert space $H = L^2(\mathcal{O})$, with domain $D(A) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$, and let A_n be its Yosida regularized $A_n = nI - n^2R(n, -A) =$

$nI - n^2(nI + A)^{-1}$. We know that

$$\lim_{n\to+\infty} A_n h = Ah, \quad \forall h \in D(A).$$

Each A_n is a linear, bounded, self adjoint and positive operator in H, and so we can define $A_n^{\frac{1}{2}}$, the positive square root of A_n, which is a linear, bounded, self adjoint and positive operator, which also verifies:

$$\left\| A_n^{\frac{1}{2}} \right\|_{\mathcal{L}(H)} = \|A_n\|_{\mathcal{L}(H)}^{\frac{1}{2}}, \forall n \in \mathbb{N}. \tag{3.2}$$

Let P_n be the solution of the corresponding Riccati equation:

$$\frac{dP_n}{dx} + P_n^2 = A_n, \quad P_n(0) = 0. \tag{3.3}$$

From [4], the equation (3.3) admits a solution given by:

$$P_n(x) = A_n^{\frac{1}{2}}(\exp(2xA_n^{\frac{1}{2}}) - I)(\exp(2xA_n^{\frac{1}{2}}) + I)^{-1}. \tag{3.4}$$

First of all, we notice that each operator $A_n^{\frac{1}{2}}$ is a linear bounded operator in H, and so it is the infinitesimal generator of an uniformly continuous semigroup of bounded linear operators in H:

$$T_n(x) = \exp(xA_n^{\frac{1}{2}}), \ x \geq 0, \ n \in \mathbb{N}, \tag{3.5}$$

and taking into account that $A_n^{\frac{1}{2}}$ is self adjoint $\forall n \in \mathbb{N}$, then $T_n(x)$ is also self adjoint $\forall n \in \mathbb{N}$.

Theorem 3.1 *For each $x \geq 0$, $P_n(x)$ is well defined and $P_n(x) \in \mathcal{L}(H)$ is a positive and self adjoint operator in H. Moreover, we have that:*

$$P_n \in C^1([0,1]; \mathcal{L}(H)). \tag{3.6}$$

Proof. First of all, we notice that for each $x \geq 0$, the linear operator $\exp(2xA_n^{\frac{1}{2}}) + I$ is invertible in H. In fact, we know that $T_n(x) = \exp(xA_n^{\frac{1}{2}})$ is self adjoint in H, $\forall x \geq 0$, and so we have that:

$$((\exp(2xA_n^{\frac{1}{2}}) + I)h, h) = (\exp(xA_n^{\frac{1}{2}})\exp(xA_n^{\frac{1}{2}})h, h) + (h, h) = \tag{3.7}$$

$$= (\exp(xA_n^{\frac{1}{2}})h, \exp(xA_n^{\frac{1}{2}})h) + \|h\|^2 \geq \tag{3.8}$$

$$\geq \|h\|^2, \forall h \in H \tag{3.9}$$

and, taking into account that $\exp(2xA_n^{\frac{1}{2}}) + I$ is continuous in H, then, by Lax-Milgram theorem, we may conclude that it is invertible in H, and we have also that:

$$\left\|(\exp(2xA_n^{\frac{1}{2}}) + I)^{-1}\right\|_{\mathcal{L}(H)} \leq 1, \forall x \geq 0. \tag{3.10}$$

We remark that, for each $x \geq 0$, $P_n(x)$ is the product of self adjoint, positive and bounded operators in H, that commute with each other, and, consequently, we may conclude that $P_n(x)$ is a self adjoint, positive and bounded operator in H, $\forall x \geq 0$. Next we show that $P_n \in C^1([0,1]; \mathcal{L}(H))$ Let $x_0 \in [0,1]$ arbitrary. Then:

$$\begin{aligned}
&P_n(x) - P_n(x_0) = \\
&A_n^{\frac{1}{2}}[(\exp(2xA_n^{\frac{1}{2}}) - I)(\exp(2xA_n^{\frac{1}{2}}) + I)^{-1} - (\exp(2x_0A_n^{\frac{1}{2}}) - I)(\exp(2x_0A_n^{\frac{1}{2}}) + I)^{-1}] = \\
&= A_n^{\frac{1}{2}}[(\exp(2xA_n^{\frac{1}{2}}) - I)(\exp(2x_0A_n^{\frac{1}{2}}) + I)(\exp(2x_0A_n^{\frac{1}{2}}) + I)^{-1}(\exp(2xA_n^{\frac{1}{2}}) + I)^{-1} - \\
&-(\exp(2x_0A_n^{\frac{1}{2}}) - I)(\exp(2xA_n^{\frac{1}{2}}) + I)(\exp(2xA_n^{\frac{1}{2}}) + I)^{-1}(\exp(2x_0A_n^{\frac{1}{2}}) + I)^{-1}] = \\
&= A_n^{\frac{1}{2}}[(\exp(2xA_n^{\frac{1}{2}}) - I)(\exp(2x_0A_n^{\frac{1}{2}}) + I) - \\
&-(\exp(2x_0A_n^{\frac{1}{2}}) - I)(\exp(2xA_n^{\frac{1}{2}}) + I)](\exp(2xA_n^{\frac{1}{2}}) + I)^{-1}(\exp(2x_0A_n^{\frac{1}{2}}) + I)^{-1} = \\
&= 2A_n^{\frac{1}{2}}[\exp(2xA_n^{\frac{1}{2}}) - \exp(2x_0A_n^{\frac{1}{2}})](\exp(2xA_n^{\frac{1}{2}}) + I)^{-1}(\exp(2x_0A_n^{\frac{1}{2}}) + I)^{-1}.
\end{aligned}$$

Taking into account that each $A_n^{\frac{1}{2}}$ is bounded in $\mathcal{L}(H)$, $(\exp(2xA_n^{\frac{1}{2}})$ is an uniformely continuous semigroup of bounded linear operators in H and

$$\left\|(\exp(2xA_n^{\frac{1}{2}}) + I)^{-1}\right\|_{\mathcal{L}(H)} \leq 1, \forall x \geq 0, \forall n \in \mathbb{N}$$

it follows that

$$\lim_{x \to x_0} \|P_n(x) - P_n(x_0)\|_{\mathcal{L}(H)} = 0,$$

so P_n is continuous in x_0. From (3.4) we obtain:

$$\frac{dP_n}{dx} = 4A_n \exp(2xA_n^{\frac{1}{2}})(\exp(2xA_n^{\frac{1}{2}}) + I)^{-2}. \tag{3.11}$$

