Congruences on monoids of transformations preserving the orientation on a finite chain

Vítor H. Fernandes, Gracinda M. S. Gomes, Manuel M. Jesus [1]

Centro de Álgebra da Universidade de Lisboa,
Av. Prof. Gama Pinto, 2,
1649-003 Lisboa,
Portugal

E-mails: *vhf@fct.unl.pt, ggomes@cii.fc.ul.pt, mrj@fct.unl.pt*

Abstract

The main subject of this paper is the description of the congruences on certain monoids of transformations on a finite chain X_n with n elements. Namely, we consider the monoids $\mathcal{OR}_n$ and $\mathcal{POR}_n$ of all full, respectively partial, transformations on X_n that preserve or reverse the orientation, as well as their respective submonoids $\mathcal{OP}_n$ and $\mathcal{POP}_n$ of all orientation-preserving elements. The inverse monoid $\mathcal{PORI}_n$ of all injective elements of $\mathcal{POR}_n$ is also considered.

2000 Mathematics Subject Classification: 20M20, 20M05, 20M17.

Keywords: congruences, orientation-preserving, orientation-reversing, transformations.

Introduction and preliminaries

For $n \in \mathbb{N}$, let X_n be a finite chain with n elements, say $X_n = \{1 < 2 < \cdots < n\}$. As usual, we denote by $\mathcal{PT}_n$ the monoid (under composition) of all partial transformations of X_n. The submonoid of $\mathcal{PT}_n$ of all full transformations of X_n and the (inverse) submonoid of all injective partial transformations of X_n are denoted by $\mathcal{T}_n$ and $\mathcal{I}_n$, respectively.

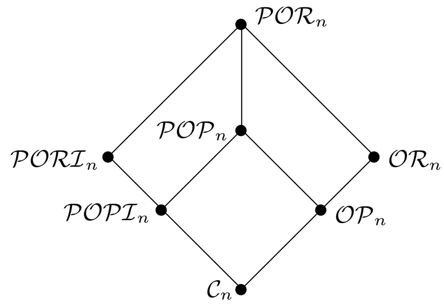
Let $a = (a_1, a_2, \ldots, a_t)$ be a sequence of t ($t \geq 0$) elements from the chain X_n. We say that a is *cyclic* [*anti-cyclic*] if there exists no more than one index $i \in \{1, \ldots, t\}$ such that $a_i > a_{i+1}$ [$a_i < a_{i+1}$], where a_{t+1} denotes a_1. Notice that, the sequence a is cyclic [anti-cyclic] if and only if a is empty or there exists $i \in \{0, 1, \ldots, t-1\}$ such that $a_{i+1} \leq a_{i+2} \leq \cdots \leq a_t \leq a_1 \leq \cdots \leq a_i$ [$a_{i+1} \geq a_{i+2} \geq \cdots \geq a_t \geq a_1 \geq \cdots \geq a_i$] (the index $i \in \{0, 1, \ldots, t-1\}$ is unique unless a is constant and $t \geq 2$). Let $s \in \mathcal{PT}_n$ and suppose that $\mathrm{Dom}(s) = \{a_1, \ldots, a_t\}$, with $t \geq 0$ and $a_1 < \cdots < a_t$. We say that s is an *orientation-preserving* [*orientation-reversing*] transformation if the sequence of its images $(a_1 s, \ldots, a_t s)$ is cyclic [anti-cyclic]. It is easy to show that the product of two orientation-preserving or of two orientation-reversing transformations is orientation-preserving and the product of an orientation-preserving transformation by an orientation-reversing transformation is clearly orientation-reversing.

[1]This work was developed within the activities of Centro de Álgebra da Universidade de Lisboa, supported by FCT and FEDER, within project POCTI/MAT/893/2003 – "Fundamental and Applied Algebra".

Denote by $\mathcal{POP}_n$ the submonoid of $\mathcal{PT}_n$ of all orientation-preserving transformations of X_n. As usual, $\mathcal{OP}_n$ denotes the monoid $\mathcal{POP}_n \cap \mathcal{T}_n$ of all full transformations of X_n that preserve the orientation. This monoid was considered by Catarino in [3] and by Arthur and Ruškuc in [2]. The injective counterpart of $\mathcal{OP}_n$, i.e. the inverse monoid $\mathcal{POPI}_n = \mathcal{POP}_n \cap \mathcal{I}_n$, was studied by the first author in [9, 11].

Comprehensiver classes of monoids are obtained when we take transformations that either preserve or reverse the orientation. In this way we get $\mathcal{POR}_n$, the submonoid of $\mathcal{PT}_n$ of all transformations that preserve or reverse the orientation. Within $\mathcal{T}_n$ sits the submonoid $\mathcal{OR}_n = \mathcal{POR}_n \cap \mathcal{T}_n$ and inside $\mathcal{I}_n$ is $\mathcal{PORI}_n = \mathcal{POR}_n \cap \mathcal{I}_n$.

The following diagram, with respect to the inclusion relation and where $\mathcal{C}_n$ denotes the cyclic group of order n, exposes the relationship between these semigroups:



The study of transformations that respect the orientation is intrinsically associated to the knowledge of the ones that respect the order. A transformation s in $\mathcal{PT}_n$ is called *order-preserving* if $x \leq y$ implies $xs \leq ys$, for all $x, y \in \mathrm{Dom}(s)$. Denote by $\mathcal{PO}_n$ the submonoid of $\mathcal{PT}_n$ of all partial order-preserving transformations of X_n. The monoid $\mathcal{PO}_n \cap \mathcal{T}_n$ of all full transformations of X_n that preserve the order is denoted by $\mathcal{O}_n$. This monoid has been largely studied by several authors (e.g. see [1, 15, 16, 18]). The injective counterpart of $\mathcal{O}_n$ is the inverse monoid $\mathcal{POI}_n = \mathcal{PO}_n \cap \mathcal{I}_n$, which is considered, for example, in [5, 7, 8, 10, 12].

In this paper, on one hand we aim to describe the Green relations on some of the monoids mentioned above and to use the descriptions obtained to calculate their sizes and ranks. This type of questions were also considered by Catarino and Higgins [4] for $\mathcal{OP}_n$ and for $\mathcal{OR}_n$; by Fernandes [9] for $\mathcal{POPI}_n$ and by the authors [13] for $\mathcal{PORI}_n$. So, it remains to study the monoids $\mathcal{POP}_n$ and $\mathcal{POR}_n$ and that is done in Section 1.

On the other hand, we want to describe the congruences of the monoids $\mathcal{OP}_n$, $\mathcal{POP}_n$, $\mathcal{OR}_n$, $\mathcal{POR}_n$ and $\mathcal{PORI}_n$. It was proved by Aĭzenštat [1], and later by Lavers and Solomon [18], that the congruences of $\mathcal{O}_n$ are exactly the Rees congruences. A similar result was proved by the first author [10] for the monoid $\mathcal{POI}_n$ and by the authors [14] for the monoid $\mathcal{PO}_n$. Fernandes [9] proved that the congruences on $\mathcal{POPI}_n$ are associated with its maximal subgroups. In Section 2, under certain conditions, on an arbitrary finite semigroup we define a class of congruences associated to its maximal subgroups. In Section 3, we show that, as in the case of $\mathcal{POPI}_n$, all congruences in the monoids referred above are of this type.

Next, for completion, we recall some notions and fix the notation.

Let M be a monoid. We denote by $E(M)$ its set of idempotents. Let $\leq_{\mathcal{J}}$ be the quasi-order on M defined by

$$u \leq_{\mathcal{J}} v \text{ if and only if } MuM \subseteq MvM,$$

for all $u, v \in M$. Denote by J_u the $\mathcal{J}$-class of an element $u \in M$. As usual, a partial order relation $\leq_{\mathcal{J}}$ is defined on the set $M/\mathcal{J}$ by setting $J_u \leq_{\mathcal{J}} J_v$ if and only if $u \leq_{\mathcal{J}} v$, for all $u, v \in M$. For $u, v \in M$, we write $u <_{\mathcal{J}} v$ and also $J_u <_{\mathcal{J}} J_v$ if and only if $u \leq_{\mathcal{J}} v$ and $(u, v) \notin \mathcal{J}$.

