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CIRCULAR AND STAR-SHAPED DOMAINS

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Ao Vitor e à Bruna

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Abstract

This thesis concerns the factorization of elliptic operators, namely the decomposition of a second order boundary value problem, defined in an open bounded regular domain, in an uncoupled system of two first order initial value problems. The method presented here is inspired on the theory of Optimal Control. It is a return, in a new spatial approach, to the technique of the invariant temporal embedding, defined originally in the context of Dynamic Programming, used in Control Theory for the computation of the optimal feedback. This technique consists in embedding the initial problem in a family of similar problems depending on a parameter, which are solved recursively. In our case, each problem is defined over a sub-domain limited by a mobile boundary depending on the parameter. We introduce an operator relating the trace of the function defined for each problem, and the trace of its normal derivative over the mobile boundary.

Without loss of generality, we particularize the study to a Poisson's equation with, for example, a Dirichlet's boundary condition. We first consider a circular domain and we present for it two approaches: first, we apply an invariant embedding that starts on the boundary of the circle and go towards its center, followed by an invariant embedding in the opposite direction. Next, we generalize the method, applying it to the case of an arbitrary star shaped domain. In all cases, the family of curves which limits the subdomains defined by the invariant embedding are homothetic to one another and homothetic to a point. This fact induces the appearing of a singularity.

Resumo

O objectivo deste trabalho é a factorização de operadores elípticos, nomeadamente a decomposição de um problema de segunda ordem com valores na fronteira, definido num domínio aberto regular e limitado, num sistema desacoplado de dois problemas de valor inicial de primeira ordem. O método utilizado é inspirado na Teoria do Controlo Óptimo. Trata-se de um retorno, numa nova abordagem espacial, à técnica da "imersão invariante" na variável tempo, que se definiu originalmente no contexto da programação dinâmica, e que é usada na Teoria do Controlo para calcular o "feedback" óptimo. Esta técnica consiste em imergir o problema inicial numa família de problemas similares dependentes de um parâmetro, que são resolvidos recursivamente. No nosso caso, cada problema está definido num subdomínio limitado por uma fronteira móvel dependente desse parâmetro. Introduzimos um operador que relaciona o traço da função definida para cada problema, com o traço da sua derivada normal sobre a fronteira móvel.

Sem perda de generalidade, particularizamos este estudo à equação de Poisson com, por exemplo, uma condição de fronteira do tipo Dirichlet. Consideramos inicialmente um domínio circular e apresentamos para este domínio duas abordagens: primeiro, aplicamos uma imersão invariante que se inicia na fronteira do círculo e que converge para o seu centro e, de seguida, usamos uma imersão invariante em sentido oposto. Posteriormente, generalizamos o método aplicando-o ao caso de um domínio estrelado arbitrário. Em todos os casos estudados, as curvas que limitam os sucessivos domínios definidos pela imersão invariante são homotéticas entre si e homotéticas a um ponto, o que induz o aparecimento de uma singularidade.

Contents

Introduction

We are going to use the technique of invariant embedding ([3]), in order to factorize a second order elliptic boundary value problem in a system of uncoupled first order initial value problems. This technique ([2]) has been used to derive analytic and numerical results in a number of different fields as atmospheric physics, transport theory and wave propagation, to mention a few, and consists in embedding the initial problem in a family of similar problems depending on a parameter, which are solved recursively. Particularly, it is used ([23, 5]) in the decoupling of systems arising from Optimal Control problems associated to evolution equations of parabolic and hyperbolic type. In these cases the parameter used is the time variable. In our case, we follow the same steps using a spatial embedding, that is, we embed our initial problem in a family of similar problems each one defined over a sub-domain limited by a mobile boundary depending on the parameter. From a Control Theory point of view, we consider the equation of the problem as the optimality system of a control problem, where we substitute the time variable with one of the space variables and the embedding allow us to decouple the optimality system in the same way as to obtain the optimal feedback. Therefore, the factorization method that we use in this thesis has the following key points: first, we fractionate the initial domain (for clarification of the procedure, we may suppose that it is a rectangular domain) by the introduction of a mobile boundary over which we impose a Dirichelet or a Neumann boundary condition (each type of conditions will lead to a different factorization); next, we define an operator relating the value of the solution, or its derivative, with the mobile boundary condition and, finally, we displace this boundary from one extremity to the other of the domain. A similar approach was developed in ([19]) for the case of an elliptic operator in a cylindrical domain. We point out that, in this particular study, the geometry of the moving boundary is always the same. It was also shown in ([19]) that the obtained factorization could be viewed as an extension to the infinite dimensional problem of the block Gauss LU factorization.

In the course of this work, we want to generalize the method to more general geometries and, in particular, to the case where the family of surfaces which limits the sub-domains has no longer invariant geometry but are homothetic to one another. We study the case where the moving boundary starts on the outside boundary of the domain and shrinks to a point or vice versa. This means that we must deal with the singularity that will necessarily appear at that point.

The first chapter of this thesis makes a panoramic view over the method and contains the concepts and results that we need for the succeeding chapters. In the first section, we present the state of the art and in section 2 and 3 we describe the factorization method by invariant embedding. Afterwards, in section 4 we introduce the general problem in study and an auxiliary problem, needed to deal with the singularities originated by the method. In the last section, the convergence of the auxiliary problem to the initial one is achieved, which is a key result throughout this work.

Our main goal in Chapter 2 and 3 is to factorize the Laplace operator in a circular domain - in chapter 2 the factorization starts in the boundary of the domain and shrinks to the center of the circle and in chapter 3 it starts in that center and spreads to the circumference. In both cases we consider the moving boundary to be a family of concentric circles which radii or decrease to zero or increase from zero. We present results that, in the first case, deal with the singularities appearing on the origin and, in the second case, handle the definition of the initial condition for the decomposition. The material of this two chapters can be found in ([16, 17]). In the last chapter, we are going to generalize the previous results to a star shaped domain. Again, the subdomains defined by the invariant embedding are homothetic to one another. The final step of this path will be, naturally, the generalization to the case of an arbitrary open regular bounded domain. However, it still remains, for the time being, an open problem.

Chapter 1

Preliminaries

The aim of this chapter is to make a global presentation to the method of invariant embedding as well as to the problem in study. In the first section we present the state of the art, which also includes the present situation regarding other studies in course. A short description of the invariant embedding method is given in second section, and the third section is entirely dedicated to the presentation of the foundations of the technique of factorization by invariant embedding, for a parabolic operator, following J.L.Lions ([23]). In section 4, we define our problem which, due to the singularity originated by the method, will imply the definition of an auxiliary problem. According to a density result we prove, in section 5, the convergence of the auxiliary problem to the initial one, which ends this chapter and is a fundamental result throughout this work.

1.1. State of the art

The technique of invariant embedding was first proposed by Bellman ([3]), in the context of optimal control theory, and was formally used by Angel and Bellman ([2]) in the resolution of a Laplace's problem defined over a rectangle. We can succinctly explain this technique on saying that, considering the equation $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ on the rectangle $0 \le x \le a$, $0 \le y \le b$ (u being specified on all sides of the rectangle), Angel and Bellman seek for solutions of the form $u(x,y)=\displaystyle\int_0^b r(x,y,z)\frac{\partial u}{\partial x}(x,z)\,\mathrm{d}z+s(x,y)$. Then, differentiating this equality two times with respect to y and using the obtained expression into its derivative with respect to x, they find

$\int_0^b \delta(y-z)\frac{\partial u}{\partial x}(x,z)\,\mathrm{d}z = \int_0^b \frac{\partial r}{\partial x}(x,y,z)\frac{\partial u}{\partial x}(x,z)\,\mathrm{d}z - \int_0^b\int_0^b r(x,y,w)\frac{\partial^2 r}{\partial w^2}(x,w,z)\,\mathrm{d}w\,\mathrm{d}z - \int_0^b r(x,y,z)\frac{\partial^2 s}{\partial z^2}(x,z)\,\mathrm{d}z + \frac{\partial s}{\partial x}(x,y)$. From this equality they derive $\frac{\partial r}{\partial x}(x,y,z) = \delta(y-z) + \int_0^b r(x,y,w)\frac{\partial^2 r}{\partial w^2}(x,w,z)\,\mathrm{d}w$ and $\frac{\partial s}{\partial x}(x,y) = \int_0^b r(x,y,z)\frac{\partial^2 s}{\partial z^2}(x,z)\,\mathrm{d}z$, equating coefficients of $\frac{\partial u}{\partial x}$. Requiring the same form of solutions to hold at the boundaries, they must also have $r(x,y,z) = 0$ and $s(x,y) = u(x,y)$ at $x = a$, $y = 0$ and $y = b$, which gives initial and auxiliary equations to determine r and s. Knowing $r(0,y,z)$ and $s(0,y)$, since $u(0,y)$ is given, again from the form of the solutions, they can determine the missing initial condition $\frac{\partial u}{\partial x}(0,y)$. Finally, they find an initial value problem for $\frac{\partial u}{\partial x}$, which uses the stored values of r and s, and permits, back to the equality $u(x,y) = \int_0^b r(x,y,z)\frac{\partial u}{\partial x}(x,z)\,\mathrm{d}z + s(x,y)$, to determine the desired values of u.

As described, in brief, in the introduction, J.L. Lions ([23]) gave a justification for this invariant embedding for the computation of the optimal feedback in the framework of Optimal Control of evolution equations of parabolic type. The method gives rise to a Riccati equation, that is, a differential equation with quadratic terms, which is justified through the Galarkin method. It is also similarly used in Bensoussan ([5]). We notice that, in the kernel notation of Angel-Bellman, the Riccati equation appears in the term $\frac{\partial r}{\partial x}(x,y,z) = \delta(y-z) + \int_0^b r(x,y,w)\frac{\partial^2 r}{\partial w^2}(x,w,z)\,\mathrm{d}w$.

On the direct study of Riccati equations in infinite dimension, we can also recommend an extensive bibliography. We stand out Temam ([31]), where it can be found the Hilbert-Schmidt solutions of the equations; Tartar ([30]), that uses the method of fixed point; Bensoussan-Da Prato-Delfour-Mitter ([6]) and Lasiecka-Triggiani ([22]), on the study through Semigroup Theory. In all these quotations the operator appearing in the Riccati equation is continuous from a certain space into itself.

In a chapter of A. M. Ramos PhD Thesis ([26]) it was presented the resolution of a second order elliptic problems in an open cylindrical domain. Afterwards, the method was developed by Henry and Ramos ([19]) which presented a complete justification for the invariant embedding of a Poisson's problem in a cylindrical domain, adapting the method of Lions. Here, they have no longer that property on the continuity of the operator. Traditionally, the chosen parameter of the invariant embedding was the variable time but, in this new line of work, they use a spatial invariant embedding, that is, they embedded the initial problem in a family of similar problems, each one defined over a subcylinder bounded by a variable section. In this case, the embedding is naturally done in the direction of the axis of the cylinder and allows the factorization of the second order operator in a product of first order operators with respect to this coordinate. They obtained a factorization in two uncoupled problems of parabolic type, in opposite directions, that requires the computation of an operator, which is solution of a Riccati equation. They showed, as well, that the same method applied to the discretized problem (e.g. through finite differences) can be interpreted as a Gauss block factorization of the matrix of the problem. This means that the method can be seen as a generalization, up to infinite dimension, of the LU block factorization of matrices: solving the Riccati equation is analogous to computing the L and U factors for a block tridiagonal matrix and solving the two parabolic problems is related to solving the lower and upper triangular systems. A different approach to the method was also made by the same authors in ([20]), where was directly studied the solution of the Riccati equation which appears in the factorization process, using an Hilbert-Schmidt framework, in the same line of ([31]).

The invariant embedding method was also applied by Henry-Yvon ([18]) to the case of a control problem in order to determine explicitly the solution, and also by Henry ([15]) on the resolution of certain inverse problems. More recently, the application to a problem of wave propagation was made by I. Champagne in her PhD thesis ([8]).

1.2. A brief sketch of the method

As far as we are concerned, the main feature of the invariant embedding method is the transformation of a second order elliptic boundary value problem in a decoupled system of first order initial value problems which can be solved recursively. According to this method, and in the particular case of a rectangular domain as considered in ([2]), we first introduce a mobile boundary corresponding to a transversal section of the rectangle, in which we choose an arbitrary condition. A priori, this condition is of the same type of the initial boundary condition. We solve the problem in the subdomain defined between one of the sides of the rectangle and the mobile boundary. Next, we extend the process along the propagation axis, until we find the whole domain. This allows us to define an operator connecting the solution of the equation with the arbitrary boundary condition. This way, we define a family of operators on functions of the section satisfying a Riccati equation and relating the boundary conditions on the section (Dirichlet-Neumann or Neumann-Dirichlet, for example). In the resultant decoupled system and besides this operator, the two variables involved are the solution of the problem and the affine part appearing in the relation between the solution and the operator. The solution is now achieved by a two steps process: first, we solve the Riccati equation and the differential equation of the affine part and this computation is done in the same direction as the displacement of the boundary; then, we look for the solution of the system following the path in the opposite direction.

For a given problem, the invariant embedding method is not unique. On the one hand, we can apply the method either to the family of subdomains described above, either to the family of complementary subdomains and, in this thesis, we will do both approaches, for the same domain, respectively on chapter 2 and 3. In this last case, the boundary will move in the opposite direction and the method will give rise to another operator. On the other hand, it is possible to change the type of condition that we impose over the mobile boundary.

1.3. Global methodology

In this section we present, following Lions ([23]), the general scheme of proof for the factorization by invariant embedding of the optimality system for the control problem of a parabolic operator. We assume the following framework: V and H are Hilbert spaces where V' is the dual of V, V is dense in H, H' is identified with H and such that $V \subset H \subset V'$; the variable t denotes time and we suppose $t \in]0,T[, T < \infty$; $a(t;y,p)$, for each $t \in]0,T[$, is a continuous and coercive bilinear form on V, and can be written in the form $a(t;y,p) = (A(t)y,p)$, $A(t)y \in V'$; in addition, $A(.) \in \mathcal{L}(L^2(0,T;V);L^2(0,T;V'))$, where $L^2(0,T;V(\text{resp. } V'))$ stands for the set of functions $t \to f(t)$ of $]0,T[\to V(\text{resp. } V')$, measurable and such that $\left(\int_0^T \|f(t)\|^2_{V(\text{ resp.} V')} \,\mathrm{d}t\right)^{\frac{1}{2}} < \infty$. Further, we consider $\mathcal{U} = L^2(0,T;E)$ (space of controls) and $\mathcal{H} = L^2(0,T;F)$ (space of observations), where E and F are separable Hilbert spaces. We are given an operator $B \in \mathcal{L}(\mathcal{U};L^2(0,T;V'))$ and f and y_0, with $f \in L^2(0,T;V)$ and $y_0 \in H$.

Within the above notations and denoting A^* the adjoint of A, we consider the system of equations

$$\frac{\partial y}{\partial t} + A(t)y + D_1(t)p = f,\ y(0) = y_0;\quad -\frac{\partial p}{\partial t} + A^*(t)p - D_2(t)y = g,\ p(T) = 0 \qquad (1.1)$$

for all $t \in]0, T[$, which is the optimality system for

$$\frac{\partial y(v)}{\partial t} + A(t)y(v) = f + Bv$$
$$y(v)_{|t=0} = y_0$$
$$y(v) \in L^2(0, T; V),$$

where the cost function is given by

$$J(v) = \|Cy(v) - z_d\|^2_{\mathcal{H}} + (Nv, v)_{\mathcal{U}}.$$

N is given such that $N \in \mathcal{L}(\mathcal{U}; \mathcal{U})$ and $(Nu, u)_{\mathcal{U}} \geq \mu \|u\|^2_{\mathcal{U}}$, $\mu > 0$, $C \in \mathcal{L}(L^2(0, T; V); \mathcal{H})$, and z_d is a given element in $\mathcal{H}$. Also, $D_1, D_2 \in \mathcal{L}(V; V')$, with $D_1 = B(t)N(t)^{-1}{\Lambda_E}^{-1}B(t)^*$, $D_2(t) = C(t)^*\Lambda_F C(t)$ (Λ_E (resp, Λ_F) being the canonical isomorphism of E (resp, F)) and $g(t) = -C^*(t)\Lambda_F z_d(t)$.

Then, we embed (1.1) in a family of similar problems depending on the present time s, which defines the "moving boundary", and the state h at that time. The resulting system of equations

$$\frac{d\varphi}{dt} + A(t)\varphi + D_1(t)\psi = f,\ \varphi(s) = h;\ \ -\frac{d\psi}{dt} + A^*(t)\psi - D_2(t)\varphi = g,\ \psi(T) = 0 \qquad (1.2)$$

where $t \in]s, T[$, $0 < s < T$, and h is given in H, has a unique solution. For φ and ψ this way defined, it can be proved that the mapping $h \to \psi(t)$ is a continuous affine mapping of $H \to H$ and consequently this mapping can be written in a unique way as $\psi(s) = P(s)h + r(s)$, where $P(s) \in \mathcal{L}(H; H)$ and $r(s) \in H$.

Follows the fundamental result. Considering $\{y, p\}$ to be a solution of (1.1), we have $p(t) = P(t)y(t) + r(t)$, $\forall t \in]0, T[$, where $P(t)$ and $r(t)$ are given, respectively, by $P(s)h = \gamma(s)$, where γ is the solution, in $]s, T[$, of

$$\frac{d\beta}{dt} + A(t)\beta + D_1(t)\gamma = 0,\ \beta(s) = h;\ \ -\frac{d\gamma}{dt} + A^*(t)\gamma - D_2(t)\beta = 0,\ \gamma(T) = 0$$

and $r(s) = \xi(s)$, where ξ is the solution, in $]s, T[$, of

$$\frac{d\eta}{dt} + A(t)\eta + D_1(t)\xi = f,\ \eta(s) = 0;\ \ -\frac{d\xi}{dt} + A^*(t)\xi - D_2(t)\eta = g,\ \xi(T) = 0.$$

Moreover, taking $f \in L^2(0, T; H)$, then P and r have the following properties: $P(t) \in \mathcal{L}(H; H)$; $P(t) = P^*(t)$; if $\eta \in W(0, T) = \{f : f \in L^2(0, T; V), \frac{df}{dt} \in L^2(0, T; V')\}$ with $\frac{d\eta}{dt} + A(t)\eta \in L^2(0, T; H)$, then $P(t)\eta \in W(0, T)$; P satisfies the Riccati equation $-\frac{dP}{dt} + PA + A^*P + PD_1P = D_2$, in $]0, T[$, in the sense that $-\left(\frac{dP}{dt}\right)\eta + PA\eta + A^*P\eta + PD_1P\eta = D_2\eta$, for all $\eta \in W(0, T)$ with $\frac{d\eta}{dt} + A(t)\eta \in L^2(0, T; H)$ and $A\eta \in L^2(0, T; H)$ and we have $P(T) = 0$; r is the solution in $W(0, T)$ of $-\frac{dr}{dt} + A^*r + PD_1r = Pf + g$, and we have $r(T) = 0$. P and r thus defined are unique.

This last result is first obtained in a formal way, by using the main identity $p = Py+r$ and the equations of system (1.1). Next, these formal calculations can be justified, using a finite dimensional approximation of the original problem. In fact, in finite dimension we can prove the existence of a global solution (that is, for $t \in]0,T[$) to the decoupled system. Afterwards, we pass to the limit, when the dimension tends to infinity, leading to the conclusions above.

Adapting this general method to the factorization of a second order elliptic boundary value problem, it can be found in ([26, 19]) a presentation of the case where the domain is a cylinder whose axis is parallel to the x_1 coordinate. Considering that Ω is the cylinder $\Omega =]0,a[\times\mathcal{O}$, $\mathcal{O}$ is a bounded open set of $\mathbb{R}^{n-1}$, $\Sigma = \partial\mathcal{O}\times]0,a[$ and denoting $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} = \frac{\partial^2}{\partial x_1^2} + \Delta_z$, where z represents the independent variables $x_2, \ldots, x_n$, they showed that the problem

$$\begin{cases} -\Delta y = f, & \text{in } \Omega \\ y = 0, & \text{on } \Sigma \\ y = y_0, & \text{on } \Gamma_0 \\ \dfrac{\partial y}{\partial x_1} = y_a, & \text{on } \Gamma_a \end{cases}$$

can be factorized as

$$\begin{cases} \dfrac{\partial P}{\partial x_1} + P\Delta_z P + I = 0, & P(0) = 0 \\ \\ \dfrac{\partial r}{\partial x_1} + P\Delta_z r = -Pf, & r(0) = y_0 \\ \\ P\dfrac{\partial y}{\partial x_1} + y = r, & y(a) = -P(a)y_a + r(a). \end{cases}$$

An alternative factorization or the same problem is

$$\begin{cases} \dfrac{\partial Q}{\partial x_1} - Q^2 - \Delta_z = 0, & Q(a) = 0 \\ \dfrac{\partial w}{\partial x_1} - Qw = f, & w(a) = y_a \\ \dfrac{\partial y}{\partial x_1} + Qy = -w, & y(0) = y_0. \end{cases}$$

We can also find in ([26, 19]) a justification of the derivation of the Riccati equation $\frac{\partial P}{\partial x_1} + P\Delta_z P + I = 0$, $P(0) = 0$, using the fact that P was defined as a Neumann-Dirichlet operator on the boundary of the subdomains defined by the invariant embedding. Similarly to the method used by Lions, it was used a Galarkin method to study the problem in finite dimension and, afterwards, passing to the limit to the infinite dimensional problem.

1.4. Definition of the problem and regularization

Let Ω be an open bounded regular domain of $\mathbb{R}^2$. We consider the Poisson problem with Dirichlet data

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u_{|\Gamma} = 0 \end{cases} \tag{1.3}$$

where Γ is the boundary of Ω and $f \in L^2(\Omega)$. In spite of the particularization to the Laplacian operator in this definition, we believe that the same procedure could be applied to any strongly elliptic self-adjoint problem.

Applying the (spatial) invariant embedding method to this problem, we must start defining a family of subdomais sweeping the initial domain Ω. Unlike the case study we just described, we find that the correspondent moving boudary do not have, necessarily, the same geometry.

We start dealing with the case where the family of surfaces which limits the sub-domains, starts on the boundary of the domain and shrinks homothetically to a point. Since the mobile boundary reduces to a point, a singularity will necessary appear at that point. We must make, as a consequence, a regularization around this point and a possible way to do it, is to define an auxiliary domain, where we introduce a fictitious boundary around that singular point. In this case, however, we introduce a perturbation of the solution so, naturally, we must choose the new boundary condition, in a way that we can obtain the convergence of this auxiliary problem to the initial one. With this purpose, we will consider the following auxiliary problem:

$$\begin{cases} -\Delta u_\varepsilon = f, & \text{in } \Omega \setminus \Omega_\varepsilon \\ u_{\varepsilon|\Gamma} = 0 \\ \displaystyle\int_{\Gamma_\varepsilon} \frac{\partial u_\varepsilon}{\partial n} \, \mathrm{d}\Gamma_\varepsilon = 0 \\ u_{\varepsilon|\Gamma_\varepsilon} \text{is constant.} \end{cases} \tag{1.4}$$

Here, Ω_ε is an open regular domain verifying $\overline{\Omega}_\varepsilon \subset \Omega$ and Γ_ε is the boundary of Ω_ε. We can justify the choice of the boundary conditions on Γ_ε with the fact that the condition $\displaystyle\int_{\Gamma_\varepsilon} \frac{\partial u_\varepsilon}{\partial n} \, \mathrm{d}\Gamma_\varepsilon = 0$ corresponds to a null total flux. Notice that Ω_ε is in the situation previously described: it shrinks homothetically to a point, when $\varepsilon \to 0$.

There is a natural link between this line of work and the work of, for example, Sokolowski ([27]) for topological derivatives. In that study, is intended to obtain the variation of the solution of the problem, when a small hole is created on the domain. We will return to this subject at the end of Chapter 2.

The variational formulation of problem (1.4) is obtained through the following propositions:

Proposition 1.4.1. *Let* $U_\varepsilon = \{u_\varepsilon \in H^1(\Omega \setminus \Omega_\varepsilon) : u_{\varepsilon_{|\Gamma}} = 0 \ \wedge \ u_{\varepsilon_{|\Gamma_\varepsilon}}$ *is constant*$\}$. U_ε *is an Hilbert space.*

Proof. We start defining $U_\varepsilon^{aux} = \{u_\varepsilon \in H^1(\Omega \setminus \Omega_\varepsilon) : u_{\varepsilon_{|\Gamma}} = 0\}$ and a sequence $u_{n_\varepsilon} \in U_\varepsilon^{aux}$ such that $u_{n_\varepsilon} \to u_\varepsilon$ in $H^1(\Omega \setminus \Omega_\varepsilon)$. Is obvious that $u_\varepsilon \in H^1(\Omega \setminus \Omega_\varepsilon)$, as a consequence of the completeness of the space $H^1(\Omega \setminus \Omega_\varepsilon)$. Also, having $u_{n_\varepsilon} \to u_\varepsilon$ in $H^1(\Omega \setminus \Omega_\varepsilon)$ implies, by trace theorem, that $u_{n_\varepsilon} \to u_\varepsilon$ in $L^2(\Gamma \cup \Gamma_\varepsilon)$. Therefore we obtain, in particular, that $u_{\varepsilon_{|\Gamma}} = 0$. Then, this is a closed subspace of $H^1(\Omega \setminus \Omega_\varepsilon)$ and therefore it is itself an Hilbert space for the same norm. In the same way, we can prove that U_ε is a closed subspace of U_ε^{aux}, which means that it's again itself an Hilbert space for the same norm. We can therefore conclude that U_ε is an Hilbert space associated with the norm of $H^1(\Omega \setminus \Omega_\varepsilon)$. ■

Remark 1.4.2. *Considering in* U_ε *the norm* $\|u_\varepsilon\|^2_{U_\varepsilon} = \|u_\varepsilon\|^2_{H^1(\Omega\setminus\Omega_\varepsilon)} + \int_{\Gamma_\varepsilon} u_{\varepsilon_{|\Gamma_\varepsilon}}^2$ *, which is equivalent to the usual norm* $\|u_\varepsilon\|^2_{H^1(\Omega\setminus\Omega_\varepsilon)}$ *by means of trace theorem, we can also prove that* U_ε *is an Hilbert space.*

Proposition 1.4.3. *The variational formulation of problem (1.4) is*

$$\begin{cases} u_\varepsilon \in U_\varepsilon \\ \displaystyle\int_{\Omega\setminus\Omega_\varepsilon} \nabla u_\varepsilon \nabla v_\varepsilon = \int_{\Omega\setminus\Omega_\varepsilon} f\, v_\varepsilon, \forall v_\varepsilon \in U_\varepsilon. \end{cases} \tag{1.5}$$

Proof. After multiplying by $v_\varepsilon \in U_\varepsilon$ and integrating in $\Omega \setminus \Omega_\varepsilon$ both sides of $-\Delta u_\varepsilon = f$, we integrate by parts the left-hand side of the resultant equality and obtain

$$\int_{\Omega\setminus\Omega_\varepsilon} \nabla u_\varepsilon \nabla v_\varepsilon - \int_\Gamma \frac{\partial u_\varepsilon}{\partial n} v_\varepsilon - \int_{\Gamma_\varepsilon} \frac{\partial u_\varepsilon}{\partial n} v_\varepsilon = \int_{\Omega\setminus\Omega_\varepsilon} f\, v_\varepsilon.$$

Taking in account that $v_\varepsilon = 0$ in Γ, v_ε is constant in Γ_ε and $\displaystyle\int_{\Gamma_\varepsilon} \frac{\partial u_\varepsilon}{\partial n} = 0$, we find (1.5). ■

Then, problem (1.4) is well posed (that is, it has a unique solution which depends continuously on the initial conditions) as a consequence of the next proposition:

Proposition 1.4.4. *For each $\varepsilon > 0$, problem (1.5) has a unique solution.*

Proof. Consider $a(u_\varepsilon, v_\varepsilon) = \int_{\Omega\setminus\Omega_\varepsilon} \nabla u_\varepsilon \nabla v_\varepsilon$ and $(f, v_\varepsilon) = \int_{\Omega\setminus\Omega_\varepsilon} f\ v_\varepsilon$. The existence and uniqueness of a solution $u_\varepsilon \in U_\varepsilon$ for the equation $a(u_\varepsilon, v_\varepsilon) = (f, v_\varepsilon)$ is obtained by a direct application of Lax-Milgram's theorem. The continuity and coercivity of a is a consequence, respectively, of Holder's inequality and Poincaré's inequality, since

$$|a(u_\varepsilon, v_\varepsilon)| = \left| \int_{\Omega\setminus\Omega_\varepsilon} \nabla u_\varepsilon \nabla v_\varepsilon \right| \leq \|\nabla u_\varepsilon\|_{L^2(\Omega\setminus\Omega_\varepsilon)} \|\nabla v_\varepsilon\|_{L^2(\Omega\setminus\Omega_\varepsilon)} \leq \|u_\varepsilon\|_{H^1(\Omega\setminus\Omega_\varepsilon)} \|v_\varepsilon\|_{H^1(\Omega\setminus\Omega_\varepsilon)}$$

and

$$\|u_\varepsilon\|^2_{H^1(\Omega\setminus\Omega_\varepsilon)} = \|u_\varepsilon\|^2_{L^2(\Omega\setminus\Omega_\varepsilon)} + \|\nabla u_\varepsilon\|^2_{L^2(\Omega\setminus\Omega_\varepsilon)} \leq (c+1)\|\nabla u_\varepsilon\|^2_{L^2(\Omega\setminus\Omega_\varepsilon)} = (c+1)\left|a(u_\varepsilon, u_\varepsilon)\right|,$$

where c is the Poincaré constant.

Furthermore, continuity of the linear form (f, v_ε) is also a consequence of Holder's inequality and all the other hypotheses can be easily verified. ■

1.5. A convergence result

Let us consider for each $u_\varepsilon \in U_\varepsilon$ the function $\tilde{u}_\varepsilon$ defined, in Ω, by:

$$\tilde{u}_\varepsilon = \begin{cases} u_\varepsilon, \text{ in } \Omega \setminus \Omega_\varepsilon \\ u_\varepsilon = u_{\varepsilon|_{\Gamma_\varepsilon}}, \text{ in } \Omega_\varepsilon. \end{cases} \tag{1.6}$$

Obviously, $\tilde{u}_\varepsilon \in H^1_0(\Omega)$.

We consider the situation where Ω_ε shrinks to a point $p \in \Omega$ when ε goes to zero. More precisely, let $\{\varepsilon_n\}$ be a strictly decreasing sequence of real numbers such that $\varepsilon_n \to 0$. We assume that for $n \in \mathbb{N}, n \geq n_0$, Ω_{ε_n} is a regular open set such that $\overline{\Omega}_{\varepsilon_n} \subset \mathcal{B}_{\varepsilon_n}(p) \subset \overline{\mathcal{B}}_{\varepsilon_n}(p) \subset \Omega$ (where $\mathcal{B}_{\varepsilon_n}(p)$ is the open ball with center p and radius ε_n).

We define

$$\tilde{U} = \{\tilde{u} \in H^1_0(\Omega) : \text{there exists } \varepsilon_n \text{ such that } \tilde{u}_{|_{\overline{\Omega}_{\varepsilon_n}}} \text{ is constant}\}. \tag{1.7}$$

Theorem 1.5.1. *$\tilde{U}$ is dense in $H_0^1(\Omega)$.*

Proof. For simplicity, we supress the index n on ε_n. Let $f \in C_0^1(\Omega)$, where $C_0^1(\Omega)$ is the set of all $C^1(\Omega)$ functions which are zero on Γ and, without loss of generality, suppose that $\overline{\mathcal{B}}_{2\varepsilon}(p) \subset \Omega$. For $x \in \mathcal{B}_{2\varepsilon}(p) \setminus \mathcal{B}_\varepsilon(p)$ we write, in polar coordinates, $x = (x_1, x_2) = (p_1 + r\,\cos\theta, p_2 + r\,\sin\theta)$, where $p = (p_1, p_2)$. Then, consider the function u_ε given by

$$u_\varepsilon(x)=\begin{cases} f(p), & x \in \mathcal{B}_\varepsilon(p) \\ f(x), & x \notin \mathcal{B}_{2\varepsilon}(p) \\ f(p) + \dfrac{f(p_1 + 2\,\varepsilon\,\cos\theta, p_2 + 2\,\varepsilon\,\sin\theta) - f(p)}{\varepsilon}(r-\varepsilon), & \text{if } \varepsilon < r < 2\varepsilon, \\ & 0 \le \theta \le 2\pi. \end{cases}$$

It is obvious that $u_\varepsilon \in \tilde{U}$. Since $\Gamma_\varepsilon \subset \mathcal{B}_\varepsilon(p)$, u_ε has zero normal derivative a.e. on $\Gamma_\varepsilon = \partial\Omega_\varepsilon$ and consequently satisfies $\displaystyle\int_{\Gamma_\varepsilon} \frac{\partial u_\varepsilon}{\partial n} = 0$.

On the other hand, we have

$$\begin{aligned} & \|u_\varepsilon - f\|^2_{H^1(\Omega)} = \|u_\varepsilon - f\|^2_{H^1(\mathcal{B}_{2\varepsilon}(p))} \\ = & \int_{\mathcal{B}_{2\varepsilon}(p)} |u_\varepsilon - f|^2 + \int_{\mathcal{B}_{2\varepsilon}(p)} |\nabla u_\varepsilon - \nabla f|^2 \\ \le & \; 2\int_{\mathcal{B}_{2\varepsilon}(p)} |u_\varepsilon|^2 + 2\int_{\mathcal{B}_{2\varepsilon}(p)} |f|^2 + 2\int_{\mathcal{B}_{2\varepsilon}(p)} |\nabla u_\varepsilon|^2 + 2\int_{\mathcal{B}_{2\varepsilon}(p)} |\nabla f|^2. \end{aligned}$$

In $\mathcal{B}_{2\varepsilon}(p)\setminus\mathcal{B}_\varepsilon(p)$, seeing that $0 \le r-\varepsilon \le \varepsilon$, we obtain $|u_\varepsilon| \le 2\,|f(p)| + \max_{x\in\partial\mathcal{B}_{2\varepsilon}(p)} |f(x)|$. Then, since $f \in C_0^1(\Omega)$, $|f|^2$, $|\nabla f|^2$ and $|u_\varepsilon|^2$ are bounded by a constant not depending on ε, we have $\displaystyle\int_{\mathcal{B}_{2\varepsilon}(p)} |f|^2 \to 0$, $\displaystyle\int_{\mathcal{B}_{2\varepsilon}(p)} |\nabla f|^2 \to 0$ and $\displaystyle\int_{\mathcal{B}_{2\varepsilon}(p)} |u_\varepsilon|^2 \to 0$, as $\varepsilon \to 0$. It remains to analyze the term $\displaystyle\int_{\mathcal{B}_{2\varepsilon}(p)} |\nabla u_\varepsilon|^2 = \int_{\mathcal{B}_{2\varepsilon}(p)\setminus\mathcal{B}_\varepsilon(p)} |\nabla u_\varepsilon|^2$.

For $\varepsilon < r < 2\varepsilon$, we have

$$\frac{\partial u_\varepsilon}{\partial r}(p_1 + r\cos\theta, p_2 + r\sin\theta) = \frac{f(p_1 + 2\,\varepsilon\,\cos\theta, p_2 + 2\,\varepsilon\,\sin\theta) - f(p)}{\varepsilon}$$

and, using the notation $\xi_1 = p_1 + 2\,\varepsilon\,\cos\theta$, $\xi_2 = p_2 + 2\,\varepsilon\,\sin\theta$,

$$\begin{aligned} & \lim_{\varepsilon\to 0} \frac{\partial u_\varepsilon}{\partial r}(p_1 + r\cos\theta, p_2 + r\sin\theta) \\ = & \lim_{\varepsilon\to 0} \frac{\partial f}{\partial x_1}(p_1 + 2\,\varepsilon\,\cos\theta, p_2 + 2\,\varepsilon\,\sin\theta)\frac{\partial \xi_1}{\partial\varepsilon} + \frac{\partial f}{\partial x_2}(p_1 + 2\,\varepsilon\,\cos\theta, p_2 + 2\,\varepsilon\,\sin\theta)\frac{\partial \xi_2}{\partial\varepsilon} \\ = & \; \nabla f(p).(2\cos\theta, 2\sin\theta), \end{aligned}$$

so $\left|\dfrac{\partial u_\varepsilon}{\partial r}\right|$ is bounded by a constant not depending on ε; also, $\left|\dfrac{\partial r}{\partial x_1}\right| = \dfrac{|x_1 - p_1|}{\sqrt{(x_1-p_1)^2+(x_2-p_2)^2}}$ ≤ 1 and $\left|\dfrac{\partial r}{\partial x_2}\right| \leq 1$. Further, $\dfrac{\partial u_\varepsilon}{\partial \theta}(p_1 + r\cos\theta, p_2 + r\sin\theta) = \nabla f(p_1 + 2\,\varepsilon\,\cos\theta, p_2 + 2\,\varepsilon\,\sin\theta).(-2(r-\varepsilon)\sin\theta, 2(r-\varepsilon)\cos\theta)$, so $\left|\dfrac{\partial u_\varepsilon}{\partial \theta}\right| \leq |\nabla f|\, 2\varepsilon$, in $\mathcal{B}_{2\varepsilon}(p) \setminus \mathcal{B}_\varepsilon(p)$. Since

$$\left|\frac{\partial \theta}{\partial x_1}\right| = \frac{|x_2-p_2|}{(x_1-p_1)^2+(x_2-p_2)^2} \leq \frac{1}{\sqrt{(x_1-p_1)^2+(x_2-p_2)^2}} = \frac{1}{r} \leq \frac{1}{\varepsilon}$$

and $\left|\dfrac{\partial \theta}{\partial x_2}\right| = \dfrac{|x_1-p_1|}{(x_1-p_1)^2+(x_2-p_2)^2} \leq \dfrac{1}{\varepsilon}$ in $\mathcal{B}_{2\varepsilon}(p) \setminus \mathcal{B}_\varepsilon(p)$, finally, we can conclude that both $\left|\dfrac{\partial u_\varepsilon}{\partial x_1}\right|$ and $\left|\dfrac{\partial u_\varepsilon}{\partial x_2}\right|$ are bounded by a constant not depending on ε, that is, $|\nabla u_\varepsilon|$ is bounded in $\mathcal{B}_{2\varepsilon}(p) \setminus \mathcal{B}_\varepsilon(p)$ by a constant not depending on ε, which implies that, as $\varepsilon \to 0$, $\displaystyle\int_{\mathcal{B}_{2\varepsilon}(p)\setminus\mathcal{B}_\varepsilon(p)} |\nabla u_\varepsilon|^2 \to 0$.

We have proved that each $f \in C_0^1(\Omega)$ is approached by functions of $\tilde{U}$.

On the other hand the space $\{f \in C^1(\Omega) : \|f\|_{H^1(\Omega)} < +\infty\}$ is dense in $H^1(\Omega)$ (see [1], Theorem 3.16, page 52). So, $C_0^1(\Omega)$ is dense in $H_0^1(\Omega)$, which concludes the proof. ■

As stated before, we intend to prove that when $\varepsilon \to 0$, problem (1.4) reduces to problem (1.3), that is, u_ε, the solution of problem (1.4), converges to u, the solution of problem (1.3).

Lemma 1.5.2. $\|\tilde{u}_\varepsilon\|_{H_0^1(\Omega)}$ *is bounded independently of* ε.

Proof. Considering $v_\varepsilon = u_\varepsilon$ in (1.5) we obtain

$$\int_{\Omega\setminus\Omega_\varepsilon} |\nabla u_\varepsilon|^2 = \int_{\Omega\setminus\Omega_\varepsilon} f\ u_\varepsilon. \tag{1.8}$$

Since $\tilde{u}_\varepsilon$ is constant in Ω_ε,

$$\begin{aligned}\|\tilde{u}_\varepsilon\|^2_{H_0^1(\Omega)} &= \int_\Omega |\nabla \tilde{u}_\varepsilon|^2 = \int_{\Omega\setminus\Omega_\varepsilon} f\ u_\varepsilon \leq \|f\|_{L^2(\Omega\setminus\Omega_\varepsilon)} \|u_\varepsilon\|_{L^2(\Omega\setminus\Omega_\varepsilon)} \leq \|f\|_{L^2(\Omega)} \|\tilde{u}_\varepsilon\|_{L^2(\Omega)} \\ &\leq \|f\|_{L^2(\Omega)} \sqrt{c}\, \|\nabla \tilde{u}_\varepsilon\|_{L^2(\Omega)} = \|f\|_{L^2(\Omega)}\, \sqrt{c}\, \|\tilde{u}_\varepsilon\|_{H_0^1(\Omega)}\end{aligned}$$

where c is the Poincaré constant. So, $\|\tilde{u}_\varepsilon\|_{H_0^1(\Omega)} \leq k$ where $k = \|f\|_{L^2(\Omega)}\, \sqrt{c}$ is independent of ε. ■

As a consequence of the previous proposition, we can extract from $(\tilde{u}_\varepsilon)$ a subsequence, still denoted by $(\tilde{u}_\varepsilon)$, such that $\tilde{u}_\varepsilon \to \tilde{u}$, $H_0^1(\Omega)$-weak, when $\varepsilon \to 0$.