Let $x_0 \in [0,1]$ arbitrary. Then:

$$\frac{dP_n}{dx} - \frac{dP_n}{dx_0} =$$
$$= 4A_n[\exp(2xA_n^{\frac{1}{2}})(\exp(2xA_n^{\frac{1}{2}}) + I)^{-2} - \exp(2x_0A_n^{\frac{1}{2}})(\exp(2x_0A_n^{\frac{1}{2}}) + I)^{-2}] =$$
$$= 4A_n[\exp(2xA_n^{\frac{1}{2}})(\exp(2x_0A_n^{\frac{1}{2}}) + I)^2(\exp(2x_0A_n^{\frac{1}{2}}) + I)^{-2}(\exp(2xA_n^{\frac{1}{2}}) + I)^{-2}-$$
$$- \exp(2x_0A_n^{\frac{1}{2}})(\exp(2xA_n^{\frac{1}{2}}) + I)^2(\exp(2xA_n^{\frac{1}{2}}) + I)^{-2}(\exp(2x_0A_n^{\frac{1}{2}}) + I)^{-2}] =$$
$$= 4A_n[(\exp(2xA_n^{\frac{1}{2}})\exp(2x_0A_n^{\frac{1}{2}}) - I)(\exp(2x_0A_n^{\frac{1}{2}}) - \exp(2xA_n^{\frac{1}{2}}))].$$
$$.(\exp(2x_0A_n^{\frac{1}{2}}) + I)^{-2}(\exp(2xA_n^{\frac{1}{2}}) + I)^{-2}$$

As before, we know that:

$$\lim_{x\to x_0} \| \exp(2xA_n^{\frac{1}{2}}) - \exp(2x_0A_n^{\frac{1}{2}})\|_{\mathcal{L}(H)} = 0. \tag{3.12}$$

Moreover, all the other factors present in the expression of $\frac{dP_n}{dx} - \frac{dP_n}{dx_0}$ are bounded operators in $\mathcal{L}(H)$, so we deduce that:

$$\lim_{x\to x_0} \|\frac{dP_n}{dx} - \frac{dP_n}{dx_0}\|_{\mathcal{L}(H)} = 0 \tag{3.13}$$

which shows that $\frac{dP_n}{dx}$ is continuous at x_0. Since $x_0 \in [0,1]$ is arbitrary, we deduce that $\frac{dP_n}{dx} \in C([0,1];\mathcal{L}(H))$. ■

Theorem 3.2 *For each $h \in D(A)$ there exists a constant $c(h) \geq 0$, such that*

$$\|P_n(x)h\| \leq c(h), \forall x \in [0,1], \forall n \in \mathbb{N}.$$

Proof. By (3.11) and, being the product of positive operators that commute with each other, we may conclude that:

$$\frac{dP_n}{dx} \geq 0, \quad \forall n \in \mathbb{N}, \quad \forall x \geq 0.$$

From the Riccati equation (3.3) we have:

$$(\frac{dP_n}{dx}h, h) + (P_n^2(x)h, h) = (A_n h, h),$$

P_n being self adjoint,

$$(\frac{dP_n}{dx}h, h) + \|P_n(x)h\|^2 = (A_n h, h), \quad \forall h \in H, \forall x \geq 0,$$

and so we may conclude that:

$$\|P_n(x)h\|^2 \leq (A_n h, h) \longrightarrow (Ah, h), \forall h \in D(A), \forall x \in [0, 1],$$

and, consequently, for each $h \in D(A)$ there exists a constant $c(h) \geq 0$, such that:

$$\|P_n(x)h\| \leq c(h), \forall x \in [0, 1], \forall n \in \mathbb{N}. \qquad \blacksquare$$

Theorem 3.3 *For each $h \in D(A)$, the following limit: $\lim_{n \to +\infty} P_n(x)h$ exists strongly in H, uniformly in $x \in [0, 1]$.*

Proof. For each $h \in H$ and each $x \in [0, 1]$, we consider the sequence $\{P_n(x)h\}_{n \in \mathbb{N}}$. Now, it can be shown that for each $x \geq 0$, $P_n(x)$ and $P_m(x)$ commute with each other, $\forall m, n \in \mathbb{N}$, and so we have that:

$$\frac{d}{dx}(P_n(x) - P_m(x))h + (P_n(x) + P_m(x))(P_n(x) - P_m(x))h = (A_n - A_m)h. \qquad (3.14)$$

Multiplying by $(P_n(x) - P_m(x))h$ it results that

$$(\tfrac{d}{dx}(P_n(x) - P_m(x))h, (P_n(x) - P_m(x))h)+$$

$$+((P_n(x) + P_m(x))(P_n(x) - P_m(x))h, (P_n(x) - P_m(x))h) =$$

$$= ((A_n - A_m)h, (P_n(x) - P_m(x))h) \Rightarrow$$

$$\Rightarrow \tfrac{1}{2}\tfrac{d}{dx}((P_n(x) - P_m(x))h, (P_n(x) - P_m(x))h) \leq$$

$$\leq ((A_n - A_m)h, (P_n(x) - P_m(x))h) \leq$$

$$\leq \|(A_n - A_m)h\| \, \|(P_n(x) - P_m(x))h\| \Leftrightarrow$$

$$\Leftrightarrow \tfrac{1}{2}\tfrac{d}{dx} \|(P_n(x) - P_m(x))h\|^2$$

$$= \|(P_n(x) - P_m(x))h\| \tfrac{d}{dx} \|(P_n(x) - P_m(x))h\| \leq$$

$$\leq \|(A_n - A_m)h\| \, \|(P_n(x) - P_m(x))h\|$$

$$\Rightarrow \tfrac{d}{dx} \|(P_n(x) - P_m(x))h\| \leq \|(A_n - A_m)h\|$$

$$\Rightarrow \|(P_n(x) - P_m(x))h\| \leq x \|(A_n - A_m)h\|$$

$$\leq \|(A_n - A_m)h\|, \forall x \in [0,1], \forall h \in H.$$

We obtain the estimate

$$\|(P_n(x) - P_m(x))h\| \leq \|(A_n - A_m)h\|, \forall x \in [0,1], \forall h \in H. \tag{3.15}$$

Since $A_n h \to Ah, \forall h \in D(A)$, and remarking that $\|(A_n - A_m)h\|$ does not depend on x, we conclude that, for each $h \in D(A)$, the sequence $\{P_n(x)h\}_{n\in\mathbb{N}}$ is a Cauchy sequence, uniformly in $x \in [0,1]$, and, consequently, for each $h \in D(A)$, the sequence $\{P_n(x)h\}_{n\in\mathbb{N}}$ is strongly convergent in H, uniformly in $x \in [0,1]$. ■

3.2 Passing to the limit

We now define, for each $h \in D(A)$ and each $x \in [0,1]$: $P(x)h = \lim\limits_{n\to+\infty} P_n(x)h$, which, by linearity, defines the operator $P(x)$ from $D(A)$ to H.

Theorem 3.4 *The operator P is a solution of the Riccati equation (3.1) in the following sense:*

$$\frac{d}{dx}(P(x)h,\overline{h})+(P(x)h,P(x)\overline{h})=(-\Delta_y h,\overline{h}),\quad \forall h,\overline{h}\in H^2(\mathcal{O})\cap H_0^1(\mathcal{O}), \tag{3.16}$$

and $P(0)=0$.

