The cardinal and the idempotent number of various monoids of transformations on a finite chain

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Abstract

We consider various classes of monoids of transformations on a finite chain, in particular transformations that preserve or reverse either the order or the orientation. Being finite monoids we are naturally interested in computing both their cardinals and the number of their idempotents.

In this note we present a short survey on these questions which have been approached by various authors and close the problem by computing the number of idempotents of those monoids not considered before. Fibonacci and Lucas numbers play an essential role in the last computations.

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Introduction

Let X_n be a finite chain with n elements, say $X_n = \{1 < \cdots < n\}$. We denote by $\mathcal{PT}_n$ the monoid (under composition) of all partial transformations of X_n. The submonoids of $\mathcal{PT}_n$ of all full transformations of X_n and of all injective partial transformations of X_n are denoted by $\mathcal{T}_n$ and $\mathcal{I}_n$, respectively.

For general background on monoids, we refer the reader to Howie's book [9]. Given $s \in \mathcal{PT}_n$, we denote by $\mathrm{Dom}(s)$ its *domain* and by $\mathrm{Im}(s)$ its *image* and define its *rank* as being the cardinal of $\mathrm{Im}(s)$.

We say that a transformation s in $\mathcal{PT}_n$ is *order-preserving* (resp., *order-reversing*) if $x \le y$ implies $xs \le ys$ (resp., $xs \ge ys$), for all $x, y \in \mathrm{Dom}(s)$. Clearly the product of two order-preserving transformations or of two order-reversing transformations is order-preserving and the product of an order-preserving transformation by an order-reversing transformation is order-reversing.

Denote by $\mathcal{PO}_n$ the submonoid of $\mathcal{PT}_n$ of all partial order-preserving transformations of X_n. As usual, we denote by $\mathcal{O}_n$ the monoid $\mathcal{PO}_n \cap \mathcal{T}_n$ of all full transformations of X_n that preserve the order. The cardinal and the number of idempotents of $\mathcal{O}_n$ were calculated by Howie [8]. Later on, Gomes and Howie [7] considered these questions for $\mathcal{PO}_n$. The injective counterpart of $\mathcal{O}_n$ is the inverse monoid $\mathcal{POI}_n = \mathcal{PO}_n \cap \mathcal{I}_n$ of all injective order-preserving partial transformations of X_n. The cardinal of $\mathcal{POI}_n$ was calculated by Fernandes [2]. Obviously $\mathcal{POI}_n$ and $\mathcal{I}_n$ have the same idempotents, which are the partial identities on X_n, and there are exactly 2^n such elements.

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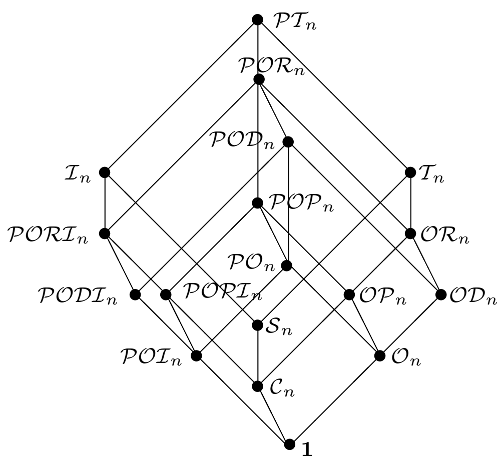
Wider classes of monoids are obtained when we take transformations that either preserve or reverse the order. In this way, we get $\mathcal{POD}_n$, the submonoid of $\mathcal{PT}_n$ of all partial transformations that preserve or reverse the order. Naturally, we may also consider its submonoids $\mathcal{OD}_n = \mathcal{POD}_n \cap \mathcal{T}_n$ and $\mathcal{PODI}_n = \mathcal{POD}_n \cap \mathcal{I}_n$. The cardinals of these monoids were calculated by the authors [4, 5].