Theorem 1.5.3. *Suppose that $\varepsilon_n \to 0$ and $\Omega_{\varepsilon_q} \subset \Omega_{\varepsilon_p}$, if $q > p$. If, for each ε_n, u_{ε_n} is the solution of (1.4), then $\tilde{u}_{\varepsilon_n} \to u$, strongly in $H_0^1(\Omega)$, where u is the solution of (1.3).*

Proof. Let, for some p, $\tilde{v}_{\varepsilon_p} \in \tilde{U}$, $\tilde{v}_{\varepsilon_p}$ constant on Ω_{ε_p}; of course $\tilde{v}_{\varepsilon_p}$ is constant on Ω_{ε_n}, if $n > p$. For a moment, we fix p and take $n > p$. From (1.4),

$$\int_\Omega \nabla\tilde{u}_{\varepsilon_n}\nabla\tilde{v}_{\varepsilon_p} = \int_{\Omega\setminus\Omega_{\varepsilon_n}} \nabla u_{\varepsilon_n}\nabla v_{\varepsilon_p} = \int_{\Omega\setminus\Omega_{\varepsilon_n}} f\, v_{\varepsilon_p} = \int_\Omega f\tilde{v}_{\varepsilon_p} - \int_{\Omega_{\varepsilon_n}} f\tilde{v}_{\varepsilon_p}.$$

If u is the weak limit of $\tilde{u}_{\varepsilon_n}$, when $\varepsilon_n \to 0$, then

$$\int_\Omega \nabla\tilde{u}_{\varepsilon_n}\nabla\tilde{v}_{\varepsilon_p} \to \int_\Omega \nabla u\nabla\tilde{v}_{\varepsilon_p}$$

and, naming k the value of $\tilde{v}_{\varepsilon_p}$ in Ω_{ε_p},

$$\left|\int_{\Omega_{\varepsilon_n}} f\tilde{v}_{\varepsilon_p}\right| = |k|\left|\int_{\Omega_{\varepsilon_n}} f\right| \leq |k|\,|\Omega_{\varepsilon_n}|^{\frac{1}{2}}\,\|f\|_{L^2(\Omega)} \to 0$$

so,

$$\int_\Omega \nabla u\nabla\tilde{v}_{\varepsilon_p} = \int_\Omega f\tilde{v}_{\varepsilon_p}.$$

By Theorem 1.5.1, for every $v \in H_0^1(\Omega)$, there is a sequence $(\tilde{v}_{\varepsilon_p}) \subset \tilde{U}$ such that $\tilde{v}_{\varepsilon_p} \to v$ in $H_0^1(\Omega)$ so we can take limits in both sides of last equality and obtain

$$\int_\Omega \nabla u\nabla v = \int_\Omega fv,\ \forall v \in H_0^1(\Omega),$$

which means that u is the solution of (1.3).

Further, we have

$$\begin{aligned}\int_\Omega |\nabla(u-\tilde{u}_{\varepsilon_n})|^2 &= \int_\Omega \nabla u\nabla(u-\tilde{u}_{\varepsilon_n}) - \int_\Omega \nabla u\nabla\tilde{u}_{\varepsilon_n} + \int_{\Omega\setminus\Omega_\varepsilon} \nabla\tilde{u}_{\varepsilon_n}\nabla\tilde{u}_{\varepsilon_n} \\ &= \int_\Omega \nabla u\nabla(u-\tilde{u}_{\varepsilon_n}) - \int_\Omega \nabla u\nabla\tilde{u}_{\varepsilon_n} + \int_\Omega f\tilde{u}_{\varepsilon_n} - \int_{\Omega_\varepsilon} f\tilde{u}_{\varepsilon_n}\end{aligned}$$

and, by Holder's inequality,

$$\left|\int_{\Omega_\varepsilon} f\tilde{u}_{\varepsilon_n}\right| \leq \left(\int_{\Omega_\varepsilon} f^2\right)^{1/2}\left(\int_{\Omega_\varepsilon} \tilde{u}_{\varepsilon_n}^2\right)^{1/2} \leq \left(\int_{\Omega_\varepsilon} f^2\right)^{1/2}\left(\int_\Omega \tilde{u}_{\varepsilon_n}^2\right)^{1/2} \leq \sqrt{k}\left(\int_\Omega f^2\chi_{\Omega_\varepsilon}\right)^{1/2}.$$

When $\varepsilon \to 0$ we have $f^2\chi_{\Omega_\varepsilon} \to 0$ ($0 \leq f^2\chi_{\Omega_\varepsilon} \leq f^2$, in Ω) and consequently, using Lebesgue's theorem, $\int_{\Omega_\varepsilon} f\tilde{u}_{\varepsilon_n} \to 0$. As $\tilde{u}_{\varepsilon_n} \rightharpoonup u$ in $H_0^1(\Omega)$, we also have, $\int_\Omega \nabla u\nabla(u-\tilde{u}_{\varepsilon_n}) \to 0$, $\int_\Omega \nabla u\nabla\tilde{u}_{\varepsilon_n} \to \int_\Omega |\nabla u|^2$ and $\int_\Omega f\tilde{u}_{\varepsilon_n} \to \int_\Omega fu$. Thus, we obtain the strong limit. ∎

Chapter 2

The factorization method in a circular domain

In this chapter we apply the method presented in Chapter 1 to problem (1.3), in order to factorize this second order elliptic boundary value problem in the product of two first order decoupled initial value problems. We present here the simple situation where Ω is a disk of $\mathbb{R}^2$ with radius a and centered on the origin. In this case, the sub-domains defined by the invariant embedding are the annuli $\Omega \setminus \Omega_s$, $s \in (0, a)$.

2.1. Definition of the framework

Let us begin recalling problem (1.3):

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u_{|\Gamma} = 0. \end{cases}$$

We now assume that Ω is a circle centered at the origin. As in the case of the cylinder referred in Section 1.1., through the invariant embedding technique, the embedding parameter appears in a natural way as the direction of the radius of the circle. Therefore, we can define a family of similar problems, each one defined over the annuli $\Omega \setminus \Omega_s$, $s \in (0, a)$, choosing, for instance, a Neumann boundary condition on the moving boundary. However, this approach implies not having the solution (of each problem) always defined over the same class of functions. Besides that, we already comment upon the singularity that this method generates on the origin. To avoid these difficulties, we are going to use polar coordinates: for all function $v \in \Omega$ we associate a function $\hat{v} \in \widehat{\Omega}$, through the polar coordinates transformation $v(x,y) = \hat{v}(\rho,\theta)$, with $x = \rho\cos\theta$, $y = \rho\sin\theta$, $\rho \in (0,a]$ and $\theta \in [0,2\pi]$. Then, problem (1.3) becomes

$$\begin{cases} -\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial\hat{u}}{\partial\rho}\right) - \dfrac{1}{\rho^2}\dfrac{\partial^2\hat{u}}{\partial\theta^2} = f, \text{ in } \widehat{\Omega} = (0,a)\times[0,2\pi] \\ \hat{u}_{|\Gamma_a} = 0 \\ \hat{u}_{|\theta=0} = \hat{u}_{|\theta=2\pi} \\ \dfrac{\partial\hat{u}}{\partial\theta}_{|\theta=0} = \dfrac{\partial\hat{u}}{\partial\theta}_{|\theta=2\pi}, \end{cases} \tag{2.1}$$

and in place of the auxiliary problem chosen in Section 1.4., we can find now

$$\begin{cases} -\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial\hat{u}_\varepsilon}{\partial\rho}\right) - \dfrac{1}{\rho^2}\dfrac{\partial^2\hat{u}_\varepsilon}{\partial\theta^2} = f, \text{ in } \widehat{\Omega}\setminus\widehat{\Omega}_\varepsilon = (\varepsilon,a)\times[0,2\pi] \\ \hat{u}_{\varepsilon_{|\Gamma_a}} = 0 \\ \hat{u}_{\varepsilon_{|\Gamma_\varepsilon}} \text{ constant}, \quad \displaystyle\int_{\Gamma_\varepsilon} \frac{\partial\hat{u}_\varepsilon}{\partial\rho}\,\mathrm{d}\theta = 0 \\ \hat{u}_{\varepsilon_{|\theta=0}} = \hat{u}_{\varepsilon_{|\theta=2\pi}} \\ \dfrac{\partial\hat{u}_\varepsilon}{\partial\theta}_{|\theta=0} = \dfrac{\partial\hat{u}_\varepsilon}{\partial\theta}_{|\theta=2\pi}, \end{cases} \tag{2.2}$$

where $\Omega\setminus\Omega_\varepsilon$ represents now the annulus delimited by two concentric circumferences, one with radius ε and the other with radius a, $\varepsilon < a$.

Due to this transformation of coordinates, $\displaystyle\int_{\Omega\setminus\Omega_\varepsilon} |v(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y = \int_{\widehat{\Omega}\setminus\widehat{\Omega}_\varepsilon} |\hat{v}(\rho,\theta)|^2\rho\,\mathrm{d}\rho\,\mathrm{d}\theta = \int_0^{2\pi}\int_\varepsilon^a |\hat{v}(\rho,\theta)|^2\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$. Then, to the space $L^2(\Omega\setminus\Omega_\varepsilon)$ corresponds the space $L^2_\rho(\varepsilon,a;L^2(0,2\pi))$, where $\|\hat{v}\|^2_{L^2_\rho(\varepsilon,a;L^2(0,2\pi))} = \displaystyle\int_0^{2\pi}\int_\varepsilon^a |\hat{v}(\rho,\theta)|^2\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$ and $L^2_\rho(\varepsilon,a)$ denotes the L^2-space of functions of ρ, with the measure $\rho\,\mathrm{d}\rho$. Further, we denote by $H^1_\rho(\varepsilon,a)$ the space of functions $\hat{v}$ of ρ, such that $\hat{v} \in L^2_\rho(\varepsilon,a)$ and $\dfrac{\partial\hat{v}}{\partial\rho} \in L^2_\rho(\varepsilon,a)$ and we denote by $H^1_{\rho,P}(0,2\pi)$ the space of functions $\hat{v}$ of θ, verifying $\hat{v} \in L^2(0,2\pi)$, $\dfrac{1}{\rho}\dfrac{\partial\hat{v}}{\partial\theta} \in L^2(0,2\pi)$ and such that $\hat{v}$ has periodic boundary conditions $\hat{v}(0) = \hat{v}(2\pi)$. Therefore, we are going to consider the following definitions of norm: $\|\hat{v}(\theta)\|^2_{L^2_\rho(\varepsilon,a)} = \displaystyle\int_\varepsilon^a |\hat{v}|^2\rho\,\mathrm{d}\rho$; $\|\hat{v}(\theta)\|^2_{H^1_\rho(\varepsilon,a)} = \displaystyle\int_\varepsilon^a \left(|\hat{v}|^2 + \left(\frac{\partial\hat{v}}{\partial\rho}\right)^2\right)\rho\,\mathrm{d}\rho$; $\|\hat{v}(\rho)\|^2_{L^2(0,2\pi)} = \displaystyle\int_0^{2\pi} |\hat{v}|^2\,\mathrm{d}\theta$; $\|\hat{v}(\rho)\|^2_{H^1_{\rho,P}(0,2\pi)} = \displaystyle\int_0^{2\pi}\left(|\hat{v}|^2 + \frac{1}{\rho^2}\left(\frac{\partial\hat{v}}{\partial\theta}\right)^2\right)\mathrm{d}\theta$.

According to the previous notations, to the Hilbert space $H^1(\Omega\setminus\Omega_\varepsilon)$ corresponds the space $\widehat{H}_\varepsilon = \{\hat{v} : \hat{v} \in L^2_\rho\left(\varepsilon,a;H^1_{\rho,P}(0,2\pi)\right), \dfrac{\partial\hat{v}}{\partial\rho} \in L^2_\rho\left(\varepsilon,a;L^2(0,2\pi)\right)\}$. In fact, to the space

$L^2_\rho(\varepsilon, a; H^1_{\rho,P}(0, 2\pi))$ belong the functions $\hat{v}$ of ρ defined a.e. on (ε, a), with values in the space of functions of θ, measurable in ρ for the measure $\rho\,\mathrm{d}\rho$, such that $\hat{v}(\rho) \in H^1_{\rho,P}(0, 2\pi)$ *a.e.* in ρ and $\displaystyle\int_\varepsilon^a \|\hat{v}\|^2_{H^1_{\rho,P}(0,2\pi)}\rho\,\mathrm{d}\rho < \infty$ - that is,

$$\|\hat{v}\|^2_{L^2_\rho(\varepsilon,a;H^1_{\rho,P}(0,2\pi))} = \int_0^{2\pi}\int_\varepsilon^a \left(|\hat{v}(\rho,\theta)|^2 + \frac{1}{\rho^2}\left(\frac{\partial\hat{v}}{\partial\theta}(\rho,\theta)\right)^2\right)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta, \text{ and}$$

$$\|v\|^2_{H^1(\Omega\backslash\Omega_\varepsilon)} = \|\hat{v}\|^2_{L^2_\rho(\varepsilon,a;H^1_{\rho,P}(0,2\pi))} + \left\|\frac{\partial\hat{v}}{\partial\rho}\right\|^2_{L^2_\rho(\varepsilon,a;L^2(0,2\pi))} = \|\hat{v}\|^2_{\widehat{H}_\varepsilon}.$$

Based on the fact that $L^2(\varepsilon, a; L^2(0, 2\pi))$ is an Hilbert space, it is easy to prove that the space $L^2_\rho(\varepsilon, a; H^1_{\rho,P}(0, 2\pi))$, is also an Hilbert space, for all $\varepsilon \geq 0$.

In this framework, the following remark is an immediate consequence of Proposition 1.4.3:

Remark 2.1.1. *Let* $\widehat{U}_\varepsilon = \{\hat{u}_\varepsilon \in \widehat{H}_\varepsilon : \hat{u}_{\varepsilon|_{\Gamma_a}} = 0 \wedge \hat{u}_{\varepsilon|_{\Gamma_\varepsilon}}$ *is constant*$\}$. *As previously,* $\widehat{U}_\varepsilon$ *being a closed subspace of* $\widehat{H}_\varepsilon$*, is itself an Hilbert space, for the same norm. Then, the variational formulation of problem (2.2) is*

$$\begin{cases} \hat{u}_\varepsilon \in \widehat{U}_\varepsilon \\ \displaystyle\int_\varepsilon^a \int_0^{2\pi} \left(\frac{\partial\hat{u}_\varepsilon}{\partial\rho}\frac{\partial\hat{v}_\varepsilon}{\partial\rho}\rho + \frac{1}{\rho}\frac{\partial\hat{u}_\varepsilon}{\partial\theta}\frac{\partial\hat{v}_\varepsilon}{\partial\theta}\right)\mathrm{d}\theta\,\mathrm{d}\rho = \int_\varepsilon^a\int_0^{2\pi} f\hat{v}_\varepsilon\rho\;\mathrm{d}\theta\,\mathrm{d}\rho,\ \forall\hat{v}_\varepsilon \in \widehat{U}_\varepsilon. \end{cases} \tag{2.3}$$

Analogously, to the space $H^1_0(\Omega)$ *corresponds the space* $\widehat{U}_0 = \{\hat{v} \in \widehat{H}_0 : \hat{v}_{|_{\Gamma_a}} = 0,\ \hat{v}_{|_{\Gamma_0}}$ *constant*$\}$ *and the variational formulation of problem (2.1) is*

$$\begin{cases} \hat{u} \in \widehat{U}_0 \\ \displaystyle\int_0^a \int_0^{2\pi} \left(\frac{\partial\hat{u}}{\partial\rho}\frac{\partial\hat{v}}{\partial\rho}\rho + \frac{1}{\rho}\frac{\partial\hat{u}}{\partial\theta}\frac{\partial\hat{v}}{\partial\theta}\right)\mathrm{d}\theta\,\mathrm{d}\rho = \int_0^a\int_0^{2\pi} f\hat{v}\rho\;\mathrm{d}\theta\,\mathrm{d}\rho,\ \forall\hat{v} \in \widehat{U}_0. \end{cases} \tag{2.4}$$

We end this section with the presentation of an essencial trace theorem, which is a direct application of Theorem 3.1, page 19 of [24]:

Proposition 2.1.2. *We have* $\hat{v} \in \mathcal{C}\left(\varepsilon, a; H^{1/2}_{\rho,P}(0,2\pi)\right)$, *for all* $\hat{v} \in \widehat{H}_\varepsilon$, *where the space* $H^{1/2}_{\rho,P}(0,2\pi)$ *represents the* $1/2$ *interpolate between* $H^1_{\rho,P}(0,2\pi)$ *and* $L^2(0,2\pi)$. *Also, for all* $\hat{v} \in \widehat{X}_\varepsilon = \left\{\hat{v} \in \widehat{H}_\varepsilon : \dfrac{\partial^2 \hat{v}}{\partial \rho^2} \in L^2_\rho\Big(\varepsilon, a; \big(H^1_{\rho,P}(0,2\pi)\big)'\Big)\right\}$, *we also have* $\dfrac{\partial \hat{v}}{\partial \rho} \in \mathcal{C}\Big(\varepsilon, a; \Big(H^{1/2}_{\rho,P}(0,2\pi)\Big)'\Big)$, *where* $\left(H^{1/2}_{\rho,P}(0,2\pi)\right)'$ *represents the* $1/2$ *interpolate between* $\left(H^1_{\rho,P}(0,2\pi)\right)'$ *and* $L^2(0,2\pi)$. *Furthermore, the trace mapping* $\hat{v} \to \left(\hat{v}_{|\Gamma_\varepsilon}, \dfrac{\partial \hat{v}}{\partial \rho}|_{\Gamma_\varepsilon}\right)$ *is continuous from* $\widehat{X}_\varepsilon$ *onto* $H^{1/2}_{\rho,P}(0,2\pi) \times \left(H^{1/2}_{\rho,P}(0,2\pi)\right)'$.

2.2. Invariant embedding

Using the technique of invariant embedding, we embed problem (2.2) in a family of similar problems defined on $[s,a] \times [0,2\pi]$, for $s \in [\varepsilon, a)$. For each problem we impose the boundary condition $\dfrac{\partial \hat{u}_s}{\partial \rho}|_{\Gamma_s} = h$, where Γ_s is the moving boundary:

$$\begin{cases} -\dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\rho \dfrac{\partial \hat{u}_s}{\partial \rho}\right) - \dfrac{1}{\rho^2}\dfrac{\partial^2 \hat{u}_s}{\partial \theta^2} = f, \text{ in } \widehat{\Omega} \setminus \widehat{\Omega}_s \\ \hat{u}_{s_{|\Gamma_a}} = 0 \\ \dfrac{\partial \hat{u}_s}{\partial \rho}|_{\Gamma_s} = h \\ \hat{u}_{s_{|\theta=0}} = \hat{u}_{s_{|\theta=2\pi}} \\ \dfrac{\partial \hat{u}_s}{\partial \theta}|_{\theta=0} = \dfrac{\partial \hat{u}_s}{\partial \theta}|_{\theta=2\pi}. \end{cases} \tag{2.5}$$

In (2.5) we take $h \in \left(H^{1/2}_{\rho,P}(0,2\pi)\right)'$. Since $\dfrac{\partial \hat{u}_\varepsilon}{\partial \rho}|_{\Gamma_\varepsilon}$ is well determined through the conditions "$\hat{u}_{\varepsilon_{|\Gamma_\varepsilon}}$ constant" and "$\displaystyle\int_{\Gamma_\varepsilon} \frac{\partial \hat{u}_\varepsilon}{\partial \rho}\, d\theta = 0$", it is clear that (2.2) is exactly (2.5), for $s = \varepsilon$ and $h = \dfrac{\partial \hat{u}_\varepsilon}{\partial \rho}|_{\Gamma_\varepsilon}$.

The variational formulation of the embedded problem can be now directly achieved:

Proposition 2.2.1. *Considering the Hilbert space* $\widehat{U}_s = \{\hat{u}_s \in \widehat{H}_s : \hat{u}_{s_{|\Gamma_a}} = 0\}$, *the variational formulation of problem (2.5) is*

$$\begin{cases} \hat{u}_s \in \widehat{U}_s \\ \displaystyle\int_s^a\!\!\int_0^{2\pi} \left(\frac{\partial \hat{u}_s}{\partial \rho} \frac{\partial \hat{v}_s}{\partial \rho} \rho + \frac{1}{\rho} \frac{\partial \hat{u}_s}{\partial \theta} \frac{\partial \hat{v}_s}{\partial \theta} \right) \mathrm{d}\theta \, \mathrm{d}\rho = -\int_0^{2\pi} h \hat{v}_s(s) s \, \mathrm{d}\theta + \int_s^a\!\!\int_0^{2\pi} f \hat{v}_s \rho \, \mathrm{d}\theta \, \mathrm{d}\rho, \\ \forall \hat{v}_s \in \widehat{U}_s. \end{cases} \tag{2.6}$$

Proof. Using (2.5), multiplying by $\hat{v}_s \in \widehat{U}_s$, and integrating in $\widehat{\Omega} \setminus \widehat{\Omega}_s$, we obtain:

$$\begin{aligned} &\int_0^{2\pi} \int_s^a \left(-\frac{\partial^2 \hat{u}_s}{\partial \rho^2} \hat{v}_s \rho - \frac{1}{\rho^2} \frac{\partial^2 \hat{u}_s}{\partial \theta^2} \hat{v}_s \rho - \frac{1}{\rho} \frac{\partial \hat{u}_s}{\partial \rho} \hat{v}_s \rho \right) \mathrm{d}\rho \, \mathrm{d}\theta = \int_0^{2\pi} \int_s^a f \hat{v}_s \rho \, \mathrm{d}\rho \, \mathrm{d}\theta \\ \Rightarrow \; &\int_0^{2\pi} \left. -\frac{\partial \hat{u}_s}{\partial \rho} \hat{v}_s \rho \right]_s^a \mathrm{d}\theta + \int_0^{2\pi} \int_s^a \frac{\partial \hat{u}_s}{\partial \rho} \left(\frac{\partial \hat{v}_s}{\partial \rho} \rho + \hat{v}_s \right) \mathrm{d}\rho \, \mathrm{d}\theta - \int_s^a \left. \frac{1}{\rho} \frac{\partial \hat{u}_s}{\partial \theta} \hat{v}_s \right]_0^{2\pi} \mathrm{d}\rho \\ &+ \int_0^{2\pi} \int_s^a \frac{1}{\rho} \frac{\partial \hat{u}_s}{\partial \theta} \frac{\partial \hat{v}_s}{\partial \theta} \mathrm{d}\rho \, \mathrm{d}\theta - \int_0^{2\pi} \int_s^a \frac{\partial \hat{u}_s}{\partial \rho} \hat{v}_s \, \mathrm{d}\rho \, \mathrm{d}\theta = \int_0^{2\pi} \int_s^a f \hat{v}_s \rho \, \mathrm{d}\rho \, \mathrm{d}\theta \\ \Rightarrow \; &\int_0^{2\pi} s h \hat{v}_s(s) \, \mathrm{d}\theta + \int_0^{2\pi} \int_s^a \frac{\partial \hat{u}_s}{\partial \rho} \frac{\partial \hat{v}_s}{\partial \rho} \rho \, \mathrm{d}\rho \, \mathrm{d}\theta + \int_0^{2\pi} \int_s^a \frac{1}{\rho} \frac{\partial \hat{u}_s}{\partial \theta} \frac{\partial \hat{v}_s}{\partial \theta} \mathrm{d}\rho \, \mathrm{d}\theta \\ &= \int_0^{2\pi} \int_s^a f \hat{v}_s \rho \, \mathrm{d}\rho \, \mathrm{d}\theta. \end{aligned}$$

■

Naturally, the above variational formulation reduces to (2.3), when $s = \varepsilon$. Using this variational formulation and Lax-Milgram theorem, it is easy to prove, similarly to Proposition 1.4.4, that the problem (2.5) is well posed.

In order to apply a method similar to the one used by Lions ([23]) for decoupling the optimality conditions associated to an optimal control problem of a parabolic equation, we define:

Definition 2.2.1. For every $s \in [\varepsilon, a)$ and $h \in \left(H^{1/2}_{\rho,P}(0, 2\pi) \right)'$ we define $P(s)h = \gamma_{s_{|\Gamma_s}}$, where $\gamma_s \in \left\{ \hat{v} \in \widehat{H}_s : \hat{v}_{|\Gamma_a = 0} \right\}$ is the solution of

$$\begin{cases} -\dfrac{1}{\rho} \dfrac{\partial}{\partial \rho} \left(\rho \dfrac{\partial \gamma_s}{\partial \rho} \right) - \dfrac{1}{\rho^2} \dfrac{\partial^2 \gamma_s}{\partial \theta^2} = 0, \text{ in } \widehat{\Omega} \setminus \widehat{\Omega}_s \\ \gamma_{s_{|\Gamma_a}} = 0, \quad \dfrac{\partial \gamma_s}{\partial \rho}_{|\Gamma_s} = h \\ \gamma_{s|_{\theta=0}} = \gamma_{s|_{\theta=2\pi}} \\ \dfrac{\partial \gamma_s}{\partial \theta}_{|\theta=0} = \dfrac{\partial \gamma_s}{\partial \theta}_{|\theta=2\pi}. \end{cases} \tag{2.7}$$

(hence, P is a Neumann-to-Dirichlet operator) and $r(s) = \beta_{s_{|\Gamma_s}}$, where $\beta_s \in \left\{ \hat{v} \in \widehat{H}_s : \hat{v}_{|\Gamma_a = 0} \right\}$ is the solution of

$$\begin{cases} -\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial\beta_s}{\partial\rho}\right) - \dfrac{1}{\rho^2}\dfrac{\partial^2\beta_s}{\partial\theta^2} = f, \text{ in } \widehat{\Omega}\setminus\widehat{\Omega}_s \\ \beta_{s_{|\Gamma_a}} = 0, \;\; \dfrac{\partial\beta_s}{\partial\rho}|_{\Gamma_s} = 0 \\ \beta_{s_{|\theta=0}} = \beta_{s_{|\theta=2\pi}} \\ \dfrac{\partial\beta_s}{\partial\theta}|_{\theta=0} = \dfrac{\partial\beta_s}{\partial\theta}|_{\theta=2\pi}. \end{cases} \tag{2.8}$$

In the particular case of $s = \varepsilon$, h must verify $\displaystyle\int_0^{2\pi} h = \int_0^{2\pi} \frac{\partial\gamma_\varepsilon}{\partial\rho}(\varepsilon) = 0$ and $\gamma_\varepsilon(\varepsilon) - \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\gamma_\varepsilon(\varepsilon)\,\mathrm{d}\theta = -r(\varepsilon) + \frac{1}{2\pi}\int_0^{2\pi} r(\varepsilon)\,\mathrm{d}\theta$, since $u_\varepsilon(\varepsilon) = \gamma_\varepsilon(\varepsilon) + r(\varepsilon)$ is constant.

As a direct consequence of the computations exhibited in Proposition 2.2.1, taking $f = 0$ and $h = 0$, respectively, the variational formulation of problems (2.7) and (2.8) are, respectively,

$$\begin{cases} \gamma_s \in \widehat{U}_s \\ \displaystyle\int_s^a\!\!\int_0^{2\pi}\left(\frac{\partial\gamma_s}{\partial\rho}\frac{\partial\overline{\gamma}_s}{\partial\rho}\rho + \frac{1}{\rho}\frac{\partial\gamma_s}{\partial\theta}\frac{\partial\overline{\gamma}_s}{\partial\theta}\right)\mathrm{d}\theta\,\mathrm{d}\rho = -\int_0^{2\pi} h\overline{\gamma}_s(s)s\,\mathrm{d}\theta, \;\forall\overline{\gamma}_s \in \widehat{U}_s \end{cases} \tag{2.9}$$

and

$$\begin{cases} \beta_s \in \widehat{U}_s \\ \displaystyle\int_s^a\!\!\int_0^{2\pi}\left(\frac{\partial\beta_s}{\partial\rho}\frac{\partial\overline{\beta}_s}{\partial\rho}\rho + \frac{1}{\rho}\frac{\partial\beta_s}{\partial\theta}\frac{\partial\overline{\beta}_s}{\partial\theta}\right)\mathrm{d}\theta\,\mathrm{d}\rho = \int_s^a\!\!\int_0^{2\pi} f\overline{\beta}_s\rho\;\mathrm{d}\theta\,\mathrm{d}\rho, \;\forall\overline{\beta}_s \in \widehat{U}_s. \end{cases} \tag{2.10}$$

In addition, from Proposition 2.1.2, for every $s \in [\varepsilon, a]$, $P(s) : \left(H^{1/2}_{\rho,P}(0,2\pi)\right)' \to H^{1/2}_{\rho,P}(0,2\pi)$ is a linear operator and $r(s) \in H^{1/2}_{\rho,P}(0,2\pi)$. By linearity of (2.5) we have

$$\hat{u}_{s_{|\Gamma_s}} = P(s)\frac{\partial\hat{u}_s}{\partial\rho}|_{\Gamma_s} + r(s), \forall s \in [\varepsilon, a]. \tag{2.11}$$

Furthermore, the solution $\hat{u}_\varepsilon$ of (2.2) is given by

$$\hat{u}_\varepsilon(\rho,\theta) = (P(\rho)\frac{\partial\hat{u}_\varepsilon}{\partial\rho}|_{\Gamma_\rho})(\theta) + (r(\rho))(\theta). \tag{2.12}$$

We can observe as well that we have, in fact, $\gamma_s, \beta_s \in \widehat{X}_s$:

Remark 2.2.2. *Since $\gamma_s \in \widehat{H}_s$, in particular we have $\gamma_s \in L^2_\rho(s,a;H^1_{\rho,P}(0,2\pi))$. Thus, $\dfrac{\partial^2\gamma_s}{\partial\theta^2} \in L^2_\rho\left(s,a;\left(H^1_{\rho,P}(0,2\pi)\right)'\right)$. Furthermore, $\dfrac{\partial\gamma_s}{\partial\rho} \in L^2_\rho(s,a;L^2(0,2\pi))$ and consequently,*

$\frac{\partial \gamma_s}{\partial \rho} \in L^2_\rho\left(s,a;\left(H^1_{\rho,P}(0,2\pi)\right)'\right)$, *making the usual identification of* $L^2(0,2\pi)$ *with its dual space. Therefore,* $\frac{\partial^2 \gamma_s}{\partial \rho^2} = -\frac{1}{\rho^2}\frac{\partial^2 \gamma_s}{\partial \theta^2} - \frac{1}{\rho}\frac{\partial \gamma_s}{\partial \rho} \in L^2_\rho\left(s,a;\left(H^1_{\rho,P}(0,2\pi)\right)'\right)$ *and* $\gamma_s \in \widehat{X}_s$. *Obviously, we can establish the same result for* β_s, *since we also have* $f \in L^2_\rho(s,a;L^2(0,2\pi))$.

In the next Proposition we present the first properties of the operator P:

Proposition 2.2.3. *The linear operator* $P(s) : \left(H^{1/2}_{\rho,P}(0,2\pi)\right)' \rightarrow H^{1/2}_{\rho,P}(0,2\pi)$ *is continuous, self-adjoint and negative definite, for all* $s \in [\varepsilon, a)$.

Proof. The operator $P(s)$ is continuous since it's the composition of continuous operators: $h \rightarrow \gamma_s \rightarrow \gamma_{s_{|\Gamma_s}}$, defined by (2.7), respectively in the spaces $\left(H^{1/2}_{\rho,P}(0,2\pi)\right)'$, $\widehat{H}_s$ and $H^{1/2}_{\rho,P}(0,2\pi)$. Let's consider γ_s and $\overline{\gamma}_s$ two solutions of (2.7), with $\frac{\partial \gamma_s}{\partial \rho}|_{\Gamma_s} = h$ and $\frac{\partial \overline{\gamma}_s}{\partial \rho}|_{\Gamma_s} = \overline{h}$, respectively. Then, (2.9) can be written in the form

$$-\int_{\widehat{\Omega}\backslash\widehat{\Omega}_s} \nabla\gamma_s \nabla\overline{\gamma}_s \, \rho \, \mathrm{d}\rho \, \mathrm{d}\theta = s \, \langle h, \overline{\gamma}_s(s) \rangle_{H^{1/2}_{\rho,P}(0,2\pi)\;', H^{1/2}_{\rho,P}(0,2\pi)} = s \, \left\langle h, P(s)\overline{h} \right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\;', H^{1/2}_{\rho,P}(0,2\pi)}.$$

Therefore

$$s \, \left\langle h, P(s)\overline{h} \right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\;', H^{1/2}_{\rho,P}(0,2\pi)} = s \, \left\langle \overline{h}, P(s)h \right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\;', H^{1/2}_{\rho,P}(0,2\pi)},$$

and we conclude that $P(s)$ is a self-adjoint operator.

On the other hand, taking $\gamma_s = \overline{\gamma}_s$ we have

$$s \, \langle h, P(s)h \rangle_{H^{1/2}_{\rho,P}(0,2\pi)\;', H^{1/2}_{\rho,P}(0,2\pi)} = -\int_{\widehat{\Omega}\backslash\widehat{\Omega}_s} |\nabla\gamma_s|^2 \, \rho \, \mathrm{d}\rho \, \mathrm{d}\theta \tag{2.13}$$

and consequently $P(s)$ is a negative operator. Using Poincaré's inequality, we have

$$\int_{\widehat{\Omega}\backslash\widehat{\Omega}_s} |\nabla\gamma_s|^2 \, \rho \, \mathrm{d}\rho \, \mathrm{d}\theta = \|\nabla\gamma_s\|^2_{L^2_\rho(s,a;L^2(0,2\pi))} \geq \frac{1}{c^2}\|\gamma_s\|^2_{L^2_\rho(s,a;L^2(0,2\pi))}.$$

Therefore,

$$\begin{aligned}
& \|\nabla\gamma_s\|^2_{L^2_\rho(s,a;L^2(0,2\pi))} + \frac{1}{c^2}\|\nabla\gamma_s\|^2_{L^2_\rho(s,a;L^2(0,2\pi))} \\
\geq \;& \frac{1}{c^2}\|\gamma_s\|^2_{L^2_\rho(s,a;L^2(0,2\pi))} + \frac{1}{c^2}\|\nabla\gamma_s\|^2_{L^2_\rho(s,a;L^2(0,2\pi))} \\
\Rightarrow \;& \left(1 + \frac{1}{c^2}\right)\|\nabla\gamma_s\|^2_{L^2_\rho(s,a;L^2(0,2\pi))} \geq \frac{1}{c^2}\|\gamma_s\|^2_{\widehat{H}_s} \\
\Rightarrow \;& \|\nabla\gamma_s\|^2_{L^2_\rho(s,a;L^2(0,2\pi))} \geq \frac{1}{c^2+1}\|\gamma_s\|^2_{\widehat{H}_s} \\
\Rightarrow \;& -\int_{\widehat{\Omega}\backslash\widehat{\Omega}_s} |\nabla\gamma_s|^2 \, \rho \, \mathrm{d}\rho \, \mathrm{d}\theta \leq -\frac{1}{c^2+1}\|\gamma_s\|^2_{\widehat{H}_s}.
\end{aligned} \tag{2.14}$$

Since $s\langle h, P(s)h\rangle_{H^{1/2}_{\rho,P}(0,2\pi)'\,,\,H^{1/2}_{\rho,P}(0,2\pi)} = -\int_{\widehat{\Omega}\setminus\widehat{\Omega}_s} |\nabla\gamma_s|^2\,\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$ we then have

$$s\langle h, P(s)h\rangle_{H^{1/2}_{\rho,P}(0,2\pi)'\,,\,H^{1/2}_{\rho,P}(0,2\pi)} \leq -c_1\|\gamma_s\|^2_{\widehat{H}_s}.$$

Now, since $\Delta\gamma_s = 0$, by Lemma 1, page 381 of [12], follows that $\exists\, k_s > 0$ (the constant should depend on s, due to the utilization of polar coordinates) such that

$$\left\|\frac{\partial\gamma_s}{\partial\rho}|_{\Gamma_s}\right\|_{H^{1/2}_{\rho,P}(0,2\pi)'} \leq k_s\,\|\gamma_s\|_{H(\Delta,\widehat{\Omega}\setminus\widehat{\Omega}_s)} = k_s\,\|\gamma_s\|_{\widehat{H}_s}$$

$$\Rightarrow\quad -\|\gamma_s\|^2_{\widehat{H}_s} \leq -\frac{1}{k_s^2}\left\|\frac{\partial\gamma_s}{\partial\rho}|_{\Gamma_s}\right\|^2_{H^{1/2}_{\rho,P}(0,2\pi)'}.$$

Then,

$$s\langle h, P(s)h\rangle_{H^{1/2}_{\rho,P}(0,2\pi)'\,,\,H^{1/2}_{\rho,P}(0,2\pi)} \leq -\frac{c_1}{k_s^2}\left\|\frac{\partial\gamma_s}{\partial\rho}|_{\Gamma_s}\right\|^2_{H^{1/2}_{\rho,P}(0,2\pi)'} = -c_2\|h\|^2_{H^{1/2}_{\rho,P}(0,2\pi)'},$$

which proves that $P(s)$ is a negative definite operator.

Furthermore, from (2.13), Poincaré's inequality and Holder's inequality, we have

$$c_1\|\gamma_s\|^2_{\widehat{H}_s} \leq \|\nabla\gamma_s\|^2_{L^2_\rho(s,a;L^2(0,2\pi))} \leq s\,\|h\|_{H^{1/2}_{\rho,P}(0,2\pi)'}\|\gamma_s(s)\|_{H^{1/2}_{\rho,P}(0,2\pi)},$$

and, on the other hand, due to trace theorem, $\exists c_s > 0$ (again, c_s should depend on s) such that

$$\|\gamma_s(s)\|_{H^{1/2}_{\rho,P}(0,2\pi)} \leq c_s\|\gamma_s\|_{\widehat{H}_s}$$

and consequently

$$\frac{c_1}{c_s^2}\|\gamma_s(s)\|^2_{H^{1/2}_{\rho,P}(0,2\pi)} \leq c_1\|\gamma_s\|^2_{\widehat{H}_s} \leq s\,\|h\|_{H^{1/2}_{\rho,P}(0,2\pi)'}\|\gamma_s(s)\|_{H^{1/2}_{\rho,P}(0,2\pi)}$$

$$\Rightarrow\quad \|\gamma_s(s)\|_{H^{1/2}_{\rho,P}(0,2\pi)} \leq \frac{s\,c_s^2}{c_1}\|h\|_{H^{1/2}_{\rho,P}(0,2\pi)'}.$$

■

From (2.12) taking the derivative, in a formal way, with respect to ρ we obtain

$$\begin{aligned}
\frac{\partial\hat{u}_\varepsilon}{\partial\rho} &= \frac{\partial P}{\partial\rho}\frac{\partial\hat{u}_\varepsilon}{\partial\rho} + P\frac{\partial^2\hat{u}_\varepsilon}{\partial\rho^2} + \frac{\partial r}{\partial\rho}\\
&= \frac{\partial P}{\partial\rho}\frac{\partial\hat{u}_\varepsilon}{\partial\rho} + P\left(-f - \frac{1}{\rho^2}\frac{\partial^2\hat{u}_\varepsilon}{\partial\theta^2} - \frac{1}{\rho}\frac{\partial\hat{u}_\varepsilon}{\partial\rho}\right) + \frac{\partial r}{\partial\rho}\\
&= \frac{\partial P}{\partial\rho}\frac{\partial\hat{u}_\varepsilon}{\partial\rho} - Pf - P\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\left(P\frac{\partial\hat{u}_\varepsilon}{\partial\rho} + r\right) - P\frac{1}{\rho}\frac{\partial\hat{u}_\varepsilon}{\partial\rho} + \frac{\partial r}{\partial\rho}\\
&= \left(\frac{\partial P}{\partial\rho} - P\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}P - P\frac{1}{\rho}\right)\frac{\partial\hat{u}_\varepsilon}{\partial\rho} - Pf - P\frac{1}{\rho^2}\frac{\partial^2 r}{\partial\theta^2} + \frac{\partial r}{\partial\rho}
\end{aligned} \tag{2.15}$$

and consequently, since $\dfrac{\partial \hat{u}_\varepsilon}{\partial \rho}$ is arbitrary (see Remark 2.4.1, with $m = \infty$), we have the system

$$\begin{cases} \dfrac{\partial P}{\partial \rho} - \dfrac{1}{\rho^2} P \dfrac{\partial^2}{\partial \theta^2} P - P \dfrac{1}{\rho} - I = 0 \\ -Pf - P \dfrac{1}{\rho^2} \dfrac{\partial^2 r}{\partial \theta^2} + \dfrac{\partial r}{\partial \rho} = 0 \\ P \dfrac{\partial \hat{u}_\varepsilon}{\partial \rho} - \hat{u}_\varepsilon = -r. \end{cases}$$

Again from (2.12) and considering the Γ_a initial condition in (2.2) we obtain

$$P(a) = 0 \quad \text{and} \quad r(a) = 0.$$

From the first two equations of the previous system, and respective initial conditions, we can obtain P and r. Knowing $P(\varepsilon)$ and $r(\varepsilon)$ we want to determine uniquely $\hat{u}_\varepsilon(\varepsilon)$ satisfying "$\hat{u}_{\varepsilon|_{\Gamma_\varepsilon}}$ constant" and "$\int_{\Gamma_\varepsilon} \dfrac{\partial \hat{u}_\varepsilon}{\partial \rho} \, \mathrm{d}\theta = 0$". For this, we need to prove that the operator P preserve both constant functions and functions of null mean.