Proof. We know that each $P_n$verifies:

$$(\frac{dP_n}{dx}h,\overline{h})+(P_n(x)h,P_n(x)\overline{h})=(A_nh,\overline{h}),\quad \forall h,\overline{h}\in H.$$

Let $\varphi\in\mathcal{D}(]0,1[)$. Then:

$$-\int_0^1(P_n(x)h,\overline{h})\,\varphi'(x)dx+\int_0^1(P_n(x)h,P_n(x)\overline{h})\,\varphi(x)dx=\int_0^1(A_nh,\overline{h})\,\varphi(x)dx. \tag{3.17}$$

For each h, $\overline{h}$ fixed in $D(A)=H^2(\mathcal{O})\cap H_0^1(\mathcal{O})$, we have:

$$\lim_{n\to+\infty}(P_n(x)h,\overline{h})\,\varphi'(x)=(P(x)h,\overline{h})\,\varphi'(x),\quad \forall x\in[0,1] \tag{3.18}$$

and

$$\left|(P_n(x)h,\overline{h})\,\varphi'(x)\right|\leq c(h).\left\|\overline{h}\right\|\left|\varphi'(x)\right|,\forall x\in[0,1]\,,\forall n\in\mathbb{N}. \tag{3.19}$$

We also have that

$$\lim_{n\to+\infty}(P_n(x)h,P_n(x)\overline{h})\,\varphi(x)=(P(x)h,P(x)\overline{h})\,\varphi(x),\forall x\in[0,1] \tag{3.20}$$

and

$$\left|(P_n(x)h,P_n(x)\overline{h})\,\varphi(x)\right|\leq c(h).c(\overline{h})\left|\varphi(x)\right|,\ \forall x\in[0,1]\,,\forall n\in\mathbb{N}. \tag{3.21}$$

Finally, we notice that:

$$\lim_{n\to+\infty}\int_0^1(A_nh,\overline{h})\,\varphi(x)dx=(Ah,\overline{h})\int_0^1\varphi(x)dx. \tag{3.22}$$

Consequently, we may use the Lebesgue dominated convergence theorem to pass to the limit for each term of (3.17). Then (3.16) is satisfied in the sense of distributions on $]0,1[$ for each $h,\overline{h}\in D(A)$. Furthermore, since $P_n(0)h=0$ for all $n\in\mathbb{N}$ and $h\in D(A)$ then, by Theorem 3.3, $P(0)=0$. ∎

Theorem 3.5 *The operator $P(x)$ verifies:*

$$(\frac{dP}{dx}h,h) \geq 0, \forall h \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}). \tag{3.23}$$

Proof. From Theorem 3.4, we know that $P(x)$ verifies:

$$(\frac{dP}{dx}h,h) + \|P(x)h\|^2 = (-\Delta_y h,h), \forall h \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}). \tag{3.24}$$

We also know that, for each $n \in \mathbb{N}$, $P_n(x)$ verifies:

$$(\frac{dP_n}{dx}h,h) + \|P_n(x)h\|^2 = (A_n h,h), \forall h \in L^2(\mathcal{O}).$$

Now, taking into account that, for each $h \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$, $P_n(x)h \underset{n\to+\infty}{\longrightarrow} P(x)h$, strongly in $L^2(\mathcal{O})$, uniformly in $x \in [0,1]$, and

$$(A_n h,h) \underset{n\to+\infty}{\longrightarrow} (-\Delta_y h,h), \forall h \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}), \tag{3.25}$$

we may conclude that:

$$(\frac{dP_n}{dx}h,h) \underset{n\to+\infty}{\longrightarrow} (\frac{dP}{dx}h,h), \forall h \in D(A).$$

On the other hand, we know that $\frac{dP_n}{dx} \geq 0$ in $L^2(\mathcal{O})$, and so we conclude that:

$$(\frac{dP}{dx}h,h) \geq 0, \forall h \in D(A). \qquad \blacksquare$$

Theorem 3.6 *The operator $P(x)$ verifies*

$$\|P(x)h\|_{L^2(\mathcal{O})} \leq \|h\|_{H_0^1(\mathcal{O})}, \forall h \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}), \forall x \in [0,1],$$

and, consequently, it also verifies

$$P \in L^\infty(0,1;\mathcal{L}(H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}), L^2(\mathcal{O}))). \tag{3.26}$$

Proof. We know that $P(x)$ verifies

$$\begin{aligned}(\tfrac{dP}{dx}h,h) + \|P(x)h\|^2 &= (-\Delta_y h,h) = \|\nabla_y h\|^2 \\ &= \|h\|^2_{H_0^1(\mathcal{O})} \leq \|h\|^2_{H^2(\mathcal{O})}, \quad \forall h \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})\end{aligned}$$

and, taking into account that $(\frac{dP}{dx}h,h) \geq 0, \forall h \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$ the conclusion is now obvious. $\blacksquare$

Theorem 3.7 *There exists an operator*

$$P \in L^\infty(0,1;\mathcal{L}(H_0^1(\mathcal{O}),L^2(\mathcal{O}))) \tag{3.27}$$

solution of the Riccati equation

$$(\frac{dP}{dx}h,\overline{h}) + (P(x)h,P(x)\overline{h}) = (\nabla_y h,\nabla_y \overline{h}), \forall h,\overline{h} \in H_0^1(\mathcal{O}) \tag{3.28}$$

verifying $P(0) = 0$. This solution is strongly continuous in the sense that $P(x)h \in C([0,1];L^2(\mathcal{O}))$, for all $h \in H_0^1(\mathcal{O})$.

Proof. The operator P, referred in the previous theorems, can be extended by density, taking Theorem 3.6 into account, to an operator $\overline{P}$ from $H_0^1(\mathcal{O})$ to $L^2(\mathcal{O})$. This extension is unique and we name $\overline{P}$ by P. From Theorem 3.6, we may conclude that:

$$\|P(x)h\|_{L^2(\mathcal{O})} \leq \|h\|_{H_0^1(\mathcal{O})}, \forall h \in H_0^1(\mathcal{O}), \forall x \in [0,1], \tag{3.29}$$

and, consequently, $P \in L^\infty(0,1;\mathcal{L}(H_0^1(\mathcal{O}),L^2(\mathcal{O})))$.
Following a similar way to the one we used in the proof of Theorem 3.4, it can be shown that $P(x)$ verifies the Riccati equation (3.28). By Theorem 3.3, $P(x)h \in C([0,1];L^2(\mathcal{O}))$, for all $h \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$. The remaining property follows from (3.29). ■

Remark 3.8 *The uniqueness of the solution of (3.1) can be proved. The easiest way is to use the interpretation of P as a Dirichlet-Neumann operator and to use the uniqueness of the solution of the boundary value problem.*

Remark 3.9 *If instead of $A = -\Delta_y$ we consider the operator*

$$A = -\sum_{i,j=1}^{n-1} \frac{\partial}{\partial y_j}\left(a_{i,j}(y)\frac{\partial}{\partial y_i}\right)$$

with $a_{i,j}(y) = a_{j,i}(y)$, $\forall i,j$ and $\forall y \in \mathcal{O}$, $a_{i,j}$ being continuously differentiable in $\overline{\mathcal{O}}$, and A being $H_0^1(\mathcal{O})$-elliptic, we can do everything as before with almost no changes. In fact, A remains self adjoint, positive and $-A$ is the infinitesimal generator of a strongly continuous semigroup. The explicit formula at the begining remains. The norm of P in $L^\infty(0,1;\mathcal{L}(H_0^1(\mathcal{O}),L^2(\mathcal{O})))$ is now bounded by M for some M, instead of 1. This is the unique little change.