The *Rees congruence* ρ_I on M associated to an ideal I of M is defined by $(u, v) \in \rho_I$ if and only if $u = v$ or $u, v \in I$, for all $u, v \in M$. For convenience, we admit the empty set as an ideal. In what follows the identity congruence will be denoted by 1 and the universal congruence by ω. The *rank* of M is, by definition, the minimum of the set $\{|X| : X \subseteq M \text{ and } X \text{ generates } M\}$. For more details, see e.g. [17].

A subset C of the chain X_m is said to be *convex* if $x, y \in C$ and $x \leq z \leq y$ imply that $z \in C$. An equivalence ρ on X_m is *convex* if its classes are convex. We say that ρ is of *weight* k if $|X_m/\rho| = k$. Clearly, the number of convex equivalences of weight k on X_m is $\binom{m-1}{k-1}$.

Now let G be a cyclic group of order n. It is well known that there exists a one-to-one correspondence between the subgroups of G and the (positive) divisors of n. Since G is abelian, all subgroups are normal, and so there is a one-to-one correspondence between the congruences of G and the (positive) divisors of n. These correspondences are, in fact, lattice isomorphisms.

The dihedral group D_n of order $2n$ ($n \geq 3$) can be defined by the group presentation

$$\langle x, y | x^n = 1, y^2 = 1, yx = x^{-1}y \rangle$$

and its proper normal subgroups are:

(1) $\langle x^2, y \rangle$, $\langle x^2, xy \rangle$ and $\langle x^{\frac{n}{p}} \rangle$, with p a divisor of n, when n is even;

(2) $\langle x^{\frac{n}{p}} \rangle$, with p a divisor of n, when n is odd.

See [6] for more details.

The following concept will be used in Section 3. Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be two disjoint posets. The *ordinal sum* of P_1 and P_2 (in this order) is the poset $P_1 \oplus P_2$ with universe $P_1 \cup P_2$ and partial order $\leq$ defined by: for all $x, y \in P_1 \cup P_2$, we have $x \leq y$ if and only if $x \in P_1$ and $y \in P_2$; or $x, y \in P_1$ and $x \leq_1 y$; or $x, y \in P_2$ and $x \leq_2 y$. Observe that this operator on posets is associative but not commutative.

1 The monoids $\mathcal{POP}_n$ and $\mathcal{POR}_n$

In this section we describe the Green relations and calculate the sizes and the ranks of the monoids $\mathcal{POP}_n$ and $\mathcal{POR}_n$. We show that their structure is similar to the one of the monoids $\mathcal{POPI}_n$, $\mathcal{PORI}_n$, $\mathcal{PT}_n$ and $\mathcal{I}_n$. In particular, in all of them, the $\mathcal{J}$-classes are the sets of all elements with the same rank and form a chain, with respect to the partial order $\leq_{\mathcal{J}}$. Notice also that all these monoids are regular.

In what follows, we must have in mind that an element of $\mathcal{POR}_n$ is either in $\mathcal{POP}_n$ or it reverses the orientation. Denote by $\mathcal{P}or_n$ the set of all orientation-reversing partial transformations of X_n. Clearly, $\mathcal{POR}_n = \mathcal{POP}_n \cup \mathcal{P}or_n$. In view of the next lemma, we have $\mathcal{POP}_n \cap \mathcal{P}or_n = \{s \in \mathcal{POP}_n : |\operatorname{Im}(s)| \leq 2\}$.

Lemma 1.1 [4] *Let a be a cyclic [anti-cyclic] sequence. Then a is (also) anti-cyclic [cyclic] if and only if a has no more than two distinct values.* $\square$

It is easy to show that $E(\mathcal{POR}_n) = E(\mathcal{POP}_n)$.

Let us consider the permutation of order two

$$h = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}.$$

Clearly, $h^2 = 1$ and h is an orientation-reversing full transformation. We showed in [13] that $\mathcal{POPI}_n$ together with h form a set of generators of $\mathcal{PORI}_n$. Similarly, by just noticing that, given an orientation-reversing transformation s, the product sh is an orientation-preserving transformation, it follows that $\mathcal{POR}_n$ is generated by $\mathcal{POP}_n \cup \{h\}$.

We prove that $\mathcal{POR}_n$ is regular, using the fact that $\mathcal{POP}_n$ is already known to be regular [9]. It remains to show that all the elements of $\mathcal{P}or_n$ are regular. Let s be an orientation-reversing transformation. Then $sh \in \mathcal{POP}_n$ and so there exists $s' \in \mathcal{POP}_n$ such that $(sh)s'(sh) = sh$. Thus, multiplying on the right by h, we obtain $s(hs')s = s$ and so s is a regular element of $\mathcal{POR}_n$.

Next consider the following permutation

$$g = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix},$$

which is an element of $\mathcal{POP}_n$ such that $g^n = 1$. As in [9, Proposition 3.1], it is a routine matter to prove the following (non unique) factorisation of an element of $\mathcal{POP}_n$:

Proposition 1.2 *Let $s \in \mathcal{POP}_n$. Then there exist $i \in \{0, 1, \dots, n-1\}$ and $u \in \mathcal{PO}_n$ such that $s = g^i u$.* $\square$

As an immediate consequence of this proposition, we have:

Corollary 1.3 *The monoid $\mathcal{POP}_n$ is generated by $\mathcal{PO}_n \cup \{g\}$.* $\square$

Corollary 1.4 *Let $s \in \mathcal{POR}_n$. Then there exist $i \in \{0, 1, \dots, n-1\}$, $j \in \{0, 1\}$ and $u \in \mathcal{PO}_n$ such that $s = g^i u h^j$.* $\square$

Notice that, with the notation of the last corollary, we can always take:

(1) $j = 0$, if $s \in \mathcal{POP}_n$;

(2) $u \in \mathcal{O}_n$, if $s \in \mathcal{OR}_n$.

Therefore, wherever in this paper we take such a factorisation of an element s of $\mathcal{POR}_n$, we will consider j and u as above.

Denote by M_n either the monoid $\mathcal{POR}_n$ or the monoid $\mathcal{POP}_n$.

Proposition 1.5 *Let s and t be elements of M_n. Then:*

(1) $s \mathcal{R} t$ if and only if $\operatorname{Ker}(s) = \operatorname{Ker}(t)$;

(2) $s \mathcal{L} t$ if and only if $\operatorname{Im}(s) = \operatorname{Im}(t)$;

(3) $s \leq_{\mathcal{J}} t$ if and only if $|\operatorname{Im}(s)| \leq |\operatorname{Im}(t)|$.

Proof. Since M_n is a regular submonoid of $\mathcal{PT}_n$, conditions (1) and (2) follow immediately from well known results on regular semigroups (e.g. see [17]).