Lemma 2.2.4. *Let*

$$M = \left\{ v \in \left(H^{1/2}_{\rho,P}(0, 2\pi) \right)' : \int_0^{2\pi} v \, \mathrm{d}\theta = 0 \right\}.$$

Then M is an Hilbert space.

Proof. It's easy to prove that M is a subspace of $\left(H^{1/2}_{\rho,P}(0, 2\pi) \right)'$. Moreover, M is closed, since it is the kernel of a continuous linear form. ∎

Lemma 2.2.5. *Let*

$$N = \left\{ v \in H^{1/2}_{\rho,P}(0, 2\pi) : v \text{ is constant} \right\}.$$

Then N is an Hilbert space. Moreover, any $v \in H^{1/2}_{\rho,P}(0, 2\pi)$ may be written in a unique way in the form $v = v_M + v_N$, where $v_M \in M \cap H^{1/2}_{\rho,P}(0, 2\pi)$ and $v_N \in N$.

Proof. It's evident that N is a subspace of $H^{1/2}_{\rho,P}(0,2\pi)$. In order to prove that N is closed, we consider a sequence $(v_n)_{n\in\mathbb{N}} \in N$ such that $v_n \to v$ in $H^{1/2}_{\rho,P}(0,2\pi)$. To conclude that $v \in N$, we only need to prove that v is constant. Now, since $v_n \to v$ in $H^{1/2}_{\rho,P}(0,2\pi)$ (that is, $\|v_n - v\|_{H^{1/2}_{\rho,P}(0,2\pi)} \to 0$) and $\|v_n - v\|^2_{L^2(0,2\pi)} \leq \|v_n - v\|^2_{H^{1/2}_{\rho,P}(0,2\pi)}$, we have $\|v_n - v\|^2_{L^2(0,2\pi)} \to 0$, which implies that $v_n - v \to 0$ *a.e.* in $(0,2\pi)$. Therefore, since v_n is constant we also have v constant and N is a closed subspace of $H^{1/2}_{\rho,P}(0,2\pi)$.

The second part of the proof is a direct consequence of Theorem 3.4, page 7 of [21], noticing that $H^{1/2}_{\rho,P}(0,2\pi) \subset L^2(0,2\pi)$. ■

Proposition 2.2.6. *The operator P is such that $P : M \to M$ and $P : N \to N$.*

Proof. For each $s \in [\varepsilon, a)$ and $h \in N$, we define $P(s)h = \gamma_{s_{|\Gamma_s}}$, where $\gamma_s \in \widehat{X}_s$ is the solution of (2.7) (that is, we consider a solution of (2.7) verifying also $\frac{\partial \gamma_s}{\partial \rho}|_{\Gamma_s}$ constant in θ). Considering $\alpha(\rho)$ the solution of the linear two points boundary value problem, $\alpha''(\rho) + \frac{1}{\rho}\alpha'(\rho) = 0$, $\alpha(a) = 0$, $\alpha'(s) = h$ (in fact, it's easy to prove that $\alpha(\rho) = -s\,h \log a + s\,h \log \rho$), then $\gamma_s(\rho,\theta) = \alpha(\rho)$ is the solution of problem (2.7), since

$$\begin{aligned} & -\frac{\partial^2 \gamma_s}{\partial \rho^2}(\rho,\theta) - \frac{1}{\rho^2}\frac{\partial^2 \gamma_s}{\partial \theta^2}(\rho,\theta) - \frac{1}{\rho}\frac{\partial \gamma_s}{\partial \rho}(\rho,\theta) \\ = & -\frac{\partial^2 \alpha}{\partial \rho^2}(\rho) - \frac{1}{\rho^2}\frac{\partial^2 \alpha}{\partial \theta^2}(\rho) - \frac{1}{\rho}\frac{\partial \alpha}{\partial \rho}(\rho) \\ = & -\frac{\partial^2 \alpha}{\partial \rho^2}(\rho) - 0 - \frac{1}{\rho}\frac{\partial \alpha}{\partial \rho}(\rho) \\ = & \ 0. \end{aligned}$$

Then, we can conclude that considering $\frac{\partial \gamma_s}{\partial \rho}|_{\Gamma_s} = h$ constant in θ, we also have $\gamma_s(\rho,\theta)$ constant in θ and therefore $\gamma_{s_{|\Gamma_s}}$ has the same property. Consequently, $P(s)h = \gamma_{s_{|\Gamma_s}}$ is constant in θ and $P : N \to N$.

Now, for each $s \in [\varepsilon, a)$ and $h \in M$, we define $P(s)h = \gamma_{s_{|\Gamma_s}}$, where $\gamma_s \in \widehat{X}_s$ is the solution of (2.7) (that is, we consider a solution of (2.7) verifying also $\int_0^{2\pi} \frac{\partial \gamma_s}{\partial \rho}|_{\Gamma_s}\, \mathrm{d}\theta = 0$). We have

$$
\begin{aligned}
& -\frac{\partial^2\gamma_s}{\partial\rho^2}(\rho,\theta) - \frac{1}{\rho^2}\frac{\partial^2\gamma_s}{\partial\theta^2}(\rho,\theta) - \frac{1}{\rho}\frac{\partial\gamma_s}{\partial\rho}(\rho,\theta) = 0 \\
\Rightarrow \quad & -\int_0^{2\pi}\frac{\partial^2\gamma_s}{\partial\rho^2}(\rho,\theta)\,\mathrm{d}\theta - \int_0^{2\pi}\frac{1}{\rho^2}\frac{\partial^2\gamma_s}{\partial\theta^2}(\rho,\theta)\,\mathrm{d}\theta - \int_0^{2\pi}\frac{1}{\rho}\frac{\partial\gamma_s}{\partial\rho}(\rho,\theta)\,\mathrm{d}\theta = 0 \\
\Rightarrow \quad & -\frac{\partial^2}{\partial\rho^2}\int_0^{2\pi}\gamma_s(\rho,\theta)\,\mathrm{d}\theta - \frac{1}{\rho^2}\left.\frac{\partial\gamma_s}{\partial\theta}(\rho,\theta)\right]_0^{2\pi} - \frac{1}{\rho}\frac{\partial}{\partial\rho}\int_0^{2\pi}\gamma_s(\rho,\theta)\,\mathrm{d}\theta = 0 \\
\Rightarrow \quad & -\frac{\partial^2}{\partial\rho^2}\int_0^{2\pi}\gamma_s(\rho,\theta)\,\mathrm{d}\theta - \frac{1}{\rho}\frac{\partial}{\partial\rho}\int_0^{2\pi}\gamma_s(\rho,\theta)\,\mathrm{d}\theta = 0.
\end{aligned}
$$

Considering $\alpha(\rho) = \int_0^{2\pi}\gamma_s(\rho,\theta)\,\mathrm{d}\theta$, since $\gamma_{s_{|\Gamma_a}} = 0 \Rightarrow \int_0^{2\pi}\gamma_{s_{|\Gamma_a}}\,\mathrm{d}\theta = 0$ and $\int_0^{2\pi}\frac{\partial\gamma_s}{\partial\rho}|_{\Gamma_s}\,\mathrm{d}\theta = \frac{\partial}{\partial\rho}\int_0^{2\pi}\gamma_{s_{|\Gamma_s}}\,\mathrm{d}\theta = 0$, we obtain the two points boundary value problem, $\alpha''(\rho) + \frac{1}{\rho}\alpha'(\rho) = 0$, $\alpha(a) = 0$, $\alpha'(s) = 0$, which has the zero solution. Then, we can conclude that considering $\int_0^{2\pi}\frac{\partial\gamma_s}{\partial\rho}|_{\Gamma_s}\,\mathrm{d}\theta = 0$, we also have $\int_0^{2\pi}\gamma_s(\rho,\theta)\,\mathrm{d}\theta = 0$ for each ρ, and therefore $\int_0^{2\pi}\gamma_{s_{|\Gamma_s}}\,\mathrm{d}\theta$ has the same property, that is, $P(s)h = \gamma_{s_{|\Gamma_s}} \in M$.

■

We can now establish the aimed uniqueness result:

Proposition 2.2.7. *For any $\psi \in N$, there exists a unique solution $\phi \in M$ such that $\psi = P(\varepsilon)\phi + r(\varepsilon)$, for given $r(\varepsilon)$ and $P(\varepsilon)$.*

Proof. Let $\hat{v} \in M$. Then,

$$
\begin{aligned}
& P\phi\hat{v} = \psi\hat{v} - r\hat{v} \\
\Rightarrow \quad & \int_0^{2\pi} P\phi\hat{v}\,\mathrm{d}\theta = \int_0^{2\pi}\psi\hat{v}\,\mathrm{d}\theta - \int_0^{2\pi} r\hat{v}\,\mathrm{d}\theta \\
\Rightarrow \quad & \int_0^{2\pi} P\phi\hat{v}\,\mathrm{d}\theta = \psi\int_0^{2\pi}\hat{v}\,\mathrm{d}\theta - \int_0^{2\pi} r\hat{v}\,\mathrm{d}\theta \quad (\psi \text{ is constant}) \\
\Rightarrow \quad & \int_0^{2\pi} P\phi\hat{v}\,\mathrm{d}\theta = -\int_0^{2\pi} r\hat{v}\,\mathrm{d}\theta \quad (\hat{v} \in M).
\end{aligned}
$$

Considering

$$
a(\phi,\hat{v}) = \int_0^{2\pi} -P\phi\hat{v}\,\mathrm{d}\theta \quad \text{and} \quad (r,\hat{v}) = \int_0^{2\pi} r\hat{v}\,\mathrm{d}\theta
$$

the former equation can be written in the form

$$a(\phi, \hat{v}) = (r, \hat{v}), \text{ with } \phi, \hat{v} \in M. \tag{2.16}$$

It is immediate that we have a bilinear form in the left-hand side of the previous equality, and a linear one in the right-hand side. Furthermore, we have

$$\begin{aligned} \left| \langle P\phi, \hat{v} \rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' \right| &\leq \|P\phi\|_{H^{1/2}_{\rho,P}(0,2\pi)} \|\hat{v}\|_{H^{1/2}_{\rho,P}(0,2\pi)}{}' \\ &\leq c\,\|\phi\|_{H^{1/2}_{\rho,P}(0,2\pi)}{}' \,\|\hat{v}\|_{H^{1/2}_{\rho,P}(0,2\pi)}{}' \end{aligned}$$

where the first inequality is a consequence of Holder's inequality and the second one is a consequence of the continuity of P. Therefore, since the Hilbert space M is closed, $\exists\, c > 0$ such that

$$|a(\phi, \hat{v})| \leq c\,\|\phi\|_M \,\|\hat{v}\|_M$$

and a is continuous. The form a is also coercive because, attending to the negative definiteness of P in ε, $\exists\, c_2 > 0$ such that

$$-\int_0^{2\pi} P\phi.\phi \,\mathrm{d}\theta \geq c_2 \,\|\phi\|_M^2 \,.$$

Further, the linear form is continuous since

$$\begin{aligned} \left| \int_0^{2\pi} r\hat{v}\,\mathrm{d}\theta \right| &\leq \|r\|_{H^{1/2}_{\rho,P}(0,2\pi)} \|\hat{v}\|_M \\ &\leq c\,\|\hat{v}\|_M. \end{aligned}$$

Therefore, according to Lax-Milgram's theorem, there exists a unique solution $\phi \in M$ for the equation (2.16). ■

At this point, we can also conclude that in order to determine the unknown constant $\hat{u}_\varepsilon(\varepsilon)$ of Proposition 2.2.7 we only need to compute the projection $r(\varepsilon)_{|N}$ of $r(\varepsilon)$ over the set N. In fact, we have $\hat{u}_\varepsilon(\varepsilon)_{|N} = \left(P(\varepsilon)\dfrac{\partial \hat{u}_\varepsilon}{\partial \rho}(\varepsilon)\right)_{|N} + r(\varepsilon)_{|N}$. Then, since $P : N \to N$, we obtain $\hat{u}_\varepsilon(\varepsilon)_{|N} = P(\varepsilon)\,\dfrac{\partial \hat{u}_\varepsilon}{\partial \rho}(\varepsilon)_{|N} + r(\varepsilon)_{|N}$. Since $\displaystyle\int_0^{2\pi} \frac{\partial \hat{u}_\varepsilon}{\partial \rho}(\varepsilon)\,\mathrm{d}\theta = 0$, the projection of $\dfrac{\partial \hat{u}_\varepsilon}{\partial \rho}(\varepsilon)$ over the set N is zero and finally we obtain $\hat{u}_\varepsilon(\varepsilon) = r(\varepsilon)_{|N}$.

Therefore, we obtain the following system:

$$\begin{cases} \dfrac{\partial P}{\partial \rho} - \dfrac{1}{\rho^2} P \dfrac{\partial^2}{\partial \theta^2} P - P\dfrac{1}{\rho} - I = 0, \ P(a) = 0 \\ -Pf - P\dfrac{1}{\rho^2}\dfrac{\partial^2 r}{\partial \theta^2} + \dfrac{\partial r}{\partial \rho} = 0, \ r(a) = 0 \\ P\dfrac{\partial \hat{u}_\varepsilon}{\partial \rho} - \hat{u}_\varepsilon = -r, \ \hat{u}_\varepsilon(\varepsilon) = r(\varepsilon)_{|_N}. \end{cases} \tag{2.17}$$

2.3. Semi discretization

We consider $\{w_1, w_2, \ldots, w_n, \ldots\}$ an Hilbert basis of $L^2(0, 2\pi)$ formed by the eigenfunctions of the problem $-\dfrac{d^2 w_i}{d\theta^2} = \lambda_i w_i$ (see Theorem IX.31, pag 192, of [7]), with periodic boundary conditions (that is, $w_i(0) = w_i(2\pi)$ and $\dfrac{\partial w_i}{\partial \theta}(0) = \dfrac{\partial w_i}{\partial \theta}(2\pi)$). This basis satisfies the following properties:

(a) $\forall i, j \in \mathbb{N}, \displaystyle\int_0^{2\pi} \frac{\partial w_i}{\partial \theta}\frac{\partial w_j}{\partial \theta}\, \mathrm{d}\theta = \lambda_i\, \delta_{i,j}$;

(b) $\forall i, j \in \mathbb{N}, \displaystyle\int_0^{2\pi} w_i\, w_j\, \mathrm{d}\theta = \delta_{i,j}$;

(c) The finite linear combinations $\sum \eta_i w_i$ with $\eta_i \in \mathbb{R}$ are a dense subset of $H^1_{\rho,P}(0, 2\pi)$.

Therefore, we have an orthonormal basis of $L^2(0, 2\pi)$ and an orthogonal basis of $H^1_{\rho,P}(0, 2\pi)$. In our particular case, it is easy to prove that the elements of the Hilbert basis have the form $\sin(i\theta)$ or $\cos(i\theta)$. We are going to assume that the eigenvalues verify $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$.

Remark 2.3.1. *The first eigenvector, associated to the zero eigenvalue, is constant. Moreover, since*

$$\begin{aligned} & -\frac{d^2 w_i}{d\theta^2} = \lambda_i w_i \\ \Rightarrow \quad & 0 = -\left.\frac{\partial w_i}{\partial \theta}(\theta)\right|_0^{2\pi} = \lambda_i \int_0^{2\pi} w_i\, \mathrm{d}\theta \\ \Rightarrow \quad & \int_0^{2\pi} w_i\, \mathrm{d}\theta = 0, \ i \geq 2 \end{aligned}$$

we can conclude that all eigenvectors have null mean, excepting the first one.

Using this basis, we can write all $\hat{u}_\varepsilon \in \widehat{U}_\varepsilon$ in the form

$$\hat{u}_\varepsilon(\rho,\theta) = \sum_1^\infty u_i(\rho) w_i(\theta). \tag{2.18}$$

Substituting (2.18) in the norms previously defined and using again the properties of the Hilbert basis, we obtain respectively:

$$\|\hat{u}_\varepsilon(\rho)\|^2_{L^2(0,2\pi)} = \sum_{i=1}^\infty u_i^2,$$

$$\|\hat{u}_\varepsilon(\rho)\|^2_{H^1_{\rho,P}(0,2\pi)} = \sum_{i=1}^\infty \left(1 + \frac{\lambda_i}{\rho^2}\right) u_i^2 = u_1^2 + \sum_{i=2}^\infty \left(1 + \frac{\lambda_i}{\rho^2}\right) u_i^2,$$

$$\|\hat{u}_\varepsilon\|^2_{L^2_\rho(\varepsilon,a;L^2(0,2\pi))} = \int_\varepsilon^a \sum_1^\infty u_i^2 \rho \, \mathrm{d}\rho \tag{2.19}$$

and

$$\|\hat{u}_\varepsilon\|^2_{\widehat{H}_\varepsilon} = \int_\varepsilon^a \sum_1^\infty \left(\rho + \frac{\lambda_i}{\rho}\right) u_i^2 \, \mathrm{d}\rho + \int_\varepsilon^a \sum_1^\infty \rho \left(\frac{\partial u_i}{\partial \rho}\right)^2 \mathrm{d}\rho. \tag{2.20}$$

By interpolation we also have

$$\|\hat{u}_\varepsilon(\rho)\|^2_{H^{1/2}_{\rho,P}(0,2\pi)} = u_1^2 + \sum_{i=2}^\infty \sqrt{1 + \frac{\lambda_i}{\rho^2}} \; u_i^2 \tag{2.21}$$

and

$$\|\hat{u}_\varepsilon(\rho)\|^2_{H^{3/2}_{\rho,P}(0,2\pi)} = u_1^2 + \sum_{i=2}^\infty \sqrt{\left(1 + \frac{\lambda_i}{\rho^2}\right)^3} \; u_i^2. \tag{2.22}$$

It follows some basic properties on the defined norms:

Proposition 2.3.2. $\|\hat{u}_\varepsilon(\rho)\|^2_{H^{1/2}_{\rho,P}(0,2\pi)} \leq \|\hat{u}_\varepsilon(\rho)\|_{H^1_{\rho,P}(0,2\pi)} \|\hat{u}_\varepsilon(\rho)\|_{L^2(0,2\pi)}$.

Proof.

$$\begin{aligned}
\|\hat{u}_\varepsilon\|^2_{H^{1/2}_{\rho,P}} &= \sum_1^\infty \sqrt{1+\frac{\lambda_i}{\rho^2}}\ u_i \cdot u_i \\
&\leq \left(\sum_1^\infty \left(1+\frac{\lambda_i}{\rho^2}\right) u_i^2\right)^{1/2} \left(\sum_1^\infty u_i^2\right)^{1/2} \\
&= \|\hat{u}_\varepsilon\|_{H^1_{\rho,P}(0,2\pi)} \|\hat{u}_\varepsilon\|_{L^2(0,2\pi)}.
\end{aligned}$$

■

Proposition 2.3.3. *The norm*

$$\|\hat{u}_\varepsilon\|^2_{\widehat{H}_\varepsilon} = \int_\varepsilon^a \sum_1^\infty \left(\rho + \frac{\lambda_i}{\rho}\right) u_i^2 \,\mathrm{d}\rho + \int_\varepsilon^a \sum_1^\infty \rho \left(\frac{\partial u_i}{\partial \rho}\right)^2 \mathrm{d}\rho$$

is uniformly equivalent, with respect to ε*, to the norm*

$$\|\hat{u}_\varepsilon\|^2_{\widehat{H}_\varepsilon} = \int_\varepsilon^a \sum_2^\infty \frac{\lambda_i}{\rho} u_i^2 \,\mathrm{d}\rho + \int_\varepsilon^a \sum_1^\infty \rho \left(\frac{\partial u_i}{\partial \rho}\right)^2 \mathrm{d}\rho. \tag{2.23}$$

Proof. Obviously

$$\begin{aligned}
&\int_\varepsilon^a \sum_1^\infty \left(\rho + \frac{\lambda_i}{\rho}\right) u_i^2 \,\mathrm{d}\rho + \int_\varepsilon^a \sum_1^\infty \rho \left(\frac{\partial u_i}{\partial \rho}\right)^2 \mathrm{d}\rho \\
> \ &\int_\varepsilon^a \sum_2^\infty \frac{\lambda_i}{\rho} u_i^2 \,\mathrm{d}\rho + \int_\varepsilon^a \sum_1^\infty \rho \left(\frac{\partial u_i}{\partial \rho}\right)^2 \mathrm{d}\rho.
\end{aligned}$$

On the other hand, using Poincaré's inequality in $\widehat{\Omega} \setminus \widehat{\Omega}_\varepsilon$, $\exists\, c > 0$ such that

$$\|\hat{u}_\varepsilon\|^2_{L^2_\rho(\varepsilon,a;L^2(0,2\pi))} \leq c \|\nabla \hat{u}_\varepsilon\|^2_{L^2_\rho(\varepsilon,a;L^2(0,2\pi))},$$

which means,

$$\int_\varepsilon^a \sum_1^\infty \rho u_i^2 \,\mathrm{d}\rho \leq c \left(\int_\varepsilon^a \sum_2^\infty \frac{\lambda_i}{\rho} u_i^2 \,\mathrm{d}\rho + \int_\varepsilon^a \sum_1^\infty \rho \left(\frac{\partial u_i}{\partial \rho}\right)^2 \mathrm{d}\rho\right).$$

Finally, using this last inequality we have,

$$\begin{aligned}
&\int_\varepsilon^a \sum_1^\infty \rho\, u_i^2\,\mathrm{d}\rho + \int_\varepsilon^a \sum_2^\infty \frac{\lambda_i}{\rho}\, u_i^2\,\mathrm{d}\rho + \int_\varepsilon^a \sum_1^\infty \rho \left(\frac{\partial u_i}{\partial \rho}\right)^2 \mathrm{d}\rho \\
\le\ & c\int_\varepsilon^a \sum_2^\infty \frac{\lambda_i}{\rho} u_i^2\,\mathrm{d}\rho + c\int_\varepsilon^a \sum_1^\infty \rho \left(\frac{\partial u_i}{\partial \rho}\right)^2 \mathrm{d}\rho + \int_\varepsilon^a \sum_2^\infty \frac{\lambda_i}{\rho}\, u_i^2\,\mathrm{d}\rho \\
+\ & \int_\varepsilon^a \sum_1^\infty \rho \left(\frac{\partial u_i}{\partial \rho}\right)^2 \mathrm{d}\rho \\
=\ & (c+1)\left(\int_\varepsilon^a \sum_2^\infty \frac{\lambda_i}{\rho} u_i^2\,\mathrm{d}\rho + \int_\varepsilon^a \sum_1^\infty \rho \left(\frac{\partial u_i}{\partial \rho}\right)^2 \mathrm{d}\rho\right).
\end{aligned}$$

The constants $k_1 = 1$ e $k_2 = c+1$ do not depend on ε . ■

Proposition 2.3.4. *The following pairs of norms are equivalent to each other, uniformly with respect to* ε*:*

$$\|\hat{u}_\varepsilon(\rho)\|^2_{H^1_{\rho,P}(0,2\pi)} = \sum_1^\infty \left(1+\frac{\lambda_i}{\rho^2}\right) u_i^2 \quad \textit{and} \quad \|\hat{u}_\varepsilon(\rho)\|^2_{H^1_{\rho,P}(0,2\pi)} = u_1^2 + \sum_2^\infty \frac{\lambda_i}{\rho^2} u_i^2; \tag{2.24}$$

$$\|\hat{u}_\varepsilon(\rho)\|^2_{H^{1/2}_{\rho,P}(0,2\pi)} = \sum_1^\infty \sqrt{1+\frac{\lambda_i}{\rho^2}}\, u_i^2 \quad \textit{and} \quad \|\hat{u}_\varepsilon(\rho)\|^2_{H^{1/2}_{\rho,P}(0,2\pi)} = u_1^2 + \sum_2^\infty \frac{\sqrt{\lambda_i}}{\rho}\, u_i^2; \tag{2.25}$$

$$\|\hat{u}_\varepsilon(\rho)\|^2_{H^{3/2}_{\rho,P}(0,2\pi)} = \sum_1^\infty \sqrt{\left(1+\frac{\lambda_i}{\rho^2}\right)^3}\, u_i^2 \quad \textit{and} \quad \|\hat{u}_\varepsilon(\rho)\|^2_{H^{3/2}_{\rho,P}(0,2\pi)} = u_1^2 + \sum_2^\infty \frac{\lambda_i^{3/2}}{\rho^3}\, u_i^2. \tag{2.26}$$

The pair of semi-norms

$$\left\|\frac{\partial \hat{u}_\varepsilon}{\partial \theta}(\rho)\right\|^2_{H^1_{\rho,P}(0,2\pi)} = \sum_2^\infty \left(\lambda_i + \left(\frac{\lambda_i}{\rho}\right)^2\right) u_i^2 \ \textit{and} \ \left\|\frac{\partial \hat{u}_\varepsilon}{\partial \theta}(\rho)\right\|^2_{H^1_{\rho,P}(0,2\pi)} = \sum_2^\infty \left(\frac{\lambda_i}{\rho}\right)^2 u_i^2 \tag{2.27}$$

are also uniformly equivalent, with respect to ε*.*

Proof. Obviously

$$\sum_1^\infty \left(1+\frac{\lambda_i}{\rho^2}\right)u_i^2 \;=\; u_1^2+\sum_2^\infty u_i^2+\frac{1}{\rho^2}\sum_2^\infty \lambda_i\,u_i^2 \geq u_1^2+\frac{1}{\rho^2}\sum_2^\infty \lambda_i\,u_i^2.$$

On the other hand,

$$\begin{aligned}\sum_2^\infty u_i^2 &= \frac{1}{\lambda_2}\sum_2^\infty \lambda_2 u_i^2\\ &\leq \frac{1}{\lambda_2}\sum_2^\infty \lambda_i u_i^2 \quad (\lambda_2\leq\lambda_i,\ \forall i\geq 3)\\ &\leq \frac{a^2}{\lambda_2}\sum_2^\infty \frac{\lambda_i}{\rho^2}u_i^2 \quad (\text{since } \rho<a).\end{aligned}$$

Finally, using this last inequality, we have

$$\begin{aligned}&u_1^2+\sum_2^\infty u_i^2+\frac{1}{\rho^2}\sum_2^\infty \lambda_i\,u_i^2 \leq u_1^2+\frac{a^2}{\lambda_2}\sum_2^\infty \frac{\lambda_i}{\rho^2}u_i^2+\sum_2^\infty \frac{\lambda_i}{\rho^2}\,u_i^2\\ &\leq \left(1+\frac{a^2}{\lambda_2}\right)\left(u_1^2+\frac{1}{\rho^2}\sum_2^\infty \lambda_i\,u_i^2\right),\end{aligned}$$

which completes the proof of (2.24). The constants $k_1=1$ e $k_2=1+\dfrac{a^2}{\lambda_2}$ do not depend on ε.

The equivalences (2.25), (2.26) and (2.27) can be obtained similarly. The equivalence constants, which are respectively $k_1'=1, k_2'=1+\dfrac{a}{\sqrt{\lambda_2}}$, $k_1''=1, k_2''=\dfrac{a^3}{\lambda_2^{3/2}}+\dfrac{\sqrt{3}a^2}{\lambda_2}+\dfrac{\sqrt{3}a}{\sqrt{\lambda_2}}+1$ and $k_1'''=1, k_2'''=\dfrac{a^2}{\lambda_2}+1$ do not depend on ε. ■

Remark 2.3.5. *In order to have* $<.\,,.>_{H^{1/2}_{\rho,P}(0,2\pi)}{}'{}_{,H^{1/2}_{\rho,P}(0,2\pi)}=(.\,,.)_{L^2(0,2\pi)}$ *(whenever this last inner product makes sense), we define*

$$\|\hat{u}_\varepsilon(\rho)\|^2_{H^{1/2}_{\rho,P}(0,2\pi)}{}'=u_1^2+\sum_2^\infty \frac{\rho}{\sqrt{\lambda_i}}\,u_i^2.$$

2.4. Finite dimension

With the purpose of establishing an approximation of $\hat{u}_\varepsilon$, the solution of (2.3), in the framework of the last section, we define $\widehat{U}_\varepsilon^m = \left\{ v \in H_\rho^1(\varepsilon, a; V^m) : v_{|\Gamma_a} = 0,\ v_{|\Gamma_\varepsilon} \text{constant} \right\}$, where $V^m = \langle w_1, \dots, w_n \rangle$. Then, the approximation $\hat{u}_\varepsilon^m \in \widehat{U}_\varepsilon^m$ of $\hat{u}_\varepsilon$ is the solution of

$$\begin{cases} \hat{u}_\varepsilon^m \in \widehat{U}_\varepsilon^m \\ \displaystyle\int_\varepsilon^a \int_0^{2\pi} \left(\frac{\partial \hat{u}_\varepsilon^m}{\partial \rho} \frac{\partial \hat{v}_\varepsilon^m}{\partial \rho} \rho + \frac{1}{\rho} \frac{\partial \hat{u}_\varepsilon^m}{\partial \theta} \frac{\partial \hat{v}_\varepsilon^m}{\partial \theta} \right) \mathrm{d}\theta \, \mathrm{d}\rho = \int_\varepsilon^a \int_0^{2\pi} f \hat{v}_\varepsilon^m \rho \, \mathrm{d}\theta \, \mathrm{d}\rho,\ \forall \hat{v}_\varepsilon^m \in \widehat{U}_\varepsilon^m. \end{cases}$$

Obviously, we can also define the approximation $\hat{u}^m$ of $\hat{u}$ (see (2.4)) by the solution of

$$\begin{cases} \hat{u}^m \in \widehat{U}_0^m = \left\{ v \in H_\rho^1(0, a; V^m) : v_{|\Gamma_a} = 0,\ v_{|\Gamma_0} \text{ constant} \right\} \\ \displaystyle\int_0^a \int_0^{2\pi} \left(\frac{\partial \hat{u}^m}{\partial \rho} \frac{\partial \hat{v}^m}{\partial \rho} \rho + \frac{1}{\rho} \frac{\partial \hat{u}^m}{\partial \theta} \frac{\partial \hat{v}^m}{\partial \theta} \right) \mathrm{d}\theta \, \mathrm{d}\rho = \int_0^a \int_0^{2\pi} f \hat{v}^m \rho \, \mathrm{d}\theta \, \mathrm{d}\rho,\ \forall \hat{v}^m \in \widehat{U}_0^m. \end{cases} \tag{2.29}$$

Since we can write all $\hat{u}_\varepsilon^m \in \widehat{U}_\varepsilon^m$ in the form

$$\hat{u}_\varepsilon^m(\rho, \theta) = \sum_1^m u_i(\rho) w_i(\theta), \tag{2.30}$$

and we have $\hat{u}_\varepsilon^m(a, \theta) = \sum_{i=1}^m u_i(a) w_i(\theta) = 0$, we can conclude that $u_i(a) = 0$, for $i = 1, \dots, m$. In the same way, from the initial condition $\hat{u}_\varepsilon^m(\varepsilon, \theta) = \sum_{i=1}^m u_i(\varepsilon) w_i(\theta)$ constant, since w_1 is constant, we also obtain $u_i(\varepsilon) = 0, i \geq 2$. Furthermore, from the initial condition

$$\int_0^{2\pi} \frac{\partial \hat{u}_\varepsilon^m}{\partial \rho}(\varepsilon, \theta) \, \mathrm{d}\theta = \int_0^{2\pi} \frac{\partial \sum_{i=1}^m u_i(\varepsilon) w_i(\theta)}{\partial \rho} \, \mathrm{d}\theta = \sum_{i=1}^m \frac{\partial u_i}{\partial \rho}(\varepsilon) \int_0^{2\pi} w_i(\theta) \, \mathrm{d}\theta = 0,$$

and Remark 2.3.1 we can conclude that $\frac{\partial u_1}{\partial \rho}(\varepsilon) \int_0^{2\pi} w_1(\theta) \, \mathrm{d}\theta = 0$ and consequently, $\frac{\partial u_1}{\partial \rho}(\varepsilon) = 0$, since the first eigenvector is constant.

For $\hat{u}_\varepsilon^m, \hat{v}_\varepsilon^m$ of the form (2.30), using the properties of the Hilbert basis and this initial conditions, from (2.28) we obtain successively:

$$\int_0^{2\pi}\int_\varepsilon^a \left[\frac{\partial}{\partial\rho}\left(\sum_{i=1}^m u_i(\rho)w_i(\theta)\right)\frac{\partial}{\partial\rho}\left(\sum_{j=1}^m v_j(\rho)w_j(\theta)\right)\rho\right.$$
$$\left.+\frac{1}{\rho}\frac{\partial}{\partial\theta}\left(\sum_{i=1}^m u_i(\rho)w_i(\theta)\right)\frac{\partial}{\partial\theta}\left(\sum_{j=1}^m v_j(\rho)w_j(\theta)\right)\right]\mathrm{d}\rho\,\mathrm{d}\theta$$
$$=\int_0^{2\pi}\int_\varepsilon^a f\sum_{j=1}^m v_j(\rho)w_j(\theta)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$$
$$\Leftrightarrow\quad \sum_{i,j=1}^m\int_0^{2\pi} w_i(\theta)w_j(\theta)\,\mathrm{d}\theta\int_\varepsilon^a \frac{\partial u_i}{\partial\rho}(\rho)\frac{\partial v_j}{\partial\rho}(\rho)\rho\,\mathrm{d}\rho$$
$$+\sum_{i,j=1}^m\int_0^{2\pi}\frac{\partial w_i}{\partial\theta}(\theta)\frac{\partial w_j}{\partial\theta}(\theta)\,\mathrm{d}\theta\int_\varepsilon^a\frac{1}{\rho}\,u_i(\rho)v_j(\rho)\,\mathrm{d}\rho$$
$$=\sum_{j=1}^m\int_\varepsilon^a\left(\int_0^{2\pi} fw_j(\theta)\rho\,\mathrm{d}\theta\right)v_j(\rho)\,\mathrm{d}\rho$$
$$\Leftrightarrow\quad \sum_{i=1}^m\int_\varepsilon^a\frac{\partial u_i}{\partial\rho}(\rho)\frac{\partial v_i}{\partial\rho}(\rho)\rho\,\mathrm{d}\rho+\sum_{i=1}^m\lambda_i\int_\varepsilon^a\frac{1}{\rho}\,u_i(\rho)v_i(\rho)\,\mathrm{d}\rho$$
$$=\sum_{i=1}^m\int_\varepsilon^a\left(\int_0^{2\pi} fw_i(\theta)\rho\,\mathrm{d}\theta\right)v_i(\rho)\,\mathrm{d}\rho$$
$$\Leftrightarrow\quad \sum_{i=1}^m\left[v_i(\rho)\frac{\partial u_i}{\partial\rho}(\rho)\rho\Big|_\varepsilon^a-\int_\varepsilon^a\left(\frac{\partial^2u_i}{\partial\rho^2}(\rho)\rho+\frac{\partial u_i}{\partial\rho}(\rho)\right)v_i(\rho)\,\mathrm{d}\rho\right]+\sum_{i=1}^m\int_\varepsilon^a\frac{\lambda_i}{\rho}\,u_i(\rho)v_i(\rho)\,\mathrm{d}\rho$$
$$=\sum_{i=1}^m\int_\varepsilon^a\left(\int_0^{2\pi} fw_i(\theta)\rho\,\mathrm{d}\theta\right)v_i(\rho)\,\mathrm{d}\rho$$
$$\Leftrightarrow\quad \sum_{i=1}^m\int_\varepsilon^a\left(-\frac{\partial^2u_i}{\partial\rho^2}(\rho)\rho-\frac{\partial u_i}{\partial\rho}(\rho)\right)v_i(\rho)\,\mathrm{d}\rho+\sum_{i=1}^m\int_\varepsilon^a\left(\frac{\lambda_i}{\rho}\,u_i(\rho)\right)v_i(\rho)\,\mathrm{d}\rho$$
$$=\sum_{i=1}^m\int_\varepsilon^a\left(\int_0^{2\pi} fw_i(\theta)\rho\,\mathrm{d}\theta\right)v_i(\rho)\,\mathrm{d}\rho.$$

Consequently, the coordinates $\{u_i(\rho)\}_{i=1}^m$ of $\hat{u}_\varepsilon^m$ must verify the following system:

$$\begin{cases} -\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial u_i(\rho)}{\partial\rho}\right)+\dfrac{\lambda_i}{\rho^2}u_i(\rho) \\ =\displaystyle\int_0^{2\pi} fw_i(\theta)\,\mathrm{d}\theta=\hat{f}_i(\rho),\ \varepsilon<\rho<a, i=1,\dots,m \\ u_i(a)=0,\ i=1,\dots,m \\ u_i(\varepsilon)=0,\ i=2,\dots,m \\ \dfrac{\partial u_1}{\partial\rho}(\varepsilon)=0. \end{cases}\tag{2.31}$$

Once again we are going to embed the problem (in this case the approximated problem (2.28)) in a family of problems depending on h^m and s. For all $s \in [\varepsilon, a)$ we consider the finite dimension approximation defined on $\widehat{\Omega}\backslash\widehat{\Omega}_s = (s,a)\times(0,2\pi)$ and, for each problem, we impose the boundary condition $\dfrac{\partial \hat{u}_s^m}{\partial \rho}(s) = h^m$. We define $\widehat{U}_s^m = \left\{ v \in H^1_\rho(s,a;V^m) : v_{|\Gamma_a} = 0 \right\}$ and denote by $\beta_s^m, \gamma_s^m \in \widehat{U}_s^m$, respectively, the part of $\hat{u}_s^m$ independent on h^m and linearly dependent on h^m, that is, we define the finite dimension operator $P^m(s)$ by $\gamma_s^m(s) = P^m(s)h^m$ and fix $P^m(a) = 0$; we also define $r^m(s) = \beta_s^m(s)$ and fix $r^m(a) = 0$.