Now, let $a = (a_1, a_2, \ldots, a_t)$ be a sequence of t ($t \geq 0$) elements from the chain X_n. We say that a is *cyclic* (resp., *anti-cyclic*) if there exists no more than one index $i \in \{1, \ldots, t\}$ such that $a_i > a_{i+1}$ (resp., $a_i < a_{i+1}$), where a_{t+1} denotes a_1. Let $s \in \mathcal{PT}_n$ and suppose that $\mathrm{Dom}(s) = \{a_1, \ldots, a_t\}$, with $t \geq 0$ and $a_1 < \cdots < a_t$. We say that s is an *orientation-preserving* (resp., *orientation-reversing*) transformation if the sequence of its images $(a_1 s, \ldots, a_t s)$ is cyclic (resp., anti-cyclic). These notions were introduced by McAlister [10] and, independently, by Catarino and Higgins [1]. As before the product of two orientation-preserving or of two orientation-reversing transformations is orientation-preserving and the product of an orientation-preserving transformation by an orientation-reversing transformation is orientation-reversing.

We denote by $\mathcal{POP}_n$ the submonoid of $\mathcal{PT}_n$ of all orientation-preserving transformations and by $\mathcal{POR}_n$ the submonoid of $\mathcal{PT}_n$ of all orientation-preserving transformations together with all orientation-reversing transformations. Denote by $\mathcal{OP}_n$ the submonoid of $\mathcal{POP}_n$ of all (orientation-preserving) full transformations, by $\mathcal{OR}_n$ the submonoid of $\mathcal{POR}_n$ of all (orientation-preserving or orientation-reversing) full transformations, by $\mathcal{POPI}_n$ the inverse submonoid of $\mathcal{I}_n$ whose elements are all orientation-preserving transformations and by $\mathcal{PORI}_n$ the inverse submonoid of $\mathcal{I}_n$ whose elements belong to $\mathcal{POR}_n$. The cardinals of $\mathcal{OP}_n$ and $\mathcal{OR}_n$ were calculated by McAlister [10] and, independently, by Catarino and Higgins [1], who also computed the number of their idempotents. In [3] Fernandes calculated the cardinal of $\mathcal{POPI}_n$ and the authors, in [4], answered this same question for $\mathcal{PORI}_n$. Again, it is easy to show that the idempotents of $\mathcal{PODI}_n$, $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$ are exactly the 2^n idempotents of $\mathcal{I}_n$.

In what follows we denote by $\mathbf{1}$ the trivial monoid, by $\mathcal{S}_n$ the symmetric group and by $\mathcal{C}_n$ the cyclic group of order n. The diagram bellow, with respect to the inclusion relation, clarifies the relationship between these various monoids:



The first section of this note is dedicated to recalling known formulas to calculate the cardinal of the monoids defined above. In the second section we consider the analogous question for the set of idempotents of those monoids. Here we revise the known results and complete the study by computing the remaining cases.

1 Cardinals

Let $\mathcal{PD}_n$ be the set of all order-reversing partial transformations of X_n and let $\mathcal{ID}_n$ and $\mathcal{D}_n$ be the subsets of $\mathcal{PD}_n$ of all injective transformations and of all full transformations, respectively. Clearly, $\mathcal{POD}_n = \mathcal{PO}_n \cup \mathcal{PD}_n$ and so $\mathcal{PODI}_n = \mathcal{POI}_n \cup \mathcal{ID}_n$ and $\mathcal{OD}_n = \mathcal{O}_n \cup \mathcal{D}_n$. Furthermore, $\mathcal{PO}_n \cap \mathcal{PD}_n = \{s \in \mathcal{PT}_n : |\operatorname{Im}(s)| \leq 1\}$ and so $\mathcal{POI}_n \cap \mathcal{ID}_n = \{s \in \mathcal{I}_n : |\operatorname{Im}(s)| \leq 1\}$ and $\mathcal{O}_n \cap \mathcal{D}_n = \{s \in \mathcal{T}_n : |\operatorname{Im}(s)| = 1\}$.

