APPLICATIONS OF FOURIER METHODS TO THE ANALYSIS OF SOME STOCHASTIC PROCESSES

MANUEL L. ESQUÍVEL

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 42C15, 46F25; Secondary 60G17, 60G20.

Author addresses:

Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Quinta da Torre, 2825 Monte de Caparica, Portugal., C.M.A.F.-U.L. Av. Prof. Gama Pinto, 2, 1699 Lisboa Codex, Portugal

Dedicatory

To Isabel, with all my love

Acknowledgments

... Hilel...solía decir asimismo: si yo no estoy para mí, quíen estará?,
y si yo estoy para mí, qué soy yo? y si ahora no, cuando?
Pirqe Abot I.14, La Misna,
Editora Nacional, Madrid 1981.

The work presented in this thesis was made under the direction of Jean-Pierre Kahane. For all the beautiful and inspiring pages he has been writing all his life on very deep and powerful theorems, for the patience he has always shown on my very modest attempts to produce some Mathematics, for the kind and precise words of correction and encouragement, for the fatherly and benevolent look in his eyes I could always find at the end of each of our encounters, my warmest and everlasting gratitude.

António César de Freitas acted as a co-director of this thesis work. For the encouraging tone of his words and for the enthusiastic energy he is always ready to share, goes my admiration and respect.

Alfredo Pereira Gomes sponsored my entrance as a member of the Center for Mathematics and Fundamental Applications of the University of Lisbon. For the support and consideration permanently shown, my sincere appreciation.

Elvira Coimbra has been head of the Mathematics department for the last five years. The smoothness of his guiding hand, despite the not always favorable circumstances, has been a constant impulse for trying an improvement of my work in the lecturing and managing sides of my activity, as a member of this department.

João Tiago Mexia has been a trustworthy and devoted friend for more than nine years. His unbreakable enthusiasm for mathematics, knowledge and people has been a constant source of strong faith in the accomplishment of our common dreams about Science and Mathematics in Portugal.

I have been blessed with the friendship and respect of too many people so as to be considered a very lucky person but also, so as to be certain of not mentioning someone, if given the opportunity to try to do it. Nevertheless, let me recall Jorge Salazar, António Ornelas, Carlos Braumann, and José Carlos Tiago de Oliveira at Évora, Jaime Carvalho e Silva, Natália Bebiano, Paulo Eduardo de Oliveira, João Filipe Queiró, Graciano Neves de Oliveira, Carlos Duarte Gamas, António Ribeiro Gomes and Eduardo Marques de Sá at Coimbra, José Francisco Rodrigues, João Paulo Dias, Teresa Monteiro Fernandes, Luís Saraiva, Isabel Simão, António Bívar, Manuel Monteiro Marques, José Sousa Ramos and Anabela Cruzeiro at Lisbon, Fernando Bastos, Luís e Rita Monteiro, Maria de Fátima Martins, Rui Rodrigues, João de Deus Marques, Teresa Pecorelli, Vítor Hugo Fernandes, José Paulo Lampreia, Alice Silva, Bento Louro, Ana de Sá, Pedro Côrte-Real, João Pedro Cabral, Carlos Dias and Manuel Costa Leite at Monte de Caparica, Ely Merzbach and Ken Hochberg at Tel Aviv, Robert Adler and Michael Stiasnie at Haifa, Serge Dubuc at Montréal, Christian Houdré at Atlanta, Victor Shira at Moscow, Guido Weiss at St. Louis, Elias Saab and Nigel Kalton at Columbia, Kazimierz Jesuita at Warsaw, Tomasz Mazur at Radom, Ryszard Szwarc at Wroclaw, Victoria Collado at Granada, Catherine Leduc, Dominique Cellier, Etienne Ménard, Yves Suprin, Mahmoud Haurani, Sinh and Pascale Nguyen Luong, Thierry Didier, Laurent Decoux, Michel Giersh and Cristophe in Rouen, Maklouff Derridj, José de Sam Lazaro, Jean-Pierre Raoult and Fan Hi Hua in Paris, Eckart Platten and David Heath in Canberra, Andy Obaldestein in Loughborough, Angelos Yokaris in Athens, Yoshio Miyahara in Nagoya and Wolfgang Runggaldier in Padova.

To Francisca and João Esquível my eternal remembrance for all the love, intelligence, and understanding. To Mercedes and Lopo Leote Tavares the souvenir of the tenderness of the early ages. To Maria das Mercedes Leote Tavares, my love and thanks for the stubbornness and thirst for life. To Manuel Esquível my love and thanks for having taught me so many things and for showing me most of the good truths in which I still believe. To Adolfo and Manuela Cabral and to Virgínia Brito, my gratitude for having given me a family, these last four years.

Francisca, Joaquim e Vitória, three solid anchors whenever navigating the rough seas of conflicts and ignorance of each day, are also the three most relevant, but yet unproven, theorems of my life. If there was no other reason to live for, a smile of each one of them would give me enough justification for every effort and pain.

Isabel you are the sunshine of my life, the bread and salt of each new discovery, the joy of every moment, the companion for all fights. I lay here these only simple words of thanks as I don't feel able to express what I owe to you.

Financial helps

The work for this dissertation took great benefit from various sources of financial and other material help. It is a pleasure, more than a duty, to recall them at this moment.

- Until 1981, my father helped me, much beyond his natural obligations. I am most grateful to him for the freedom he gave me to choose a career path in accordance with my deepest way of feeling.
- A French government scholarship was essential in the process of overcoming some of my limitations of knowledge and know-how, even, if it did produce only a tiny exterior sign of academic accomplishment.
- The Mathematics Department of the Faculty of Science and Technology of the New University of Lisbon sponsored, partially, some of the participations in scientific gatherings. These participations helped us to have a more equilibrated perspective on some aspects of our work and gave us a lot of enthusiasm to continue.
- The Center of Mathematics and Fundamental Applications of the University of Lisbon has complemented the help of the Mathematics Department and, at some instances, has even replaced it completely.
- The library of the Mathematics Department of the University of Paris XI gave us the opportunity of collecting most of the references we needed.
- The invitation of Bar-Ilan University and the French government, for participating in the symposium on the brownian sheet, was a turning point in the development of our work.
- The sponsorship of Columbia University allowed for the concentration from which stemmed the final version of the second paper on the local behavior of the brownian sheet.
- In 1996, the invitations of the University of Radom, in Poland, to present some of our most recent work and of the Australian National University, in Canberra, for

a participation in a workshop in quantitative methods in finance, on the basis of published work which is not contemplated in this dissertation, greatly contributed to produce self-confidence. This state of mind helped in the overcoming of some obstacles to the conclusion of this work, which remained until the very last moments.

Summary

In the first chapter, a class of random periodic Schwartz distributions is introduced, some examples, elementary properties and a characterization result are studied and, three applications are presented. A random Schwartz periodic distribution is, for us, just a function defined in a complete probability space and taking values in the space of Schwartz distributions over the line, that are left invariant by an integer translation, endowed with the natural algebraic and topological structures.

The second chapter deals, primarily, with an extension of the methods of Kahane, as applied to the brownian sheet, in what concerns analogs of the rapid points. After presenting the brownian sheet process, by way of gaussian white noise, some results, on the local behavior of this process and for some other processes associated with the sheet, are derived using the Schauder series representation.

In the third chapter, we prove a formula essentially due to Frostman, we look at the behavior at infinity of the Fourier transform of some remarkable functions and measures and, finally, we study the asymptotic behavior of the second moment of the Fourier transform of a random measure that appears in the theory of multiplicative chaos.

In the last chapter, a class of random tempered distributions on the line is introduced by considering random series, in the usual Hermite functions, having as coefficients random variables which satisfy certain growth conditions. This class, is shown to be exactly the class of random Schwartz distributions having a mean. We present also a study, on a possible converse of a result on brownian distributions, that leads to a moment problem.

Preface

... In the field of mathematics in general, it is mostly
by reading the works of other scholars that one comes
upon ideas for one's independent research.
S. Kovalevskaya
in The mathematics of Sonya Kovalevskaya by Roger Cooke,
Springer Verlag, New York 1984.

...pour que le disciple puisse donner sa mesure
il doit s'arracher à l'emprise de son Maître.
Elie Wiesel
in Célébration hassidique by Elie Wiesel, Seuil, Paris 1972.

This dissertation has four chapters, each one dealing with particular instances of the idea expressed in the title.

In the first chapter, we study the class of random *periodic* distributions having a first moment, using the Fourier series representation for such objects. This is done, after showing that this representation holds in a reasonable sense for the class of random distributions under consideration. In the fourth chapter, we present a study similar to the one presented in chapter one, but this time, using the Hermite series representation for random *tempered* distributions. This first and fourth chapters contain material related to paragraph I.3, of the plan of work for the thesis studies, which reads:

Des questions un peu plus abstraites se posent quand nous prennons comme point de départ, pour l'étude des processus gaussiens généralisés, le cadre que l'on peut abstraire de l'exposé sur les distributions browniennes fait par J.-P. Kahane...

The second chapter, deals with the study of some local properties of the brownian sheet using the Schauder series representation of this process. The problem considered is stated, in the plan of work referred above, in the following form.

L'existence de versions continues pour certains processus gaussiens stationnaires, sur la droite ou le tore, permet l'explicitation de comportements locaux différents et une classification des points de chaque trajectoire en points rapides, ordinaires et lents. Peut-on obtenir une classification semblable pour des processus gaussiens plus généraux?

Finally, in the third chapter, we study the asymptotic behavior of the second moment of a random measure using, for that purpose, some remarks on the asymptotic behavior of functions and measures in euclidean space. The problem addressed is stated, again, in the plan of work, as quoted below.

Dans le cadre um peu plus large, où l'on considère le comportement à l'infini des transformés de Fourier des mesures aléatoires qui apparaissent dans l'étude des multiplications aléatoires itérées, on a des espoirs, basés sur des résultats positifs dans quelques cas particuliers.

The text presented in the following has, as its main sources and references, the publications that we now indicate in some detail. The first chapter is a rearrangement of the articles *Sur une classe de distributions aléatoires périodiques* and *A characterization of the class of random periodic distributions having a mean* ([14] and [15] respectively) [1]. The idea, to study the harmonic analysis of random Fourier Schwartz series, came from [31]. In this text, some properties of such random series are evoked in the context of random point masses on the circle.

The fourth chapter, is a faithful reproduction of the article *On the space of random tempered distributions having a mean* [18]. It is a natural complement to the first chapter, as an extension to the random tempered distributions' context of the harmonic analysis already done for random periodic distributions. The first reference to the crucial example presented is, of course, [30].

The main part of the second chapter is a fusion of the articles *On the local behavior of the brownian sheet* and *Points of rapid oscillation for the brownian sheet via Fourier Schauder series representation*([16] and [17] respectively). ,The presentations of the brownian sheet in the introduction as well as the remark on the tensorial brownian process in the appendix, were written for this dissertation. Our aim, not yet attained, is to mimic, for the brownian sheet, the study of the local behavior of the Wiener process as done in [31].

Finally, the third chapter is taken from the article *The asymptotic behavior of the second moment of a random measure* [19]. The main root of this work is the text in [32].

[1]The numbers in brackets, point to bibliographic references completely identified in the Bibliography.

Contents

Dedicatory **iii**

Acknowledgments **v**

Financial helps **vii**

Summary **ix**

Preface **xi**

Chapter I. Amenable periodic distributions **1**

1. Summary **1**

2. Introduction **2**

2.1. Notation 2

2.2. A first characterization of the class of distributions under study 3

3. Examples of random periodic Distributions **4**

3.1. Random point masses on the circle 4

3.2. The brownian processes 5

3.3. Classical Schwartz distributions randomized by translation 7

4. Properties **7**

4.1. The uniqueness of the Fourier Schwartz representation 7

4.2. The derivative of a random periodic distribution 8

4.3. The mean of a random periodic distribution 9

5. Applications **13**

5.1. The asymptotic behavior of the Fourier coefficients of the brownian processes. 15

5.2. Particular solutions for a generalized equation. 17

6. Appendix **18**

Chapter II. Local behavior of the brownian sheet **21**

1. Summary **21**

2. Introduction **22**

3. On the local behavior **29**

3.1. Modulus of continuity 29

3.2. The location homogeneous part 31

3.3. The usual increments 34

3.4. Rapid points for the independent increments 36

4. Appendix **41**

4.1. Estimations for gaussian and pseudogaussian random variables 41

4.2. A nongaussian example of a pseudogaussian random variable 44

4.3. A brownian tensorial product process 46

Chapter III. The asymptotic behavior of a Fourier transform **49**

1. Summary **49**

2. Introduction **49**

2.1. Asymptotic behavior and uniform continuity 50

3. The asymptotic behavior of some classic functions **53**

4. Some Parseval formulas and tempered distributions **57**

5. A random measure obtained from multiplicative chaos **59**

5.1. Asymptotic behavior of the second moment 60

Chapter IV. Amenable random Schwartz distributions **65**

1. Summary **65**

2. Introduction **65**

3. The Hermite Functions **66**

3.1. Laurent Schwartz's fundamental result 66

3.2. The generating function for the square of Hermite functions 69

4. Random Schwartz tempered distributions having a mean **69**

4.1. Introduction 69

4.2. A characterization 73

5. Two examples **77**

5.1. A brownian type process on $\mathbb{R}$ 77

5.2. The brownian distributions 82

6. Some remarks on the brownian distribution example **87**

Bibliography **95**

Table of Notations **99**

Index **101**

CHAPTER I

On the class of random periodic distributions having a mean

1. Summary

A class of random periodic Schwartz distributions is introduced, some examples and elementary properties are studied and three applications are presented.

In order to define a random objet we follow [31, p. 9]. A random Schwartz periodic distribution is just a function defined in Ω (where $(\Omega, \mathcal{A}, \mathbb{P})$ is a complete probability space) and taking values in $\mathcal{D}'(\mathbb{R}/\mathbb{Z})$. This last space, can be taken as the set of Schwartz distributions over $\mathbb{R}$ which are left invariant by an integer translation, endowed with the natural algebraic and topological structures. $\mathcal{D}'(\mathbb{R}/\mathbb{Z})$ is isomorphic to the space of sequences (indexed by $\mathbb{Z}$) of slow growth at infinity ([55, p. 225], [65, II, p. 69]).

For a random distribution T, subjected to some measurability condition, there exists two random variables A and K, defined in Ω and taking values in $\mathbb{R}_+$ and $\mathbb{R}$ respectively, and such that if, for all ω in Ω and n in $\mathbb{Z}$, we denote by $< T(\omega), e^{2\pi inx} >$ the Fourier coefficient at the frequency n, of the Schwartz distribution $T(\omega)$, we have:

$$| < T(\omega), e^{2\pi inx} > | \quad \leq \quad A(\omega)\ (1 + |n|)^{K(\omega)}\ .$$

The class of random Schwartz distributions under study is obtained by taking A in $L^1(\Omega)$ and K having an upper bound. These conditions are equivalent to the one obtained by supposing that the sequence $(\mathbb{E}[| < T(\omega), e^{2\pi inx} > |])_{n\in\mathbb{Z}}$ (where $\mathbb{E}$ denotes the expectation in $(\Omega, \mathcal{A}, \mathbb{P})$) is of slow growth at infinity. This equivalence is proved in theorem 2.1.

In the second section, we show that the class introduced contains, as examples, some classic random objects such as the "random point-masses on the circle" ([31, p. 109]) and the brownian motions (classical of Wiener and Lévy and fractional of Kolmogoroff and Mandelbrot). In order to verify the assertion just made, on the brownian motions, we prove that a random process, with trajectories in a Banach space, which is continuously embedded in the

Lebesgue space $L^1([0,1])$, may be used to define, by periodic repetition, a random periodic distribution with Fourier coefficients in the class under study. As a third kind of example, we show that if we randomize by translation a classical Schwartz periodic distribution we obtain, also, an element of the class.

In the third section, we present some simple results on the harmonic analysis, a result on the statistics of the random periodic distributions introduced and a characterization of the class under study. Namely, the uniqueness of the the representation by Fourier series, derivation, expected value, with the correspondent representations by Fourier series and, the crucial result that shows that the class of random distributions introduced is, exactly, the class of random periodic distributions having a mean. For illustration, we compute the Fourier series of the expected value of a classical Schwartz distribution randomized by translation.

There are three applications in the fourth section. The first one is an extension, to the class introduced, of the Fourier method for building particular solutions for ordinary differential equations with constant coefficients. The second application, deals with the computation of the Fourier coefficients of the fractional brownian motions using a stochastic integral with respect to the white noise gaussian Fourier series. Finally, using this last result, we build a particular solution for a generalized Langevin equation in which the second member is a fractional brownian motion.

In the appendix, we prove that the following condition is sufficient for a random distribution T to belong to the class under scrutiny: "for some p in the interval $]1,+\infty[$, the sequence of p-moments $((\mathbb{E}[| < T(\omega), e^{2\pi inx} > |^p])^{1/p})_{n\in\mathbb{Z}}$ is of slow growth". This result is, of course, weaker than the result stated in theorem 2.1 but the proof is quite different and could be of independent interest. In order to control the growth of the random variables $< T, e^{2\pi inx} >$ we use a natural random variable and the proof that this random variable is integrable is the main issue. To that end, an associated stopping time is of crucial importance.

2. Introduction

2.1. Notation. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space which we will suppose to be a complete probability space. Let us denote by $\mathcal{M}$, the space of random variables defined in Ω and taking values in $\mathbb{C}$ and by $\mathcal{M}^{\mathbb{Z}}$, the space of sequences of elements of $\mathcal{M}$, indexed by the integers $\mathbb{Z}$. For an element A, of $\mathcal{M}$, we will denote:

$$\mathbb{E}[|A|] = \int_\Omega |A| d\mathbb{P}\,.$$

Let $\mathcal{C}_m$ be the subspace of $\mathcal{M}^{\mathbb{Z}}$ defined by:

$$\mathcal{C}_m = \{(C_n)_{n\in\mathbb{Z}} \in \mathcal{M}^{\mathbb{Z}} : \quad (\exists A \in \mathcal{M}\,,\, A \geq 0\,,\, \mathbb{E}[A] < +\infty)\,(\exists k \in \mathbb{Z})$$
$$(\forall n \in \mathbb{Z})\; |C_n| \leq A(1+|n|)^k \text{ a.s. on } \Omega\} \quad . \tag{1}$$

2.2. A first characterization of the class of distributions under study. To begin with, we will look at two conditions which describe completely this space, the last one being a condition which can be easily verified. Next, a random periodic distribution T, will be associated to an arbitrary element $(C_n)_{n\in\mathbb{Z}}$ of $\mathcal{C}_m$, by taking the C_n, for n in $\mathbb{Z}$, as the Fourier coefficients of T.

THEOREM 2.1. *Consider $(C_n)_{n\in\mathbb{Z}}$, the following conditions are equivalent :*

(1) $(C_n)_{n\in\mathbb{Z}} \in \mathcal{C}_m$.

(2) *There exists an integrable random variable A and another one K bounded, defined on Ω, such that:*

$$(\forall n \in \mathbb{Z})\;\; |C_n| \leq A\,(1+|n|)^K \;\; a.s. \;\; on\; \Omega\,.$$

(3) *The sequence $(\mathbb{E}[|C_n|])_{n\in\mathbb{Z}}$is a sequence of slow growth, or, otherwise stated:*

$$(\exists a \geq 0)\;(\exists k \in \mathbb{Z})\;(\forall n \in \mathbb{Z})\;\; \mathbb{E}[|C_n|] \leq a\,(1+|n|)^k\;.$$

PROOF. Proposition (2) is a trivial consequence of (1), by taking for K, a constant random variable. The conditions taken on A and K in proposition (2), allow us to write:

$$(\forall n \in \mathbb{Z})\;\; \mathbb{E}[|C_n|] \leq \mathbb{E}[A]\,(1+|n|)^k\,,$$

where k denotes an upper bound of the random variable K. This shows that (3) follows from (2). The fact that (3) $\Rightarrow$ (1) is obtained as follows. Let (3) be valid and take $\alpha > 1$. We have then:

$$(\forall n \in \mathbb{Z})\;\; \mathbb{E}[\frac{|C_n|}{(1+|n|)^{k+\alpha}}] \leq \frac{a}{(1+|n|)^{\alpha}}\,.$$

As a consequence we get:

$$\mathbb{E}[\sum_{n\in\mathbb{Z}} \frac{|C_n|}{(1+|n|)^{k+\alpha}}] < +\infty\,.$$

The random variable

$$A = \sum_{n\in\mathbb{Z}} \frac{|C_n|}{(1+|n|)^{k+\alpha}}\,,$$

is then almost surely real and integrable and

$$(\forall n \in \mathbb{Z})\;\; \frac{|C_n|}{(1+|n|)^{k+\alpha}} \leq A \text{ a.s. on } \Omega\,,$$

from which the desired results follows. □

REMARK 1. In the appendix, we present the proof of a result which is weaker then (3) $\Rightarrow$ (1). In this proof, the random variable A is built by using a classical stopping time.

The main difficulty with this approach is to prove that A is integrable. We believe that the technique used has an independent interest.

It is possible now, to associate, to an arbitrary element $(C_n)_{n\in\mathbb{Z}}$ of the space $\mathcal{C}_m$, a random periodic distribution having period 1, ([31, p. 9]) that is to say, an element of $(\mathcal{D}'(\mathbb{R}/\mathbb{Z}))^\Omega$, defined almost surely on Ω by:

$$T(\omega) = \sum_{n\in\mathbb{Z}} C_n(\omega) e^{2\pi inx} .$$

Conversely, if T belongs to $(\mathcal{D}'(\mathbb{R}/\mathbb{Z}))^\Omega$ and has as Fourier-Schwartz coefficients:

$$(\forall n \in \mathbb{Z})\ (\forall \omega \in \Omega)\ \ C_n^T(\omega) =< T(\omega), e^{-2\pi inx} > ,$$

and if the sequence $(C_n)_{n\in\mathbb{Z}}$ is in the space $\mathcal{C}_m$, then, the random periodic distribution:

$$V = \sum_{n\in\mathbb{Z}} C_n^T e^{2\pi inx} .$$

coincides with T, almost surely on Ω.

The class of random periodic distributions, which is the object of the study presented in the following, is thus defined by the conditions stated in Theorem 2.1, conditions applied to the sequence of Fourier-Schwartz coefficients of the random distribution.

3. Examples of random periodic Distributions

Point masses on the circle, (classical and fractional) brownian processes and classical Schwartz distributions randomized by translation.

3.1. Random point masses on the circle. We consider $(\epsilon_j)_{j\in\mathbb{N}^*}$ a Rademacher standard sequence, that is, a sequence of independent random variables defined on Ω and taking only the values $+1$ or -1 with the same probability. We consider also $(\theta_j)_{j\in\mathbb{N}^*}$, a sequence of independent random variables defined on Ω, taking values in $[0,1]$ and, having all the uniform distribution. We suppose that the distribution of the first sequence is independent of the distribution of the second one. Finally, we consider $(m_j)_{j\in\mathbb{N}^*}$ a sequence of nonnegative numbers, the masses, such that:

$$\sum_{j=1}^{\infty} m_j^2 < +\infty .$$

Let now $T \in \mathcal{D}'(\mathbb{R}/\mathbb{Z})$, be the random periodic distribution defined by:

$$(\forall \omega \in \Omega)\ \ T(\omega) = \lim_{N\to+\infty} \sum_{j=1}^{N} \epsilon_j(\omega) m_j \delta_{\theta_j(\omega)} .$$

We have that:

$$(\forall n \in \mathbb{Z}) \;\; C_n^T = \lim_{N \to +\infty} \sum_{j=1}^{N} \epsilon_j m_j e^{-2\pi i n \theta_j} \text{ a.s. on } \Omega\,.$$

Each one of the C_n^T is a random variable. In fact, it is given as a simple limit, defined almost surely on Ω, of a sequence of random variables. The sequence C_n^T is in the class $\mathcal{C}_m$, as a result of an application of theorem 2.1 to the following fact:

$$(\forall n \in \mathbb{Z}) \;\; \mathbb{E}[|C_n^T|] \leq (\mathbb{E}[|C_n^T|^2])^{1/2} \leq (\sum_{j=1}^{\infty} m_j^2)^{1/2} < +\infty\,.$$

3.2. The brownian processes. We start by establishing a theorem that gives a condition ensuring that a random distribution with *function trajectories* has the correspondent Fourier coefficients in the space $\mathcal{C}_m$. Consider B a separable Banach space, subspace of $L^1([0,1], \mathcal{B}, dx)$, such that the natural injection is a continuous map. Let T be defined on Ω and taking its values in B, be a vectorial random variable. That means that, $\mathcal{E}$ being the norm topology on B, we have $T^{-1}(\mathcal{E}) \subset \mathcal{A}$ where, we recall, $\mathcal{A}$ is the given σ-algebra on Ω (see [31, p. 3] or [20, p. 14]).

THEOREM 3.1.

$$\int_{\Omega} ||T(\omega)||_B d\mathbb{P}(\omega) < +\infty \;\;\Rightarrow\;\; (C_n^T)_{n \in \mathbb{Z}} \in \mathcal{C}_m\,.$$

PROOF. B being a Polish space is a standard space ([20, p. 15, I.42]) and, as a consequence, T is given as a simple limit, in B, of a sequence of simple functions, otherwise stated: T is measurable or strongly measurable. Being so, T is also weakly $\mathbb{P}$ measurable ([9, p. 41]) and, as by the hypothesis made $L^\infty \subset B^*$, we have that: for each $n \in \mathbb{Z}$, $C_n^T =< T, e^{-2\pi i n x} >$ is a numeric random variable. As a consequence of the conditions taken on T, we have that T is a Bochner integrable random variable.The following easy estimate:

$$(\forall n \in \mathbb{Z}) \;\; \mathbb{E}[|C_n^T|] = \int_{\Omega} | \int_0^1 T(\omega) e^{-2\pi i n x} dx \,| d\mathbb{P}(\omega) \leq$$

$$\leq \int_{\Omega} ||T(\omega)||_{L^1([0,1])} d\mathbb{P}(\omega) \leq m \int_{\Omega} ||T(\omega)||_B d\mathbb{P}(\omega) < +\infty\,, \tag{2}$$

with m being a constant, shows that the conclusion of the theorem holds. □

We will see now that the preceding theorem can be applied to the brownian processes. We will have then proved that these brownian processes define random distributions, in the class under study, with no calculation of the correspondent Fourier coefficients, done, with that purpose.

Observe, incidentally, that we will establish an asymptotic formula for these Fourier coefficients in subsection 5.1, as a preliminary remark on the construction of a particular solution for a generalized Langevin equation.

Consider the generalization of the Wiener process, given by a gaussian process, $X^\gamma = (X^\gamma_t)_{t\in[0,1]}$, taking complex values and, verifying:

$$(\forall s,t \in [0,1])\ \ \mathbb{E}[|X^\gamma_t - X^\gamma_s|^2] = |t-s|^\gamma\,. \tag{3}$$

The construction of such processes is known for $\gamma \in]0,2]$(see [31, p. 136, 263, 279]). For $\gamma = 1$, it is the Wiener process or the classical brownian process and for $\gamma \neq 1$, these process are the fractional brownian processes . The random distribution X^γ defined on Ω by the map that associates to each $\omega \in \Omega$ the (continuous) trajectory $(X^\gamma_t(\omega))_{t\in[0,1]}$, has its Fourier coefficients in the class $\mathcal{C}_m$. This follows as a consequence of the next result.

THEOREM 3.2. *The Banach space* $B = C^0([0,1]) \cong C^0(\mathbb{R}/\mathbb{Z})$*, with the norm given by:*

$$(\forall f \in B)\ \ ||f||_B = \sup_{t\in[0,1]} |f(t)|\,,$$

verifies the hypothesis made for theorem 3.1 *and:*

$$\int_\Omega ||X^\gamma_t||_B \, d\mathbb{P} < +\infty\ .$$

PROOF. We will apply Dudley theorem on the existence of continuous versions for gaussian processes. (See [31, p. 219]). On $[0,1]$ we introduce the semimetric given by:

$$d(s,t) = \mathbb{E}[|X^\gamma_s - X^\gamma_t|^2])^{\frac{1}{2}} = |t-s|^{\frac{\gamma}{2}}\ .$$

Denote by $N_d(\epsilon)$, the minimum number of balls having radius ϵ, for the semimetric d, which are needed to cover $[0,1]$. Following Dudley,

$$\int_\Omega ||X^\gamma_t||_B \, d\mathbb{P} \le cte\, (\int_0^{+\infty} \sqrt{\log(N_d(\epsilon))}\, d\epsilon + \inf_{t\in[0,1]} \mathbb{E}[|X^\gamma_t|^2]^{\frac{1}{2}})\,.$$

In order to conclude, it is enough to prove that the entropy integral (in the right-hand side of the inequality) is convergent. Observe that, the convergence issue for this integral is in the inferior limit of integration, namely, on zero. The property which defines d, that is (3), has as a consequence that, for $t \in [0,1]$, the ball centered at t and with ϵ as radius for the semimetric d, coincides, with the ball centered at t with $\epsilon^{2/\gamma}$ as radius for the usual distance on $[0,1]$. We have then to study, for some constant $c > 1$, the integral:

$$\int_0^c \sqrt{\log(N_d(\epsilon))}\, d\epsilon \le \int_0^c \sqrt{\log(\frac{1}{2\epsilon^{2/\gamma}})}\, d\epsilon\,.$$

The change of variable, given by $2\epsilon^{2/\gamma} = e^{-x}$, leads to the study of an integral which converges as soon as the integral:

$$\int_{c'}^{+\infty} \sqrt{x}\, e^{-\frac{\gamma}{2}x} dx\,,$$

converges. This happens as long as $\gamma > 0$. □

3.3. Classical Schwartz distributions randomized by translation. Let F be a random variable defined on Ω and taking values in $[0,1]$. Consider f, a Schwartz periodic distribution having 1 as period (this will be denoted by $f \in \mathcal{D}^{'}(\mathbb{R}/\mathbb{Z})$). Define the random distribution $T \in (\mathcal{D}^{'}(\mathbb{R}/\mathbb{Z}))^{\Omega}$ by:

$$(\forall \omega \in \Omega) \;\; T(\omega) = \tau_{F(\omega)} f \, ,$$

where τ_a, for $a \in [0,1]$, is the classic translation operator. We then have $(C_n^T)_{n\in\mathbb{Z}} \in \mathcal{C}_m$. In fact,

$$\begin{aligned} C_n^T = \; & < \tau_F f, e^{-2\pi i n x} > = < f, \tau_{-F} e^{-2\pi i n x} > = < f, e^{-2\pi i n(x+F)} > \\ & = e^{-2\pi i n F} < f, e^{-2\pi i n x} > = e^{-2\pi i n F} \hat{f}(n) \, . \end{aligned} \tag{4}$$

As $f \in \mathcal{D}^{'}$, the sequence $(\hat{f}(n))_{n\in\mathbb{Z}}$, of the correspondent Fourier coefficients, is of slow growth. As for n in $\mathbb{Z}$, C_n^T differs from $\hat{f}(n)$ by a factor of modulus 1, the announced result is thus verified. The next section will present an interesting statistical property of these random distributions.

4. Properties

Uniqueness of the Fourier series representation for a random periodic distribution. Derivation and harmonic analysis of the derivative. The mean of a random periodic distribution.

4.1. The uniqueness of the Fourier Schwartz representation. The results presented next, which are essential for the applications, are given an easy formulation in the context of the random distributions whose coefficients belong to the class $\mathcal{C}_m$.

THEOREM 4.1. *Let T be a random distribution which has its Fourier coefficients in the space $\mathcal{C}_m$. The following propositions are equivalent:*

(1) $T = 0_{\mathcal{D}^{'}(\mathbb{R}/\mathbb{Z})}$ *a.s. on* Ω.
(2) $(\forall n \in \mathbb{Z}) \;\; C_n^T = 0$ *a.s. on* Ω.

PROOF. Suppose that (2) is verified. For each $n \in \mathbb{Z}$, consider Ω_n a measurable subset of Ω such that $\mathbb{P}[\Omega_n] = 1$ and $C_n^T = 0$ on Ω_n. The set $\Omega^* = \bigcap_{n\in\mathbb{Z}} \Omega_n$ is measurable and verifies $\mathbb{P}[\Omega^*] = 1$. Naturally we have:

$$(\forall n \in \mathbb{Z}) \; (\forall \omega \in \Omega^*) \;\; C_n^T(\omega) = 0 \, .$$

Besides, at least on Ω^*, we have that:

$$(\forall \varphi \in C^{\infty}(\mathbb{R}/\mathbb{Z})) \quad < T, \varphi > = \sum_{n\in\mathbb{Z}} C_n^T \hat{\varphi}(-n) = 0 \, .$$

Thus showing that T is, almost surely, equal to zero on Ω. Suppose now that (1) is verified. We will then have that, on the set $\Omega_1 \subset \Omega$, which we can take measurable and verifying $\mathbb{P}[\Omega_1] = 1$:

$$(\forall n \in \mathbb{Z}) \;\; C_n^T = < T, e^{-2\pi i n x} > = 0 \,,$$

This can be read as saying that the Fourier coefficients are, random variables, almost surely equal to zero. □

Corollary 4.2. *For T and V, random periodic distributions having, $(C_n^T)_{n\in\mathbb{Z}}$ and $(C_n^V)_{n\in\mathbb{Z}}$ respectively, the sequences of the correspondent Fourier coefficients in the class $\mathcal{C}_m$, we have the following equivalent propositions.*

(1) $T = V$ *a.s. on* Ω .
(2) $(\forall n \in \mathbb{Z}) \;\; C_n^T = C_n^V$ *a.s. on* Ω .

Proof. It is enough to check that $T - V \in (\mathcal{D}'(\mathbb{R}/\mathbb{Z}))^\Omega$ and that

$$(\forall n \in \mathbb{Z}) \;\; C_n^{T-V} = C_n^T - C_n^V \text{ a.s. on } \Omega \,,$$

which is obviously verified. □

4.2. The derivative of a random periodic distribution. Given a random distributions $T \in (\mathcal{D}'(\mathbb{R}/\mathbb{Z}))^\Omega$, is is natural to define T', the derivative of T, as the random distribution that, for each $\omega \in \Omega$, associates the derivative, in the usual sense of the derivative of a distribution, of $T(\omega)$. More precisely, we have the following definition.

Definition 4.1. The *derivative* of a random distribution T is, by definition, the random distribution T', given by:

$$(\forall \omega \in \Omega)(\forall \varphi \in C^\infty(\mathbb{R}/\mathbb{Z}) \quad < T'(\omega), \varphi > = - < T(\omega), \varphi' > \;.$$

The following easy theorem shows that the class of random distributions under study is stable by derivation.

Theorem 4.3. *Let $T \in (\mathcal{D}'(\mathbb{R}/\mathbb{Z}))^\Omega$ having the corresponding Fourier coefficients in the space $\mathcal{C}_m$. Then T', the derivative of T, has as Fourier coefficients the sequence defined by:*

$$(\forall n \in \mathbb{Z}) \;\; C_n^{T'} = 2\pi i n C_n^T \,,$$

this sequence being also in the space $\mathcal{C}_m$.

Proof. This result is an immediate consequence of the definitions. In fact:

$$(\forall n \in \mathbb{Z}) \;\; C_n^{T'} = < T', e^{-2\pi i n x} > = - < T, (e^{-2\pi i n x})' > = 2\pi i n C_n^T \,.$$

In order to show the last statement made in the theorem, consider A, a non negative integrable random variable, such that for k, an integer, we have

$$(\forall n \in \mathbb{Z}) \;\; |C_n^T| \leq A(1+|n|)^k \text{ a.s. on } \Omega \,.$$

Observing that:

$$(\forall n \in \mathbb{Z}) \;\; |C_n^{T'}| \leq (2\pi A)(1+|n|)^{(k+1)} \text{ a.s. on } \Omega \,,$$

the result announced is seen to be true. □

4.3. The mean of a random periodic distribution. The study of the first moments of an usual random variable is useful is most applications. In the context of periodic random distributions under study one can always define a first generalized moment, that is, a mean. The following definition is similar to the one that is written, for instance, in the classic reference [23].

Definition 4.2. Let T be a periodic random distribution. T, is said to admit the classic Schwartz distribution $\overline{T} \in \mathcal{D}'(\mathbb{R}/\mathbb{Z})$ as a *mean*, if and only if:

(1) $(\forall \varphi \in C^\infty(\mathbb{R}/\mathbb{Z})) \quad < T, \varphi > \;\in\; \mathcal{M} \cap L^1(\Omega)\,.$
(2) $(\forall \varphi \in C^\infty(\mathbb{R}/\mathbb{Z})) \quad \mathbb{E}[< T, \varphi >] = < \overline{T}, \varphi > \;.$

A random distribution having a mean will be called *Amenable*[1].

Theorem 4.4 (On the mean of random distributions). *Consider T, a random periodic distribution, such that the corresponding sequence of Fourier coefficients is in the class $\mathcal{C}_m$. Then, T admits*

$$\overline{T} = \sum_{n\in\mathbb{Z}} \mathbb{E}[C_n^T] e^{2\pi i n x} \,,$$

as a mean.