As before, for every $s \in [\varepsilon, a]$, $P^m(s) : V^m \to V^m$ (on which we consider in the first set the norm of $\left(H^{1/2}_{\rho,P}(0,2\pi)\right)'$ and in the second one the norm of $H^{1/2}_{\rho,P}(0,2\pi)$) and is a linear operator and $r^m(s) \in V^m$. Then we have

$$\hat{u}^m_{s_{|\Gamma_s}} = P^m(s)\frac{\partial \hat{u}_s^m}{\partial \rho}|_{\Gamma_s} + r^m(s), \forall s \in [\varepsilon, a]. \tag{2.32}$$

Furthermore, the solution $\hat{u}_\varepsilon^m$ of (2.28) is given by

$$\hat{u}_\varepsilon^m(\rho,\theta) = (P^m(\rho)\frac{\partial \hat{u}_\varepsilon^m}{\partial \rho}|_{\Gamma_\rho})(\theta) + (r^m(\rho))(\theta). \tag{2.33}$$

From the last equality we can easily derive the following system:

$$\begin{cases} \dfrac{\partial P^m}{\partial \rho} - \dfrac{1}{\rho^2}P^m\dfrac{\partial^2}{\partial \theta^2}P^m - \dfrac{1}{\rho}P^m - I = 0, \ P^m(a) = 0 \\ -P^m f^m - P^m\dfrac{1}{\rho^2}\dfrac{\partial^2 r^m}{\partial \theta^2} + \dfrac{\partial r^m}{\partial \rho} = 0, \ r^m(a) = 0 \\ P^m\dfrac{\partial \hat{u}_\varepsilon^m}{\partial \rho} - \hat{u}_\varepsilon^m = -r^m, \ \hat{u}_\varepsilon^m(\varepsilon) = r^m(\varepsilon)_{|N}, \end{cases} \tag{2.34}$$

where $f^m = \sum_{i=1}^m \hat{f}_i(\rho)w_i(\theta)$. In fact, from (2.33), taking the formal derivative with respect to ρ, we obtain

$$\begin{aligned} &\frac{\partial u_\varepsilon^m}{\partial \rho} = \frac{\partial P^m}{\partial \rho}\frac{\partial u_\varepsilon^m}{\partial \rho} + P^m\frac{\partial^2 u_\varepsilon^m}{\partial \rho^2} + \frac{\partial r^m}{\partial \rho} \\ \Rightarrow &\int_0^{2\pi} \frac{\partial u_\varepsilon^m}{\partial \rho}w_j \,\mathrm{d}\theta = \int_0^{2\pi} \frac{\partial P^m}{\partial \rho}\frac{\partial u_\varepsilon^m}{\partial \rho}w_j \,\mathrm{d}\theta + \int_0^{2\pi} \frac{\partial^2 u_\varepsilon^m}{\partial \rho^2}P^m w_j \,\mathrm{d}\theta + \int_0^{2\pi} \frac{\partial r^m}{\partial \rho}w_j \,\mathrm{d}\theta. \end{aligned}$$

Since

$$\begin{aligned}
&\int_0^{2\pi} \frac{\partial^2 u_\varepsilon^m}{\partial\rho^2} P^m w_j \,\mathrm{d}\theta \\
=\;& \int_0^{2\pi} \sum_{i=1}^m \left(\frac{1}{\rho^2}\lambda_i u_i(\rho) w_i(\theta) - \frac{1}{\rho}\frac{\partial u_i}{\partial\rho}(\rho) w_i(\theta) - w_i(\theta) \int_0^{2\pi} f w_i(\theta)\,\mathrm{d}\theta \right) P^m w_j \,\mathrm{d}\theta \\
=\;& \int_0^{2\pi} \sum_{i=1}^m \left(\frac{1}{\rho^2}\lambda_i u_i(\rho) w_i(\theta) \right) P^m w_j \,\mathrm{d}\theta - \int_0^{2\pi} \sum_{i=1}^m \left(\frac{1}{\rho}\frac{\partial u_i}{\partial\rho}(\rho) w_i(\theta) \right) P^m w_j \,\mathrm{d}\theta \\
&- \int_0^{2\pi} \sum_{i=1}^m \left(w_i(\theta) \int_0^{2\pi} f w_i(\theta)\,\mathrm{d}\theta \right) P^m w_j \,\mathrm{d}\theta \\
=\;& -\int_0^{2\pi} \sum_{i=1}^m \frac{1}{\rho^2} u_i(\rho) \frac{\partial^2 w_i(\theta)}{\partial\theta^2} P^m w_j \,\mathrm{d}\theta - \int_0^{2\pi} \frac{1}{\rho}\frac{\partial u_\varepsilon^m}{\partial\rho} P^m w_j \,\mathrm{d}\theta \\
&- \int_0^{2\pi} f^m P^m w_j \,\mathrm{d}\theta \\
=\;& -\int_0^{2\pi} \frac{1}{\rho^2}\frac{\partial^2 u_\varepsilon^m}{\partial\theta^2} P^m w_j \,\mathrm{d}\theta - \int_0^{2\pi} \frac{1}{\rho}\frac{\partial u_\varepsilon^m}{\partial\rho} P^m w_j \,\mathrm{d}\theta - \int_0^{2\pi} f^m P^m w_j \,\mathrm{d}\theta \\
=\;& -\int_0^{2\pi} \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} \left(P^m \frac{\partial u_\varepsilon^m}{\partial\rho} + r^m \right) P^m w_j \,\mathrm{d}\theta - \int_0^{2\pi} \frac{1}{\rho}\frac{\partial u_\varepsilon^m}{\partial\rho} P^m w_j \,\mathrm{d}\theta \\
&- \int_0^{2\pi} f^m P^m w_j \,\mathrm{d}\theta \\
=\;& -\int_0^{2\pi} \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} P^m \frac{\partial u_\varepsilon^m}{\partial\rho} P^m w_j \,\mathrm{d}\theta - \int_0^{2\pi} \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} r^m P^m w_j \,\mathrm{d}\theta \\
&- \int_0^{2\pi} \frac{1}{\rho}\frac{\partial u_\varepsilon^m}{\partial\rho} P^m w_j \,\mathrm{d}\theta - \int_0^{2\pi} f^m P^m w_j \,\mathrm{d}\theta,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\int_0^{2\pi} \frac{\partial u_\varepsilon^m}{\partial\rho} w_j \,\mathrm{d}\theta \\
=& \int_0^{2\pi} \frac{\partial P^m}{\partial\rho}\frac{\partial u_\varepsilon^m}{\partial\rho} w_j \,\mathrm{d}\theta - \int_0^{2\pi} \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} P^m \frac{\partial u_\varepsilon^m}{\partial\rho} P^m w_j \,\mathrm{d}\theta - \int_0^{2\pi} \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} r^m P^m w_j \,\mathrm{d}\theta \\
&- \int_0^{2\pi} \frac{1}{\rho}\frac{\partial u_\varepsilon^m}{\partial\rho} P^m w_j \,\mathrm{d}\theta - \int_0^{2\pi} f^m P^m w_j \,\mathrm{d}\theta + \int_0^{2\pi} \frac{\partial r^m}{\partial\rho} w_j \,\mathrm{d}\theta.
\end{aligned}$$

Now, from the equality

$$\begin{aligned}
&\int_0^{2\pi} \frac{\partial u_\varepsilon^m}{\partial\rho} w_j \,\mathrm{d}\theta \\
=& \int_0^{2\pi} \left(\frac{\partial P^m}{\partial\rho}\frac{\partial u_\varepsilon^m}{\partial\rho} - P^m \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} P^m \frac{\partial u_\varepsilon^m}{\partial\rho} - P^m \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} r^m - P^m \frac{1}{\rho}\frac{\partial u_\varepsilon^m}{\partial\rho} - P^m f^m + \frac{\partial r^m}{\partial\rho} \right) w_j \mathrm{d}\theta
\end{aligned}$$

follows the desired result, as in (2.15), since $\frac{\partial u_\varepsilon^m}{\partial\rho}$ is arbitrary.

The fact that $\dfrac{\partial u_\varepsilon^m}{\partial\rho}|_{\Gamma_\rho}$ is arbitrary can be easily achieved through the following observation:

Remark 2.4.1. *Using the equation on $u_i(\rho)$ (with $f = 0$) of (2.31), we consider the system*

$$\begin{cases} \rho^2 u_i''(\rho) + \rho\, u_i'(\rho) - \lambda_i\, u_i(\rho) = 0 \\ u_i'(s) = h_i \\ u_i(a) = 0 \end{cases}$$

which has a solution of the form $u_i(\rho) = c_1 \rho^{\sqrt{\lambda_i}} + c_2 \rho^{-\sqrt{\lambda_i}}$. On determining the constants c_1 and c_2 we find $u_i(\rho) = h_i \dfrac{s}{\sqrt{\lambda_i}} \left(\dfrac{\left(\frac{\rho}{a}\right)^{\sqrt{\lambda_i}} - \left(\frac{a}{\rho}\right)^{\sqrt{\lambda_i}}}{\left(\frac{s}{a}\right)^{\sqrt{\lambda_i}} + \left(\frac{a}{s}\right)^{\sqrt{\lambda_i}}} \right)$ and $u_i'(\rho) = h_i \dfrac{s}{\rho} \left(\dfrac{\left(\frac{\rho}{a}\right)^{\sqrt{\lambda_i}} + \left(\frac{a}{\rho}\right)^{\sqrt{\lambda_i}}}{\left(\frac{s}{a}\right)^{\sqrt{\lambda_i}} + \left(\frac{a}{s}\right)^{\sqrt{\lambda_i}}} \right)$, which means that, being h_i arbitrary, $u_i'(\rho)$ is also arbitrary.

From now on, we will denote by Λ the diagonal matrix formed by the eigenvalues λ_i, $i = 1, \ldots, m$. To go further we need to discuss the existence and uniqueness of a local solution for the system (2.34):

Proposition 2.4.2. *The system*

$$\begin{cases} \dfrac{\partial P^m}{\partial \rho} = \dfrac{1}{\rho^2} P^m \Lambda P^m + \dfrac{1}{\rho} P^m + I, \; P^m(a) = 0 \\ \dfrac{\partial r^m}{\partial \rho} + \left(-P^m \dfrac{1}{\rho^2} \Lambda \right) r^m = P^m f^m, \; r^m(a) = 0 \end{cases}$$

has a unique local solution in $[a-\alpha, a]$, for a certain $\alpha > 0$. Moreover, $P^m \in C^1([a-\alpha, a]; \mathcal{L}(V^m, V^m))$ and $r^m \in H^1((a-\alpha, a); V^m)$.

Proof. The function $F(P^m, \rho) = \dfrac{1}{\rho^2} P^m \Lambda P^m + \dfrac{1}{\rho} P^m + I$ is bounded on the rectangle $|\rho - a| \le b_1$, $\|P^m\| \le b_2$, with $b_1 = a - \varepsilon$ and for any fixed constant b_2. Let $M = \max \|F(P^m, \rho)\|$ on this rectangle. Further,

$$\begin{aligned}
& \|F(P_1^m, \rho) - F(P_2^m, \rho)\| \\
= & \left\| \frac{1}{\rho^2} P_1^m \Lambda P_1^m + \frac{1}{\rho} P_1^m - \frac{1}{\rho^2} P_2^m \Lambda P_2^m - \frac{1}{\rho} P_2^m \right\| \\
= & \left\| \frac{1}{\rho^2} P_1^m \Lambda P_1^m - \frac{1}{\rho^2} P_1^m \Lambda P_2^m + \frac{1}{\rho^2} P_1^m \Lambda P_2^m - \frac{1}{\rho^2} P_2^m \Lambda P_2^m + \frac{1}{\rho} P_1^m - \frac{1}{\rho} P_2^m \right\| \\
= & \left\| \frac{1}{\rho^2} P_1^m \Lambda \left(P_1^m - P_2^m\right) + \left(P_1^m - P_2^m\right) \frac{1}{\rho^2} \Lambda P_2^m + \frac{1}{\rho} \left(P_1^m - P_2^m\right) \right\| \\
\le & \left(\frac{1}{\rho^2} \|P_1^m\| \|\Lambda\| + \frac{1}{\rho^2} \|\Lambda\| \|P_2^m\| + \frac{1}{\rho} \right) \|P_1^m - P_2^m\| \\
\le & \left(\frac{2\, b_2}{\varepsilon^2} \|\Lambda\| + \frac{1}{\varepsilon} \right) \|P_1^m - P_2^m\|
\end{aligned}$$

and since the constant $\frac{2\,b_2}{\varepsilon^2}\|\Lambda\| + \frac{1}{\varepsilon}$ is independent of ρ, the function $F(P^m, \rho)$ is uniformly Lipschitzian (with respect to P^m) on the rectangle. Also, $\frac{1}{\rho^2}P^m\Lambda P^m + \frac{1}{\rho}P^m + I$, as a function of P^m and ρ, is continuous.

Therefore, from the theory of ordinary differential equations (see Theorem 2.3 of [10], page 10), there exists a unique local solution P^m to

$$\begin{cases} \dfrac{\partial P^m}{\partial \rho} = \dfrac{1}{\rho^2}P^m\Lambda P^m + \dfrac{1}{\rho}P^m + I \\ P^m(a) = 0 \end{cases} \tag{2.35}$$

on $|\rho - a| \leq \alpha$, $\alpha = \min\left(a - \varepsilon, \frac{b_2}{M}\right)$. Moreover, P^m is C^1 from $[a - \alpha, a]$, with values in $\mathcal{L}(V^m, V^m)$.

Thus, since $-P^m\frac{1}{\rho^2}\Lambda$ and $P^m f^m$ are continuous (again, as functions of P^m), from the theory of nonhomogeneous linear systems, there exists also a unique local solution r^m to

$$\begin{cases} \dfrac{\partial r^m}{\partial \rho} + \left(-P^m\dfrac{1}{\rho^2}\Lambda\right) r^m = P^m f^m \\ r^m(a) = 0 \end{cases}$$

in $[a - \alpha, a]$. Moreover, $r^m \in H^1(a - \alpha, a; V^m)$. ■

Since in the equation

$$\frac{\partial P^m}{\partial \rho} = \frac{1}{\rho^2}P^m\Lambda P^m + \frac{1}{\rho}P^m + I \tag{2.36}$$

the matrices Λ and I are diagonal, is natural that this equation has a diagonal solution. We are going to suppose that this is the case. Then, denoting by p_i the $i \times i$ - component of the P^m matrix, we conclude that the coordinates of P^m must satisfy, for $i \geq 1$, the equation

$$\frac{\partial p_i}{\partial \rho}(\rho) + p_i^2(\rho)\frac{1}{\rho^2}\lambda_i - \frac{p_i(\rho)}{\rho} - 1 = 0, \tag{2.37}$$

with $p_i(a) = 0$, $i \geq 1$. Then, for $i \geq 2$, taking $\frac{p_i}{\rho} = q_i$, we obtain

$$\frac{\partial q_i}{\partial \rho}\rho + q_i^2\lambda_i - 1 = 0,\ q_i(a) = 0$$

and, making the change of variables $\varphi = \log\rho$, follows

$$\frac{\partial \mathsf{q}_i}{\partial \varphi} + (\mathsf{q}_i)^2\,\lambda_i - 1 = 0,\ \mathsf{q}_i(\log(a)) = 0.$$

One can show, using the method of separable variables, that this last equation has the solution

$$\mathsf{q}_i(\varphi) = -\frac{1}{\sqrt{\lambda_i}} \frac{a^{2\sqrt{\lambda_i}} e^{-2\sqrt{\lambda_i}\varphi} - 1}{a^{2\sqrt{\lambda_i}} e^{-2\sqrt{\lambda_i}\varphi} + 1}. \tag{2.38}$$

Consequently,

$$q_i(\rho) = -\frac{1}{\sqrt{\lambda_i}} \frac{\left(\frac{a}{\rho}\right)^{2\sqrt{\lambda_i}} - 1}{\left(\frac{a}{\rho}\right)^{2\sqrt{\lambda_i}} + 1}$$

and

$$p_i(\rho) = -\frac{\rho}{\sqrt{\lambda_i}} \left(\frac{\left(\frac{a}{\rho}\right)^{\sqrt{\lambda_i}} - \left(\frac{\rho}{a}\right)^{\sqrt{\lambda_i}}}{\left(\frac{a}{\rho}\right)^{\sqrt{\lambda_i}} + \left(\frac{\rho}{a}\right)^{\sqrt{\lambda_i}}} \right). \tag{2.39}$$

For $i = 1$, the equation (2.37) simply becomes $\frac{\partial p_1}{\partial \rho} - \frac{p_1}{\rho} = 1$, since $\lambda_1 = 0$. Taking $q_1 = \frac{p_1}{\rho}$ we obtain $\frac{\partial q_1}{\partial \rho}\rho = 1$, which can be integrated as an equation of separable variables, obtaining $q_1 = \log\rho + c$, where c is an arbitrary constant. Then, $p_1 = \rho\log\rho + c\rho$ and since $p_1(a) = 0$ we can determine c (as $-\log a$) and conclude that

$$p_1(\rho) = \rho \log\left(\frac{\rho}{a}\right). \tag{2.40}$$

It is now easy to see that the diagonal matrix formed by this $(p_i(\rho))$, $i \geq 1$ is in fact a solution of the equation (2.36). Since we have seen, in Proposition 2.4.2, that (2.35) has a unique local solution, we can therefore conclude that we have found that solution, at least on the interval $[a-\alpha, a]$. Furthermore, taking $\|P\| = \sum_{i=1}^{m} |p_i|$ we have

$$\|P\| < \left|\rho\log\left(\frac{\rho}{a}\right)\right| + \sum_{i=2}^{m} \frac{\rho}{\sqrt{\lambda_i}} < \frac{a}{e} + \frac{(m-1)\,a}{\sqrt{\lambda_2}} = b_2',\ \forall\rho \in [a-\alpha, a]. \tag{2.41}$$

Next, we present a very important result on the operator P^m:

Proposition 2.4.3. *The system (2.35) has a unique global solution on (ε, a).*

Proof. With the previous reasoning we have presented an explicit solution of the equation (2.35), defined on the interval (ε, a). Nevertheless, we must prove that this solution is unique.

We already know that we have local uniqueness on $|\rho - a| \leq \alpha$, $\alpha = \min\left(a - \varepsilon, \dfrac{b_2}{M}\right)$, where $M = \max|F(P^m, \rho)|$, on the rectangle $|\rho - a| \leq a - \varepsilon$, $\|P^m\| \leq b_2$. If $\alpha < a - \varepsilon$, we know by (2.41) that $\|P^m(\alpha)\| \leq b_2'$. We can repeat the reasoning of the proof of Proposition 2.4.2 for the rectangle $|\rho - (a - \alpha)| < a - \alpha - \varepsilon$, $\|P^m - P^m(\alpha)\| \leq b_2'$ and get $M' = \max\|F(P^m, \rho)\| \leq \dfrac{4}{\varepsilon^2}(b_2')^2\|\Lambda\| + \dfrac{2}{\varepsilon}b_2' + 1$. Then, $\alpha_1 = \min(a - \alpha - \varepsilon, \dfrac{b_2'}{M'})$. By Proposition 2.4.2 we have a unique solution of (2.35) in $[a - \alpha - \alpha_1, a - \alpha]$. We remark that we can repeat the process, with the same constants, as many times as we need and so, we have a unique solution of (2.35) in $[\varepsilon, a]$. ■

In the justification presented above, it is also possible to present a Lipschitz constant which is independent of ε. In fact, with $\dfrac{P}{\rho} = Q$, we obtain (remark that P^m and Λ are diagonal)

$$\begin{aligned} &\frac{\partial P^m}{\partial \rho} - \frac{1}{\rho^2}P^m\Lambda P^m - \frac{1}{\rho}P^m - I = \frac{\partial P^m}{\partial \rho} - \frac{1}{\rho^2}\Lambda(P^m)^2 - \frac{1}{\rho}P^m - I = 0 \\ \Rightarrow \quad &\rho\frac{\partial Q^m}{\partial \rho} - \Lambda(Q^m)^2 - I = 0, \end{aligned}$$

and using the change of variables $\varphi = \log\rho$, we get $\dfrac{\partial Q^m}{\partial \varphi} = \Lambda(Q^m)^2 + I$. Thus, with $G(Q^m, \varphi) = \Lambda(Q^m)^2 + I$, $\varphi \in (-\infty, \log a]$, we obtain

$$\begin{aligned} \|G(Q_1^m, \varphi) - G(Q_2^m, \varphi)\| &= \|\Lambda(Q_1^m)^2 - \Lambda(Q_2^m)^2\| \\ &= \|\Lambda\|\|Q_1^m + Q_2^m\|\|Q_1^m - Q_2^m\|. \end{aligned}$$

From (2.38) we have $|\mathsf{q}_i(\varphi)| < \dfrac{1}{\sqrt{\lambda_i}}$, for $i \geq 2$. Also, we see easily that $\mathsf{q}_1(\varphi) = \varphi - \log a$. Thus, $\|Q^m\| = \displaystyle\sum_{i=1}^{m}|\mathsf{q}_i| < 2\log a + \sum_{i=2}^{m}\frac{1}{\sqrt{\lambda_i}}$ and

$$\|\Lambda\|\|Q_1^m + Q_2^m\| < \left(\sum_{i=1}^{m}\lambda_i\right)\left(4\log a + 2\sum_{i=2}^{m}\frac{1}{\sqrt{\lambda_i}}\right),$$

which is constant, since m is finite. Then,

$$\|F(P_1^m, \rho) - F(P_2^m, \rho)\| = \|G(Q_1^m, \varphi) - G(Q_2^m, \varphi)\| < c\|Q_1^m - Q_2^m\|, \forall\rho \in (0, a].$$

2.5. On the definition of $\hat{u}^m(0)$

From the equation $\hat{u}_\varepsilon^m(\varepsilon) = P^m(\varepsilon)\dfrac{\partial \hat{u}_\varepsilon^m}{\partial \rho}(\varepsilon) + r^m(\varepsilon)$, using (2.30) and the initial conditions on (2.31), we obtain

$$
\begin{aligned}
\hat{u}_\varepsilon^m(\varepsilon,\theta) &= \sum_{i=1}^{m} u_i(\varepsilon)w_i(\theta) = u_1(\varepsilon)w_1(\theta) \\
&= \left(P^m(\varepsilon)\frac{\partial u_1}{\partial \rho}(\varepsilon) + r_1(\varepsilon)\right) w_1(\theta) \\
&= r_1(\varepsilon)w_1(\theta) = \frac{1}{\sqrt{2\pi}} r_1(\varepsilon),
\end{aligned}
\tag{2.42}
$$

since $P^m\,0 = 0$ (P^m is linear).

This way, to determine the constant $\hat{u}_\varepsilon^m(\varepsilon)$ we must compute (as seen, for infinite dimension, in the end of Section 2.2.) the value of $r_1(\varepsilon)$. From (2.31) and analogously to what has been done in infinite dimension, knowing that $\hat{u}_\varepsilon^m = \beta^m + \gamma^m$, the coordinates of β^m verify, for $\varepsilon < \rho < a$ and $i = 1,\ldots,m$

$$
\begin{cases}
-\dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\rho \dfrac{\partial \beta_i(\rho)}{\partial \rho}\right) + \dfrac{\lambda_i}{\rho^2}\beta_i(\rho) \\
= \displaystyle\int_0^{2\pi} \hat{f} w_i(\theta)\,\mathrm{d}\theta = \hat{f}_i(\rho),\ \varepsilon < \rho < a, i = 1,\ldots,m \\
\beta_i(a) = 0,\ i = 1,\ldots,m \\
\dfrac{\partial \beta_i}{\partial \rho}(\varepsilon) = 0,\ i = 1,\ldots,m.
\end{cases}
\tag{2.43}
$$

Consequently, for $i = 1$, we have successively:

$$
\begin{aligned}
& \frac{\partial}{\partial \rho}\left(\rho\frac{\partial \beta_1}{\partial \rho}\right) = -\rho \hat{f}_1(\rho) \\
\Rightarrow\quad & \int_\varepsilon^t \frac{\partial}{\partial \rho}\left(\rho\frac{\partial \beta_1}{\partial \rho}\right)\mathrm{d}\rho = -\int_\varepsilon^t \rho \hat{f}_1(\rho)\,\mathrm{d}\rho \\
\Rightarrow\quad & \rho\frac{\partial \beta_1}{\partial \rho}\Big]_\varepsilon^t = -\int_\varepsilon^t \rho \hat{f}_1(\rho)\,\mathrm{d}\rho \\
\Rightarrow\quad & \frac{\partial \beta_1}{\partial t}(t) = -\frac{1}{t}\int_\varepsilon^t \rho \hat{f}_1(\rho)\,\mathrm{d}\rho \\
\Rightarrow\quad & \int_\varepsilon^a \frac{\partial \beta_1}{\partial t}(t)dt = -\int_\varepsilon^a \frac{1}{t}\int_\varepsilon^t \rho \hat{f}_1(\rho)\,\mathrm{d}\rho\,\mathrm{d}t \\
\Rightarrow\quad & \beta_1(t)]_\varepsilon^a = -\int_\varepsilon^a \frac{1}{t}\int_\varepsilon^t \rho \hat{f}_1(\rho)\,\mathrm{d}\rho\,\mathrm{d}t \\
\Rightarrow\quad & r_1(\varepsilon) = \int_\varepsilon^a \frac{1}{t}\int_\varepsilon^t \rho \hat{f}_1(\rho)\,\mathrm{d}\rho\,\mathrm{d}t \quad \text{(using the notation } r(s) = \beta_{s_{|\Gamma_s}}\text{).}
\end{aligned}
$$

Using this fact, we can obtain the value of the constant $\hat{u}_\varepsilon^m(0)$:

Proposition 2.5.1. *Let $\hat{f}_1(\rho)$ be a bounded function on $(0,a)$. Then, we have* $\lim_{\varepsilon\to 0} \hat{u}^m_\varepsilon(\varepsilon) = \frac{1}{\sqrt{2\pi}}\int_0^a \frac{1}{t}\int_0^t \rho\hat{f}_1(\rho)\,\mathrm{d}\rho\,\mathrm{d}t$.

Proof. Considering $r_1(\varepsilon) = \int_\varepsilon^a \frac{1}{t}\int_\varepsilon^t \rho\hat{f}_1(\rho)\,\mathrm{d}\rho\,\mathrm{d}t$, if there exists a positive constant c, such that $|\hat{f}_1(\rho)| < c$, then

$$\int_\varepsilon^t \rho|\hat{f}_1(\rho)|\,\mathrm{d}\rho < \int_\varepsilon^t \rho\, c\,\mathrm{d}\rho = c\left(\frac{t^2}{2} - \frac{\varepsilon^2}{2}\right) < c\frac{t^2}{2}.$$

Consequently,

$$\frac{1}{t}\int_\varepsilon^t \rho|\hat{f}_1(\rho)|\,\mathrm{d}\rho < c\frac{1}{t}\frac{t^2}{2} = c\frac{t}{2} < c\frac{a}{2}.$$

Therefore, $\lim_{\varepsilon\to 0} r_1(\varepsilon) = \lim_{\varepsilon\to 0}\int_\varepsilon^a \frac{1}{t}\int_\varepsilon^t \rho\hat{f}_1(\rho)\,\mathrm{d}\rho\,\mathrm{d}t$ and from (2.42) we can conclude that

$$\lim_{\varepsilon\to 0} \hat{u}^m_\varepsilon(\varepsilon) = \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon\to 0} r_1(\varepsilon)\left(= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon\to 0} u_1(\varepsilon)\right) = \frac{1}{\sqrt{2\pi}}\int_0^a \frac{1}{t}\int_0^t \rho\hat{f}_1(\rho)\,\mathrm{d}\rho\,\mathrm{d}t. \qquad (2.44)$$

■

In what follows, we are going to consider a function $f \in \mathcal{C}^{0,\alpha}(\Omega)$ and consequently (see Proposition 9, page 291 of [11]) the solutions u of the equation $-\Delta u = f$, on Ω, are of classe $\mathcal{C}^{2,\alpha}(\Omega)$. We recall that a function $f \in \mathcal{C}^{m,\alpha}(\Omega)$ is a function of class $\mathcal{C}^m(\Omega)$ whose derivatives of order m are Holder functions of order α $(0 < \alpha < 1)$ on every compact subset K of Ω (that is, verifying the following property: there exists a constant c_K such that $|f(x) - f(y)| \le c_K\,|x-y|^\alpha, \forall x, y \in K$).

Lemma 2.5.2. *If $v \in C(\Omega)$ then $\hat{v} \in C(\widehat{\Omega} \cup \{\{0\} \times [0,2\pi]\})$, with $\hat{v}(0,\theta)$ constant, for $0 \le \theta \le 2\pi$.*

Proof. Obviously, if $v \in C(\Omega \setminus \{(0,0)\})$ then $\hat{v} \in C(\widehat{\Omega})$ (see [25]). Therefore, we only need to prove that, if v is also a continuous function on $(0,0)$, then $\hat{v}$ is still a continuous function when we consider the points $\{(0,\theta), \theta \in [0,2\pi]\}$.

We know that the function $v(x,y)$ converges to the limit b when (x,y) converges to $(0,0)$ if and only if $\hat{v}$ verifies the following condition: for all $\delta > 0$, there exists $\epsilon > 0$

such that the inequality $|\hat{v}(\rho,\theta) - b| < \delta$ is verified whenever $0 < \rho < \epsilon$ (independently of the value of θ). Then, since $v \in C(\Omega)$, in fact we have $b = v(0,0)$ and consequently, for all $\delta > 0$ exists $\epsilon > 0$ such that $0 < \rho < \epsilon \Rightarrow |\hat{v}(\rho,\theta) - v(0,0)| < \delta$, independently of θ, which means that $\lim_{\rho\to 0} \hat{v}(\rho,\theta) = v(0,0)$, independently of θ. Therefore the function

$$\bar{\hat{v}} = \begin{cases} \hat{v}(\rho,\theta), \ \rho \neq 0, \theta \in [0,2\pi] \\ v(0,0), \ \rho = 0, \theta \in [0,2\pi] \end{cases}$$

(still denoted by $\hat{v}$) is a continuous function on $\widehat{\Omega} \cup \{\{0\} \times [0,2\pi]\} = [0,a] \times [0,2\pi]$. ■

The next Lemma, that we present only in a finite dimension context, is also valid for infinite dimension.

Lemma 2.5.3. *For all* $\hat{v}(\rho,\theta) = \sum_{i=1}^{m} v_i(\rho) w_i(\theta) \in C([0,a] \times [0,2\pi])$*, we have* $v_i(\rho) \in C([0,a])$.

Proof. For each $\rho \in [0,a]$, $(\hat{v}(\rho,\theta), w_i(\theta))_{L^2(0,2\pi)} = v_i(\rho)$. Then, for each $\rho_1, \rho_2 \in [0,a]$, we obtain

$$\begin{aligned} |v_i(\rho_1) - v_i(\rho_2)| &= |\,(\hat{v}(\rho_1,\theta) - \hat{v}(\rho_2,\theta), w_i(\theta))_{L^2(0,2\pi)}\,| \\ &\leq \|\hat{v}(\rho_1,\theta) - \hat{v}(\rho_2,\theta)\|_{L^2(0,2\pi)} \|w_i(\theta)\|_{L^2(0,2\pi)} \\ &= \left(\int_0^{2\pi} |\hat{v}(\rho_1,\theta) - \hat{v}(\rho_2,\theta)|^2 \,\mathrm{d}\theta\right)^{1/2}. \end{aligned}$$

Since $\hat{v}(\rho,\theta)$ is continuous in $[0,a] \times [0,2\pi]$, then $\hat{v}(\rho)$ is continuous in $[0,a]$ (notice that every continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is separately continuous with regard to each one of its variables, since its components are the result of the composition of f with a continuous application of the type $t \to (a_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_m)$) and consequently ($[0,a]$ is closed and bounded) is uniformly continuous in $[0,a]$. Then, $\forall\, \delta > 0, \exists\, \epsilon > 0 : \forall \rho_1, \rho_2 \in [0,a]$

$$\begin{aligned} |\rho_1 - \rho_2| < \epsilon \ &\Rightarrow |\hat{v}(\rho_1,\theta) - \hat{v}(\rho_2,\theta)| < \frac{\delta}{\sqrt{2\pi}}, \forall\, \theta \in [0,2\pi] \\ &\Rightarrow |\hat{v}(\rho_1,\theta) - \hat{v}(\rho_2,\theta)|^2 < \frac{\delta^2}{2\pi} \\ &\Rightarrow \left(\int_0^{2\pi} |\hat{v}(\rho_1,\theta) - \hat{v}(\rho_2,\theta)|^2 \,\mathrm{d}\theta\right)^{\frac{1}{2}} \leq \left(\int_0^{2\pi} \frac{\delta^2}{2\pi} \,\mathrm{d}\theta\right)^{\frac{1}{2}} = \delta. \end{aligned}$$

Consequently, $|\rho_1 - \rho_2| < \epsilon \Rightarrow |v_i(\rho_1) - v_i(\rho_2)| < \delta$ and v_i are continuous functions on $[0,a]$. ■

We are now in the position of affirming that the constant $\hat{u}^m_\varepsilon(0)$ computed in Proposition 2.5.1 is in fact the searched value $\hat{u}^m(0)$:

Proposition 2.5.4. *Let, in (2.29), $f \in C^{0,\alpha}(\Omega)$. Under this hypothesis, we have* $\lim_{\varepsilon\to 0} \hat{u}^m_\varepsilon(\varepsilon) = u_1(0) = \hat{u}^m(0)$.

Proof. Denoting by u_i^0 (respectively, v_i^0) the coordinates of $\hat{u}^m$ (respectively, $\hat{v}^m$), and substituting the equalities

$$\hat{u}^m(\rho,\theta) = \sum_1^m u_i^0(\rho) w_i(\theta), \quad \hat{v}^m(\rho,\theta) = \sum_1^m v_i^0(\rho) w_i(\theta) \tag{2.45}$$

in (2.29), we obtain, by similar computations to those performed to achieve (2.31) (here we integrate in $[0,a]$ instead of $[\varepsilon,a]$), that the coordinates $\{u_i^0(\rho)\}_{i=1}^m$ of $\hat{u}^m$ must verify

$$-\frac{\partial^2 u_i^0}{\partial\rho^2}(\rho) - \frac{1}{\rho}\frac{\partial u_i^0}{\partial\rho}(\rho) + \frac{\lambda_i}{\rho^2}\, u_i^0(\rho) = \hat{f}_i(\rho),\ 0<\rho<a, i=1,\dots,m$$

and also $u_i^0(a)=0$, $i=1,\dots,m$.

Then, for $i \geq 2$, we have

$$-\rho^2\frac{\partial^2 u_i^0}{\partial\rho^2}(\rho) - \rho\frac{\partial u_i^0}{\partial\rho}(\rho) + \lambda_i\, u_i^0(\rho) = \rho^2\, \hat{f}_i(\rho).$$

Since we took $f \in C^{0,\alpha}(\Omega)$, then $u \in C^{2,\alpha}(\Omega)$ and, in particular, we have $f \in C(\Omega)$ and $u \in C^2(\Omega)$. According to Lemma 2.5.2 and Lemma 2.5.3 (in the finite dimensional particular case) we can therefore conclude that u_i^0 and $\hat{f}_i$ are continuous functions on $[0,a]$. Since $\frac{\partial\hat{u}}{\partial\rho} = \frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta)$ and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in C^1(\Omega)$, we have $\frac{\partial\hat{u}}{\partial\rho} \in C^1(\widehat{\Omega})$. Further, since $\frac{\partial\hat{u}}{\partial\rho}(0,\theta) = \frac{\partial u}{\partial x}(0,0)\cos(\theta) + \frac{\partial u}{\partial y}(0,0)\sin(\theta)$ and we have assumed enough regularity around the origin, we also have $\frac{\partial\hat{u}}{\partial\rho} \in C(\widehat{\Omega}\cup\{\{0\}\times[0,2\pi]\}) = \mathcal{C}([0,a]\times[0,2\pi])$ and consequently, by Lemma 2.5.3, $\frac{\partial u_i^0}{\partial\rho}$ is a continuous functions on $[0,a]$. In the same way, we have $\frac{\partial^2\hat{u}}{\partial\rho^2} = \cos^2(\theta)\frac{\partial^2 u}{\partial x^2} + 2\cos(\theta)\sin(\theta)\frac{\partial^2 u}{\partial x\partial y} + \sin^2(\theta)\frac{\partial^2 u}{\partial y^2}$ (whence in particular $\frac{\partial^2\hat{u}}{\partial\rho^2}(0,\theta) = \cos^2(\theta)\frac{\partial^2 u}{\partial x^2}(0,0) + 2\cos(\theta)\sin(\theta)\frac{\partial^2 u}{\partial x\partial y}(0,0) + \sin^2(\theta)\frac{\partial^2 u}{\partial y^2}(0,0)$) and $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x\partial y}, \frac{\partial^2 u}{\partial y^2} \in C(\Omega)$, from which we conclude that $\frac{\partial^2\hat{u}}{\partial\rho^2} \in C(\widehat{\Omega}\cup\{\{0\}\times[0,2\pi]\}) = \mathcal{C}([0,a]\times[0,2\pi])$ and consequently, again by Lemma 2.5.3, $\frac{\partial^2 u_i^0}{\partial\rho^2}$ is a continuous functions on $[0,a]$.

Therefore,

$$\lim_{\rho\to 0} -\rho^2 \frac{\partial^2 u_i^0}{\partial \rho^2}(\rho) - \rho \frac{\partial u_i^0}{\partial \rho}(\rho) + \lambda_i\, u_i^0(\rho) = \lim_{\rho\to 0} \rho^2\, \hat{f}_i(\rho)$$
$$\Rightarrow \quad \lambda_i\, u_i^0(0) = 0 \Rightarrow u_i^0(0) = 0.$$

For $i = 1$, we proceed like just after (2.43) and obtain $u_1^0(0) = \displaystyle\int_0^a \frac{1}{t} \int_0^t \rho\, \hat{f}_1(\rho)\, \mathrm{d}\rho\, \mathrm{d}t$. Then, $\hat{u}^m(0) = u_1^0(0) = \displaystyle\int_0^a \frac{1}{t} \int_0^t \rho\, \hat{f}_1(\rho)\, \mathrm{d}\rho\, \mathrm{d}t$ and from (2.44) we can therefore conclude that $\lim_{\varepsilon\to 0} \hat{u}_\varepsilon^m(\varepsilon) = \hat{u}^m(0)$. ■

2.6. Global nature of P^m and r^m: some estimates

In order to establish the global nature of P^m and r^m, solutions of the system (2.34), we need to develop some estimates on $P^m(s)$ and $r^m(s)$, independently of s . Naturally, we are going to use the norms defined previously, in the particular case of finite dimension.

Lemma 2.6.1. *For all $\xi \in H_\rho^1(s,a)$ such that $\xi(a) = 0$, we have*

$$\int_s^a \frac{s^2}{\rho}\, \xi^2(\rho)\, \mathrm{d}\rho \leq 4\, a^2 \int_s^a \rho \left(\frac{\partial \xi}{\partial \rho}(\rho) \right)^2 \mathrm{d}\rho.$$

Proof. Since

$$\xi^2(\rho) = -2 \int_\rho^a \xi(t) \frac{\partial \xi}{\partial t}(t)\, \mathrm{d}t$$

we have, for $s \leq \rho \leq a$,

$$\begin{aligned}
\frac{1}{\rho}\xi^2(\rho) &\leq \frac{2}{\rho} \left| \int_\rho^a \xi(t) \frac{\partial \xi}{\partial t}(t)\, \mathrm{d}t \right| \\
&\leq \frac{2}{\rho} \left(\int_\rho^a \frac{1}{t} \xi^2(t)\, \mathrm{d}t \right)^{1/2} \left(\int_\rho^a t \left(\frac{\partial \xi}{\partial t}(t) \right)^2 \mathrm{d}t \right)^{1/2} \\
&\leq \frac{2}{s} \left(\int_s^a \frac{1}{t} \xi^2(t)\, \mathrm{d}t \right)^{1/2} \left(\int_s^a t \left(\frac{\partial \xi}{\partial t}(t) \right)^2 \mathrm{d}t \right)^{1/2},
\end{aligned}$$

consequently,

$$\begin{aligned}
\int_s^a \frac{1}{\rho}\xi^2(\rho)\,\mathrm{d}\rho &\leq \int_s^a \left(\frac{2}{s}\left(\int_s^a \frac{1}{t}\xi^2(t)\,\mathrm{d}t\right)^{1/2}\left(\int_s^a t\left(\frac{\partial\xi}{\partial t}(t)\right)^2\mathrm{d}t\right)^{1/2}\right)\mathrm{d}\rho \\
&= \left(\int_s^a \frac{1}{t}\xi^2(t)\,\mathrm{d}t\right)^{1/2}\left(\int_s^a t\left(\frac{\partial\xi}{\partial t}(t)\right)^2\mathrm{d}t\right)^{1/2}\int_s^a \frac{2}{s}\,\mathrm{d}\rho \\
&= \left(\int_s^a \frac{1}{t}\xi^2(t)\,\mathrm{d}t\right)^{1/2}\left(\int_s^a t\left(\frac{\partial\xi}{\partial t}(t)\right)^2\mathrm{d}t\right)^{1/2}\frac{2(a-s)}{s} \\
&\leq \left(\int_s^a \frac{1}{t}\xi^2(t)\,\mathrm{d}t\right)^{1/2}\left(\int_s^a t\left(\frac{\partial\xi}{\partial t}(t)\right)^2\mathrm{d}t\right)^{1/2}\frac{2\,a}{s},
\end{aligned}$$

so,

$$\begin{aligned}
&\left(\int_s^a \frac{1}{\rho}\xi^2(\rho)\,\mathrm{d}\rho\right)^{1/2} \leq \left(\int_s^a \rho\left(\frac{\partial\xi}{\partial\rho}(\rho)\right)^2\mathrm{d}\rho\right)^{1/2}\frac{2\,a}{s} \\
\Rightarrow\ &\int_s^a \frac{1}{\rho}\xi^2(\rho)\,\mathrm{d}\rho \leq \frac{4\,a^2}{s^2}\int_s^a \rho\left(\frac{\partial\xi}{\partial\rho}(\rho)\right)^2\mathrm{d}\rho \\
\Rightarrow\ &\int_s^a \frac{s^2}{\rho}\xi^2(\rho)\,\mathrm{d}\rho \leq 4\,a^2\int_s^a \rho\left(\frac{\partial\xi}{\partial\rho}(\rho)\right)^2\mathrm{d}\rho.
\end{aligned}$$

■

It follows a "trace theorem", which is valid both for the functions γ_s^m and β_s^m:

Theorem 2.6.2. *For all $\rho \in [s,a)$ $(s \in [\varepsilon,a])$, there exists $k>0$ (independent of ρ) such that*

$$\sqrt{\rho}\|\xi^m(\rho)\|_{H^{1/2}_{\rho,P}(0,2\pi)} \leq k\|\xi^m\|_{\widehat{H}_s},$$

for all $\xi^m \in \widehat{H}_s$ verifying $\xi^m(a)=0$.