Next we prove condition (3). First, suppose that $s \leq_{\mathcal{J}} t$. Then there exist $x, y \in M_n$ such that $s = xty$. Since $\operatorname{Im}(s) \subseteq \operatorname{Im}(ty)$ and $|\operatorname{Im}(ty)| = |\operatorname{Im}(t)y| \leq |\operatorname{Im}(t)|$, then $|\operatorname{Im}(s)| \leq |\operatorname{Im}(t)|$. Conversely, let $s, t \in M_n$ be such that $|\operatorname{Im}(s)| \leq |\operatorname{Im}(t)|$. By Corollary 1.4, there exist $i_1, i_2 \in \{0, \ldots, n-1\}$, $j_1, j_2 \in \{0, 1\}$ and $u, v \in \mathcal{PO}_n$ such that $s = g^{i_1} u h^{j_1}$ and $t = g^{i_2} v h^{j_2}$. Thus $|\operatorname{Im}(s)| = |\operatorname{Im}(u)|$ and $|\operatorname{Im}(t)| = |\operatorname{Im}(v)|$, since $g^{i_1}, g^{i_2}, h^{j_1}, h^{j_2}$ are permutations. Hence $u \leq_{\mathcal{J}} v$ in $\mathcal{PO}_n$ (see [15]) and so there exist $x, y \in \mathcal{PO}_n$ such that $u = xvy$. Then

$$s = g^{i_1} u h^{j_1} = g^{i_1} x v y h^{j_1} = (g^{i_1} x g^{n-i_2}) g^{i_2} v h^{j_2} (h^{2-j_2} y h^{j_1}) = (g^{i_1} x g^{n-i_2}) t (h^{2-j_2} y h^{j_1}),$$

with $g^{i_1} x g^{n-i_2}, h^{2-j_2} y h^{j_1} \in M_n$, and so $s \leq_{\mathcal{J}} t$ in M_n, as required. $\square$

It follows, from condition (3), that

$$M_n/\mathcal{J} = \{J_0 <_{\mathcal{J}} J_1 <_{\mathcal{J}} \cdots <_{\mathcal{J}} J_n\},$$

where $J_k = \{s \in M_n \mid |\operatorname{Im}(s)| = k\}$, for all $0 \leq k \leq n$.

On the other hand, given an element $s \in M_n \cap \mathcal{I}_n$, from conditions (1) and (2) above and from the corresponding descriptions for the monoids $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$ ([9, Proposition 2.4] and [13, Proposition 5.3], respectively), it follows that the $\mathcal{H}$-class of s in $M_n \cap \mathcal{I}_n$ coincides with its $\mathcal{H}$-class in M_n. Thus, as for the monoid $\mathcal{POPI}_n$ ([9, Proposition 2.6]), we have:

Proposition 1.6 *Let $s \in \mathcal{POP}_n$ be such that $1 \leq |\operatorname{Im}(s)| = k \leq n$. Then $|H_s| = k$. Moreover, if s is an idempotent then H_s is a cyclic group of order k.* $\square$

Since a transformation $s \in \mathcal{POR}_n$ is both orientation-preserving and orientation-reversing if and only if $|\operatorname{Im}(s)| \leq 2$, we have the following:

Corollary 1.7 *Let $s \in \mathcal{POR}_n$ be such that $1 \leq |\operatorname{Im}(s)| = k \leq 2$. Then $|H_s| = k$. Moreover, if s is an idempotent then H_s is a cyclic group of order k.* $\square$

Also, as for the monoid $\mathcal{PORI}_n$ ([13, Proposition 5.3]), we have:

Proposition 1.8 *Let $s \in \mathcal{POR}_n$ be such that $3 \leq |\operatorname{Im}(s)| = k \leq n$. Then $|H_s| = 2k$. Moreover, if s is an idempotent then H_s is a dihedral group of order $2k$.* $\square$

Let s be an element of $\mathcal{POR}_n$ with rank k, $0 \leq k \leq n$. Suppose that $\operatorname{Im}(s) = \{b_1, \ldots, b_k\}$. Then, considering the kernel classes of s, we obtain two types of partitions of the domain of s into intervals:

(a) $\operatorname{Dom}(s) = \dot{\bigcup}_{i=1}^{k} P_i$ with $s = \left(\begin{array}{c|c|c} P_1 & \cdots & P_k \\ b_1 & \cdots & b_k \end{array}\right)$; or

(b) $\operatorname{Dom}(s) = \dot{\bigcup}_{i=1}^{k+1} P_i$ with $s = \left(\begin{array}{c|c|c|c} P_1 & \cdots & P_k & P_{k+1} \\ b_1 & \cdots & b_k & b_1 \end{array}\right)$.

Notice that, in the first case, $P_1, \ldots, P_k$ are precisely the kernel classes of s whereas, in the second one, the kernel classes are $P_1 \cup P_{k+1}, P_2, \ldots, P_k$.

Now let $k \in \{2, \ldots, n\}$ and suppose that s is an element of $\mathcal{POP}_n$ with rank k. If s verifies (a) then $\operatorname{Ker}(s)$ is a convex equivalence on $\operatorname{Dom}(s)$ of weight k. On the other hand, if s verifies (b) then we can associate to s a convex relation of weight $k+1$ (with classes $P_1, \ldots, P_k, P_{k+1}$). Therefore the number of $\mathcal{R}$-classes of rank k with the same domain as s is given by

$$\binom{|\operatorname{Dom}(s)|-1}{k-1} + \binom{|\operatorname{Dom}(s)|-1}{k} = \binom{|\operatorname{Dom}(s)|}{k},$$

whence the total number of $\mathcal{R}$-classes of rank k is equal to $\sum_{j=k}^{n} \binom{n}{j}\binom{j}{k}$. As $\binom{n}{j}\binom{j}{k} = \binom{n}{k}\binom{n-k}{j-k}$, we have $\sum_{j=k}^{n} \binom{n}{j}\binom{j}{k} = \binom{n}{k}\sum_{j=k}^{n}\binom{n-k}{j-k} = \binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j} = \binom{n}{k}2^{n-k}$. Since the number of $\mathcal{L}$-classes of rank k is, clearly, equal to $\binom{n}{k}$ and, by Proposition 1.6, each $\mathcal{H}$-class of rank k has k elements, the monoid $\mathcal{POP}_n$ has precisely $k\binom{n}{k}\binom{n}{k}2^{n-k}$ elements of rank k. Furthermore, by noticing that the number of transformations of rank 1 of $\mathcal{PT}_n$ (and so of $\mathcal{POP}_n$) is equal to $(2^n-1)n$, we conclude the following result:

Proposition 1.9 $|\mathcal{POP}_n| = 1 + (2^n-1)n + \sum_{k=2}^{n} k\binom{n}{k}^2 2^{n-k}$. $\square$

As there is a natural bijection between $\mathcal{POP}_n$ and $\mathcal{P}or_n$ (obtained by simply reversing the sequence of the images), we have $|\mathcal{POR}_n| = 2|\mathcal{POP}_n| - |\{s \in \mathcal{POP}_n \mid |\operatorname{Im}(s)| \leq 2\}|$, whence:

Proposition 1.10 $|\mathcal{POR}_n| = 1 + (2^n-1)n + 2\binom{n}{2}^2 2^{n-2} + \sum_{k=3}^{n} 2k\binom{n}{k}^2 2^{n-k}$. $\square$

Naturally, at this point, we would like to compute the rank of these monoids.

Let us consider the following elements $s_0, s_1, \ldots, s_{n-1}$ of $\mathcal{POI}_n$:

$$s_0 = \begin{pmatrix} 2 & \cdots & n-1 & n \\ 1 & \cdots & n-2 & n-1 \end{pmatrix}$$

and

$$s_i = \left(\begin{array}{ccc|c|ccc} 1 & \cdots & n-i-1 & n-i & n-i+2 & \cdots & n \\ 1 & \cdots & n-i-1 & n-i+1 & n-i+2 & \cdots & n \end{array}\right),$$

for $i \in \{1, 2, \ldots, n-1\}$. Consider also the elements $u_1, \ldots, u_{n-1}$ of $\mathcal{O}_n$ defined by

$$u_i = \left(\begin{array}{ccc|c|ccc} 1 & \cdots & i-1 & i & i+1 & \cdots & n \\ 1 & \cdots & i-1 & i+1 & i+1 & \cdots & n \end{array}\right),$$

for $1 \leq i \leq n-1$. Since $\mathcal{PO}_n = \langle s_0, \ldots, s_{n-1}, u_1, \ldots, u_{n-1} \rangle$ (see [15]), it follows from Corollary 1.3 that:

Corollary 1.11 $\mathcal{POP}_n = \langle s_0, \ldots, s_{n-1}, u_1, \ldots, u_{n-1}, g \rangle$. $\square$