Chapter 4

Discretization

In this chapter we are going to obtain matrix formulae for the operators P and Q written with the aid of an orthonormal basis of $H_0^1(\mathcal{O})$.

4.1 An expression for P

From ([6]), Theorem IX.31, pag.192, we know that there exists an orthogonal basis $(e_n)_{n\in\mathbb{N}}$ of $L^2(\mathcal{O})$, which is an orthonormal basis of $H_0^1(\mathcal{O})$, formed by the eigenfunctions of the problem:

$$\begin{cases} -\Delta_y e_n = \lambda_n e_n \quad \text{in } \mathcal{O}, \\ e_n|_{\partial\mathcal{O}} = 0. \end{cases} \tag{4.1}$$

where the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is a positive and non-decreasing sequence in $\mathbb{R}$, such that: $\lim\limits_{n\to+\infty} \lambda_n = +\infty$.

It has the following properties:

$$\begin{aligned} &1.\ \|e_n\|_{H_0^1(\mathcal{O})} = \left(\int_{\mathcal{O}} |\nabla_y e_n|^2 dy\right)^{\frac{1}{2}} = 1, \quad \forall n \in \mathbb{N}. \\ &2.\ (e_n, e_m)_{H_0^1(\mathcal{O})} = \int_{\mathcal{O}} \nabla_y e_n . \nabla_y e_m dy = 0, \quad \forall n, m \in \mathbb{N}, n \neq m. \\ &3.\ \text{The set } \{\sum_{\text{finite}} \lambda_i e_i, \lambda_i \in \mathbb{R}\} \text{ is a dense subset of } H_0^1(\mathcal{O}). \end{aligned} \tag{4.2}$$

In addition, the spaces $L^2(\mathcal{O})$, $H_0^1(\mathcal{O})$, $H_{00}^{\frac{1}{2}}(\mathcal{O})$ and $H_{00}^{\frac{1}{2}}(\mathcal{O})'$ may be defined as follows:

$$L^2(\mathcal{O}) = \{u = \sum_n u_n e_n : \sum_n |u_n|^2 < +\infty\} \tag{4.3}$$

$$H_0^1(\mathcal{O}) = \{u = \sum_n u_n e_n : \sum_n (1+\lambda_n)|u_n|^2 < +\infty\} \tag{4.4}$$

$$H_{00}^{\frac{1}{2}}(\mathcal{O}) = \{u = \sum_n u_n e_n : \sum_n (1+\lambda_n)^{\frac{1}{2}}|u_n|^2 < +\infty\} \tag{4.5}$$

$$H_{00}^{\frac{1}{2}}(\mathcal{O})' = \{u = \sum_n u_n e_n : \sum_n (1+\lambda_n)^{-\frac{1}{2}}|u_n|^2 < +\infty\}. \tag{4.6}$$

Given $h \in (H_{00}^{1/2}(\mathcal{O}))'$, by the definition of the operator P, $P(s)h = \frac{\partial u}{\partial x}|_{\Gamma_s}$ where u is the solution of

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega_s, \\ u|_\Sigma = 0, \\ -\frac{\partial u}{\partial x}|_{\Gamma_0} = 0, \quad u|_{\Gamma_s} = h. \end{cases} \tag{4.7}$$

But $u \in L^2(0,1; H_0^1(\mathcal{O})) \cap H^1(0,1; L^2(\mathcal{O}))$, and so its trace in each section Γ_x, for a fixed x, may be written in the basis $(e_n)_{n\in\mathbb{N}}$ in the following way:

$$u(x,y) = \sum_{n=1}^{+\infty} u_n(x)e_n(y), \quad \text{where } u_n(x) = \int_{\Gamma_x} u(x,y)e_n(y)dy. \tag{4.8}$$

We then have:

$$\begin{aligned} &\Delta u = 0 \quad \text{in } \Omega_s \Leftrightarrow \sum_{n=1}^{+\infty} \frac{d^2u_n}{dx^2}(x)e_n(y) + u_n(x)\Delta_y e_n(y) = 0 \ \text{ in } \ \Omega_s \\ &\Leftrightarrow \frac{d^2u_n}{dx^2}(x) - \lambda_n u_n(x) = 0 \quad \text{in } \Omega_s, \ \ \forall n \in \mathbb{N}. \end{aligned} \tag{4.9}$$

From the boundary conditions we derive that:

$$-\frac{\partial u}{\partial x}|_{\Gamma_0} = 0 \Leftrightarrow -\sum_{n=1}^{+\infty} \frac{du_n}{dx}(0)e_n(y) = 0 \Leftrightarrow \frac{du_n}{dx}(0) = 0, \ \ \forall n \in \mathbb{N}, \tag{4.10}$$

and

$$u(s) = h \Leftrightarrow \sum_{n=1}^{+\infty} u_n(s)e_n = \sum_{n=1}^{+\infty} h_n e_n \Leftrightarrow u_n(s) = h_n, \ \ \forall n \in \mathbb{N}. \tag{4.11}$$

Thus, for each $n \in \mathbb{N}$, u_n is the solution of

$$\begin{cases} \dfrac{d^2 u_n}{dx^2} - \lambda_n u_n = 0, \\ \dfrac{du_n}{dx}(0) = 0, \quad u_n(s) = h_n. \end{cases} \tag{4.12}$$

The solution of this problem is given by:

$$u_n(x) = h_n \frac{e^{\sqrt{\lambda_n} x} + e^{-\sqrt{\lambda_n} x}}{e^{\sqrt{\lambda_n} s} + e^{-\sqrt{\lambda_n} s}}, \quad \forall n \in \mathbb{N}, \tag{4.13}$$

and so we conclude that:

$$P(s)h = \frac{\partial u}{\partial x}|_{\Gamma_s} = \sum_{n=1}^{+\infty} h_n \sqrt{\lambda_n} \frac{e^{\sqrt{\lambda_n} s} - e^{-\sqrt{\lambda_n} s}}{e^{\sqrt{\lambda_n} s} + e^{-\sqrt{\lambda_n} s}} e_n. \tag{4.14}$$