Proof. Since

$$\xi_i^2(\rho) = -2\int_\rho^a \xi_i(t)\frac{\partial\xi_i}{\partial t}(t)\,\mathrm{d}t \tag{2.46}$$

we have

$$\begin{aligned}
\sqrt{\lambda_i}\xi_i^2(\rho) &= 2\sqrt{\lambda_i}\left|\int_\rho^a \xi_i(t)\frac{\partial \xi_i}{\partial t}(t)\,\mathrm{d}t\right| \\
&\le 2\left(\int_\rho^a \frac{\lambda_i}{t}\,\xi_i^2\,\mathrm{d}t\right)^{1/2}\left(\int_\rho^a t\left(\frac{\partial \xi_i}{\partial t}\right)^2\mathrm{d}t\right)^{1/2} \\
&\le 2\left(\int_s^a \frac{\lambda_i}{\rho}\,\xi_i^2\,\mathrm{d}\rho\right)^{1/2}\left(\int_s^a \rho\left(\frac{\partial \xi_i}{\partial \rho}\right)^2\mathrm{d}\rho\right)^{1/2} \\
&\le \int_s^a \frac{\lambda_i}{\rho}\,\xi_i^2\,\mathrm{d}\rho + \int_s^a \rho\left(\frac{\partial \xi_i}{\partial \rho}\right)^2\mathrm{d}\rho.
\end{aligned} \tag{2.47}$$

Adding up from 2 to m, we obtain:

$$\sum_2^m \sqrt{\lambda_i}\xi_i^2(\rho) \le \sum_2^m \int_s^a \frac{\lambda_i}{\rho}\,\xi_i^2\,\mathrm{d}\rho + \sum_2^m \int_s^a \rho\left(\frac{\partial \xi_i}{\partial \rho}\right)^2\mathrm{d}\rho. \tag{2.48}$$

On the other hand, using Lemma 2.6.1,

$$\begin{aligned}
\rho\xi_1^2(\rho) &\le 2\rho\left|\int_\rho^a \xi_1(t)\frac{\partial \xi_1}{\partial t}(t)\,\mathrm{d}t\right| \\
&\le \int_\rho^a \frac{\rho^2}{t}\xi_1^2\,\mathrm{d}t + \int_\rho^a t\left(\frac{\partial \xi_1}{\partial t}\right)^2\mathrm{d}t \\
&\le (4\,a^2+1)\int_\rho^a t\left(\frac{\partial \xi_1}{\partial t}\right)^2\mathrm{d}t \\
&\le k^2\int_s^a \rho\left(\frac{\partial \xi_1}{\partial \rho}\right)^2\mathrm{d}\rho.
\end{aligned} \tag{2.49}$$

Therefore, from (2.48) and (2.49) we obtain

$$\begin{aligned}
&\rho\xi_1^2(\rho) + \sum_2^m \sqrt{\lambda_i}\xi_i^2(\rho) \\
\le\ & k^2\int_s^a \rho\left(\frac{\partial \xi_1}{\partial \rho}\right)^2\mathrm{d}\rho + \sum_2^m \int_s^a \frac{\lambda_i}{\rho}\,\xi_i^2\,\mathrm{d}\rho + \sum_2^m \int_s^a \rho\left(\frac{\partial \xi_i}{\partial \rho}\right)^2\mathrm{d}\rho \\
\le\ & k^2\left(\sum_2^m \int_s^a \frac{\lambda_i}{\rho}\,\xi_i^2\,\mathrm{d}\rho + \sum_1^m \int_s^a \rho\left(\frac{\partial \xi_i}{\partial \rho}\right)^2\mathrm{d}\rho\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\rho\left(\xi_1^2(\rho) + \sum_2^m \frac{\sqrt{\lambda_i}}{\rho}\xi_i^2(\rho)\right) = \rho\xi_1^2(\rho) + \sum_2^m \sqrt{\lambda_i}\xi_i^2(\rho) \\
\le\ & k^2\left(\sum_2^m \int_s^a \frac{\lambda_i}{\rho}\,\xi_i^2(\rho)\,\mathrm{d}\rho + \sum_1^m \int_s^a \rho\left(\frac{\partial \xi_i}{\partial \rho}(\rho)\right)^2\mathrm{d}\rho\right).
\end{aligned}$$

The constant $k = \sqrt{4\,a^2+1}$ does not depend on ρ. ■

From (2.31) and similarly as done to achieve (2.43), we conclude that the coordinates of the state equation verify, for $s<\rho<a$ and $i=1,\dots,m$

$$\begin{cases} -\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial\gamma_i(\rho)}{\partial\rho}\right)+\dfrac{\lambda_i}{\rho^2}\gamma_i(\rho)=0 \\ \gamma_i(a)=0 \\ \dfrac{\partial\gamma_i}{\partial\rho}(s)=h_i. \end{cases} \tag{2.50}$$

From (2.31) and (2.43) we know that, in the particular case of $s=\varepsilon$, h must be such that $\gamma_i(\varepsilon)=-r_i(\varepsilon)$, for $i\geq 2$ and $h_1=0$.

Proposition 2.6.3. *For γ_s^m solution of (2.50), we have*

$$\|\gamma_s^m\|^2_{\widehat{H}_s}\leq s\,\|\gamma_s^m(s)\|_{H^{1/2}_{s,P}(0,2\pi)}\,\|h^m\|_{H^{1/2}_{s,P}(0,2\pi)\,'},$$

for all $s\in[\varepsilon,a]$.

Proof. From (2.50)

$$\begin{aligned} &-\int_s^a\frac{\partial^2\gamma_i}{\partial\rho^2}\gamma_i\rho\,\mathrm{d}\rho+\int_s^a\frac{1}{\rho^2}\lambda_i\gamma_i^2\rho\,\mathrm{d}\rho-\int_s^a\frac{1}{\rho}\frac{\partial\gamma_i}{\partial\rho}\gamma_i\rho\,\mathrm{d}\rho=0\\ \Rightarrow\;&\frac{\partial\gamma_i}{\partial\rho}(s)\;\gamma_i(s)\;s+\int_s^a\left(\frac{\partial\gamma_i}{\partial\rho}\right)^2\rho\,\mathrm{d}\rho+\int_s^a\frac{1}{\rho}\lambda_i\gamma_i^2\,\mathrm{d}\rho=0\\ \Rightarrow\;&-\sum_1^m h_i\;\gamma_i(s)\;s=\sum_1^m\int_s^a\left(\frac{\partial\gamma_i}{\partial\rho}\right)^2\rho\,\mathrm{d}\rho+\sum_1^m\int_s^a\frac{1}{\rho}\lambda_i\gamma_i^2\,\mathrm{d}\rho=\|\gamma_s^m\|^2_{\widehat{H}_s}. \end{aligned} \tag{2.51}$$

On the other hand,

$$\begin{aligned} -\sum_1^m h_i\gamma_i(s) &= -\left(h_1\gamma_1(s)+\sum_2^m\frac{\sqrt{s}}{\sqrt[4]{\lambda_i}}h_i\;\frac{\sqrt[4]{\lambda_i}}{\sqrt{s}}\gamma_i(s)\right)\\ &\leq \left(h_1^2+\sum_2^m\frac{s}{\sqrt{\lambda_i}}\,h_i^2\right)^{1/2}\left(\gamma_1^2(s)+\sum_2^m\frac{\sqrt{\lambda_i}}{s}\gamma_i^2(s)\right)^{1/2}\\ &= \|h^m\|_{H^{1/2}_{s,P}(0,2\pi)\,'}\,\|\gamma_s^m(s)\|_{H^{1/2}_{s,P}(0,2\pi)}, \end{aligned}$$

and we obtain

$$\|\gamma_s^m\|^2_{\widehat{H}_s}\leq s\,\|h^m\|_{H^{1/2}_{s,P}(0,2\pi)\,'}\,\|\gamma_s^m(s)\|_{H^{1/2}_{s,P}(0,2\pi)}.$$

■

The next Theorem, is now a direct consequence of Theorem 2.6.2 and Proposition 2.6.3:

Theorem 2.6.4. *There exists $k = \sqrt{4a^2+1} > 0$ (independent of s) such that*

$$\|\gamma_s^m(s)\|_{H^{1/2}_{s,P}(0,2\pi)} \leq k\,\|h^m\|_{\left(H^{1/2}_{s,P}(0,2\pi)\right)'}. \tag{2.52}$$

The above result tell us that the operator

$$\begin{array}{rccc} P^m(s): & \left(H^{1/2}_{s,P}(0,2\pi)\right)' & \longrightarrow & H^{1/2}_{s,P}(0,2\pi) \\ & h^m & \longrightarrow & P^m(s)h^m = \gamma_s^m(s) \end{array}$$

is continuous and

$$\|P^m(s)\| \leq k, \tag{2.53}$$

so the operator P^m is bounded by a constant which does not depend on s.

Theorem 2.6.5. *There exists $k = \sqrt{4a^2+1} > 0$ (independent of s) such that*

$$\|\gamma_s^m(s)\|_{H^{3/2}_{s,P}(0,2\pi)} \leq k\,\|h^m\|_{H^{1/2}_{s,P}(0,2\pi)}. \tag{2.54}$$

Proof. This proof follows initially the same steps of Theorem 2.6.2. Multiplying (2.47) by λ_i, for the particular case of $\rho = s$ in the left-hand side, we obtain

$$\sum_2^m \lambda_i^{3/2}\gamma_i^2(s) \leq \sum_2^m \int_s^a \frac{1}{\rho}\lambda_i^2\gamma_i^2(\rho)\,\mathrm{d}\rho + \sum_2^m \int_s^a \rho\lambda_i\left(\frac{\partial\gamma_i}{\partial\rho}(\rho)\right)^2\mathrm{d}\rho. \tag{2.55}$$

Furthermore, through (2.49), we find

$$s^3\gamma_1^2(s) \leq \int_s^a s^4\frac{\gamma_1^2(\rho)}{\rho}\,\mathrm{d}\rho + \int_s^a s^2\rho\left(\frac{\partial\gamma_1}{\partial\rho}(\rho)\right)^2\mathrm{d}\rho.$$

Using Lemma 2.6.1 we have

$$s^3\gamma_1^2(s) \leq c_1\int_s^a s^2\rho\left(\frac{\partial\gamma_1}{\partial\rho}(\rho)\right)^2\mathrm{d}\rho,$$

with $c_1 = 4a^2+1$. Then,

$$\begin{aligned} s^3\|\gamma_s^m(s)\|^2_{H^{3/2}_{s,P}(0,2\pi)} &= s^3\gamma_1^2(s) + \sum_2^m \lambda_i^{3/2}\gamma_i^2(s) \\ \leq\ & c_1\int_s^a s^2\rho\left(\frac{\partial\gamma_1}{\partial\rho}(\rho)\right)^2\mathrm{d}\rho + \sum_2^m\int_s^a\frac{1}{\rho}\lambda_i^2\gamma_i^2(\rho)\,\mathrm{d}\rho + \sum_2^m\int_s^a\rho\lambda_i\left(\frac{\partial\gamma_i}{\partial\rho}(\rho)\right)^2\mathrm{d}\rho \\ \leq\ & c_1\left(\int_s^a s^2\rho\left(\frac{\partial\gamma_1}{\partial\rho}(\rho)\right)^2\mathrm{d}\rho + \sum_2^m\int_s^a\frac{1}{\rho}\lambda_i^2\gamma_i^2(\rho)\,\mathrm{d}\rho + \sum_2^m\int_s^a\rho\lambda_i\left(\frac{\partial\gamma_i}{\partial\rho}(\rho)\right)^2\mathrm{d}\rho\right). \end{aligned} \tag{2.56}$$

On the other hand, from (2.51), we have:

$$\int_s^a \lambda_i \left(\frac{\partial \gamma_i}{\partial \rho}\right)^2 \rho \, \mathrm{d}\rho + \int_s^a \frac{1}{\rho} \lambda_i^2 \gamma_i^2 \, \mathrm{d}\rho = -\lambda_i h_i \, \gamma_i(s) \, s \text{ and } \int_s^a s^2 \left(\frac{\partial \gamma_1}{\partial \rho}\right)^2 \rho \, \mathrm{d}\rho = -h_1 \, \gamma_1(s) \, s^3, \quad (2.57)$$

for $i \geq 2$ and $i = 1$, respectively.

Consequently

$$\begin{aligned}
& \int_s^a s^2 \left(\frac{\partial \gamma_1}{\partial \rho}\right)^2 \rho \, \mathrm{d}\rho + \sum_2^m \int_s^a \lambda_i \left(\frac{\partial \gamma_i}{\partial \rho}\right)^2 \rho \, \mathrm{d}\rho + \sum_2^m \int_s^a \frac{1}{\rho} \lambda_i^2 \gamma_i^2 \, \mathrm{d}\rho \\
= \; & -h_1 \, \gamma_1(s) \, s^3 - \sum_2^m \lambda_i \, h_i \, \gamma_i(s) \, s \\
\leq \; & \left| h_1 \, \gamma_1(s) \, s^3 + \sum_2^m \lambda_i \, h_i \, \gamma_i(s) \, s \right| \\
= \; & s^3 \left| h_1 \, \gamma_1(s) + \sum_2^m \lambda_i^{3/4} \lambda_i^{1/4} \, h_i \, \gamma_i(s) \, \frac{1}{s^{3/2}} \frac{1}{s^{1/2}} \right| \\
\leq \; & s^3 \left(h_1^2 + \sum_2^m \frac{\lambda_i^{1/2}}{s} h_i^2 \right)^{1/2} \left(\gamma_1^2(s) + \sum_2^m \frac{\lambda_i^{3/2}}{s^3} \gamma_i^2(s) \right)^{1/2} \\
= \; & s^3 \| h^m \|_{H^{1/2}_{s,P}(0,2\pi)} \| \gamma_s^m(s) \|_{H^{3/2}_{s,P}(0,2\pi)}.
\end{aligned}$$

From (2.56) we can therefore conclude that

$$\begin{aligned}
& s^3 \| \gamma_s^m(s) \|^2_{H^{3/2}_{s,P}(0,2\pi)} \leq c_1 s^3 \| h^m \|_{H^{1/2}_{s,P}(0,2\pi)} \| \gamma_s^m(s) \|_{H^{3/2}_{s,P}(0,2\pi)} \\
\Rightarrow \quad & \| \gamma_s^m(s) \|_{H^{3/2}_{s,P}(0,2\pi)} \leq c_1 \| h^m \|_{H^{1/2}_{s,P}(0,2\pi)},
\end{aligned}$$

as required. ■

The following Corollary is a direct consequence (by interpolation) of Propositions 2.6.4 and 2.6.5:

Corollary 2.6.6. *There exists $k > 0$ (independent of s) such that*

$$\| \gamma_s^m(s) \|_{H^1_{s,P}(0,2\pi)} \leq k \, \| h^m \|_{L^2(0,2\pi)}. \quad (2.58)$$

Proposition 2.6.7. *There exists $k > 0$ (independent of s) such that*

$$\| \gamma_s^m(s) \|_{L^2(0,2\pi)} \leq k \, \| h^m \|_{\left(H^1_{s,P}(0,2\pi)\right)'}. \quad (2.59)$$

Proof. Using the same reasoning as to get (2.55) we can obtain

$$\sum_2^m \gamma_i^2(s) \le \sum_2^m \int_s^a \frac{\sqrt{\lambda_i}}{\rho}\,\gamma_i^2\,\mathrm{d}\rho + \sum_2^m \int_s^a \frac{1}{\sqrt{\lambda_i}}\rho\left(\frac{\partial\gamma_i}{\partial\rho}\right)^2\,\mathrm{d}\rho. \tag{2.60}$$

On the other hand, dividing (2.51) by $\sqrt{\lambda_i}$, gives

$$\sum_2^m \int_s^a \frac{1}{\sqrt{\lambda_i}}\left(\frac{\partial\gamma_i}{\partial\rho}\right)^2\rho\,\mathrm{d}\rho + \sum_2^m \int_s^a \frac{1}{\rho}\sqrt{\lambda_i}\gamma_i^2\,\mathrm{d}\rho = -\sum_2^m \frac{1}{\sqrt{\lambda_i}}h_i\gamma_i(s)\;s. \tag{2.61}$$

So, from (2.60) and (2.61),

$$\sum_2^m \gamma_i^2(s) \le -\sum_2^m \frac{1}{\sqrt{\lambda_i}}h_i\gamma_i(s)\;s.$$

Further, as it was showed on the proof of Proposition 2.2.6 (or by solving (2.37)), for $i = 1$, we have $\gamma_1(s) = h_1 s\log\left(\dfrac{s}{a}\right)$. Then, $\gamma_1^2(s) = \gamma_1(s)h_1 s\log\left(\dfrac{s}{a}\right)$ and since $s < a$ we have $\gamma_1^2(s) = |\gamma_1(s)||h_1|\left(-s\log\left(\frac{s}{a}\right)\right) \le |\gamma_1(s)||h_1|\dfrac{a}{e} < |\gamma_1(s)||h_1|a$. Consequently,

$$\begin{aligned}\sum_1^m \gamma_i^2(s) \quad &\le -\sum_2^m \frac{1}{\sqrt{\lambda_i}}h_i\gamma_i(s)\;s + |\gamma_1(s)||h_1|a\\ &\le \max\{a,1\}\left(h_1^2 + \sum_2^m \frac{s^2}{\lambda_i}h_i^2\right)^{1/2}\left(\sum_1^m \gamma_i^2\right)^{1/2}\end{aligned}$$

and

$$\begin{aligned} &\|\gamma_s^m\|^2_{L^2(0,2\pi)} \le a\|h^m\|_{\left(H^1_{s,P}(0,2\pi)\right)'}\|\gamma_s^m\|_{L^2(0,2\pi)}\\ \Rightarrow\quad &\|\gamma_s^m\|_{L^2(0,2\pi)} \le a\|h^m\|_{\left(H^1_{s,P}(0,2\pi)\right)'}.\end{aligned}$$

■

In the next Propositions we can find some estimations on the function β_s^m.

In a similar way to (2.43), we can deduce that the coordinates of β_s^m verify, for $s < \rho < a$ and $i = 1,\ldots,m$:

$$\left\{\begin{array}{l} -\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial\beta_i(\rho)}{\partial\rho}\right) + \dfrac{\lambda_i}{\rho^2}\beta_i(\rho)\\ = \displaystyle\int_0^{2\pi} \hat{f}w_i(\theta)\,\mathrm{d}\theta,\; s<\rho<a, i = 1,\ldots,m\\ \beta_i(a) = 0,\; i = 1,\ldots,m\\ \dfrac{\partial\beta_i}{\partial\rho}(s) = 0,\; i = 1,\ldots,m. \end{array}\right. \tag{2.62}$$

Proposition 2.6.8. *For all $\rho \in (s,a)$, there exists $c > 0$ (independent of ρ) such that*

$$\|\beta_s^m\|_{L^2_\rho(s,a;H^{1/2}_{\rho,P}(0,2\pi))} \leq c.$$

Proof. From (2.62), we have

$$\begin{aligned}
& -\int_s^a \frac{\partial^2 \beta_i}{\partial \rho^2}\beta_i \rho \,\mathrm{d}\rho - \int_s^a \frac{\partial \beta_i}{\partial \rho}\beta_i \,\mathrm{d}\rho + \int_s^a \frac{\lambda_i}{\rho}\,\beta_i^2 \,\mathrm{d}\rho = \int_s^a \left(\int_0^{2\pi} \hat{f} w_i(\theta)\,\mathrm{d}\theta\right)\beta_i \rho \,\mathrm{d}\rho \\
\Rightarrow\ & -\frac{\partial \beta_i}{\partial \rho}\beta_i \rho\Big]_s^a + \int_s^a \frac{\partial \beta_i}{\partial \rho}\left(\frac{\partial \beta_i}{\partial \rho}\rho + \beta_i\right)\mathrm{d}\rho - \int_s^a \frac{\partial \beta_i}{\partial \rho}\beta_i \,\mathrm{d}\rho + \int_s^a \frac{\lambda_i}{\rho}\,\beta_i^2 \,\mathrm{d}\rho \\
& = \int_s^a \left(\int_0^{2\pi} \hat{f} w_i(\theta)\,\mathrm{d}\theta\right)\beta_i \rho \,\mathrm{d}\rho \\
\Rightarrow\ & \int_s^a \left(\frac{\partial \beta_i}{\partial \rho}\right)^2 \rho \,\mathrm{d}\rho + \int_s^a \frac{\lambda_i}{\rho}\,\beta_i^2 \,\mathrm{d}\rho = \int_s^a \left(\int_0^{2\pi} \hat{f} w_i(\theta)\,\mathrm{d}\theta\right)\beta_i \rho \,\mathrm{d}\rho \\
\Rightarrow\ & \sum_1^m \left(\int_s^a \left(\frac{\partial \beta_i}{\partial \rho}\right)^2 \rho \,\mathrm{d}\rho + \int_s^a \frac{\lambda_i}{\rho}\,\beta_i^2 \,\mathrm{d}\rho\right) = \sum_1^m \int_s^a \left(\int_0^{2\pi} \hat{f} w_i(\theta)\,\mathrm{d}\theta\right)\beta_i \rho \,\mathrm{d}\rho \\
\Rightarrow\ & \|\beta_s^m\|^2_{\widehat{H}_s} = \int_s^a \int_0^{2\pi} \hat{f}\left(\sum_1^m \beta_i w_i(\theta)\right)\rho \,\mathrm{d}\rho \,\mathrm{d}\theta = \int_s^a \int_0^{2\pi} \hat{f} \beta_s^m \rho \,\mathrm{d}\rho \,\mathrm{d}\theta.
\end{aligned}$$

As

$$\begin{aligned}
\int_s^a \int_0^{2\pi} \hat{f} \beta_s^m \rho \,\mathrm{d}\rho \,\mathrm{d}\theta &\leq \|\hat{f}\|_{L^2_\rho(s,a;L^2(0,2\pi))}\|\beta_s^m\|_{L^2_\rho(s,a;L^2(0,2\pi))} \\
&\leq \|\hat{f}\|_{L^2_\rho(0,a;L^2(0,2\pi))}\|\beta_s^m\|_{L^2_\rho(s,a;H^{1/2}_{\rho,P}(0,2\pi))},
\end{aligned}$$

we find

$$\|\beta_s^m\|^2_{\widehat{H}_s} \leq \|\hat{f}\|_{L^2_\rho(0,a;L^2(0,2\pi))}\|\beta_s^m\|_{L^2_\rho(s,a;H^{1/2}_{\rho,P}(0,2\pi))} \tag{2.63}$$

and from Theorem 2.6.2, we obtain

$$\begin{aligned}
& \int_s^a \rho \|\beta_s^m(\rho)\|^2_{H^{1/2}_{\rho,P}(0,2\pi)} \,\mathrm{d}\rho \leq \int_s^a (4a^2+1)\|\hat{f}\|_{L^2_\rho(0,a;L^2(0,2\pi))}\|\beta_s^m\|_{L^2_\rho(s,a;H^{1/2}_{\rho,P}(0,2\pi))}\,\mathrm{d}\rho \\
\Rightarrow\ & \|\beta_s^m\|^2_{L^2_\rho(s,a;H^{1/2}_{\rho,P}(0,2\pi))} \leq (4a^2+1)\,(a-s)\|\hat{f}\|_{L^2_\rho(0,a;L^2(0,2\pi))}\|\beta_s^m\|_{L^2_\rho(s,a;H^{1/2}_{\rho,P}(0,2\pi))} \\
\Rightarrow\ & \|\beta_s^m\|_{L^2_\rho(s,a;H^{1/2}_{\rho,P}(0,2\pi))} \leq (4a^2+1)\,a\,\|\hat{f}\|_{L^2_\rho(0,a;L^2(0,2\pi))} = c.
\end{aligned}$$

■

From Theorem 2.6.2 and Proposition 2.6.8, we can also conclude that

$$\rho \|\beta_s^m(\rho)\|^2_{H^{1/2}_{\rho,P}(0,2\pi)} \leq a\,\|\beta_s^m(\rho)\|^2_{H^{1/2}_{\rho,P}(0,2\pi)} \leq c_1.$$

In the sequence of Propositions 2.4.2 and 2.4.3, and as a direct consequence of Propositions 2.6.4, 2.6.5, 2.6.8 and Corollary 2.6.6, we can now establish:

Proposition 2.6.9. *P^m, the global solution of (2.34), verifies $P^m \in C^1([\varepsilon, a]; \mathcal{L}(V^m, V^m))$; consequently, r^m is a global solution of (2.34) and $r^m \in H^1(\varepsilon, a; V^m)$.*

2.7. Some more estimates

Lemma 2.7.1. *There exists a continuous lifting from $H^{1/2}_{s,P}(0, 2\pi)$ into $\widehat{H}_s$.*

Proof. Let $z^m \in H^{1/2}_{s,P}(0, 2\pi)$, $z^m = \sum_{i=1}^{m} z_i w_i$ and let us consider, for $i > 1$

$$v_i(\rho) = z_i\, s\, \varphi(\lambda_i^{1/2}(\rho - s))$$

and

$$v_1(\rho) = z_1\, s\, \varphi(\lambda_2^{1/2}(\rho - s))$$

(notice that $\lambda_2 < \lambda_i$, $\forall i > 2$), where $\varphi \in \mathcal{D}_{[0,b[}([0, +\infty[)$ (where $\mathcal{D}_X(Y)$ represents the C^∞ functions with values in Y which have compact support in X) and $b = \lambda_2^{1/2}(a - s)$. Obviously $\varphi(b) = 0$ and we impose $\varphi(0) = 1$. We then have

$$v_1(s) = z_1\, s\, \varphi(0) = z_1\, s;$$

$$v_i(s) = z_i\, s\, \varphi(0) = z_i\, s,\ \forall i > 1;$$

$$v_1(a) = z_1\, s\, \varphi(\lambda_2^{1/2}(a - s)) = z_1\, s\varphi(b) = 0;$$

$$v_2(a) = z_2\, s\, \varphi(\lambda_2^{1/2}(a - s)) = z_2\, s\varphi(b) = 0;$$

$$v_3(a) = z_3\, s\, \varphi(\lambda_3^{1/2}(a - s)) = z_3\, s\, 0 = 0.$$

Notice that $\lambda_3^{1/2}(a - s) > b$, since $\lambda_3 > \lambda_2$. For the same reason,

$$v_i(a) = 0,\ \forall i > 3.$$

For $i = 1$, we consider $x = \lambda_2^{1/2}(\rho - s)$. Consequently, we have $\mathrm{d}x = \lambda_2^{1/2}\, \mathrm{d}\rho$ and $\frac{\partial v_1}{\partial \rho}(\rho) = z_1\, s \frac{\partial \varphi}{\partial x} \lambda_2^{1/2}$. In the same way, for $i > 1$, we consider $x = \lambda_i^{1/2}(\rho - s)$. Then $\mathrm{d}x = \lambda_i^{1/2}\, \mathrm{d}\rho$ and $\frac{\partial v_i}{\partial \rho}(\rho) = z_i\, s \frac{\partial \varphi}{\partial x} \lambda_i^{1/2}$. Then,

$$\begin{aligned}\|v_s^m\|^2_{\widehat{H}_s} &= \int_s^a \sum_2^m \frac{\lambda_i}{\rho} v_i^2 \,\mathrm{d}\rho + \int_s^a \sum_1^m \rho \left(\frac{\partial v_i}{\partial \rho}\right)^2 \mathrm{d}\rho \\ &\le \frac{1}{s}\int_s^a \sum_2^m \lambda_i v_i^2 \,\mathrm{d}\rho + a\int_s^a \left(\frac{\partial v_1}{\partial \rho}\right)^2 \mathrm{d}\rho + \frac{a^2}{s}\int_s^a \sum_2^m \left(\frac{\partial v_i}{\partial \rho}\right)^2 \mathrm{d}\rho.\end{aligned}$$

Making the change of variables, we obtain

$$\begin{aligned}& \frac{1}{s}\sum_2^m \int_0^{\lambda_i^{1/2}(a-s)} \lambda_i z_i^2 s^2 \varphi^2 \lambda_i^{-1/2}\,\mathrm{d}x + a\int_0^{\lambda_2^{1/2}(a-s)} z_1^2 s^2 \left(\frac{\partial \varphi}{\partial x}\right)^2 \lambda_2 \lambda_2^{-1/2}\,\mathrm{d}x \\ & + \frac{a^2}{s}\sum_2^m \int_0^{\lambda_i^{1/2}(a-s)} z_i^2 s^2 \left(\frac{\partial \varphi}{\partial x}\right)^2 \lambda_i \lambda_i^{-1/2}\,\mathrm{d}x \\ = {} & s\sum_2^m \sqrt{\lambda_i} z_i^2 \int_0^{\lambda_i^{1/2}(a-s)} \varphi^2\,\mathrm{d}x + a\sqrt{\lambda_2} s^2 z_1^2 \int_0^b \left(\frac{\partial \varphi}{\partial x}\right)^2 \mathrm{d}x \\ & + a^2 s\sum_2^m \sqrt{\lambda_i} z_i^2 \int_0^{\lambda_i^{1/2}(a-s)} \left(\frac{\partial \varphi}{\partial x}\right)^2 \mathrm{d}x \\ \le {} & s\sum_2^m \sqrt{\lambda_i} z_i^2 \underbrace{\int_0^{+\infty} \varphi^2\,\mathrm{d}x}_{c_1} + a\sqrt{\lambda_2} s^2 z_1^2 \underbrace{\int_0^b \left(\frac{\partial \varphi}{\partial x}\right)^2 \mathrm{d}x}_{c_2} \\ & + a^2 s\sum_2^m \sqrt{\lambda_i} z_i^2 \underbrace{\int_0^{+\infty} \left(\frac{\partial \varphi}{\partial x}\right)^2 \mathrm{d}x}_{c_2} \\ \le {} & c\,s^2\left(\sum_2^m \frac{\sqrt{\lambda_i}}{s} z_i^2 + z_1^2 + a^2\sum_2^m \frac{\sqrt{\lambda_i}}{s} z_i^2\right) \qquad (\text{with } c = \max\{c_1, a\sqrt{\lambda_2}\,c_2, c_2\}) \\ \le {} & c\,s^2\,(a^2+1)\left(\sum_2^m \frac{\sqrt{\lambda_i}}{s} z_i^2 + z_1^2\right) = c\,s^2\,(a^2+1)\|z^m\|^2_{H^{1/2}_{s,P}(0,2\pi)}.\end{aligned}$$

We can therefore conclude that $\|v_s^m\|^2_{\widehat{H}_s} \le k\,s^2\|z^m\|^2_{H^{1/2}_{s,P}(0,2\pi)}$, with $k = c\,(a^2+1)$. ■

In the following Proposition, let $H(\Delta, \widehat{\Omega}\setminus\widehat{\Omega}_s) = \left\{\xi \in H^1(\widehat{\Omega}\setminus\widehat{\Omega}_s) : \Delta\xi \in L^2(\widehat{\Omega}\setminus\widehat{\Omega}_s)\right\}$ be provided with the norm $\|\xi\|_{H(\Delta,\widehat{\Omega}\setminus\widehat{\Omega}_s)} = \left(\|\xi\|^2_{H^1(\widehat{\Omega}\setminus\widehat{\Omega}_s)} + \|\Delta\xi|^2_{L^2(\widehat{\Omega}\setminus\widehat{\Omega}_s)}\right)^{1/2}$, as in Lemma 1, page 381 of [12].

Proposition 2.7.2. *For $\xi^m \in H(\Delta, \widehat{\Omega}\setminus\widehat{\Omega}_s)$ and $z^m \in H^{1/2}_{s,P}(0, 2\pi)$, there exists a constant $k > 0$, independent of s, such that*

$$\left|\sum_{i=1}^m -\frac{\partial \xi_i}{\partial \rho}(s) z_i\right| \le \|\xi^m\|_{H(\Delta,\widehat{\Omega}\setminus\widehat{\Omega}_s)}\sqrt{k}\|z^m\|_{H^{1/2}_{s,P}(0,2\pi)}.$$

Proof. Let $\xi^m \in \mathcal{D}(\overline{\widehat{\Omega} \setminus \widehat{\Omega}_s})$. For $z^m \in H^{1/2}_{s,P}(0,2\pi)$ we put

$$l(z^m) = \sum_{i=1}^{m} -\frac{\partial \xi_i}{\partial \rho}(s) z_i \, s.$$

According to Lemma 2.7.1, there exists $v_s^m \in \widehat{H}_s$ having z^m for its trace and such that $\|v_s^m\|_{\widehat{H}_s} \leq \sqrt{k}\, s \|z^m\|_{H^{1/2}_{s,P}(0,2\pi)}$. Then, using the properties of the Hilbert basis previously defined and applying Green's formula, we have

$$\begin{aligned}
& \int_s^a \int_0^{2\pi} \left(\frac{\partial^2 \xi^m}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 \xi^m}{\partial \theta^2} + \frac{1}{\rho}\frac{\partial \xi^m}{\partial \rho} \right) v_s^m \, \rho \, \mathrm{d}\theta \, \mathrm{d}\rho \\
= & \sum_{i,j=1}^{m} \left(\int_s^a \rho \frac{\partial^2 \xi_i}{\partial \rho^2} v_j \, \mathrm{d}\rho \int_0^{2\pi} w_i w_j \, \mathrm{d}\theta \right) + \sum_{i,j=1}^{m} \left(\int_s^a \frac{1}{\rho} \xi_i v_j \, \mathrm{d}\rho \int_0^{2\pi} \frac{\partial^2 w_i}{\partial \theta^2} w_j \, \mathrm{d}\theta \right) \\
& + \sum_{i,j=1}^{m} \left(\int_s^a \frac{\partial \xi_i}{\partial \rho} v_j \, \mathrm{d}\rho \int_0^{2\pi} w_i w_j \, \mathrm{d}\theta \right) \\
= & \sum_{i=1}^{m} \left(\int_s^a \rho \frac{\partial^2 \xi_i}{\partial \rho^2} v_i \, \mathrm{d}\rho \right) + \sum_{i=1}^{m} \left(\int_s^a -\frac{\lambda_i}{\rho} \xi_i v_i \, \mathrm{d}\rho \right) + \sum_{i=1}^{m} \left(\int_s^a \frac{\partial \xi_i}{\partial \rho} v_i \, \mathrm{d}\rho \right) \\
= & \sum_{i=1}^{m} \left(\frac{\partial \xi_i}{\partial \rho}(a) v_i(a)\, a - \frac{\partial \xi_i}{\partial \rho}(s) v_i(s)\, s \right) - \sum_{i=1}^{m} \int_s^a \frac{\partial \xi_i}{\partial \rho}\frac{\partial v_i}{\partial \rho} \rho \, \mathrm{d}\rho \\
& + \sum_{i=1}^{m} \left(\int_s^a -\frac{\lambda_i}{\rho} \xi_i v_i \, \mathrm{d}\rho \right).
\end{aligned}$$

Consequently, since $v_i(a) = 0$ and $v_i(s) = z_i$, we have:

$$\begin{aligned}
& \sum_{i=1}^{m} \left(\int_s^a \rho \frac{\partial^2 \xi_i}{\partial \rho^2} v_i \, \mathrm{d}\rho \right) + \sum_{i=1}^{m} \left(\int_s^a -\frac{\lambda_i}{\rho} \xi_i v_i \, \mathrm{d}\rho \right) + \sum_{i=1}^{m} \left(\int_s^a \frac{\partial \xi_i}{\partial \rho} v_i \, \mathrm{d}\rho \right) \\
= & \sum_{i=1}^{m} \left(-\frac{\partial \xi_i}{\partial \rho}(s) z_i \, s \right) - \sum_{i=1}^{m} \int_s^a \frac{\partial \xi_i}{\partial \rho}\frac{\partial v_i}{\partial \rho} \rho \, \mathrm{d}\rho + \sum_{i=1}^{m} \left(\int_s^a -\frac{\lambda_i}{\rho} \xi_i v_i \, \mathrm{d}\rho \right).
\end{aligned}$$

Therefore, using Schwartz's inequality, we have

$$\begin{aligned}
& |l(z^m)| \\
= & \left| \sum_{i=1}^{m} \int_s^a \left(\rho \frac{\partial^2 \xi_i}{\partial \rho^2} v_i - \frac{\lambda_i}{\rho} \xi_i v_i + \frac{\partial \xi_i}{\partial \rho} v_i \right) \mathrm{d}\rho + \sum_{i=1}^{m} \int_s^a \left(\frac{\partial \xi_i}{\partial \rho}\frac{\partial v_i}{\partial \rho} \rho + \frac{\lambda_i}{\rho} \xi_i v_i \right) \mathrm{d}\rho \right| \\
\leq & \left| \sum_{i=1}^{m} \int_s^a \underbrace{\left(\frac{\partial^2 \xi_i}{\partial \rho^2} - \frac{\lambda_i}{\rho^2} \xi_i + \frac{1}{\rho}\frac{\partial \xi_i}{\partial \rho} \right)}_{\Delta \xi^m} v_i \rho \, \mathrm{d}\rho \right| + \left| \sum_{i=1}^{m} \int_s^a \underbrace{\left(\frac{\partial \xi_i}{\partial \rho}\frac{\partial v_i}{\partial \rho} + \frac{\lambda_i}{\rho^2} \xi_i v_i \right)}_{\nabla \xi^m . \nabla \hat{v}_s^m} \rho \, \mathrm{d}\rho \right|
\end{aligned}$$

$$\leq \|\Delta\xi^m\|_{L^2_\rho(s,a;L^2(0,2\pi))}\|v_s^m\|_{L^2_\rho(s,a;L^2(0,2\pi))}+\|\nabla\xi^m\|_{L^2_\rho(s,a;L^2(0,2\pi))}\|\nabla v_s^m\|_{L^2_\rho(s,a;L^2(0,2\pi))}$$
$$\leq \left(\|\Delta\xi^m\|_{L^2_\rho(s,a;L^2(0,2\pi))}+\|\xi^m\|_{\widehat{H}_s}\right)\|v_s^m\|_{\widehat{H}_s}$$
$$= \|\xi^m\|_{H(\Delta,\widehat{\Omega}\setminus\widehat{\Omega}_s)}\|v_s^m\|_{\widehat{H}_s}$$
$$\leq \|\xi^m\|_{H(\Delta,\widehat{\Omega}\setminus\widehat{\Omega}_s)}\sqrt{k}\, s\|z^m\|_{H^{1/2}_{s,P}(0,2\pi)}.$$

∎

Proposition 2.7.3. *Considering $\xi^m \in H(\Delta,\widehat{\Omega}\setminus\widehat{\Omega}_s)$, there exists a constant $c>0$ such that* $\left\|\dfrac{\partial\xi^m}{\partial\rho}(s)\right\|_{H^{1/2}_{s,P}(0,2\pi)'} \leq c.$

Proof. It is a direct consequence of Proposition 2.7.2:

$$\left|\sum_{i=1}^{m} -\frac{\partial\xi_i}{\partial\rho}(s)z_i\right| = \left|\left\langle -\frac{\partial\xi^m}{\partial\rho}(s), z^m(s)\right\rangle_{H^{1/2}_{s,P}(0,2\pi)',H^{1/2}_{s,P}(0,2\pi)}\right|$$
$$\leq \|\xi^m\|_{H(\Delta,\widehat{\Omega}\setminus\widehat{\Omega}_s)}\sqrt{k}\,\|z^m\|_{H^{1/2}_{s,P}(0,2\pi)} = c\,\|z^m\|_{H^{1/2}_{s,P}(0,2\pi)}.$$

∎

Since, in particular, $f\in L^2(\widehat{\Omega}\setminus\widehat{\Omega}_s)$, it is obvious that $\hat{u}_s^m$, the finite dimension solution of (2.5), belongs to $H(\Delta,\widehat{\Omega}\setminus\widehat{\Omega}_s)$. In this case, the norm $\|\hat{u}_s^m\|_{H(\Delta,\widehat{\Omega}\setminus\widehat{\Omega}_s)}$, used in Proposition 2.7.2, is independent of s, since

$$\|\hat{u}_s^m\|^2_{H(\Delta,\widehat{\Omega}\setminus\widehat{\Omega}_s)} = \|\Delta\hat{u}_s^m\|^2_{L^2_\rho(s,a;L^2(0,2\pi))}+\|\hat{u}_s^m\|^2_{\widehat{H}_s} = \|f^m\|^2_{L^2_\rho(s,a;L^2(0,2\pi))}+\|\hat{u}_s^m\|^2_{\widehat{H}_s}$$
$$\leq \|f^m\|^2_{L^2_\rho(0,a;L^2(0,2\pi))}+\|\hat{u}_s^m\|^2_{\widehat{H}_s},$$

and $\|f^m\|_{L^2_\rho(0,a;L^2(0,2\pi))}$, $\|\hat{u}_s^m\|_{\widehat{H}_s}$ are independent the s (see Lemma (1.5.2), particularized for finite dimension).