Also, as $g^{n-1}u_i g = u_{i+1}$, for $1 \leq i \leq n-2$, $s_0 = g^{n-1}(s_1 g)^{n-1}$ and $s_i = g^{i-1}s_1 g^{n-i+1}$, for $1 \leq i \leq n-1$, we get:

Corollary 1.12 $\mathcal{POP}_n = \langle s_1, u_1, g \rangle$. $\square$

Finally, since any generating set of $\mathcal{POP}_n$ must clearly contain a permutation, a non-permutation full transformation and a non-full transformation, we must have:

Theorem 1.13 *$\mathcal{POP}_n$ has rank 3.* $\square$

Next we observe that, given an orientation-reversing partial transformation s, we have $sh \in \mathcal{POP}_n$, whence $sh = x_1x_2\cdots x_k$, for some $x_1, x_2, \ldots, x_k \in \{s_1, u_1, g\}$ and $k \in \mathbb{N}$. Thus $s = sh^2 = x_1x_2\cdots x_kh$ and so we may conclude the following:

Corollary 1.14 $\mathcal{POR}_n = \langle s_1, u_1, g, h\rangle$. $\square$

Let A be a set of generators of $\mathcal{POR}_n$. As for $\mathcal{POP}_n$, the set A must contain at least one non-permutation full transformation and one non-full transformation. On the other hand, for $n \geq 3$, the group of units of $\mathcal{POR}_n$ is the dihedral group D_n, which has rank two. Hence we must also have two permutations in A. We have proved the next result.

Theorem 1.15 *For $n \geq 3$ the monoid $\mathcal{POR}_n$ has rank 4.* $\square$

2 Congruences associated to maximal subgroups

In this section we construct a family of congruences associated to maximal subgroups of a $\mathcal{J}$-class that satisfies certain conditions. As we will show in Section 3, this family provides a description for the congruences of the monoids we want to consider.

We start with a simple technical lemma.

Lemma 2.1 *Let S be a semigroup and let $s, t, u \in S$ be such that s is regular and $s\,\mathcal{H}\,t$. Then there exist $v_1, v_2 \in S$ such that $v_1s = us$, $v_1t = ut$, $sv_2 = su$, $tv_2 = tu$, $v_1s\,\mathcal{R}\,v_1\,\mathcal{R}\,v_1t$ and $sv_2\,\mathcal{L}\,v_2\,\mathcal{L}\,tv_2$.*

Proof. It is well known (e.g. see [17]) that $ss' = tt'$ and $s's = t't$, for some inverses s' of s and t' of t. Let $v_1 = uss' = utt'$ and $v_2 = s'su = t'tu$. Then $v_1s = us$, $v_1t = ut$, $sv_2 = su$ and $tv_2 = tu$. On the other hand, as $ss'\,\mathcal{R}\,s$, $tt'\,\mathcal{R}\,t$, $s's\,\mathcal{L}\,s$ and $t't\,\mathcal{L}\,t$, we obtain $v_1 = uss'\,\mathcal{R}\,us = v_1s$, $v_1 = utt'\,\mathcal{R}\,ut = v_1t$, $v_2 = s'su\,\mathcal{L}\,su = sv_2$ and $v_2 = t'tu\,\mathcal{L}\,tu = tv_2$, as required. $\square$

Let S be a finite semigroup and let J be a $\mathcal{J}$-class of S. Denote by $B(J)$ the set of all elements $s \in S$ such that $J \not\leq_{\mathcal{J}} J_s$. It is clear that $B(J)$ is an ideal of S. We associate to J a relation π_J on S defined by: for all $s, t \in S$, we have $s\,\pi_J\,t$ if and only if

(a) $s = t$; or

(b) $s, t \in B(J)$; or

(c) $s, t \in J$ and $s\,\mathcal{H}\,t$.

Lemma 2.2 [9] *The relation π_J is a congruence on S.* $\square$

Assume that J is regular and take a group $\mathcal{H}$-class H_0 of J. Also, suppose that there exists a mapping

$$\begin{array}{rcccc} \varepsilon & : & J & \longrightarrow & H_0 \\ & & s & \longmapsto & \tilde{s} \end{array}$$

which satisfies the following property: given $s,t \in J$ such that $st \in J$, there exist $x,y \in H_0$ such that

$$b\,\mathcal{H}\,t \text{ implies } \widetilde{sb} = x\tilde{s}\tilde{b} \tag{1}$$

$$a\,\mathcal{H}\,s \text{ implies } \widetilde{at} = \tilde{a}\tilde{t}y. \tag{2}$$

The existence of such a map for the monoid $\mathcal{I}_n$ (and for some of its submonoids) was showed by the first author in [9].

To each congruence π on H_0, we associate a relation ρ_π on S defined by: given $s,t \in S$, we have

$$s\,\rho_\pi\,t \text{ if and only if } s\,\pi_J\,t \text{ and } s,t \in J \text{ implies } \tilde{s}\,\pi\,\tilde{t}.$$

Theorem 2.3 *The relation ρ_π is a congruence on S.*

Proof. First, observe that ρ_π is an equivalence relation, since $\mathcal{H}$ and π are equivalence relations and $B(J) \cap J = \emptyset$. So, it remains to prove that ρ_π is compatible with the multiplication.

Let $s,t \in S$ be such that $s\,\rho_\pi\,t$ and assume that $s \neq t$. Let $u \in S$. As $s\,\pi_J\,t$ and π_J is a congruence, we have $us\,\pi_J\,ut$ and $su\,\pi_J\,tu$. In order to prove that $us\,\rho_\pi\,ut$, suppose that $us, ut \in J$. Then $us, ut \notin B(J)$ and, as $B(J)$ is an ideal, $s,t \notin B(J)$. Since $s \neq t$, we must therefore have $s,t \in J$ and $s\,\mathcal{H}\,t$. Also we get $\tilde{s}\,\pi\,\tilde{t}$. Now, by Lemma 2.1, there exists $v_1 \in S$ such that $v_1 s = us$, $v_1 t = ut$ and $v_1 s\,\mathcal{R}\,v_1\,\mathcal{R}\,v_1 t$. Hence we have $s, v_1, v_1 s \in J$. As $t\,\mathcal{H}\,s$, it follows from condition (1) that $\widetilde{v_1 s} = x\tilde{v_1}\tilde{s}$ and $\widetilde{v_1 t} = x\tilde{v_1}\tilde{t}$, for some $x \in H_0$. Thus, as π is a congruence,

$$\widetilde{us} = \widetilde{v_1 s} = x\tilde{v_1}\tilde{s}\,\pi\,x\tilde{v_1}\tilde{t} = \widetilde{v_1 t} = \widetilde{ut}$$

and so $us\,\rho_\pi\,ut$. Similarly, we prove that $su\,\rho_\pi\,tu$, as required. $\square$

3 On the congruences of $\mathcal{OP}_n$, $\mathcal{POP}_n$, $\mathcal{OR}_n$, $\mathcal{PORI}_n$ and $\mathcal{POR}_n$

The goal of this section is to describe the congruences of the monoids $\mathcal{OP}_n$, $\mathcal{POP}_n$, $\mathcal{OR}_n$, $\mathcal{PORI}_n$ and $\mathcal{POR}_n$. We will use a method that generalises the process developed by the first author to describe the congruences of the monoid $\mathcal{POPI}_n$ [9]. In fact, this new technique will also comprise that case.

Although there are details that differ from one case to the other, we will present the proof in a way that solves the problem simultaneously for all these monoids.

To prove our main result, Theorem 3.3, we need to fix some notation and recall some properties of the monoids $\mathcal{OP}_n$, $\mathcal{OR}_n$, $\mathcal{POPI}_n$, $\mathcal{PORI}_n$, $\mathcal{POP}_n$ and $\mathcal{POR}_n$ presented in [4, 9, 13] or in this paper.