From the last expression, we conclude that the operator P is diagonal, and its components are:

$$P^n(s) = \sqrt{\lambda_n} \frac{e^{\sqrt{\lambda_n} s} - e^{-\sqrt{\lambda_n} s}}{e^{\sqrt{\lambda_n} s} + e^{-\sqrt{\lambda_n} s}}. \tag{4.15}$$

4.2 An expression for Q

Given $k \in (H_{00}^{1/2}(\mathcal{O}))'$, let $u \in H^2(\Omega)$ be the solution of the problem:

$$\begin{cases} \Delta^2 u = 0 \quad \text{in } \Omega_s, \\ u|_\Sigma = \Delta u|_\Sigma = 0, \\ -\dfrac{\partial u}{\partial x}|_{\Gamma_0} = 0, \quad \dfrac{\partial \Delta u}{\partial x}|_{\Gamma_0} = 0 \\ u|_{\Gamma_s} = 0, \quad \Delta u|_{\Gamma_s} = k. \end{cases} \tag{4.16}$$

Let $w = \Delta u$. Then w is the solution of

$$\begin{cases} \Delta w = 0 \quad \text{in } \Omega_s, \\ w|_\Sigma = 0, \\ -\dfrac{\partial w}{\partial x}|_{\Gamma_0} = 0, \quad w|_{\Gamma_s} = k. \end{cases} \tag{4.17}$$

and, by the previous section, we may conclude that:

$$w(x,y) = \sum_{n=1}^{+\infty} w_n(x) e_n(y) \tag{4.18}$$

with:

$$w_n(x) = k_n \frac{e^{\sqrt{\lambda_n}x} + e^{-\sqrt{\lambda_n}x}}{e^{\sqrt{\lambda_n}s} + e^{-\sqrt{\lambda_n}s}}, \quad \forall n \in \mathbb{N}. \tag{4.19}$$

If $u(x,y) = \sum_{n=1}^{+\infty} u_n(x) e_n(y)$, then

$$\begin{aligned} &\Delta u = w \text{ in } \Omega_s \Leftrightarrow \sum_{n=1}^{+\infty} \left(\frac{d^2 u_n}{dx^2}(x) - \lambda_n u_n(x) \right) e_n(y) = \sum_{n=1}^{+\infty} w_n(x) e_n(y) \Leftrightarrow \\ &\Leftrightarrow \frac{d^2 u_n}{dx^2}(x) - \lambda_n u_n(x) = w_n(x) = k_n \frac{e^{\sqrt{\lambda_n}x} + e^{-\sqrt{\lambda_n}x}}{e^{\sqrt{\lambda_n}s} + e^{-\sqrt{\lambda_n}s}}, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{4.20}$$

From the boundary conditions we derive that:

$$\begin{aligned} &u|_{\Gamma_s} = 0 \Leftrightarrow u_n(s) = 0, \quad \forall n \in \mathbb{N}, \\ &-\frac{\partial u}{\partial x}|_{\Gamma_0} = 0 \Leftrightarrow -\frac{du_n}{dx}(0) = 0. \end{aligned} \tag{4.21}$$

The solution of this problem is given by:

$$\begin{aligned} u_n(x) = {} & \frac{k_n s}{2\sqrt{\lambda_n}} \frac{e^{-\sqrt{\lambda_n}s} - e^{\sqrt{\lambda_n}s}}{(e^{\sqrt{\lambda_n}s} + e^{-\sqrt{\lambda_n}s})^2} (e^{\sqrt{\lambda_n}x} + e^{-\sqrt{\lambda_n}x}) + \\ & + \frac{k_n}{2\sqrt{\lambda_n}(e^{\sqrt{\lambda_n}s} + e^{-\sqrt{\lambda_n}s})} x (e^{\sqrt{\lambda_n}x} - e^{-\sqrt{\lambda_n}x}). \end{aligned} \tag{4.22}$$

We finally deduce the formula for the operator Q:

$$\begin{aligned} Q(s)k = \frac{\partial u}{\partial x}|_{\Gamma_s} = \sum_{n=1}^{+\infty} \frac{du_n}{dx} e_n(y) = \sum_{n=1}^{+\infty} \Bigg[& \frac{k_n s (e^{-\sqrt{\lambda_n}s} - e^{\sqrt{\lambda_n}s})(e^{\sqrt{\lambda_n}s} - e^{-\sqrt{\lambda_n}s})}{2(e^{\sqrt{\lambda_n}s} + e^{-\sqrt{\lambda_n}s})^2} + \\ \frac{k_n}{2\sqrt{\lambda_n}(e^{\sqrt{\lambda_n}s} + e^{-\sqrt{\lambda_n}s})} & \left(e^{\sqrt{\lambda_n}s} - e^{-\sqrt{\lambda_n}s} + \sqrt{\lambda_n}s \left(e^{\sqrt{\lambda_n}s} + e^{-\sqrt{\lambda_n}s} \right) \right) \Bigg] e_n = \\ = \sum_{n=1}^{+\infty} k_n \frac{4\sqrt{\lambda_n}s + e^{2\sqrt{\lambda_n}s} - e^{-2\sqrt{\lambda_n}s}}{2\sqrt{\lambda_n}(e^{\sqrt{\lambda_n}s} + e^{-\sqrt{\lambda_n}s})^2} e_n. & \end{aligned} \tag{4.23}$$

Thus $Q(s)$ is diagonal, and its components are:

$$Q_n(s) = \frac{4\sqrt{\lambda_n}s + e^{2\sqrt{\lambda_n}s} - e^{-2\sqrt{\lambda_n}s}}{2\sqrt{\lambda_n}(e^{\sqrt{\lambda_n}s} + e^{-\sqrt{\lambda_n}s})^2}, \quad \forall n \in \mathbb{N}. \tag{4.24}$$

In particular, Q is an invertible operator, and the components of the inverse operator are:

$$Q_n^{-1}(s) = \frac{2\sqrt{\lambda_n}(e^{\sqrt{\lambda_n}s} + e^{-\sqrt{\lambda_n}s})^2}{4\sqrt{\lambda_n}s + e^{2\sqrt{\lambda_n}s} - e^{-2\sqrt{\lambda_n}s}}, \quad \forall n \in \mathbb{N}. \tag{4.25}$$