From Theorem 2.6.4 and Proposition 2.7.3, we remark also that

$$\left\|P^m(s)\frac{\partial\hat{u}_s^m}{\partial\rho}(s)\right\|_{H^{1/2}_{s,P}(0,2\pi)} \leq k\left\|\frac{\partial\hat{u}_s^m}{\partial\rho}(s)\right\|_{H^{1/2}_{s,P}(0,2\pi)'} \leq kc.$$

2.8. Passing to the limit

We begin this section with a very important result, as we intend to pass to the limit both when $m\to\infty$ and $\varepsilon\to 0$.

Proposition 2.8.1. *For $\hat{u}_\varepsilon^m$ solution of (2.28), the norm $\|\hat{u}_\varepsilon^m\|_{\widehat{H}_\varepsilon}$ is bounded, independently of ε and m.*

Proof. From (2.28), considering $u_\varepsilon^m = v_\varepsilon^m$, and using Holder's inequality, we obtain

$$\begin{aligned}\|\nabla \hat{u}_\varepsilon^m\|^2_{L^2_\rho(\varepsilon,a;L^2(0,2\pi))} &= \int_\varepsilon^a \int_0^{2\pi} \left(\frac{\partial \hat{u}_\varepsilon^m}{\partial \rho}\right)^2 \rho + \frac{1}{\rho}\left(\frac{\partial \hat{u}_\varepsilon^m}{\partial \theta}\right)^2 \mathrm{d}\theta\, \mathrm{d}\rho = \int_\varepsilon^a \int_0^{2\pi} f \hat{u}_\varepsilon^m \rho \, \mathrm{d}\theta\, \mathrm{d}\rho \\ &\leq \|f\|_{L^2_\rho(\varepsilon,a;L^2(0,2\pi))} \|\hat{u}_\varepsilon^m\|_{L^2_\rho(\varepsilon,a;L^2(0,2\pi))}.\end{aligned}$$

Therefore, by a reasoning similar to (2.14) we have

$$\|\hat{u}_\varepsilon^m\|^2_{\widehat{H}_\varepsilon} \leq (c^2+1)\|f\|_{L^2_\rho(\varepsilon,a;L^2(0,2\pi))} \|\hat{u}_\varepsilon^m\|_{L^2_\rho(\varepsilon,a;L^2(0,2\pi))},$$

where c is the Poincaré's constant, and consequently

$$\|\hat{u}_\varepsilon^m\|_{\widehat{H}_\varepsilon} \leq (c^2+1)\|f\|_{L^2_\rho(\varepsilon,a;L^2(0,2\pi))} \leq (c^2+1)\|f\|_{L^2_\rho(0,a;L^2(0,2\pi))} \leq k.$$

∎

We can now pass to the limit on (2.28), when $m \to \infty$:

Proposition 2.8.2. *Let $\hat{u}_\varepsilon^m$ and $\hat{u}_\varepsilon$ be the solutions of (2.28) and (2.3), respectively. Then $\hat{u}_\varepsilon^m \to \hat{u}_\varepsilon$, strongly in $\widehat{U}_\varepsilon$, when $m \to \infty$. Moreover, $\hat{u}_\varepsilon^m(\rho) \to \hat{u}_\varepsilon(\rho)$, strongly in $H^{1/2}_{\rho,P}(0,2\pi)$, when $m \to \infty$, for all $\rho \in [\varepsilon, a]$.*

Proof. From Proposition 2.8.1, we can extract from $(\hat{u}_\varepsilon^m)$ a subsequence, still denoted by $(\hat{u}_\varepsilon^m)$, such that $\hat{u}_\varepsilon^m \to \hat{v}$, weakly in $\widehat{U}_\varepsilon$, when $m \to \infty$.

From (2.28) we have

$$\int_\varepsilon^a \int_0^{2\pi} \frac{\partial \hat{u}_\varepsilon^m}{\partial \rho}\frac{\partial \hat{\varphi}}{\partial \rho}\rho + \frac{1}{\rho}\frac{\partial \hat{u}_\varepsilon^m}{\partial \theta}\frac{\partial \hat{\varphi}}{\partial \theta}\,\mathrm{d}\theta\,\mathrm{d}\rho = \int_\varepsilon^a \int_0^{2\pi} f\hat{\varphi}\rho\,\mathrm{d}\theta\,\mathrm{d}\rho, \forall \varphi \in \widehat{U}_\varepsilon^m.$$

Then, since $\hat{v}$ is the weak limit of $\hat{u}_\varepsilon^m$, we obtain

$$\int_\varepsilon^a \int_0^{2\pi} \frac{\partial \hat{v}}{\partial \rho}\frac{\partial \hat{\varphi}}{\partial \rho}\rho + \frac{1}{\rho}\frac{\partial \hat{v}}{\partial \theta}\frac{\partial \hat{\varphi}}{\partial \theta}\,\mathrm{d}\theta\,\mathrm{d}\rho = \int_\varepsilon^a \int_0^{2\pi} f\hat{\varphi}\rho\,\mathrm{d}\theta\,\mathrm{d}\rho, \forall \varphi \in \widehat{U}_\varepsilon^m,$$

and by density

$$\int_{\varepsilon}^{a}\int_{0}^{2\pi} \frac{\partial \hat{v}}{\partial \rho}\frac{\partial \hat{\varphi}}{\partial \rho}\rho + \frac{1}{\rho}\frac{\partial \hat{v}}{\partial \theta}\frac{\partial \hat{\varphi}}{\partial \theta}\,\mathrm{d}\theta\,\mathrm{d}\rho = \int_{\varepsilon}^{a}\int_{0}^{2\pi} f\hat{\varphi}\rho\,\mathrm{d}\theta\,\mathrm{d}\rho, \forall \varphi \in \widehat{U}_{\varepsilon},$$

which means, by uniqueness, that $v = \hat{u}_{\varepsilon}$, solution of (2.3).

Then, using (2.3) and (2.28), we have

$$\begin{aligned}
&\int_{\widehat{\Omega}\backslash\widehat{\Omega}_{\varepsilon}} |\nabla(\hat{u}_{\varepsilon}^{m} - \hat{u}_{\varepsilon})|^{2}\rho\,\mathrm{d}\rho\,\mathrm{d}\theta \\
= &\int_{\widehat{\Omega}\backslash\widehat{\Omega}_{\varepsilon}} \nabla\hat{u}_{\varepsilon}^{m}\nabla\hat{u}_{\varepsilon}^{m}\rho\,\mathrm{d}\rho\,\mathrm{d}\theta - \int_{\widehat{\Omega}\backslash\widehat{\Omega}_{\varepsilon}} \nabla\hat{u}_{\varepsilon}^{m}\nabla\hat{u}_{\varepsilon}\rho\,\mathrm{d}\rho\,\mathrm{d}\theta - \int_{\widehat{\Omega}\backslash\widehat{\Omega}_{\varepsilon}} \nabla\hat{u}_{\varepsilon}\nabla(\hat{u}_{\varepsilon}^{m} - \hat{u}_{\varepsilon})\rho\,\mathrm{d}\rho\,\mathrm{d}\theta \\
= &\int_{\widehat{\Omega}\backslash\widehat{\Omega}_{\varepsilon}} f\hat{u}_{\varepsilon}^{m}\rho\,\mathrm{d}\rho\,\mathrm{d}\theta - \int_{\widehat{\Omega}\backslash\widehat{\Omega}_{\varepsilon}} \nabla\hat{u}_{\varepsilon}^{m}\nabla\hat{u}_{\varepsilon}\rho\,\mathrm{d}\rho\,\mathrm{d}\theta - \int_{\widehat{\Omega}\backslash\widehat{\Omega}_{\varepsilon}} \nabla\hat{u}_{\varepsilon}\nabla(\hat{u}_{\varepsilon}^{m} - \hat{u}_{\varepsilon})\rho\,\mathrm{d}\rho\,\mathrm{d}\theta \\
\to &\int_{\widehat{\Omega}\backslash\widehat{\Omega}_{\varepsilon}} f\hat{u}_{\varepsilon}\rho\,\mathrm{d}\rho\,\mathrm{d}\theta - \int_{\widehat{\Omega}\backslash\widehat{\Omega}_{\varepsilon}} \nabla\hat{u}_{\varepsilon}\nabla\hat{u}_{\varepsilon}\rho\,\mathrm{d}\rho\,\mathrm{d}\theta - 0 = 0.
\end{aligned}$$

We can therefore conclude that $\hat{u}_{\varepsilon}^{m}$, solution of (2.28), converges to $\hat{u}_{\varepsilon}$, solution of (2.3), strongly in $\widehat{U}_{\varepsilon}$. From $\hat{u}_{\varepsilon}^{m} \to \hat{u}_{\varepsilon}$, strongly in $\widehat{H}_{\varepsilon}$, by Proposition 2.1.2 we also have $\hat{u}_{\varepsilon}^{m}(\rho) \to \hat{u}_{\varepsilon}(\rho)$, strongly in $H^{1/2}_{\rho,P}(0,2\pi)$, for all $\rho \in [\varepsilon, a]$. ■

Corollary 2.8.3. *For all $s \in (\varepsilon, a)$, $r^{m}(s) \to r(s)$ strongly in $H^{1/2}_{\rho,P}(0,2\pi)$, when $m \to \infty$. Also, for all $s \in (\varepsilon, a)$ and for a fixed h, $P^{m}(s)h \to P(s)h$, strongly in $H^{1/2}_{\rho,P}(0,2\pi)$, weakly in $H^{3/2}_{\rho,P}(0,2\pi)$ and strongly in $H^{1}_{\rho,P}(0,2\pi)$, when $m \to \infty$.*

Proof. Applying Proposition 2.8.2 for all $s \in (\varepsilon, a)$, we obtain $\hat{u}_{\varepsilon}^{m}(s) = P^{m}(s)h + r^{m}(s) \to P(s)h + r(s) = \hat{u}_{\varepsilon}(s)$, strongly in $H^{1/2}_{\rho,P}(0,2\pi)$. Taking $h = 0$, we obtain $r^{m}(s) \to r(s)$ and consequently $P^{m}(s)h \to P(s)h$, strongly in $H^{1/2}_{\rho,P}(0,2\pi)$. Now, from Proposition 2.6.5, $P^{m}(s)h$ is bounded in $H^{3/2}_{\rho,P}(0,2\pi)$ and consequently we can extract a subsequence converging weakly. By density (since $P^{m}(s)h \to P(s)h$, strongly in $H^{1/2}_{\rho,P}(0,2\pi)$) that subsequence converges also to $P(s)h$. Since $H^{3/2}_{\rho,P}(0,2\pi) \subset H^{1}_{\rho,P}(0,2\pi)$, with $H^{3/2}_{\rho,P}(0,2\pi)$ dense in $H^{1}_{\rho,P}(0,2\pi)$, then $P^{m}(s)h \to P(s)h$ strongly in $H^{1}_{\rho,P}(0,2\pi)$, for all $s \in (\varepsilon, a)$. ■

Going back to the equation on P^{m} of (2.34), we obtain the following result:

Proposition 2.8.4. *For every $h, \bar{h}$ in $L^2(0,2\pi)$, the operator $P \in L^\infty\big((\varepsilon,a);\mathcal{L}\big(L^2(0,2\pi), H^1_{\rho,P}(0,2\pi)\big)\big)$ satisfies $P(a)=0$ and the following equation*

$$\left(\frac{\partial P}{\partial\rho}h,\bar{h}\right)_{L^2(0,2\pi)}+\left(\frac{1}{\rho^2}\frac{\partial}{\partial\theta}Ph,\frac{\partial}{\partial\theta}P\bar{h}\right)_{L^2(0,2\pi)}-\left(\frac{1}{\rho}h,P\bar{h}\right)_{L^2(0,2\pi)}=\big(h,\bar{h}\big)_{L^2(0,2\pi)},$$

in $\mathcal{D}'(\varepsilon,a)$.

Proof. For a fixed m_0, let $h,\bar{h}\in V^{m_0}$. Then, from (2.34), we obtain, for $m\geq m_0$

$$\left(\frac{\partial P^m}{\partial\rho}h,\bar{h}\right)_{L^2(0,2\pi)}-\left(P^m\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}P^m h,\bar{h}\right)_{L^2(0,2\pi)}-\left(P^m\frac{1}{\rho}h,\bar{h}\right)_{L^2(0,2\pi)}=\big(h,\bar{h}\big)_{L^2(0,2\pi)}.$$

Considering $\phi\in\mathcal{C}^1_0(\varepsilon,a]$ (that is, $\phi(\varepsilon)=0$ and we can have $\phi(a)\neq 0$), we have:

$$\begin{aligned}&\int_\varepsilon^a\left(\frac{\partial P^m}{\partial\rho}h,\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho-\int_\varepsilon^a\left(P^m\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}P^m h,\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho\\&-\int_\varepsilon^a\left(P^m\frac{1}{\rho}h,\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho=\int_\varepsilon^a\big(h,\bar{h}\big)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho.\end{aligned}$$

Integrating by parts the first term, since $P^m(a)=0$ (and $\phi(\varepsilon)=0$), we obtain

$$\begin{aligned}&\int_\varepsilon^a-\big(P^m h,\bar{h}\big)_{L^2(0,2\pi)}\phi'\rho\,\mathrm{d}\rho-\int_\varepsilon^a\left(\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}P^m h,P^m\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho\\&-2\int_\varepsilon^a\left(\frac{1}{\rho}h,P^m\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho=\int_\varepsilon^a\big(h,\bar{h}\big)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho.\end{aligned}$$

Now, integrating by parts the second term, and taking into account the periodic boundary conditions, we achieve

$$\begin{aligned}&\int_\varepsilon^a-\big(P^m h,\bar{h}\big)_{L^2(0,2\pi)}\phi'\rho\,\mathrm{d}\rho+\int_\varepsilon^a\left(\frac{1}{\rho^2}\frac{\partial}{\partial\theta}P^m h,\frac{\partial}{\partial\theta}P^m\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho\\&-2\int_\varepsilon^a\left(\frac{1}{\rho}h,P^m\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho=\int_\varepsilon^a\big(h,\bar{h}\big)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho.\end{aligned}$$

In the previous equality all the integrands are bounded, as a consequence of Corollary 2.6.6. In fact, for $h\in L^2(0,2\pi)$ we have $\|P^m h\|_{H^1_{\rho,P}(0,2\pi)}$ bounded and consequently both $\|P^m h\|_{L^2(0,2\pi)}$ and $\left\|\frac{1}{\rho}\frac{\partial}{\partial\theta}(P^m h)\right\|_{L^2(0,2\pi)}$ are bounded. Then, we can use Lebesgue's theorem and according to Corollary 2.8.3, we can pass to the limit and obtain

$$\begin{aligned}&\int_\varepsilon^a-\big(Ph,\bar{h}\big)_{L^2(0,2\pi)}\phi'\rho\,\mathrm{d}\rho+\int_\varepsilon^a\left(\frac{1}{\rho^2}\frac{\partial}{\partial\theta}Ph,\frac{\partial}{\partial\theta}P\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho\\&-2\int_\varepsilon^a\left(\frac{1}{\rho}h,P\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho=\int_\varepsilon^a\big(h,\bar{h}\big)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho.\end{aligned}\tag{2.64}$$

In fact, since $P^m h \to Ph$ strongly in $H^1_{\rho,P}(0,2\pi)$, then $\dfrac{\partial}{\partial\theta}P^m h \to \dfrac{\partial}{\partial\theta}Ph$ strongly in $L^2(0,2\pi)$.

Now, since $\mathcal{D}(\varepsilon,a) \subset \mathcal{C}^1_0(\varepsilon,a]$, we can take $\phi \in \mathcal{D}(\varepsilon,a)$ in the previous equality and integrate backwards the first term, obtaining

$$\begin{aligned}&\int_\varepsilon^a \left(\frac{\partial P}{\partial\rho}h,\bar h\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho + \int_\varepsilon^a \left(\frac{1}{\rho^2}\frac{\partial}{\partial\theta}Ph, \frac{\partial}{\partial\theta}P\bar h\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho\\ &-\int_\varepsilon^a \left(\frac{1}{\rho}h, P\bar h\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho = \int_\varepsilon^a \left(h,\bar h\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho,\end{aligned}$$

for $h,\bar h \in V^{m_0}$. Then, by density (see (c) in section 2.3.), when $m_0 \to \infty$, we obtain

$$\begin{aligned}&\int_\varepsilon^a \left(\frac{\partial P}{\partial\rho}h,\bar h\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho + \int_\varepsilon^a \left(\frac{1}{\rho^2}\frac{\partial}{\partial\theta}Ph, \frac{\partial}{\partial\theta}P\bar h\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho\\ &-\int_\varepsilon^a \left(\frac{1}{\rho}h, P\bar h\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho = \int_\varepsilon^a \left(h,\bar h\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho,\end{aligned}$$

for $h,\bar h \in L^2(0,2\pi)$. Thus, from the equality in $\mathcal{D}'(\varepsilon,a)$

$$\left(\frac{\partial P}{\partial\rho}h,\bar h\right)_{L^2(0,2\pi)} = -\left(\frac{1}{\rho^2}\frac{\partial}{\partial\theta}Ph, \frac{\partial}{\partial\theta}P\bar h\right)_{L^2(0,2\pi)} + \left(\frac{1}{\rho}h, P\bar h\right)_{L^2(0,2\pi)} + \left(h,\bar h\right)_{L^2(0,2\pi)},$$

and using again Corollary 2.6.6 (notice that the result is independent of m), we see that $\left(\dfrac{\partial P}{\partial\rho}h,\bar h\right)_{L^2(0,2\pi)} \in L^\infty(\varepsilon,a)$. Then, from $\left(\dfrac{\partial P}{\partial\rho}h,\bar h\right)_{L^2(0,2\pi)} \in L^2_\rho(\varepsilon,a)$ and $\left(P(\rho)h,\bar h\right)_{L^2(0,2\pi)} \in L^2_\rho(\varepsilon,a)$ (again $\left(P(\rho)h,\bar h\right)_{L^2(0,2\pi)} \in L^\infty(\varepsilon,a)$), we deduce that $\left(P(\rho)h,\bar h\right)_{L^2(0,2\pi)}$ is continuous in ρ. Consequently, for $\phi \in \mathcal{C}^1_0(\varepsilon,a]$ we can integrate (2.64) backwards to obtain $P(a) = 0$. ∎

Similarly, recalling the equation on r^m of (2.34), we obtain the following result:

Proposition 2.8.5. *The function r belongs to $\mathcal{C}\left(\varepsilon,a,L^2(0,2\pi)\right)$, satisfies $r(a)=0$, and for every h in $H^{1/2}_{\rho,P}(0,2\pi)$ verifies the following equation*

$$\left\langle \frac{1}{\rho^2}\frac{\partial r}{\partial\theta}, \frac{\partial}{\partial\theta}Ph\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)} + \left\langle \frac{\partial r}{\partial\rho}, h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)} = (f,Ph)_{L^2(0,2\pi)},$$

in $\mathcal{D}'(\varepsilon,a)$.

Proof. For a fixed m_0, let $h \in V^{m_0}$. Then, from (2.34), we obtain, for $m \geq m_0$

$$\begin{aligned}&\left\langle -P^m\frac{1}{\rho^2}\frac{\partial^2 r^m}{\partial\theta^2}, h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)} + \left\langle \frac{\partial r^m}{\partial\rho}, h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)}\\ &= (P^m f,h)_{L^2(0,2\pi)}.\end{aligned}$$

Considering $\phi \in \mathcal{C}_0^1(\varepsilon, a]$, we have:

$$\int_\varepsilon^a \left\langle -P^m \frac{1}{\rho^2}\frac{\partial^2 r^m}{\partial\theta^2}, h\right\rangle_{{H^{1/2}_{\rho,P}(0,2\pi)}', H^{1/2}_{\rho,P}(0,2\pi)} \phi\rho\,\mathrm{d}\rho + \int_\varepsilon^a \left\langle \frac{\partial r^m}{\partial\rho}, h\right\rangle_{{H^{1/2}_{\rho,P}(0,2\pi)}', H^{1/2}_{\rho,P}(0,2\pi)} \phi\rho\,\mathrm{d}\rho = \int_\varepsilon^a (P^m f, h)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho.$$

Integrating by parts the second term, since $r^m(a) = 0$ (and $\phi(\varepsilon) = 0$), we obtain

$$\int_\varepsilon^a \left\langle -\frac{1}{\rho^2}\frac{\partial^2 r^m}{\partial\theta^2}, P^m h\right\rangle_{{H^{3/2}_{\rho,P}(0,2\pi)}', H^{3/2}_{\rho,P}(0,2\pi)} \phi\rho\,\mathrm{d}\rho - \int_\varepsilon^a (r^m, h)_{L^2(0,2\pi)}\,\phi'\rho\,\mathrm{d}\rho - \int_\varepsilon^a \left(r^m, \frac{1}{\rho}h\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho = \int_\varepsilon^a (P^m f, h)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho.$$

Integrating by parts the first term and according to the periodic boundary conditions, we have

$$\int_\varepsilon^a \left\langle \frac{1}{\rho^2}\frac{\partial r^m}{\partial\theta}, \frac{\partial}{\partial\theta}P^m h\right\rangle_{{H^{1/2}_{\rho,P}(0,2\pi)}', H^{1/2}_{\rho,P}(0,2\pi)} \phi\rho\,\mathrm{d}\rho - \int_\varepsilon^a (r^m, h)_{L^2(0,2\pi)}\,\phi'\rho\,\mathrm{d}\rho - \int_\varepsilon^a \left(r^m, \frac{1}{\rho}h\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho = \int_\varepsilon^a (f, P^m h)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho.$$

From Corollary 2.8.3 and Lebesgue's theorem (again all the integrands are bounded as a consequence of Proposition 2.6.8), we can pass to the limit in the previous equality. Then,

$$\int_\varepsilon^a \left\langle \frac{1}{\rho^2}\frac{\partial r}{\partial\theta}, \frac{\partial}{\partial\theta}Ph\right\rangle_{{H^{1/2}_{\rho,P}(0,2\pi)}', H^{1/2}_{\rho,P}(0,2\pi)} \phi\rho\,\mathrm{d}\rho - \int_\varepsilon^a (r, h)_{L^2(0,2\pi)}\,\phi'\rho\,\mathrm{d}\rho - \int_\varepsilon^a \left(r, \frac{1}{\rho}h\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho = \int_\varepsilon^a (f, Ph)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho. \tag{2.65}$$

In fact, since $r^m \to r$ strongly in $H^{1/2}_{\rho,P}(0, 2\pi)$, we have $\frac{\partial}{\partial\theta}r^m \to \frac{\partial}{\partial\theta}r$ strongly in $\left(H^{1/2}_{\rho,P}(0, 2\pi)\right)'$. We also have $\frac{\partial}{\partial\theta}P^m h \to \frac{\partial}{\partial\theta}Ph$ weakly in $H^{1/2}_{\rho,P}(0, 2\pi)$. Now, since $\mathcal{D}(\varepsilon, a) \subset \mathcal{C}_0^1(\varepsilon, a]$, we can take $\phi \in \mathcal{D}(\varepsilon, a)$ in the previous equality and integrate backwards the second term, obtaining

$$\int_\varepsilon^a \left\langle \frac{1}{\rho^2}\frac{\partial r}{\partial\theta}, \frac{\partial}{\partial\theta}Ph\right\rangle_{{H^{1/2}_{\rho,P}(0,2\pi)}', H^{1/2}_{\rho,P}(0,2\pi)} \phi\rho\,\mathrm{d}\rho + \int_\varepsilon^a \left\langle \frac{\partial r}{\partial\rho}, h\right\rangle_{{H^{1/2}_{\rho,P}(0,2\pi)}', H^{1/2}_{\rho,P}(0,2\pi)} \phi\rho\,\mathrm{d}\rho = \int_\varepsilon^a (f, Ph)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho,$$

for $h \in V^{m_0}$.

Then, by density, when $m_0 \to \infty$, we have

$$\int_\varepsilon^a \left\langle \frac{1}{\rho^2}\frac{\partial r}{\partial\theta}, \frac{\partial}{\partial\theta}Ph\right\rangle_{{H^{1/2}_{\rho,P}(0,2\pi)}', H^{1/2}_{\rho,P}(0,2\pi)} \phi\rho\,\mathrm{d}\rho + \int_\varepsilon^a \left\langle \frac{\partial r}{\partial\rho}, h\right\rangle_{{H^{1/2}_{\rho,P}(0,2\pi)}', H^{1/2}_{\rho,P}(0,2\pi)} \phi\rho\,\mathrm{d}\rho = \int_\varepsilon^a (f, Ph)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho,$$

for $h \in H^{1/2}_{\rho,P}(0,2\pi)$ (notice that with this choice for h, the first term is well defined).

Again by Proposition 2.6.8 (the result is independent of m), from the equality $\left\langle \frac{\partial r}{\partial \rho}, h \right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)} = -\left\langle \frac{1}{\rho^2}\frac{\partial r}{\partial \theta}, \frac{\partial}{\partial \theta}Ph \right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)} + (f, Ph)_{L^2(0,2\pi)}$, in $\mathcal{D}'(\varepsilon,a)$, we obtain $\frac{\partial r}{\partial \rho} \in L^\infty\left(\varepsilon, a, \left(H^{1/2}_{\rho,P}(0,2\pi)\right)'\right)$. Then, from $\frac{\partial r}{\partial \rho} \in L^2_\rho\left(\varepsilon, a, \left(H^{1/2}_{\rho,P}(0,2\pi)\right)'\right)$ and $r \in L^2_\rho\left(\varepsilon, a, H^{1/2}_{\rho,P}(0,2\pi)\right)$, we deduce that $r \in \mathcal{C}\left(\varepsilon, a, L^2(0,2\pi)\right)$. Consequently, for $\phi \in \mathcal{C}^1_0(\varepsilon, a]$ we can integrate (2.65) backwards to obtain $r(a) = 0$. ∎

Finally, with respect to the last equation of (2.34), we obtain:

Proposition 2.8.6. *For every h in $\left(H^{1/2}_{\rho,P}(0,2\pi)\right)'$, $\hat{u}_\varepsilon$ satisfies the following equation*

$$\langle \hat{u}_\varepsilon, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} = \left\langle P\frac{\partial \hat{u}_\varepsilon}{\partial \rho}, h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} + \langle r, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '}$$

in $\mathcal{D}'(\varepsilon, a)$.

Proof. For a fixed m_0 let $h \in V^{m_0}$. Then, from (2.34), we obtain, for $m \geq m_0$

$$\begin{aligned} &\left\langle P^m \frac{\partial \hat{u}^m_\varepsilon}{\partial \rho}, h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} - \langle \hat{u}^m_\varepsilon, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} \\ =\ &\langle -r^m, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '}. \end{aligned}$$

Considering $\phi \in \mathcal{D}(\varepsilon, a)$, we have:

$$\begin{aligned} &\int_\varepsilon^a \left\langle P^m \frac{\partial \hat{u}^m_\varepsilon}{\partial \rho}, h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} \phi\rho\,\mathrm{d}\rho \\ &- \int_\varepsilon^a \langle \hat{u}^m_\varepsilon, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} \phi\rho\,\mathrm{d}\rho = \int_\varepsilon^a \langle -r^m, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} \phi\rho\,\mathrm{d}\rho \\ \Rightarrow\ &\int_\varepsilon^a \left\langle \frac{\partial \hat{u}^m_\varepsilon}{\partial \rho}, P^m h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} \phi\rho\,\mathrm{d}\rho \\ &- \int_\varepsilon^a \langle \hat{u}^m_\varepsilon, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} \phi\rho\,\mathrm{d}\rho = \int_\varepsilon^a \langle -r^m, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} \phi\rho\,\mathrm{d}\rho. \end{aligned}$$

Then, by Proposition 2.8.2, Corollary 2.8.3 and Lebesgue's theorem, we can pass to the limit and obtain

$$\begin{aligned} &\int_\varepsilon^a \left\langle \frac{\partial \hat{u}_\varepsilon}{\partial \rho}, Ph\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} \phi\rho\,\mathrm{d}\rho - \int_\varepsilon^a \langle \hat{u}_\varepsilon, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} \phi\rho\,\mathrm{d}\rho \\ &= \int_\varepsilon^a \langle -r, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} \phi\rho\,\mathrm{d}\rho, \end{aligned}$$

for $h \in V^{m_0}$. Then, by density we have

$$\int_\varepsilon^a \left\langle \frac{\partial \hat{u}_\varepsilon}{\partial \rho}, Ph \right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)'} \phi\rho \,\mathrm{d}\rho - \int_\varepsilon^a \langle \hat{u}_\varepsilon, h \rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)'} \phi\rho \,\mathrm{d}\rho =$$
$$\int_\varepsilon^a \langle -r, h \rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)'} \phi\rho \,\mathrm{d}\rho,$$

for $h \in \left(H^{1/2}_{\rho,P}(0,2\pi)\right)'$. ■

Since P^m and r^m do not depend on ε, further we have:

Remark 2.8.7. *The convergence established in Corollary 2.8.3 for all (ε, a), is valid for ε arbitrarily small. Consequently, $P^m(\rho)h \to P(\rho)h$ and $r^m(\rho) \to r(\rho)$, strongly in $H^{1/2}_{\rho,P}(0,2\pi)$, for all $\rho \in (0,a)$.*

In the next two Propositions we are going to establish the comportment of P and r in a neighborhood of the origin. Using Proposition 2.8.4 and the late remark we can conclude that the coordinates of P, in the interval $(0,a)$, are exactly the ones previously achieved in (2.39).

Proposition 2.8.8. *For P satisfying $\frac{\partial}{\partial\rho}P - \frac{1}{\rho^2}P\frac{\partial^2}{\partial\theta^2}P - \frac{1}{\rho}P - I = 0$ and $P(a) = 0$, we have $\lim_{\rho\to 0} \|P(\rho)\|_{\mathcal{L}\left(H^{1/2}_{\rho,P}(0,2\pi)',H^{1/2}_{\rho,P}(0,2\pi)\right)} = 1$ and $\lim_{\rho\to 0} \|P(\rho)\|_{\mathcal{L}(L^2(0,2\pi),L^2(0,2\pi))} = 0$. Moreover, we have $\lim_{\rho\to 0} \left\|P(\rho) - \rho\left(P_\infty \, o \, P_{|M}\right)\right\|_{\mathcal{L}\left(H^{1/2}_{\rho,P}(0,2\pi)',H^{1/2}_{\rho,P}(0,2\pi)\right)} = 0$, where P_∞ is the operator satisfying $-P_\infty \frac{\partial^2}{\partial\theta^2}P_\infty = I$, and $P_{|M}$ is the projection operator on the space M.*

Proof. From (2.40), we have

$$\lim_{\rho\to 0} p_1(\rho) = \lim_{\rho\to 0} \rho \log\left(\frac{\rho}{a}\right) = 0.$$

Also, for $i \geq 2$, and using (2.39) it comes,

$$\lim_{\rho\to 0} p_i(\rho) = \lim_{\rho\to 0} -\frac{\rho}{\sqrt{\lambda_i}} \left(\frac{\left(\frac{a}{\rho}\right)^{\sqrt{\lambda_i}} - \left(\frac{\rho}{a}\right)^{\sqrt{\lambda_i}}}{\left(\frac{a}{\rho}\right)^{\sqrt{\lambda_i}} + \left(\frac{\rho}{a}\right)^{\sqrt{\lambda_i}}} \right) = 0,$$

and

$$\lim_{\rho\to 0}\frac{p_i(\rho)}{\rho}=\lim_{\rho\to 0}-\frac{1}{\sqrt{\lambda_i}}\left(\frac{\left(\frac{a}{\rho}\right)^{\sqrt{\lambda_i}}-\left(\frac{\rho}{a}\right)^{\sqrt{\lambda_i}}}{\left(\frac{a}{\rho}\right)^{\sqrt{\lambda_i}}+\left(\frac{\rho}{a}\right)^{\sqrt{\lambda_i}}}\right)=-\frac{1}{\sqrt{\lambda_i}}, \tag{2.66}$$

which means that $p_i(\rho)\sim-\rho$ $(i\geq 2)$.

Then, we have

$$\|P(\rho)\|_{\mathcal{L}\left(H^{1/2}_{\rho,P}(0,2\pi)'\!,\,H^{1/2}_{\rho,P}(0,2\pi)\right)}=\sup_{\substack{h\neq 0\\ h\in H^{1/2}_{\rho,P}(0,2\pi)'}}\frac{\|P(\rho)h\|_{H^{1/2}_{\rho,P}(0,2\pi)}}{\|h\|_{H^{1/2}_{\rho,P}(0,2\pi)'}}$$

$$=\sup_{\substack{h\neq 0\\ h\in H^{1/2}_{\rho,P}(0,2\pi)'}}\frac{\left(h_1^2p_1^2+\sum_2^\infty\frac{\sqrt{\lambda_i}}{\rho}h_i^2p_i^2\right)^{1/2}}{\left(h_1^2+\sum_2^\infty\frac{\rho}{\sqrt{\lambda_i}}h_i^2\right)^{1/2}}=\sup_{\substack{h\neq 0\\ h\in H^{1/2}_{\rho,P}(0,2\pi)'}}\left(\frac{h_1^2p_1^2+\sum_2^\infty\frac{\rho}{\sqrt{\lambda_i}}h_i^2\frac{\lambda_i}{\rho^2}p_i^2}{h_1^2+\sum_2^\infty\frac{\rho}{\sqrt{\lambda_i}}h_i^2}\right)^{1/2}$$

$$=\left(\sup_{\substack{h\neq 0\\ h\in H^{1/2}_{\rho,P}(0,2\pi)'}}\frac{h_1^2p_1^2+\sum_2^\infty\frac{\rho}{\sqrt{\lambda_i}}h_i^2\frac{\lambda_i}{\rho^2}p_i^2}{h_1^2+\sum_2^\infty\frac{\rho}{\sqrt{\lambda_i}}h_i^2}\right)^{1/2}.$$

Since we have (notice that $\rho<a$)

$$\lim_{i\to\infty}\frac{\lambda_i}{\rho^2}p_i^2=\lim_{i\to\infty}\left(\frac{\left(\frac{a}{\rho}\right)^{\sqrt{\lambda_i}}-\left(\frac{\rho}{a}\right)^{\sqrt{\lambda_i}}}{\left(\frac{a}{\rho}\right)^{\sqrt{\lambda_i}}+\left(\frac{\rho}{a}\right)^{\sqrt{\lambda_i}}}\right)^2=1,$$

the quantity $\frac{\lambda_i}{\rho^2}p_i^2$ remains bounded, for increasing values of i. For this reason we obtain

$$\sup_{\substack{h\neq 0\\ h\in H^{1/2}_{\rho,P}(0,2\pi)'}}\frac{h_1^2p_1^2+\sum_2^\infty\frac{\rho}{\sqrt{\lambda_i}}h_i^2\frac{\lambda_i}{\rho^2}p_i^2}{h_1^2+\sum_2^\infty\frac{\rho}{\sqrt{\lambda_i}}h_i^2}=\max\left\{p_1^2,\left(\frac{\lambda_i}{\rho^2}p_i^2\right)_{i\geq 2}\right\}.$$

Consequently,

$$\lim_{\rho\to 0}\|P(\rho)\|^2_{\mathcal{L}\left(H^{1/2}_{\rho,P}(0,2\pi)'\!,\,H^{1/2}_{\rho,P}(0,2\pi)\right)}=\lim_{\rho\to 0}\max\left\{p_1^2,\left(\frac{\lambda_i}{\rho^2}p_i^2\right)_{i\geq 2}\right\}$$

$$=\max\left\{\lim_{\rho\to 0}p_1^2,\lim_{\rho\to 0}\left(\frac{\lambda_i}{\rho^2}p_i^2\right)_{i\geq 2}\right\}=\max\left\{0,\left(\frac{\lambda_i}{\lambda_i}\right)_{i\geq 2}\right\}=1.$$

Using the same arguments, we have

$$\lim_{\rho\to 0}\|P(\rho)\|^2_{\mathcal{L}(L^2(0,2\pi),L^2(0,2\pi))}=\lim_{\rho\to 0}\left(\sup_{\substack{h\neq 0\\ h\in L^2(0,2\pi)}}\frac{\|P(\rho)h\|_{L^2(0,2\pi)}}{\|h\|_{L^2(0,2\pi)}}\right)=\lim_{\rho\to 0}\left(\sup_{\substack{h\neq 0\\ h\in L^2(0,2\pi)}}\frac{\sum_1^\infty h_i^2p_i^2}{\sum_1^\infty h_i^2}\right)$$

$$=\lim_{\rho\to 0}\max\left\{\left(p_i^2\right)_{i\geq 1}\right\}=\max\left\{\lim_{\rho\to 0}p_1^2,\lim_{\rho\to 0}\left(p_i^2\right)_{i\geq 2}\right\}=\max\left\{0,(0)_{i\geq 2}\right\}=0.$$

Furthermore, considering the operator P_∞ defined by $-P_\infty\frac{\partial^2}{\partial\theta^2}P_\infty=I$ (notice that P satisfies the equation $\rho\frac{\partial}{\partial\rho}\left(\frac{P}{\rho}\right)-\frac{P}{\rho}\frac{\partial^2}{\partial\theta^2}\frac{P}{\rho}=I$ and that $\lim_{\rho\to 0}\frac{p_i(\rho)}{\rho}=-\frac{1}{\sqrt{\lambda_i}}$, for $i\geq 2$), the coordinates of P_∞ satisfy, for $i\geq 2$, $p_{\infty i}^2\lambda_i=1$, and consequently, satisfies $p_{\infty i}=-\frac{1}{\sqrt{\lambda_i}}$. Then, if we consider the operator P_∞^M, defined as the result of the composition of P_∞ with the projection operator on the space M, the coordinates of P_∞^M satisfy $p^M_{\infty 1}=0$ and $p^M_{\infty i}=p_{\infty i}$, for $i\geq 2$. Then,

$$\|P(\rho)-\rho P_\infty^M\|_{\mathcal{L}\left(H^{1/2}_{\rho,P}(0,2\pi)',H^{1/2}_{\rho,P}(0,2\pi)\right)}$$

$$=\sup_{\substack{h\neq 0\\ h\in H^{1/2}_{\rho,P}(0,2\pi)'}}\frac{\left(h_1^2\left(p_1-\rho p^M_{\infty 1}\right)^2+\sum_2^\infty\frac{\sqrt{\lambda_i}}{\rho}h_i^2(p_i-\rho p^M_{\infty i})^2\right)^{1/2}}{\left(h_1^2+\sum_2^\infty\frac{\rho}{\sqrt{\lambda_i}}h_i^2\right)^{1/2}}$$

$$=\left(\sup_{\substack{h\neq 0\\ h\in H^{1/2}_{\rho,P}(0,2\pi)'}}\frac{h_1^2p_1^2+\sum_2^\infty\frac{\rho}{\sqrt{\lambda_i}}h_i^2\frac{\lambda_i}{\rho^2}(p_i-\rho p_{\infty i})^2}{h_1^2+\sum_2^\infty\frac{\rho}{\sqrt{\lambda_i}}h_i^2}\right)^{1/2}$$

$$=\left(\max\left\{p_1^2,\left(\frac{\lambda_i}{\rho^2}(p_i-\rho p_{\infty i})^2\right)_{i\geq 2}\right\}\right)^{1/2}=\left(\max\left\{p_1^2,\left(\frac{\lambda_i}{\rho^2}(p_i+\rho\frac{1}{\sqrt{\lambda_i}})^2\right)_{i\geq 2}\right\}\right)^{1/2}.$$

Then,

$$\begin{aligned}\lim_{\rho\to 0}\|P(\rho)-\rho P_\infty^M\|^2_{\mathcal{L}\left(H^{1/2}_{\rho,P}(0,2\pi)',H^{1/2}_{\rho,P}(0,2\pi)\right)}&=\max\left\{\lim_{\rho\to 0}p_1^2,\lim_{\rho\to 0}\left(\frac{\lambda_i}{\rho^2}(p_i+\rho\frac{1}{\sqrt{\lambda_i}})^2\right)_{i\geq 2}\right\}\\&=\max\{0,0\}=0.\end{aligned}$$

■

Remark 2.8.9. *From the equality (2.66), since* $\frac{\left(\frac{a}{\rho}\right)^{\sqrt{\lambda_i}}-\left(\frac{\rho}{a}\right)^{\sqrt{\lambda_i}}}{\left(\frac{a}{\rho}\right)^{\sqrt{\lambda_i}}+\left(\frac{\rho}{a}\right)^{\sqrt{\lambda_i}}}\leq 1$, *we also have*

$$\left|\frac{p_i(\rho)}{\rho}\right| \leq \frac{1}{\sqrt{\lambda_i}},\ \forall \rho \in (0,a]. \tag{2.67}$$

We can remark also that, for $i \geq 2$, the function

$$q_i(\rho) = \frac{p_i(\rho)}{\rho} = -\frac{1}{\sqrt{\lambda_i}}\left(\frac{1-\left(\frac{\rho}{a}\right)^{2\sqrt{\lambda_i}}}{1+\left(\frac{\rho}{a}\right)^{2\sqrt{\lambda_i}}}\right)$$

is $\rho-$increasing on the interval $[0,a]$. In fact, it's easy to see that

$$q_i'(\rho) = \frac{4}{\rho}\frac{\left(\frac{\rho}{a}\right)^{2\sqrt{\lambda_i}}}{\left(1+\left(\frac{\rho}{a}\right)^{2\sqrt{\lambda_i}}\right)^2} > 0,\ \forall \rho \in (0,a].$$

Further, from $q_i(0) = -\dfrac{1}{\sqrt{\lambda_i}}$ *and* $\dfrac{\left(\frac{a}{\rho}\right)^{\sqrt{\lambda_i}} - \left(\frac{\rho}{a}\right)^{\sqrt{\lambda_i}}}{\left(\frac{a}{\rho}\right)^{\sqrt{\lambda_i}} + \left(\frac{\rho}{a}\right)^{\sqrt{\lambda_i}}} \leq 1$, *we also have* $q_i(0) < q_i(\rho), \forall \rho \in (0,a]$ *so we can extend the result to the interval* $[0,a]$.