First, remember that $\mathcal{OP}_n = \langle \mathcal{O}_n, g \rangle$, $\mathcal{POPI}_n = \langle \mathcal{POI}_n, g \rangle$, $\mathcal{POP}_n = \langle \mathcal{PO}_n, g \rangle$, $\mathcal{OR}_n = \langle \mathcal{O}_n, g, h \rangle$, $\mathcal{PORI}_n = \langle \mathcal{POI}_n, g, h \rangle$ and $\mathcal{POR}_n = \langle \mathcal{PO}_n, g, h \rangle$.

Let us fix $T \in \{\mathcal{O}_n, \mathcal{POI}_n, \mathcal{PO}_n\}$ and let M be either the monoid $\langle T, g \rangle$ or the monoid $\langle T, g, h \rangle$. Both T and M are regular monoids (moreover, if $T = \mathcal{POI}_n$ then M and T are inverse monoids) and, for the partial order $\leq_{\mathcal{J}}$, the quotients $T/\mathcal{J}$ and $M/\mathcal{J}$ are chains. More precisely, for $S \in \{T, M\}$, we have

$$S/\mathcal{J} = \{J_0^S <_{\mathcal{J}} J_1^S <_{\mathcal{J}} \cdots <_{\mathcal{J}} J_n^S\}, \qquad \text{if } T \in \{\mathcal{POI}_n, \mathcal{PO}_n\},$$

and

$$S/\mathcal{J} = \{J_1^S <_{\mathcal{J}} \cdots <_{\mathcal{J}} J_n^S\}, \qquad \text{if } T = \mathcal{O}_n,$$

where J_k^S denotes the $\mathcal{J}$-class of S of the elements of rank k, for k suitably defined. Since $S/\mathcal{J}$ is a chain, the sets $I_k^S = \{s \in S \mid |\operatorname{Im}(s)| \leq k\}$, with $0 \leq k \leq n$, together with the empty set (if necessary), constitute all the ideals of S (see [10]). Observe also that T is an aperiodic monoid (i.e. T has only trivial subgroups); the $\mathcal{H}$-classes of rank k of $\langle T, g\rangle$ have precisely k elements, for $1 \leq k \leq n$; and the $\mathcal{H}$-classes of rank k of $\langle T, g, h\rangle$ have precisely $2k$ elements, for $3 \leq k \leq n$, and k elements, for $k = 1, 2$.

For a $\mathcal{J}$-class J_k^M of M (necessarily regular, since M is regular), with $1 \leq k \leq n$, we want to find a particular group $\mathcal{H}$-class H_k in J_k^M and a mapping $\varepsilon : J_k^M \longrightarrow H_k$ satisfying the conditions of Theorem 2.3. Notice that, we have $B(J_k^M) = J_0^M \cup J_1^M \cup \cdots \cup J_{k-1}^M$ or $B(J_k^M) = J_1^M \cup \cdots \cup J_{k-1}^M$.

Given $s \in \mathcal{PT}_n$ with $\operatorname{Dom}(s) = \{i_1 < \cdots < i_k\}$, where $1 \leq k \leq n$, define $\overline{s} \in \mathcal{T}_n$ by, for every $x \in X_n$,

$$(x)\overline{s} = \begin{cases} (i_1)s, & \text{if } 1 \leq x \leq i_1 \\ (i_j)s, & \text{if } i_{j-1} < x \leq i_j \text{ and } 2 \leq j \leq k \\ (i_k)s, & \text{if } i_k < x \leq n. \end{cases}$$

It is clear that $\overline{s}$ and s have the same rank. Moreover:

(a) If $s \in \mathcal{PO}_n$ then $\overline{s} \in \mathcal{O}_n$;

(b) If $s \in \mathcal{POP}_n$ then $\overline{s} \in \mathcal{OP}_n$; and

(c) If $s \in \mathcal{POR}_n$ then $\overline{s} \in \mathcal{OR}_n$.

Fix $1 \leq k \leq n$ and consider the following elements of $\mathcal{I}_n$ (which are permutations of $\{1, \ldots, k\}$):

$$\epsilon = \begin{pmatrix} 1 & \cdots & k \\ 1 & \cdots & k \end{pmatrix}, \ \gamma = \begin{pmatrix} 1 & 2 & \cdots & k-1 & k \\ 2 & 3 & \cdots & k & 1 \end{pmatrix} \text{ and } \eta = \begin{pmatrix} 1 & 2 & \cdots & k-1 & k \\ k & k-1 & \cdots & 2 & 1 \end{pmatrix}.$$

Let J_k be the $\mathcal{J}$-class of M of the elements of rank k. If $T \in \{\mathcal{POI}_n, \mathcal{PO}_n\}$, take the following elements:

$$e_k = \epsilon, \ g_k = \gamma \text{ and } h_k = \eta.$$

When $T = \mathcal{O}_n$, consider the following full transformations of X_n:

$$e_k = \overline{\epsilon}, \ g_k = \overline{\gamma} \text{ and } h_k = \overline{\eta}.$$

Notice that $h_k g_k^i = g_k^{k-i} h_k$, for $1 \leq i \leq k$.

Denote by H_k the $\mathcal{H}$-class of M of the idempotent e_k and observe that:

(a) If $M = \langle T, g\rangle$, then H_k is the cyclic group of order k generated by g_k; and

(b) If $k \geq 3$ and $M = \langle T, g, h\rangle$, then H_k is the dihedral group of order $2k$ generated by g_k and h_k.

Let $s \in J_k$. Suppose that $\{a_1 < \cdots < a_k\}$ is the transversal of the kernel of s formed by the minimum element of each kernel class. Let $\mathrm{Im}(s) = \{b_1 < \cdots < b_k\}$ and take the injective partial order-preserving transformations

$$\sigma_L = \begin{pmatrix} 1 & \cdots & k \\ a_1 & \cdots & a_k \end{pmatrix}, \quad \sigma_R = \begin{pmatrix} b_1 & \cdots & b_k \\ 1 & \cdots & k \end{pmatrix}$$

and

$$\sigma'_L = \begin{pmatrix} a_1 & \cdots & a_k \\ 1 & \cdots & k \end{pmatrix}, \quad \sigma'_R = \begin{pmatrix} 1 & \cdots & k \\ b_1 & \cdots & b_k \end{pmatrix}.$$

Define $s_L, s_R, s'_L, s'_R \in T$ by:

(a) $s_L = \sigma_L$, $s_R = \sigma_R$, $s'_L = \sigma'_L$ and $s'_R = \sigma'_R$, if $T \in \{\mathcal{POI}_n, \mathcal{PO}_n\}$;

(b) $s_L = \overline{\sigma_L}$, $s_R = \overline{\sigma_R}$, $s'_L = \overline{\sigma'_L}$ and $s'_R = \overline{\sigma'_R}$, if $T = \mathcal{O}_n$.

Clearly, $s_L \,\mathcal{R}\, e_k \,\mathcal{L}\, s_R$ and $s_L s'_L = e_k = s'_R s_R$.

Now let $b_{p_1}, \ldots, b_{p_k} \in \{b_1, \ldots, b_k\}$ be such that $b_{p_\ell} = a_\ell s$, for $1 \le \ell \le k$. There exists $i \in \{0, \ldots, k-1\}$ such that $b_{p_{i+1}} < \cdots < b_{p_k} < b_{p_1} < \cdots < b_{p_i}$ if s is orientation-preserving, or $b_{p_{i+1}} > \cdots > b_{p_k} > b_{p_1} > \cdots > b_{p_i}$ if s is orientation-reversing. It can be proved in a routine manner that $s_L s s_R = g_k^{k-i}$ if s is orientation-preserving, or $s_L s s_R = g_k^{k-i} h_k$ if s is orientation-reversing.