Proof. From the definition, it is clear that if T admits $\overline{T}$ as a mean, then, this mean is unique. As a consequence, we have only to verify that $\overline{T}$, given in the statement of the theorem, is a mean for T . The sequence $(\mathbb{E}[C_n^T])_{n\in\mathbb{Z}}$ is a sequence of slow growth, as a result of the hypothesis taken and, thus, defines a random periodic Schwartz distribution. Let $\varphi \in C^\infty(\mathbb{R}/\mathbb{Z})$ be an arbitrary test function. From Parseval formula , we get that:

$$< T, \varphi > = \sum_{n\in\mathbb{Z}} C_n^T \hat{\varphi}(-n) \text{ a.s. on } \Omega \,.$$

[1]Amenable: capable of, or agreeable to, being tested, tried, analyzed, etc.

As this series converges, absolutely and almost surely on Ω, we have that condition (1), of the definition, is verified and, moreover, that:

$$\mathbb{E}[< T, \varphi >] = \sum_{n\in\mathbb{Z}} \mathbb{E}[C_n^T]\hat{\varphi}(-n) =< \sum_{n\in\mathbb{Z}} \mathbb{E}[C_n^T]e^{2\pi inx}, \varphi > ,$$

as wanted. □

We next show that the property of having a mean characterizes the class i.e., the random periodic Schwartz distributions having an expected value are, exactly, those in the class introduced.

THEOREM 4.5 (A CHARACTERIZATION OF AMENABLE RANDOM DISTRIBUTIONS). *Let T be a random measurable periodic Schwartz distribution, such that :*

$$\forall \varphi \in \mathcal{C}^\infty(\Pi) \quad < T, \varphi >\in L^1(\Omega) .$$

Then T, is in the class $\mathcal{C}_m$.

PROOF. Let us show first, using the closed graph theorem, that the map Λ_T, defined for every $\varphi \in \mathcal{C}^\infty(\Pi)$ by $\Lambda_T(\varphi) =< T, \varphi >$, is continuous from $\mathcal{C}^\infty(\Pi)$ into $L^1(\Omega)$. As almost surely, T is a random periodic distribution if, $(\varphi_l)_{l\in\mathbb{N}}$ is a sequence of test functions converging to zero in $\mathcal{C}^\infty(\Pi)$ then, the sequence of random variables $(U_l)_{l\in\mathbb{N}}$, defined for $l \in \mathbb{N}$, by:

$$U_l =< T, \varphi_l > ,$$

converges a.s. to zero. So, this sequence $(U_l)_{l\in\mathbb{N}}$, converges also in probability to zero. Now, suppose that $(U_l)_{l\in\mathbb{N}}$ converges to U in $L^1(\Omega)$. Then, the sequence converges also in probability to U and so, $U = 0$. By the closed graph theorem, [53, p. 51] the map Λ_T is continuous. Taking in account the topologies of $\mathcal{C}^\infty(\Pi)$ and $L^1(\Omega)$, the continuity of Λ_T can be expressed in the following way:

$$\exists k \in \mathbb{N}, \;\; c_k > 0 \;\; \forall \varphi \in \mathcal{C}^\infty(\Pi) \quad || < T, \varphi > ||_{L^1(\Omega)} \;\leq\; c_k \sup_{n\in\mathbb{Z}} (1 + |n|)^k \hat{\varphi}_n . \tag{5}$$

As a second step, we use the sequence of Rademacher functions [67, p. 212], defined in $[0, 1]$, by:

$$\forall t \in [0, 1] \quad \forall n \in \mathbb{N} \quad r_n(t) = \operatorname{sign}(\sin(2^{n+1}\pi t)) .$$

And, we also consider $s(\mathbb{Z})$, the space of rapidly decreasing complex sequences with the topology induced by the quasi-norms $|.|_k$ $k \in \mathbb{N}$, which are defined for $s = (s_n)_{n\in\mathbb{Z}}$, an element of $s(\mathbb{Z})$, by:

$$|s|_k = \sup_{n\in\mathbb{Z}} (1 + |n|)^k |s_n| .$$

Then, for every $t \in [0,1]$, the map from $s(\mathbb{Z})$ to $s(\mathbb{Z})$ which associates, to each sequence $s \in s(\mathbb{Z})$, the sequence $w = (w_n)_{n\in\mathbb{Z}}$, defined by:

$$\forall n \in \mathbb{Z} \quad w_n = r_{|n|}(t)s_n \ ,$$

is an homeomorphism such that:

$$\forall k \in \mathbb{Z} \quad |w|_k = |s|_k \ .$$

As a consequence of this observation, of the expression of the continuity of Λ_T given by (5) and, of Parseval formula, we have that:

$$\begin{aligned} &\exists k \in \mathbb{N},\ c_k > 0\ \forall \varphi \in \mathcal{C}^\infty(\Pi),\ \forall t \in [0,1] \\ &\mathbb{E}|\sum_{n=-\infty}^{+\infty} r_{|n|}(t)\hat{T}_{-n}\hat{\varphi}_n| \ \le\ c_k \sup_{n\in\mathbb{Z}}(1+|n|)^k\hat{\varphi}_n \ . \end{aligned} \tag{6}$$

In the third step of the proof, we will show that the left-hand side of the inequality in (6) can be replaced by the expression:

$$\mathbb{E}[(\sum_{n=-\infty}^{+\infty} |\hat{T}_{-n}|^2|\hat{\varphi}_n|^2)^{\frac{1}{2}}] \ .$$

In order to do as stated, we observe that, as the sequence $(\hat{\varphi}_n)_{n\in\mathbb{Z}}$ is rapidly decreasing and, almost surely $(\hat{T}_{-n})_{n\in\mathbb{Z}}$ is a sequence of slow growth, we have almost surely:

$$\sum_{n=-\infty}^{+\infty} |\hat{T}_{-n}|^2|\hat{\varphi}_n|^2 \ <\ +\infty \ .$$

Now, by the standard inequality for Rademacher functions [67, p. 213], we have, for some constant c, a. s. :

$$(\sum_{n=-\infty}^{+\infty} |\hat{T}_{-n}|^2|\hat{\varphi}_n|^2)^{\frac{1}{2}} \ \le\ c \int_0^1 |\sum_{n=-\infty}^{+\infty} r_{|n|}(t)\hat{T}_{-n}\hat{\varphi}_n|dt \ .$$

To conclude as desired, it is enough to apply Fubini theorem to get, for k, c_k and φ as in (6), that:

$$\mathbb{E}[(\sum_{n=-\infty}^{+\infty} |\hat{T}_{-n}|^2|\hat{\varphi}_n|^2)^{\frac{1}{2}}] \le c\ c_k \sup_{n\in\mathbb{Z}}(1+|n|)^k\hat{\varphi}_n \ . \tag{7}$$

In the fourth step, we remark that the left-hand side of the inequality in 7, can be replaced by:

$$\mathbb{E}[\sum_{n=-\infty}^{+\infty} \sqrt{\alpha_n}|\hat{T}_{-n}||\hat{\varphi}_n|] \ ,$$

where $(\alpha_n)_{n\in\mathbb{Z}}$ is an arbitrary sequence of strictly positive numbers such that $\sum_{n=-\infty}^{n=+\infty} \alpha_n = 1$. This statement results from the fact that, for almost every $\omega \in \Omega$, the expression:

$$\sum_{n=-\infty}^{+\infty} |\hat{T}_{-n}|^2|\hat{\varphi}_n|^2 = \sum_{n=-\infty}^{+\infty} \alpha_n|\hat{T}_{-n}|^2\frac{|\hat{\varphi}_n|^2}{\alpha_n} \ ,$$

can be considered as an integral, over $\mathbb{Z}$, of the function defined by:

$$\forall n \in \mathbb{Z} \quad |\hat{T}_{-n}|^2 \frac{|\hat{\varphi}_n|^2}{\alpha_n} \ ,$$

with respect to the measure over $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}))$, that puts a mass α_n on each integer n. Applying Jensen inequality, to the convex function $-\sqrt{x}$ on the interval $[0, +\infty[$, we have that a. s.:

$$(\sum_{n=-\infty}^{+\infty} |\hat{T}_{-n}|^2 |\hat{\varphi}_n|^2)^{\frac{1}{2}} \ \geq \ \sum_{n=-\infty}^{+\infty} \alpha_n |\hat{T}_{-n}| \frac{|\hat{\varphi}_n|}{\sqrt{\alpha_n}} \ .$$

As a consequence, for k, c_k and φas in (6):

$$\mathbb{E}[\sum_{n=-\infty}^{+\infty} \sqrt{\alpha_n} |\hat{T}_{-n}| |\hat{\varphi}_n|] \ \leq \ c \ c_k \sup_{n \in \mathbb{Z}} (1 + |n|)^k |\hat{\varphi}_n| \ . \tag{8}$$

This expression shows that the sequence $(\mathbb{E}[|\hat{T}_{-n}|\sqrt{\alpha_n}])_{n\in\mathbb{Z}}$ is of slow growth at infinity. In order to conclude now, it is enough to consider, for instance, the sequence $(\alpha_n)_{n\in\mathbb{Z}}$ defined by:

$$\alpha_n = \begin{cases} \frac{1}{A} & n = 0 \\ \frac{1}{n^2 A} & n \neq 0 \end{cases}$$

where $A = 1 + \frac{\pi^2}{3}$. This sequence obviously satisfies the hypothesis made in the fourth step and, it is clear that if with this sequence, $(\mathbb{E}[|\hat{T}_{-n}|\sqrt{\alpha_n}])_{n\in\mathbb{Z}}$ is a sequence of slow growth at infinity then, $(\mathbb{E}[|\hat{T}_{-n}|])_{n\in\mathbb{Z}}$ is also of slow growth, thus showing that the random Schwartz periodic distribution T is in the class $\mathcal{C}_m$. □

Remark 2 (Example on the mean of classic randomized distributions). With the notations of example 3.3 above, consider μ_F the law of the random variable F. The calculation of the Fourier coefficients of the Schwartz distribution, which is the mean of the randomized periodic distribution, goes as follows.

$$(\forall n \in \mathbb{Z}) \ \ \mathbb{E}[C_n^T] = \hat{f}(n)\mathbb{E}[e^{-2\pi i n F}] = \hat{f}(n) \int_\Omega e^{-2\pi i n F} d\mathbb{P} =$$

$$= \hat{f}(n) \int_0^1 e^{-2\pi i n x} d\mu_F(x) = \hat{f}(n)\hat{\mu}_F(n) \, . \tag{9}$$

Observe now that, as f is a distribution and μ_F is a measure, the convolution of the two $f * \mu_F$, is a well defined distributions whose Fourier coefficients verify:

$$(\forall n \in \mathbb{Z}) \ \ \widehat{f * \mu_F}(n) = \hat{f}(n)\hat{\mu}_F(n) \, .$$

The preceding theorem and the observation just made, now show that T admits $\overline{T} = f * \mu_F$ as a mean.

As a conclusion for this section, let us mention an immediate consequence of this theorem, which will be useful later on.

Corollary 4.6. *Let T and V be random periodic distributions having the sequence of the corresponding Fourier coefficients in the space $\mathcal{C}_m$. Let α and β be complex numbers, $\overline{T}$ the mean of T and $\overline{V}$ the mean of V. Then $\alpha T + \beta V$ admits $\alpha\overline{T} + \beta\overline{V}$ as a mean.*

5. Applications

Necessary and sufficient conditions for the existence of a particular solution. The calculation of the Fourier coefficients of the brownian processes. Construction of a particular solution for a Langevin generalized solution.

The representation of distributions as a Fourier-Schwartz series, allows the construction of particular solutions for ordinary differential equations with constant coefficients, by means of a compatibility condition between, the roots of the polynomial which is naturally associated with the differential operator in question and, the null Fourier-Schwartz coefficients of the distribution appearing in the second member of the differential equation[2]. The compatibility condition mentioned, also holds in the class of random periodic distributions under study. This result is the purpose of theorem 5.1. But first, we are going to fix some notations. We denote by $P(z) = \sum_{k=0}^{r} a_k z^k$, a polynomial in one complex variable having complex coefficients. By induction, we define the following differential operators:

$$D^0 = Id\,, \quad D = \frac{1}{2\pi i}\frac{d}{dt}\,, \quad (\forall k \geq 2)\;\; D^k = D \circ D^{k-1}\,.$$

The factor $(1/2\pi i)$, in the expression of D, is there with the only purpose of simplifying a little the calculations. To $P(z)$, a polynomial as above, we associate in one, and only one, way the differential operator: $P(D) = \sum_{k=0}^{r} a_k D^k$.

Theorem 5.1 (On the existence of particular solutions). *Let T be a random periodic distribution, such that $(C_n^T)_{n\in\mathbb{Z}}$, the sequence of the correspondent Fourier coefficients, is in $\mathcal{C}_m$. The following propositions are equivalent.*

(1) $(\exists V \in \mathcal{D}'(\mathbb{R}/\mathbb{Z})^{\Omega}) \quad (C_n^V)_{n\in\mathbb{Z}} \in \mathcal{C}_m$ *et* $P(D)V = T$ *a.s. on* Ω.

(2) $(\forall n \in \mathbb{Z}) \quad P(n) = 0 \;\Rightarrow\; C_n^T = 0$ *a.s. on* Ω.

Proof. To begin with, suppose (1). Having $P(D)V = T$, the uniqueness of the representation in Fourier series allow us to write:

$$(\forall n \in \mathbb{Z})\;\; C_n^{P(D)V} = C_n^T \text{ a.s. on } \Omega\,.$$

[2]See for instance [11, II, p. 126] or [12]. In this last reference we have shown that, by a local change of the distribution in the second member of the differential equation, one can always get a local representation in a Fourier-Schwartz series. In [13] it is shown how to use this technique for solving a particular Cauchy problem.

An easy calculation also shows that:

$$(\forall n \in \mathbb{Z})\ \ C_n^{P(D)V} = P(n) C_n^V \ ,$$

Condition (2), as a consequence, is verified. Suppose now (2). Denote by $Z_p = \{n_1, ..., n_p\}$ the set of integer roots of the polynomial $P(z)$. Let $\Lambda_{n_1}, ..., \Lambda_{n_p}$ be p complex random variables defined on Ω and integrable. Define now V, the particular solution of the differential equation under study, by:

$$V = \sum_{n \in \mathbb{Z} - Z_p} \frac{C_n^T}{P(n)} e^{2\pi i n x} + \sum_{k=1}^{p} \Lambda_{n_k} e^{2\pi i n_k x} \text{ a.s. on } \Omega \, .$$

Set for simplicity:

$$\begin{cases} (\forall n \in \mathbb{Z} - Z_p) & C_n = \frac{C_n^T}{P(n)} \text{ a.s. on } \Omega \\ (\forall k \in \{1, ..., p\}) & C_{n_k} = \Lambda_{n_k} \text{ on } \Omega \end{cases}$$

Let us show that the sequence $(C_n)_{n \in \mathbb{Z}}$ is in the class $\mathcal{C}_m$. By the hypotheses made, there exist $A \in L^1(\Omega)$, A being non negative and $s \in \mathbb{Z}$, such that:

$$(\forall n \in \mathbb{Z})\ \ |C_n^T| \leq A(1 + |n|)^s \text{ a.s. on } \Omega \, .$$

It can also be seen easily, that there exists $c \in \mathbb{R}_+^*$ such that, if r is the degree of the polynomial P, then:

$$(\forall n \in \mathbb{Z} - Z_p)\ \ |\frac{1}{P(n)}| \leq \frac{c}{(1 + |n|)^r} \, .$$

As conclusion, it follows that:

$$(\forall n \in \mathbb{Z} - Z_p)\ \ |C_n| \leq cA(1 + |n|)^{s-r} \text{ a.s. on } \Omega \, .$$

It order to take care of the set indexes n, in Z_p, we can consider the integrable random variable

$$\Lambda = \sup_{k \in \{1, ..., p\}} \frac{|\Lambda_{n_k}|}{(1 + |n_k|)^{s-r}}$$

and get this other integrable random variable almost surely defined by:

$$A_1 = \sup(\Lambda, cA) \, .$$

It is easily verified that:

$$(\forall n \in \mathbb{Z})\ \ |C_n| \leq A_1(1 + |n|)^{s-r} \text{ a.s. on } \Omega \, .$$

For a conclusion of the proof, we just have to check that $P(D)V = T$, almost surely on Ω. As we can write, $V = \sum_{n \in \mathbb{Z}} C_n e^{2\pi i n x}$ almost surely on Ω, we get almost surely:

$$P(D)V = \sum_{n \in \mathbb{Z}} P(n) e^{2\pi i n x} = \sum_{n \in \mathbb{Z} - Z_p} C_n^T e^{2\pi i n x} = T \text{ a.s. on } \Omega \, .$$

and this ends the proof of the theorem. □

The following easy result shows that, in order to obtain the mean of a solution which is a random distribution, of an equation, such as the one referred to in the previous theorem, one has only to solve the deterministic differential equation, having as second member, $\overline{T}$, the random distribution which is the mean of T.

THEOREM 5.2. *Under the conditions of the previous theorem, there exists $\overline{V} \in \mathcal{D}'(\mathbb{R}/\mathbb{Z})$ such that $P(D)\overline{V} = \overline{T}$ on $C^\infty(\mathbb{R}/\mathbb{Z})$ and, such that, V admits $\overline{V}$ as a mean.*

PROOF. Knowing that the random distribution V admits $\overline{V} = \sum_{n\in\mathbb{Z}} \mathbb{E}[C_n^V]e^{2\pi inx}$ as a mean, we have that on $C^\infty(\mathbb{R}/\mathbb{Z})$: .

$$P(D)\overline{V} = \sum_{n\in\mathbb{Z}} P(n)\mathbb{E}[C_n^V]e^{2\pi inx} = \sum_{n\in\mathbb{Z}} \mathbb{E}[P(n)C_n^V]e^{2\pi inx} = \sum_{n\in\mathbb{Z}} \mathbb{E}[C_n^T]e^{2\pi inx} = \overline{T}\,,$$

thus ending the proof. □

5.1. The asymptotic behavior of the Fourier coefficients of the brownian processes. The asymptotic behavior of the Fourier coefficients of the fractional brownian processes, introduced in the subsection 3.2, lays essentially on the representation of these processes as images of white noise by convolution operators.

THEOREM 5.3. *For $\gamma \in]0,2[$ and $\gamma \neq 1$ the process $X^\gamma = (X_t^\gamma)_{t\in[0,1]}$ admits the following representation as a Fourier series.*

$$(\forall \gamma \in]0,1[)\quad X_t^\gamma = \frac{2t^{\frac{1+\gamma}{2}}}{1+\gamma} + \sum_{n\in\mathbb{Z}^*} \frac{g_1(\gamma,n)}{(2\pi|n|)^{\frac{1+\gamma}{2}}} Z_n e^{2\pi int}\,,$$

$$(\forall \gamma \in]1,2[)\quad X_t^\gamma = \frac{2t^{\frac{1+\gamma}{2}}}{1+\gamma} + \sum_{n\in\mathbb{Z}^*} \Big(\frac{|n|}{n}\frac{(\gamma-1)}{2i}\frac{g_2(\gamma,n)}{(2\pi|n|)^{\frac{1+\gamma}{2}}} - \frac{1}{2\pi in}\Big) Z_n e^{2\pi int}\,,$$

The convergence of these series takes place for all t, almost surely on Ω where, for $j=1,2$, $(g_j(\gamma,n))_{n\in\mathbb{Z}}$ is a sequence of complex numbers verifying:

$$(\forall \gamma \in]0,1[)\quad \lim_{n\to\pm\infty} g_1(\gamma,n) = \Gamma\Big(\frac{1+\gamma}{2}\Big)\exp\Big(\mp i\frac{(1+\gamma)\pi}{4}\Big)\,,$$

$$(\forall \gamma \in]1,2[)\quad \lim_{n\to\pm\infty} g_2(\gamma,n) = \Gamma\Big(\frac{\gamma-1}{2}\Big)\exp\Big(\mp i\frac{(\gamma-1)\pi}{4}\Big)\,,$$

and where $(Z_n)_{n\in\mathbb{Z}}$ is a normal sequence, that is, a sequence of independent centered gaussian variables with variance one.

PROOF. Take, as a starting point, the following alternative definition for the generalized Wiener process X^γ for $\gamma \in]0,2[$ and $\gamma \neq 1$:

$$(\forall t\in[0,1])\ X_t^\gamma = \int_0^t \frac{1}{(t-s)^{\frac{1-\gamma}{2}}} dX_s^1 \quad \text{a.s. on } \Omega\,, \tag{10}$$

(see [31, p. 279] and [45]). In fact, the integral in the right-hand side of this equality is a stochastic integral, with respect to the Wiener process, of a non random function and so, it defines a gaussian random variable having the law $\mathcal{N}(0, t^\gamma/\gamma)$. We also use the representation of the Wiener process, as a random Fourier series, given by:

$$X_t^1 = \sum_{n \neq 0} \frac{Z_n}{2\pi i n} e^{2\pi i n t} + Z_0 t \,,$$

where $(Z_n)_{n\in\mathbb{Z}}$, is a normal sequence (see [31, p. 235] et [25, p. 23]). The integral representation 10 of the processX^γ, can be interpreted as the convolution of the function f_γ, defined by:

$$(\forall u \in [0,1]) \;\; f_\gamma(u) = \frac{\mathbb{I}_{]0,+\infty[}(u)}{u^{\frac{1-\gamma}{2}}} \,,$$

with the random periodic distribution $(X^1)'$, having as representation in a Fourier series form, the series given by:

$$\sum_{n\neq 0} Z_n e^{2\pi i n t} + Z_0 \,,$$

series which converges in the sense of distributions (see [44, p. 355]). This interpretation, has a consequence that the Fourier coefficients of X^γ are, for $n \in \mathbb{Z}^*$, the corresponding Fourier coefficients of f_γ.

So, after a clear change of variables and after considering separately the case where $n > 0$ from the case where $n < 0$, we have:

$$(\forall n \in \mathbb{Z}^*) \; \hat{f}_\gamma(n) = \int_0^1 \frac{e^{-2\pi i n u}}{u^{\frac{1-\gamma}{2}}} \, du = \frac{1}{(2\pi|n|)^{\frac{1+\gamma}{2}}} \int_0^{2\pi|n|} \frac{e^{-i\frac{|n|}{n}v}}{v^{\frac{1-\gamma}{2}}} \, dv \,.$$

Consider first the case where $\gamma \in]0,1[$. For a non zero integer n, denote by:

$$g_1(\gamma, n) = \int_0^{2\pi|n|} \frac{e^{-i\frac{|n|}{n}v}}{v^{\frac{1-\gamma}{2}}} dv \,.$$

Knowing that for $a \in]0,1[$:

$$\int_0^{+\infty} \frac{cos(v)}{v^a} \, dv = \Gamma(1-a) \, \cos(\frac{(1-a)\pi}{2})$$

$$\int_0^{+\infty} \frac{sin(v)}{v^a} \, dv = \Gamma(1-a) \, \sin(\frac{(1-a)\pi}{2}) \,,$$

(see [63, p. 181]). The announced results follows by applying, to the real and imaginary parts of $g_1(\gamma, n)$, the above formulae. For $\gamma \varepsilon]1,2[$, an easy integration by parts, of $g_1(\gamma, n)$, shows that:

$$\int_0^{2\pi|n|} \frac{e^{-i\frac{|n|}{n}v}}{v^{\frac{1-\gamma}{2}}} \, dv = i\frac{|n|}{n}(2\pi|n|)^{\frac{\gamma-1}{2}} - i\frac{|n|}{n}(\frac{\gamma-1}{2}) \int_0^{2\pi|n|} \frac{e^{-i\frac{|n|}{n}v}}{v^{\frac{3-\gamma}{2}}} \, dv \,.$$

Take now:

$$g_2(\gamma, n) = \int_0^{2\pi|n|} \frac{e^{-i\frac{|n|}{n}v}}{v^{\frac{3-\gamma}{2}}} \, dv \,,$$

and finish as the previous case. The Fourier coefficient for $n = 0$ is obtained by the convolution of f_γ with the constant function equal to 1. The case where $\gamma = 2$ is trivial if one recalls that, by considering,

$$(\forall t \in [0,1]) \quad X_t^2 = \sqrt{t} Z \ , \quad Z \in \mathcal{N}(0,1) \ ,$$

we get a version of X^2, defined almost surely. $\square$

Remark 3. The form of the Fourier coefficients of the fractional brownian processes for $\gamma \in]1,2[$, reflects the method used to build them. We have applied a periodic transformation (see [65, II, p. 63]) to the process defined by formula 10. The trajectories over $\mathbb{R}$ are accordingly almost surely discontinuous for every t in $\mathbb{Z}$. The derivative of the trajectories aren't functions and the order of growth at infinity of the Fourier coefficients just shows that fact.

5.2. Particular solutions for a generalized equation. Let us consider a process $Y^\gamma = (Y_t^\gamma)_{t\in[0,1]}$ satisfying the equation given by:

$$D^2 Y_t^\gamma + KDY_t^\gamma + LY_t^\gamma = X_t^\gamma \ ,$$

where K and L are complex constants and $X^\gamma = (X_t^\gamma)_{t\in[0,1]}$ represents a generalized Wiener process describing a fractional brownian process. Theorem 5.1 gives us the assurance that, if the equation $n^2 + Kn + L = 0$ has no integer roots then, a almost sure representation, for the process Y^γ, is given by:

$$Y_t^\gamma = \frac{2Z_0 h_\gamma(t)}{1+\gamma} + \sum_{n\neq 0} \frac{\widehat{X^\gamma}(n)}{(n^2+Kn+L)} e^{2\pi i n t} \quad p.s.\ sur\ \Omega \, ,$$

where $h_\gamma(t)$ is a particular solution for the ordinary differential equation given by:

$$D^2 y(t) + K\, Dy(t) + L\, y(t) = t^{\frac{1+\gamma}{2}} \ .$$

when equation $n^2 + Kn + L = 0$ admits an integer solution we still can, by means of a local change of the right-hand side of the differential equation, represent (locally only) a particular solution of the equation (see [12]).

Remark 4 (On the regularity of the process). The Fourier coefficients of Y^γ are equivalent (at infinity) to $1/n^{((5+\gamma)/2)}$ for $\gamma \in]0,1[$ and to $1/n^3$ for $\gamma \in]1,2[$. As a conclusion, we have that as a consequence of theorem 4.3 and of the results on gaussian Fourier series in [31, p. 199] that Y^γ has its trajectories in $C^2([0,1])$ almost surely in Ω.

6. Appendix

We present here a proof of a result, which is weaker then the one studied in theorem 2.1 above. The proof evolves on a different line of reasoning and is slightly more complicated. The notations followed are those of the introduction.

THEOREM 6.1. *Let $(C_n)_{n\in\mathbb{Z}}$ be an element of the space $M^{\mathbb{Z}}$. If for some p in the interval $]1,+\infty[$, the sequence of moments of order p, namely $(||C_n||_{L^p})_{n\in\mathbb{Z}}$, is of slow growth then, the sequence $(C_n)_{n\in\mathbb{Z}}$ is in the class $\mathcal{C}_m$.*

This theorem shows that the following condition on $(C_n)_{n\in\mathbb{Z}}$, an element of the space $M^{\mathbb{Z}}$,

$$(\exists p\in]1,+\infty[)\,(\exists a>0)\,(\exists r\in\mathbb{Z})\,(\forall n\in\mathbb{Z})\ \ \mathbb{E}[|C_n|^p]^{1/p}\leq a(1+|n|)^r\,,$$

is sufficient for having the sequence in the space $\mathcal{C}_m$.

PROOF. Let us notice that applying Hölder inequality:

$$(\forall n\in\mathbb{Z})\ \mathbb{E}[|C_n|]\leq\mathbb{E}[|C_n|^p]^{1/p}\leq a(1+|n|)^r\,.$$

Taking α positive and $n\in\mathbb{Z}^*$, Markov inequality gives

$$\mathbb{P}[\{|C_n|>(1+|n|)^\alpha\,.\,\mathbb{E}[|C_n|\}]]\leq\frac{1}{(1+|n|)^\alpha}\,.$$

For $\alpha>1$, Borel-Cantelli lemma tells us that:

$$\mathbb{P}[\overline{\lim}_{n\to+\infty}\{|C_n|>(1+|n|)^\alpha\,.\,\mathbb{E}[|C_n|\}]]=$$

$$=\mathbb{P}[\overline{\lim}_{n\to+\infty}\{|C_{-n}|>(1+|n|)^\alpha\,.\,\mathbb{E}[|C_{-n}|\}]]=0\,,$$

or, in an equivalent way,

$$\mathbb{P}[\underline{\lim}_{n\to+\infty}\{|C_n|\leq(1+|n|)^\alpha\,.\,\mathbb{E}[|C_n|\}]]=$$

$$=\mathbb{P}[\underline{\lim}_{n\to+\infty}\{|C_{-n}|\leq(1+|n|)^\alpha\,.\,\mathbb{E}[|C_{-n}|\}]]=1\,.$$

This allow us to conclude that there exists Ω_1, a measurable subset of Ω, such that $\mathbb{P}[\Omega_1]=1$ and verifying:

$$(\exists n_1\in\mathbb{N}\,,\,n_1=n_1(\omega))\ (\forall n\in\mathbb{N})\ \ n>n_1\ \Rightarrow\ |C_n(\omega)|\leq a\,(1+n)^{\alpha+r}\,,$$

$$(\exists n_2\in\mathbb{N}\,,\,n_2=n_2(\omega))\ (\forall n\in\mathbb{N})\ \ n>n_2\ \Rightarrow\ |C_{-n}(\omega)|\leq a\,(1+n)^{\alpha+r}\,,$$

By posing $m(\omega)=\sup(n_1(\omega),n_2(\omega))$ we have that:

$$(\forall\omega\in\Omega_1)\,(\forall n\in\mathbb{Z})|n|>m(\omega)\ \Rightarrow\ |C_n(\omega)|\leq a(1+|n|)^{\alpha+r}\,.$$

Let then be, for $\omega\in\Omega_1$, $m^*(\omega)$ the first positive integer for which the implication above is verified. The correspondence that associates to each $\omega\in\Omega_1$, the integer $m^*(\omega)$, defines a random variable, as we have that, for each $n \in \mathbb{Z}$, C_n is a random variable. Define also, for $\omega \in \Omega_1$:

$$A(\omega) = \sup(a, \sup_{|n| \leq m^*(\omega)} |C_n(\omega)|) .$$

As the map $S_M = \sup_{|n| \leq M} |C_n|$, defined for $M \in \mathbb{N}$ is a random variable, an easy exercise show that $A = S_{m^*}$ is also a random variable, which moreover verifies for $\omega \in \Omega_1$ arbitrary *and as soon as* $\alpha + r \geq 0$:

$$|n| > m^*(\omega) \;\Rightarrow\; |C_n(\omega)| \leq a(1+|n|)^{\alpha+r} \leq A(\omega)(1+|n|)^{\alpha+r} ,$$

$$|n| \leq m^*(\omega) \;\Rightarrow\; |C_n(\omega)| \leq \sup_{|n| \leq m^*(\omega)} |C_n(\omega)| \leq A(\omega) \leq A(\omega)(1+|n|)^{\alpha+r} .$$

We are now going to show that it is possible to choose α in a coherent way with the conditions already posed on this parameter, in order that $A \in L^1(\Omega)$. We have always that:

$$\int_\Omega A \, d\mathbb{P} \leq a + \int_\Omega \sup_{|n| \leq m^*(\omega)} |C_n| \, d\mathbb{P} .$$

Moreover:

$$\int_\Omega \sup_{|n| \leq m^*(\omega)} |C_n| \, d\mathbb{P} = \int_{\cup_{k=0}^{\infty}\{m^*=k\}} \sup_{|n| \leq m^*(\omega)} |C_n| \, d\mathbb{P} =$$

$$= \sum_{k=0}^{+\infty} \int_{\{m^*=k\}} \sup_{|n| \leq k} |C_n| \, d\mathbb{P} \leq \sum_{k=0}^{+\infty} \sum_{|n| \leq k} \int_{\{m^*=k\}} |C_n| \, d\mathbb{P} .$$

By an application of Hölder inequality and using the hypotheses

$$\int_{\{m^*=k\}} |C_n| \, d\mathbb{P} \leq (\int_\Omega |C_n|^p d\mathbb{P})^{1/p} . (\int_\Omega \mathbb{I}_{\{m^*=k\}} d\mathbb{P})^{1-\frac{1}{p}} \leq$$

$$\leq a(1+|n|)^r . (\mathbb{P}[m^* = k])^{1-\frac{1}{p}} .$$

The probability appearing in the right hand side above can be estimated as:

$$\{m^* = k\} \subset \{|C_k| > (1+|k|)^\alpha . \mathbb{E}[|C_k|\} \cup \{|C_{-k}| > (1+|k|)^\alpha . \mathbb{E}[|C_{-k}|\} .$$

By an application of Markov inequality we have:

$$\mathbb{P}[\{m^* = k\}] \leq \frac{2}{(1+k)^\alpha} .$$

Taking in consideration this estimate we get

$$\int_\Omega A \, d\mathbb{P} \leq a + a2^{(1-\frac{1}{p})} \sum_{k=0}^{+\infty} (\sum_{|n| \leq k} (1+|n|)^r) \frac{1}{(1+k)^{\alpha(1-\frac{1}{p})}} .$$

In order to conclude let us discuss now, over the values of parameter r.

- Suppose that $r \leq 0$. The series of the second member of the inequality is dominated by:

$$\sum_{k=0}^{+\infty} \frac{(2k+1)}{(1+k)^{\alpha(1-\frac{1}{p})}} \ .$$

This series converges if $\alpha > \frac{2}{1-\frac{1}{p}}$. The conditions on α are thus $(\alpha > 1)$, $(\alpha \geq -r, r \leq 0)$ and the one obtained just above. Finally the condition

$$\alpha > \sup(1, -r, \frac{2}{1-\frac{1}{p}}) \ ,$$

ensures that A is integrable.

- Suppose that $r > 0$. The series of the second member is now dominated by:

$$\sum_{k=0}^{+\infty} \frac{(2k+1)(k+1)^r}{(1+k)^{\alpha(1-\frac{1}{p})}} \ .$$

This series converges if $\alpha > \frac{r+2}{1-\frac{1}{p}}$. In this case, the conditions imposed on α are $(\alpha > 1)$, $(\alpha \geq -r, r > 0)$ and the one obtained. The condition

$$\alpha > \sup(1, \frac{2+r}{1-\frac{1}{p}}) \ ,$$

is thus enough to guarantee that A is integrable.

□

CHAPTER II

On the local behavior of the brownian sheet

1. Summary

In his seminal book, *"Some Random Series of Functions"* [31], J.-P. Kahane has shown, in a systematic way, how to take advantage of Paul Lévy's construction of the Wiener process, using the Haar functions, in order to study the local behavior of this process. To reach this purpose, Kahane, looks at the Haar's interpolation of the brownian process done by Lévy, as a series expansion in the Schauder system, having gaussian random variables as coefficients. This representation of the brownian process as a sum of a series, which converges uniformly almost surely, of Schauder functions having as coefficients normal random variables, is a simple consequence of the definition of the brownian process using gaussian white noise. Kahane, then exploits this series representation with sharp estimates of the distribution function of the maximum of a finite subfamily of a normal sequence. With this method Kahane gets, easily, the results corresponding to the existence of rapid points and slow points (which were first discovered by Orey and Taylor [50] and Kahane [33], respectively).

Our work, deals with an extension of the methods of Kahane, as applied to the Brownian sheet, in what concerns analogs of the rapid points. After presenting the brownian sheet process, by way of gaussian white noise, some results on the local behavior of the brownian sheet and for some other processes associated with the sheet, are derived, using this series representation. Namely, a uniform modulus of continuity, nondifferentiability results and, at some points, *faster* oscillation than the one prescribed by the laws of iterated logarithm for the usual increments. We present next results on rapid points and almost sure everywhere nondifferentiability for the *location homogeneous* part of the Fourier-Schauder series representation. In the sequel, we show the existence of *rapid* points for the independent increments of the brownian sheet, using the same method. In an appendix, we observe that a random field, baring some resemblances to the brownian sheet process, can be obtained by the tensorial product of two brownian processes. The method applied to study the sheet process is shown to work but only as far as the uniform modulus of continuity result.

2. Introduction

On the usual Lebesgue space $L^2([0,1])$, the Haar system is given by:

$$e_0 = \mathbb{I}_{[0,1]} \ ,$$

$$e_{in} = 2^{\frac{i}{2}} (\mathbb{I}_{[n\ 2^{-i},(n+\frac{1}{2})2^{-i}[} - \mathbb{I}_{[(n+\frac{1}{2})2^{-i},(n+1)2^{-i}[}) \ ,$$

for $i = 0, 1, 2, ...\ ; n = 0, 1, ..., 2^i - 1$. For us, here, the relevance of this set of functions follows from the next proposition.

PROPOSITION 2.1. *The Haar system is a complete orthonormal set of functions in the space* $L^2([0,1])$.

PROOF. A trivial calculation shows that the functions of the Haar system are normalized. In order to verify that two different Haar functions, e_{in} and e_{jm}, are orthogonal we consider two cases. If $i = j$ then the supports of the two functions are disjoint sets and so the integral of the product of these two functions is zero. If $i \neq j$ suppose that $i > j$. Then, e_{jm} is identically zero or has constant sign on the support of e_{in}. The integral of the product is then zero as the integral of e_{in} over $[0,1]$ is zero.