Consider the following order-reversing permutation of order two:

$$h = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}.$$

To calculate the cardinals of $\mathcal{POD}_n$, $\mathcal{PODI}_n$ and $\mathcal{OD}_n$, we will make use of the mapping $\varphi : \mathcal{PO}_n \longrightarrow \mathcal{PD}_n$ defined by $(s)\varphi = sh$, for all $s \in \mathcal{PO}_n$. Clearly, this mapping φ is a bijection and so we have $|\mathcal{PD}_n| = |\mathcal{PO}_n|$. On the other hand, φ maps $\mathcal{O}_n$ onto $\mathcal{D}_n$ and $\mathcal{POI}_n$ onto $\mathcal{ID}_n$, whence $|\mathcal{D}_n| = |\mathcal{O}_n|$ and $|\mathcal{ID}_n| = |\mathcal{POI}_n|$.

Now, we recall that Howie [8] showed that

$$|\mathcal{O}_n| = \binom{2n-1}{n-1}$$

and since we have $|\mathcal{O}_n \cap \mathcal{D}_n| = |\{s \in \mathcal{T}_n : |\operatorname{Im}(s)| = 1\}| = n$, the next result followed.

Theorem 1.1 [5] $|\mathcal{OD}_n| = 2\binom{2n-1}{n-1} - n.$ □

As $|\mathcal{POI}_n \cap \mathcal{ID}_n| = |\{s \in \mathcal{I}_n : |\operatorname{Im}(s)| \leq 1\}| = n^2 + 1$ and Fernandes [2] proved that

$$|\mathcal{POI}_n| = \binom{2n}{n},$$

we deduced the cardinal of $\mathcal{PODI}_n$.

Theorem 1.2 [4] $|\mathcal{PODI}_n| = 2\binom{2n}{n} - n^2 - 1.$ □

Next, we noticed that Gomes and Howie [7] established that

$$|\mathcal{PO}_n| = \sum_{i=1}^{n} \binom{n}{i}\binom{n+i-1}{i} + 1$$

and using the fact that $|\mathcal{PO}_n \cap \mathcal{PD}_n| = |\{s \in \mathcal{PT}_n : |\operatorname{Im}(s)| \leq 1\}| = n\sum_{i=1}^{n}\binom{n}{i} + 1$, we computed $|\mathcal{POD}_n|$.

Theorem 1.3 [5] $|\mathcal{POD}_n| = \sum_{i=1}^{n} \binom{n}{i}\left(2\binom{n+i-1}{i} - n\right) + 1.$ □

The cardinal of $\mathcal{POP}_n$ was also calculated by the authors.

Theorem 1.4 [6] $|\mathcal{POP}_n| = 1 + (2^n - 1)n + \sum_{k=2}^{n} k\binom{n}{k}^2 2^{n-k}.$ □

Denote by $\mathcal{P}or_n$ the set of all orientation-reversing partial transformations of X_n. By definition, we have $\mathcal{POR}_n = \mathcal{POP}_n \cup \mathcal{P}or_n$. Catarino and Higgins proved the following fact:

Lemma 1.5 [1] *Let a be a cyclic (resp., anti-cyclic) sequence. Then a is also anti-cyclic (resp., cyclic) if and only if a has no more than two distinct values.* □

This allows us to conclude that $\mathcal{POP}_n \cap \mathcal{P}or_n = \{s \in \mathcal{POP}_n : |\operatorname{Im}(s)| \leq 2\}$. As the mapping $\Psi : \mathcal{POP}_n \longrightarrow \mathcal{P}or_n$ defined by $(s)\Psi = sh$, for all $s \in \mathcal{POP}_n$, is a bijection, we get $|\mathcal{POP}_n| = |\mathcal{P}or_n|$ and so $|\mathcal{POR}_n| = 2|\mathcal{POP}_n| - |\{s \in \mathcal{POP}_n \mid |\operatorname{Im}(s)| \leq 2\}|$. Therefore we obtain the cardinal of $\mathcal{POR}_n$.