Considering now r, the solution of $-Pf - P\dfrac{1}{\rho^2}\dfrac{\partial^2 r}{\partial\theta^2} + \dfrac{\partial r}{\partial\rho} = 0$ and $r(a) = 0$, its coordinates satisfy,

$$-p_i\,\hat{f}_i + p_i\frac{1}{\rho^2}\lambda_i r_i + \frac{\partial r_i}{\partial\rho} = 0, \tag{2.68}$$

with $r_i(a) = 0$, for $i \geq 1$.

Since, for $i \geq 2$, $\dfrac{\partial r_i}{\partial\rho} + \left(p_i\dfrac{1}{\rho^2}\lambda_i\right) r_i = p_i\,\hat{f}_i$ is a linear differential equation with non constant coefficients, it has the explicit solution

$$\begin{aligned}
r_i(\rho) &= e^{-\int_a^\rho p_i(\varrho)\frac{\lambda_i}{\varrho^2}\,\mathrm{d}\varrho}\left(\int_a^\rho e^{\int_a^t p_i(\varrho)\frac{\lambda_i}{\varrho^2}\,\mathrm{d}\varrho} p_i(t)\hat{f}_i(t)\,\mathrm{d}t\right) \\
&= \int_a^\rho e^{\int_\rho^t p_i(\varrho)\frac{\lambda_i}{\varrho^2}\,\mathrm{d}\varrho} p_i(t)\hat{f}_i(t)\,\mathrm{d}t = -\int_\rho^a e^{\int_\rho^t \frac{p_i(\varrho)\lambda_i}{\varrho^2}\,\mathrm{d}\varrho} p_i(t)\,\hat{f}_i(t)\,\mathrm{d}t \\
&= -\int_0^a \chi_{[\rho,a]} e^{\int_\rho^t \frac{p_i(\varrho)\lambda_i}{\varrho^2}\,\mathrm{d}\varrho} p_i(t)\,\hat{f}_i(t)\,\mathrm{d}t.
\end{aligned} \tag{2.69}$$

For $i = 1$ we obtain the equation $-p_1(\rho)\hat{f}_1 + \dfrac{\partial r_1}{\partial\rho} = 0$, which again can be integrated as an equation of separable variables. Therefore, we obtain $r_1(\rho) = \displaystyle\int_a^\rho p_1(t)\hat{f}_1(t)\,\mathrm{d}t$ and consequently

$$r_1(0) = \lim_{\rho\to 0} r_1(\rho) = -\int_0^a p_1(t)\hat{f}_1(t)\,\mathrm{d}t. \tag{2.70}$$

One can see that this is the value obtained before in (2.44). In fact, we have

$$r_1(\rho) = \int_a^\rho p_1(t)\hat{f}_1(t)\,\mathrm{d}t = \int_a^\rho t\,\log\left(\frac{t}{a}\right)\hat{f}_1(t)\,\mathrm{d}t$$

and, on the other hand, integrating by parts, we get

$$\begin{aligned} r_1(\rho) = \int_\rho^a \frac{1}{t}\int_\rho^t \varrho\hat{f}_1(\varrho)\,\mathrm{d}\varrho\,\mathrm{d}t &= \left(\log(t)\int_\rho^t \left(\varrho\hat{f}_1(\varrho)\right)\,\mathrm{d}\varrho\right)\Big]_\rho^a - \int_\rho^a \log(t)\,t\,\hat{f}_1(t)\,\mathrm{d}t \\ &= \int_\rho^a \log(a)\,t\hat{f}_1(t)\,\mathrm{d}t - \int_\rho^a \log(t)\,t\,\hat{f}_1(t)\,\mathrm{d}t. \end{aligned}$$

Before establishing the behavior of r near the origin, we need two auxiliary results.

Lemma 2.8.10. *The series* $\sum_2^\infty r_i^2(\rho)$ *is uniformly convergent on* $[0,a]$.

Proof. For all $\rho \in [0,a]$, from $\hat{f}_i(\rho) = \int_0^{2\pi} \hat{f}(\rho,\theta)\,w_i(\theta)\,\mathrm{d}\theta$, for $i \geq 2$, and considering, without loss of generality, $w_i(\theta) = \sin(i\theta)$, we obtain

$$\begin{aligned} \hat{f}_i(\rho) &= -\frac{\cos(i\theta)}{i}\hat{f}\Big|_0^{2\pi} + \int_0^{2\pi} \frac{\partial\hat{f}}{\partial\theta}\frac{\cos(i\theta)}{i}\,\mathrm{d}\theta \\ &= -\frac{\cos(i\,2\pi)}{i}\hat{f}(\rho,2\pi) + \frac{\cos(0)}{i}\hat{f}(\rho,0) + \int_0^{2\pi}\frac{\partial\hat{f}}{\partial\theta}\frac{\cos(i\theta)}{i}\,\mathrm{d}\theta \\ &= -\frac{1}{i}\hat{f}(\rho,2\pi) + \frac{1}{i}\hat{f}(\rho,0) + \int_0^{2\pi}\frac{\partial\hat{f}}{\partial\theta}\frac{\cos(i\theta)}{i}\,\mathrm{d}\theta \\ &= \int_0^{2\pi}\frac{\partial\hat{f}}{\partial\theta}\frac{\cos(i\theta)}{i}\,\mathrm{d}\theta, \end{aligned}$$

since $\hat{f}(\rho,\theta)$ is $\theta-$ periodic. Then

$$\left|\hat{f}_i(\rho)\right| \leq \int_0^{2\pi}\left|\frac{\partial\hat{f}}{\partial\theta}\right|\frac{|\cos(i\theta)|}{i}\,\mathrm{d}\theta \leq \int_0^{2\pi} c_1\frac{1}{i}\,\mathrm{d}\theta = 2c_1\pi\frac{1}{i} = \frac{c}{i} \tag{2.71}$$

since $\frac{\partial\hat{f}}{\partial\theta}$ is bounded (notice that $f \in C^1(\Omega)$ and consequently $\left|\frac{\partial\hat{f}}{\partial\theta}\right| \leq \left|\frac{\partial f}{\partial x}\right|\rho|\sin(\theta)| + \left|\frac{\partial f}{\partial y}\right|\rho|\cos(\theta)| < c_1$). Furthermore, from (2.69), using (2.71) and the fact that $p_i(\rho) < 0$ $(i \geq 1)$, we obtain, again for $i \geq 2$ and for all $\rho \in [0,a]$,

$$|r_i(\rho)| = \left|\int_\rho^a e^{\int_\rho^t \frac{p_i(\varrho)\lambda_i}{\varrho^2}\,\mathrm{d}\varrho} p_i(t)\,\hat{f}_i(t)\,\mathrm{d}t\right| \leq \int_\rho^a e^{\int_\rho^t \frac{p_i(\varrho)\lambda_i}{\varrho^2}\,\mathrm{d}\varrho}|p_i(t)|\,|\hat{f}_i(t)|\,\mathrm{d}t$$

$$\leq \int_\rho^a |p_i(t)|\,|\hat{f}_i(t)|\,\mathrm{d}t \leq \int_\rho^a \frac{t}{\sqrt{\lambda_i}}\frac{c}{i}\,\mathrm{d}t \leq \frac{a}{\sqrt{\lambda_i}}\frac{c}{i}(a-\rho) \leq \frac{a^2}{\sqrt{\lambda_i}}\frac{c}{i} \leq \frac{a^2}{\sqrt{\lambda_2}}\frac{c}{i} = \frac{\sqrt{c_2}}{i}.$$

Since $\sum_{2}^{\infty} c_2 \frac{1}{i^2}$ is a numerical convergent series, the series $\sum_{2}^{\infty} r_i^2(\rho)$ is uniformly convergent on $[0, a]$. ■

Lemma 2.8.11. *The series* $\sum_{2}^{\infty} \lambda_i \frac{r_i^2(\rho)}{\rho^2}$ *is uniformly convergent on* $[0, \frac{a}{2}]$.

Proof. Since $q_i(\rho)$ is an increasing function (see Remark 2.8.9), in particular on the interval $[0, \frac{a}{2}]$, we have $\frac{p_i(\rho)}{\rho} \leq \frac{p_i(\frac{a}{2})}{\frac{a}{2}} = -\frac{1}{\sqrt{\lambda_i}} \left(\frac{1 - \left(\frac{1}{2}\right)^{2\sqrt{\lambda_i}}}{1 + \left(\frac{1}{2}\right)^{2\sqrt{\lambda_i}}} \right)$, $\forall \rho \in [0, \frac{a}{2}]$, $\forall i \geq 2$.

We are going to consider $i^\star$ such that $\lambda_i \geq 4, \forall i \geq i^\star$. Then, for $i \geq i^\star$, we have $\frac{1 - \left(\frac{1}{2}\right)^{2\sqrt{\lambda_i}}}{1 + \left(\frac{1}{2}\right)^{2\sqrt{\lambda_i}}} \geq \frac{15}{17} \geq \frac{2}{3}$. Consequently, $p_i(\rho) \leq -\frac{2}{3}\frac{\rho}{\sqrt{\lambda_i}}$, for all $\rho \in [0, \frac{a}{2}]$ and for all $i \geq i^\star$.

Now, for $t \leq \frac{a}{2}$ (and $\rho < t$), we have

$$\begin{aligned} \int_\rho^t \lambda_i \frac{p_i(\varrho)}{\varrho^2} \, \mathrm{d}\varrho &\leq \int_\rho^t -\frac{2}{3} \frac{\varrho}{\sqrt{\lambda_i}} \frac{\lambda_i}{\varrho^2} \, \mathrm{d}\varrho = \int_\rho^t -\frac{2}{3} \frac{\sqrt{\lambda_i}}{\varrho} \, \mathrm{d}\varrho \\ &= -\frac{2\sqrt{\lambda_i}}{3} \log(\varrho)]_\rho^t = \frac{2\sqrt{\lambda_i}}{3} \log\left(\frac{\rho}{t}\right) \end{aligned}$$

and

$$e^{\int_\rho^t \lambda_i \frac{p_i(\varrho)}{\varrho^2} \, \mathrm{d}\varrho} \leq e^{\frac{2\sqrt{\lambda_i}}{3} \log\left(\frac{\rho}{t}\right)} = \left(\frac{\rho}{t}\right)^{\frac{2\sqrt{\lambda_i}}{3}}.$$

In the same way, for $t \geq \frac{a}{2}$ and since $p_i(\varrho) < 0$, we have

$$\begin{aligned} \int_\rho^t \lambda_i \frac{p_i(\varrho)}{\varrho^2} \, \mathrm{d}\varrho &= \int_\rho^{\frac{a}{2}} \lambda_i \frac{p_i(\varrho)}{\varrho^2} \, \mathrm{d}\varrho + \int_{\frac{a}{2}}^t \lambda_i \frac{p_i(\varrho)}{\varrho^2} \, \mathrm{d}\varrho \\ &\leq \int_\rho^{\frac{a}{2}} \lambda_i \frac{p_i(\varrho)}{\varrho^2} \, \mathrm{d}\varrho \leq \frac{2\sqrt{\lambda_i}}{3} \log\left(\frac{\rho}{\frac{a}{2}}\right) \end{aligned}$$

and

$$e^{\int_\rho^t \lambda_i \frac{p_i(\varrho)}{\varrho^2} \, \mathrm{d}\varrho} \leq \left(\frac{\rho}{\frac{a}{2}}\right)^{\frac{2\sqrt{\lambda_i}}{3}}.$$

Therefore, using (2.71), (2.69) and (2.67), we obtain, for $i \geq i^\star$ and $\rho \leq \frac{a}{2}$,

$$
\begin{aligned}
&\left|\frac{r_i(\rho)}{\rho}\right| \\
= \;& \frac{1}{\rho}\left|\int_\rho^a e^{\int_\rho^t \frac{p_i(\varrho)\lambda_i}{\varrho^2}\,\mathrm{d}\varrho} p_i(t)\,\hat{f}_i(t)\,\mathrm{d}t\right| \\
\leq \;& \frac{1}{\rho}\int_\rho^a e^{\int_\rho^t \frac{p_i(\varrho)\lambda_i}{\varrho^2}\,\mathrm{d}\varrho} |p_i(t)|\,|\hat{f}_i(t)|\,\mathrm{d}t \\
= \;& \frac{1}{\rho}\left(\int_\rho^{\frac{a}{2}} e^{\int_\rho^t \frac{p_i(\varrho)\lambda_i}{\varrho^2}\,\mathrm{d}\varrho} |p_i(t)|\,|\hat{f}_i(t)|\,\mathrm{d}t + \int_{\frac{a}{2}}^a e^{\int_\rho^t \frac{p_i(\varrho)\lambda_i}{\varrho^2}\,\mathrm{d}\varrho} |p_i(t)|\,|\hat{f}_i(t)|\,\mathrm{d}t\right) \\
\leq \;& \frac{1}{\rho}\left(\int_\rho^{\frac{a}{2}} \left(\frac{\rho}{t}\right)^{\frac{2\sqrt{\lambda_i}}{3}} |p_i(t)|\,|\hat{f}_i(t)|\,\mathrm{d}t + \int_{\frac{a}{2}}^a \left(\frac{\rho}{\frac{a}{2}}\right)^{\frac{2\sqrt{\lambda_i}}{3}} |p_i(t)|\,|\hat{f}_i(t)|\,\mathrm{d}t\right) \\
\leq \;& \frac{1}{\rho}\left(\int_\rho^{\frac{a}{2}} \left(\frac{\rho}{t}\right)^{\frac{2\sqrt{\lambda_i}}{3}} \frac{t}{\sqrt{\lambda_i}}\frac{c}{i}\,\mathrm{d}t + \int_{\frac{a}{2}}^a \left(\frac{\rho}{\frac{a}{2}}\right)^{\frac{2\sqrt{\lambda_i}}{3}} \frac{t}{\sqrt{\lambda_i}}\frac{c}{i}\,\mathrm{d}t\right) \\
\leq \;& \int_\rho^{\frac{a}{2}} \rho^{\frac{2\sqrt{\lambda_i}}{3}-1} t^{-\frac{2\sqrt{\lambda_i}}{3}} \frac{a}{\sqrt{\lambda_i}}\frac{c}{i}\,\mathrm{d}t + \int_{\frac{a}{2}}^a \left(\frac{\rho}{\frac{a}{2}}\right)^{\frac{2\sqrt{\lambda_i}}{3}} \frac{a}{\sqrt{\lambda_i}}\frac{c}{i}\,\mathrm{d}t \\
\leq \;& \left.\frac{t^{-\frac{2\sqrt{\lambda_i}}{3}+1}}{-\frac{2\sqrt{\lambda_i}}{3}+1}\right]_\rho^{\frac{a}{2}} \rho^{\frac{2\sqrt{\lambda_i}}{3}-1}\frac{a}{\sqrt{\lambda_i}}\frac{c}{i} + \left(\frac{\rho}{\frac{a}{2}}\right)^{\frac{2\sqrt{\lambda_i}}{3}} \frac{a^2}{2\sqrt{\lambda_i}}\frac{c}{i} \\
= \;& \left(\frac{\left(\frac{a}{2}\right)^{-\frac{2\sqrt{\lambda_i}}{3}+1}}{-\frac{2\sqrt{\lambda_i}}{3}+1} - \frac{\rho^{-\frac{2\sqrt{\lambda_i}}{3}+1}}{-\frac{2\sqrt{\lambda_i}}{3}+1}\right)\rho^{\frac{2\sqrt{\lambda_i}}{3}-1}\frac{a}{\sqrt{\lambda_i}}\frac{c}{i} + \left(\frac{\rho}{\frac{a}{2}}\right)^{\frac{2\sqrt{\lambda_i}}{3}-1} \frac{a}{\sqrt{\lambda_i}}\frac{c}{i} \\
= \;& \left(\frac{1-\left(\frac{\rho}{\frac{a}{2}}\right)^{\frac{2\sqrt{\lambda_i}}{3}-1}}{\frac{2\sqrt{\lambda_i}}{3}-1}\right)\frac{a}{\sqrt{\lambda_i}}\frac{c}{i} + \left(\frac{\rho}{\frac{a}{2}}\right)^{\frac{2\sqrt{\lambda_i}}{3}-1} \frac{a}{\sqrt{\lambda_i}}\frac{c}{i} \\
\leq \;& 3\left(1-\left(\frac{\rho}{\frac{a}{2}}\right)^{\frac{2\sqrt{\lambda_i}}{3}-1}\right)\frac{a}{\sqrt{\lambda_i}}\frac{c}{i} + \left(\frac{\rho}{\frac{a}{2}}\right)^{\frac{2\sqrt{\lambda_i}}{3}-1} \frac{a}{\sqrt{\lambda_i}}\frac{c}{i} \\
\leq \;& \frac{3\,a}{\sqrt{\lambda_i}}\frac{c}{i} \leq \frac{3\,a}{\sqrt{\lambda_2}}\frac{c}{i} \\
\leq \;& \sqrt{c_1}\frac{1}{i}.
\end{aligned}
$$

Since $\displaystyle\sum_{i^\star}^{\infty} c_1\frac{1}{i^2}$ is a numerical convergent series, then the series $\displaystyle\sum_{i^\star}^{\infty} \lambda_i\frac{r_i^2(\rho)}{\rho^2}$ is uniformly convergent on $[0, \frac{a}{2}]$ and so it is $\displaystyle\sum_{2}^{\infty} \lambda_i\frac{r_i^2(\rho)}{\rho^2}$. ■

Proposition 2.8.12. *For r, solution of $-Pf - P\dfrac{1}{\rho^2}\dfrac{\partial^2 r}{\partial\theta^2} + \dfrac{\partial r}{\partial\rho} = 0$ and $r(a) = 0$, we have $\lim_{\rho\to 0}\|r(\rho)-r(0)\|_{L^2(0,2\pi)}= 0$ and $\lim_{\rho\to 0}\|r(\rho)-r(0)\|_{H^1_{\rho,P}(0,2\pi)}= 0$ (in particular, $\lim_{\rho\to 0}\|r(\rho)\|_{L^2(0,2\pi)} = |r_1(0)|$ and $\lim_{\rho\to 0}\|r(\rho)\|_{H^1_{\rho,P}(0,2\pi)} = |r_1(0)|$, respectively).*

Proof. From (2.69), and using Lebesgue's (dominated convergence) theorem, we find, for $i \geq 2$,

$$\begin{aligned}\lim_{\rho\to 0} r_i(\rho) &= \lim_{\rho\to 0} -\int_0^a \chi_{[\rho,a]} e^{\int_\rho^t \frac{p_i(\varrho)\lambda_i}{\varrho^2}\,\mathrm{d}\varrho} p_i(t)\,\hat{f}_i(t)\,\mathrm{d}t \\ &= -\int_0^a \lim_{\rho\to 0}\left(e^{\int_\rho^t \frac{p_i(\varrho)}{\varrho}\frac{\lambda_i}{\varrho}\,\mathrm{d}\varrho} \chi_{[\rho,a]} p_i(t)\,\hat{f}_i(t)\right)\mathrm{d}t.\end{aligned}$$

On the other hand, since

$$\lim_{\rho\to 0^+} \frac{\frac{|p_i(\rho)|}{\rho}\frac{\lambda_i}{\rho}}{\frac{1}{\rho}} = \lim_{\rho\to 0^+}\frac{|p_i(\rho)|}{\rho}\lambda_i = \frac{1}{\sqrt{\lambda_i}}\lambda_i = \sqrt{\lambda_i}.$$

we have that the improper integral $\int_0^t \frac{|p_i(\varrho)|}{\varrho}\frac{\lambda_i}{\varrho}\,\mathrm{d}\varrho$ (being of the same nature of $\int_0^t \frac{1}{\rho}$) is divergent and consequently

$$\lim_{\rho\to 0} e^{\int_\rho^t \frac{p_i(\varrho)}{\varrho}\frac{\lambda_i}{\varrho}\,\mathrm{d}\varrho} = e^{-\lim_{\rho\to 0}\int_\rho^t \frac{|p_i(\varrho)|}{\varrho}\frac{\lambda_i}{\varrho}\,\mathrm{d}\varrho} = 0.$$

Hence

$$r_i(0) = \lim_{\rho\to 0} r_i(\rho) = -\int_0^a 0 = 0,$$

since p_i, χ and $\hat{f}_i$ are bounded. Also, we have already seen, in (2.70), that

$$r_1(0) = \lim_{\rho\to 0} r_1(\rho) = -\int_0^a p_1(t)\hat{f}_1(t)\,\mathrm{d}t.$$

Furthermore, using (2.68), we obtain for $i \geq 2$

$$\begin{aligned} &\lim_{\rho\to 0}\frac{r_i(\rho)}{\rho^2} = \lim_{\rho\to 0}\frac{r_i'(\rho)}{2\rho} \\ = \; &\lim_{\rho\to 0}\frac{p_i(\rho)\hat{f}_i(\rho)}{2\rho} - \lim_{\rho\to 0}\frac{p_i(\rho)}{\rho}\frac{\lambda_i}{2}\frac{r_i(\rho)}{\rho^2} \\ = \; &\lim_{\rho\to 0}\frac{p_i(\rho)\hat{f}_i(\rho)}{2\rho} + \frac{\sqrt{\lambda_i}}{2}\lim_{\rho\to 0}\frac{r_i(\rho)}{\rho^2} \\ \Rightarrow \; &\lim_{\rho\to 0}\frac{r_i(\rho)}{\rho^2} = \frac{1}{2-\sqrt{\lambda_i}}\lim_{\rho\to 0}\frac{p_i(\rho)\hat{f}_i(\rho)}{\rho} = \frac{\hat{f}_i(0)}{\lambda_i - 2\sqrt{\lambda_i}}\end{aligned}$$

which means that $r_i(\rho) \sim \rho^2$ $(i \geq 2)$. We also can conclude that $\lim\limits_{\rho\to 0} \dfrac{r_i(\rho)}{\rho} = 0$.

Then, since the series $\sum\limits_2^\infty r_i^2$ is uniformly convergent on $[0,a]$ by Lemma 2.8.10, we have,

$$\begin{aligned}
\lim_{\rho\to 0} \|r(\rho) - r(0)\|^2_{L^2(0,2\pi)} &= \lim_{\rho\to 0} \left(\int_a^\rho p_1(t)\hat{f}_1(t)\,\mathrm{d}t - r_1(0) \right)^2 + \lim_{\rho\to 0} \sum_2^\infty r_i^2 \\
&= \left(- \int_0^a p_1(t)\hat{f}_1(t)\,\mathrm{d}t - r_1(0) \right)^2 + \sum_2^\infty 0 \\
&= (r_1(0) - r_1(0))^2 = 0.
\end{aligned}$$

Further, since the series $\sum\limits_2^\infty \lambda_i \dfrac{r_i^2}{\rho^2}$ is uniformly convergent on $[0,a/2]$ by Lemma 2.8.11, we have

$$\begin{aligned}
\lim_{\rho\to 0} \|r(\rho) - r(0)\|^2_{H^1_{\rho,P}(0,2\pi)} &= \lim_{\rho\to 0} \left(\int_a^\rho p_1(t)\hat{f}_1(t)\,\mathrm{d}t - r_1(0) \right)^2 + \lim_{\rho\to 0} \sum_2^\infty \lambda_i \frac{r_i^2}{\rho^2} \\
&= \left(- \int_0^a p_1(t)\hat{f}_1(t)\,\mathrm{d}t - r_1(0) \right)^2 + \sum_2^\infty 0 \\
&= (r_1(0) - r_1(0))^2 = 0.
\end{aligned}$$

■

Remark 2.8.13. *In ([28]), Sokolowski-Zochowski look for a solution of the obstacle problem*

$$u = u(\Omega) \in K : \int_\Omega \nabla u . \nabla (v - u) \geq 0, \; \forall v \in K,$$

where $K(\Omega) = \{v \in H^1(\Omega) : v = g \text{ on } \Gamma_0, v \geq 0 \text{ in } \Omega)$. *They considered a domain* Ω_ρ*, with a small hole* $B(\rho)$ *in the form of a disc* $B(\rho) = \{x : |x - \mathcal{O}| < \rho\} \subset \Omega$*,* $\mathcal{O}$ *being the center of the hole and assumed to be the origin. In addition they assume that the (unique) solution of the obstacle problem, denoted by* $u = u(\Omega_\rho)$ *satisfies the homogeneous Neumann conditions on the boundary* Γ_ρ *of the hole* $B(\rho)$*. They are interested in the asymptotic behavior of* $u(\Omega_\rho) \in H^1(\Omega_\rho)$*, for* $\rho \to 0^+$*. For this problem they find* u_ρ*, which is an outer approximation of the solution* $u(\Omega_\rho)$*, and they prove that* $u_\rho = u(\Omega) + \rho^2 q + o(\rho^2)$*, for some function* q*. This can be seen as an expansion of the form*

$$u_\rho = u(\Omega) + \rho \frac{\partial u_\rho}{\partial \rho}(0) + \frac{\rho^2}{2} \frac{\partial^2 u_\rho}{\partial \rho^2}(0) + o(\rho^2), \tag{2.72}$$

which means that, in the approach of Sokolowski-Zochowski, $\dfrac{\partial u_\rho}{\partial \rho}(0) = 0$.

On the other hand, in our framework, we can write $u(s) = P(s)\dfrac{\partial u}{\partial s}(s) + r(s), \forall s,$ *and, using the Neumann boundary condition, we obtain, on* Γ_ρ, $u_\rho(\rho) = r(\rho)$. *Differentiating the previous equality with respect to* ρ, *we find* $\dfrac{\partial u_\rho}{\partial \rho}(\rho) = \dfrac{\partial r}{\partial \rho}(\rho)$.

From (2.68) we find $\dfrac{\partial r_i}{\partial \rho}(\rho) = p_i(\rho)\,\hat{f}_i(\rho) - p_i(\rho)\lambda_i \dfrac{r_i(\rho)}{\rho^2}$, *for* $i \geq 1$. *Also for* $i \geq 1$, *from the proof of Proposition 2.8.8, we know that* $\lim_{\rho\to 0} p_i(\rho) = 0$. *Then, since* $\dfrac{r_i(\rho)}{\rho^2}$ *and* $\hat{f}_i(\rho)$ *are bounded, we have* $\lim_{\rho\to 0} \dfrac{\partial r_i}{\partial \rho}(\rho) = 0$ *and consequently* $\dfrac{\partial u_\rho}{\partial \rho}(0) = 0$ *which, as we saw, is in agreement with the first approach.*

Now, we aim to pass to the limit when $\varepsilon \to 0$, which means that we are going to pass to the limit in $\hat{u}_\varepsilon$, using the results obtained in Chapter 1. Considering $\tilde{\hat{u}}_\varepsilon$ as in (1.6), as a consequence of Lemma 1.5.2, we have:

Lemma 2.8.14. $\|\tilde{\hat{u}}_\varepsilon\|_{\widehat{U}_0}$ *is bounded independently of* ε.

Therefore, as a consequence of Theorem 1.5.3, we also have:

Proposition 2.8.15. $\tilde{\hat{u}}_\varepsilon \to \hat{u}$, *when* $\varepsilon \to 0$, *strongly in* $\widehat{U}_0$, *where* $\hat{u}_\varepsilon$ *and* $\hat{u}$ *are the solutions of (2.2) and (2.1), respectively.*

Proposition 2.8.16. *For every* h *in* $\left(H^{1/2}_{\rho,P}(0,2\pi)\right)'$, $\hat{u}$, *the solution of (2.1), satisfies the following equation*

$$\langle \hat{u}, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} = \left\langle P\frac{\partial \hat{u}}{\partial \rho}, h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} + \langle r, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '}$$

in $\mathcal{D}'(0,a)$.

Proof. Let $\phi \in \mathcal{D}(0,a)$. Since $\phi(0) = 0$, in a neighborhood of the origin, and $\dfrac{\partial \tilde{\hat{u}}_\varepsilon}{\partial \rho} = 0$, for $\rho \in (0,\varepsilon)$ ($\tilde{\hat{u}}_\varepsilon$ is constant in Ω_ε), considering (2.12) extended to the interval $(0,a)$, we have

$$\int_0^a \left\langle \frac{\partial \tilde{\hat{u}}_\varepsilon}{\partial \rho}, Ph \right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)'} \phi\rho \,\mathrm{d}\rho$$

$$- \int_0^a \left\langle \tilde{\hat{u}}_\varepsilon, h \right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)'} \phi\rho \,\mathrm{d}\rho + \int_0^\varepsilon \left\langle \tilde{\hat{u}}_\varepsilon, h \right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)'} \phi\rho \,\mathrm{d}\rho$$

$$= \int_0^a \langle -r, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)'} \phi\rho \,\mathrm{d}\rho - \int_0^\varepsilon \langle -r, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)'} \phi\rho \,\mathrm{d}\rho.$$

Now, since $\left(\tilde{\hat{u}}_\varepsilon, h\right)_{L^2(0,2\pi)} \phi\rho$ and $(-r, h)_{L^2(0,2\pi)} \phi\rho$ are bounded in $[0, \varepsilon)$ by a constant not depending on ε (the result for $\tilde{\hat{u}}_\varepsilon$ is due to Proposition 2.5.1 and the result for r is a consequence of Proposition 2.8.12), for ε arbitrarily small, we have $\int_0^\varepsilon \left\langle \tilde{\hat{u}}_\varepsilon, h \right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)'} \phi\rho \,\mathrm{d}\rho \to 0$ and $\int_0^\varepsilon \langle r, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)'} \phi\rho \,\mathrm{d}\rho \to 0$, as $\varepsilon \to 0$. This way, passing to the limit when $\varepsilon \to 0$ (using Proposition 2.8.15), we obtain

$$\int_0^a \left\langle \frac{\partial \hat{u}}{\partial \rho}, Ph \right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)'} \phi\rho \,\mathrm{d}\rho - \int_0^a \langle \hat{u}, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)'} \phi\rho \,\mathrm{d}\rho$$

$$= \int_0^a \langle -r, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)'} \phi\rho \,\mathrm{d}\rho,$$

from what follows the desired result. ■

The coordinates of $\hat{u}$, solution of $\hat{u} = P\dfrac{\partial \hat{u}}{\partial \rho} + r$ and $\hat{u}(0) = u_0$ verify, for $i \geq 1$, $u_i^0(\rho) = p_i(\rho)\dfrac{\partial u_i^0}{\partial \rho}(\rho) + r_i(\rho)$. Further, from Proposition 2.5.4, considering that $\hat{u}_\varepsilon^m(\varepsilon)$ and $\hat{u}^m(0)$ are both constants, in fact we have

$$\lim_{\varepsilon \to 0} \hat{u}_\varepsilon(\varepsilon) = \hat{u}(0) = u_0. \tag{2.73}$$

So, we have $u_i^0(0) = 0$, for $i \geq 2$, and $u_1^0(0) = u_0$.

Considering again $\dfrac{p_i(\rho)}{\rho} = q_i(\rho)$, we obtain

$$u_i^0(\rho) = \rho\, q_i(\rho) \frac{\partial u_i^0}{\partial \rho}(\rho) + r_i(\rho),$$

and through the change of variables $\varphi = \log \rho$, that can be written as

$$\mathtt{u}_i(\varphi) = \mathtt{q}_i(\varphi) \frac{\partial \mathtt{u}_i}{\partial \varphi}(\varphi) + \mathtt{r}_i(\varphi). \tag{2.74}$$

Then, since $\dfrac{\partial \mathtt{u}_i}{\partial \varphi} - \dfrac{1}{\mathtt{q}_i}\mathtt{u}_i = -\dfrac{\mathtt{r}_i}{\mathtt{q}_i}$ is a linear differential equation, we obtain for $i \geq 2$ (since $\mathtt{u}_i(-\infty) = 0$, for $i \geq 2$)

$$\mathtt{u}_i(\varphi) = e^{\int_{-\infty}^{\varphi} \frac{1}{\mathtt{q}_i(t)} \,\mathrm{d}t} \left(\int_{-\infty}^{\varphi} e^{\int_{-\infty}^{\varrho} -\frac{1}{\mathtt{q}_i(t)} \,\mathrm{d}t} \left(-\frac{\mathtt{r}_i(\varrho)}{\mathtt{q}_i(\varrho)} \right) \mathrm{d}\varrho \right). \tag{2.75}$$

In this last expression, we have $\lim_{\varphi\to-\infty}\frac{1}{\mathtt{q}_i(\varphi)}=\lim_{\rho\to 0}\frac{1}{q_i(\rho)}=-\sqrt{\lambda_i}$ and $\lim_{\varphi\to-\infty}\frac{\mathtt{r}_i(\varphi)}{\mathtt{q}_i(\varphi)}=\lim_{\rho\to 0}\frac{r_i(\rho)}{q_i(\rho)}=0.(-\sqrt{\lambda_i})=0$, which means that both the quantities $\frac{1}{\mathtt{q}_i(t)}$ (for $t\in[-\infty,\varphi]$) and $e^{\int_{-\infty}^{\varrho}-\frac{1}{\mathtt{q}_i(t)}\,\mathrm{d}t}\left(-\frac{\mathtt{r}_i(\varrho)}{\mathtt{q}_i(\varrho)}\right)$ (for $\varrho\in[-\infty,\varphi]$) are bounded, for $i\geq 2$. Then, $\lim_{\varphi\to-\infty}\mathtt{u}_i(\varphi)=0$, as pretended.

For $i=1$, since $\mathtt{u}_1(-\infty)=u_0$, we obtain as the solution of the respective linear differential equation

$$\mathtt{u}_1(\varphi)=e^{\int_{-\infty}^{\varphi}\frac{1}{\mathtt{q}_1(t)}\,\mathrm{d}t}\left(\int_{-\infty}^{\varphi}e^{\int_{-\infty}^{\varrho}-\frac{1}{\mathtt{q}_1(t)}\,\mathrm{d}t}\left(-\frac{\mathtt{r}_1(\varrho)}{\mathtt{q}_1(\varrho)}\right)\mathrm{d}\varrho\right)+u_0\,e^{\int_{-\infty}^{\varphi}\frac{1}{\mathtt{q}_1(t)}\,\mathrm{d}t}.$$

As previously, $\lim_{\varphi\to-\infty}\frac{1}{\mathtt{q}_1(\varphi)}=\lim_{\rho\to 0}\frac{1}{q_1(\rho)}=\lim_{\rho\to 0}\frac{1}{\log(\frac{\rho}{a})}=0$ and $\lim_{\varphi\to-\infty}\frac{\mathtt{r}_1(\varphi)}{\mathtt{q}_1(\varphi)}=\lim_{\rho\to 0}\frac{r_1(\rho)}{q_1(\rho)}=\lim_{\rho\to 0}\frac{r_1(\rho)}{\log(\frac{\rho}{a})}=0$, which means that, in fact, $\lim_{\varphi\to-\infty}\mathtt{u}_1(\varphi)=u_0$.

Before setting out the behavior of $\hat{u}$ near the origin, we need an auxiliary result.

Lemma 2.8.17. *The series* $\sum_1^{\infty}(\mathrm{u}_i(\varphi))^2$ *is uniformly convergent on* $[-\infty,\log\left(\frac{a}{2}\right)]$.

Proof. Once again, we are going to consider $i^\star$ such that $\lambda_i\geq 4$, $\forall i\geq i^\star$. Then, using the computations exhibited in the proof of Lemma 2.8.11, we obtain:

$$\begin{aligned}&\left|\frac{r_i(\rho)}{\rho\, q_i(\rho)}\right|\\ \leq\ &\frac{1}{\rho}\left(\int_{\rho}^{\frac{a}{2}}e^{\int_{\rho}^{t}\frac{p_i(\varrho)\lambda_i}{\varrho^2}\,\mathrm{d}\varrho}\frac{t|q_i(t)|}{|q_i(\rho)|}|\hat{f}_i(t)|\,\mathrm{d}t+\int_{\frac{a}{2}}^{a}e^{\int_{\rho}^{t}\frac{p_i(\varrho)\lambda_i}{\varrho^2}\,\mathrm{d}\varrho}\frac{t|q_i(t)|}{|q_i(\rho)|}|\hat{f}_i(t)|\,\mathrm{d}t\right).\end{aligned}$$

Since $q_i(\rho)$ is an increasing function on $[0,a]$ (see Remark 2.8.9) and negative on $[0,a]$, then the function $|q_i(\rho)|$ is decreasing on $[0,a]$. So, $\frac{|q_i(t)|}{|q_i(\rho)|}\leq 1$, for $\rho<t$. Consequently, using again the computations of the proof of Lemma 2.8.11,

$$\begin{aligned}&\left|\frac{r_i(\rho)}{\rho\, q_i(\rho)}\right|\\ \leq\ &\frac{1}{\rho}\left(\int_{\rho}^{\frac{a}{2}}e^{\int_{\rho}^{t}\frac{p_i(\varrho)\lambda_i}{\varrho^2}\,\mathrm{d}\varrho}t|\hat{f}_i(t)|\,\mathrm{d}t+\int_{\frac{a}{2}}^{a}e^{\int_{\rho}^{t}\frac{p_i(\varrho)\lambda_i}{\varrho^2}\,\mathrm{d}\varrho}t|\hat{f}_i(t)|\,\mathrm{d}t\right)\\ \leq\ &\sqrt{c_1}\frac{1}{i},\ \forall\rho\in[0,\frac{a}{2}].\end{aligned}$$

Therefore, for $i \geq i^\star$ (notice that obviously $i^\star \geq 2$) and using the fact that $p_i(\varrho) < 0$, we obtain, for all $\rho \in [0, \frac{a}{2}]$,

$$\begin{aligned} & \frac{\partial u_i}{\partial \rho}(\rho) - \frac{1}{p_i(\rho)} u_i(\rho) = -\frac{r_i(\rho)}{p_i(\rho)} \\ \Rightarrow \quad & u_i(\rho) = \int_0^\rho e^{\int_t^\rho \frac{1}{p_i(\varrho)} \mathrm{d}\varrho} \left(-\frac{r_i(t)}{p_i(t)} \right) \mathrm{d}t \\ \Rightarrow \quad & |u_i(\rho)| \leq \int_0^\rho \left| \frac{r_i(t)}{t\, q_i(t)} \right| \mathrm{d}t \leq \int_0^\rho \sqrt{c_1} \frac{1}{i} \,\mathrm{d}t \leq \sqrt{c_1} \frac{\rho}{i} \leq \sqrt{c_1} \frac{a}{i}. \end{aligned}$$

Then, as in Lemma 2.8.11, since the numerical series $\sum_{i^\star}^{\infty} \frac{a^2 c_1}{i^2}$ is convergent, the series $\sum_2^\infty (u_i(\rho))^2$ is uniformly convergent on $[0, \frac{a}{2}]$ and consequently the series $\sum_2^\infty (\mathtt{u}_i(\varphi))^2$ is uniformly convergent on $[-\infty, \log\left(\frac{a}{2}\right)]$. ∎

Proposition 2.8.18. *For $\hat{u}$, solution of $\hat{u} = P \frac{\partial \hat{u}}{\partial \rho} + r$ and $\hat{u}(0) = u_0$, where $\hat{u}(0)$ is given by (2.73), we have $\lim_{\rho \to 0} \|\hat{u}(\rho) - \hat{u}(0)\|^2_{L^2(0,2\pi)} = 0$ (in particular, $\lim_{\rho \to 0} \|\hat{u}(\rho)\|_{L^2(0,2\pi)} = |u_0|$).*

Proof. Since the series $\sum_2^\infty (\mathtt{u}_i(\varphi))^2$ is uniformly convergent on $[-\infty, \log\left(\frac{a}{2}\right)]$ by Lemma 2.8.17, we have

$$\begin{aligned} \lim_{\varphi \to -\infty} \|\mathtt{u}(\varphi) - \mathtt{u}(-\infty)\|^2_{L^2(0,2\pi)} &= \lim_{\varphi \to -\infty} (\mathtt{u}_1(\varphi) - u_0)^2 + \lim_{\varphi \to -\infty} \sum_2^\infty (\mathtt{u}_i(\varphi))^2 \\ &= (\mathtt{u}_1(-\infty) - u_0)^2 + \sum_2^\infty 0 = (u_0 - u_0)^2 = 0. \end{aligned}$$

∎

For $\mathtt{u}_i(\varphi)$ given by (2.74), we have, for $i \geq 2$,

$$\begin{aligned} & \lim_{\varphi \to -\infty} \frac{\mathtt{u}_i(\varphi)}{e^{2\varphi}} = \lim_{\varphi \to -\infty} \frac{(\mathtt{u}_i(\varphi))'}{2\, e^{2\varphi}} = \lim_{\varphi \to -\infty} \left(\frac{1}{\mathtt{q}_i(\varphi)} \frac{\mathtt{u}_i(\varphi)}{2\, e^{2\varphi}} - \frac{\mathtt{r}_i(\varphi)}{2\, \mathtt{q}_i(\varphi)\, e^{2\varphi}} \right) \\ \Rightarrow \quad & \lim_{\varphi \to -\infty} \frac{\mathtt{u}_i(\varphi)}{e^{2\varphi}} = \lim_{\varphi \to -\infty} \frac{-\mathtt{r}_i(\varphi)}{(2\, \mathtt{q}_i(\varphi) - 1)\, e^{2\varphi}}. \end{aligned}$$

On the other hand, $\lim_{\varphi \to -\infty} \frac{-\mathtt{r}_i(\varphi)}{(2\, \mathtt{q}_i(\varphi) - 1)\, e^{2\varphi}} = \lim_{\rho \to 0} \frac{-r_i(\rho)}{(2\, q_i(\rho) - 1)\, \rho^2} = \lim_{\rho \to 0} \frac{-\frac{r_i(\rho)}{\rho^2}}{2\, q_i(\rho) - 1} = \frac{-\frac{\hat{f}_i(0)}{\lambda_i - 2\sqrt{\lambda_i}}}{2\frac{-1}{\sqrt{\lambda_i}} - 1} = \frac{\hat{f}_i(0)}{\lambda_i - 4}$ and consequently $\lim_{\varphi \to -\infty} \frac{\mathtt{u}_i(\varphi)}{e^{2\varphi}} = \frac{\hat{f}_i(0)}{\lambda_i - 4}$, which means that $\mathtt{u}_i(\varphi) \sim e^{2\varphi}$.