Chapter 5

Homographic transformation

In this chapter we are going to obtain a simple factorized formula for the Dirichlet-to-Neumann operator P, solution of the Riccati operator equation. We begin by showing that the Riccati equation of P is equivalent to a first order linear coupled system. Next, by changing variables, we show that the former system is equivalent to a first order decoupled linear system. Finally we obtain an explicit formula for the operator P, which verifies the Riccati equation (see section (4.2)):

$$\frac{dP}{dx} + P^2 + \Delta_y = 0, \quad P(0) = 0. \tag{5.1}$$

5.1 Linear system for the Riccati equation

Let $u \in H^1(\Omega)$ be the solution of $(\mathcal{P}_0)$ with $f = 0$, $u_0 = 0$ and $u_1 \in H^{1/2}_{00}(\mathcal{O})$. We define the operators $X(x) \in \mathcal{L}(H^{1/2}_{00}(\mathcal{O}), H^{1/2}_{00}(\mathcal{O})')$ and $Y(x) \in \mathcal{L}(H^{1/2}_{00}(\mathcal{O}), H^{1/2}_{00}(\mathcal{O}))$ in the following way:

$$X(x): \quad u(1) \to \frac{\partial u}{\partial x}(x), \tag{5.2}$$

$$Y(x): \quad u(1) \to u(x). \tag{5.3}$$

Taking into account the definition of the operator P, we have:

$$P(x)Y(x) = X(x), \quad \forall x \in [0,1]. \tag{5.4}$$

Furthermore, X and Y verify $X(0) = 0$, $Y(1) = I$. Taking the derivative with respect to x on both sides of the identity

$$X(x)u(1) = \frac{\partial u}{\partial x}(x) \tag{5.5}$$

and due to the fact that $\Delta u = 0$ in Ω, it follows

$$\frac{\partial X}{\partial x}(x)u(1) = \frac{\partial^2 u}{\partial x^2}(x) = -\Delta_y u(x). \tag{5.6}$$

Then

$$\frac{\partial X}{\partial x}(x)u(1) = -\Delta_y Y(x)u(1), \tag{5.7}$$

and, due to the fact that $u(1)$ is arbitrary, we derive that

$$\frac{\partial X}{\partial x}(x) = -\Delta_y Y(x). \tag{5.8}$$

Similarly, taking the derivative with respect to x on both sides of

$$Y(x)u(1) = u(x) \tag{5.9}$$

we derive

$$\frac{\partial Y}{\partial x}(x)u(1) = \frac{\partial u}{\partial x}(x) = X(x)u(1) \tag{5.10}$$

and, because $u(1)$ is arbitrary, we conclude that

$$\frac{\partial Y}{\partial x}(x) = X(x). \tag{5.11}$$

We have thus obtained the following linear system:

$$\begin{cases} \dfrac{dX}{dx}(x) = -\Delta_y Y(x), \quad X(0) = 0, \\ \dfrac{dY}{dx}(x) = X(x), \quad Y(1) = I. \end{cases} \tag{5.12}$$

Next we show that problem (5.12) and the Riccati equation of P are equivalent, by means of the relation (5.4). Let P be the solution of (5.1). Taking into account (5.4) and recalling that P is the Dirichlet-to-Neumann operator, then, formally, we have:

$$\begin{aligned} &\frac{dX}{dx}(x) = \frac{dP}{dx}(x)Y(x) + P(x)\frac{dY}{dx}(x) = \\ &= (-P^2 - \Delta_y)Y(x) + P(x)P(x)Y(x) = -\Delta_y Y(x), \end{aligned}$$

so X and Y verify the linear system (5.12).

Conversely, let X and Y be solutions of (5.12) for $x \in]0,1[$. From the definition it is easy to show that $Y(x)$ is an injective operator with dense image, for each $x \in]0,1[$, and so we can define the operator $P(x)$ through the relation:

$$P(x)Y(x) = X(x), \quad \forall x \in]0,1[. \tag{5.13}$$

Formally taking the derivative with respect to x on both sides of this equation, we derive

$$\frac{dP}{dx}(x)Y(x) + P(x)\frac{dY}{dx}(x) = \frac{dX}{dx}(x). \tag{5.14}$$

Taking into account the equations verified by X and Y, it follows that:

$$\frac{dP}{dx}(x)Y(x) + P^2(x)Y(x) = -\Delta_y Y(x). \tag{5.15}$$

Finally, again due to the fact that $Y(x)$ is an operator with dense image, for each $x \in]0,1[$, we conclude that P verifies the Riccati equation, and from $P(0)Y(0) = X(0) = 0$ we derive that $P(0) = 0$.

5.2 Diagonalization of the linear system

In this section, we consider as an orthonormal basis of $L^2(\mathcal{O})$ the set $\{e_n\}_{n\in\mathbb{N}}$ of eigenfunctions of the Dirichlet problem

$$\begin{cases} -\Delta_y e_n(y) = \lambda_n e_n(y), \quad \forall y \in \mathcal{O}, \quad n \in \mathbb{N} \\ e_n|_\Sigma = 0, \quad \forall n \in \mathbb{N}, \end{cases} \tag{5.16}$$

with (λ_i) a positive and non decreasing sequence such that $\lim_{i\to+\infty} \lambda_i = +\infty$. Let $P_0 \in \mathcal{L}(H_0^1(\mathcal{O}), L^2(\mathcal{O})) \cap \mathcal{L}(H_{00}^{1/2}(\mathcal{O}), H_{00}^{1/2}(\mathcal{O})')$ be the positive stationary solution of the Riccati equation (5.1):

$$P_0 = +(-\Delta_y)^{\frac{1}{2}}. \tag{5.17}$$

P_0 is a particular diagonal solution of the Riccati equation, which is invertible, because $\lambda_n > 0$, $\forall n \in \mathbb{N}$.

Theorem 5.1 *The linear system*

$$\begin{cases} \dfrac{dX}{dx}(x) = -\Delta_y Y(x), \quad X(0) = 0, \\ \dfrac{dY}{dx}(x) = X(x), \quad Y(1) = I, \end{cases} \tag{5.18}$$

is equivalent to the diagonal system

$$\begin{cases} \dfrac{d\varphi}{dx}(x) = P_0\varphi(x), \quad \varphi(0) - P_0\psi(0) = 0, \\ \dfrac{d\psi}{dx}(x) = -P_0\psi(x), \quad P_0^{-1}\varphi(1) + \psi(1) = I, \end{cases} \tag{5.19}$$

where the operators $\varphi(x) \in \mathcal{L}(H_{00}^{1/2}(\mathcal{O}), H_{00}^{1/2}(\mathcal{O})')$ *and* $\psi(x) \in \mathcal{L}(H_{00}^{1/2}(\mathcal{O}), H_{00}^{1/2}(\mathcal{O}))$ *are related with* X *and* Y *through the equations*

$$\begin{cases} X = \varphi - P_0\psi, \\ Y = P_0^{-1}\varphi + \psi, \end{cases} \tag{5.20}$$

$$\begin{cases} \varphi = \frac{1}{2}(P_0 Y + X), \\ \psi = \frac{1}{2}(Y - P_0^{-1}X). \end{cases} \tag{5.21}$$