Theorem 1.6 [6] $|\mathcal{POR}_n| = 1 + (2^n - 1)n + 2\binom{n}{2}^2 2^{n-2} + \sum_{k=3}^{n} 2k\binom{n}{k}^2 2^{n-k}.$ □

The cardinal of $\mathcal{POPI}_n$, computed by Fernandes [3], is given by the following formula

$$|\mathcal{POPI}_n| = 1 + \frac{n}{2}\binom{2n}{n}.$$

As, in this case Ψ maps $\mathcal{POPI}_n$ onto the set of all injective orientation-reversing transformations, we conclude that $|\mathcal{PORI}_n| = 2|\mathcal{POPI}_n| - |\{s \in \mathcal{POPI}_n \mid |\operatorname{Im}(s)| \leq 2\}|$. We then proved the following.

Theorem 1.7 [4] $|\mathcal{PORI}_n| = 1 + n\binom{2n}{n} - \frac{n^2}{2}(n^2 - 2n + 3)$. $\square$

The cardinals of $\mathcal{OP}_n$ and $\mathcal{OR}_n$ were computed by McAlister [10] and, independently, by Catarino and Higgins [1], who proved that

$$|\mathcal{OP}_n| = n\binom{2n-1}{n-1} - n(n-1) \quad \text{and} \quad |\mathcal{OR}_n| = n\binom{2n}{n} - \frac{n^2}{2}(n^2 - 2n + 5) + n.$$

Just to complete the picture recall that

$$|\mathcal{C}_n| = n,\ |\mathcal{S}_n| = n!,\ |\mathcal{I}_n| = \sum_{k=0}^{n}\binom{n}{k}^2 k!,\ |\mathcal{T}_n| = n^n \quad \text{and} \quad |\mathcal{PT}_n| = (n+1)^n.$$

2 Number of idempotents

For a given monoid M, we denote by $E(M)$ its set of idempotents.

First we will consider the order case. Let $M \in \{\mathcal{OD}_n, \mathcal{PODI}_n, \mathcal{POD}_n\}$. Let $e \in E(M)$. As $e^2 = e$ and the product of two order-preserving transformations or of two order-reversing transformations is an order-preserving transformation, we conclude that e must be order-preserving. Thus $E(\mathcal{OD}_n) = E(\mathcal{O}_n)$ and $E(\mathcal{POD}_n) = E(\mathcal{PO}_n)$.

In [8] Howie showed that

$$|E(\mathcal{O}_n)| = F_{2n},$$

where F_n is the nth Fibonacci number.

Recall that, the Fibonacci numbers are recursively defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{k+1} = F_k + F_{k-1}, k \geq 1.$$

Another interesting sequence of numbers is the Lucas numbers, which are also recursively defined as follows

$$L_0 = 2, \quad L_1 = 1, \quad L_{k+1} = L_k + L_{k-1}, k \geq 1.$$

Fibonacci and Lucas numbers are intrinsically related, in fact, for any $n \in \mathbb{N}_0$,

$$F_n = \frac{\tau^n - \theta^n}{\tau - \theta} \quad \text{and} \quad L_n = \tau^n + \theta^n,$$

where τ is the golden number and θ is its rational conjugate, that is $\tau = \frac{1+\sqrt{5}}{2}$ and $\theta = \frac{1-\sqrt{5}}{2}$, and we have $F_{2n} = F_n L_n$. For further details, see e.g. [11].

In view of the above observations, we conclude that

$$|E(\mathcal{OD}_n)| = |E(\mathcal{O}_n)| = F_{2n} = \frac{\tau^{2n} - \theta^{2n}}{\tau - \theta}.$$

Concerning the correspondent classes of partial transformations, we prove the following.