We now show that the Haar system is complete. Let $f \in L^2([0,1]]$. If, for all indices (j,n) in $\mathbb{N}^* \times \{0,\ldots,2^j-1\}$, we have

$$2^{\frac{j}{2}} \left(\int_{[\frac{n}{2^j},(n+\frac{1}{2})\frac{1}{2^j}]} f(x))dx - \int_{[(n+\frac{1}{2})\frac{1}{2^j},\frac{n+1}{2^j}]} f(x)dx \right) = 0 \ ,$$

we can say, for instance, that:

$$\int_0^1 f(x)dx = 2 \int_{[0,\frac{1}{2}]} f(x)dx = 2 \int_{[\frac{1}{2},1]} f(x)dx$$

and, for $j = 2$ that:

$$\int_{[0,\frac{1}{2}]} f(x)dx = 2 \int_{[0,\frac{1}{4}]} f(x)dx = 2 \int_{[\frac{1}{4},\frac{1}{2}]} f(x)dx \ .$$

By induction, we will have then that:

$$\forall j \in \mathbb{N}^* \ \ \forall n \in \{0,\ldots 2^j - 1\} \ \int_{[0,1]} f(x)dx = 2^j \int_{[\frac{n}{2^j},\frac{n+1}{2^j}]} f(x)dx \ . \tag{11}$$

We want to prove that f is constant almost everywhere. If that is the case, we should have a.e.:

$$f(x) = \int_0^1 f(x)dx \ .$$

As a consequence, it is natural to proceed estimating $|f(x) - \int_0^1 f(x)dx|$. Let $x \in [0,1]$ and consider $([\frac{m}{2^i}, \frac{m+1}{2^i}])_{(m,i)\in\mathcal{N}\times\mathcal{I}}$, a family of dyadic intervals containing x, with $\mathcal{N} \subset \mathbb{N}^*$ and $\mathcal{I} \subset \{0,\ldots,2^j-1\}$. We will have then, as a consequence of 11,

$$|f(x) - \int_0^1 f(x)dx| = |f(x) - 2^i \int_{[\frac{m}{2^i},\frac{m+1}{2^i}]} f(t)dt| = 2^i| \int_{[\frac{m}{2^i},\frac{m+1}{2^i}]} (f(x) - f(t))dt| \leq$$

$$\frac{1}{\frac{1}{2^i}} \int_{[\frac{m}{2^i},\frac{m+1}{2^i}]} |f(x) - f(t)|dt \leq$$

$$2\frac{1}{\frac{1}{2^{i-1}}} \int_{[x-\frac{1}{2^i},x+\frac{1}{2^i}]} |f(x) - f(t)|dt \; .$$

The last inequality resulting from the observation, easily verified, that:

$$[\frac{m}{2^i}, \frac{m+1}{2^i}] \subset [x - \frac{1}{2^i}, x + \frac{1}{2^i}] \; .$$

Now, for almost all $x \in [0,1]$, x is a Lebesgue point of f (see [54, p. 138]) and that implies:

$$\lim_{i\to+\infty} \frac{1}{\frac{1}{2^{i-1}}} \int_{[x-\frac{1}{2^i},x+\frac{1}{2^i}]} |f(x) - f(t)|dt = 0 \; .$$

As so, the conclusion desired now follows. □

In order to build a complete orthonormal system for $L^2([0,1]^2)$, we can consider the tensor products having as factors the functions of the Haar system, and we get the functions:

$$e_0^0(s,t) = e_0(s)\, e_0(t) \; ,$$

$$e_0^{jm}(s,t) = e_0(s)\, e_{jm}(t) \; ,$$

$$e_{in}^0(s,t) = e_{in}(s)\, e_0(t) \; ,$$

$$e_{in}^{jm}(s,t) = e_{in}(s)\, e_{jm}(t) \; ,$$

for $i,j = 0,1,2,...\,; n = 0,1,...,2^i-1\,; m = 0,1,...,2^j-1$. By Fubini theorem, or using the theory of tensor products of L^2 spaces see [46], it can he checked that the family of functions:

$$(e_0^0,\, e_0^{jm},\, e_{in}^0,\, e_{in}^{jm})_{injm} \; , \tag{12}$$

is a complete orthonormal set of functions in the Lebesgue space $L^2([0,1]^2)$. We have in particular, with the equality holding in the L^2 sense,

$$\mathbb{I}_{[0,s[\times[0,t[} = st\, e_0^0 + s \sum_{j=0}^{+\infty} \sum_{m=0}^{2^j-1} \triangle_{jm}(t) e_0^{jm} +$$

$$+ t \sum_{i=0}^{+\infty} \sum_{n=0}^{2^i-1} \triangle_{in}(s) e_{in}^0 + \sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} \sum_{n=0}^{2^i-1} \sum_{m=0}^{2^j-1} \triangle_{in}(s)\, \triangle_{jm}\,(t)\, e_{in}^{jm} \; . \tag{13}$$

where the Schauder function, $\triangle_{in}$, defined for all i,n by:

$$\triangle_{in}(s) = \int_0^s e_{in}(x)dx \, ,$$

is a triangular shaped continuous function having $[n\, 2^{-i}, (n+1)2^{-i}]$ as its compact support.

The main object of the studies presented in this chapter, the brownian sheet, can readily be introduced as the unique, in law, gaussian centered stochastic process $D(s,t)$ having as parameter set $\mathbb{R}^2_+$ and, as covariance function:

$$\mathbb{E}[D(s,t)D(u,v)] = \min(s,u)\min(t,v)\ .$$

The purpose of allowing a deeper understanding of the brownian sheet and of other naturally associated processes defined in the following is, nevertheless, better served with an introduction given by means of another basic object namely, gaussian white noise. This procedure is both classical [6, p. 111] and up to date [2, p. 6].

For a starting point, we suppose we are given a gaussian white noise based on the Lebesgue space $(\mathbb{R}^2_+, \mathcal{B}(\mathbb{R}^2_+), \lambda)$. That will have a meaning, for us, as described in the next definition.

Definition 2.1. A *gaussian white noise* based on the Lebesgue space $(\mathbb{R}^2_+, \mathcal{B}(\mathbb{R}^2_+), \lambda)$ is a family $(W(A))_{A\in\mathcal{B}'}$ of centered gaussian random variables on a common complete probability space with no atoms $(\Omega, \mathcal{A}, \mathbb{P})$ indexed by $\mathcal{B}' = \{A \in \mathcal{B}(\mathbb{R}^2_+)\ : \lambda(A) < +\infty\}$ such that:

(1) $W(A)$ is a centered gaussian variable with variance $\lambda(A)$. This will be denoted by: $W(A) \in \mathcal{N}(0, \lambda(A))$.
(2) If, for $A, B \in \mathcal{B}'$, we have $A \cap B = \emptyset$ then, $W(A)$ and $W(B)$ will be independent random variables. This relation will be denoted by $W(A)\ i\ W(B)$.
(3) W is additive in the sense that if $A, B \in \mathcal{B}'$ and $A \cap B = \emptyset$ then, $W(A + B) = W(A) + W(B)$.

Of course, the question of the existence of such an object is a most pertinent one and will be addressed further on. The conditions imposed on such a gaussian white noise have important consequences which are stated in the next proposition.

Proposition 2.2. *Let* $(W(A))_{A\in\mathcal{B}'}$ *be a gaussian white noise as in definition* 2.1. *Then, the following properties are satisfied.*

(1) *For* $A, B \in \mathcal{B}'$, $A \subset B$ *and* $\lambda(B) < +\infty$ *we have that:*

$$||W(A) - W(B)||^2_{L^2(\Omega,\mathcal{A},\mathbb{P})} = \lambda(B) - \lambda(A)\ .$$

(2) *For all increasing sequences, in the sense of inclusion,* $(A_n)_{n\in\mathbb{N}}$ *in* $\mathcal{B}'$ *such that* $\cup_{n=0}^{\infty} A_n = A$ *with* $\lambda(A) < +\infty$, *we have:*

$$\lim_{n\to+\infty} W(A_n) = W(A)\ .$$

The limit being taken in $L^2(\Omega, \mathcal{A}, \mathbb{P})$.

(3) *For all sequences* $(A_n)_{n\in\mathbb{N}}$ *in* $\mathcal{B}'$ *such that, if* $m \neq n$ *then* $A_n \cap A_m = \emptyset$ *and* $\lambda(\cup_{n=0}^{\infty} A_n) < +\infty$*, we have:*

$$W(\bigcup_{n=0}^{\infty} A_n) = \sum_{n=0}^{\infty} W(A_n) \ . \tag{14}$$

The equality being taken in $L^2(\Omega, \mathcal{A}, \mathbb{P})$ *and also almost surely.*

(4) *For all* $A, B \in \mathcal{B}'$ *we have:*

$$\mathbb{E}[W(A)W(B)] = \lambda(A \cap B) \ .$$

Proof. The first statement in the proposition is a consequence of axiom 3 in the definition if one observes that $B = (B - A) \cup A$. The second statement follows from the first statement and from the continuity of the Lebesgue measure in $\mathbb{R}^2_+$. The second statement implies the third by considering the increasing sequence of sets in $\mathcal{B}'$ defined by $B_m = \cup_{n=0}^{m} A_n$ for $m \in \mathbb{N}$ and using again axiom 3 in the definition. The almost surely convergence follows from a known result in [61, p. 58]. Finally, the fourth statement is proved by the easy calculation that follows.

$$\begin{aligned}\mathbb{E}[W(A)\ W(B)] = \mathbb{E}[W\left((A - B) \cup (A \cap B)\right)\ W(B)] = \\ \mathbb{E}[W(A \cap B)\, W(B)] + \mathbb{E}[W(A - B)\, W(B)] = \\ \mathbb{E}[W(A \cap B)\ (W(B - A) + W(A \cap B))] = \\ \mathbb{E}[W(A \cap B)^2] + \mathbb{E}[W(B - A)\, W(A \cap B)] = \lambda(A \cap B) \ ,\end{aligned}$$

where we used axiom 2 of the definition and the fact that for $C \in \mathcal{B}'$, $W(C)$ is centered. □

Remark 5. As a consequence of this proposition, it is certain that a gaussian white noise, based on $(\mathbb{R}^2_+, \mathcal{B}(\mathbb{R}^2_+), \lambda)$, is a gaussian centered stochastic process W, with $\mathcal{B}'$ as index set and covariance function Γ given for $A, B \in \mathcal{B}'$ by $\Gamma(A, B) = \lambda(A \cap B)$ such that, W is additive i.e., for $A, B \in \mathcal{B}'$ with $A \cap B = \emptyset$ then $W(A + B) = W(A) + W(B)$. Conversely, a stochastic process enjoying the properties just referred [1] can be shown to verify the properties of definition 2.1 (see [47, p. 81]. Moreover, as a consequence for instance, of the theory of reproducing kernels (see [46, p. 38] or [47, p. 80]) or of the extension theorem of Kolmogoroff (see [29, p. 407D]), such a process is assured to exist. The question mentioned, about the existence of a gaussian white noise, in the sense of definition 2.1, is thus settled.

Let $\mathfrak{H}$ denote the closure, in $L^2(\Omega, \mathcal{A}, \mathbb{P})$, of the vector space generated by the set $\{W(B) : B \in \mathcal{B}'\}$. The next proposition shows that W can be extended, as an isometry, between $L^2(\mathbb{R}^2_+, \mathcal{B}(\mathbb{R}^2_+), \lambda)$ and $\mathfrak{H}$.

[1] Such a stochastic process, receives in [2, p. 6] the denomination of gaussian white noise based on λ.

PROPOSITION 2.3. *There exists $\widetilde{W}$, a map defined in $L^2(\mathbb{R}^2_+)$ and taking values in $\mathfrak{H}$, such that:*

$$\forall A \in \mathcal{B}' \quad \widetilde{W}(\mathbb{I}_A) = W(A) \tag{15}$$

and also

$$\forall f, g \in L^2(\mathbb{R}^2_+) \quad < \widetilde{W}(f), \widetilde{W}(g) >_{\mathfrak{H}} = < f, g >_{L^2(\mathbb{R}^2_+)} \quad . \tag{16}$$

As a consequence, $\widetilde{W}$ is an isometry such that:

$$\forall f, g \in L^2(\mathbb{R}^2_+) \ \forall A, B \in \mathcal{B}' \ \ A \cap B = \emptyset \Rightarrow \widetilde{W}(f\mathbb{I}_A) \, i \, \widetilde{W}(g\mathbb{I}_B) \ .$$

PROOF. The map $\widetilde{W}$ is obtained in the usual way, extending $\widetilde{W}$ first defined on the indicator functions of the elements of $\mathcal{B}'$, as in condition 15 above then, by linearity on the simple functions and finally, by continuity on the general measurable L^2 functions.

We define $\widetilde{W}(\alpha \mathbb{I}_A) = \alpha W(A)$ for every $A \in \mathcal{B}'$ and $\alpha \in \mathbb{R}$. Let s and r be simple $L^2(\mathbb{R}^2_+)$ functions and $s = \sum_{i=1}^n \alpha_i \mathbb{I}_{A_i}$ (respectively $r = \sum_{j=1}^m \beta_j \mathbb{I}_{B_j}$) be the correspondent canonical representation of s (respectively r) such that $0 \notin \{\alpha_1, \ldots, \alpha_n\}$ (respectively $0 \notin \{\beta_1, \ldots, \beta_m\}$). Observe that, as $s, r \in L^2(\mathbb{R}^2_+)$ the condition imposed on zero, with respect to the coefficients, implies that for all i, j, $\lambda(A_i) < +\infty$ and $\lambda(B_j) < +\infty$. Define $\widetilde{W}(s)$ (respectively $\widetilde{W}(r)$) as $\widetilde{W}(s) = \sum_{i=1}^n \alpha_i W(A_i)$ (respectively $\widetilde{W}(r) = \sum_{j=1}^m \beta_j W(B_j)$. It is then clear that $\widetilde{W}(s) \in \mathfrak{H}$ (respectively $\widetilde{W}(r) \in \mathfrak{H}$) and that by the linearity of the scalar products in L^2 spaces:

$$< \widetilde{W}(s), \widetilde{W}(r) >_{\mathfrak{H}} = \sum_{(i,j)} \alpha_i \beta_j \ < \widetilde{W}(A_i), \widetilde{W}(B_j) >_{\mathfrak{H}} = \sum_{(i,j)} \alpha_i \beta_j \, \lambda(A_i \cap B_j) =$$
$$\int_{\mathbb{R}^2_+} \sum_{(i,j)} \alpha_i \beta_j \, \mathbb{I}_{A_i} \mathbb{I}_{B_j} d\lambda = < s, r >_{L^2(\mathbb{R}^2_+)} \quad .$$

Let now f (respectively g) be a measurable positive function in $L^2(\mathbb{R}^2_+)$ and $(s_n)_{n \in \mathbb{N}}$ (respectively $(r_n)_{n \in \mathbb{N}}$) an increasing sequence of simple functions converging almost everywhere and in L^2 to f (respectively to g). Observe that, as

$$||\widetilde{W}(s_n) - \widetilde{W}(s_m)||_{\mathfrak{H}} = ||s_n - s_m||_{L^2(\mathbb{R}^2_+)} \ ,$$

with the same expression being verified with the sequence $(s_n)_{n \in \mathbb{N}}$ replaced by the sequence $(r_n)_{n \in \mathbb{N}}$ then, $(\widetilde{W}(s_n))_{n \in \mathbb{N}}$ (respectively $(\widetilde{W}(r_n))_{n \in \mathbb{N}}$) is a Cauchy sequence in $\mathfrak{H}$. Define $\widetilde{W}(f)$ (respectively $\widetilde{W}(g)$) to be the limit of $(\widetilde{W}(s_n))_{n \in \mathbb{N}}$ (respectively of $(\widetilde{W}(r_n))_{n \in \mathbb{N}}$) in $\mathfrak{H}$. By the continuity of the scalar products in L^2 spaces we have that:

$$< \widetilde{W}(f), \widetilde{W}(g) >_{\mathfrak{H}} = \lim_{n \to +\infty} \lim_{m \to +\infty} < \widetilde{W}(s_n), \widetilde{W}(r_m) >_{\mathfrak{H}} =$$
$$\lim_{n \to +\infty} \lim_{m \to +\infty} < s_n, r_m >_{L^2(\mathbb{R}^2_+)} = < f, g >_{L^2(\mathbb{R}^2_+)} \quad .$$

Finally, for f, g general measurable functions in $L^2(\mathbb{R}^2_+)$, we have the usual decomposition in positive and negative parts of these functions denoted by $f = f^+ - f^-$ and $g = g^+ - g^-$. Defining naturally, $\widetilde{W}(f) = \widetilde{W}(f^+) - \widetilde{W}(f^-)$ and $\widetilde{W}(g) = \widetilde{W}(g^+) - \widetilde{W}(g^-)$, we get, after some trivial algebraic computations, that:

$$< \widetilde{W}(f), \widetilde{W}(g) >_{\mathfrak{H}} = < f, g >_{L^2(\mathbb{R}^2_+)} \ .$$

As a consequence, observe that if $A, B \in \mathcal{B}'$ with $A \cap B = \emptyset$ then:

$$< \widetilde{W}(f\mathbb{I}_A), \widetilde{W}(g\mathbb{I}_B) >_{\mathfrak{H}} = \int_{A\cap B} (fg) d\lambda = 0$$

and, as in $\mathfrak{H}$, independence is equivalent to orthogonality, see [46, p. 23], we get that $\widetilde{W}(f\mathbb{I}_A) \, i \, \widetilde{W}(g\mathbb{I}_B)$. By construction, $\widetilde{W}$ preserves the scalar product and so $\widetilde{W}$ is linear. By construction, also, $\widetilde{W}$ is surjective and so the proof is completed. □

Remark 6. There is a converse to proposition 2.3. In fact, given $\widetilde{W}$ an isometry between $L^2(\mathbb{R}^2_+)$ and $\mathfrak{H}$, a closed gaussian subspace of $L^2(\Omega)$ of centered random variables verifying 15 and 16, that is a centered gaussian stochastic process $(\widetilde{W}(f))_{f\in L^2(\mathbb{R}^2_+)}$ having as a covariance function $\Gamma(f, g) =< f, g >_{L^2}$, the restriction of $\widetilde{W}$ to the indicator functions of the sets in $\mathcal{B}'$ is a gaussian white noise (see [47, p. 80] or [46, p. 67]). In the sequel, we will denote by W either the gaussian white noise or the associated isometry.

Definition 2.2. Given W, a gaussian white noise based on the Lebesgue space $L^2(\mathbb{R}^2_+)$, we will call *brownian sheet* the stochastic process $D(s, t)_{(s,t)\in\mathbb{R}^2_+}$ defined, for each couple (s, t), by:

$$D(s, t) = W(R(s, t)) \, (= \widetilde{W}(\mathbb{I}_{\mathbb{I}_{[0,s[\times[0,t[}}) \ , \tag{17}$$

where $R(s, t)$ denotes the rectangle in $\mathbb{R}^2_+$ having as upper right corner (s, t) and as a lower left corner $(0, 0)$.

The brownian sheet is thus, as already referred, a gaussian centered stochastic process indexed by $\mathbb{R}^2_+$ with a covariance function given for $(s, t), (u, v) \in \mathbb{R}^2_+$ by:

$$\mathbb{E}[D(s, t)\, D(u, v)] = \lambda(R(s, t) \cap R(u, v)) = \min(s, u) \min(t, v) \ .$$

Detailed information on some aspects of the local behavior of this process, not mentioned in this study, can be read in [1]. A picture of a simulation of a typical trajectory of this process is also presented there.

Observe that we have also, that if: $0 < s_1 < s_2 < s_3 < s_4 < 1$ and $0 < t_1 < t_2 < t_3 < t_4 < 1$, then:

$$D(s_4, t_4) - D(s_4, t_3) - D(s_3, t_4) + D(s_3, t_3)\ , \tag{18}$$

is independent from:

$$D(s_2, t_2) - D(s_2, t_1) - D(s_1, t_2) - D(s_1, t_1)\ .$$

Let us note also that the processes

$$\sqrt{ab}\ D(\frac{s}{a}, \frac{t}{b})\ ,$$

for $a, b > 0$ and

$$st\ D(\frac{1}{s}, \frac{1}{t})\ ,$$

are equal in distribution to D and that, for $s_0, t_0 \varepsilon [0, 1]$ fixed, the process

$$D(s + s_0, t + t_0) - D(s_0, t_0)\ ,$$

is equal in distribution to the process:

$$D(s, t) + \sqrt{s_0} B_1(t) + \sqrt{t_0} B_2(s)\ ,$$

with B_1, B_2, Brownian unidimensional processes such that D, B_1 and B_2 are independent from one another.

Remark 7. For $u < s$ and $v < t$, define:

$$\Delta D((s, t), (u, v)) = W(\mathbb{I}_{[u,s[} \times \mathbb{I}_{[v,t[}) = \Delta D((s, v), (u, t)) \tag{19}$$

The gaussian process ΔD, indexed by the rectangles with sides parallel to the axes, is naturally associated with the independent increments of the brownian sheet . As this class of index sets, is a Vapnik-Červonenkis class of sets in the Borel σ- algebra of the unit square, it can be shown, see [2], that ΔD is a continuous process.

Definition 17 as applied to representation 13 gives the following L^2 series representation for the brownian sheet:

$$\begin{aligned} D(s, t) = st\ \zeta_0^0\ + s \sum_{j=0}^{+\infty} \sum_{m=0}^{2^j - 1} \triangle_{jm}(t)\ \zeta_0^{jm} + \\ + t \sum_{i=0}^{+\infty} \sum_{n=0}^{2^i - 1} \triangle_{in}(s)\ \zeta_{in}^0\ + \sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} \sum_{n=0}^{2^i - 1} \sum_{m=0}^{2^j - 1} \triangle_{in}(s)\ \triangle_{jm}\ (t)\ \zeta_{in}^{jm}\ , \end{aligned} \tag{20}$$

where

$$(\zeta_0^0,\ \zeta_0^{jm},\ \zeta_{in}^0,\ \zeta_{in}^{jm})_{injm}\ , \tag{21}$$

is a normal sequence obtained as the image of the system 12, by the isometric mapping W.

The representation in 20, holds in fact, in a stronger sense than the L^2 sense.

THEOREM 2.4. *All the three series in representation* 20, *converge, on* $[0,1]^2$, *uniformly almost surely.*

PROOF. We take, for instance, the double series, as the proof for any one of the other two series can be written along the same lines. As:

$$\sup_{s\varepsilon[0,1]} \triangle_{in}(s) = 2^{-\frac{i}{2}-1}\,, \tag{22}$$

we have that:

$$\sup_{(s,t)\varepsilon[0,1]^2} \Big(\sum_{n,m=0}^{2^i-1,2^j-1} \triangle_{in}(s)\, \triangle_{jm}\,(t)\, |\zeta_{in}^{jm}| \Big) = 2^{-\frac{i}{2}-1}\, 2^{-\frac{j}{2}-1} \sup_{0\leq n\leq 2^i-1, 0\leq m\leq 2^j-1} |\zeta_{in}^{jm}|\,,$$

if we take in account that for i, $j=2$ fixed and for $n=0,1,...,2^i-1$; $m=0,1,...,2^j-1$ the supports of $\triangle_{in}(s)$ and $\triangle_{jm}(t)$ are disjoint. Using now the standard estimates for the tail values of gaussian variables, see 4.4 in the appendix, we have for any $\lambda > 1$ and some constant C:

$$\mathbb{P}[\sup_{0\leq n\leq 2^i-1, 0\leq m\leq 2^j-1} |\zeta_{in}^{jm}| > \sqrt{2\lambda \log 2^{i+j}}] \leq C\, 2^{(i+j)(1-\lambda)}\,. \tag{23}$$

By Borel-Cantelli, we have almost surely, for i, j large enough:

$$\sup_{0\leq n\leq 2^i-1, 0\leq m\leq 2^j-1} |\zeta_{in}^{jm}| \leq \sqrt{2\lambda(i+j)\log 2}\,, \tag{24}$$

and so the double series converges as the series:

$$\sum_{i,j=0}^{+\infty} \sqrt{\frac{i+j}{2^{(i+j)}}}\,,$$

and the result stated is seen to hold. □

3. On the local behavior

In the following, we present some properties, of the local behavior of the Brownian sheet, obtained from the formula which gives the representation of the sheet as a series in the Schauder system 20, having as coefficients, gaussian random variables.

3.1. Modulus of continuity. A first and easy result is a uniform modulus of continuity, in the sense of ([50, pg. 174], for the Brownian sheet.

THEOREM 3.1. *For almost all $\omega \in \Omega$ there exists $E = E(\omega)$ such that for $0 < h, k < 1$ small enough we have:*

$$\omega_D(h,k) = \sup_{(s,t)\varepsilon[0,1]^2} |D(s+h,t+k) - D(s,t)| \leq$$

$$E\,(\sqrt{h\log(\frac{1}{hk})} + \sqrt{k\log(\frac{1}{kh})})\,. \tag{25}$$

PROOF. Take again the double series in representation 20. The general term is given by:

$$G_i^j(s,t) = \sum_{n=0}^{n=2^i-1} \sum_{m=0}^{m=2^j-1} \triangle_{in}(s)\,\triangle_{jm}\,(t)\,\zeta_{in}^{jm}\,. \tag{26}$$

For the dyadic points as follows:

$$h = \frac{1}{2^{u+1}} \quad ,k = \frac{1}{2^{v+1}} \quad ,s = \frac{p}{2^u} \quad ,t = \frac{q}{2^v}\ ,$$

and the correspondent increments, for the processes $G_i^j(s,t)$, taken at these points, we get:

$$|G_i^j(s+h,t+k) - G_i^j(s,t)| \leq$$
$$\sum_{n=0}^{n=2^i-1} \sum_{m=0}^{m=2^j-1} |\,\triangle_{in}\,(s+h)\,\triangle_{jm}\,(t+k) - \triangle_{in}(s)\,\triangle_{jm}\,(t)|\,\zeta_{in}^{jm}\,.$$

The sum on the right, can be estimated, by taking instead of the terms corresponding to the functions $\triangle_{in}\triangle_{jm}$, the sum:

$$\triangle_{in}(s+h)|\,\triangle_{jm}\,(t+k) - \triangle_{jm}(t)| + \triangle_{jm}(t)|\,\triangle_{in}\,(s+h) - \triangle_{im}(s)|\,,$$

Now, the form of the functions $\triangle_{in}\triangle_{jm}$ (which are linear by parts) gives us the assurance that we can estimate the differences in the following way:

$$|\,\triangle_{in}\,(s+h) - \triangle_{in}(s)| \leq$$
$$\begin{array}{ll} 0 & i > u \\ 2^{-i/2-1} & i = u \\ 2^{i/2}\,\times\,2^{-u-1} & i \leq u\,. \end{array}$$

$$|\,\triangle_{jm}\,(t+k) - \triangle_{jm}(t)| \leq$$
$$\begin{array}{ll} 0 & j > v \\ 2^{-j/2-1} & j = v \\ 2^{j/2}\,\times\,2^{-v-1} & j \leq v\,. \end{array}$$

As we can use $\triangle_{in}(s+h) \leq 2^{-i/2-1}$, $\triangle_{jm}(t) \leq 2^{-j/2-1}$ and again the estimate (24), valid almost surely, we can say that, almost surely for u, v large enough:

(27)
$$\begin{aligned}
&\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}|G_i^j(s+h,t+k) - G_i^j(s,t)| \leq \\
&\leq E\sum_{i=0}^{u}\sum_{j=0}^{v}\sqrt{2\lambda(i+j)\log 2}\left(2^{\frac{-i}{2}-1}2^{\frac{j}{2}}2^{-v-1} + 2^{\frac{-j}{2}-1}2^{\frac{i}{2}}2^{-u-1}\right) \leq \\
&\leq E\sqrt{(u+v)\log 2}\left(2^{\frac{-(v+1)}{2}}2^{\frac{-1}{2}} - 2^{\frac{-(v+1+u+1)}{2}}2^{\frac{-1}{2}} + 2^{\frac{-(u+1)}{2}}2^{\frac{-1}{2}} - 2^{\frac{-(v+1+u+1)}{2}}2^{\frac{-1}{2}}\right) \leq \\
&\leq E\sqrt{\log(\frac{1}{hk})}\left(\sqrt{k}+\sqrt{h}-2\sqrt{hk}\right) .
\end{aligned}$$

The inequality shown between the first and the last term of (27) is exactly the result we pretend to prove, stated for dyadic points and the double series part of the representation (20). With a similar proof as the one just shown, we can take care of the two simple series terms of the representation, getting for dyadic points an estimate of the form:

$$\omega_D(h,k) \leq A\sqrt{h\log(\frac{1}{h})} + B\sqrt{k\log(\frac{1}{k})} + C\sqrt{h\log(\frac{1}{hk})} + D\sqrt{k\log(\frac{1}{hk})} ,$$

with A, B, C, D constants, h, k small enough. Now, the result, stated in the theorem, holds as ω_D is a continuous function which is also non decreasing in each variable. □

3.2. The location homogeneous part. The law of iterated logarithm (L.I.L.) for the Brownian sheet, as stated in [66, pg. 1237], reads:

$$\mathbb{P}[\limsup_{s,t\to+\infty}\frac{D(s,t)}{\sqrt{4st\log\log(st)}} = 1] = 1 ,$$

and this can be translated to a L.I.L., describing the behavior at $(0,0)$ using the time inverted equal in distribution process $stD(1/s,1/t)$, to give:

$$\mathbb{P}[\limsup_{h,k\to 0, h,k\geq 0}\frac{D(h,k)}{\sqrt{4hk\log\log(\frac{1}{hk})}} = 1] = 1 ,$$

Now, as already stressed, for any fixed $s, t\varepsilon[0,1]$ the process:

$$D(s+h,t+k) - D(s,t) , \tag{28}$$

is equal, in distribution, to the process:

$$D(h,k) + \sqrt{t}B_1(h) + \sqrt{s}B_2(k) ,$$

and being so, it is impossible to obtain a complete analogy with the unidimensional Brownian process case, for which a L.I.L. describing the almost sure almost everywhere behavior, for the increments of the type (28), would hold. We have nevertheless the following positive result. Recall that G, the location homogeneous part, denotes the process having as a Schauder series representation, the part of representation (20) corresponding to the double series, and that is:

$$(\forall s, t\varepsilon[0,1]) \quad G(s,t) = \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{n=0}^{2^i-1}\sum_{m=0}^{2^j-1} \triangle_{in}(s)\,\triangle_{jm}\,(t)\,\zeta_{in}^{jm}\,.$$

THEOREM 3.2. *For almost all trajectories there exist points* $(s,t)\varepsilon[0,1]^2$ *such that:*

$$\limsup_{h,k\to 0, h,k\geq 0} \frac{|G(s+h,t+k)-G(s,t)|}{\sqrt{4hk\log(\frac{1}{hk})}} > 0\,.$$

PROOF. Take the dyadic points given by:

$$s_{in} = (n+\frac{1}{2})\frac{1}{2^i} \quad , t_{jm} = (m+\frac{1}{2})\frac{1}{2^j}\ ,$$

and the increments:

$$\delta_i = \frac{1}{2^{i+1}} \quad , \epsilon_j = \frac{1}{2^{j+1}}\ ,$$

then, for the correspondent increments of the process G, we have the following interpolation result:

LEMMA 3.3.

$$\sum_{\alpha,\beta\varepsilon\{+1,-1\}} G(s_{in},t_{jm}) - G(s_{in}+\alpha\,\delta_i, t_{jm}+\beta\,\epsilon_j) = 2^{-\frac{i+j}{2}}\zeta_{in}^{jm}\ , \tag{29}$$

PROOF. Writing each term of the sum in the form:

$$G(s_{in},t_{jm}) - G(s_{in}-\delta_i, t_{jm}-\epsilon_j) =$$
$$G(s_{in},t_{jm}) - G(s_{in},t_{jm}-\epsilon_j) + G(s_{in},t_{jm}-\epsilon_j) - G(s_{in}-\delta_i,t_{jm}-\epsilon_j)\,,$$

$$G(s_{in},t_{jm}) - G(s_{in}+\delta_i, t_{jm}+\epsilon_j) =$$
$$G(s_{in},t_{jm}) - G(s_{in},t_{jm}+\epsilon_j) + G(s_{in},t_{jm}+\epsilon_j) - G(s_{in}+\delta_i,t_{jm}+\epsilon_j)\,,$$

$$G(s_{in},t_{jm}) - G(s_{in}-\delta_i, t_{jm}+\epsilon_j) =$$
$$G(s_{in},t_{jm}) - G(s_{in},t_{jm}+\epsilon_j) + G(s_{in},t_{jm}+\epsilon_j) - G(s_{in}-\delta_i,t_{jm}+\epsilon_j)\,,$$

$$G(s_{in},t_{jm}) - G(s_{in}+\delta_i, t_{jm}-\epsilon_j) =$$
$$G(s_{in},t_{jm}) - G(s_{in},t_{jm}-\epsilon_j) + G(s_{in},t_{jm}-\epsilon_j) - G(s_{in}+\delta_i,t_{jm}-\epsilon_j)\,,$$

we only have to prove that the sum of the terms right above is zero at the *resolutions* $l < i$ and $k < j$, that is to say, the sum is zero when, instead of G, we take:

$$G_l^k(s,t) = \sum_{u=0}^{2^l-1}\sum_{v=0}^{2^k-1} \triangle_{lu}(s)\,\triangle_{kv}\,(t)\,\zeta_{lu}^{kv}$$

with $l = 1, 2, ..., i-1$, $k = 1, 2, ..., j-1$. Now, for each l, k as above, the terms appearing in the sum and obtained from the process G_l^k, are the sum for $u = 0, ..., 2^l - 1$ and $v = 0, 1, ..., 2^h - 1$, of terms of the form:

$$\triangle_{lu}(s_{in})(\triangle_{kv}(t_{jm}) - \triangle_{kv}(t_{jm} - \epsilon_j)) + (\triangle_{lu}(s_{in}) - \triangle_{lu}(s_{in} - \delta_i)) \triangle_{kv} (t_{jm} - \epsilon_j)\,,$$

$$\triangle_{lu}(s_{in})(\triangle_{kv}(t_{jm}) - \triangle_{kv}(t_{jm} + \epsilon_j)) + (\triangle_{lu}(s_{in}) - \triangle_{lu}(s_{in} + \delta_i)) \triangle_{kv} (t_{jm} + \epsilon_j)\,,$$

$$\triangle_{lu}(s_{in})(\triangle_{kv}(t_{jm}) - \triangle_{kv}(t_{jm} + \epsilon_j)) + (\triangle_{lu}(s_{in}) - \triangle_{lu}(s_{in} - \delta_i)) \triangle_{kv} (t_{jm} + \epsilon_j)\,,$$

$$\triangle_{lu}(s_{in})(\triangle_{kv}(t_{jm}) - \triangle_{kv}(t_{jm} - \epsilon_j)) + (\triangle_{lu}(s_{in}) - \triangle_{lu}(s_{in} + \delta_i)) \triangle_{kv} (t_{jm} - \epsilon_j)\,.$$

By regrouping the first two terms in each of the first two sums respectively, we shall have to evaluate, for instance, a term of the form:

$$\triangle_{lu}(s_{in})(2 \triangle_{kv} (t_{jm}) - \triangle_{kv}(t_{jm} - \epsilon_j) - \triangle_{kv}(t_{jm} + \epsilon_j))\,,$$

and the part in brackets is zero because it is given by the scalar product (in $L^2([0,1])$) of e_{jm} by e_{kv}. The same procedure can be applied for all the other regroupments and the result stated follows. □

Returning to the course of the main argument, we use another standard estimation for tail values of finite subfamilies of a normal sequence, see proposition 4.5 in the appendix. For $0 < \lambda < 1$ and for some $0 < c < 1 - \lambda$ we have, if i, j are large enough:

$$\mathbb{P}[\sup_{0 \le n \le 2^i - 1, 0 \le m \le 2^j - 1} \zeta_{in}^{jm} > \sqrt{2\lambda \log 2^{i+j}}] \ > \ 1 - \exp(-2^{(i+j)\,c})\,. \tag{30}$$

Then again, by Borel-Cantelli, for almost all ω:

$$\exists r(\omega) \ \ \forall i, j \geq r \ \ \exists n \varepsilon \{0, ..., 2^i - 1\} \ \ \exists m \varepsilon \{0, ..., 2^j - 1\} \quad \zeta_{in}^{jm}(\omega) > \sqrt{2(i+j)\lambda \log 2}\,.$$

Let (s, t) be an accumulation point of the sequence $(s_{in(\omega)}, t_{jm(\omega)})_{i,n(\omega),j,m(\omega)}$ just exhibited. For such a point and, as a consequence of the lemma, at least one of the five differences:

$$|G(s,t) - G(s_{in}, t_{jm})|\,,$$

and for $\alpha, \beta \varepsilon \{+1, -1\}$,

$$|G(s,t) - G(s_{in} + \alpha\,\delta_i, t_{jm} + \beta\,\epsilon_j)|\,,$$

is greater than:

$$2^{-3} 2^{\frac{-(i+j)}{2}} \sqrt{2\lambda \log(2^{(i+j)})}\,.$$

This gives of course:

$$\limsup_{h,k \to 0, h,k \geq 0} \frac{|G(s+h, t+k) - G(s,t)|}{\sqrt{4hk \log(\frac{1}{hk})}} \geq \sqrt{\frac{\lambda}{2^7}} > 0\,,$$

as stated in the theorem. □

As a last observation we can formulate a nondifferentiability result similar to the one given for the unidimensional Brownian process.