Proof. Taking the derivative with respect to x on both sides of (5.21), we have:

$$\frac{d\varphi}{dx} = \frac{1}{2}\frac{d}{dx}(P_0Y + X) = \frac{1}{2}(P_0X - \Delta_y Y) = \frac{1}{2}[P_0(\varphi - P_0\psi) - \Delta_y(P_0^{-1}\varphi + \psi)] =$$
$$= \frac{1}{2}2P_0\varphi = P_0\varphi.$$

and

$$\frac{d\psi}{dx} = \frac{1}{2}\left(\frac{dY}{dx} - P_0^{-1}\frac{dX}{dx}\right) = \frac{1}{2}[X - P_0^{-1}(-\Delta_y Y)] = \frac{1}{2}(X - P_0Y) =$$
$$= \frac{1}{2}[\varphi - P_0\psi - P_0(P_0^{-1}\varphi + \psi)] = -P_0\psi.$$

Finally, we have:

$$X(0) = 0 \Leftrightarrow \varphi(0) - P_0\psi(0) = 0,$$

$$Y(1) = I \Leftrightarrow P_0^{-1}\varphi(1) + \psi(1) = I.$$

Conversely, if (φ, ψ) are solutions of (5.19), then, using (5.20), we can easily show that (X, Y) verify (5.18). ■

We have thus showed that the operators

$$\begin{pmatrix} 0 & -\Delta_y \\ I & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} P_0 & 0 \\ 0 & -P_0 \end{pmatrix}$$

are similar. In fact, in matrix form:

$$\begin{pmatrix} \dfrac{dX}{dx} \\ \dfrac{dY}{dx} \end{pmatrix} = \begin{pmatrix} 0 & -\Delta_y \\ I & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix},$$

$$\begin{pmatrix} \dfrac{d\varphi}{dx} \\ \dfrac{d\psi}{dx} \end{pmatrix} = \begin{pmatrix} P_0 & 0 \\ 0 & -P_0 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix},$$

and

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} I & -P_0 \\ P_0^{-1} & I \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}.$$

Now, due to the fact that P_0 is independent of x, integrating (5.19), and representing by $\exp(P_0 x)$ the semigroup generated by P_0, we obtain:

$$\begin{cases} \varphi(x) = \varphi(0)\exp(P_0 x), \\ \psi(x) = \psi(1)\exp\left(-P_0(x-1)\right). \end{cases} \tag{5.22}$$

and consequently

$$\begin{cases} \varphi(1) = \varphi(0)\exp(P_0), \\ \psi(0) = \psi(1)\exp(P_0). \end{cases} \tag{5.23}$$

Taking into account the initial conditions verified by φ and ψ, it follows that

$$\begin{cases} \varphi(0) = P_0\psi(0) = P_0\psi(1)\exp(P_0), \\ P_0^{-1}\varphi(1) + \psi(1) = P_0^{-1}\varphi(0)\exp(P_0) + \psi(1) = \psi(1)\exp(2P_0) + \psi(1) = I. \end{cases} \tag{5.24}$$

From the last equality, and due to the fact that P_0, P_0^{-1} and $\exp(P_0)$ comute with each other, we derive that

$$\psi(1) = (\exp(2P_0) + I)^{-1} \tag{5.25}$$

and

$$\varphi(0) = P_0\left(\exp(2P_0) + I\right)^{-1}\exp(P_0). \tag{5.26}$$

Finally, we deduce:

$$\begin{cases} \varphi(x) = P_0\left(\exp(2P_0) + I\right)^{-1}\exp\left(P_0(1+x)\right), \\ \psi(x) = \left(\exp(2P_0) + I\right)^{-1}\exp\left(-P_0(x-1)\ \right) \end{cases} \tag{5.27}$$

5.3 Homographic representation of P

Theorem 5.2 *The operator P may be written in the following way:*

$$P(x) = (\varphi(x)\psi^{-1}(x) - P_0)(P_0^{-1}\varphi(x)\psi^{-1}(x) + I)^{-1} \tag{5.28}$$

Proof.

$$\begin{aligned} P(x) &= X(x)Y^{-1}(x) = (\varphi(x) - P_0\psi(x))(P_0^{-1}\varphi(x) + \psi(x))^{-1} = \\ &= (\varphi(x)\psi^{-1}(x)\psi(x) - P_0\psi(x))(P_0^{-1}\varphi(x)\psi^{-1}(x)\psi(x) + \psi(x))^{-1} = \\ &= (\varphi(x)\psi^{-1}(x) - P_0)\psi(x)((P_0^{-1}\varphi(x)\psi^{-1}(x) + I)\psi(x))^{-1} = \\ &= (\varphi(x)\psi^{-1}(x) - P_0)(P_0^{-1}\varphi(x)\psi^{-1}(x) + I)^{-1}. \end{aligned} \tag{5.29}$$

Let

$$W(x) = \varphi(x)\psi^{-1}(x). \tag{5.30}$$

From the definition, we derive an explicit formula for W:

$$W(x) = \varphi(x)\psi^{-1}(x) = P_0\left(\exp(2P_0) + I\right)^{-1} \exp\left(P_0(1+x)\right)\left[(\exp 2P_0 + I)^{-1} \exp\left(-P_0(x-1)\right)\right]^{-1} =$$

$$= P_0\left(\exp(2P_0) + I\right)^{-1} \exp\left(P_0(1+x)\right) \exp\left(P_0(x-1)\right)\left(\exp(2P_0) + I\right) =$$

$$= P_0 \exp(2P_0 x). \tag{5.31}$$

In addition, $W(0) = \varphi(0)\psi^{-1}(0)$ and, from (5.26) and (5.27) we deduce that $W(0) = P_0$.

■

Theorem 5.3 *The operator W is solution of the linear differential equation:*

$$\frac{dW}{dx} = P_0 W + W P_0, \quad W(0) = P_0. \tag{5.32}$$

Proof. In fact, formally taking the derivative with respect to x on both sides of (5.30) we have:

$$\frac{dW}{dx}(x) = \frac{d\varphi}{dx}(x)\psi^{-1}(x) - \varphi(x)\psi^{-1}(x)\frac{d\psi}{dx}(x)\psi^{-1}(x) \tag{5.33}$$

and, from (5.19) the conclusion is obvious. ■

We finally deduce an explicit formula for the operator P:

Theorem 5.4 *The operator P may be defined in the following way:*

$$P(x) = \left[(\exp(P_0 x)P_0 \exp(P_0 x) - P_0)\right]\left[P_0^{-1}\exp(P_0 x)P_0 \exp(P_0 x) + I\right]^{-1} \tag{5.34}$$

Proof. The result is obvious: we just have to replace $W(x) = \exp(P_0 x)P_0 \exp(P_0 x)$ in (5.29). ■

Chapter 6

Factorization of other elliptic problems

In this chapter, we apply the method of factorization by invariant embedding to factorize more general elliptic problems, by proceeding in the straightforward way.