Theorem 2.1 $|E(\mathcal{POD}_n)| = |E(\mathcal{PO}_n)| = \displaystyle\sum_{j=1}^{n} \binom{n}{j} F_{2j} + 1 = \sum_{j=1}^{n} \binom{n}{j} \left(\frac{\tau^{2j} - \theta^{2j}}{\tau - \theta} \right) + 1.$

Proof. First, notice that an element $s \in \mathcal{PT}_n$ is idempotent if and only if

$$\mathrm{Im}(s) \subseteq \mathrm{Fix}(s) = \{x \in \mathrm{Dom}(s) \mid xs = x\}.$$

Next, observe that for each nonempty subset A of X_n, the number of idempotents of $\mathcal{PO}_n$ with domain A coincides with $|E(\mathcal{O}_{|A|})|$. Putting these facts together, we get

$$|E(\mathcal{POD}_n)| = |E(\mathcal{PO}_n)| = \textstyle\sum_{j=1}^{n} \binom{n}{j} |E(\mathcal{O}_j)| + 1 = \sum_{j=1}^{n} \binom{n}{j} F_{2j} + 1 = \sum_{j=1}^{n} \binom{n}{j} \left(\frac{\tau^{2j} - \theta^{2j}}{\tau - \theta} \right) + 1,$$

as required. □

We now look into the orientation case. Let $M \in \{\mathcal{OR}_n, \mathcal{PORI}_n, \mathcal{POR}_n\}$. Let $e \in E(M)$. As for the order case, the product of two orientation-preserving or of two orientation-reversing elements of M is an orientation-preserving transformation, whence e preserves the orientation. Thus $E(\mathcal{OR}_n) = E(\mathcal{OP}_n)$ and $E(\mathcal{POR}_n) = E(\mathcal{POP}_n)$.

Recall that Catarino and Higgins [1] showed that

$$|E(\mathcal{OR}_n)| = |E(\mathcal{OP}_n)| = L_{2n} - (n^2 - n + 2) = \tau^{2n} + \theta^{2n} - (n^2 - n + 2).$$

We finish this note by computing the number of idempotents of $\mathcal{POP}_n$ and of $\mathcal{POR}_n$.

Theorem 2.2 $|E(\mathcal{POR}_n)| = |E(\mathcal{POP}_n)| = \displaystyle\sum_{j=1}^{n} \binom{n}{j} [L_{2j} - (j^2 - j + 2)] + 1 = \sum_{j=1}^{n} \binom{n}{j} [\tau^{2j} + \theta^{2j} - (j^2 - j + 2)] + 1.$

Proof. Once again, as an element $s \in \mathcal{PT}_n$ is idempotent if and only if $\mathrm{Im}(s) \subseteq \mathrm{Fix}(s)$, and for each nonempty subset A of X_n, the number of idempotents of $\mathcal{POP}_n$ with domain A coincides with $|E(\mathcal{OP}_{|A|})|$, we obtain

$$\begin{aligned} |E(\mathcal{POR}_n)| = |E(\mathcal{POP}_n)| &= \textstyle\sum_{j=1}^{n} \binom{n}{j} |E(\mathcal{OP}_j)| + 1 \\ &= \textstyle\sum_{j=1}^{n} \binom{n}{j} [L_{2j} - (j^2 - j + 2)] + 1 \\ &= \textstyle\sum_{j=1}^{n} \binom{n}{j} \left[\tau^{2j} + \theta^{2j} - (j^2 - j + 2)\right] + 1, \end{aligned}$$

as required. □

To complete the picture recall that $|E(\mathcal{C}_n)| = |E(\mathcal{S}_n)| = 1$, $|E(\mathcal{POI}_n)| = |E(\mathcal{PODI}_n)| = |E(\mathcal{PORI}_n)| = |E(\mathcal{I}_n)| = 2^n$, $|E(\mathcal{T}_n)| = \sum_{j=1}^{n} \binom{n}{j} j^{n-j}$ and $|E(\mathcal{PT}_n)| = \sum_{j=1}^{n} \binom{n}{j} \left(\sum_{k=1}^{j} \binom{j}{k} k^{j-k} \right) + 1$.

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