THEOREM 3.4. *Almost surely for each* $(s,t)\varepsilon[0,1]^2$*:*

$$\limsup_{h,k\to 0,h,k\geq 0} \frac{|G(s+h,t+k)-G(s,t)|}{\sqrt{hk}} > 0\,,$$

PROOF. The proof of this result, follows exactly the same steps as the proof, for the Brownian process, given in [31, pg. 239,240], replacing everywhere dyadic interval by dyadic rectangle and using lemma 3.3 presented in this text. □

3.3. The usual increments. We are not yet aware of a L.I.L., for the usual increments of the brownian sheet, similar to the one we have stated for the independent increments. Being so, we don't have a description of the almost everywhere almost sure behavior of those increments. As a consequence, the following theorem should only be seen as an analog of the theorem for the independent increments and under the perspective given by the result on the uniform modulus of continuity 3.1. A preliminary observation, is that we have an interpolation lemma for the usual increments as the one stated previously for the independent increments. That is, with the same method as the one used for proving lemma 3.3, the following interpolation result for the whole Brownian sheet representation, can be established.

LEMMA 3.5.

$$\sum_{\alpha,\beta\varepsilon\{+1,-1\}} D(s_{in},t_{jm}) - D(s_{in}+\alpha\,\delta_i, t_{jm}+\beta\,\eta_j) =$$
$$= s_{in}\,2^{-\frac{j}{2}+1}\zeta_0^{jm} + t_{jm}\,2^{-\frac{i}{2}+1}\zeta_{in}^{0} + 2^{-\frac{i+j}{2}}\zeta_{in}^{jm}\,.$$

PROOF. Looking at the Fourier-Schauder series representation 20, there are three kinds of series we have to deal with, namely, the series in Schauder functions of the variable t, the series in the Schauder functions of the variable s and the series in the product of two Schauder functions, one in the t variable and the other in the s variable. This last kind of series was treated in [16] and accounts for the last term of the sum in the right side of the formula in the lemma. Each one of the series in one variable s or t can be treated in a similar way. Take for instance the dyadic points given by:

$$s_{pr} = (r+\frac{1}{2})\frac{1}{2^p} \quad , t_{qs} = (s+\frac{1}{2})\frac{1}{2^q}\ ,$$

and the increments,

$$\delta_p = \frac{1}{2^{p+1}} \quad , \eta_q = \frac{1}{2^{q+1}}\ ,$$

and consider,

$$H(s,t) = s \sum_{j=0}^{+\infty} \sum_{m=0}^{2^j-1} \triangle_{jm}(t)\, \zeta_0^{jm}\,.$$

As we have that,

$$\begin{aligned} &H(s_{pr}, t_{qs}) - H(s_{pr} + \alpha\,\delta_p, t_{qs} + \beta\,\eta_q) = \\ &\quad = s_{pr} \sum_{j=0}^{+\infty} \sum_{m=0}^{2^j-1} (\triangle_{jm}(t_{qs}) - \triangle_{jm}(t_{qs} + \beta\,\eta_q))\, \zeta_0^{jm} - \alpha \sum_{j=0}^{+\infty} \sum_{m=0}^{2^j-1} \triangle_{jm}(t_{qs} + \beta\,\eta_q)\, \zeta_0^{jm}\,. \end{aligned}$$

when summing over $\alpha, \beta\, \varepsilon \{-1, +1\}$ the terms multiplied by α have a sum equal to zero and we are left with:

$$2 s_{pr} \sum_{\beta\ \varepsilon\{+1,-1\}} \sum_{j=0}^{+\infty} \sum_{m=0}^{2^j-1} (\triangle_{jm}(t_{qs}) - \triangle_{jm}(t_{qs} + \beta\,\eta_q))\, \zeta_0^{jm}\,,$$

which we can easily see, using the same kind of arguments as used in [16], to be equal to:

$$s_{pr}\, 2^{-\frac{q}{2}+1}\, \zeta_0^{qs}\,.$$

It takes a simple computation to show that when summing over $\alpha, \beta\, \varepsilon \{-1, +1\}$ the terms corresponding to$(s_{pr}\, t_{qs} - (s_{pr} + \alpha\,\delta_p, t_{qs} + \beta\,\eta_q))\, \zeta_0^0$, the sum is zero. The lemma is now proved. $\square$

THEOREM 3.6. *For almost all trajectories there exist points $s, t \varepsilon [0,1]^2$ such that:*

$$\limsup_{h,k\to 0} \frac{|D(s,t) - D(s+h, t+k))|}{s\,\sqrt{|k| \log(\frac{1}{|k|})} + t\,\sqrt{|h| \log(\frac{1}{|h|})}} > 0\,.$$

PROOF. We shall use the same method as in the proof of the result on the independent increments process. Take the dyadic points given by:

$$s_{in} = (n + \frac{1}{2})\frac{1}{2^i} \quad, t_{jm} = (m + \frac{1}{2})\frac{1}{2^j}\,,$$

and the correspondent increments:

$$\delta_i = \frac{1}{2^{i+1}} \quad, \eta_j = \frac{1}{2^{j+1}}\,,$$

Using again (30), for any $0 < \lambda < 1$ and for a constant $0 < c < 1 - \lambda$, we have that :

$$\mathbb{P}[\sup_{0 \le n \le 2^i - 1} \zeta_{in}^0 > \sqrt{2\lambda \log 2^i}] \; > \; 1 - \exp(-2^{i\,c})\,. \tag{31}$$

and,

$$\mathbb{P}[\sup_{0 \le m \le 2^j - 1} \zeta_0^{jm} > \sqrt{2\lambda \log 2^j}] \; > \; 1 - \exp(-2^{j\,c})\,. \tag{32}$$

Then again, by Borel-Cantelli, for almost all ω we have a set of indexes $in(\omega), jm(\omega)$ such that:

$$\zeta_0^{jm}(\omega) > \sqrt{2j\lambda \log 2}\ ,\ \ \zeta_{in}^0(\omega) > \sqrt{2i\lambda \log 2}\,.$$

For any one of these indexes: $\zeta_{in}^{jm}(\omega) < 0$ or $\zeta_{in}^{jm}(\omega) > 0$ with probability 1/2. The random variables ζ_{in}^{jm} being independent, for almost all ω there is an infinite subset of our initial set of indexes $in(\omega), jm(\omega)$, such that, for any index in it, we have: $\zeta_{in}^{jm}(\omega) > 0$. Let (s,t) be an accumulation point, of the corresponding sequence of points in the unit square $(s_{in(\omega)}, t_{jm(\omega)})$, and suppose, with no loss of generality, that the sequence converges to the point (s,t). Now, for such a point and, as a consequence of the lemma, at least one of the five differences:

$$|D(s,t) - D(s_{in}, t_{jm})|\,,$$

and for $\alpha, \beta \varepsilon \{+1, -1\}$,

$$|D(s,t) - D(s_{in} + \alpha\,\delta_i, t_{jm} + \beta\,\eta_j)|\,,$$

is greater than:

$$2^{-3}(s_{in}\,2^{-\frac{j}{2}+1}\,\sqrt{2\lambda \log 2^j} + t_{jm}\,2^{-\frac{i}{2}+1}\,\sqrt{2\lambda \log 2^i})\,.$$

If not, we should have the following contradiction:

$$\begin{aligned}
& s_{in}\,2^{-\frac{j}{2}+1}\zeta_0^{jm} + t_{jm}\,2^{-\frac{i}{2}+1}\zeta_{in}^0 \leq \\
& \quad \leq |\sum_{\alpha,\beta\varepsilon\{+1,-1\}} D(s_{in}, t_{jm}) - D(s_{in} + \alpha\,\delta_i, t_{jm} + \beta\,\eta_j)| \leq \\
& \quad \leq \sum_{\alpha,\beta\varepsilon\{+1,-1\}} |D(s_{in}, t_{jm}) - D(s,t) + D(s,t) - D(s_{in} + \alpha\,\delta_i, t_{jm} + \beta\,\eta_j)| \leq \\
& \quad \leq\ s_{in}\,2^{-\frac{j}{2}+1}\,\sqrt{2\lambda \log 2^j} + t_{jm}\,2^{-\frac{i}{2}+1}\,\sqrt{2\lambda \log 2^i} < \\
& \quad < s_{in}\,2^{-\frac{j}{2}+1}\zeta_0^{jm} + t_{jm}\,2^{-\frac{i}{2}+1}\zeta_{in}^0\,.
\end{aligned}$$

We can conclude by writing that for $(\alpha, \beta)\varepsilon\{-1, +1\}^2$ or $(\alpha, \beta) = (0, 0)$ and, for some constant $C > 0$:

$$|D(s,t) - D(s_{in} + \alpha\,\delta_i, t_{jm} + \beta\,\eta_j)| > C\,(s_{in}\sqrt{2\eta_j \log(\frac{1}{\eta_j})} + t_{jm}\sqrt{2\delta_i \log(\frac{1}{\delta_i})}\,)\,.$$

This inequality obviously implies the inequality stated in the theorem. □

3.4. Rapid points for the independent increments. The law of iterated logarithm (L.I.L.) for the process ΔD, associated to the independent increments of the brownian sheet, as stated in [66] reads:

$$\mathbb{P}[\limsup_{h,k\to 0, h,k>0} \frac{\Delta D((s,t)(s+h, t+k))}{\sqrt{2hk \log\log(\frac{1}{hk})}} = 1] = 1\,.$$

This formula describes the almost sure almost everywhere behavior of the independent increments process. The next result shows that in the zero Lebesgue measure set of points where the L.I.L. fails, there are points where the oscillation is more rapid than the almost sure almost everywhere oscillation described by the L.I.L.

THEOREM 3.7. *For almost all trajectories, there exist points* $s, t\varepsilon[0,1]^2$*, such that:*

$$\limsup_{h,k\to 0} \frac{|\Delta D((s,t),(s+h,t+k))|}{\sqrt{|hk|\log(\frac{1}{|hk|})}} > 0\,.$$

PROOF. Take the dyadic points given by:

$$s_{in} = (n+\frac{1}{2})\frac{1}{2^i} \quad , t_{jm} = (m+\frac{1}{2})\frac{1}{2^j} \; ,$$

and the increments:

$$\delta_i = \frac{1}{2^{i+1}} \quad , \eta_j = \frac{1}{2^{j+1}} \; ,$$

then considering the decomposition of the dyadic rectangle centered at the point (s_{in}, t_{jm}) and side's length $2\delta_i$ and $2\eta_j$, as shown in the next figure , we have the following result:

LEMMA 3.8. Interpolation Lemma

$$\Delta D(R1) + \Delta D(R2) - \Delta D(R3) - \Delta D(R4) = 2^{-\frac{i+j}{2}}\,\zeta_{in}^{jm} \tag{33}$$

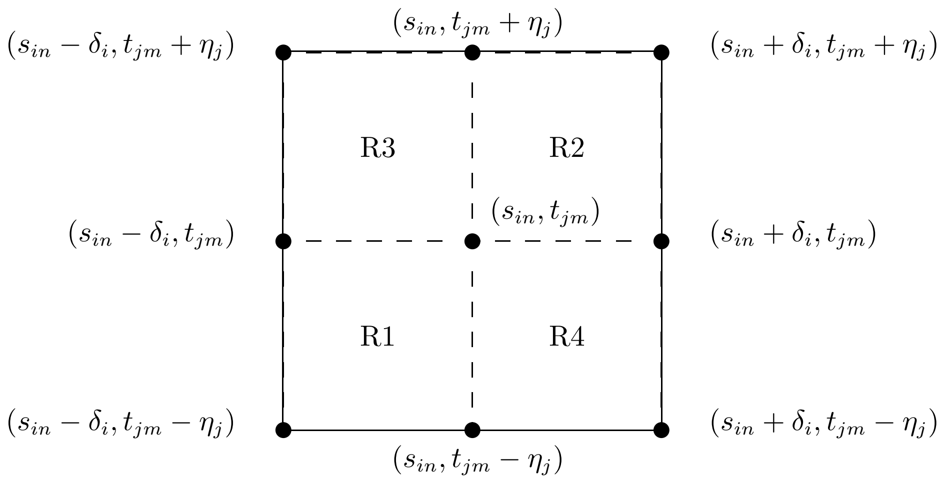

Figure 1.

PROOF. It is a direct consequence of the definitions that:

$$2^{-\frac{i+j}{2}}\,\zeta_{in}^{jm} = W(\mathbb{I}_{R1} + \mathbb{I}_{R2} - \mathbb{I}_{R3} - \mathbb{I}_{R4})\,,$$

and this equality is the equality stated in the lemma. □

Let us return to the course of the main argument. A standard estimation for tail values of finite subfamilies of a normal sequence, see proposition 4.5 in the appendix, gives that for $0 < \lambda < 1$ and for some $0 < c < 1 - \lambda$ we have if i, j are large enough:

$$\mathbb{P}[\sup_{0 \leq n \leq 2^i - 1, 0 \leq m \leq 2^j - 1} \zeta_{in}^{jm} > \sqrt{2\lambda \log 2^{i+j}}] > 1 - \exp(-2^{(i+j)\, c}) \,. \tag{34}$$

Then again by Borel-Cantelli for almost all ω:

$$\exists r(\omega) \;\; \forall i, j \geq r \;\; \exists n \varepsilon \{0, ..., 2^i - 1\} \;\; \exists m \varepsilon \{0, ..., 2^j - 1\} \quad \zeta_{in}^{jm}(\omega) > \sqrt{2(i+j)\lambda \log 2} \,.$$

Let (s, t) be an accumulation point, of the sequence $(s_{in(\omega)}, t_{jm(\omega)})_{i,n(\omega),j,m(\omega)}$, where the indexes are given by the condition just above and suppose, with no loss of the generality, that the sequence is convergent to this point. As a consequence of the lemma, at least one of the values,

$$|\Delta D((s,t), (s_{in} + \alpha\, \delta_i, t_{jm} + \beta\, \eta_j))(\omega)|$$

for $\alpha, \beta, \; \varepsilon \; \{0, -1, 1\}$, is greater or equal then:

$$2^{-4}\, 2^{-\frac{i+j}{2}} \sqrt{2\lambda \, \log(2^{i+j})} \,.$$

Let us prove this assertion. By the lemma, we have always that:

$$2^{-\frac{i+j}{2}}\, \zeta_{in}^{jm}(\omega) \leq \sum_{k=1}^{4} |\Delta D(Rk(in, jm))(\omega)| \,, \tag{35}$$

where $Rk(in, jm)(\omega)$ stands for each one of the four rectangles, decomposing the rectangle centered at the point $(s_{in}, t_{jm})(\omega)$, as in the figure above. Now, if the assertion made is not true, each one of the four terms of the sum is less than:

$$2^2\, 2^{-4}\, 2^{-\frac{i+j}{2}} \sqrt{2\lambda \, \log(2^{i+j})} \,. \tag{36}$$

This is easily seen if we consider that for a given $Rk(in, jm)(\omega)$, the accumulation point (s, t) can be located either inside, either outside, either on the frontier but not coinciding with one of the dyadic points or, finally, coinciding with one of the dyadic points. In each one of these cases, each one of the rectangles $Rk(in, jm)(\omega)$ can be further decomposed, in rectangles, using the point (s, t) and some (up to four in any case) of the dyadic points. This implies that, for some choices of $\alpha, \beta, \; \varepsilon \; \{0, -1, 1\}$:

$$|\Delta D(Rk(in, jm))| \leq 4\, |\Delta D((s,t), (s_{in} + \alpha\, \delta_i, t_{jm} + \beta\, \eta_j))| \,. \tag{37}$$

The decompositions are schematized in the next figures, for the inside and outside situations and, for each of the four rectangles $Rk(in, jm)(\omega)$. It is easy to verify that the same kind of decompositions can be made for the remaining cases (the point (s,t) being on the frontier of one of the rectangles $Rk(in,jm)(\omega)$). Inequalities 35, 36 and 37 combined, show that:

$$\zeta_{in}^{jm}(\omega) \leq \sqrt{2(i+j)\lambda \log 2}\,,$$

contradicting our choice of the indexes $in(\omega), jm(\omega)$. The assertion just proved is that, for some choice $\alpha, \beta, \varepsilon \{0,-1,1\}$, we have:

$$|\Delta D((s,t),(s_{in}+\alpha\,\delta_i, t_{jm}+\beta\,\eta_j))(\omega)| \geq 2^{-4}\sqrt{\frac{2\,\lambda}{2^{i+j}}\log(2^{i+j})}\,,$$

and this implies the assertion made in the theorem. □

.

.

Figure 2.

4. Appendix

This appendix is presented here with a threefold purpose. Primarily, in the first subsection, we state and prove the essential estimates used in this chapter. This is done for pseudogaussian random variables, a larger class than the class of gaussian random variables. The proofs are straightforward rewriting of the original proofs in [31, p. 219]. Next, in following subsection, we present an example which shows that the class of pseudogaussian random variables is, in fact, larger than the class of gaussian random variables. Finally, in the last subsection, we just observe that the tensorial product of two independent brownian processes defines a two parameter stochastic process which admits a random Schauder series representation, having as coefficients, pseudogaussian random variables of the type previously introduced. The methods used for studying the local behavior of the brownian sheet, are show to work as far as the modulus continuity result.

4.1. Estimations for gaussian and pseudogaussian random variables. Let us make precise the concept of pseudogaussian random variable.

DEFINITION 4.1. A random variable will be called *reduced pseudogaussian* if it has, as a density f, a real function defined almost everywhere, with respect to the Lebesgue measure in $\mathbb{R}$, by:

$$f(x) = \exp(-\frac{x^2}{2})\varphi(x) \ ,$$

where φ is a even positive function defined in $\mathbb{R}^*$ and nonincreasing on $\mathbb{R}^*_+$.

Let ζ be a reduced pseudogaussian random variable. We have the following estimates on the tail behavior of such a random variable.

PROPOSITION 4.1. *For $u \geq 1$:*

$$\frac{\varphi(u+1)}{u}\,(1-\frac{1}{e})\,\frac{1}{\sqrt{e}}\,e^{-\frac{u^2}{2}} \leq \mathbb{P}[\zeta \geq u] \leq \frac{\varphi(u)}{u}\,e^{-\frac{u^2}{2}} \tag{38}$$

PROOF. By the change of variables $y = x - u$ we get:

$$\begin{aligned}\mathbb{P}[\zeta \geq u] &= \int_u^{+\infty} f(x)dx = \int_0^{+\infty} e^{-\frac{(y+u)^2}{2}}\varphi(y+u)dy\\ &= e^{-\frac{u^2}{2}}\int_0^{+\infty} e^{-\frac{y^2}{2}}e^{-yu}\varphi(y+u)dy\ .\end{aligned}$$

Now, by observing that for $y \geq 0$ we have $e^{-y^2/2} \leq 1$ and $\varphi(y+u) \leq \varphi(u)$, we get the right-hand side bound. To obtain the lower bound, remark that as for $y \in [0,1]$, we have $e^{-y^2/2} \geq e^{-1/2}$ and $\varphi(y+u) \geq \varphi(1+u)$, then:

$$\int_0^{+\infty} e^{-\frac{y^2}{2}}e^{-yu}\varphi(y+u)dy \geq \frac{\varphi(1+u)}{\sqrt{e}}\int_0^1 e^{-yu}dy\ ,$$

the stated result being thus obtained. □

The correspondent result for gaussian random variables goes as follows:

COROLLARY 4.2. *If ξ is a reduced gaussian random variable then $\varphi(x) = \frac{1}{\sqrt{2\pi}}$ and we get for $u \geq 1$:*

$$\frac{1}{\sqrt{2\pi e}}\,(1-\frac{1}{e})\,\frac{e^{-\frac{u^2}{2}}}{u} \leq \mathbb{P}[\xi \geq u] \leq \frac{1}{\sqrt{2\pi}}\,\frac{e^{-\frac{u^2}{2}}}{u}$$

The next proposition gives an estimation of the supreme of a finite family of modules of real pseudogaussian random variables.

PROPOSITION 4.3. *Let $n \geq 2$, $\zeta_1, \ldots, \zeta_n$ be reduced pseudogaussian random variables and $\lambda > 1$ a parameter. We have then, for a constant γ:*

$$\mathbb{P}[\sup_{1\leq j\leq n} |\zeta_j| > \sqrt{2\lambda\log n}] \leq \gamma\, n^{(1-\lambda)}$$

PROOF. Applying the estimate 38 to $|\zeta_j|$ gives:

$$\mathbb{P}[|\zeta_j| > \sqrt{2\lambda\log n}] \leq 2\frac{\varphi(u)}{u}\,e^{-\frac{u^2}{2}}|_{u=\sqrt{2\lambda\log n}} = 2n^{-\lambda}\frac{\varphi(\sqrt{2\lambda\log n})}{\sqrt{2\lambda\log n}}\ .$$

Now, as

$$\mathbb{P}[\sup_{1\leq j\leq n}|\zeta_j| > \sqrt{2\lambda\log n}] = \mathbb{P}[\bigcup_{j=1}^n\{|\zeta_j| > \sqrt{2\lambda\log n}\}] \leq 2n^{-\lambda}\frac{\varphi(\sqrt{2\lambda\log 2})}{\sqrt{2\lambda\log 2}}\ \sum_{j=1}^n 1\ ,$$

the announced result now follows by taking $\gamma = 2\frac{\varphi(\sqrt{2\lambda\log 2})}{\sqrt{2\lambda\log 2}}$ □

In the particular case of gaussian random variables, we get the proposition under the following form.

COROLLARY 4.4. *For $n \geq 2$, $\xi_1, \ldots, \xi_n$, centered gaussian random variables, identically distributed with variance σ^2 bounded by 1, and for a parameter $\lambda \geq 1$, we have with $\gamma = \frac{2\sigma\sqrt{\pi}}{\sqrt{\lambda \log 2}}$:*

$$\mathbb{P}[\sup_{1 \leq j \leq n} |\xi_j| > \sqrt{2\lambda \log n}] \leq \gamma \, n^{(1-\lambda)}$$

The next proposition, gives an estimate for the lower bound probability, of a tail event, for a finite subfamily of a sequence of pseudogaussian variables. As already said, we follow the general line of reasoning of the proof, for the correspondent result for gaussian variables, in [31, p. 221].

PROPOSITION 4.5. *Let $\zeta_1, \ldots, \zeta_n, \ldots$ be a sequence of independent, reduced, identically distributed, pseudogaussian random variables such that, with the notations given in the definition, for a given α there exist β and γ such that:*

$$\forall n \in \mathbb{N} \quad \forall \lambda > 0 \quad \varphi(\alpha\sqrt{2\lambda \log n}) \geq \beta n^{-\gamma\lambda} \; .$$

For $0 < \lambda < 1/(\gamma+1)$ and $0 < c < 1 - \lambda(\gamma+1)$, parameters, we have for, n large enough, that:

$$\mathbb{P}[\sup_{1 \leq j \leq n} \zeta_j > \sqrt{2\lambda \log n}] > 1 - e^{-n^c} \; .$$

PROOF. Observe that the condition of independence, on the sequence, allow us to write for $u > 1$:

$$\mathbb{P}[\sup_{1 \leq j \leq n} \zeta_j < u] = \mathbb{P}[\bigcap_{j=1}^{n} \{\zeta_j < u\}] = \prod_{j=1}^{n} \mathbb{P}[\zeta_j < u] \; .$$

Now, by using the lower bound in 38,we have that:

$$\prod_{j=1}^{n} \mathbb{P}[\zeta_j < u] \leq (1 - \gamma_1(u)\frac{e^{-\frac{u^2}{2}}}{u})^n \; ,$$

with $\gamma_1(u) = \frac{\varphi(1+u)(e-1)}{e\sqrt{e}}$. Suppose now that $u = \sqrt{2\lambda \log n}$. As for $0 \leq a < 1$, we have that $(1-a)^n \leq e^{-na}$, we get, using the hypothesis on φ and, for a certain constant β:

$$(1 - \gamma_1(u)\frac{e^{-\frac{u^2}{2}}}{u})^n|_{u=\sqrt{2\lambda \log n}} \leq e^{-\frac{\gamma_1(\sqrt{2\lambda \log n})}{\sqrt{2\lambda \log n}} n^{(1-\lambda)}} \leq e^{\frac{-\beta n^{1-\lambda(\gamma+1)}}{\sqrt{\log n}}} \; . \tag{39}$$

Consider $0 < c < (1 - \lambda(\gamma+1))$. As we have that $(1 - \lambda(\gamma+1)) = c + ((1 - \lambda(\gamma+1)) - c)$ and $\lim_{n \to +\infty}[\frac{\beta}{\sqrt{\log n}} n^{((1-\lambda(\gamma+1))-c)}] = +\infty$, we get, for the right-hand side of 39 that, for n big enough:

$$\exp(-\frac{\beta}{\sqrt{\log n}} n^{(1-\lambda(\gamma+1))}) = \exp(-[\frac{\beta}{\sqrt{\log n}} n^{(1-\lambda(\gamma+1))-c}] n^c) \leq e^{-n^c} \; .$$

The result announced now follows from:

$$\mathbb{P}[\sup_{1\leq j\leq n} \zeta_j < \sqrt{2\lambda \log n}] \leq e^{-n^c} ,$$

by observing that $\mathbb{P}[\sup_{1\leq j\leq n} \zeta_j = \sqrt{2\lambda \log n}] = 0$ □

4.2. A nongaussian example of a pseudogaussian random variable. We will observe first, the general form of the density of the product of two independent normal variables then, we will show that a random variable, such as the one just described, is a pseudogaussian random variable and finally, we will present some estimates that show a relation between the density of this pseudogaussian random variable and, the density of a normal variable.

PROPOSITION 4.6. *Let ξ_1, ξ_2 be two independent reduced normal random variables. Then, $\zeta = \xi_1\xi_2$ is a random variable, that admits as a density over $\mathbb{R}^*$, the function f defined by:*

$$\forall x \in \mathbb{R}^* \quad f(x) = \frac{\exp(-2x^2)}{2\pi} \int_0^{+\infty} \frac{e^{-u}}{\sqrt{u(x^2+u)}} du .$$

PROOF. As a first step, we compute the characteristic function of ζ, using the independence of ξ_1 and ξ_2.

$$\mathbb{E}[e^{it\zeta}] = \int_{\mathbb{R}^2} e^{itx_1x_2} dF_{\xi_1\xi_2}(x_1,x_2) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{i(tx_1)x_2} dF_{\xi_2}(x_2) \right) dF_{\xi_1}(x_1) .$$

Now, as the inner integral is given by $e^{-\frac{1}{2}t^2x_1^2}$, we have:

$$\mathbb{E}[e^{it\zeta}] = \int_{\mathbb{R}} e^{-\frac{1}{2}t^2x_1^2} dF_{\xi_1}(x_1) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}t^2x_1^2} e^{-\frac{x_1^2}{2}} dx_1 = \frac{1}{\sqrt{1+t^2}}$$

Now, instead of inverting directly this characteristic function, we make the following trivial observations. Suppose that η is a χ^2 random variable with one degree of freedom then, $\mathbb{E}[e^{it\eta}] = \frac{1}{\sqrt{1-2it}}$. Suppose that η_1 and η_2 are independent and have the same distribution as η. For $\delta = \frac{1}{2}(\eta_1 - \eta_2)$, the corresponding characteristic functions will be given by:

$$\mathbb{E}[e^{it\delta}] = \mathbb{E}[e^{i\frac{t}{2}\eta_1}]\,\mathbb{E}[e^{i\frac{-t}{2}\eta_2}] = \frac{1}{\sqrt{1-it}} \frac{1}{\sqrt{1+it}} = \frac{1}{\sqrt{1+t^2}} .$$

The observations just made, show that δ and ζ have the same law. In order to calculate the law of δ, we remark that η_1 (respectively $(-\eta_2)$) has a density f_1 (respectively f_2), defined by:

$$f_1(x) = \frac{1}{\sqrt{2}\Gamma(\frac{1}{2})} \frac{e^{-\frac{x^2}{2}}}{\sqrt{x}} \mathbb{I}_{]0,+\infty[}(x) \quad (\text{ respectivamente } f_2(x) = \frac{1}{\sqrt{2}\Gamma(\frac{1}{2})} \frac{e^{-\frac{x^2}{2}}}{\sqrt{-x}} \mathbb{I}_{]-\infty,0[}(x)).$$

Then, for $x \neq 0$, g, the density of 2δ will be given by:

$$\begin{aligned} f(x) = (f_1 * f_2)(x) &= \frac{1}{2\pi}\int_0^{+\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{y}}\frac{e^{-\frac{(x-y)^2}{2}}}{\sqrt{y-x}}\mathbb{I}_{]-\infty,0[}(x-y)dy \\ &= \frac{1}{2\pi}\int_x^{+\infty} \frac{e^{-\frac{y^2}{2}}\, e^{-\frac{x^2}{2}+\frac{2xy}{2}-\frac{y^2}{2}}}{\sqrt{y(y-x)}}dy \\ &= \frac{e^{-\frac{x^2}{2}}}{2\pi}\int_x^{+\infty} \frac{e^{-y(y-x)}}{\sqrt{y(y-x)}}dy \ . \end{aligned}$$

Now, with the change of variables $y(y-x) = u$, we get finally the result stated in the proposition, if we use the fact that $f(x) = 2g(2x)$. For $x = 0$, the convolution above is seen to be a nonconvergent integral. □

PROPOSITION 4.7. *The random variable* ζ *as in proposition* 4.6 *is a pseudogaussian random variable.*

PROOF. In fact, for $x \neq 0$, we have, with the notations of definition 4.1, that

$$\varphi(x) = \frac{\exp(\frac{-3x^2}{2})}{2\pi}\int_0^{+\infty}\frac{e^{-u}}{\sqrt{u(u+x^2)}}du = \frac{\exp(\frac{-3x^2}{2})}{2\pi}\psi(x) \ ,$$

which is an even positive function such that, by the Lebesgue convergence theorem, verifies:

$$\psi^{'}(x) = -x\int_0^{+\infty}\frac{e^{-u}u}{(u(u+x^2))^{\frac{3}{2}}}du \ .$$

This shows that $\psi^{'}$ is negative for $x > 0$ and, as a consequence, that ψ and then φ as defined above, fall under the scope of definition 4.1. □

The next proposition, allow us to compare the general form of the graph of the density f, with the known graph of the density of a normal random variable. Moreover it shows that proposition 4.5 can be applied to the current example of pseudogaussian random variable.

PROPOSITION 4.8. *Let* $f_N(x) = \frac{\exp(\frac{-x^2}{2})}{\sqrt{2\pi}}$ *be the density of a normal random variable. We have the following estimates.*

$$\frac{\exp(\frac{-3x^2}{2})}{|x|}\left[\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}\frac{e^{-u}}{\sqrt{u}}du\right] \geq \frac{f(x)}{f_N(x)} \geq \begin{cases} \frac{\exp(\frac{-3x^2}{2})}{|x|^{\frac{3}{2}}}\left[\frac{1}{\sqrt{10\pi}}(1-e^{-|x|})\right] & para\ |x| \geq 1 \\ \log(\frac{1}{|x|})\left[\sqrt{\frac{2}{5\pi}}\right]\exp(\frac{-3x^2}{2}) & para\ |x| < 1 \end{cases}$$

PROOF. For the upper bound just observe that for $x \neq 0$:

$$\int_0^{+\infty}\frac{e^{-u}}{\sqrt{u(u+x^2)}}du \leq \frac{1}{|x|}\int_0^{+\infty}\frac{e^{-u}}{\sqrt{u}}du \ .$$

For the lower bound, when $|x| \geq 1$, consider:

$$\int_0^{+\infty} \frac{e^{-u}}{\sqrt{u(u+x^2)}} du \geq \int_0^{+|x|} \frac{e^{-u}}{\sqrt{u(u+x^2)}} du$$
$$\geq \frac{1}{\sqrt{5|x|(4|x|+x^2)}} \int_0^{+|x|} e^{-u} du \geq \frac{1}{\sqrt{5}|x|^{\frac{3}{2}}}(1-e^{-|x|}) \ .$$

For the remaining case, that is the lower bound when $|x| < 1$, it is enough to take in account that:

$$\int_0^{+\infty} \frac{e^{-u}}{\sqrt{u(u+x^2)}} du \geq \int_{x^2}^{1} \frac{e^{-u}}{\sqrt{u(u+x^2)}} du \geq \frac{1}{\sqrt{5}} \int_{x^2}^{1} \frac{du}{u} = \frac{1}{\sqrt{5}} 2 \log(\frac{1}{|x|}) \ .$$

This last estimation, shows the logarithmic singularity at zero of the density of the pseudo-gaussian random variable under scrutiny. □

For illustration purposes we present next a representation of the graph of the density f.

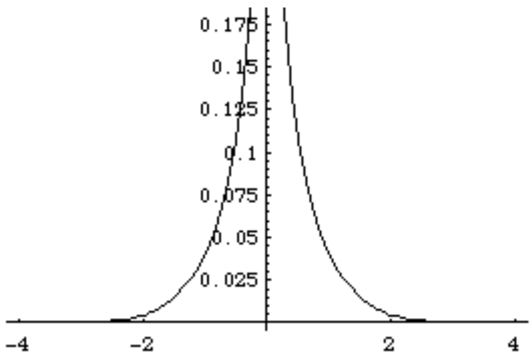

Figure 3.

4.3. A brownian tensorial product process. A natural way of building functions of two variables, is to consider the tensorial product of functions of one variable. The property of being gaussian, being essentially preserved by the action of linear operations, it is not to be expected that such a procedure will yield gaussian processes whenever operating over trajectories of gaussian processes. Nevertheless, using the additional hypothesis of independence, of the factor processes, some similarities remain, with the gaussian case, that allow us to obtain, at least, a modulus of continuity result.

Let $(B_1(s))_{s\in[0,1]}$ and $(B_2(t))_{t\in[0,1]}$ be two independent brownian processes, defined on the same probability space. Let:

$$B_1(s) = s\ \xi_0\ + \sum_{i=0}^{+\infty} \sum_{n=0}^{2^i-1} \triangle_{in}(s)\ \xi_{im}\ ,$$

and

$$B_2(t) = t\ \zeta_0\ + \sum_{j=0}^{+\infty} \sum_{m=0}^{2^j-1} \triangle_{jm}(t)\ \zeta_{jm}\ ,$$

be the corresponding expansions as Schauder gaussian series, which we know to be convergent uniformly almost surely. Then, the tensorial product process $(B_1 \otimes B_2(s,t))_{(s,t)\in[0,1]^2}$ defined almost surely by:

$$B_1 \otimes B_2(s,t) = B_1(s)B_2(t) \ ,$$

admits a Schauder series representation given by:

$$\begin{aligned} B_1 \otimes B_2(s,t) = st \ \zeta_0\xi_0 \ + \ s \sum_{j=0}^{+\infty}\sum_{m=0}^{2^j-1} \triangle_{jm}(t) \ \zeta_{jm}\xi_0 + \\ + t \sum_{i=0}^{+\infty}\sum_{n=0}^{2^i-1} \triangle_{in}(s) \ \xi_{in}\zeta_0 \ + \ \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{n=0}^{2^i-1}\sum_{m=0}^{2^j-1} \triangle_{in}(s) \triangle_{jm} \ (t) \ \xi_{in}\zeta_{jm} \ , \end{aligned} \tag{40}$$

Observe that, for all m, n, i, j, the random variables $\xi_0\zeta_0$, $\xi_0\zeta_{jm}$, $\xi_{in}\zeta_0$ $\xi_{in}\zeta_{jm}$, are pseudogaussian random variables. This representation is similar to the one already used for the brownian sheet in formula 20. The main differences are, of course, that the series is a Schauder pseudogaussian series, instead of a Schauder gaussian series of *independent* random variables. The process $(\frac{B_1 \otimes B_2(s,t)}{\sqrt{st}})_{(s,t)\in[0,1]^2}$ is a centered pseudogaussian process and, due to the independence hypothesis,the process $B_1 \otimes B_2$ has the same covariance function as the brownian sheet.

The methods used for the series representation, given for the brownian sheet, namely the estimations presented in the first subsection just above, will work for the series representation of $B_1 \otimes B_2$, as long as the independence of the random variables in the series representation is not invoked. Being so, we have, with exactly the same proof as already presented in theorem 2.4 and theorem 3.1, the following analog results.

THEOREM 4.9. *All the three series in representation* 40*, converge uniformly almost surely.*

THEOREM 4.10. *Almost surely for h and k small enough, we have for some constant E:*

$$\begin{aligned} \omega_{B_1 \otimes B_2}(h,k) = \sup_{(s,t)\varepsilon[0,1]^2} |B_1 \otimes B_2(s+h,t+k) - B_1 \otimes B_2(s,t)| \leq \\ E \, (\sqrt{h \log(\frac{1}{hk})} + \sqrt{k \log(\frac{1}{kh})}) \, . \end{aligned} \tag{41}$$

CHAPTER III

The asymptotic behavior of the second moment of the Fourier transform of a random measure

1. Summary

The behavior, at infinity, of the Fourier transform of the random measures, that appear in the theory of multiplicative chaos of J.-P. Kahane, is an area quite unexplored. In this chapter, we present first, some reflections on the connection between uniform continuity and behavior at infinity, for an integrable function. Then, after proving a formula essentially due to Frostman and, after looking at the behavior at infinity of the Fourier transform of some remarkable functions and measures, we study the asymptotic behavior of the second moment of the Fourier transform of a random measure that appears in the theory of multiplicative chaos. The result obtained generalizes slightly a result previously announced by J.-P. Kahane.