Next let $a_{q_1}, \ldots, a_{q_k} \in \{a_1, \ldots, a_k\}$ be such that $b_\ell = a_{q_\ell} s$, for $1 \le \ell \le k$, and consider the following injective partial transformation:

$$\sigma' = \begin{pmatrix} b_1 & \cdots & b_k \\ a_{q_1} & \cdots & a_{q_k} \end{pmatrix}.$$

Define $\hat{s} \in M$ by:

(a) $\hat{s} = \sigma'$, if $T \in \{\mathcal{POI}_n, \mathcal{PO}_n\}$;

(b) $\hat{s} = \overline{\sigma'}$, if $T = \mathcal{O}_n$.

Clearly, $\hat{s}$ is an inverse of s and it is easy to show that $s = s\hat{s}s'_L s_L s s_R s'_R$ and $s_L s \hat{s} s'_L = e_k$.

Next consider the mapping

$$\begin{array}{rccc} \varepsilon \; : & J_k & \longrightarrow & H_k \\ & s & \longmapsto & \tilde{s} = s_L s s_R. \end{array}$$

Observe that, given $s, t \in J_k$ such that $s \,\mathcal{H}\, t$, we have $s_L = t_L$, $s_R = t_R$, $s'_L = t'_L$ and $s'_R = t'_R$. Moreover, since $s\hat{s}$ and $t\hat{t}$ are idempotents of J_k with the same kernel and the same image, we also have $s\hat{s} = t\hat{t}$.

Lemma 3.1 *Let $s, t \in M$ be such that $s, t, st \in J_k$. Then there exist $\ell_1, \ell_2 \in \{0, 1, k-1\}$ such that*

(1) $b \,\mathcal{R}\, t$ implies $\widetilde{sb} = g_k^{\ell_1} \tilde{s}\tilde{b}$;

(2) $a \,\mathcal{L}\, s$ implies $\widetilde{at} = \tilde{a}\tilde{t} g_k^{\ell_2}$.

Proof. Suppose that s reverses the orientation and t preserves the orientation. Let

$$s = \left(\begin{array}{c|c|c|c|c|c|c} P_1 & \cdots & P_i & P_{i+1} & \cdots & P_k & P_{k+1} \\ s_1 & \cdots & s_i & s_{i+1} & \cdots & s_k & s_1 \end{array} \right)$$

and

$$t = \left(\begin{array}{c|c|c|c|c|c|c} Q_1 & \cdots & Q_j & Q_{j+1} & \cdots & Q_k & Q_{k+1} \\ t_1 & \cdots & t_j & t_{j+1} & \cdots & t_k & t_1 \end{array} \right)$$

with possibly $P_{k+1} = \emptyset$ or $Q_{k+1} = \emptyset$. Then $\mathrm{Im}(s) = \{s_{i+1} > \cdots > s_k > s_1 > \cdots > s_i\}$ and $\mathrm{Im}(t) = \{t_{j+1} < \cdots < t_k < t_1 < \cdots < t_j\}$, for some $0 \le i, j \le k-1$. Hence we have $\tilde{s} = g_k^{k-i} h_k$ and $\tilde{t} = g_k^{k-j}$. As $s, t, st \in J_k$, then $\mathrm{Im}(s)$ is a transversal of $\mathrm{Ker}(t)$ and we have two possible situations:

(a) If $s_i \in Q_1, \ldots, s_1 \in Q_i, s_k \in Q_{i+1}, \ldots, s_{i+1} \in Q_k$, then

$$st = \left(\begin{array}{c|c|c|c|c|c|c} P_1 & \cdots & P_i & P_{i+1} & \cdots & P_k & P_{k+1} \\ t_i & \cdots & t_1 & t_k & \cdots & t_{i+1} & t_i \end{array} \right).$$

Since $(P_{i-j})st = \{t_{j+1}\}$ if $0 \le j \le i-1$ and $(P_{i-j+k})st = \{t_{j+1}\}$ if $i \le j \le k-1$, it follows that $\widetilde{st} = g_k^{k-i+j} h_k$, whence $\widetilde{st} = \tilde{s}\tilde{t}$.

(b) If $s_i \in Q_2, \ldots, s_1 \in Q_{i+1}, s_k \in Q_{i+2}, \ldots, s_{i+1} \in Q_{k+1}$, then

$$st = \left(\begin{array}{c|c|c|c|c|c|c} P_1 & \cdots & P_{i+1} & P_{i+2} & \cdots & P_k & P_{k+1} \\ t_{i+1} & \cdots & t_1 & t_k & \cdots & t_{i+2} & t_{i+1} \end{array} \right)$$

and so $(P_{i-j+1})st = \{t_{j+1}\}$ if $0 \le j \le i$ and $(P_{k+i-j+1})st = \{t_{j+1}\}$ if $i+1 \le j \le k-1$. Therefore, in both cases, $\widetilde{st} = g_k^{k+j-i-1} h_k$ and so $\widetilde{st} = g_k^{k-1}\tilde{s}\tilde{t} = \tilde{s}\tilde{t}g_k$.

If s preserves the orientation or t reverses the orientation, it is a routine matter to show that, in the situation analogous to (a), we always have $\widetilde{st} = \tilde{s}\tilde{t}$. On the other hand, the situation analogous to (b) can be summarised by the following table:

s	t	ℓ_1	ℓ_2
orientation-preserving	orientation-preserving	1	1
orientation-reversing	orientation-preserving	$k-1$	1
orientation-preserving	orientation-reversing	1	$k-1$
orientation-reversing	orientation-reversing	$k-1$	$k-1$

Finally, as elements $\mathcal{R}$-related have the same kernel and elements $\mathcal{L}$-related have the same image, it is clear that ℓ_1 does not depend of the element of the $\mathcal{R}$-class of t (s fixed) and ℓ_2 does not depend of the element of the $\mathcal{L}$-class of s (t fixed), as required. $\square$

The next proposition follows from this lemma and Theorem 2.3.

Proposition 3.2 *Let $k \in \{1, \ldots, n\}$ and let π be a congruence on H_k. Then ρ_π is a congruence on M.* $\square$

Notice that, if π is the universal congruence of H_k, then the relation ρ_π is the congruence π_{J_k} of M. On the other hand, if π is the identity congruence of H_k, then the relation ρ_π is the Rees congruence of M associated to the ideal I^M_{k-1}. Thus, for $k = 1$, the relation ρ_π is the identity congruence of M and, for $2 \le k \le n$, there exist $s \in B(J_k) = I^M_{k-1}$ and $t \in J_k$, whence $(s,t) \notin \rho_\pi$ and so ρ_π is not the universal congruence of M.

At this point, we can state our main result.

Theorem 3.3 *The congruences of* $M \in \{\mathcal{OP}_n, \mathcal{POPI}_n, \mathcal{POP}_n, \mathcal{OR}_n, \mathcal{PORI}_n, \mathcal{POR}_n\}$ *are exactly the congruences* ρ_π*, where* π *is a congruence on* H_k*, for* $k \in \{1, \ldots, n\}$*, and the universal congruence.*

Denote by $\mathrm{Con}(S)$ the lattice of the congruences on a semigroup S.

Recall that $\mathrm{Con}(T)$ is formed only by the Rees congruences of $T \in \{\mathcal{O}_n, \mathcal{POI}_n, \mathcal{PO}_n\}$ (see [1, 10, 14]).

On the other hand, it is clear that $\mathrm{Con}(\mathcal{OP}_1) = \mathrm{Con}(\mathcal{OR}_1) = \{1\}$ and $\mathrm{Con}(M) = \{1, \omega\}$ if $M \in \{\mathcal{POPI}_1, \mathcal{PORI}_1, \mathcal{POP}_1, \mathcal{POR}_1\}$.

To prove Theorem 3.3 we start by establishing some auxiliary results.

Let $c_1, \ldots, c_n \in \mathcal{T}_n$ be the constant mappings such that $\mathrm{Im}(c_i) = \{i\}$, for all $1 \le i \le n$. Notice that, if $s, t \in \mathcal{T}_n$ are such that $c_i s = c_i t$, for all $1 \le i \le n$, then we must have $s = t$.