Using the same notations and conditions, we consider the problem defined in section 1.3. We embed problem (1.13) in the family of similar problems defined in Ω_s, $s \in]0,1]$, $h \in H^{1/2}_{00}(\mathcal{O})$:

$$\begin{cases} Lu = f \quad \text{in } \Omega_s, \\ u|_\Sigma = 0, \\ \left(-\displaystyle\sum_{j=1}^{n} a_{1,j} \frac{\partial u}{\partial x_j} \right) |_{\Gamma_0} = u_0, \quad u|_{\Gamma_s} = h. \end{cases} \tag{6.1}$$

From Theorem 1.14, we know that these are well posed problems. For each $s \in]0,1]$ we define

$$\bar{P}(s)h = \left(\sum_{j=1}^{n} a_{1,j} \frac{\partial \gamma_{s,h}}{\partial x_j} \right) |_{\Gamma_s} \tag{6.2}$$

where $\gamma_{s,h}$ is the solution of the problem:

$$\begin{cases} L\gamma_{s,h} = 0 \quad \text{in } \Omega_s, \\ \gamma_{s,h}|_\Sigma = 0, \\ \left(-\sum_{j=1}^n a_{1,j}\dfrac{\partial \gamma_{s,h}}{\partial x_j}\right)|_{\Gamma_0} = 0, \quad \gamma_{s,h}|_{\Gamma_s} = h, \end{cases} \tag{6.3}$$

and $\bar{P}(0) = 0$. Moreover, for each $s \in]0,1]$, we set

$$\bar{r}(s) = \left(\sum_{j=1}^n a_{1,j}\frac{\partial \beta_{s,h}}{\partial x_j}\right)|_{\Gamma_s} \tag{6.4}$$

where $\beta_{s,h}$ is the solution of

$$\begin{cases} L\beta_{s,h} = f \quad \text{in } \Omega_s, \\ \beta_{s,h}|_\Sigma = 0, \\ \left(-\sum_{j=1}^n a_{1,j}\dfrac{\partial \beta_{s,h}}{\partial x_j}\right)|_{\Gamma_0} = u_0, \quad \beta_{s,h}|_{\Gamma_s} = 0, \end{cases} \tag{6.5}$$

and $\bar{r}(0) = -u_0$. Then the solution $\bar{u}_{s,h}$ of (6.1) verifies:

$$\left(\sum_{j=1}^n a_{1,j}\frac{\partial \bar{u}_{s,h}}{\partial x_j}\right)|_{\Gamma_s} = \bar{P}(s)h + \bar{r}(s), \quad \forall s \in [0,1]. \tag{6.6}$$

Furthermore, the solution u of (1.13) verifies (6.1) with $h = \bar{u}|_{\Gamma_s}$, for each $s \in [0,1]$ so we have the following identity:

$$\left(\sum_{j=1}^n a_{1,j}\frac{\partial u}{\partial x_j}\right)|_{\Gamma_{x_1}} = \bar{P}(x_1)u|_{\Gamma_{x_1}} + \bar{r}(x_1), \quad \forall x_1 \in [0,1]. \tag{6.7}$$

Formal derivation of (6.7) with respect to x_1 leads to:

$$-\frac{\partial}{\partial x_1}\left(\sum_{j=1}^n a_{1,j}\frac{\partial u}{\partial x_j}\right) = -\frac{d\bar{P}}{dx_1}u - \bar{P}\frac{\partial u}{\partial x_1} - \frac{d\bar{r}}{dx_1}, \quad \forall x_1 \in [0,1]. \tag{6.8}$$

By substitution from (1.13) and (6.7), and noting that

$$\frac{\partial u}{\partial x_1} = a_{11}^{-1}\left(\bar{P}u + \bar{r} - \sum_{j=2}^{n} a_{1,j}\frac{\partial u}{\partial x_j}\right) \tag{6.9}$$

we obtain:

$$\begin{aligned} &-\frac{d\bar{P}}{dx_1}u - \bar{P}a_{11}^{-1}\left(\bar{P}u + \bar{r} - \sum_{j=2}^{n} a_{1,j}\frac{\partial u}{\partial x_j}\right) - \frac{\partial \bar{r}}{\partial x_1} = \\ &= \sum_{i=2}^{n}\frac{\partial}{\partial x_i}\left(a_{i,1}a_{11}^{-1}\left(\bar{P}u + \bar{r} - \sum_{j=2}^{n} a_{1,j}\frac{\partial u}{\partial x_j}\right)\right) + \sum_{i,j=2}^{n}\frac{\partial}{\partial x_j}\left(a_{i,j}\frac{\partial u}{\partial x_j}\right) + f, \end{aligned} \tag{6.10}$$

so, we deduce:

$$\begin{aligned} &\left(\frac{d\bar{P}}{dx_1} + \bar{P}a_{11}^{-1}\bar{P} - \bar{P}a_{11}^{-1}\sum_{j=2}^{n} a_{1,j}\frac{\partial}{\partial x_j} + \sum_{i,j=2}^{n}\frac{\partial}{\partial x_i}\left(a_{i,1}a_{11}^{-1}\bar{P}\right) - \right. \\ &\left. -\sum_{i,j=2}^{n}\frac{\partial}{\partial x_i}\left(a_{i,1}a_{11}^{-1}a_{1,j}\right)\frac{\partial}{\partial x_j} + \sum_{i,j=2}^{n}\frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial}{\partial x_j}\right)\right)u + \\ &+\bar{P}a_{11}^{-1}\bar{r} + \frac{\partial \bar{r}}{\partial x_1} + \sum_{i=2}^{n}\frac{\partial}{\partial x_i}\left(a_{i,1}a_{11}^{-1}\bar{r}\right) + f = 0 \end{aligned} \tag{6.11}$$

and taking into account that $u|_{\Gamma_s} = h$ is arbitrary, we finally conclude that:

$$\begin{cases} \dfrac{d\bar{P}}{dx_1} + \displaystyle\sum_{i,j=2}^{n}\frac{\partial}{\partial x_i}\left(a_{i,1}a_{11}^{-1}\bar{P}\right) + \bar{P}a_{11}^{-1}\bar{P} - \bar{P}a_{11}^{-1}\sum_{j=2}^{n} a_{1,j}\frac{\partial}{\partial x_j} - \\ -\displaystyle\sum_{i,j=2}^{n}\frac{\partial}{\partial x_i}\left(a_{i,1}a_{11}^{-1}a_{1,j}\right)\frac{\partial}{\partial x_j} + \sum_{i,j=2}^{n}\frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial}{\partial x_j}\right) = 0, \quad \bar{P}(0) = 0, \\ \dfrac{\partial \bar{r}}{\partial x} + \bar{P}a_{11}^{-1}\bar{r} + \displaystyle\sum_{i=2}^{n}\frac{\partial}{\partial x_i}\left(a_{i,1}a_{11}^{-1}\bar{r}\right) + f = 0, \quad \bar{r}(0) = -u_0, \\ -\dfrac{\partial u}{\partial x_1} - \displaystyle\sum_{j=2}^{n} a_{11}^{-1}a_{1,j}\frac{\partial u}{\partial x_j} + a_{11}^{-1}\bar{P}(x)u = -a_{11}^{-1}\bar{r}, \quad u(1) = u_1. \end{cases} \tag{6.12}$$

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