2. Introduction

The problem considered, in the last sections of this work, admits a general formulation that can be stated as follows. A random measure is defined, in the sense of a random object (see [31, p. 9]), by the action of a random operator on an usual Borel measure, in a way such that, its Fourier transform is almost surely a uniformly continuous and bounded function. A natural conjecture to be made, is that, the almost sure behavior, at infinity, of the Fourier transform of the random measure, is somehow related to the behavior at infinity of the Fourier transform of the Borel measure, used to build the random measure. A technique that has given good results, in problems such as the one here presented, goes as follows (see [31, p. 257]). One gets first, good estimates on the behavior of the moments of the random functions and then, by an accumulation argument, the almost sure behavior is obtained. The study of the asymptotic behavior of the second moment, besides the instrumental usefulness for the technique described, can give an idea of what to expect on the almost sure behavior.

2.1. Asymptotic behavior and uniform continuity. Let us consider, for instance, the Lebesgue space on the line, denoted by $(\mathbb{R}, \mathcal{L}, \lambda)$ and a numeric integrable function f. If, $(a_n)_{n\in\mathbb{N}}$ is a sequence of non negative numbers, such that $\lim_{n\to\infty} a_n = +\infty$ then, by Markov inequality:

$$\forall n \in \mathbb{N} \;\; \lambda(|f| \geq a_n) \leq \frac{||f||_1}{a_n} \; .$$

As a consequence, $\lim_{n\to\infty} \lambda(|f| \geq a_n) = 0$. In order to have a stronger result, namely that $\lim_{x\to+\infty} f(x) = 0$, some additional hypotheses are needed. In fact, let f be defined by:

$$f = \sum_{i=0}^{+\infty} 2^{i-1} \mathbb{I}_{[i, i+2^{-2i}[} \; .$$

For this function, $||f||_1 = 1$ and, nevertheless, $\limsup_{x\to+\infty} f(x) = +\infty$. Let us observe that, continuity, or even differentiability, is not the main issue in what concerns the additional hypothesis needed. One can consider, for instance, the continuous function given by:

$$\forall x \in \mathbb{R} \quad g(x) = \sum_{i=1}^{+\infty} 2^i (1 - 2^{2i}|x - i|) \mathbb{I}_{[i-2^{-2i}, i+2^{-2i}]}(x) \; .$$

Noticing that g is an affine function by pieces, that takes the value zero for $x = i - 2^{-2i}$ and $x = i + 2^{-2i}$ and, the value 2^i for $x = i$, a quick calculation shows that $||g||_1 = 1$ with, of course, $\lim_{i\to+\infty} g(i) = +\infty$. Even a $\mathcal{C}^\infty(\mathbb{R})$ function exists with the same behavior. Take the classic test function defined by

$$\forall x \in \mathbb{R} \;\; f(x) = e^{\frac{-1}{1-x^2}} \mathbb{I}_{\{|x|<1\}}(x)$$

and, for each $\epsilon \in]0, 1]$, the function h_ϵ defined by:

$$\forall x \in \mathbb{R} \;\; h_\epsilon(x) = \frac{1}{f(0)\, ||f||_1 \sqrt{\epsilon}} f(\frac{x}{\epsilon}) \; .$$

It is easy to verify that:

$$h_\epsilon(0) = \frac{1}{||f||_1 \sqrt{\epsilon}} \; ,$$

that the support of h_ϵ is given by

$$\text{supp}(h_\epsilon) = \epsilon \, \text{supp}(f) = [-\epsilon, +\epsilon] \; ,$$

and finally, that $||h_\epsilon||_1 = \frac{\sqrt{\epsilon}}{f(0)} = e\sqrt{\epsilon}$. Our $\mathcal{C}^\infty(\mathbb{R})$ function, is defined by:

$$\forall x \in \mathbb{R} \;\; h(x) = \sum_{i=1}^{+\infty} h_{2^{-i}}(x - i) \; .$$

The function h is the sum of regular disjointedly supported bumps $(h_{2^{-i}})$, centered at the points $i \in \mathbb{N}$, and so:

$$||h||_1 = \sum_{i=1}^{+\infty} ||h_{2^{-i}}||_1 = \frac{e}{\sqrt{2} - 1} \; .$$

Nevertheless, we still have:

$$\lim_{i\to+\infty} h(i) = \lim_{i\to+\infty} h_{2^{-i}}(0) = \lim_{i\to+\infty} \frac{(\sqrt{2})^i}{||f||_1} = +\infty \ .$$

The next proposition shows that, under the hypothesis of uniform continuity, the integrability of the function is enough to ensure that the limit at infinity is zero. This result, can be taken in consideration when dealing with the Fourier transform of a measure, which is a uniformly continuous function on $\mathbb{R}$ (see [37, p. 132]).

PROPOSITION 2.1. *Let u be an integrable uniformly continuous function on $\mathbb{R}$. Then* $\lim_{|x|\to+\infty} u(x) = 0$.

PROOF. By considering the positive and negative parts of u, we can restrain the proof to the case in which the function u is non negative. Furthermore, we will only consider the case of the limit in $+\infty$, as the proof is similar for the other case, the limit in $-\infty$. For $\delta > 0$, the integral of u can be decomposed in the following way:

$$\int_0^{+\infty} u(x)dx = \sum_{n=0}^{+\infty} \int_{n\delta}^{(n+1)\delta} u(x)dx < +\infty \ . \tag{42}$$

Using the mean value theorem of Cauchy (see [24, p. 321]), we get a real sequence $(x_n)_{n\in\mathbb{N}}$ such that:

$$\forall n \in \mathbb{N} \quad x_n \in [n\delta, (n+1)\delta] \ \ , \ \ u(x_n)\delta = \int_{n\delta}^{(n+1)\delta} u(x)dx \ ,$$

Observe that the decomposition of the integral, in 42, implies that $\lim_{n\to+\infty} u(x_n) = 0$ Now, for $\epsilon > 0$ given, we have, as a consequence of the uniform continuity of u that, there exists $\delta_0 > 0$ such that:

$$\forall x, y \in \mathbb{R}_+ \ \ |x - y| \leq \delta_0 \ \Rightarrow \ |u(x) - u(y)| \leq \epsilon \ .$$

For any $x\epsilon\mathbb{R}_+$, let n_0 be the integer such that $x \in [n_0\delta_0, (n_0+1)\delta_0[$. Then, for x big enough, such that $u(x_{n_0}) \leq \epsilon$, we get:

$$|u(x)| \leq |u(x) - u(x_{n_0})| + u(x_{n_0}) \leq 2\epsilon$$

and so, the result announced follows. □

COROLLARY 2.2. *Let u and φ be uniformly continuous functions, such that u is bounded and φ is non increasing. If, for some $A > 0$*

$$\int_A^{+\infty} |\varphi(x)u(x)|dx < +\infty \ ,$$

then, $\lim_{x\to+\infty} \varphi(x)u(x) = 0$.

PROOF. This result follows from proposition 2.1, by noticing first that the product of two bounded uniformly continuous functions is an uniformly (and bounded) continuous function and then, that, the proposition referred, remains true if the requirement, of uniform continuity of the function, is made only in a neighborhood of $+\infty$ or $-\infty$. □

REMARK 8. If, in the corollary, u is taken to be uniformly continuous and, φ is such that $1/\varphi$ is well defined and decreasing to zero then, the conclusion doesn't stand anymore.

As an example of such a situation, suppose that $\varphi \geq 1$ and consider a function u defined by following the general idea used above to define g. So, let it be:

$$\forall x \in \mathbb{R}\ u(x) = \sum_{i=1}^{+\infty} \frac{1}{\varphi(i)} \ (1 - |x - i| 2^i \varphi(i + 2^{-i}))\, \mathbb{I}_{[i - \frac{1}{2^i \varphi(i+2^{-i})}, i + \frac{1}{2^i \varphi(i+2^{-i})}[}(x) \ .$$

Then, by construction, $\lim_{i \to +\infty} u(i)\varphi(i) = 1$ and, moreover:

$$\int_0^{+\infty} u(x)\varphi(x)\, dx \leq \sum_{i=1}^{i=+\infty} \varphi(i + \frac{1}{2^i \varphi(i + 2^{-i})}) \ \frac{1}{2^{i-1}\varphi(i + 2^{-i})} \leq 2 \ .$$

Observe that u is uniformly continuous, as soon as:

$$\sum_{i=1}^{+\infty} \frac{1}{\varphi(i)} < +\infty \ .$$

In fact, under this condition and by Weierstrass criterion, u can be seen as a uniform limit of uniformly continuous functions. A function as ψ, defined by:

$$\forall x \in \mathbb{R} \quad \psi(x) = (1 - \frac{|x - a|}{M}) \mathbb{I}_{[a-M, a+M]}(x) \, .$$

is uniformly continuous on $\mathbb{R}$ because, its Fourier transform, which a quick calculation shows to be equal to:

$$\widehat{\psi}(\xi) = \frac{2}{M} \ \frac{1 - \cos(M\xi)}{(2\pi\xi)^2} \ e^{-2\pi i a \xi} \ ,$$

is Lebesgue integrable. Being so, by the inversion formula (see [37, p. 126]):

$$\psi = \frac{1}{2\pi} \ (\widehat{\psi})^{\vee} \, .$$

As a consequence, ψ is uniformly continuous.

REMARK 9. Our main motivation, for embarking in the study presented above, was a possible application to the asymptotic behavior of integrals of Fourier transform of measures, similar to the following one:

$$\int_{\mathbb{R}^n} \frac{|\sigma(x)|^2}{||x - \xi||^a} dx \ .$$

Unfortunately, this kind of approach is doomed to fail as $\widehat{\sigma}$ need not go to infinity for Radon measures σ, even for Radon measures with finite energy. See [43, p. 169] for positive partial results concerning the asymptotic behaviors of Fourier transforms of Radon measures.

3. The asymptotic behavior of some classic functions

For completeness sake, we next state and prove a classical result (see [57, p. 51]), on the asymptotic behavior of the Fourier transform of the indicator function of the unit ball in $\mathbb{R}^n$. The method used in the proof, will be next applied to study a similar, but less classical, result. Hereafter, in this chapter, letters c, d and e will denote constants not necessarily the same at every instance.

PROPOSITION 3.1. *Let $B = B(0,1) = \{x \in \mathbb{R}^n : ||x|| < 1\}$ be the unit ball of $\mathbb{R}^n$ and $U = \mathbb{I}_B$ the indicator function of B. Then, for some constant c, we have that:*

$$|\widehat{U}(\xi)| \leq \frac{c}{|\xi|^{\frac{n+1}{2}}} \tag{43}$$

PROOF. As U is a radial function, its Fourier transform is given by (see [60, p. 430] and, for a proof, [59, p. 155]):

$$\widehat{U}(\xi) = 2\pi ||\xi||^{\frac{2-n}{2}} \int_0^{+\infty} J_{\frac{n-2}{2}}(2\pi||\xi||r)\, U_0(r)\, r^{\frac{n}{2}}\, dr\ , \tag{44}$$

where U_0, is such that $U(x) = U_0(||x||)$ and, $J_{\frac{n-2}{2}}$ is a Bessel function. As U_0 is the indicator function of $[0,1[$, then:

$$\widehat{U}(\xi) = 2\pi ||\xi||^{\frac{2-n}{2}} \int_0^{1} J_{\frac{n-2}{2}}(2\pi||\xi||r)\, r^{\frac{n}{2}}\, dr\ .$$

By a change of variables, given by $2\pi||\xi r|| = \rho$, this expression is turned on the following one:

$$\widehat{U}(\xi) = (2\pi)^{-\frac{n}{2}} ||\xi||^{-n} \int_0^{2\pi||\xi||} J_{\frac{n-2}{2}}(\rho)\, \rho^{\frac{n}{2}}\, d\rho\ .$$

Now, a classical relation of Bessel functions (see [41, p. 141]) states that:

$$t^{\nu+1}\, J_\nu(t) = \frac{d}{dt}[\, t^{\nu+1} J_{\nu+1}(t)]\ . \tag{45}$$

Applying this relation, with $\nu = \frac{n}{2} - 1$, gives:

$$\widehat{U}(\xi) = ||\xi||^{-\frac{n}{2}}\, J_{\frac{n}{2}}(2\pi||\xi||)$$

The asymptotic behavior of $J_\nu(t)$ is known (see [41, p. 134]). In fact, as for some constant c, the following estimate holds,

$$|J_\nu(t)| \leq \frac{c}{\sqrt{t}}\ , \tag{46}$$

we have finally that

$$|\widehat{U}(\xi)| \leq (\frac{c}{\sqrt{2\pi}})\, \frac{1}{||\xi||^{\frac{n+1}{2}}}\ ,$$

which is the result requested. □

A similar result is got, when, the unit ball is replaced by $B(0,\delta)$ and, the indicator function appears multiplied by a remarkable locally integrable radial function.

Proposition 3.2. *Let* $B_\delta = B(0,\delta) = \{x \in \mathbb{R}^n : ||x|| < \delta\}$ *be a ball of* $\mathbb{R}^n$*, centered in zero with radius* $\delta > 0$ *and, for* $0 < \alpha < n$*, the function defined by:*

$$U_\delta^\alpha(x) = \frac{\mathbb{I}_{B_\delta}(x)}{||x||^\alpha} \ .$$

Then, for some constants, which we denote always, by c*, we have that:*

$$|\widehat{U_\delta^\alpha}(\xi)| \leq \begin{cases} \frac{c}{||\xi||^{n-\alpha}} & \text{if} \quad \alpha > \frac{n-1}{2} \\ \frac{c}{||\xi||^{\frac{n+1}{2}}} & \text{if} \quad \alpha \leq \frac{n-1}{2} \end{cases} \tag{47}$$

Proof. We will consider that $\delta = 1$. In fact, the change of variables, given by $x = \delta y$, shows that:

$$\int_{||x||<\delta} \frac{e^{-2\pi i x\cdot\xi}}{||x||^\alpha} \, dx = \delta^{n-\alpha} \int_{||y||<1} \frac{e^{-2\pi i y\cdot(\delta\xi)}}{||y||^\alpha} \, dy \ .$$

As that means that

$$\widehat{U_\delta^\alpha}(\xi) = \delta^{n-\alpha} \, \widehat{U_1^\alpha}(\delta\xi) \ , \tag{48}$$

if, for some constant c, we have:

$$|\widehat{U_1^\alpha}(\xi)| \leq \frac{c}{||\xi||^{\min(n-\alpha, \frac{n+1}{2})}} \ ,$$

we will have also that

$$|\widehat{U_\delta^\alpha}(\xi)| \leq \frac{c \cdot \delta^{\max(0, \frac{n-1}{2} - \alpha)}}{||\xi||^{\min(n-\alpha, \frac{n+1}{2})}} \ .$$

We will use this remark later on. By using formula 44 again, and then, a change of variables, given by $2\pi||\xi||r = \rho$ in the integral obtained, the Fourier transform of $\widehat{U_1^\alpha}$, can be written in the following form:

$$\widehat{U_1^\alpha}(\xi) = \frac{(2\pi)^{\alpha - \frac{n}{2}}}{||\xi||^{n-\alpha}} \int_0^{2\pi||\xi||} J_{\frac{n-2}{2}}(\rho) \, \rho^{\frac{n}{2}-\alpha} \, d\rho \ . \tag{49}$$

We observe that, by using the estimate given by 46, the integral, in the right-hand side above, converges absolutely at $+\infty$ if $\alpha > \frac{n+1}{2}$. Under that condition on α, the first line in formula 47 is obtained. In order to deal with a second instance, of the first line of the result and, with the second line of the result we have to prove, we notice that, as a consequence of formula 45 and of a trivial integration by parts:

$$\int_0^z \rho^\mu \, J_\nu(\rho) \, d\rho = z^\mu J_{\nu+1}(z) - (\mu - \nu - 1) \int_0^z \rho^{\mu-1} \, J_{\nu+1}(\rho) \, d\rho \ . \tag{50}$$

(See again, [41, p. 141]). Suppose now that:

$$\frac{n-1}{2} < \alpha \leq \frac{n+1}{2} \ .$$

Applying formula 50, to the integral in formula 49, we get, for some constants c and d, that:

$$|\widehat{U_1^\alpha}(\xi)| \leq \frac{c}{||\xi||^{\frac{n+1}{2}}} + \frac{d}{||\xi||^{n-\alpha}} \int_0^{2\pi||\xi||} J_{\frac{n}{2}}(\rho) \, \rho^{\frac{n}{2}-\alpha-1} \, d\rho \ , \tag{51}$$

where, the integral in the right-hand side of the expression converges absolutely by force of the condition on α. This condition, obviously, implies also that $n-\alpha < \frac{n+1}{2}$ and so, the first line of the statement of the proposition also holds in this case. Suppose now that:

$$\frac{n-3}{2} < \alpha \leq \frac{n-1}{2} \,.$$

We apply again formula 50 but, this time, to the integral in the right-hand side of the expression 51 and, we get, for some constants c, d and e:

$$|\widehat{U_1^\alpha}(\xi)| \leq \frac{c}{||\xi||^{\frac{n+1}{2}}} + \frac{d}{||\xi||^{\frac{n+3}{2}}} + \frac{e}{||\xi||^{n-\alpha}} \int_0^{2\pi||\xi||} J_{\frac{n}{2}+1}(\rho)\,\rho^{\frac{n}{2}-\alpha-2}\,d\rho \,, \tag{52}$$

the integral, on the right-hand side, being absolutely convergent by the conditions imposed on α. These conditions also imply that $\frac{n+1}{2} \leq n-\alpha < \frac{n+3}{2}$ and so, in this case, we have as a conclusion, the second line of the statement made in the proposition. If we suppose that the conditions on α, are now given by:

$$\frac{n-5}{2} < \alpha \leq \frac{n-3}{2} \,,$$

the same method gives a new term in formula 52 namely, for some constants c, d, e, f:

$$|\widehat{U_1^\alpha}(\xi)| \leq \frac{c}{||\xi||^{\frac{n+1}{2}}} + \frac{d}{||\xi||^{\frac{n+3}{2}}} + \frac{e}{||\xi||^{\frac{n+5}{2}}} + \frac{f}{||\xi||^{n-\alpha}} \int_0^{2\pi||\xi||} J_{\frac{n}{2}+2}(\rho)\,\rho^{\frac{n}{2}-\alpha-3}\,d\rho \,, \tag{53}$$

where the integral is absolutely convergent and, $\frac{n+3}{2} \leq n-\alpha < \frac{n+5}{2}$, by force of the conditions imposed on α. In order to conclude, its only necessary to proceed by induction, observing that, after a finite number of steps, α will be close to its inferior limit, namely zero, and the integral appearing as a remainder is absolutely convergent. Observe that the integrals are all convergent in zero, by virtue of the fact that, for small ρ:

$$J_\nu(\rho) \approx \frac{\rho^\nu}{2^\nu\,\Gamma(1+\nu)}$$

(See [41, p. 134], again). □

The following consequence, of the last proposition, will prove useful when dealing with a particular case of our ultimate goal in section 5.1.

PROPOSITION 3.3. *Let g be a $C^\infty(\mathbb{R}^n)$ function, with compact support, such that g is strictly positive in a neighborhood of zero. Let $0<\alpha<n$ and define $I_{\alpha,g}$, a Fourier transform, by:*

$$I_{\alpha,g}(\xi) = \int_{\mathbb{R}^n} \frac{e^{-2\pi i\,\xi\cdot x}}{||x||^\alpha}\, g(x)\,dx \,. \tag{54}$$

The asymptotic behavior of $I_{\alpha,g}$ is the same as the asymptotic behavior of $\widehat{U_1^\alpha}$, that is, for some constant c,

$$|I_{\alpha,g}(\xi)| \leq \frac{c}{||\xi||^{min(n-\alpha,\frac{n+1}{2})}} . \tag{55}$$

PROOF. The conditions imposed on g, insure the existence of $M > 0$ and $0 < \eta < 2$ such that, for $v \in B(0,\eta)$, we have that $g(0) > 0$,

$$g(v) = g(0) + dg(0)(v) + ||v||\epsilon(v) , \tag{56}$$

where $dg(0)$ is the differential of g at zero, $\epsilon(v)$ tends to zero with v and:

$$|dg(0)(v) + ||v||\epsilon(v)| \leq M . \tag{57}$$

Let $R > 0$ be such that, the support of g is contained in $B(0,R)$. We can consider now, a standard partition of unity subordinated to the open cover, of the support of g, given by $B(0,\eta)$ and $B(0,2R) \cup (B(0,\frac{\eta}{2}))^c$, (see [53, p. 162] for instance). More precisely, let ϕ_1, ϕ_2 be $C^\infty(\mathbb{R}^n)$ functions with compact support such that $\phi_1 \equiv 1$ in $B(0,\frac{\eta}{2})$ and $\mathrm{supt}(\phi_1) \subset B(0,\eta)$, $\phi_2 \equiv 1$ in $B(0,R)$ and $\mathrm{supt}(\phi_2) \subset B(0,2R)$. It is clear, that if we define ψ_1, ψ_2 by $\phi_1 \equiv \psi_1$ and $\psi_2 \equiv (1-\phi_1)\cdot\phi_2$ then, ψ_1, ψ_2 are $C^\infty(\mathbb{R}^n)$ functions with compact support, verifying: $\psi_1 + \psi_2 \equiv 1$ on $B(0,R)$ and $\mathrm{supt}(\psi_1+\psi_2) \subset B(0,2R)$.

The Fourier transform, whose asymptotic behavior we pretend to study, can now be written, using this partition of unity, as:

$$I_{\alpha,g}(\xi) = \int_{\mathbb{R}^n} \frac{e^{-2\pi i\,\xi\cdot x}}{||x||^\alpha}\, g(v)\cdot\psi_1(v)\,dv + \int_{\mathbb{R}^n} \frac{e^{-2\pi i\,\xi\cdot x}}{||x||^\alpha}\, g(v)\cdot\psi_2(v)\,dv . \tag{58}$$

Denote by $I^l_{\alpha,g}(\xi)$ (respectively $I^r_{\alpha,g}(\xi)$), the integral on the left (respectively on the right), in the right-hand side of equality 58. Observe that, as $f(v) = \frac{g(v)\cdot\psi_2(v)}{||x||^\alpha}$ is a $C^\infty(\mathbb{R}^n)$ function with compact support, its Fourier transform, as given by $I^r_{\alpha,g}(\xi)$, has a decay at infinity as fast as we want, as a consequence of, for instance, Paley-Wiener theorem (see [53, p. 198]). There exists then a constant, c, such that:

$$|I^r_{\alpha,g}(\xi)| \leq \frac{c}{||\xi||^n} . \tag{59}$$

The integral $I^l_{\alpha,g}(\xi)$, can be further decomposed as follows:

$$I^l_{\alpha,g}(\xi) = \int_{||v||<\frac{\eta}{2}} \frac{e^{-2\pi i\,\xi\cdot x}}{||x||^\alpha}\, g(v)\,dv + \int_{\frac{\eta}{2}\leq||v||<\eta} \frac{e^{-2\pi i\,\xi\cdot x}}{||x||^\alpha}\, g(v)\cdot\psi_1(v)\,dv . \tag{60}$$

Using now 56, the integral on the left can be rewritten as:

$$\int_{||v||<\frac{\eta}{2}} \frac{e^{-2\pi i\,\xi\cdot x}}{||x||^\alpha} g(v)\,dv = g(0)\cdot\widehat{U^\alpha_{\frac{\eta}{2}}}(\xi) + \int_{||v||<\frac{\eta}{2}} \frac{e^{-2\pi i\,\xi\cdot x}}{||x||^\alpha}(dg(0)(v) + ||v||\epsilon(v))\,dv . \tag{61}$$

By passing to polar coordinates and, on account of 57, we can estimate the integral, on the right-hand side of this equality, by:

$$|\int_{||v||<\frac{\eta}{2}} \frac{e^{-2\pi i\,\xi\cdot x}}{||x||^{\alpha}}\,(dg(0)(v)+||v||\epsilon(v))\,dv| \leq \frac{M}{2^{n-\alpha+1}\,(n-\alpha+1)}\,\eta^{n-\alpha+1}\,. \tag{62}$$

By passing to polar coordinates, we can also take care of the integral on the right in 60. In fact, for some constant c:

$$|\int_{\frac{\eta}{2}\leq||v||<\eta} \frac{e^{-2\pi i\,\xi\cdot x}}{||x||^{\alpha}}\,g(v)\psi_2(v)\,dv| \leq c\sup_{||v||\leq\eta}|g(v)\psi_2(v)|\int_{\eta}^{\frac{\eta}{2}}\rho^{n-\alpha-1}\,d\rho \leq c\,\eta^{n-\alpha}\,. \tag{63}$$

In order to conclude, it is sufficient to collect the estimates made on 59, 61, 62 and 63, to recall proposition 3.2, in particular the remark at the beginning of the corresponding proof in conjunction with the fact that we took $0<\eta<2$, and we get, for some constants c and d:

$$|I_{\alpha,g}(\xi)| \leq \frac{c}{||\xi||^{\min(n-\alpha,\frac{n+1}{2})}} + d\,\eta^{n-\alpha}\,. \tag{64}$$

Observing that, in obtaining estimates 62 and 63, we only consider the modulus of the function we were integrating, we can take the parameter η, as small as we want in 64, thus getting the result stated. □

4. Some Parseval formulas and tempered distributions

For $0<\alpha<n$ let U^{α}, denote the locally integrable function , defined by:

$$U^{\alpha}(x) = \frac{1}{||x||^{\alpha}}\,\mathbb{I}_{\mathbb{R}^{*n}}(x)\,.$$

This function defines a tempered distribution, whose Fourier transform, which we denote by $\widehat{U^{\alpha}}$ but also by $\mathcal{F}U^{\alpha}$, is represented again by a locally integrable function, given by:

$$\widehat{U^{\alpha}}(\xi) = \frac{c(\alpha)}{||\xi||^{n-\alpha}}\,\mathbb{I}_{\mathbb{R}^{*n}}(\xi)\ \ ,\ \ c(\alpha) = \frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n}{2}-\alpha}\Gamma(\frac{\alpha}{2})}\,.$$

(See [58, p. 117] or [65, p. 52, 278]).

The following result, essentially due to Frostman, is usually formulated for real measures and without the exponential term in formula 66 (see [7, p. 22]).

Theorem 4.1. *: Let σ be a positive Radon measure over $\mathbb{R}^n$ with compact support and $0<\alpha<n$ such that E_α, the α energy of σ, is finite. That is:*

$$E_\alpha = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{d\sigma(t)\,d\sigma(s)}{||t-s||^{\alpha}} < +\infty\,. \tag{65}$$

We then have:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{e^{2i\pi\xi(t-s)}}{||t-s||^{\alpha}}d\sigma(t)\,d\sigma(s) = c(\alpha)\int_{\mathbb{R}^n}\frac{|\widehat{\sigma}(x)|^2}{||x-\xi||^{n-\alpha}}dx\,, \tag{66}$$

whenever the integral on the right is finite.

PROOF. The formula we have to prove, is verified for measures given by $d\sigma(t) = \phi(t)dt$, where $\phi \in \mathcal{S}$, $\mathcal{S}$ being the Schwartz test function space of rapidly decreasing functions. Indeed, for such a measure, the integral on the left-hand side of formula 66, which we denote by I, is written as:

$$I = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{e^{2i\pi\xi t}}{||t-s||^\alpha} \phi(t)dt \right) e^{-2i\pi\xi s}\phi(s)ds = \int_{\mathbb{R}^n} \frac{(2\pi)^{n-\alpha}}{c(n-\alpha)} I_{n-\alpha}(h)(s)\overline{h}(s)ds \ ,$$

where, for $0 < \beta < n$, the β Riesz potential of f is given by:

$$I_\beta(f)(x) = \frac{c(\beta)}{(2\pi)^\beta} \int_{\mathbb{R}^n} \frac{f(y)}{||x-y||^{n-\beta}}\, dy$$

and h stands for $h(s) = e^{2i\pi\xi s}\phi(s)$. Now, given $f, g \in \mathcal{S}$, we have that:

$$\int_{\mathbb{R}^n} I_\beta(f)(x)\overline{g}(x)dx = \int_{\mathbb{R}^n} \frac{\widehat{f}(x)\overline{\widehat{g}}(x)}{(2\pi||x||)^\beta} dx \ ,$$

which is essentially Parseval formula (see [58, p. 117] for all the properties of the notion of Riesz potential used). Observing that $\widehat{h}(y) = \widehat{\phi}(y-\xi)$ and $\overline{\widehat{h}} = \widehat{\phi}(\xi - y)$, we have:

$$I = \frac{1}{c(n-\alpha)} \int_{\mathbb{R}^n} \frac{\widehat{\phi}(y-\xi)\widehat{\phi}(\xi-y)}{||y||^{n-\alpha}} dy \,,$$

which gives the result, claimed in the statement of the theorem, by a trivial change of variables, noticing that, as ϕ is real:

$$\widehat{\phi}(u)\widehat{\phi}(-u) = \widehat{\phi}(u)\overline{\widehat{\phi}}(u) = |\widehat{\phi}(u)|^2 \ ,$$

and that $C(\alpha) = 1/c(n-\alpha)$. Let now μ, be a complex measure with compact support and, for $0 < \beta < n$, define I_β, the β Riesz potential of μ, by:

$$I_\beta(\mu)(x) = \frac{c(\beta)}{(2\pi)^\beta} \left(\mu * U^{n-\alpha}\right) .$$

This definition makes good sense as we are considering the convolution of μ, a distribution with compact support, with $U^{n-\alpha}$, which is a tempered distribution. Observe that, as a consequence of a theorem of Soboleff (see [55, p. 181]), $I_\beta(\mu)$ is locally in L^q for $q < \frac{n}{n-\beta}$ and, in particular, $I_\beta(\mu)$ is integrable over any compact of $\mathbb{R}^n$. Moreover, the Fourier transform of $I_\beta(\mu)$, in the sense of distributions, is easily computed (see for instance [27, p. 21]) to give:

$$\widehat{I_\beta(\mu)} = \frac{c(\beta)}{(2\pi)^\beta} \widehat{\mu} \cdot \widehat{U^{n-\beta}} = \frac{1}{(\pi)^\beta} \widehat{\mu} \cdot \widehat{U^{n-\beta}} \ .$$

Considering now $d\mu(t) = e^{2\pi i\xi\cdot t} d\sigma(t)$, we have that:

$$I = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{d\mu(t)}{||t-s||^\alpha} \right) d\overline{\mu}(s) = \frac{(2\pi)^\alpha}{c(n-\alpha)} \int_{\mathbb{R}^n} I_{n-\alpha}(\mu)(s) d\overline{\mu}(s) \ .$$

As μ has compact support and, as a consequence of the hypotheses, done on the integral on the right of formula 66, we can apply Parseval formula (as given for instance in [37, p. 132] or [42, p. 121]), to obtain

$$I = \frac{(2\pi)^\alpha}{c(n-\alpha)} \int_{\mathbb{R}^n} \widehat{I_{n-\alpha}(\mu)}(x)\widehat{\overline{\mu}}(x)dx = c(\alpha) \int_{\mathbb{R}^n} \widehat{\mu}(x)U^\alpha(x)\widehat{\overline{\mu}}(-x)dx \ ,$$

which, after some computations of Fourier transforms and a change of variables, is exactly the desired result. □

5. A random measure obtained from multiplicative chaos

We consider given the context of indexed martingales ([32], [34] and [35]) and the notions related to multiplicative chaos. The purpose of this section is to point some remarks on the random measure of multiplicative chaos and the corresponding Fourier transform. Recalling the notations of [35] we have, for a given Radon measure σ on T, a locally compact space and for $\varphi \in \mathcal{C}_0(T)$, the sequence of random variables $(X_n(\varphi))_{n\in\mathbb{N}}$ where:

$$\forall n \in \mathbb{N} \ \ X_n(\varphi) = \int_T \varphi(t) \ Q_n\sigma(dt) \ ,$$

is a L^1 martingale, converging almost surely to a limit denoted by:

$$S(\varphi) = \int \varphi dS \ .$$

Observe now that the measures, of the sequence $(Q_n\sigma)_{n\in\mathbb{N}}$, are positive measures that, they converge almost surely weakly and, in the case when Q *lives on* σ, ([35], p. 13) that is to say when

$$\forall \varphi \in \mathcal{C}_0(T) \ \ \mathbb{E}[\int \varphi dS] = \int_T \varphi(t)q(t)\sigma(dt) \ ,$$

we have in an equivalent way:

$$\mathbb{E}[\int dS] = \int_T q(t)\sigma(dt) \ .$$

This is a finite quantity by the hypothesis made. As a consequence, almost surely:

$$\lim_{n\to+\infty} \int_T Q_n\sigma(dt) = S(1) < +\infty \ .$$

We can conclude that there is convergence over the bounded continuous functions on T ([42], p. 92). As a consequence, the definition of the Fourier transform of the random measure S, can be given plain sense, at least in the case when T is a locally compact commutative group. From now on we will suppose that $T = \mathbb{R}^\nu$.

DEFINITION 5.1. The *Fourier transform* $\hat{S}$, of the random measure S is, by definition, the map defined almost surely by:

$$\forall \xi \in \mathbb{R}^\nu \ \ \hat{S}(\xi) = \lim_{n \to +\infty} \int_{\mathbb{R}^\nu} \exp(2\pi i \xi t) Q_n \sigma(dt) \ .$$

It is easily verified that, almost surely, $\hat{S}$ is uniformly continuous and that, for a bounded and positive Radon measure σ, the map $\hat{S}$, is almost surely bounded.

REMARK 10. : With additional hypothesis, it can be seen that S is a random measure in the sense of a measurable map taking its values in a (measurable) space of measures.

More precisely, in the case of the theory of multiplicative chaos ([32]), if the operator Q (given as a limit of Q_n having a particular form) is strongly nondegenerate, then the martingale $(X_n(\varphi))_{n \in \mathbb{N}}$ is a L^2 martingale. In fact:

$$\forall n \in \mathbb{N} \ \ \mathbb{E}[|X_n|^2] \leq CTE \int_T \int_T \exp(q_n(t,s)) \sigma(dt) \sigma(ds) \leq A < +\infty \ .$$

Following [48], we can say that the sequence of random measures $(Q_n\sigma)_{n \in \mathbb{N}}$, converges in quadratic mean (see [48, p. 49]. This shows that S is a random measure, when considered as a map defined on a probability space and taking its values in the space of the Radon signed measures, a measurable space, whenever endowed with the Borel σ algebra associated with the topology of *vague* convergence.

The random measures associated to the multiplicative chaos, are, in this way, under some restrictive hypothesis, nontrivial examples of random signed measures in the sense of Kallemberg and Oliveira.

5.1. Asymptotic behavior of the second moment. In this section, we will state and prove a generalization of a theorem of J.-P. Kahane ([31, p. 30]) on the behavior at infinity of $\mathbb{E}[|\hat{S}(\xi)|^2]$. We will suppose that the random measure S is associated with a kernel given by:

$$\forall s,t \in \mathbb{R}^\nu \ \ q(t,s) = u \log(\frac{1}{||t-s||}) + \bigcirc(1) \ .$$

THEOREM 5.1. *Let σ be a positive measure on $\mathbb{R}$, such that:*

$$\int_{\mathbb{R}^\nu} |\hat{\sigma}(x)|^2 dx < +\infty \ \ , \ \ \forall \xi \in \mathbb{R}^\nu \ \int_{\mathbb{R}^\nu} \frac{|\hat{\sigma}(x)|^2}{||x-\xi||^{\nu-u}} dx < +\infty \ .$$

and such that, the operator Q is strongly nondegenerate on σ. We have then for some constants c and d that:

$$\mathbb{E}[|\hat{S}(\xi)|^2] \leq \frac{1}{||\xi||^{\nu-u}} \ (c + d\,||\xi||^\nu \sup_{||x-\xi||<\frac{||\xi||}{2}} |\widehat{\sigma}(x)|^2) \ .$$

Proof. We use the result, in [31], which says that:

$$\mathbb{E}[|\hat{S}(\xi)|^2] = \mathbb{E}[|\int \exp(2\pi i t\xi) dS(t)|^2] = \int_{(\mathbb{R}^\nu)^2} \exp(2\pi i(t-s)\xi) e^{q(t,s)} d\sigma(t)\, d\sigma(s)\ . \tag{67}$$

We will deal first, with the special case where σ admits a C^∞ density f, with compact support. Suppose then that, $d\sigma(t) = f(t)dt$. By a trivial change of variables and, by Fubini theorem, we get:

$$\int_{\mathbb{R}^{2\nu}} \frac{e^{-2\pi i(t-s)\xi}}{||t-s||^u} d\sigma(t)\, d\sigma(s) = \int_{\mathbb{R}^{2\nu}} \frac{e^{-2\pi i v\xi}}{||v||^u} f(v+s) f(s) dv\, ds =$$
$$\int_{\mathbb{R}^\nu} \frac{e^{-2\pi i v\xi}}{||v||^u} (f * f^\vee)(s) dv\ .$$

The hypotheses on f imply that $(f * f^\vee)$ is a C^∞ function, with compact support, strictly positive in a neighborhood of zero. As a consequence of proposition 3.3, we get, for some constant c:

$$\mathbb{E}[|S(\xi)|^2] \le \frac{c}{||\xi||^{\min(u, \frac{n+1}{2})}}\ ,$$

which is a weaker result than the one announced. The general case needs another kind of approach. We apply theorem 4.1, to the last term in formula 67, to get for some constant d:

$$\mathbb{E}[|\hat{S}(\xi)|^2] = d \int_{\mathbb{R}^\nu} \frac{|\hat{\sigma}(x)|^2}{||x-\xi||^{\nu-u}} dx\ .$$

Denote by I the following integral:

$$I = \int_{\mathbb{R}^n} \frac{|\widehat{\sigma}(x)|^2}{||x-\xi||^a}\, dx\ . \tag{68}$$

In order to obtain the asymptotic behavior of this integral, we consider a point ξ, fixed in $\mathbb{R}^n$ and, the partition, of the domain of integration, given by:

$$\mathbb{R}^n = B(0, \frac{||\xi||}{2}) \cup B(\xi, \alpha) \cup \{x \in \mathbb{R}^n : ||x|| \ge \frac{||\xi||}{2}\ , ||x-\xi|| \ge \alpha\}\ , \tag{69}$$

where α is a parameter we will deal with, below. Let I_1 (respectively I_2, I_3) be the integral of the function $\frac{|\widehat{\sigma}(x)|^2}{||x-\xi||^a}$ over the set on the left (respectively on the middle, on the right) of the partition 69. Then, it is clear that:

$$I_1 \le \frac{1}{||\xi||^a} \int_{\frac{||x||}{||\xi||}<\frac{1}{2}} \frac{|\widehat{\sigma}(x)|^2}{|1-\frac{||x||}{||\xi||}|^a}\, dx \le \frac{1}{||\xi||^a}\, 2^a ||\widehat{\sigma}||_2^2\ , \tag{70}$$

$$I_2 \le \sup_{||x-\xi||<\alpha} |\widehat{\sigma}(x)|^2 \int_{||x||<\alpha} \frac{dx}{||x||^a} \le \alpha^{\nu-a} \sup_{||x-\xi||<\alpha} |\widehat{\sigma}(x)|^2 \int_{||x||<\alpha} \frac{dx}{||x||^a}\ , \tag{71}$$

$$I_3 \le \frac{1}{\alpha^a} \int_{||x||>\frac{||\xi||}{2}} |\widehat{\sigma}(x)|^2\, dx \le \frac{||\widehat{\sigma}||_2^2}{\alpha^a}\ . \tag{72}$$

As a consequence, for some constants c and d, and choosing $\alpha = \frac{||\xi||}{2}$ in 71 and in 72, we have that:

$$I \leq \frac{1}{||\xi||^a} \left(c + d\,||\xi||^\nu \sup_{||x-\xi||<\frac{||\xi||}{2}} |\widehat{\sigma}(x)|^2\right) , \tag{73}$$

as desired. □

Remark 11. Let σ be the Lebesgue measure concentrated on the unit ball of $\mathbb{R}^\nu$. As a consequence of formula 43, we will have that, for some constant c:

$$|\hat{\sigma}(x)|^2 \leq \frac{c}{||x||^{\nu+1}} ,$$

And, as a consequence:

$$\mathbb{E}[|\hat{S}(\xi)|^2] \leq \frac{c}{||\xi||^{\nu-u}} ,$$

in agreement with the result stated in ([31, p. 30]).