The version of this property for partial transformation is the following: let $c_1, \ldots, c_n \in \mathcal{PT}_n$ be the n partial identities of rank one such that $\mathrm{Dom}(c_i) = \mathrm{Im}(c_i) = \{i\}$, for all $1 \le i \le n$. Then, given $s, t \in \mathcal{PT}_n$ such that $c_i s = c_i t$, for all $1 \le i \le n$, we must also have $s = t$.

In what follows, $c_1, \ldots, c_n$ denote the constant mappings of $\mathcal{T}_n$ if $T = \mathcal{O}_n$, and the partial identities of rank one of $\mathcal{PT}_n$ if $T \in \{\mathcal{PO}_n, \mathcal{POI}_n\}$. In both cases $c_1, \ldots, c_n \in T$. Moreover, for all $1 \le i \le n$ and $s \in M$, we have $c_i s \in T$. In fact, $c_i s$ is either a constant map of image $\{(i)s\}$ or the empty map.

Let ρ be a congruence on M and consider $\overline{\rho} = \rho \cap (T \times T)$. Then $\overline{\rho}$ is a Rees congruence of T and so $\overline{\rho} = \rho_{I^T_{k-1}}$, for some $1 \le k \le n+1$.

This notation will be used in the sequel.

Lemma 3.4 *If* $k = 1$ *then* $\rho = 1$.

Proof. First notice that $k = 1$ if and only if $\overline{\rho} = 1$. Let $s, t \in M$ be such that $s \,\rho\, t$. Then $c_i s \,\rho\, c_i t$ and, since $c_i s, c_i t \in T$, we have $c_i s \,\overline{\rho}\, c_i t$, whence $c_i s = c_i t$, for all $1 \le i \le n$. Thus $s = t$ and so $\rho = 1$, as required. $\square$

From now on, consider $k \ge 2$.

Lemma 3.5 $\rho_{I^M_{k-1}} \subseteq \rho$.

Proof. It suffices to show that $s \,\rho\, t$, for all $s, t \in I^M_{k-1}$. Let $s, t \in I^M_{k-1}$.

(1) If $s, t \in T$ then $s, t \in I^T_{k-1}$ and so $s \,\overline{\rho}\, t$, whence $s \,\rho\, t$.

(2) If $s \in M \setminus T$ and $t \in T$ then, by Corollary 1.4, there exist $i \in \{0, 1, \ldots, n-1\}$, $j \in \{0, 1\}$ and $u \in T$ such that $s = g^i u h^j$. Since $s \,\mathcal{J}\, u$, we get $g^{n-i} s h^{2-j} = u \in I^T_{k-1}$. As $c_1 \in I^T_{k-1}$, we have $u \,\overline{\rho}\, c_1$, whence $u \,\rho\, c_1$. Then $s = g^i u h^j \,\rho\, g^i c_1 h^j$. On the other hand, since $g^i c_1 h^j \in I^T_{k-1}$ (in fact, $g^i c_1 h^j$ is a constant map), we also have $g^i c_1 h^j \,\overline{\rho}\, t$. Hence $g^i c_1 h^j \,\rho\, t$ and so $s \,\rho\, t$.

(3) Finally, suppose that $s,t\in M\setminus T$. By Corollary 1.4, there exist $i_1,i_2\in\{0,1,\ldots,n-1\}$, $j_1,j_2\in\{0,1\}$ and $u,v\in T$ such that $s=g^{i_1}uh^{j_1}$ and $t=g^{i_2}vh^{j_2}$. Since $s\,\rho_{I^M_{k-1}}\,t$, it follows that $u\,\rho_{I^M_{k-1}}\,g^{n-i_1+i_2}vh^{2-j_1+j_2}$. If $g^{n-i_1+i_2}vh^{2-j_1+j_2}\in M\setminus T$ then, by (2), we have $u\,\rho\,g^{n-i_1+i_2}vh^{2-j_1+j_2}$. On the other hand, if $g^{n-i_1+i_2}vh^{2-j_1+j_2}\in T$ then, by (1), again we have $u\,\rho\,g^{n-i_1+i_2}vh^{2-j_1+j_2}$. Hence $s=g^{i_1}uh^{j_1}\,\rho\,g^{i_2}vh^{j_2}=t$, as required. $\square$

Lemma 3.6 *Let $s,t\in M$ be such that $s\,\rho\,t$. Then $|\operatorname{Im}(s)|\geq k$ if and only if $|\operatorname{Im}(t)|\geq k$.*

Proof. It suffices to show that $|\operatorname{Im}(s)|\geq k$ implies $|\operatorname{Im}(t)|\geq k$. So, suppose that $|\operatorname{Im}(s)|\geq k$.

(1) If $s,t\in T$ then $s\,\overline{\rho}\,t$. Since $s\notin I^T_{k-1}$, we have $s=t$, whence $|\operatorname{Im}(t)|\geq k$.

(2) Next consider $s\in T$ and $t\in M\setminus T$. If $t\in I^M_{k-1}$ then $t\,\rho_{I^M_{k-1}}\,c_1$ and so $t\,\rho\,c_1$, by Lemma 3.5. Hence $s\,\rho\,c_1$. By (1), we obtain $s=c_1$ and this is a contradiction, for c_1 has rank one. Therefore $|\operatorname{Im}(t)|\geq k$.

(3) Finally, if $s\in M\setminus T$, by Corollary 1.4, there exist $i\in\{0,1,\ldots,n-1\}$, $j\in\{0,1\}$ and $u\in T$ such that $s=g^iuh^j$. Then $g^{n-i}sh^{2-j}=u\in T$ and $u\,\rho\,g^{n-i}th^{2-j}$. Since $u\,\mathcal{J}\,s$ and $|\operatorname{Im}(s)|\geq k$, by (1) or (2), we deduce that $|\operatorname{Im}(t)|=|\operatorname{Im}(g^{n-i}th^{2-j})|\geq k$, as required. $\square$

Lemma 3.7 *Let $s\in M$. Then there exists an inverse $s'\in M$ of s such that $s's\in E(T)$.*

Proof. Let $s\in M$. By Corollary 1.4, there exist $i\in\{0,1,\ldots,n-1\}$, $j\in\{0,1\}$ and $u\in T$ such that $s=g^iuh^j$. Let $u'\in T$ be an inverse of u and consider $s'=h^{2-j}u'g^{n-i}$. Then s' is an inverse of s and $s's=h^{2-j}u'g^{n-i}g^iuh^j=h^{2-j}u'uh^j\in E(T)$, as required. $\square$

Lemma 3.8 *Let $t\in M$ and let D be a transversal of* $\operatorname{Ker}(t)$*. Then there exists an inverse $t'\in M$ of t such that* $\operatorname{Im}(t')=D$*.*

Proof. Consider the injective partial transformation ξ defined by $\operatorname{Dom}(\xi)=\operatorname{Im}(t)$ and, for all $x\in\operatorname{Dom}(\xi)$, $(x)\xi$ is the unique element in $D\cap(x)t^{-1}$. Define t' by:

(a) $t'=\xi$, if $T\in\{\mathcal{POI}_n,\mathcal{PO}_n\}$;

(b) $t'=\overline{\xi}$, if $T=\mathcal{O}_n$.

It is a routine matter to show that $t'\in M$ and t' is an inverse of t with image D, as required. $\square$

Lemma 3.9 *Let $s,t\in M$ be such that $s\,\rho\,t$ and $|\operatorname{Im}(s)|\geq k$. Then $s\,\mathcal{H}\,t$.*

Proof. Let s' and t' be inverses of s and t, respectively, such that $s's,t't\in T$, by Lemma 3.7. As $s\,\rho\,t$, then $s'st't\,\rho\,s'tt't=s't\,\rho\,s's$. Since $s's,s'st't\in T$, we have $s's\,\overline{\rho}\,s'st't$. Now, as $|\operatorname{Im}(s's)|=|\operatorname{Im}(s)|\geq k$, it follows that $s's=s'st't$ and so $s=(st')t$. Similarly, as $|\operatorname{Im}(t)|\geq k$, by Lemma 3.6, we also have $t=(ts')s$, whence $s\,\mathcal{L}\,t$.