Remark 12. The final conclusion, in the statement of theorem 5.1, clearly depends on the asymptotic behavior of the Fourier transform of the measure σ. We present next an example, (see [56]), that shows that, in general, the integral in 68 has no rate of decay better than $\bigcirc(1)$. We will see that the measure under scrutiny, hasn't compact support. As a consequence, a natural question is to find an example, such as the one presented, but with a measure with compact support.

Consider a sequence of functions $(\varphi_n)_{n\in\mathbb{N}}$ defined by:

$$\forall n \in \mathbb{N} \;\; \varphi_n = \mathbb{I}_{[-n,1-n]} + \mathbb{I}_{[n-1,n]} .$$

As φ_n is an even function, its Fourier $\widehat{\varphi_n}$ transform is real valued. A quick computation shows that:

$$\forall n \in \mathbb{N} \;\; \widehat{\varphi_n}(\xi) = \frac{2\sin(\pi\xi)\cos((2n-1)\pi\xi)}{\pi\xi} .$$

Define now a sequence of functions $(\psi_n)_{n\in\mathbb{N}}$ by:

$$\forall n \in \mathbb{N} \;\; \psi_n = \varphi_n * \varphi_n .$$

A simple, but tedious computation, shows that $\psi_n(x)$ is a linear by pieces continuous function with compact support, simply described as the sum of three tent functions given by:

$$\begin{aligned}\psi_n(x) = {}& (2n+x)\mathbb{I}_{[-2n,-2n+1]}(x) + (-x-2n+2)\mathbb{I}_{[-2n+1,-2n+2]}(x) + \\ & (2x+2)\mathbb{I}_{[-1,0]}(x) + (2-2x)\mathbb{I}_{[0,1]}(x) + \\ & (x-2n+2)\mathbb{I}_{[2n-2,2n-1]}(x) + (2n-x)\mathbb{I}_{[2n-1,2n]}(x) .\end{aligned}$$

As $\widehat{\psi_n} = (\widehat{\varphi_n})^2$, we have that:

$$\forall n \in \mathbb{N} \;\; \widehat{\psi_n}(\xi) = \frac{4\sin^2(\pi\xi)\cos^2((2n-1)\pi\xi)}{\pi^2\xi^2}$$

This shows that ψ_n belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Take a sequence $(a_n)_{n\in\mathbb{N}^*}$, of nonnegative numbers, such that $\sum_{n=1}^{+\infty} a_n < +\infty$ and, define a measure $d\sigma(\xi) = f(\xi)d\xi$ with:

$$\forall \xi \in \mathbb{R} \;\; f(\xi) = \sum_{n=1}^{+\infty} a_n \widehat{\psi_n}(\xi) \; .$$

As $\sigma(\mathbb{R}) = 2\sum_{n=1}^{+\infty} a_n$, the measure σ is finite. Moreover, as a consequence of Cauchy-Schwarz inequality, the density of σ, with respect to the Lebesgue measure, is in $L^2(\mathbb{R})$. In fact, we have:

$$(\int_{\mathbb{R}} f^2(\xi)d\xi)^{\frac{1}{2}} = \left(\sum_{n,m=1}^{+\infty} a_n a_m \int_{\mathbb{R}} \widehat{\psi_n}(\xi)\widehat{\psi_m}(\xi)d\xi \right)^{\frac{1}{2}} \leq$$

$$||\psi_n||_2||\psi_m||_2(\sum_{n=1}^{+\infty} a_n^2)^{\frac{1}{2}}(\sum_{m=1}^{+\infty} a_m^2)^{\frac{1}{2}} < +\infty \; .$$

Observe now that, for every $p \in \mathbb{N}$:

$$|\widehat{\sigma}(x)| = \frac{1}{2\pi}\sum_{n=1}^{+\infty} a_n \psi_n(x) \geq a_p \psi_p(x) \; .$$

As a consequence, for some constant c:

$$I(2n-1) = \int_{\mathbb{R}} \frac{|\widehat{\sigma}(x)|^2}{|x-(2n-1)|^a}dx \geq \int_{2n-2}^{2n} |\widehat{\sigma}(x)|^2 \geq ca_n^2$$

Finally, by choosing, for example:

$$a_n = \begin{cases} \frac{1}{k^2}, & \text{if } n = 2^{2^k} \\ 0, & \text{otherwise} \end{cases} ,$$

we see that I has no rate of decay better than $\bigcirc(1)$.

CHAPTER IV

On the space of random tempered distributions having a mean

1. Summary

A class of random tempered distributions, on $\mathbb{R}$, is introduced by considering random series, in the usual Hermite functions, having, as coefficients, random variables which satisfy certain growth conditions. This class is shown to be, exactly, the class of random Schwartz tempered distributions having a mean. Otherwise stated, we obtain a characterization of the stochastic processes with a first moment and having as trajectories tempered distributions. As important examples of this class, we introduce a brownian type process on $\mathbb{R}$ and recall, the brownian distributions of J.-P. Kahane. We present a study on a possible converse, of a result on brownian distributions, which leads to a moment problem.

2. Introduction

A natural starting point to study random tempered distributions is, to consider as a definition that, such an object is just a map from Ω into $\mathcal{S}^{'}(\mathbb{R})$, the space of tempered or Schwartz distributions on $\mathbb{R}$, the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ being complete (see [31, p. 9] or [26, p. 13]).

A complementary *point de vue* is, to define from the start, a random distribution as a linear continuous map from $\mathcal{S}(\mathbb{R})$, the Schwartz space of test functions, into some space of random variables, endowed with some topological (and hence measurable) structure, usually, convergence in probability (see [10, p. 210]).

Some price has to be paid for the simplicity of the first approach. In fact, as soon as some result is to be proved, requiring the map between Ω and $\mathcal{S}^{'}(\mathbb{R})$ to be a random variable, the existence of the law of such a random variable being an instance, the topological structure of $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}^{'}(\mathbb{R})$ has to be considered (see [26, p. 115] or [28, p. 21]).

In this work, we adopt first the simplest approach in order to define and study a particular class of tempered random distributions. The characterization result (theorem 4.7), proved in the sequel, which amounts to show that this class recovers exactly the space of random tempered distributions having a mean [1] uses only a trivial consequence of the measurability condition. Namely, we use the fact that the sequence of Fourier Hermite coefficients, of the random distribution, is a sequence of random variables. Using the denominations of [26], the characterization of amenable random distributions can be read as a structure result on the generalized stochastic processes having as trajectories tempered distributions.

Results on the Minlos' support of the law of a generalized stochastic process having (generalized) moments are given in [10, p. 212]. In [3], the idea of considering gaussian random fields whose covariance is the parametrix of an elliptic pseudo-differential operator, gives results on the sample path properties of such a gaussian field. This idea could perhaps be used, in a modified form, to obtain more information on the structure of generalized processes whose trajectories are tempered distributions.

3. The Hermite Functions

3.1. Laurent Schwartz's fundamental result. In this introductory section, we summarize, from [55, p. 261], some definitions and results on the Hermite functions. The most important one is theorem 3.7, which establishes that the space of tempered distributions on $\mathbb{R}$ is isomorphic to the space of slowly increasing sequences indexed by the integers. The formula presented in lemma 3.8 will be used in the study of brownian distributions.

Definition 3.1. The Hermite polynomials on $\mathbb{R}$ are the polynomials $H_m(x)$, for $m \in \mathbb{N}$, defined by the equations:

$$\frac{d^m}{dx^m} e^{-2\pi x^2} = (-1)^m \sqrt{m!}\, 2^{m-\frac{1}{4}} \pi^{\frac{m}{2}} H_m(x)\, e^{-2\pi x^2} \ .$$

Associated with these polynomials are the Hermite functions given, for $m \in \mathbb{N}$, by:

$$\mathcal{H}_m(x) = H_m(x) e^{-\pi x^2} \ .$$

Proposition 3.1. *The family $(\mathcal{H}_m)_{m\in\mathbb{N}}$, is a complete orthonormal set of functions in $L^2(\mathbb{R})$.*

The Fourier transform of an Hermite function is simply given by the following proposition.

Proposition 3.2.

$$\mathcal{F}(\mathcal{H}_m)(\xi) = (-1)^m \mathcal{H}_m(\xi) \ .$$

[1] Amenable random distributions was the designation proposed by J.-P. Kahane for tempered random distributions having a mean.

As a consequence, we have the representation for the elements of $L^2(\mathbb{R})$, as a series of Hermite functions.

PROPOSITION 3.3. *For every $\varphi \in L^2(\mathbb{R})$, the following equality in the L^2 sense holds:*

$$\varphi(x) = \sum_{m=0}^{+\infty} a_m(\varphi)\, \mathcal{H}_m(x) \ ,$$

where the coefficients $(a_m(\varphi))_{m\in\mathbb{N}}$ are given by:

$$a_m(\varphi) = \int_{\mathbb{R}} \varphi(x)\mathcal{H}_m(x)\, dx$$

and, the following Plancherel type result holds:

$$\sum_{m=0}^{+\infty} |a_m(\varphi)|^2 = \int_{\mathbb{R}} |\varphi(x)|^2 dx \ .$$

The natural operators for the Hermite functions are given by:

$$(\tau_+ f)(x) = (\frac{d}{dx} f)(x) + 2\pi x f(x) \quad (\tau_- f)(x) = -(\frac{d}{dx} f)(x) + 2\pi x f(x) \ . \tag{74}$$

where f is some regular function (or a distribution). The recurrence relations that hold among the Hermite function imply the next result about the action of τ_+ and τ_- on the Hermite functions.

PROPOSITION 3.4. *For every $m \geq 1$ or $p \geq 0$:*

$$\tau_+(\mathcal{H}_m) = 2\sqrt{\pi m}\ \ \mathcal{H}_{m-1} \quad , \quad \tau_-(\mathcal{H}_p) = 2\sqrt{\pi(p+1)}\ \ \mathcal{H}_{p+1} \ . \tag{75}$$

As a consequence, if φ, $\varphi^{'}$, $x\varphi$, are functions in $L^2(\mathbb{R})$ then, as τ_+ and τ_- are adjoint operators of each other,

$$a_m(\tau_+\varphi) = \int_{\mathbb{R}} (\tau_+\varphi)(x)\ \mathcal{H}_m(x) dx = \int_{\mathbb{R}} \varphi(x)\ (\tau_+\mathcal{H}_m)(x) dx \ .$$

This relation obviously implies that:

$$a_m(\tau_+\varphi) = 2\sqrt{\pi(m+1)}\ a_{m+1}(\varphi) \ .$$

And, by the same line of reasoning, we also have that, for $m \geq 1$:

$$a_m(\tau_-\varphi) = 2\sqrt{\pi m}\ a_{m-1}(\varphi) \ ,$$

with the natural convention that $a_0(\tau_-\varphi) = 0$. These formulae now imply the next proposition.

PROPOSITION 3.5. *The following conditions are equivalent.*

(1) *φ, $\varphi^{'}$, $x\varphi$, are in $L^2(\mathbb{R})$.*
(2) $\sum_{m=0}^{+\infty} |a_m(\varphi)|^2 < +\infty$.

By induction, using the iterates of the operators τ_+ and τ_-, we get the following theorem which gives a characterization of the space $\mathcal{S}$ of rapidly decreasing functions.

THEOREM 3.6. *A necessary and sufficient condition for having $\varphi \in \mathcal{S}$, is that the sequence $(a_m(\varphi))_{m\in\mathbb{N}}$ is a rapidly decreasing sequence. The natural map, that gives the sequence $(a_m(\varphi))_{m\in\mathbb{N}}$ as a function of φ, is an isomorphism of topological vector spaces between the space $\mathcal{S}$ and, the space of rapidly decreasing sequences.*

Given a tempered distribution T, the Hermite coefficients $(a_m(T))_{m\in\mathbb{N}}$ of T, can naturally be calculated by:

$$a_m(T) =< T, \mathcal{H}_m > , \tag{76}$$

as the functions $\mathcal{H}_m$ all belong to $\mathcal{S}$. Using the representation of T, as a finite sum of distributions which in turn are images of functions in $L^2(\mathbb{R})$, by repeated applications of the operators τ_+ and τ_-, one can prove that the sequence $(a_m(T))_{m\in\mathbb{N}}$ is a slowly increasing sequence.

Conversely, given a slowly increasing sequence $(b_m)_{m\in\mathbb{N}}$, the series $\sum_{m=0}^{+\infty} b_m\mathcal{H}_m$ converges in $\mathcal{S}^{'}$ to a limit, let it be T, such that:

$$\forall m \in \mathbb{N} \quad b_m = a_m(T)$$

In short, we have the following important result.

THEOREM 3.7 (LAURENT SCHWARTZ). *A necessary and sufficient condition for having $T \in \mathcal{S}^{'}$, is that the sequence $(a_m(T))_{m\in\mathbb{N}}$ is a slowly increasing sequence. The natural map that associates the sequence of the Hermite coefficients of the distribution T, to the distribution, is an isomorphism between $\mathcal{S}^{'}$ and the space of slowly increasing sequences.*

The representation of the duality between $\mathcal{S}$ an $\mathcal{S}^{'}$, given the Hermite coefficients of $T \in \mathcal{S}^{'}$ and $\varphi \in \mathcal{S}$, is expressed by:

$$< T, \varphi >= \sum_{m=0}^{+\infty} a_m(T)\, a_m(\varphi) . \tag{77}$$

The expression of the Fourier transform of an object in $\mathcal{S}$ or in $\mathcal{S}^{'}$ and, represented by a Hermite series:

$$\sum_{m=0}^{+\infty} c_m\, \mathcal{H}_m ,$$

is given by:

$$\mathcal{F}(\sum_{m=0}^{+\infty} c_m\, \mathcal{H}_m) = \sum_{m=0}^{+\infty} (-1)^m\, c_m\, \mathcal{H}_m ,$$

where the series converge in $\mathcal{S}$, if the sequence $(c_m)_{m\in\mathbb{N}}$ is rapidly decreasing and the series converge in $\mathcal{S}'$, if the sequence is slowly increasing.

3.2. The generating function for the square of Hermite functions. The formula presented in the next lemma, will be instrumental in the proof of a theorem of J. P. Kahane on brownian distributions as well as, in our investigation about a converse of the proposition stated in that theorem.

LEMMA 3.8. *For $|t| < 1$:*

$$\sum_{m=0}^{+\infty} t^m \mathcal{H}_m^2(x) = \sqrt{\frac{2}{1-t^2}}\ e^{-2\pi x^2 \frac{1-t}{1+t}} \tag{78}$$

PROOF. This formula, which we took without proof from [30], can be derived from a similar one already proved in [63, p. 77, 78]. In fact, theorem 43 of the monograph just quoted reads that, for $|t| < 1$:

$$\sum_{n=0}^{+\infty} t^n \frac{e^{-x^2}}{2^n\, n!}(P_n(x))^2 = \frac{1}{\sqrt{1-t^2}}\ e^{-x^2 \frac{1-t}{1+t}}\ , \tag{79}$$

where the Hermite polynomials $P_n(x)$ are defined as those polynomials which satisfy for $n \in \mathbb{N}$:

$$P_n(x) = (-1)^n\ e^{x^2} \frac{d^n}{dx^n} e^{-x^2}\ .$$

This formula shows that:

$$\frac{d^n}{dx^n} e^{-x^2} = (-1)^n\ e^{-x^2}\ P_n(x)\ .$$

We observe that trivially:

$$\frac{d^n}{dx^n} e^{-2\pi x^2} = (2\pi)^{\frac{n}{2}} \frac{d^n}{dy^n}(e^{-y^2})|_{y=\sqrt{2\pi}\,x}\ .$$

After making the appropriate substitutions, we have the following relation between Titchmarch Hermite polynomials P_n and the Schwartz Hermite polynomials H_n:

$$\forall n \in \mathbb{N}\ \ P_n(\sqrt{2\pi}\,x) = 2^{\frac{2n-1}{4}}\ \sqrt{n!}\ H_n(x)\ .$$

Using this relation in formula 79, we get the result announced. □

4. Random Schwartz tempered distributions having a mean

4.1. Introduction. In this section, we introduce a space of random Schwartz distributions over $\mathbb{R}$ by means of series of Hermite functions having as coefficients random variables satisfying a certain growth condition. The results presented here, are similar to those presented in [14], where the particular case of periodic distributions was studied.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space. Let $\mathcal{M}$ denote the space of random variables taking complex values and $\mathcal{M}^{\mathbb{N}}$ the space of sequences of elements of $\mathcal{M}$ indexed by the integers $\mathbb{N}$. For A, an element of $\mathcal{M}$, let $\mathbb{E}[|A|]$ be defined by:

$$\mathbb{E}[|A|] = \int_\Omega |A| \, d\mathbb{P} \, .$$

Let

$$\begin{aligned} \mathcal{C}_m =&\{(c_n)_{n\in\mathbb{N}} \in \mathcal{M}^{\mathbb{N}} \, : \, (\exists A \in \mathcal{M} \, , \, A \geq 0 \, , \, \mathbb{E}[A] < +\infty) \, (\exists k \in \mathbb{N}) \\ &(\forall n \in \mathbb{N}) \, |c_n| \leq A(1+n)^k \text{ a. s. on } \Omega\} \, , \end{aligned} \tag{80}$$

be a space of sequences of random variables. This space can be described by the equivalent conditions given in the next theorem.

THEOREM 4.1. *For $(c_n)_{n\in\mathbb{N}} \in \mathcal{M}^{\mathbb{N}}$ the following are equivalent:*

(1) $(c_n)_{n\in\mathbb{N}} \in \mathcal{C}_m$

(2) *There exists an integrable random variable A and a real bounded random variable K, defined on Ω, such that:*

$$\forall n \in \mathbb{N} \quad |c_n| \leq A\,(1+n)^K \, a. \ s. \ on \ \Omega$$

(3) *The sequence $(\mathbb{E}[|c_n|])_{n\in\mathbb{N}}$ is a sequence of slow growth or, in an equivalent rephrasing:*

$$(\exists a > 0) \, (\exists k \in \mathbb{N}) \, (\forall n \in \mathbb{N}) \quad \mathbb{E}[|c_n|] \leq a\,(1+n)^k$$

PROOF. The proof goes exactly as in the proof of theorem 2.1, replacing $\mathbb{Z}$ by $\mathbb{N}$ in every instance where the first set is the set of indices of the sequence. □

Associated with every sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathcal{C}_m$, there is a random Schwartz distribution T on $\mathbb{R}$, defined for a $\varphi \in \mathcal{S}$ by:

$$< T, \varphi >= \sum_{m=0}^{+\infty} c_m \, a_m(\varphi) \, . \tag{81}$$

The distribution T is well defined because, by the second condition of the theorem 4.1, the sequence $(c_n)_{n\in\mathbb{N}}$ is almost surely a sequence of slow growth. Besides, the sequence $(a_m(\varphi))_{m\in\mathbb{N}}$, of the Hermite coefficients of φ, is a rapidly decreasing sequence and so, the series, in the left-hand side of 81, converges almost surely.

We will present some examples after mentioning some properties of the random Schwartz distributions just defined.

THEOREM 4.2 (ON THE UNIQUENESS OF THE REPRESENTATION). *Let T be given, as in formula 81, by a sequence $(c_n)_{n\in\mathbb{N}}$. Then, the following are equivalent:*

(1) $T = 0$ *a. s. on* Ω .

(2) $\forall m \in \mathbb{N} \quad c_m = 0 \quad$ *a. s. on* Ω .

PROOF. Same proof as in theorem 4.1 which is the corresponding result in the first chapter. □

As a consequence, for every random Schwartz distribution, defined this way, there is, up to equality almost sure, an unique sequence of random variables such that formula 81 is verified almost surely. Such a sequence will be denoted by: .

$$(a_m(T))_{m\in\mathbb{N}} \ .$$

An obvious corollary of this theorem gives a necessary and sufficient condition for equality of two random Schwartz distributions obtained by the way just explained.

THEOREM 4.3. *Let T, U, be two random Schwartz distributions obtained by formula* 81 *from the sequences $(a_m(T))_{m\in\mathbb{N}}$ and $(a_m(U))_{m\in\mathbb{N}}$, respectively. Then, the following are equivalent:*

(1) $T = U$ *a. s. on* Ω .

(2) $\forall m \in \mathbb{N} \quad a_m(T) = a_m(U) \quad$ *a. s. on* Ω .

One important property of the class of random tempered distributions here introduced, by means of the space $\mathcal{C}_m$, is that, this class is stable by derivation.

DEFINITION 4.1. For a random Schwartz distribution T, the derivative $T^{'}$ of T, is a random Schwartz distribution defined by:

$$\forall \omega \in \Omega \ \ \forall \varphi \in \mathcal{S}(\mathbb{R}) \quad < T^{'}, \varphi >= - < T, \varphi^{'} > \ . \tag{82}$$

If a random tempered distribution has the sequence of its Hermite coefficients, given by 76, in the class $\mathcal{C}_m$ then, the Hermite coefficients of the derivative are also in the class $\mathcal{C}_m$. This fact follows from the next result.

THEOREM 4.4. *Let $T = \sum_{m=0}^{+\infty} a_m(T)\,\mathcal{H}_m$ be a random Schwartz distribution. Then, the derivative of T has a representation as a Hermite series given by:*

$$T^{'} = a_1(T)\,\sqrt{\pi}\,\mathcal{H}_0 + \sum_{m=1}^{+\infty} \left(a_{m+1}(T)\sqrt{\pi(m+1)} - a_{m-1}(T)\sqrt{\pi m} \right) \,\mathcal{H}_m \ . \tag{83}$$

PROOF. As a consequence of the definition 82 and, of duality formula 77, we have that, for any test function $\varphi \in \mathcal{S}$:

$$< T^{'}, \varphi >=< T, (-\varphi^{'}) >= \sum_{m=0}^{+\infty} a_m(T)\,(-a_m(\varphi^{'})) \ . \tag{84}$$

As a result of an integration by parts, the Hermite coefficients of $-\varphi^{'}$ are given, for $m \in \mathbb{N}$, by:

$$- a_m(\varphi^{'}) = \int_{\mathbb{R}} \varphi(u)\, \mathcal{H}_m^{'}(u)\, du \, . \tag{85}$$

Now, the derivative of any Hermite function can be simply obtained with the operators τ_+ and τ_-. In fact, it follows from the definition of these operators (74) that:

$$\frac{d}{dx} = \frac{1}{2}(\tau_+ - \tau_-) \, .$$

Then, proposition 75, imply that, for $m \geq 1$,

$$\mathcal{H}_m^{'} = \sqrt{\pi m}\, \mathcal{H}_{m-1} - \sqrt{\pi(m+1)}\, \mathcal{H}_{m+1}$$

and, as $\tau_+(\mathcal{H}_0) = 0$ that, $\mathcal{H}_0^{'} = -\sqrt{\pi}\mathcal{H}_1$. These expressions, for the derivatives of Hermite functions, together with formula 85, imply that for $m \geq 1$:

$$- a_m(\varphi^{'}) = a_{m-1}(\varphi)\sqrt{\pi m} - a_{m+1}(\varphi)\sqrt{\pi(m+1)}, \qquad -a_0(\varphi^{'}) = -a_1(\varphi)\sqrt{\pi} \, . \tag{86}$$

After replacing formula 86 in formula 84, we get:

$$< T^{'}, \varphi >= a_1(T)\sqrt{\pi}a_0(\varphi) + \sum_{p=1}^{+\infty} \left(a_{p+1}(T)\sqrt{\pi(p+1)} - a_{p-1}(T)\sqrt{\pi p} \right) a_p(\varphi) \, ,$$

which is, exactly, the formula stated in the theorem □

Due to the hypotheses made on a sequence, in $\mathcal{C}_m$, a random Schwartz distribution given, by way of formula 81, has a mean in the sense of the next definition.

Definition 4.2. Let T be a random Schwartz distribution associated to a complex sequence $(a_m(T))_{m \in \mathbb{N}}$, by formula 81. Then, T admits $\bar{T}$ as a mean if and only if:

(1) $\forall \varphi \in \mathcal{S} \quad < T, \varphi > \in \mathcal{M} \cap \mathrm{L}^1(\Omega)$.
(2) $\forall \varphi \in \mathcal{S} \quad \mathbb{E}[< T, \varphi >] =< \bar{T}, \varphi >$.

A random Schwartz distribution, built with a sequence $(c_m)_{m \in \mathbb{N}} \in \mathcal{C}_m$ by formula 81, does admit a mean. This mean has a representation as a Hermite function series having, as coefficients, the sequence of expectations $(\mathbb{E}[c_m])_{m \in \mathbb{N}}$. This simple result follows from a common application of Lebesgue dominated convergence theorem as we will show next.

Theorem 4.5. *If* $T = \sum_{m=0}^{+\infty} a_m(T)\, \mathcal{H}_m$ *is a random Schwartz distribution then,* T *admits the usual Schwartz distribution* $\bar{T} = \sum_{m=0}^{+\infty} \mathbb{E}[a_m(T)]\, \mathcal{H}_m$ *as a mean.*

Proof. For every $\varphi \in \mathcal{S}$, the sequence $(a_m(\varphi))_{m\in\mathbb{N}}$ is a rapidly decreasing sequence. By theorem 4.1, the sequence $(\mathbb{E}[|a_m(T)|])_{m\in\mathbb{N}}$ is a slow growth sequence and so, the series:

$$\sum_{m=0}^{+\infty} \mathbb{E}[|a_m(T)|]\ |a_m(\varphi)|\ ,$$

converges almost surely. Now, by the Lebesgue dominated convergence theorem:

$$\begin{aligned} < \bar{T}, \varphi > &= \sum_{m=0}^{+\infty} \mathbb{E}[a_m(T)]\ a_m(\varphi) \\ &= \mathbb{E}[\sum_{m=0}^{+\infty} a_m(T)\ a_m(\varphi)] \\ &= \mathbb{E}[< T, \varphi >]\ , \end{aligned}$$

where the last equality results from the duality formula 77. □

4.2. A characterization. A noticeable converse of theorem 4.5 holds, in a sense that we now proceed to explain. Let T be a measurable map from $(\Omega, \mathcal{A})$ into $\mathcal{S}^{'}$, which we consider endowed with the Kolmogoroff σ-algebra associated to the dual countably hilbertian (or Schwartz) topology on $\mathcal{S}$ (see [28, p. 6, 16]). Then, for almost every $\omega \in \Omega$ the Hermite coefficients of $T(\omega)$, which we denote, as usual, by

$$a_m(T(\omega)) =< T(\omega), \mathcal{H}_m >\ ,$$

for $m \in \mathbb{N}$, are all well defined and furthermore, we can consider $(a_m(T))_{m\in\mathbb{N}}$ as a well defined sequence of random variables. For a general T, no growth condition on the sequence $(a_m(T))_{m\in\mathbb{N}}$ is verified so as to ensure that this sequence is in the class $\mathcal{C}_m$. For instance, consider a sequence in which only a finite number of terms are non zero and having one of these as a nonintegrable random variable.

Another example is given by a sequence of random variables taking small values with a big probability, and big values with small probability such as, $(a_m(T))_{m\in\mathbb{N}^*}$ verifying:

$$\mathbb{P}[a_m(T) = me^m] = \frac{1}{m^2}\ ,\quad \mathbb{P}[a_m(T) = 0] = 1 - \frac{1}{m^2}\ .$$

As the series $\sum_{m=1}^{+\infty} \mathbb{P}[a_m(T) = me^m]$ converges, then, by Borel-Cantelli lemma, T is almost surely given by a finite sum of terms. That is, for almost all $\omega \in \Omega$, there exists a $N \in \mathbb{N}^*$, $N = N(\omega)$ such that:

$$T(\omega) = \sum_{m=1}^{N} a_m(T)\ \mathcal{H}_m\ .$$

Now, as we have:

$$\forall m \in \mathbb{N}^*\ \ \mathbb{E}[a_m(T)] = \frac{e^m}{m}\ ,$$

which does not define a sequence of slow growth, the sequence $(a_m(T))_{m\in\mathbb{N}^*}$ can not be in the class $\mathcal{C}_m$, as a result of theorem 4.1.

The next theorem shows that the first condition in definition 4.2 is a sufficient condition on T for the sequence $(a_m(T))_{m\in\mathbb{N}}$ to be in the class $\mathcal{C}_m$.

THEOREM 4.6. *Let T be defined (almost surely) in $\mathcal{S}$, by a sequence of random variables $(a_m(T))_{m\in\mathbb{N}}$, such that:*

$$\forall\varphi\in\mathcal{S}\quad <T,\varphi>=\sum_{m=0}^{+\infty} a_m(T)\,a_m(\varphi)\ \ a.\ s.\ on\ \omega\ .$$

Then, if:

$$\forall\varphi\in\mathcal{S}\quad <T,\varphi>\in L^1(\Omega)\ ,$$

the sequence $(a_m(T))_{m\in\mathbb{N}}$ is in the class $\mathcal{C}_m$.

PROOF. The proof is a straightforward adaptation of the proof given to a similar result for periodic Schwartz distributions in [15]. Let us show first, using the closed graph theorem that, the map Λ_T defined for every test function $\varphi\in\mathcal{S}(\mathbb{R})$ by $\Lambda_T(\varphi)=<T,\varphi>$, is continuous from $\mathcal{S}(\mathbb{R})$ into $L^1(\Omega)$. As T is, almost surely, a random tempered distribution, if $(\varphi_l)_{l\in\mathbb{N}}$ is a sequence of test functions converging to zero in $\mathcal{S}(\mathbb{R})$ then, the sequence of random variables $(U_l)_{l\in\mathbb{N}}$, defined for $l\in\mathbb{N}$, by:

$$U_l=<T,\varphi_l>\ ,$$

converges a.s. to zero. So, this sequence $(U_l)_{l\in\mathbb{N}}$ converges also in probability to zero. Now, suppose that $(U_l)_{l\in\mathbb{N}}$ converges to U in $L^1(\Omega)$; then, the sequence converges also in probability to U and so $U=0$. By the closed graph theorem [53, p. 51], the map Λ_T is continuous. Taking in account the topologies of $\mathcal{S}(\mathbb{R})$ and $L^1(\Omega)$, the continuity of Λ_T can be expressed in the following way:

$$\exists k\in\mathbb{N},\ \exists c_k>0\ \ \forall\varphi\in\mathcal{S}(\mathbb{R})\quad ||<T,\varphi>||_{L^1(\Omega)}\ \le\ c_k\sup_{n\in\mathbb{N}}(1+|n|)^k\,|a_n(\varphi)|\ . \tag{87}$$

As a second step, we use the sequence of Rademacher functions [67, p. 212] defined in $[0,1]$ by:

$$\forall t\in[0,1]\quad\forall n\in\mathbb{N}\quad r_n(t)=\sigma(\sin(2^{n+1}\pi t))\ ,$$

where $\sigma(t)$ denotes the sign of t defined by:

$$\sigma(t)=\begin{cases}\frac{|t|}{t} & \text{if } t\neq 0\\ 0 & \text{otherwise .}\end{cases} \tag{88}$$

And, we also consider $s(\mathbb{N})$, the space of rapidly decreasing complex sequences with the topology induced by the quasi-norms $|.|_k$, $k\in\mathbb{N}$, which are defined for $s=(s_n)_{n\in\mathbb{N}}$, an element of $s(\mathbb{N})$, by:

$$|s|_k=\sup_{n\in\mathbb{N}}(1+n)^k|s_n|\ .$$

Then, for every $t \in [0,1]$, the map from $s(\mathbb{N})$ to $s(\mathbb{N})$ which associates to each sequence $s \in s(\mathbb{N})$, the sequence $w = (w_n)_{n\in\mathbb{Z}}$ defined by

$$\forall n \in \mathbb{Z} \quad w_n = r_n(t)s_n \ ,$$

is an homeomorphism such that:

$$\forall k \in \mathbb{N} \quad |w|_k = |s|_k \ .$$

As a consequence of this observation, of the expression of the continuity of Λ_T, given by (87) and, of Parseval formula, we have that:

$$\begin{gathered}(\exists k \in \mathbb{N},\ c_k > 0) \quad (\forall \varphi \in \mathcal{S}(\mathbb{R})), \quad (\forall t \in [0,1]) \\ \mathbb{E}|\sum_{n=0}^{+\infty} r_n(t)a_n(T)a_n(\varphi)| \ \leq\ c_k \sup_{n\in\mathbb{N}}(1+n)^k\ |a_n(\varphi)| \ .\end{gathered} \tag{89}$$

In the third step of the proof we will show that, the left-hand side of the inequality in (89) can be replaced by the expression:

$$\mathbb{E}[(\sum_{n=0}^{+\infty} |a_n(T)|^2|a_n(\varphi)|^2)^{\frac{1}{2}}] \ .$$

In order to do as stated, we observe that, as the sequence $(a_n(\varphi))_{n\in\mathbb{N}}$ is rapidly decreasing and, almost surely, $(a_n(T))_{n\in\mathbb{N}}$ is a sequence of slow growth, we have almost surely:

$$\sum_{n=0}^{+\infty} |a_n(T)|^2|a_n(\varphi)|^2 \ <\ +\infty \ .$$

Now, by the standard inequality for Rademacher functions [67, p. 213], we have almost surely for some constant c:

$$(\sum_{n=0}^{+\infty} |a_n(T)|^2|a_n(\varphi)|^2)^{\frac{1}{2}} \ \leq\ c \int_0^1 |\sum_{n=0}^{+\infty} r_n(t)\ \ a_n(T)a_n(\varphi)|dt \ .$$

To conclude as desired, it is enough to apply Fubini theorem, to get, for k, c_k and φ as in (89):

$$\mathbb{E}[(\sum_{n=0}^{+\infty} |a_n(T)|^2|a_n(\varphi)|^2)^{\frac{1}{2}}] \leq c\ c_k \sup_{n\in\mathbb{N}}(1+n)^k a_n(\varphi) \ . \tag{90}$$

In the fourth step, we remark that the left-hand side of the inequality, in (90), can be replaced by

$$\mathbb{E}[\sum_{n=0}^{+\infty} \sqrt{\alpha_n}|a_n(T)||a_n(\varphi)|] \ ,$$

where $(\alpha_n)_{n\in\mathbb{N}}$, is an arbitrary sequence of strictly positive numbers such that, $\sum_{n=0}^{n=+\infty} \alpha_n = 1$. This statement results from the fact that, for almost every $\omega \in \Omega$, the expression

$$\sum_{n=0}^{+\infty} |a_n(T)|^2|a_n(\varphi)|^2 = \sum_{n=0}^{+\infty} \alpha_n|a_n(T)|^2\frac{|a_n(\varphi)|^2}{\alpha_n} \ ,$$

can be considered as an integral over $\mathbb{N}$, of the function defined by:

$$\forall n \in \mathbb{N} \quad |a_n(T)|^2 \frac{|a_n(\varphi)|^2}{\alpha_n} \ ,$$

with respect to the measure over $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, that puts a mass α_n on each integer n. Applying Jensen inequality, with the convex function $-\sqrt{x}$ on the interval $[0, +\infty[$, we have that a. s.:

$$(\sum_{n=0}^{+\infty} |a_n(T)|^2 |a_n(\varphi)|^2)^{\frac{1}{2}} \ \geq \ \sum_{n=0}^{+\infty} \alpha_n |a_n(T)| \frac{|a_n(\varphi)|}{\sqrt{\alpha_n}} \ .$$

As a consequence, for k, c_k and φ as in (89):

$$\mathbb{E}[\sum_{n=0}^{+\infty} \sqrt{\alpha_n} |a_n(T)||a_n(\varphi)|] \ \leq \ c \, c_k \sup_{n\in\mathbb{N}} (1+n)^k |a_n(\varphi)| \ . \tag{91}$$

This expression shows that the sequence $(\mathbb{E}[|a_n(T)|\sqrt{\alpha_n}])_{n\in\mathbb{N}}$, is of slow growth at infinity. In order to conclude now, it is enough to consider, for instance, the sequence $(\alpha_n)_{n\in\mathbb{N}}$ defined by:

$$\alpha_n = \begin{cases} 1/A & n = 0 \\ 1/n^2 A & n \neq 0 \end{cases}$$

where $A = 1 + \frac{\pi^2}{6}$. This sequence satisfies the hypothesis made, in the fourth step and, it is clear, that if with this sequence, $(\mathbb{E}[|a_n(T)|\sqrt{\alpha_n}])_{n\in\mathbb{N}}$ is a sequence of slow growth at infinity then, $(\mathbb{E}[|a_n(T)|])_{n\in\mathbb{N}}$ is also of slow growth thus showing that, the random Schwartz distribution T is in the class $\mathcal{C}_m$. □

As a consequence of this theorem we can now formulate the result which gives a characterization of random Schwartz distributions having a mean or a first moment.

THEOREM 4.7. *Let T be a measurable random Schwartz distribution. Then T, has a first moment if and only if the sequence of its Hermite coefficients is in the class $\mathcal{C}_m$.*

PROOF. That the last condition is sufficient was already shown in theorem 4.5. The condition is necessary as a consequence of the theorem 4.6. □

REMARK 13. This last result can also be read as a characterization of the stochastic processes with a first moment which have as trajectories tempered distributions. In fact, let X be a generalized stochastic process (see [26, p. 115]). This will mean for us and, according to the reference quoted that, $X = (X_\varphi)_{\varphi\in\mathcal{S}}$ is a family of random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that, for all $\omega \in \Omega$, the map from $\mathcal{S}$ into $\mathbb{C}$, given by $X_{.}(\omega)$, is a tempered distribution. We can then consider that X, has, as trajectories, tempered distributions. Obviously, such an object defines a random distribution, in the sense we have been using and, moreover, the sequence $(a_m(X))_{m\in\mathbb{N}}$ is a sequence of random variables. As a consequence, theorem 4.7 can be applied giving the characterization mentioned.

5. Two examples

5.1. A brownian type process on $\mathbb{R}$. A gaussian white noise, on $\mathbb{R}$, can naturally be presented using the concepts studied so far. Let $(\xi_m)_{m\in\mathbb{N}}$ be a normal sequence. That is, a sequence of independent, identically distributed, centered, gaussian, random variables, with variance equal to one. As we have:

$$\forall m \in \mathbb{N} \quad \mathbb{E}[|\xi_m|] = \sqrt{\frac{2}{\pi}} \ ,$$

the sequence is in the class $\mathcal{C}_m$ and, being so:

$$W = \sum_{m=0}^{+\infty} \xi_m \, \mathcal{H}_m \ , \tag{92}$$

defines in the usual way a random Schwartz distribution. Let us observe that, as a consequence of theorem 4.5:

$$\mathbb{E}[W] = \sum_{m=0}^{+\infty} \mathbb{E}[\xi_m] \, \mathcal{H}_m = 0 \ .$$

The next easy proposition shows that, W can be seen as a white noise on $\mathbb{R}$.

THEOREM 5.1. *W can be extended, as an isometry, between $L^2(\mathbb{R})$ and the gaussian space $\mathfrak{H}$, which is the closure in $L^2(\Omega)$ of the vector space generated by $(\xi_m)_{m\in\mathbb{N}}$.*

PROOF. Let us observe first, that, for $\varphi \in \mathcal{S}$:

$$< W, \varphi >= \sum_{m=0}^{+\infty} a_m(\varphi) \, \xi_m \ ,$$

is a zero mean gaussian random variable. Using the orthonormality properties of the Hermite functions, the variance of this random variable is given by:

$$\mathbb{E}[| < W, \varphi > |^2] = \sum_{m=0}^{+\infty} |a_m(\varphi)|^2 = ||\varphi||^2 \ . \tag{93}$$

Let now be $f \in \mathrm{L}^2(\mathbb{R})$; the Hermite functions being a complete orthonormal system in $\mathrm{L}^2(\mathbb{R})$ we have that:

$$||f||^2 = (\sum_{m=0}^{+\infty} |a_m(f)|^2)^{\frac{1}{2}} \ < \ +\infty \ .$$

As a result, we can consider the extension of W to the whole $\mathrm{L}^2(\mathbb{R})$ by taking for such an f:

$$W(f) = \sum_{m=0}^{+\infty} a_m(f) \, \xi_m \ ,$$

the series being almost surely convergent, by an application of a known theorem (see [61, p. 58]). It is straightforward to verify that, W is an isometry (with the same calculation as the one which was done in formula 93). The following formulas:

$$\forall m \in \mathbb{N} \quad W(\mathcal{H}_m) = \xi_m \, ,$$

show that $W(\mathrm{L}^2(\mathbb{R})) = \mathfrak{H}$. □

In order to define a brownian type process in $\mathbb{R}$, we use, in a standard way, the gaussian white noise just constructed.

THEOREM 5.2. *There is an (unique in law) brownian type process $(B_t)_{t\in\mathbb{R}}$, having a version with continuous trajectories, such that:*

$$\forall t \in \mathbb{R} \quad B_t = \frac{1}{\sqrt{2}} \sum_{m=0}^{+\infty} (\int_{-|t|}^{+|t|} \mathcal{H}_m(u)\, du)\, \xi_m \;\; a.\ s.\ on\ \ \Omega \, .$$

where $(\xi_m)_{m\in\mathbb{N}}$ is a given normal sequence.

REMARK 14. For each $t \in \mathbb{R}$, B_t is well defined as:

$$B_t = \frac{1}{\sqrt{2}}\, W(\mathbb{I}_{[-|t|,\, +|t|]}) \, . \tag{94}$$

where W, is the gaussian white noise of the previous section and, the indicator function of the interval $[-|t|,\, +|t|]$, obviously belongs to $\mathrm{L}^2(\mathbb{R})$.

LEMMA 5.3. *For each t, B_t, is a centered gaussian random variable and, given $s, t \in \mathbb{R}$, the covariance between B_s and B_t verifies:*

$$\mathbb{E}[B_t, B_s] = min(|s|,\, |t|) \, . \tag{95}$$

PROOF. [of the lemma] The fact that B_t is a centered gaussian variable stems from the definition of the gaussian white noise W. In order to verify formula 95, let us observe that, for every $t \in \mathbb{R}$:

$$\frac{1}{\sqrt{2}}\, \mathbb{I}_{[-|t|,\, +|t|]} = \frac{1}{\sqrt{2}} \sum_{m=0}^{+\infty} (\int_{-|t|}^{+|t|} \mathcal{H}_m(u)\, du)\, \mathcal{H}_m \, .$$

As a similar representation holds replacing t by $s \in \mathbb{R}$, we then have, as there is absolute convergence of the series involved:

$$\frac{1}{2}(\mathbb{I}_{[-|t|,\, +|t|]} \times \mathbb{I}_{[-|s|,\, +|s|]}) = \frac{1}{2} \sum_{m,n\in\mathbb{N}} (\int_{-|t|}^{+|t|} \mathcal{H}_m(u)\, du)(\int_{-|s|}^{+|s|} \mathcal{H}_m(v)\, dv)\, \mathcal{H}_m\, \mathcal{H}_n \, .$$

Now, integrating both sides of this equality in $\mathbb{R}$, gives:

$$\min(|s|,\, |t|) = \frac{1}{2} \sum_{m=0}^{+\infty} (\int_{-|t|}^{+|t|} \mathcal{H}_m(u)\, du)(\int_{-|s|}^{+|s|} \mathcal{H}_m(v)\, dv) \, ,$$

as a by-product of the orthonormality relations among the Hermite functions. In order to conclude, it is enough to observe that:

$$\begin{aligned}\mathbb{E}[B_t,\, B_s] &= \frac{1}{2}\,\mathbb{E}[\sum_{m=0}^{+\infty}((\int_{-|s|}^{+|s|} \mathcal{H}_m(v)\,dv)\;\xi_m)((\int_{-|t|}^{+|t} \mathcal{H}_m(u)\,du)\;\xi_n)] \\ &= \frac{1}{2}\sum_{m,n\in\mathbb{N}}(\int_{-|t|}^{+|t|} \mathcal{H}_m(u)\,du)(\int_{-|s|}^{+|s|} \mathcal{H}_m(v)\,dv)\;\mathbb{E}[\xi_m\xi_n] \\ &= \min(|s|,\,|t|)\;.\end{aligned}$$

where the last equality results, from the fact that the sequence $(\xi_m)_{m\in\mathbb{N}}$ is a normal sequence and so, $\mathbb{E}[\xi_m\xi_n] = \delta_m^n$. □

Proof. [of the theorem] Let us notice that, the kernel $k(s,t) = \min(|s|,\,|t|)$ is a positive definite kernel, as the following easy computation shows. Take I a finite set, a family $(\alpha_i)_{i\in I}$ of complex numbers and, the counting measure μ_c over $(I, 2^I)$. Then, for any family $(s_i)_{i\in I}$ of real numbers and, by Fubini theorem:

$$\begin{aligned}\sum_{(i,j)\in I^2} k(s_i, s_j)\;\alpha_i\bar{\alpha}_j &= \int_{I^2}(\int_{-\infty}^{+\infty} \mathbb{I}_{[0,|s_i|[}(u)\;\mathbb{I}_{[0,|s_j|[}(u)du)\;\alpha_i\;\bar{\alpha}_j\;d\mu_c(i)\,d\mu_c(j) = \\ &= \int_{-\infty}^{+\infty}(\int_I \mathbb{I}_{[0,|s_i|[}(u)\,\alpha_i\,d\mu_c(i))\;(\int_I \mathbb{I}_{[0,|s_j|[}(u)\;\bar{\alpha}_j\;d\mu_c(j))du = \\ &= \int_{-\infty}^{+\infty}|\int_I \mathbb{I}_{[0,|s_i|[}(u)\,\alpha_i\,d\mu_c(i)|^2 du \;\geq\; 0\;.\end{aligned} \tag{96}$$

As a consequence, applying a well know result [46, p. 39], there is an unique (in law) gaussian process, with index set $\mathbb{R}$, having as a mean function the zero function and, as a covariance function the kernel $k(s,t)$. □

Remark 15. $(B_t)_{t\in\mathbb{R}}$ could also be obtained by considering the usual brownian process in $(\tilde{B}_t)_{t\in\mathbb{R}_+}$ (see [31, p. 233]) and, posing as definition:

$$\forall t\in\mathbb{R}\quad B_t = \tilde{B}_{|t|}\quad \text{a. s. on }\;\Omega\;.$$

In fact, the following covariance computation, for $s,t\in\mathbb{R}$:

$$\mathbb{E}[\tilde{B}_{|t|}\;\tilde{B}_{|s|}] = \min(|s|,\,|t|) = \mathbb{E}[B_t\;B_s]\;,$$

shows that $(B_t)_{t\in\mathbb{R}}$ and $(\tilde{B}_{|t|})_{t\in\mathbb{R}}$, have the same law.

As a consequence of this remark, $(B_t)_{t\in\mathbb{R}}$, can be thought as the process in $\mathbb{R}$, obtained by pasting together an usual brownian process with its symmetrized version with respect to the y-axis. As a second consequence, let's point that $(B_t)_{t\in\mathbb{R}}$ admits a continuous version. This follows from the fact that $(\tilde{B}_t)_{t\in\mathbb{R}_+}$ admits a continuous version.

The process $(B_t)_{t\in\mathbb{R}}$ has a relation with W, similar with the one that the usual brownian process has with the gaussian white noise. That relation is the content of the following proposition.

THEOREM 5.4. *The stochastic process* $\mathbf{B} = (B_t)_{t\in\mathbb{R}}$*, defines a random Schwartz distribution by taking for* $\varphi \in \mathcal{S}$*:*

$$< \mathbf{B}, \varphi >= \frac{1}{\sqrt{2}} \sum_{m=0}^{+\infty} \left(\int_{-\infty}^{+\infty} \varphi(t) (\int_{-|t|}^{+|t|} \mathcal{H}_m(u)\, du)\, dt \right) \xi_m \ , \tag{97}$$

and $\mathbf{B}'$*, the derivative of* $\mathbf{B}$ *in the sense of distributions verifies:*

$$\mathbf{B}' = \frac{1}{\sqrt{2}} \left(W + W^{\vee} \right) , \tag{98}$$

where the operator $^{\vee}$ *is the transposed operator of the operator acting on a* $\varphi \in \mathcal{S}$ *by:*

$$\forall t \in \mathbb{R} \qquad \varphi^{\vee}(t) = \varphi(-t) \ . \tag{99}$$

PROOF. By Fubini theorem, the integral on the right-hand side of definition 97 can be rewritten in the following form:

$$\begin{aligned} &\int_{-\infty}^{+\infty} \mathcal{H}_m(u) \left(\int_{-\infty}^{+\infty} \mathbb{I}_{[-|t|,+|t|]}(u)\, \varphi(t)\, dt \right) du = \\ &= \int_{-\infty}^{0} \mathcal{H}_m(u) \left(\int_{-\infty}^{0} \mathbb{I}_{]-\infty,+u]}(t)\, \varphi(t)\, dt + \int_{0}^{+\infty} \mathbb{I}_{[-u,+\infty[}(t)\, \varphi(t)\, dt \right) du + \\ &+ \int_{0}^{+\infty} \mathcal{H}_m(u) \left(\int_{-\infty}^{0} \mathbb{I}_{]-\infty,-u]}(t)\, \varphi(t)\, dt + \int_{0}^{+\infty} \mathbb{I}_{[u,+\infty[}(t)\, \varphi(t)\, dt \right) du \, . \end{aligned} \tag{100}$$

Now, let us take for instance, the first integral, in the integrand part of the right-hand side of this last formula and, define:

$$F_1(u) = \left(\int_{-\infty}^{0} \mathbb{I}_{]-\infty,+u]}(t)\, \varphi(t)\, dt \right) \mathbb{I}_{]-\infty,0]}(u) \ .$$

Using the the hypothesis made on φ, namely that $\varphi \in \mathcal{S}$ and, as a result of an easy application of Cauchy-Schwarz inequality and Fubini theorem, we get $F_1 \in \mathrm{L}^2(\mathbb{R})$. In fact, we have:

$$\begin{aligned} |F_1(u)| &\leq \left(\int_{-\infty}^{u} (1+|t|)^{\frac{3}{2}}\, |\varphi(t)| \frac{dt}{(1+|t|)^{\frac{3}{2}}} \right) \mathbb{I}_{]-\infty,0]}(u) \\ &\leq \left(\int_{-\infty}^{u} (1+|t|)^{3}\, |\varphi(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_{-\infty}^{u} \frac{dt}{(1+|t|)^{3}} \right)^{\frac{1}{2}} \mathbb{I}_{]-\infty,0]}(u) \ . \end{aligned}$$

Now, considering the function G_1 defined by:

$$G_1(u) = \left(\int_{-\infty}^{u} \frac{dt}{(1+|t|)^{3}} \right)^{\frac{1}{2}} \mathbb{I}_{]-\infty,0]}(u) \ , \tag{101}$$

we have, by Fubini theorem, that:

$$\int_{\mathbb{R}} |G_1(u)|^2\, du = \int_{-\infty}^{0} \left(\int_{-\infty}^{u} \frac{dt}{(1+|t|)^{3}} \right) du = \int_{-\infty}^{0} \frac{|t|}{(1+|t|)^{3}}\, dt \ < +\infty \ .$$

We have just shown that G_1 and, as a consequence F_1, are $\mathrm{L}^2(\mathbb{R})$ functions. Applying the same idea to the similar integral terms in the integrand part of the right-hand side of 100, we can conclude that in formula 97, the coefficient of ξ_m in the sum is a Hermite coefficient of an L^2 function an so, the series converges almost surely. Let us make this idea more precise by denoting F_2, F_3 and F_4, (respectively G_2, G_3 and G_4) the functions just referred as the integral terms, similar to F_1 in 100, (respectively to G_1 in 101). Then, denoting by F_φ the function defined by:

$$F_\varphi(u) = \int_{-\infty}^{+\infty} \mathbb{I}_{[-|t|,+|t|]}(u)\, \varphi(t)\, dt \ ,$$

we have that, $F_\varphi = \sum_{i=1}^4 F_i$ with $F_i \in \mathrm{L}^2(\mathbb{R})$ for $i \in \{1, \ldots, 4\}$. Furthermore, by using similar estimates for F_i and G_i with $i = 2, \ldots, 4$, as the estimates proved for F_1 and G_1, we have that, for some constant C:

$$||F_\varphi||_2 \leq C \sup_{t\in\mathbb{R}} \left((1+|t|)^3 \, |\varphi(t)|^2\right) \left(\sum_{i=1}^4 ||G_i||_2 \right) < +\infty \ . \tag{102}$$

Now, just observe that, by the first line in formula 100,

$$< \mathbf{B}, \varphi >= \frac{1}{\sqrt{2}} \sum_{m=0}^{+\infty} < F_\varphi, \mathcal{H}_m > \ \xi_m = \frac{1}{\sqrt{2}} \, W(F_\varphi) \ .$$

This formula shows that $< \mathbf{B}, \varphi >$ is defined by a series which converges almost surely, as a consequence of having $F_\varphi \in \mathrm{L}^2(\mathbb{R})$. Furthermore, inequality 102 shows that the map from $\mathcal{S}$ into $\mathrm{L}^2(\mathbb{R})$, which associates to the test function $\varphi \in \mathcal{S}$, the function F_φ, is continuous. As W is continuous from $\mathrm{L}^2(\mathbb{R})$ into $\mathfrak{H}$ then, $\mathbf{B}$ is continuous from $\mathcal{S}$ into $\mathfrak{H}$ and so, $\mathbf{B}$ is a random Schwartz distribution. In order to compute the derivative of $\mathbf{B}$, it is enough to recall that the derivative of $\mathbf{B}$ is given, as usually, for $\varphi \in \mathcal{S}$ by:

$$< \mathbf{B}^{'}, \varphi >= - < \mathbf{B}, \varphi^{'} > \ .$$

Using the definition of $\mathbf{B}$ given by 97 and, after some integrations by parts on the terms obtained by considering the case $t \geq 0$ and the case $t \leq 0$, we get that:

$$\mathbf{B}^{'} = \frac{1}{\sqrt{2}} \sum_{m=0}^{+\infty} \left(\mathcal{H}_m(t) + \mathcal{H}_m(-t)\right) \ \xi_m \ \text{ a. s. on } \Omega \ .$$

This expression shows, after some change of variables, to deal with the series having as coefficients $(\mathcal{H}_m(-t))_{m\in\mathbb{N}}$, that:

$$\mathbf{B}^{'} = \frac{1}{\sqrt{2}} \left(W + W^{\vee}\right) \text{ a. s. on } \Omega \ ,$$

which is the result announced in 98. □

REMARK 16. The usual Brownian process on $\mathbb{R}_+$, let it be $\tilde{\mathbf{B}} = (\tilde{B}_t)_{t\in[0,+\infty[}$, could have been studied in a similar fashion, using the gaussian white noise W, by taking as a definition:

$$\tilde{B}_t = W(\mathbb{I}_{[0,t[}) \ .$$

Instead of formula 98, in theorem 5.4, we would then have as usually:

$$\frac{d}{dt}\tilde{\mathbf{B}} = W \text{ a. s. on } \Omega \ ,$$

With this definition, the process $\tilde{\mathbf{B}}$ would be a bona-fide Brownian process satisfying all the properties of Brownian processes (see [61, p. 220]. The process $\mathbf{B}$ just studied, does not satisfy the independence and distribution conditions on the increments.

5.2. The brownian distributions. A natural generalization of the construction presented in the last subsection, consists on taking as a starting point an isometry between $\mathrm{L}^2(\mathbb{R},\mu)$, where μ is a tempered measure on $\mathbb{R}$ with the Borel σ-algebra and, a gaussian space $\mathfrak{H}$ of $\mathrm{L}^2(\Omega)$. A theorem of J.-P. Kahane, which we will state and prove next (see theorem 5.7) will show that this generalization gives raise to some random Schwartz distributions of the type we have been studying. To begin with, let us make some remarks on the properties of tempered measures needed in the sequel.

DEFINITION 5.1. A positive measure μ on $\mathbb{R}$, is a tempered measure if and only if:

$$\exists l \in \mathbb{N} \quad \int_{\mathbb{R}} \frac{d\mu}{(1+|t|^2)^l} \ < +\infty \ .$$

It is obvious that μ integrates the function $1/(1+|t|^2)^{l+\epsilon}$, with $\epsilon > 0$, whenever μ integrates $1/(1+|t|^2)^l$ and so, if we define:

$$l_\mu = \min\{\, l \in \mathbb{N} \, : \int_{\mathbb{R}} \frac{d\mu}{(1+|t|^2)^l} \ < +\infty \,\} \ , \tag{103}$$

then, l_μ is a well defined integer for every tempered measure. In some sense, this integer quantifies the growth of the measure μ at infinity.

Let us state and prove now, a classical structure result (see [65, II, p. 255]), in a form which will be useful for our purposes.

THEOREM 5.5. *Let μ be a positive tempered measure, over $\mathbb{R}$. Let $l_\mu \in \mathbb{N}$ be the integer given by formula* 103. *There exists then $f \in L^2(\mathbb{R})$ such that, in the sense of tempered distributions, we have:*

$$\mu = \left((1+|t|^2)^{l_\mu+1}\, f\right)^{'}$$

This means, by definition, that:

$$\forall \varphi \in \mathcal{S} \quad <\mu,\varphi>= \int_{\mathbb{R}} \varphi \, d\mu = \int_{\mathbb{R}} (1+|t|^2)^{l_\mu+1} f(t)\, \varphi^{'}(t)\, dt \ . \tag{104}$$

Proof. Let us show first that, μ defines in the first equality of the formula 104 a tempered distribution. Indeed, for $\varphi \in \mathcal{S}$:

$$\begin{aligned} | < \mu, \varphi > | &\leq \int_{\mathbb{R}} (1+|t|^2)^{l_\mu} \, |\varphi(t)| \, \frac{d\mu(t)}{(1+|t|^2)^{l_\mu}} \\ &\leq \sup_{t\in\mathbb{R}} \left((1+|t|^2)^{l_\mu} \, |\varphi(t)|\right) \int_{\mathbb{R}} \frac{d\mu(t)}{(1+|t|^2)^{l_\mu}} \ . \end{aligned}$$

This inequality shows that μ defines a tempered distribution. Now, it is possible to prove (see [65, II, p. 253]) that, for every $\varphi \in \mathcal{S}$:

$$\sup_{t\in\mathbb{R}} \left((1+|t|^2)^{l_\mu} \, |\varphi(t)|\right) \leq || \, (1+|t|^2)^{l_\mu} \, \varphi^{'}(t) \, ||_1 \leq \pi \, || \, (1+|t|^2)^{l_\mu+1} \, \varphi^{'}(t) \, ||_2 \ .$$

And, being so, we have:

$$| < \mu, \varphi > | \leq \left(\int_{\mathbb{R}} \frac{d\mu(t)}{(1+|t|^2)^{l_\mu}}\right) \pi \, || \, (1+|t|^2)^{l_\mu+1} \, \varphi^{'}(t) \, ||_2 \ .$$

This shows that μ is an element of the dual of X, the space of $C^\infty(\mathbb{R})$ functions with compact support, endowed with the topology associated to the pre-hilbertian norm given by:

$$||\varphi|| = || \, (1+|t|^2)^{l_\mu+1} \, \varphi^{'}(t) \, ||_2 \ .$$

As the map which associates to $\varphi \in X$ the function $(1+|t|^2)^{l_\mu+1} \, \varphi^{'}(t)$ is an isometric embedding of X into $\mathrm{L}^2(\mathbb{R})$, its transpose is an surjective mapping from $\mathrm{L}^2(\mathbb{R})$ onto $X^{'}$ (see again [65, II, p. 249]). This shows that there exists $f \in \mathrm{L}^2(\mathbb{R})$, satisfying formula 104 stated in the theorem, for every φ in the test function space $\mathcal{D}$. This space $\mathcal{D}$, is the space of $C^\infty(\mathbb{R})$ functions having compact support, with the topology of uniform convergence on compact sets, not only of the functions but also of the derivatives of all orders of the functions. As the space $\mathcal{D}$ is dense in $\mathcal{S}$ (see [65, II, p. 7]), some usual integration arguments allow us to conclude. In fact, let $\varphi \in \mathcal{S}$ and $(\psi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}$ having φ as a limit in $\mathcal{S}$. Then, as we have:

$$\lim_{n\to+\infty} (1+|t|^2)^{l_\mu} \, \psi_n = (1+|t|^2)^{l_\mu} \, \varphi \ ,$$

the convergence being uniform in $\mathbb{R}$ and, $\int (1+|t|^2)^{-l_\mu} d\mu < +\infty$, we can conclude that,

$$< \mu, \varphi > = \lim_{n\to+\infty} \int_{\mathbb{R}} \psi_n \, d\mu = \lim_{n\to+\infty} < \mu, \psi_n > \ . \tag{105}$$

By Cauchy-Schwarz inequality, it is also true that:

$$\begin{aligned} &| \int_{\mathbb{R}} (1+|t|^2)^{l_\mu+1} \, f(t) \left(\psi_n^{'}(t) - \varphi^{'}(t)\right) dt| \leq \\ &\qquad ||f||_2 \left(\int_{\mathbb{R}} \frac{dt}{1+|t|^2}\right)^{\frac{1}{2}} \sup_{t\in\mathbb{R}} \left((1+|t|^2)^{l_\mu+\frac{3}{2}} \, |\psi_n^{'}(t) - \varphi^{'}(t)|\right) \ . \end{aligned}$$

This inequality shows that:

$$\lim_{n\to+\infty} \int_{\mathbb{R}} (1+|t|^2)^{l_\mu+1} \, f(t) \, \psi_n^{'}(t) \, dt = \int_{\mathbb{R}} (1+|t|^2)^{l_\mu+1} \, f(t) \, \varphi^{'}(t) \, dt \ . \tag{106}$$

As a consequence of formulas 105 and 106, we have that, formula 104 is verified also for the test functions φ in $\mathcal{S}$. $\square$

In order to prepare a lean redaction of the proof of the theorem on brownian distributions, we will apply this result to obtain estimates of the image by μ of the generating function of the squares of Hermite functions.

LEMMA 5.6. *For $0 \leq t < 1$ we have, for some constant $c = c(\mu)$:*

$$| < \mu, e^{-2\pi x^2 \, (\frac{1-t}{1+t})} > | \leq \, c \, (\frac{1-t}{1+t})^{-(l_\mu + \frac{3}{4})} \; . \tag{107}$$

PROOF. Using formula 104 we have that:

$$| < \mu, e^{-2\pi x^2 \, (\frac{1-t}{1+t})} > | \leq \, 4\pi (\frac{1-t}{1+t}) \int_{\mathbb{R}} (1+|x|^2)^{l_\mu +1} \, |x| |f(x)| e^{-2\pi x^2 \, \frac{1-t}{1+t}} \, dx \; .$$

We decompose the integral of the right-hand term, of this inequality, into the sum of the integral over $[-1,+1]$ and, the integral over the complement of this set. The first integral to consider gives, by Cauchy-Schwarz inequality:

$$\int_{|x| \leq 1} e^{-ax^2} \, |f(x)| \, 2^{l_\mu + 1} \, dx \leq \, 2^{l_\mu +1} \, ||f||_2 \, (\int_{|x| \leq 1} e^{-ax^2} \, dx)^{\frac{1}{2}} \; ,$$

denoting by:

$$a = 2\pi \, (\frac{1-t}{1+t}) \; > 0 \; .$$

Recall now that, for every $l \in \mathbb{N}$:

$$\int_{\mathbb{R}} e^{-x^2} \, x^{2l} \, dx = \sqrt{\pi} \, \frac{(2l)!}{4^l \, l!} \; . \tag{108}$$

And, this implies, after a change of variables, that:

$$(\int_{|x| \leq 1} e^{-ax^2} \, dx)^{\frac{1}{2}} \; \leq \, 2^{-\frac{1}{4}} \, (\frac{1-t}{1+t})^{-\frac{1}{4}} \; ,$$

giving finally the estimate for the first integral:

$$4\pi (\frac{1-t}{1+t}) \int_{|x| \leq 1} (1+|x|^2)^{l_\mu +1} \, |x| \; |f(x)| e^{-2\pi x^2 \, \frac{1-t}{1+t}} \, dx \; \leq \pi \, 2^{l_\mu + \frac{11}{4}} \; ||f||_2 \, (\frac{1-t}{1+t})^{\frac{3}{4}} \; .$$

For the integral over $[-1,+1]^c$ and, with the same notations as before, we have:

$$\begin{aligned} & \int_{|x|>1} |x| \, e^{-a\,x^2} (1+|x|^2)^{l_\mu +1} \, |f(x)| \, dx \leq \\ & 2^{l_\mu +1} \int_{|x|>1} |x|^{2l_\mu +3} \, e^{-a\,x^2} \, |f(x)| \, dx \; \leq \\ & 2^{l_\mu +1} \, (\int_{|x|>1} x^{4l_\mu +6} \, e^{-2a\,x^2} \, dx)^{\frac{1}{2}} \; ||f||_2 \; . \end{aligned} \tag{109}$$

Using again formula 108, with a change of variables, we get that:

$$4\pi(\frac{1-t}{1+t})\,(\int_{|x|>1} x^{4l_\mu+6}\,e^{-2a\,x^2}\,dx)^{\frac{1}{2}}||f||_2 \leq$$

$$\frac{1}{2^{3l_\mu+\frac{7}{2}}\,\pi^{l_\mu+\frac{1}{2}}}\sqrt{\frac{(4l_\mu+6)!}{(2l_\mu+3)!}}\,\,(\frac{1-t}{1+t})^{-(l_\mu+\frac{3}{4})}||f||_2$$

In order conclude, as stated in the lemma, just consider

$$c = 2\,||f||_2\,\max\left(\pi\,2^{l_\mu+\frac{11}{4}}\,,\,\frac{1}{2^{3l_\mu+\frac{7}{2}}\,\pi^{l_\mu+\frac{1}{2}}}\sqrt{\frac{(4l_\mu+6)!}{(2l_\mu+3)!}}\right)\,,$$

and observe that:

$$(\frac{1-t}{1+t})^{\frac{3}{4}}\,\leq\,(\frac{1-t}{1+t})^{-(l_\mu+\frac{3}{4})}\,,$$

for $0 \leq t < 1$ and $l_\mu \in \mathbb{N}$. □

We are now ready to state and prove a theorem of J.-P. Kahane [30, p. 121] which, in the context in which we have been working, allow us, to construct important examples of random Schwartz distributions. The proof presented, follows the main idea of the original proof but, relies in the work presented above in order to obtain accurate estimates of the order of growth of the random Schwartz distribution built.

Definition 5.2. Given a tempered measure μ, a μ-brownian distribution is an isometry between $\mathrm{L}^2(\mathbb{R},\mu)$ and $\mathfrak{H}$ a gaussian closed subspace of $\mathrm{L}^2(\Omega)$ where $(\Omega,\mathcal{A},\mathbb{P})$, is supposed to be a complete probability space with no atoms.

This definition makes good sense, since as μ is a tempered measure on $\mathbb{R}$, the Hilbert space $\mathrm{L}^2(\mathbb{R},\mu)$ is separable. Moreover, as a consequence of the hypotheses made on the probability space $(\Omega,\mathcal{A},\mathbb{P})$, there exist gaussian, closed, separable and non trivial, subspaces of $\mathrm{L}^2(\Omega)$. By choosing complete orthonormal sequences in both $\mathrm{L}^2(\mathbb{R},\mu)$ and $\mathfrak{H}$, let them be respectively $(f_n)_{n\in\mathbb{N}}$ and $(\zeta_n)_{n\in\mathbb{N}}$ and, associating naturally f_n to ζ_n for each $n \in \mathbb{N}$, there exists an isometry Z_μ between $\mathrm{L}^2(\mathbb{R},\mu)$ and $\mathfrak{H}$ given by:

$$Z_\mu(f) = \sum_{m=0}^{+\infty} < f, f_n > \zeta_n\,, \tag{110}$$

where $(< f, f_n >)_{n\in\mathbb{N}}$ is the family of the Fourier coefficients of f, with respect to $(f_n)_{n\in\mathbb{N}}$. This family verifies:

$$f = \sum_{m=0}^{+\infty} < f, f_n > f_n\,,$$

with equality in $\mathrm{L}^2(\mathbb{R},\mu)$ sense, the series 110 being convergent in $\mathrm{L}^2(\Omega)$. Observe, incidentally, that as:

$$||f||_2^2 = \sum_{m=0}^{+\infty} |< f, f_n >|^2\,<\,+\infty\,,$$

the series 110 converges almost surely.

THEOREM 5.7 (ON BROWNIAN DISTRIBUTIONS). *Let μ be a tempered positive measure and Z_μ, a μ-brownian distribution. There exists then, a random Schwartz distribution Z^*, such that $Z_\mu = Z^*$ almost surely.*

PROOF. As we have that:

$$\forall m \in \mathbb{N} \quad \mathcal{H}_m \in \mathcal{S} \subset \mathrm{L}^2(\mathbb{R}, \mu) ,$$

it will be enough to show that the sequence $(Z_\mu(\mathcal{H}_m))_{m\in\mathbb{N}}$ is a sequence of random variables in the class $\mathcal{C}_m$. As we already know, we will then have that, Z^* defined by:

$$Z^* = \sum_{m=0}^{+\infty} Z_\mu(\mathcal{H}_m) \ \mathcal{H}_m ,$$

is a well defined random Schwartz distribution. By the duality formula 77 we will have, using the fact that Z_μ is an isometry and, being so, it is a linear and continuous map that, for $\varphi \in \mathcal{S}$:

$$< Z^*, \varphi >= \sum_{m=0}^{+\infty} Z_\mu(\mathcal{H}_m) < \varphi, \mathcal{H}_m >= Z_\mu(\sum_{m=0}^{+\infty} < \varphi, \mathcal{H}_m > \mathcal{H}_m) = Z_\mu(\varphi) .$$

In order to prove the statement made on $(Z_\mu(\mathcal{H}_m))_{n\in\mathbb{N}}$, we recall formula 78, which gives the generating function for the squares of Hermite functions. For $|t| < 1$:

$$\sum_{m=0}^{+\infty} t^m \mathcal{H}_m^2(x) = (\frac{2}{1-t^2})^{\frac{1}{2}} e^{-2\pi x^2 \frac{1-t}{1+t}} .$$

Now, by the Lebesgue monotone convergence theorem, integrating both sides of this equality with respect to μ and, using the fact that Z_μ is an isometry, we have that, for $0 < t < 1$:

$$\sum_{m=0}^{+\infty} t^m \ ||Z_\mu(\mathcal{H}_m)||_2^2 = (\frac{2}{1-t^2})^{\frac{1}{2}} \int_{\mathbb{R}} e^{-2\pi x^2 \frac{1-t}{1+t}} d\mu(x) . \tag{111}$$

As a consequence of the estimates obtained in formula 107, we get for $p \geq 0$ and $0 < t < 1$:

$$t^p \ ||Z_\mu(\mathcal{H}_p)||_2^2 \leq \sum_{m=0}^{+\infty} t^m \ ||Z_\mu(\mathcal{H}_m)||_2^2 \leq c \, (\frac{2}{1-t^2})^{\frac{1}{2}} (\frac{1-t}{1+t})^{-(l_\mu + \frac{3}{4})} . \tag{112}$$

Consider now that, for $p \geq 1$ we have: $t = 1 - 1/p$. We will have then, for some constant C and, after some computations on inequality 112, that:

$$\forall p \geq 1 \ \ ||Z_\mu(\mathcal{H}_p)||_2^2 \leq C \ (2p-1)^{l_\mu + \frac{5}{4}} . \tag{113}$$

In order to conclude, let us observe that the random variable

$$\frac{Z_\mu(\mathcal{H}_p)}{||\mathcal{H}_p||} ,$$

is a standard normal random variable; being so we have:

$$\mathbb{E}[\,|\frac{Z_\mu(\mathcal{H}_p)}{||\mathcal{H}_p||}|\,] = \frac{2}{\sqrt{2\pi}} .$$

This, together with inequality 113, shows that:

$$\exists c > 0 \ \ \forall p \geq 1 \ \ \mathbb{E}[\,|Z_\mu(\mathcal{H}_p)|\,] \leq c\,(2p-1)^{\frac{l_\mu}{2}+\frac{5}{8}} ,$$

which tells us that, the sequence $(Z_\mu(\mathcal{H}_p))_{p\in\mathbb{N}}$ is in the class $\mathcal{C}_m$, fully justifying the statement of the theorem. □

6. Some remarks on the brownian distribution example

In this last section, we are going to present some comments and results tied to the following problem.