Next let D be a transversal of $\operatorname{Ker}(t)$. By Lemma 3.8, there exists an inverse t' of t such that $\operatorname{Im}(t')=D$. As $|\operatorname{Im}(t't)|=|\operatorname{Im}(t)|\geq k$ and $t't\,\rho\,t's$, by the argument above, it follows that $t't\,\mathcal{L}\,t's$. Since $t't\,\mathcal{L}\,t\,\mathcal{L}\,s$, we get $t's\,\mathcal{L}\,s$ and so $D=\operatorname{Im}(t')$ contains a transversal of $\operatorname{Ker}(s)$. As s and t are $\mathcal{L}$-related, the transversals of $\operatorname{Ker}(s)$ and $\operatorname{Ker}(t)$ have the same number of elements, whence D must also be a transversal of $\operatorname{Ker}(s)$. We conclude that any transversal of $\operatorname{Ker}(t)$ is a transversal of $\operatorname{Ker}(s)$ and vice-versa. Thus $\operatorname{Ker}(s)=\operatorname{Ker}(t)$ and so $s\,\mathcal{R}\,t$. Therefore $s\,\mathcal{H}\,t$, as required. $\square$

Lemma 3.10 *Let $s, t \in M$ be such that $s \neq t$ and $s \,\mathcal{H}\, t$. Then there exists $z \in T$ such that $|\operatorname{Im}(zs)| = |\operatorname{Im}(zt)| = |\operatorname{Im}(s)| - 1$ and $(zs, zt) \notin \mathcal{L}$.*

Proof. First, notice that, s and t have the same image and the same kernel. Next let D be a transversal of $\operatorname{Ker}(s)$. As $s \neq t$, there exists $i \in D$ such that $is \neq it$. Let τ be the partial identity with domain $D \setminus \{i\}$ and define z by:

(a) $z = \tau$, if $T \in \{\mathcal{POI}_n, \mathcal{PO}_n\}$;

(b) $z = \overline{\tau}$, if $T = \mathcal{O}_n$.

Clearly, $z \in T$ and $|\operatorname{Im}(zs)| = |\operatorname{Im}(s)| - 1 = |\operatorname{Im}(t)| - 1 = |\operatorname{Im}(zt)|$. On the other hand, as $\operatorname{Im}(zs) = \operatorname{Im}(s) \setminus \{is\}$ and $\operatorname{Im}(zt) = \operatorname{Im}(t) \setminus \{it\} = \operatorname{Im}(s) \setminus \{it\}$ and $is \neq it$, we have $\operatorname{Im}(zs) \neq \operatorname{Im}(zt)$ and so $(zs, zt) \notin \mathcal{L}$, as required. $\square$

Finally, we can prove Theorem 3.3.

Proof of Theorem 3.3. Let ρ be a congruence of M. Let $1 \leq k \leq n+1$ be such that $\overline{\rho} = \rho \cap (T \times T) = \rho_{I^T_{k-1}}$. By Lemma 3.4, we have $\rho = 1$, for $k = 1$. Thus we can consider $k \geq 2$. On the other hand, $\rho_{I^M_{k-1}} \subseteq \rho$, by Lemma 3.5, and so if $k = n+1$ the relation ρ is the universal congruence on M. Hence, in what follows, we can also assume $k \leq n$.

Let $s, t \in M$ be such that $s \,\rho\, t$ and $|\operatorname{Im}(s)| > k$. By Lemma 3.9, we have $s \,\mathcal{H}\, t$. Suppose that $s \neq t$ and let $m = |\operatorname{Im}(s)|$. By Lemma 3.10, there exists $z \in T$ such that $|\operatorname{Im}(zs)| = |\operatorname{Im}(zt)| = m-1$ and $(zs, zt) \notin \mathcal{L}$. On the other hand, as $m - 1 \geq k$ and $zs \,\rho\, zt$, by Lemma 3.9, we have $(zs, zt) \in \mathcal{H}$, which is a contradiction. Thus $s = t$.

Now let π be the congruence of H_k induced by ρ, i.e. $\pi = \rho \cap (H_k \times H_k)$. As $\rho_{I^M_{k-1}} \subseteq \rho$, to prove that $\rho = \rho_\pi$ it remains to show that, for all $s, t \in J^M_k$, we have $s \,\rho\, t$ if and only if $s \,\rho_\pi\, t$. Take $s, t \in J^M_k$. First, suppose that $s \,\rho\, t$. By Lemma 3.9, we have $s \,\mathcal{H}\, t$ and so $s \,\pi_{J_k}\, t$. Moreover, $s_L = t_L$ and $s_R = t_R$, whence $\tilde{s} = s_L s s_R \,\rho\, s_L t s_R = t_L t t_R = \tilde{t}$ and so $\tilde{s} \,\pi\, \tilde{t}$. Thus $s \,\rho_\pi\, t$. Conversely, assume that $s \,\rho_\pi\, t$. Then $s \,\mathcal{H}\, t$ and $\tilde{s} \,\pi\, \tilde{t}$. Hence $s_L = t_L$, $s_R = t_R$, $s'_L = t'_L$, $s'_R = t'_R$ and $\tilde{s} \,\rho\, \tilde{t}$. Now consider the inverses $\hat{s}$ and $\hat{t}$ of s and t, respectively. Then $s\hat{s} = t\hat{t}$ and so

$$s = s\hat{s}s'_L(s_L s s_R)s'_R \,\rho\, s\hat{s}s'_L(s_L t s_R)s'_R = t\hat{t}t'_L(t_L t t_R)t'_R = t,$$

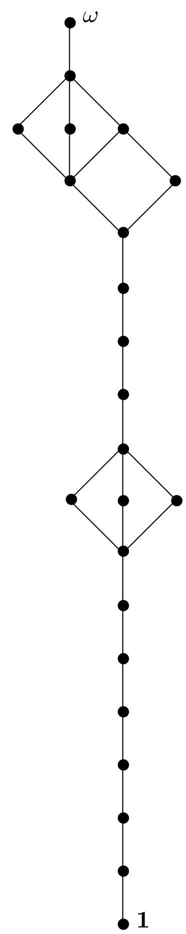
as required. $\square$

Let $k \in \{1, \ldots, n\}$. Let $\pi_1, \pi_2 \in \operatorname{Con}(H_k)$. If $\pi_1 \subset \pi_2$, it is easy to show that $\rho_{\pi_1} \subset \rho_{\pi_2}$. On the other hand, it is clear that given $k_1, k_2 \in \{1, \ldots, n\}$ such that $k_1 < k_2$, $\pi_1 \in \operatorname{Con}(H_{k_1})$ and $\pi_2 \in \operatorname{Con}(H_{k_2})$, we have $\rho_{\pi_1} \subset \rho_{\pi_2}$.

Denote by $\mathcal{D}_k$ the lattice of the congruences of the group H_k, for $1 \leq k \leq n$. By Theorem 3.3, we have the following description of $\operatorname{Con}(M)$.

Theorem 3.11 *The lattice of the congruences of the monoid M is isomorphic to the ordinal sum of lattices $\mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \cdots \oplus \mathcal{D}_n \oplus \mathcal{D}_1$.* $\square$

Example 3.12 Consider the monoid $\mathcal{POR}_6$. Applying the last result, we get the following Hasse diagram for $\operatorname{Con}(\mathcal{POR}_6)$:



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First and Third authors' second address: Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Monte da Caparica, 2829-516 Caparica, Portugal.
Second author's second address: Departamento de Matemática, Faculdade de Ciências, Universidade de Lisboa, 1749-016 Lisboa, Portugal.