Problem 6.1. *Let Z be a random Schwartz distribution of the type we have been studying. Under what conditions on Z, is there a tempered positive measure μ, such that for the associated brownian distribution Z_μ, we have $Z = Z_\mu$ almost surely ?*

As a consequence of the results presented so far, $(< Z, \mathcal{H}_m >)_{m\in\mathbb{N}}$ has to be a sequence of centered gaussian variables such that, the sequence of their variances $(\mathbb{E}[\,| < Z, \mathcal{H}_m > |^2\,])_{m\in\mathbb{N}}$, is a sequence of slow growth. Furthermore, as a consequence of formula 111, in the proof of Kahane's theorem, we know that the following Hamburger type moment problem:

$$M_m = \frac{1}{m!}\,\frac{d^m}{dx^m}[(\frac{2}{1-t^2})^{\frac{1}{2}}\int_{\mathbb{R}} e^{-2\pi x^2\frac{1-t}{1+t}}\,d\mu(x)]\,\Big|_{t=0} , \tag{114}$$

where $m \in \mathbb{N}$ and M_m stands for $\mathbb{E}[\,| < Z, \mathcal{H}_m > |^2\,]$, must have, as a solution, a tempered measure. Let us try to pursue this line of reasoning a little further. Denoting by $G(t)$, $f(t)$ and $g_\mu(t)$, respectively,

$$G(t) = \frac{1-t}{1+t} , \quad f(t) = (\frac{2}{1-t^2})^{\frac{1}{2}} \quad \text{and} \quad g_\mu(t) = \int_{\mathbb{R}} e^{-2\pi x^2\frac{1-t}{1+t}}\,d\mu(x) ,$$

we have that the moment problem, formulated in equations 114, has a solution if for every $p \in \mathbb{N}$ the lower triangular linear system, given by equations:

$$m = 0,\ldots,p \quad M_m = \sum_{k=0}^{m} \frac{1}{k!\,(m-k)!}\; f^{(m-k)}(0)\; g_\mu^{(p)}(0) , \tag{115}$$

has a solution, which we will denote by

$$(g_\mu(0), g_\mu^{'}(0), \ldots, g_\mu^{(p)}(0)) .$$

Let us observe that, as a result of a straightforward calculation, he have that for $m \in \mathbb{N}$ and $k \in \{0, 1, \ldots, m\}$:

$$f^{(m-k)}(0) = \begin{cases} 0 & \text{if } m-k \text{ is odd} \\ (1 \times 3 \times 5 \times \cdots \times (m-k-1))^2 \sqrt{2} & \text{if } m-k \text{ is even .} \end{cases}$$

As a consequence, given a sequence $(M_n)_{n\in\mathbb{N}}$, the solution of the linear system given by equations 115, always exist and, can be computed by induction. Let us detail now the computations for $(g_\mu^{(m)}(0))_{m\in\mathbb{N}}$. It easy to see that, all the derivatives of g_μ taken at the point zero exist and, using the formula of Faa di Bruno ([22, p. 261], [62, p. 248] or [4, p. I47]) which gives the derivative of order m of the composition of two maps, that:

$$g_\mu^{(m)}(0) = \sum \frac{m!\,(-1)^{p+q}\,(4\pi)^p}{n_1!\,n_2!\,\ldots n_q!} \int_{\mathbb{R}} x^{2p}\, e^{-2\pi x^2}\, d\mu(x) \; . \tag{116}$$

In formula 116, the sum is over all the integer sequences $(n_i)_{1\leq i\leq q}$ such that $\sum_{i=1}^q i n_i = m$ and with $\sum_{i=1}^q n_i = p$. This representation shows that, $g_\mu^m(0)$ is a linear combination of moments of even order of $d\nu = e^{-2\pi x^2}\, d\mu$. And so, if a tempered measure μ, solution of problem 6.1 exists, then, the even moments of $d\nu$ must satisfy equations 116. Let us observe that, for every $p \in \mathbb{N}$ the lower triangular linear system, given by equations 116 for $m = 0, \ldots, p$, always has a solution. So, problem 6.1 stated, in the introduction of this section, leads to another problem that can be rephrased in the following manner.

Problem 6.2. *Given a sequence $(M_m)_{m\in\mathbb{N}}$ of slow growth, does there exist a tempered measure μ, such that the even moments of $d\nu = e^{-2\pi x^2}\, d\mu$ satisfy equations* 116, *in which the sequence $(g_\mu^{(m)}(0))_{m\in\mathbb{N}}$ is, in turn, a solution of equations* 115*?*

At the sight of this statement of the problem, a natural question to ask is: what can go wrong since as we have already shown, a solution for the linear systems involved always exist? At the moment, the answer we can give has three main components:

(1) The sequence obtained may not be a solution of the moment problem if it has a growth which is non compatible with the condition of μ being a tempered measure. This growth is exactly defined by theorem 6.3.
(2) The Hamburger type moment problem , in itself, that is, the problem of the existence and uniqueness of a measure having given moments over $\mathbb{R}$, may not be well posed.
(3) The sequence obtained as solution of the linear systems may not be a sequence of even moments of a positive measure $d\nu$ if for instance, it contains some negative terms.

In order to deal with the first component of the answer above, we may formulate a characterization of tempered measures which will provide a useful testing criteria for measures which are solutions of problem 6.2.

THEOREM 6.3. *Let μ be a positive measure on $\mathbb{R}$. Then, given $l \in \mathbb{N}$, a necessary and sufficient condition for having,*

$$\int_{\mathbb{R}} \frac{d\mu(x)}{(1+|x|^2)^l} < +\infty \tag{117}$$

is that:

$$\mu(\{\,|x|<1\,\}) < +\infty \tag{118}$$

and

$$\begin{array}{c}(\exists (a_k)_{k\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}} \quad \sum_{k=0}^{+\infty} (2\pi)^k\, a_k < +\infty) \quad (\forall k \in \mathbb{N}) \\ \int_{\{|x|\geq 1\}} x^{2k-2l}\, e^{-2\pi x^2}\, d\mu(x) \leq k!\, a_k \;.\end{array} \tag{119}$$

PROOF. Suppose that 119 is verified. Then, by Lebesgue's monotone convergence theorem:

$$\begin{aligned} \int_{|x|\geq 1} \frac{d\mu}{(1+|x|^2)^l} &= \sum_{k=0}^{+\infty} \frac{(2\pi)^k}{k!} \int_{|x|\geq 1} x^{2k}\, e^{-2\pi x^2}\, \frac{d\mu}{(1+|x|^2)^l} \\ &\leq \sum_{k=0}^{+\infty} \frac{(2\pi)^k}{k!} \int_{|x|\geq 1} x^{2k-2l}\, e^{-2\pi x^2}\, d\mu \\ &\leq \sum_{k=0}^{+\infty} (2\pi)^k\, a_k < +\infty \;, \end{aligned}$$

as a consequence of having, for $|x| \geq 1$, that:

$$\frac{1}{(1+|x|^2)^l} \leq \frac{1}{|x|^{2l}} \;.$$

Now, observe that:

$$2^{-l}\, \mu(\{|x|<1\}) = \int_{|x|<1} \frac{d\mu}{2^l} \leq \int_{|x|<1} \frac{d\mu}{(1+|x|^2)^l} \leq \mu(\{|x|<1\}) \;, \tag{120}$$

and, obviously, we can conclude that 118 and 119 do imply 117. Suppose now that 117 is verified. Then, certainly, 118 is verified and, the same decomposition as above yields:

$$+\infty > \int_{|x|\geq 1} \frac{d\mu}{(1+|x|^2)^l} = \sum_{k=0}^{+\infty} \frac{(2\pi)^k}{k!} \int_{|x|\geq 1} x^{2k}\, e^{-2\pi x^2}\, \frac{d\mu}{(1+|x|^2)^l} \;.$$

Denoting:

$$b_k = \frac{1}{k!} \int_{|x|\geq 1} x^{2k}\, e^{-2\pi x^2}\, \frac{d\mu}{(1+|x|^2)^l} \;,$$

it is true that:

$$\sum_{k=0}^{+\infty} (2\pi)^k\, b_k < +\infty \;.$$

In order to conclude, just observe that:

$$2^l\ k!\ b_k \geq \int_{|x|\geq 1} x^{2k-2l}\, e^{-2\pi x^2}\ \frac{d\mu}{x^{2l}}\ ,$$

and so, that 119 is verified with $a_k = 2^l\, b_k$. □

To deal with the second component of our answer, we recall a result of Carleman (see [38, p. 289a] or [52, p. 292a]), on the unicity of the solution of a moment problem.

THEOREM 6.4 (CARLEMAN). *The Hamburger moment problem of asserting the existence of a measure ν, verifying for a given sequence $(N_m)_{m\in\mathbb{N}}$, the equations given by:*

$$N_m = \int_{\mathbb{R}} x^m\, d\nu(x)\ ,$$

has an unique solution if:

$$\sum_{m=0}^{+\infty} \frac{1}{\sqrt[2m]{N_{2m}}} = +\infty\ . \tag{121}$$

We observe next, that there is a connection between the Carleman's condition 121 and the condition 119 of theorem 6.3.

LEMMA 6.5. *Let μ be a tempered measure and $l \in \mathbb{N}$ be an integer for which conditions* 118 *and* 119 *are satisfied. Then, denoting for $n \in \mathbb{N}$,*

$$\mu_{2n} = \int_{\mathbb{R}} x^{2n} e^{-2\pi x^2}\ d\mu(x)\ ,$$

we have:

$$\sum_{n=0}^{+\infty} \frac{1}{\sqrt[2m]{\mu_{2m}}} = +\infty\ .$$

PROOF. As a consequence of condition 119, we have:

$$\int_{|x|\geq 1} x^{2(n+l)-2l} e^{-2\pi x^2}\ d\mu(x) = \int_{|x|\geq 1} x^{2n} e^{-2\pi x^2}\ d\mu(x) \leq a_{n+l}\,(n+l)!\ .$$

This estimate, together with condition 118, implies that:

$$\frac{1}{\mu_{2n}} \geq \frac{1}{\mu(\{|x|\geq 1\})\ +\ a_{n+l}\,(n+l)!}\ .$$

Finally, using for instance Stirling's formula, it is straightforward to prove that :

$$\frac{1}{\sqrt[2m]{\mu_{2m}}} \simeq \frac{1}{\sqrt{n}}\ ,$$

and the conclusion of the lemma is then verified. □

As a consequence of theorem 6.4 and, using a similar idea as the one presented in lemma 6.5, it is possible to establish a criteria that gives a partial conclusion to the first and second components of the answer given to the question stated after problem 6.2.

THEOREM 6.6. *Let $(N_m)_{m\in\mathbb{N}}$ be a sequence of nonnegative numbers. Then, the following propositions are equivalent.*

(1) *For a certain $l \in \mathbb{N}$, there exists $(a_k)_{k\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$, a sequence of nonnegative numbers, such that $\sum_{k=0}^{+\infty} (2\pi)^k\, a_k < +\infty$, which verifies:*

$$(\forall k \in \mathbb{N}) \qquad N_{2k} \leq a_{k+l}\,(k+l)!\ . \tag{122}$$

(2) *If there exists a measure ν such that:*

$$N_{2k} = \int_{\mathbb{R}} x^{2k}\, d\nu(x)\ ,$$

then, there exists an unique tempered measure μ such that:

$$(\forall k \in \mathbb{N}) \quad N_{2k} = \int_{\mathbb{R}} x^{2k}\, e^{-2\pi x^2}\, d\mu(x)\ . \tag{123}$$

PROOF. With the hypotheses made in proposition 1 of the theorem, we have that, for $\epsilon > 0$ given:

$$\exists k_0 \in \mathbb{N} \quad \forall k \in \mathbb{N} \quad k \geq k_0 \ \Rightarrow\ a_k \leq \frac{\epsilon}{(2\pi)^k}\ .$$

As a consequence, we get for $k \geq k_0$:

$$\frac{1}{\sqrt[2k]{N_{2k}}} \geq \frac{1}{\sqrt[2k]{a_{k+l}\,(k+l)!}} \geq \sqrt[2k]{\frac{(2\pi)^{(k+l)}}{\epsilon\,(k+l)!}} \simeq \frac{1}{\sqrt{k}}\ ,$$

and so:

$$\sum_{m=0}^{+\infty} \frac{1}{\sqrt[2k]{N_{2k}}} = +\infty\ .$$

By Carleman's theorem, this relation is now sufficient to ensure that the measure ν, on the second thesis of the theorem, is unique. Considering now the measure μ defined by:

$$d\mu(x) = e^{2\pi x^2} d\nu(x)\ ,$$

it is clear that formula 123 is verified. It remains to be proved that μ is a tempered measure. For that purpose, just observe that:

$$(\forall n \geq l) \qquad \int_{|x|\geq 1} x^{2n-2l}\, e^{-2\pi x^2}\, d\mu(x) = N_{2(n-l)} \leq a_n\, n!\ ,$$

and that:

$$e^{-2\pi} \int_{|x|\leq 1} d\mu\ \leq \int_{|x|\leq 1} e^{-2\pi x^2} d\mu(x) \leq N_0 < +\infty\ .$$

This shows that conditions 118 and 119 are verified, ensuring that μ is, in fact, a tempered measure.

Suppose now that there is a tempered measure μ such that formula 123 is verified. Then, by theorem 6.3, conditions 118 and 119 are verified and so, there exists an $l \in \mathbb{N}$ and a sequence $(a_k)_{k\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$, satisfying the summability condition given by $\sum_{k=0}^{+\infty} (2\pi)^k a_k < +\infty$, such that:

$$\forall k \in \mathbb{N} \quad \int_{|x|\geq 1} x^{2k-2l} e^{-2\pi x^2} d\mu(x) \leq a_k\, k!\ .$$

With $k = m + l$ and $m \geq 0$, this implies that:

$$\forall m \in \mathbb{N} \quad \int_{|x|\geq 1} x^{2m} e^{-2\pi x^2} d\mu(x) \leq a_{m+l}\, (m+l)!\ . \tag{124}$$

For the remaining part of the integral, that is, the integral over the interval $]-1,1[$, we have that, for $m \in \mathbb{N}$:

$$\int_{|x|<1} x^{2m} e^{-2\pi x^2} d\mu(x) \leq (m+l)! \int_{|x|<1} \frac{x^{2m}}{(m+l)!} d\mu(x) = (m+l)!\, b_{m+l}\ , \tag{125}$$

where we have written b_{m+l} for the integral in the middle term of the chain of inequalities 125. As a straightforward application of Lebesgue monotone convergence theorem, we get :

$$\begin{aligned}\sum_{m=0}^{+\infty} (2\pi)^{m+l}\, b_{m+l} &\leq (2\pi)^l \sum_{m=0}^{+\infty} (2\pi)^m \int_{|x|<1} \frac{x^{2m}}{m!} d\mu(x) = \\ &= (2\pi)^l \int_{|x|<1} \sum_{m=0}^{+\infty} \frac{(2\pi x^2)^m}{m!} d\mu(x) \leq \\ &\leq (2\pi)^l\, e^{2\pi}\, \mu(\{|x|<1\}) < +\infty\ .\end{aligned}$$

As a consequence, we have that: $\sum_{m=0}^{+\infty} (2\pi)^m b_m < +\infty$. Finally, inequalities 124 and 125 imply that for $m \in \mathbb{N}$:

$$N_{2m} = \int_{\mathbb{R}} x^{2m} e^{-2\pi x^2} d\mu(x) \leq (a_{m+l} + b_{m+l})\, (m+l)!\ .$$

This inequality together with $\sum_{m=0}^{+\infty} (2\pi)^m (a_m + b_m) < +\infty$, does prove proposition 1 in the theorem. □

In what concerns the third component of our answer to problem 6.2, we are not aware of any result which could give some assurance that the solutions are always positive. In order to better understand the possible spectrum of behaviors of the solutions, we have performed some numerical essays, using the software Mathematica. The program used to perform the calculations is reproduced next.

```
r = 20
ad[k_Integer] := ad[k] = 1/k^7; Table[ad[k], {k, 1, r}]
g1[n_Integer] := g1[n] = (1/2) * (1 − (−1)^(n + 1)) * 2^(1/2) *
Apply[Times, Table[(2 * k + 1)^2, {k, 0, Floor[n/2 − 1]}]]
a[m_Integer, k_Integer] := a[m, k] = If[k <= m, (1/((k!) * (m − k)!)) *
g1[m − k], 0]
Mtp[l_Integer] := Mtp[l] = Table[a[m, k], {m, 0, l}, {k, 0, l}]
```

$Mtp[r-1]$

$tbp[p\_Integer] := tbp[p] = Simplify[LinearSolve[Mtp[p],$

$Table[ad[k], \{k, 1, p+1\}]]]$

$tbp[r-1]$

$N[tbp[r-1]]$

$F[0, t\_] := Exp[(-2 * Pi * x^2) * ((1-t)/(1+t))]$

$F[n\_Integer, t\_] := F[n, t] = Simplify[D[F[n-1, t], t]]$

$zer[k\_Integer] := Table[0, \{i, 1, k\}]$

$For[i = 0, i < r, i++, F[i, t]; PT1[t\_] = Simplify[F[i, t]];$

$eq[i] = PT1[0] * Exp[2 * Pi * x^2];$

$Lit[i] = CoefficientList[Expand[eq[i]], x^2];$

$Print[Lit[i]]]$

$tad = Table[Join[Lit[k], zer[r-k-1]], \{k, 0, r-1\}]$

$pm[1] = tbp[r-1][[1]]/tad[[1, 1]]$

$pm[n\_Integer] := pm[n] = (1/tad[[n, n]]) * (tbp[r-1][[n]] -$

$Apply[Plus, Table[tad[[n, k]] * pm[k], \{k, 1, n-1\}]])$

$Taf = Table[Simplify[pm[n]], \{n, 1, r\}]$

$N[Taf]v$

$Table[N[((2 * Pi)^k) * pm[k]/k!], \{k, 1, r\}]$ Using this program, we have calculated, for each of the following sequences defined for $k \in \mathbb{N}$ by:

$$M[k] = 1 \quad M[k] = 1 + |\cos(k)| \quad M[k] = k$$
$$M[k] = \log(k)/k \qquad M[k] = 1/\sqrt{k} \qquad M[k] = k^2 \ ,$$

the solutions $pm[k]$, for the linear systems 115 and 116 for $N = 20$. Next, we considered the corresponding sequences given, for $k \in \{1, \ldots, 20\}$, by:

$$R[k] = \frac{pm[k]\ (2\pi)^k}{k!} \ .$$

The sequences $(\frac{R[k+1]}{R[k]})_{k \in \{1,\ldots,19\}}$ define, in a natural way, piecewise linear functions. The corresponding plots are shown in figure 4. For a sequence such as $M[k] = 1/k^2$, the solutions of the linear systems contained some negative terms and, as a consequence, can not be taken to represent even moments of a positive measure. A possible comment on the results just presented is that the necessary and sufficient condition given by theorem 6.6 is probably not verified for $M[k] = k^2$.

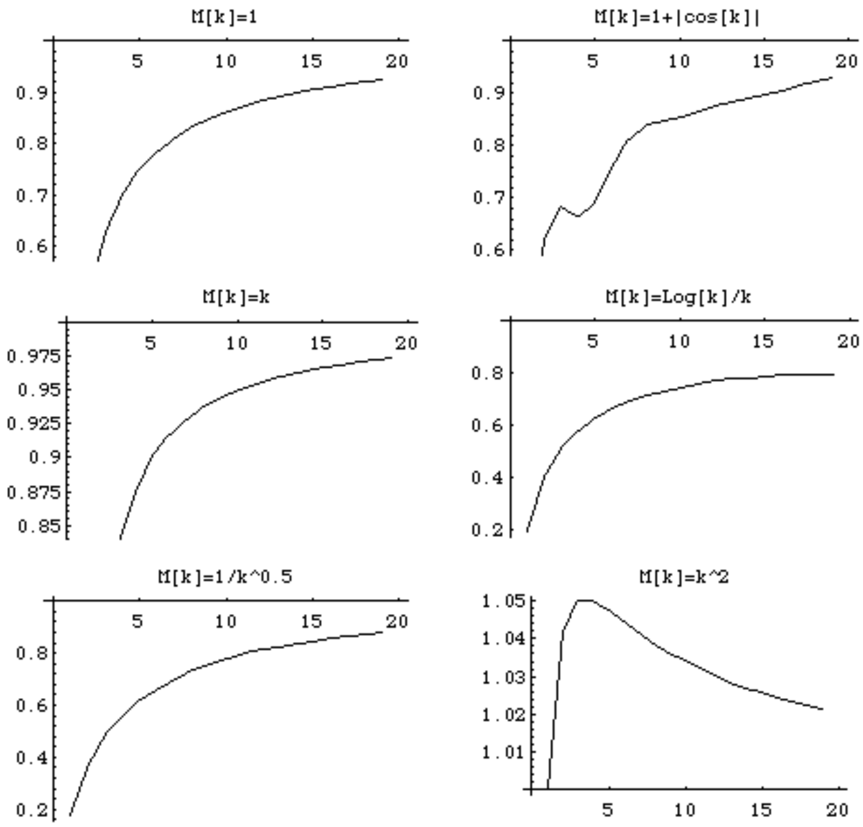

Figure 4.

So, for this sequence and for the sequence $M[k] = 1/k^2$, it is probably not possible to have corresponding tempered measures such that these sequences are the sequences of variances of the gaussian variables defined when the brownian distributions are applied to the Hermite functions. For the other sequences considered, theorem 6.6 is probably verified and so, the corresponding measures do probably exist.

Bibliography

[1] R. J. Adler, *The Geometry of Random Fields*, John Wiley and Sons, 1981.

[2] R. J. Adler, *An Introduction to Continuity, Extrema and Related Topics for General Gaussian Processes*, Institute of Mathematical Statistics, Lecture Notes Monograph Series vol. 12, 1990.

[3] A. Benassi, S. Jaffard, D. Roux, *Elliptic gaussian random fields*, to appear in Revista Mathematica Iberoamericana.

[4] N. Bourbaki, *Fonctions d'une Variable Réelle*, Diffusion C.C.L.S., Paris 1976.

[5] P. L. Butzer, U. Westphal, *An access to fractional differentiation via fractional difference quotients*, Springer Verlag New York, L.N.M. **457** (1975) 116–145.

[6] R. Cairoli, J. Walsh, *Stochastic integrals in the plane*, Acta Math. **134** (1975) 111–183.

[7] L. Carleson, *Selected Problems on Exceptional Sets*, D. Van Nostrand Company Inc., Princeton, New Jersey 1967.

[8] M. Csörgo, P. Révész, *How Big are the increments of a multi-parameter Wiener process ?*, Z. Wahrscheinlichkeitstheorie verw Gebiete **42** (1978) 1–12 .

[9] J. Diestel, J. J. Uhl. Jr., *Vector Measures*, American Mathematical Society, Providence 1977.

[10] R. M. Dudley, *Sub-stationary processes*, Pacific Jour. Math **20**, 2 (1967) 207–215.

[11] R. E. Edwards - *Fourier Series*, 2nd ed., Springer Verlag, New York 1982.

[12] M. L. Esquível, *Sur la méthode des séries de Fourier dans les équations differentielles à coefficients constants*, Trabalhos de Investigação **2**, Departamento de Matemática, F.C.T.-U.N.L., Lisboa 1987.

[13] M. L. Esquível, *Sobre o Método das Séries de Fourier nas Equações Diferenciais Ordinárias Lineares com Coeficientes Constantes*, P.A.P.C.C, Segunda Tiragem, Universidade Nova de Lisboa, Faculdade de Ciências e Tecnologia, Departamento de Matemática, Lisboa 1988.

[14] M. L. Esquível, *Sur une classe de distributions aléatoires périodiques*, Ann. Sc. Math. Québec **17**, (1993) 169–186.

[15] M. L. Esquível, *A characterization of the class of random periodic Schwartz distributions having a mean*, submitted for publication.

[16] M. L. Esquível, *On the local behavior of the brownian sheet*, Stochastic Analysis: Random Fields

and Measure-Valued Processes, K. Hochberg (ed.) et al.,. AMS, Providence Rhode Island, IMCP **10** (1996).

[17] M. L. Esquível, *Points of rapid oscillation for the Brownian sheet via Fourier-Schauder series decomposition*, N. Kalton (ed.) et al., Interaction between functional analysis, harmonic analysis and probability, Columbia MO, Marcel Dekker, Lecture Notes Pure and Appl. Math. 175, 153–162 (1996).

[18] M. L. Esquível, *On the space of random tempered distributions having a mean*, to appear in the proceedings of the 8th International Symposium on Classical Analysis, Kazimierz Dolny, Poland, in 1995.

[19] M. L. Esquível, *The asymptotic behavior of the second moment of the Fourier transform of a random measure*, submitted for publication.

[20] X. Fernique, *Processus linéaires, processus généralisés*, Ann. Inst. Fourier, Grenoble **17**, 1 (1967). 1–92.

[21] X. Fernique, *Fonctions aléatoires à valeurs dans certains espaces nucléaires,* Probab. Th. Rel. Fields **83**, (1989) 87–99.

[22] J. V. Gonçalves, *Lições de Cálculo e Geometria*, vol. 1 Coimbra 1930.

[23] I.M. Gel'fand, N Ya. Vilenkin, *Generalized Functions*, vol. 4 Academic Press, New York 1964.

[24] G. H. Hardy, *A Course in Pure Mathematics*, tenth edition, Cambridge Mathematical Library, Cambridge University Press Cambridge1952.

[25] T. Hida, *Stationary Stochastic Processes*, Princeton University Press, New Jersey 1970.

[26] T. Hida, *Brownian Motion*, Springer Verlag, New York 1980.

[27] L. Hörmander, *Linear Partial Differential Operators*, Springer Verlag, New York 1963.

[28] K. Ito, *Foundations of Stochastic Differential Equations in Infinite Dimensional Spaces*, SIAM, Philadelphia 1984.

[29] K. Ito (edt.), *Encyclopedic Dictionary of Mathematics*, second edition, The MIT Press, Cambridge Massachusetts, 1987.

[30] J.-P. Kahane, *Séries de Fourier Aléatoires*, Séminaire de Mathématiques Supérieures 4, Les Presses de L'Université de Montréal, 1963 (réédition 1966).

[31] J.-P. Kahane, *Some Random Series of Functions*, second edition, Cambridge University Press, London 1985.

[32] J.-P. Kahane, *Sur le chaos multiplicatif*, Ann. Sc. Math. Québec **9**, (1985) 105–150.

[33] J.-P. Kahane, *Slow points of gaussian processes*, The Wadsworth Math. Series, Conference on Harmonic Analysis in honor of Antoni Zygmund, vol. 1 (1983) 67-83.

[34] J.-P. Kahane, *Positive martingales et random measures*, Chin. Ann. of Math. 8 **1**, (1987), 1–12.

[35] J.-P. Kahane, *Random multiplications random coverings and multiplicative chaos*, Proceedings of the special year in modern analysis. Ed. E. Berkson, N. Tenney Pech, J. Jerry Uhl London Math. Soc. Lee. notes series 137. Cambridge University Press 1989, 196–225.

[36] J.-P. Kahane *Fractals and random measures*, Bull. Sc. math. 2ième série, **117** (1993) 153–159.

[37] Y. Katznelson, *An Introduction to Harmonic Analysis*, 2ond edition, Dover Publications, Inc. New York 1976.

[38] Yu. A. Kaz'min, *Encyclopaedia of Mathematics*, vol. 6, Kluwer Academic Publishers, Dordrecht 1990.

[39] V. P. Khavin, N. K. Nikol'skiĭ, *Commutative Harmonic Analysis IV*, Springer Verlag, Berlin Heidelberg 1992.

[40] M. T. Lacey *A remark on the multiparameter law of the iterated logarithm.*, Stochastic Processes and their Applications **32** (1989) 355–367.

[41] N. N. Lebedev, *Special Functions and their Applications*, Dover Publications, Inc. New York, revised english edition, 1972.

[42] P. Malliavin, *Intégration et Probabilités. Analyse de Fourier et Analyse Spectrale*, Masson, Paris 1982.

[43] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press Cambridge 1995.

[44] B. B. Mandelbrot, *The Fractal Geometry of Nature*, W. H. Freeman and Cpn., S. Francisco 1983.

[45] B. B. Mandelbrot and J. W. Van Ness, *Fractional brownian motions, fractional noises and applications*, SIAM Review **10**, 422–1968.

[46] J. Neveu, *Processus Aléatoires Gaussians*, Cours d'Été de l'OTAN, Les Presses de l'Université de Montreal, 1968.

[47] J. Neveu, *Bases Mathématiques du Calcul des Probabilités*, deuxième édition, Masson, Paris 1970.

[48] P. E. De Oliveira, *Infinie divisibilité, Principes d'Invariance et Estimation de Noyaux de Transition en Théorie des Mesures Aléatoires*, Thèse Docteur de l'Université de Lille, 1991 (Mathématiques Appliquées).

[49] S. Orey, W. E. Pruitt *Sample functions of the N-parameter Wiener process*, The Ann. of Prob.**1**, 1 (1973) 138–163.

[50] S. Orey, S. J. Taylor *How often on a brownian path does the law of iterated logarithm fail ?*, Proc. London Math. Soc. **3** 28 (1974) 174–192.

[51] H. O. Peitgen and D. Saupe, *The Science of Fractal Images*, Springer Verlag, New York 1988.

[52] A. V. Prokhorov, *Encyclopaedia of Mathematics*, vol. 6, Kluwer Academic Publishers, Dordrecht 1990.

[53] W. Rudin, *Functional Analysis*, Second Edition, McGraw-Hill Books, New York 1991.

[54] W. Rudin, *Real and Complex Analysis*, Third Edition, McGraw-Hill Books, New York 1986.

[55] L. Schwartz, *Théorie des Distributions*, Hermann, Paris 1966.

[56] G. Sinnamon, G. Zimmerman, personal communication.

[57] C. Sogge, *Fourier Integrals in Classical Analysis*, Cambridge University Press Cambridge 1995.

[58] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton New Jersey 1970.

[59] E. Stein, G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton New Jersey 1971.

[60] E. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscilllatory Integrals*, Princeton University Press, Princeton New Jersey 1993.

[61] K. R. Stromberg, *Probability for Analysts*, Chapman & Hall, New York, London 1994.

[62] F. G. Teixeira, *Curso de Análise Infinitésimal*, Quarta edição, Coimbra 1906.

[63] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Third edition, Chelsea Publishing Company, New York 1986.

[64] F. Treves, *Topological Vector Spaces Distributions and Kernels*, Academic Press, New York 1967.

[65] Vo-Khac Khoan, *Distributions, Analyse de Fourier, Opérateurs aux Derivées Partielles*, tomes I, II, Librairie Vuibert, Paris 1972.

[66] G.J. Zimmerman *Some sample function properties of the two parameter gaussian process*, The Ann. of Math. Stat. **43**, 4 (1972) 1235–1246.

[67] A. Zygmund, *Trigonometric Series*, Second Edition, Cambridge University Press 1959.

Table of Notations

$\mathbb{R}$, the real numbers

$\mathbb{C}$, the complex numbers

$\mathbb{Z}$, the integers

$\mathbb{N}$, the nonnegative integers

$\mathbb{N}^*$, the strictly nonnegative integers

$\mathbb{R}_+$, the nonnegative real numbers

$\mathbb{E}[A]$, the expectation of A

$\mathcal{M}$, the space of random variables defined in Ω, 2

$\mathcal{M}^{\mathbb{Z}}$, the $\mathbb{Z}$ indexed sequences of elements of $\mathcal{M}$

T,U,V, random periodic, or tempered, distributions

$\Pi = \mathbb{R}/\mathbb{Z}$, the torus

$\mathcal{P}(A)$, the set of all subsets of A

$||x||_B$, norm of the vector x in the space B

$C^\infty(\Omega)$ the indefinitely differentiable functions in Ω

$\mathcal{S}$, the space of the Schwartz test functions

$\mathbb{I}_A$, the indicator or characteristic function of the set A

$(\Omega, \mathcal{A}, \mathbb{P})$, a complete probability space, 1

$\mathcal{D}^{'}(\mathbb{R}/\mathbb{Z})$, Schwartz periodic distributions, 1

$L^1(\Omega)$, the Lebesgue space of integrable functions in $(\Omega, \mathcal{A}, \mathbb{P})$, 1

$\mathcal{C}_m$, a class of sequences of random variables, 3

$C_n^T(\omega)$, Fourier-Schwartz coefficients of the random distribution T, 4

δ_a, Dirac measure having support in $\{a\}$, 4

$L^1([0,1], \mathcal{B}, dx)$, the space of integrable functions with respect to Lebesgue measure, defined in $[0,1]$, 5

$X^\gamma = (X_t^\gamma)_{t\in[0,1]}$, Wiener and fractional brownian processes, 6

τ_a, a translation operator, 7

$\hat{f}(n)$, Fourier coefficient of f at $n \in \mathbb{Z}$, 7

$T^{'}$, the derivative of T, 8, 71

$\overline{T}$, the mean of T, 9, 72

$s(\mathbb{N})$, the space of rapidly decreasing complex $\mathbb{N}$ sequences, 10, 74

$P(D) = \sum_{k=0}^r a_k D^k$, a differential operator, 13

$\mathcal{N}(a, b)$ the set of gaussian random variables with mean a and variance b, 16

e_0, e_{in}, the Haar system in $L^2([0,1])$, 22

$(e_0^0,\, e_0^{jm},\, e_{in}^0,\, e_{in}^{jm})_{injm}$, the Haar system in $L^2([0,1]^2)$ 23

$\triangle_{in}$, a Schauder function 23

$D(s,t)$, the brownian sheet process, 24, 27

$\mathcal{B}'$, the Lebesgue sets with finite Lebesgue measure, 24

$(W(A))_{A\in\mathcal{B}'}$, gaussian white noise, 24

$W(A)\ i\ W(B)$, $W(A)$ and $W(B)$ are independent random variables, 24

$\mathit{\Gamma}$, a covariance function, 25

$\mathfrak{H}$, a gaussian space, 25, 77

$< u, v >_{\mathfrak{H}}$, inner product in $\mathfrak{H}$ 26

$\widetilde{W}$, Hilbert space isometry associated to the gaussian white noise W, 26, 77

$\Delta D((s,t)$, the independent increments' process associated with the brownian sheet, 28

$\omega_D(h,k)$, the modulus of continuity of D, 30

$G(s,t)$, the location homogeneous part of the brownian sheet, 31

$B_1 \otimes B_2(s,t)$, a brownian tensorial product process, 46

supp(f), the support of f, 50

$\widehat{\psi}$ or $\mathcal{F}\psi$, the Fourier transform of ψ, 52

J_ν, a Bessel function, 53

$B(0,\delta)$, a ball in $\mathbb{R}^n$, 54

U^α, a remarkable locally integrable function, 57

$I_\beta(f)$, the Riesz potential of f, 58

$\bigcirc(1)$, big O of one, (Hardy's notations), 60

$f^\vee(x) = f(-x)$, 61, 80

$\mathcal{S}'(\mathbb{R})$, the tempered distributions on $\mathbb{R}$, 65

$< T, \varphi >$, duality bracket for distributions, 70

H_m, a Hermite polynomial, 66

$\mathcal{H}_m$, a Hermite function, 66

$a_m(\varphi)$, a Fourier Hermite coefficient of φ, 67

τ_+ and τ_-, natural operators for the Hermite functions , 67

$a_m(T)$, a Fourier Hermite coefficient of a tempered distribution, 68

T', the derivative of a random Schwartz distribution, 71

$\bar{T}$, the mean of a random Schwartz distribution , 72

$\sigma(t)$, the sign function, 74

$\mathbf{B} = (B_t)_{t\in\mathbb{R}}$, a brownian type process, 78, 80

$\tilde{\mathbf{B}} = (\tilde{B}_t)_{t\in[0,+\infty[}$, the Wiener process, 79, 82

l_μ, an index of the growth, at infinity, of a tempered measure μ, 82

Z_μ, a μ brownian distribution, 85

Z^*, a random Schwartz distribution, 86

Index

Amenable (or having a mean), 9, 66
Brownian distributions, 85
Brownian process, 6
Brownian type process, 78
Brownian sheet, 24, 27
Fractional brownian process, 6
Frostman formula, 57
Gaussian white noise,
  on $\mathbb{R}_+^2$, 24
  on $\mathbb{R}$, 77
Haar system, 22
Hamburger moment problem, 88
Hermite,
  functions, 66
  polynomials, 66
  coefficients, 68
Independent increments process, 28
Law of iterated logarithm,
  for the brownian sheet, 31
  for the independent increments, 37
Location homogeneous part, 31
Normal sequence, 16
Parseval formula, 9
Pseudogaussian random variable, 41
Rademacher functions, 10
Rademacher standard sequence, 4
Random measure, 49, 60
Random periodic distribution, 1
  derivative of, 8
  mean of, 9
Random tempered distribution, 65
  derivative of, 71
  mean of, 72
Riesz potential,
  of a test function, 58
  of a measure, 58
Schauder function, 23
Schwartz space of test functions, 65
Second moment (of a random measure), 60
Slow growth at infinity, 1, 3
Tensorial product process, 46
Tempered distributions, 65
Tempered measure, 82
Wiener process, 6