Proposition 2.8.19. *For all $\rho \in (0,a)$ there is a unique solution $\hat{u}(\rho)$ for the boundary value problem $\hat{u}(\rho) = P(\rho)\dfrac{\partial \hat{u}}{\partial \rho}(\rho) + r(\rho)$, $\hat{u}(0) = u_0$.*

Proof. Supposing that $\hat{u}^1(\rho)$ and $\hat{u}^2(\rho)$ are two solutions of the previous problem, then $w(\rho) = \hat{u}^1(\rho) - \hat{u}^2(\rho)$ satisfies the boundary value problem $P(\rho)\dfrac{\partial w}{\partial \rho}(\rho) - w(\rho) = 0$, $w(0) = 0$. Furthermore, since $\hat{u}^1(\rho)$ and $\hat{u}^2(\rho)$ are continuous (see Lemma 2.5.2) then $w(\rho)$ is also continuous. Thus, taking the inner product with $\dfrac{\partial w}{\partial \rho}(\rho)$, in the duality $H^{1/2}_{\rho,P}(0,2\pi)$, $\left(H^{1/2}_{\rho,P}(0,2\pi)\right)'$, we obtain:

$$\left(P(\rho)\frac{\partial w}{\partial \rho}(\rho), \frac{\partial w}{\partial \rho}(\rho)\right) - \left(w(\rho), \frac{\partial w}{\partial \rho}(\rho)\right) = 0.$$

Then, we can see on page 103 of [23] that $\displaystyle\int_0^\rho \left(w(\varrho), \frac{\partial w}{\partial \varrho}(\varrho)\right) \,\mathrm{d}\varrho = \frac{1}{2}\|w(\rho)\|^2_{L^2(0,2\pi)}$ (since $w(0) = 0$ and w is continuous on $[0,\rho]$) and consequently

$$\int_0^\rho \left(-P(\varrho)\frac{\partial w}{\partial \varrho}(\varrho), \frac{\partial w}{\partial \varrho}(\varrho)\right) \,\mathrm{d}\varrho + \frac{1}{2}\|w(\rho)\|^2_{L^2(0,2\pi)} = 0.$$

Since we are summing, in the previous equation, two non negative quantities (notice that P is a negative operator), we must have $\|w(\rho)\|_{L^2(0,2\pi)} = 0$. According to the continuity previously established, we therefore conclude that $\hat{u}^1(\rho) = \hat{u}^2(\rho)$. ■

Theorem 2.8.20. *Considering $\phi \in \mathcal{D}(0,a)$ we obtain:*

1. *for every $h, \bar{h}$ in $L^2(0,2\pi)$, the operator P satisfies the equation*

$$\left(\frac{\partial P}{\partial \rho}h, \bar{h}\right)_{L^2(0,2\pi)} + \left(\frac{1}{\rho^2}\frac{\partial}{\partial \theta}Ph, \frac{\partial}{\partial \theta}P\bar{h}\right)_{L^2(0,2\pi)} - \left(\frac{1}{\rho}h, P\bar{h}\right)_{L^2(0,2\pi)} = \left(h, \bar{h}\right)_{L^2(0,2\pi)}$$

   in $\mathcal{D}'(0,a)$, with the initial condition $P(a) = 0$;

2. *for every h in $H^{1/2}_{\rho,P}(0,2\pi)$, the function r satisfies the equation*

$$\left\langle \frac{1}{\rho^2}\frac{\partial r}{\partial \theta}, \frac{\partial}{\partial \theta}Ph\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)} + \left\langle \frac{\partial r}{\partial \rho}, h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)} = (f, Ph)_{L^2(0,2\pi)}$$

   in $\mathcal{D}'(0,a)$, with the initial condition $r(a) = 0$;

3. *for every h in $\left(H^{1/2}_{\rho,P}(0,2\pi)\right)'$, $\hat{u}$ satisfies the equation*

$$(\hat{u}, h)_{L^2(0,2\pi)} = \left(P\frac{\partial \hat{u}}{\partial \rho}, h\right)_{L^2(0,2\pi)} + (r, h)_{L^2(0,2\pi)}$$

   in $\mathcal{D}'(0,a)$, with the initial condition $\hat{u}(0) = \lim\limits_{\rho \to 0} r(\rho)$.

Proof. Since P^m and r^m do not depend on ε, the first two items are a direct consequence of Propositions 2.8.4 and 2.8.5, taking into account Remark 2.8.7. The third item is a consequence of Proposition 2.8.6, considering (2.73). ■

Chapter 3

The factorization method in a circular domain: dual case

In this chapter, we consider again Ω (respectively, Ω_s) to be a disk of $\mathbb{R}^2$ with radius a (respectively, s) centered on the origin. Another factorization to the problem (1.3) could be obtained by using an invariant embedding defined by the family of disks Ω_s, $s \in (0, a)$. Here the main difficulty is to define the initial conditions for P and r at the origin.

3.1. Invariant embedding

For the reasons pointed out in Sections 1.4. and 2.1. we consider again an auxiliary problem and its formulation (2.2). As in the previous chapter, we are going to consider $f \in \mathcal{C}^{0,\alpha}(\Omega)$. Using the technique of invariant embedding, we now embed problem (2.2) in a family of similar problems defined on $\overline{\widehat{\Omega}_s \setminus \widehat{\Omega}_\varepsilon} = [\varepsilon, s] \times [0, 2\pi]$, for $s \in (\varepsilon, a]$. For each problem we impose a Robin boundary condition $\dfrac{\partial \hat{u}_\varepsilon}{\partial \rho}|_{\Gamma_s} + \alpha\, \hat{u}_{\varepsilon_{|\Gamma_s}} = h$, where $\alpha \in \mathbb{R}^+$ and Γ_s is the moving boundary:

$$\begin{cases} -\dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\rho \dfrac{\partial \hat{u}_\varepsilon}{\partial \rho}\right) - \dfrac{1}{\rho^2}\dfrac{\partial^2 \hat{u}_\varepsilon}{\partial \theta^2} = f, \text{ in } \widehat{\Omega}_s \setminus \widehat{\Omega}_\varepsilon \\ \dfrac{\partial \hat{u}_\varepsilon}{\partial \rho}|_{\Gamma_s} + \alpha\, \hat{u}_{\varepsilon_{|\Gamma_s}} = h \\ \hat{u}_{\varepsilon_{|\Gamma_\varepsilon}} \text{ constant}, \displaystyle\int_0^{2\pi} \dfrac{\partial \hat{u}_\varepsilon}{\partial \rho}|_{\Gamma_\varepsilon} \,\mathrm{d}\theta = 0 \\ \hat{u}_{\varepsilon_{|\theta=0}} = \hat{u}_{\varepsilon_{|\theta=2\pi}},\ \dfrac{\partial \hat{u}_\varepsilon}{\partial \theta}|_{\theta=0} = \dfrac{\partial \hat{u}_\varepsilon}{\partial \theta}|_{\theta=2\pi}. \end{cases} \tag{3.1}$$

In (3.1) we take $h \in \left(H^{1/2}_{\rho,P}(0,2\pi)\right)'$. Then, it is clear that (3.1) is exactly (2.2), for $s=a$ and $h = \dfrac{\partial \hat{u}_\varepsilon}{\partial \rho}|_{\Gamma_s}$.

Analogously to the previous chapter, to the Hilbert space $H^1(\widehat{\Omega}_s \setminus \widehat{\Omega}_\varepsilon)$ corresponds the space $\widehat{H}_s = \left\{ \hat{v} : \hat{v} \in L^2_\rho(\varepsilon, s; H^1_{\rho,P}(0,2\pi)), \dfrac{\partial \hat{v}}{\partial \rho} \in L^2_\rho(\varepsilon, s; L^2(0,2\pi)) \right\}$. In this space, we consider the norm

$$|||\hat{v}|||^2_{\widehat{H}_s} = \alpha s \int_0^{2\pi} (\hat{v}(s))^2 \,\mathrm{d}\theta + \int_\varepsilon^s \int_0^{2\pi} \left(\frac{1}{\rho} \left(\frac{\partial \hat{v}}{\partial \theta}\right)^2 + \rho \left(\frac{\partial \hat{v}}{\partial \rho}\right)^2 \right) \mathrm{d}\theta \,\mathrm{d}\rho.$$

Proposition 3.1.1. *The norm*

$$\|\hat{v}\|^2_{\widehat{H}_s} = \int_\varepsilon^s \int_0^{2\pi} \left((\hat{v})^2 \rho + \frac{1}{\rho} \left(\frac{\partial \hat{v}}{\partial \theta}\right)^2 + \rho \left(\frac{\partial \hat{v}}{\partial \rho}\right)^2 \right) \mathrm{d}\theta \,\mathrm{d}\rho$$

(usual norm on $\widehat{H}_s$) is equivalent to the norm

$$|||\hat{v}|||^2_{\widehat{H}_s} = \alpha s \int_0^{2\pi} (\hat{v}(s))^2 \,\mathrm{d}\theta + \int_\varepsilon^s \int_0^{2\pi} \left(\frac{1}{\rho} \left(\frac{\partial \hat{v}}{\partial \theta}\right)^2 + \rho \left(\frac{\partial \hat{v}}{\partial \rho}\right)^2 \right) \mathrm{d}\theta \,\mathrm{d}\rho.$$

Proof. We have $\|\hat{v}(s)\|^2_{L^2(0,2\pi)} \le \|\hat{v}(s)\|^2_{H^{1/2}_{\rho,P}(0,2\pi)}$ and, by trace theorem, $\|\hat{v}(s)\|^2_{H^{1/2}_{\rho,P}(0,2\pi)} \le c_s \|\hat{v}\|^2_{\widehat{H}_s}$, where c_s is a constant depending on s. Then,

$$\begin{aligned} |||\hat{v}|||^2_{\widehat{H}_s} &= \alpha s \|\hat{v}(s)\|^2_{L^2(0,2\pi)} + \|\nabla \hat{v}\|^2_{L^2_\rho(\varepsilon,s;L^2(0,\pi))} \\ &\le \alpha s\, c_s \|\hat{v}\|^2_{\widehat{H}_s} + \|\nabla \hat{v}\|^2_{L^2_\rho(\varepsilon,s;L^2(0,\pi))} \le (\alpha s\, c_s + 1)\|\hat{v}\|^2_{\widehat{H}_s}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_\rho^s t\,\hat{v}(t,\theta) \frac{\partial \hat{v}}{\partial t}(t,\theta)\,\mathrm{d}t &= \frac{1}{2} \left(\hat{v}(t,\theta)\right)^2 t\Big]_\rho^s - \frac{1}{2} \int_\rho^s \left(\hat{v}(t,\theta)\right)^2 \,\mathrm{d}t \\ &= \frac{1}{2} \left(\hat{v}(s,\theta)\right)^2 s - \frac{1}{2} \left(\hat{v}(\rho,\theta)\right)^2 \rho - \frac{1}{2} \int_\rho^s \left(\hat{v}(t,\theta)\right)^2 \,\mathrm{d}t. \end{aligned}$$

Then,

$$\begin{aligned} & \left(\hat{v}(\rho,\theta)\right)^2 \rho - \left(\hat{v}(s,\theta)\right)^2 s + \int_\rho^s \left(\hat{v}(t,\theta)\right)^2 \,\mathrm{d}t \\ = \;& -2 \int_\rho^s t\,\hat{v}(t,\theta) \frac{\partial \hat{v}}{\partial t}(t,\theta)\,\mathrm{d}t \le 2 \int_\rho^s \left| t\,\hat{v}(t,\theta) \frac{\partial \hat{v}}{\partial t}(t,\theta) \right| \mathrm{d}t \\ \le \;& 2 \left(\int_\rho^s \left(\hat{v}(t,\theta)\right)^2 \,\mathrm{d}t \right)^{1/2} \left(\int_\rho^s t^2 \left(\frac{\partial \hat{v}}{\partial t}(t,\theta) \right)^2 \mathrm{d}t \right)^{1/2} \\ \le \;& \int_\rho^s \left(\hat{v}(t,\theta)\right)^2 \,\mathrm{d}t + \int_\rho^s t^2 \left(\frac{\partial \hat{v}}{\partial t}(t,\theta) \right)^2 \mathrm{d}t. \end{aligned}$$

Consequently,

$$\begin{aligned}(\hat{v}(\rho,\theta))^2\,\rho &\le s\,(\hat{v}(s,\theta))^2 + \int_\rho^s t^2\left(\frac{\partial \hat{v}}{\partial t}(t,\theta)\right)^2\,\mathrm{d}t\\ &\le s\,(\hat{v}(s,\theta))^2 + s\int_\varepsilon^s t\left(\frac{\partial \hat{v}}{\partial t}(t,\theta)\right)^2\,\mathrm{d}t.\end{aligned}$$

Thus,

$$\begin{aligned}\int_\varepsilon^s (\hat{v}(\rho,\theta))^2\,\rho\,\mathrm{d}\rho &\le \int_\varepsilon^s s\,(\hat{v}(s,\theta))^2\,\mathrm{d}\rho + \int_\varepsilon^s\left(s\int_\varepsilon^s t\left(\frac{\partial \hat{v}}{\partial t}(t,\theta)\right)^2\,\mathrm{d}t\right)\mathrm{d}\rho\\ &= (s-\varepsilon)s\,(\hat{v}(s,\theta))^2 + (s-\varepsilon)s\int_\varepsilon^s t\left(\frac{\partial \hat{v}}{\partial t}(t,\theta)\right)^2\,\mathrm{d}t\\ &\le s^2\,(\hat{v}(s,\theta))^2 + s^2\int_\varepsilon^s t\left(\frac{\partial \hat{v}}{\partial t}(t,\theta)\right)^2\,\mathrm{d}t,\end{aligned}$$

which implies

$$\int_0^{2\pi}\int_\varepsilon^s (\hat{v}(\rho,\theta))^2\,\rho\,\mathrm{d}\rho\,\mathrm{d}\theta \le s^2\int_0^{2\pi}(\hat{v}(s,\theta))^2\,\mathrm{d}\theta + s^2\int_0^{2\pi}\int_\varepsilon^s\left(\frac{\partial \hat{v}}{\partial \rho}(\rho,\theta)\right)^2\rho\,\mathrm{d}\rho\,\mathrm{d}\theta.$$

Therefore,

$$\begin{aligned}\|\hat{v}\|^2_{\widehat{H}_s} &= \int_0^{2\pi}\int_\varepsilon^s(\hat{v})^2\rho\,\mathrm{d}\rho\,\mathrm{d}\theta + \int_0^{2\pi}\int_\varepsilon^s\frac{1}{\rho}\left(\frac{\partial \hat{v}}{\partial \theta}\right)^2\mathrm{d}\rho\,\mathrm{d}\theta + \int_0^{2\pi}\int_\varepsilon^s\left(\frac{\partial \hat{v}}{\partial \rho}\right)^2\rho\,\mathrm{d}\rho\,\mathrm{d}\theta\\ \le\ & \frac{s}{\alpha}s\,\alpha\int_0^{2\pi}(\hat{v}(s))^2\,\mathrm{d}\theta + (s^2+1)\int_0^{2\pi}\int_\varepsilon^s\left(\frac{\partial \hat{v}}{\partial \rho}\right)^2\rho\,\mathrm{d}\rho\,\mathrm{d}\theta + \int_0^{2\pi}\int_\varepsilon^s\frac{1}{\rho}\left(\frac{\partial \hat{v}}{\partial \theta}\right)^2\mathrm{d}\rho\,\mathrm{d}\theta\\ \le\ & \max\left\{s^2+1,\frac{s}{\alpha}\right\}|||\hat{v}|||^2_{\widehat{H}_s}.\end{aligned}$$

■

Furthermore, we are going to use the spaces $L^2_\rho(\varepsilon,s)$, $L^2(0,2\pi)$, $H^1_\rho(\varepsilon,s)$ and $H^1_{\rho,P}(0,2\pi)$ and respective norms, as defined in Section 2.1.

Again as a direct application of Theorem 3.1, page 19 of [24], and similarly to Proposition 2.1.2, we have the following trace theorem:

Proposition 3.1.2. *If* $\hat{v} \in \widehat{X}_s = \left\{\hat{v}\in\widehat{H}_s : \dfrac{\partial^2\hat{v}}{\partial\rho^2}\in L^2\left([\varepsilon,s];\left(H^1_{\rho,P}(0,2\pi)\right)'\right)\right\}$, *we have* $\hat{v} \in \mathcal{C}([\varepsilon,s];H^{1/2}_{\rho,P}(0,2\pi))$, $\dfrac{\partial\hat{v}}{\partial\rho} \in \mathcal{C}\left([\varepsilon,s];\left(H^{1/2}_{\rho,P}(0,2\pi)\right)'\right)$ *and the trace mapping* $\hat{v}\to(\hat{v}_{|\Gamma_s},\dfrac{\partial\hat{v}}{\partial\rho}|_{\Gamma_s})$ *is continuous from* $\widehat{X}_s$ *onto* $H^{1/2}_{\rho,P}(0,2\pi)\times\left(H^{1/2}_{\rho,P}(0,2\pi)\right)'$.

In order to decouple problem (3.1), we define:

Definition 3.1.1. For every $s \in (\varepsilon, a]$ and $h \in \left(H^{1/2}_{\rho,P}(0,2\pi)\right)'$ we define $P_\varepsilon(s)h = \gamma_{\varepsilon_{|\Gamma_s}}$, where $\gamma_\varepsilon \in \widehat{X}_s$ is the solution of

$$\begin{cases} -\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial\gamma_\varepsilon}{\partial\rho}\right) - \dfrac{1}{\rho^2}\dfrac{\partial^2\gamma_\varepsilon}{\partial\theta^2} = 0, \text{ in } \widehat{\Omega}_s \setminus \widehat{\Omega}_\varepsilon \\ \dfrac{\partial\gamma_\varepsilon}{\partial\rho}|_{\Gamma_s} + \alpha\,\gamma_{\varepsilon_{|\Gamma_s}} = h \\ \gamma_{\varepsilon_{|\Gamma_\varepsilon}} \text{ constant} \\ \displaystyle\int_0^{2\pi} \dfrac{\partial\gamma_\varepsilon}{\partial\rho}|_{\Gamma_\varepsilon}\,\mathrm{d}\theta = 0 \\ \gamma_{\varepsilon_{|\theta=0}} = \gamma_{\varepsilon_{|\theta=2\pi}} \\ \dfrac{\partial\gamma_\varepsilon}{\partial\theta}|_{\theta=0} = \dfrac{\partial\gamma_\varepsilon}{\partial\theta}|_{\theta=2\pi} \end{cases} \tag{3.2}$$

and $r_\varepsilon(s) = \beta_{\varepsilon_{|\Gamma_s}}$, where $\beta_\varepsilon \in \widehat{X}_s$ is the solution of

$$\begin{cases} -\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial\beta_\varepsilon}{\partial\rho}\right) - \dfrac{1}{\rho^2}\dfrac{\partial^2\beta_\varepsilon}{\partial\theta^2} = f, \text{ in } \widehat{\Omega}_s \setminus \widehat{\Omega}_\varepsilon \\ \dfrac{\partial\beta_\varepsilon}{\partial\rho}|_{\Gamma_s} + \alpha\,\beta_{\varepsilon_{|\Gamma_s}} = 0 \\ \beta_{\varepsilon_{|\Gamma_\varepsilon}} \text{ constant} \\ \displaystyle\int_0^{2\pi} \dfrac{\partial\beta_\varepsilon}{\partial\rho}|_{\Gamma_\varepsilon}\,\mathrm{d}\theta = 0 \\ \beta_{\varepsilon_{|\theta=0}} = \beta_{\varepsilon_{|\theta=2\pi}} \\ \dfrac{\partial\beta_\varepsilon}{\partial\theta}|_{\theta=0} = \dfrac{\partial\beta_\varepsilon}{\partial\theta}|_{\theta=2\pi}. \end{cases} \tag{3.3}$$

For every $s \in [\varepsilon, a]$, $P_\varepsilon(s) : \left(H^{1/2}_{\rho,P}(0,2\pi)\right)' \to H^{1/2}_{\rho,P}(0,2\pi)$ is a linear operator and $r_\varepsilon(s) \in H^{1/2}_{\rho,P}(0,2\pi)$. By linearity of (3.1) we have

$$\hat{u}_{\varepsilon_{|\Gamma_s}} = P_\varepsilon(s)\left(\frac{\partial\hat{u}_\varepsilon}{\partial\rho}|_{\Gamma_s} + \alpha\,\hat{u}_{\varepsilon_{|\Gamma_s}}\right) + r_\varepsilon(s), \forall s \in [\varepsilon, a]. \tag{3.4}$$

Furthermore, the solution $\hat{u}_\varepsilon$ of (2.2) is given by

$$\hat{u}_\varepsilon(\rho,\theta) = \left(P_\varepsilon(\rho)\left(\frac{\partial\hat{u}_\varepsilon}{\partial\rho}|_{\Gamma_\rho} + \alpha\,\hat{u}_{\varepsilon_{|\Gamma_\rho}}\right)\right)(\theta) + (r_\varepsilon(\rho))(\theta). \tag{3.5}$$

Proposition 3.1.3. *Considering the Hilbert space* $\widehat{U}_s = \{\hat{u}_\varepsilon \in \widehat{H}_s : \hat{u}_{\varepsilon_{|\Gamma_\varepsilon}}$ *is constant*$\}$, *the variational formulation of problem (3.1) is*

$$\left\{\begin{array}{l} \hat{u}_\varepsilon \in \widehat{U}_s \\ \displaystyle\int_0^{2\pi} \alpha \hat{u}_\varepsilon(s)\hat{v}_\varepsilon(s) s \,\mathrm{d}\theta + \int_\varepsilon^s\!\!\int_0^{2\pi} \frac{\partial \hat{u}_\varepsilon}{\partial \rho}\frac{\partial \hat{v}_\varepsilon}{\partial \rho}\rho + \frac{1}{\rho}\frac{\partial \hat{u}_\varepsilon}{\partial \theta}\frac{\partial \hat{v}_\varepsilon}{\partial \theta}\,\mathrm{d}\theta\,\mathrm{d}\rho \\ = \displaystyle\int_0^{2\pi} h\hat{v}_\varepsilon(s) s\,\mathrm{d}\theta + \int_\varepsilon^s\!\!\int_0^{2\pi} f\hat{v}_\varepsilon \rho\,\mathrm{d}\theta\,\mathrm{d}\rho,\ \forall \hat{v}_\varepsilon \in \widehat{U}_s. \end{array}\right. \tag{3.6}$$

Proof. Using (3.1), multiplying by $\hat{v}_\varepsilon \in \widehat{U}_s$, and integrating in $\widehat{\Omega}_s \setminus \widehat{\Omega}_\varepsilon$, we obtain:

$$\begin{aligned}
&\int_0^{2\pi}\int_\varepsilon^s \left(-\frac{\partial^2 \hat{u}_\varepsilon}{\partial \rho^2}\hat{v}_\varepsilon\rho - \frac{1}{\rho^2}\frac{\partial^2 \hat{u}_\varepsilon}{\partial \theta^2}\hat{v}_\varepsilon\rho - \frac{1}{\rho}\frac{\partial \hat{u}_\varepsilon}{\partial \rho}\hat{v}_\varepsilon\rho\right)\mathrm{d}\rho\,\mathrm{d}\theta = \int_0^{2\pi}\int_\varepsilon^s f\hat{v}_\varepsilon\rho\,\mathrm{d}\rho\,\mathrm{d}\theta \\
\Rightarrow\ &\int_0^{2\pi} -\frac{\partial \hat{u}_\varepsilon}{\partial \rho}\hat{v}_\varepsilon\rho\Big]_\varepsilon^s \mathrm{d}\theta + \int_0^{2\pi}\int_\varepsilon^s \frac{\partial \hat{u}_\varepsilon}{\partial \rho}\left(\frac{\partial \hat{v}_\varepsilon}{\partial \rho}\rho + \hat{v}_\varepsilon\right)\mathrm{d}\rho\,\mathrm{d}\theta - \int_\varepsilon^s \frac{1}{\rho}\,\frac{\partial \hat{u}_\varepsilon}{\partial \theta}\hat{v}_\varepsilon\Big]_0^{2\pi}\mathrm{d}\rho \\
&+\int_0^{2\pi}\int_\varepsilon^s \frac{1}{\rho}\frac{\partial \hat{u}_\varepsilon}{\partial \theta}\frac{\partial \hat{v}_\varepsilon}{\partial \theta}\mathrm{d}\rho\,\mathrm{d}\theta - \int_0^{2\pi}\int_\varepsilon^s \frac{\partial \hat{u}_\varepsilon}{\partial \rho}\hat{v}_\varepsilon\,\mathrm{d}\rho\,\mathrm{d}\theta = \int_0^{2\pi}\int_\varepsilon^s f\hat{v}_\varepsilon\rho\,\mathrm{d}\rho\,\mathrm{d}\theta \\
\Rightarrow\ &\int_0^{2\pi} -\frac{\partial \hat{u}_\varepsilon}{\partial \rho}(s)\hat{v}_\varepsilon(s)s\,\mathrm{d}\theta + \int_0^{2\pi}\frac{\partial \hat{u}_\varepsilon}{\partial \rho}(\varepsilon)\hat{v}_\varepsilon(\varepsilon)\varepsilon\,\mathrm{d}\theta + \int_0^{2\pi}\int_\varepsilon^s \frac{\partial \hat{u}_\varepsilon}{\partial \rho}\frac{\partial \hat{v}_\varepsilon}{\partial \rho}\rho\,\mathrm{d}\rho\,\mathrm{d}\theta \\
&+\int_0^{2\pi}\int_\varepsilon^s \frac{1}{\rho}\frac{\partial \hat{u}_\varepsilon}{\partial \theta}\frac{\partial \hat{v}_\varepsilon}{\partial \theta}\mathrm{d}\rho\,\mathrm{d}\theta = \int_0^{2\pi}\int_\varepsilon^s f\hat{v}_\varepsilon\rho\,\mathrm{d}\rho\,\mathrm{d}\theta \\
\Rightarrow\ &\int_0^{2\pi}(\alpha\hat{u}_\varepsilon(s) - h)\hat{v}_\varepsilon(s)s\,\mathrm{d}\theta + \int_0^{2\pi}\int_\varepsilon^s \frac{\partial \hat{u}_\varepsilon}{\partial \rho}\frac{\partial \hat{v}_\varepsilon}{\partial \rho}\rho\,\mathrm{d}\rho\,\mathrm{d}\theta \\
&+\int_0^{2\pi}\int_\varepsilon^s \frac{1}{\rho}\frac{\partial \hat{u}_\varepsilon}{\partial \theta}\frac{\partial \hat{v}_\varepsilon}{\partial \theta}\mathrm{d}\rho\,\mathrm{d}\theta = \int_0^{2\pi}\int_\varepsilon^s f\hat{v}_\varepsilon\rho\,\mathrm{d}\rho\,\mathrm{d}\theta \\
\Rightarrow\ &\int_0^{2\pi}\alpha\hat{u}_\varepsilon(s)\hat{v}_\varepsilon(s)s\,\mathrm{d}\theta + \int_0^{2\pi}\int_\varepsilon^s \frac{\partial \hat{u}_\varepsilon}{\partial \rho}\frac{\partial \hat{v}_\varepsilon}{\partial \rho}\rho\,\mathrm{d}\rho\,\mathrm{d}\theta + \int_0^{2\pi}\int_\varepsilon^s \frac{1}{\rho}\frac{\partial \hat{u}_\varepsilon}{\partial \theta}\frac{\partial \hat{v}_\varepsilon}{\partial \theta}\mathrm{d}\rho\,\mathrm{d}\theta \\
&= \int_0^{2\pi} h\hat{v}_\varepsilon(s)s\,\mathrm{d}\theta + \int_0^{2\pi}\int_\varepsilon^s f\hat{v}_\varepsilon\rho\,\mathrm{d}\rho\,\mathrm{d}\theta.
\end{aligned}$$

■

Again, the variational formulation (3.6) reduces to the variational formulation (2.3), when $s = a$ and $h = \dfrac{\partial \hat{u}_\varepsilon}{\partial \rho}|_{\Gamma_s}$. Also, it can be proved, as in Proposition 1.4.4, using the variational formulation (3.6) and Lax-Milgram theorem, that the problem (3.1) is well posed.

Now, the following corollary is a direct consequence of the computations exhibited in the previous proposition, taking $f = 0$ and $h = 0$, respectively.

Corollary 3.1.4. *The variational formulation of problem (3.2) is*

$$\begin{cases} \gamma_\varepsilon \in \widehat{U}_s \\ \displaystyle\int_0^{2\pi} \alpha\gamma_\varepsilon(s)\overline{\gamma}_\varepsilon(s)s\,\mathrm{d}\theta + \int_\varepsilon^s\int_0^{2\pi} \frac{\partial\gamma_\varepsilon}{\partial\rho}\frac{\partial\overline{\gamma}_\varepsilon}{\partial\rho}\rho + \frac{1}{\rho}\frac{\partial\gamma_\varepsilon}{\partial\theta}\frac{\partial\overline{\gamma}_\varepsilon}{\partial\theta}\,\mathrm{d}\theta\,\mathrm{d}\rho = \int_0^{2\pi} h\overline{\gamma}_\varepsilon(s)s\,\mathrm{d}\theta, \\ \forall\overline{\gamma}_\varepsilon \in \widehat{U}_s \end{cases}$$

and the variational formulation of problem (3.3) is

$$\begin{cases} \beta_\varepsilon \in \widehat{U}_s \\ \displaystyle\int_0^{2\pi} \alpha\beta_\varepsilon(s)\overline{\beta}_\varepsilon(s)s\,\mathrm{d}\theta + \int_\varepsilon^s\int_0^{2\pi} \frac{\partial\beta_\varepsilon}{\partial\rho}\frac{\partial\overline{\beta}_\varepsilon}{\partial\rho}\rho + \frac{1}{\rho}\frac{\partial\beta_\varepsilon}{\partial\theta}\frac{\partial\overline{\beta}_\varepsilon}{\partial\theta}\,\mathrm{d}\theta\,\mathrm{d}\rho = \int_\varepsilon^s\int_0^{2\pi} f\overline{\beta}_\varepsilon\rho\,\mathrm{d}\theta\,\mathrm{d}\rho, \\ \forall\overline{\beta}_\varepsilon \in \widehat{U}_s. \end{cases}$$

The following remark stands out the relation between the operators defined by (2.7) and (3.2):

Remark 3.1.5. *The operators P_1, such that $u = P_1\dfrac{\partial u}{\partial\rho} + r_1$ (as in Chapter 2), and P_2, such that $u = P_2\left(\dfrac{\partial u}{\partial\rho} + \alpha u\right) + r_2$, can be easily related. In fact, from the second equality, we obtain $(I - \alpha P_2)u = P_2\dfrac{\partial u}{\partial\rho} + r_2$. Thus, $(I - \alpha P_2)P_1 = P_2$.*

In the next proposition are collected some basic properties of the operator P_ε.

Proposition 3.1.6. *The linear operator $P_\varepsilon(s) : \left(H^{1/2}_{\rho,P}(0, 2\pi)\right)' \to H^{1/2}_{\rho,P}(0, 2\pi)$ is continuous, self adjoint and positive definite, for all $s \in [\varepsilon, a)$.*

Proof. The operator $P_\varepsilon(s)$ is continuous since it's the composition of continuous operators: $h \to \gamma_\varepsilon \to \gamma_{\varepsilon_{|\Gamma_s}}$, defined by (3.2), respectively in the spaces $\left(H^{1/2}_{\rho,P}(0, 2\pi)\right)'$, $\widehat{X}_s$ and $H^{1/2}_{\rho,P}(0, 2\pi)$. Let's consider γ_ε and $\overline{\gamma}_\varepsilon$ two solutions of (3.2), with $\dfrac{\partial\gamma_\varepsilon}{\partial\rho}|_{\Gamma_s} + \alpha\gamma_{\varepsilon_{|\Gamma_s}} = h$ and $\dfrac{\partial\overline{\gamma}_\varepsilon}{\partial\rho}|_{\Gamma_s} + \alpha\overline{\gamma}_{\varepsilon_{|\Gamma_s}} = \overline{h}$, respectively. Using the variational formulation established in Corollary 3.1.4, we have:

$$\begin{aligned} &\int_0^{2\pi} h\overline{\gamma}_\varepsilon(s)s\,\mathrm{d}\theta = \int_0^{2\pi} \overline{h}\gamma_\varepsilon(s)s\,\mathrm{d}\theta \\ \Rightarrow\quad & s\left\langle h, P_\varepsilon(s)\overline{h}\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\,',\,H^{1/2}_{\rho,P}(0,2\pi)} = s\left\langle \overline{h}, P_\varepsilon(s)h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\,',\,H^{1/2}_{\rho,P}(0,2\pi)} \end{aligned}$$

and we conclude that $P_\varepsilon(s)$ is a self adjoint operator.

On the other hand, taking $\gamma_\varepsilon = \overline{\gamma}_\varepsilon$ we have

$$s\ \langle h, P_\varepsilon(s)h\rangle_{H^{1/2}_{\rho,P}(0,2\pi)'\,,H^{1/2}_{\rho,P}(0,2\pi)} = \int_{\widehat{\Omega}_s\setminus\widehat{\Omega}_\varepsilon} |\nabla\gamma_\varepsilon|^2\,\rho\,\mathrm{d}\rho\,\mathrm{d}\theta + \alpha\, s\int_0^{2\pi}(\gamma_\varepsilon(s))^2\,\mathrm{d}\theta \qquad (3.7)$$

and consequently $P_\varepsilon(s)$ is a positive operator.

Now, from $\|\gamma_{\varepsilon_{|\Gamma_s}}\|_{H^{1/2}_{\rho,P}(0,2\pi)'} \le k_s\|\gamma_{\varepsilon_{|\Gamma_s}}\|_{H^{1/2}_{\rho,P}(0,2\pi)} \le k_{s,1}\|\gamma_\varepsilon\|_{\widehat{H}_s} \le k_{s,2}|||\gamma_\varepsilon|||_{\widehat{H}_s}$ and $\left\|\dfrac{\partial\gamma_\varepsilon}{\partial\rho}_{|\Gamma_s}\right\|_{H^{1/2}_{\rho,P}(0,2\pi)'} \le k_{s,3}\,\|\gamma_\varepsilon\|_{H(\Delta,\widehat{\Omega}_s\setminus\widehat{\Omega}_\varepsilon)} = k_{s,3}\,\|\gamma_\varepsilon\|_{\widehat{H}_s} \le k_{s,4}|||\gamma_\varepsilon|||_{\widehat{H}_s}$ (see Proposition 2.2.3), with $k_{s,1}, k_{s,2}, k_{s,3}$ and $k_{s,4}$ positive constants, we obtain

$$\begin{aligned}\left\|\alpha\gamma_{\varepsilon_{|\Gamma_s}} + \frac{\partial\gamma_\varepsilon}{\partial\rho}_{|\Gamma_s}\right\|_{H^{1/2}_{\rho,P}(0,2\pi)'} &\le \|\alpha\gamma_{\varepsilon_{|\Gamma_s}}\|_{H^{1/2}_{\rho,P}(0,2\pi)'} + \left\|\frac{\partial\gamma_\varepsilon}{\partial\rho}_{|\Gamma_s}\right\|_{H^{1/2}_{\rho,P}(0,2\pi)'}\\ &\le k_{s,5}|||\gamma_\varepsilon|||_{\widehat{H}_s}.\end{aligned}$$

Then,

$$\begin{aligned}s\ \langle h, P_\varepsilon(s)h\rangle_{H^{1/2}_{\rho,P}(0,2\pi)'\,,H^{1/2}_{\rho,P}(0,2\pi)} &= |||\gamma_\varepsilon|||^2_{\widehat{H}_s} \ge \frac{1}{k^2_{s,5}}\left\|\alpha\gamma_{\varepsilon_{|\Gamma_s}} + \frac{\partial\gamma_\varepsilon}{\partial\rho}_{|\Gamma_s}\right\|^2_{H^{1/2}_{\rho,P}(0,2\pi)'}\\ &= k_{s,6}\|h\|^2_{H^{1/2}_{\rho,P}(0,2\pi)'}.\end{aligned}$$

Again from (3.7) and Holder's inequality, we have

$$|||\gamma_\varepsilon|||^2_{\widehat{H}_s} \le s\|h\|_{H^{1/2}_{\rho,P}(0,2\pi)'}\|\gamma_{\varepsilon_{|\Gamma_s}}\|_{H^{1/2}_{\rho,P}(0,2\pi)}.$$

Then, as in Proposition 2.2.3 (and using again the inequalities $\|\gamma_{\varepsilon_{|\Gamma_s}}\|_{H^{1/2}_{\rho,P}(0,2\pi)} \le k_{s,1}\|\gamma_\varepsilon\|_{\widehat{H}_s} \le k_{s,2}|||\gamma_\varepsilon|||_{\widehat{H}_s}$), we can conclude that there exists $c_s > 0$ such that

$$\|\gamma_{\varepsilon_{|\Gamma_s}}\|_{H^{1/2}_{\rho,P}(0,2\pi)} \le c_s\|h\|_{H^{1/2}_{\rho,P}(0,2\pi)'}.$$

■

Proposition 3.1.7. *Considering M and N as in Lemma 2.2.4 and Lemma 2.2.5, respectively, the operator P_ε is such that $P_\varepsilon : M \to M$ and $P_\varepsilon : N \to N$.*

Proof. For each $s \in [\varepsilon, a)$ and $h \in N$ (constant), we define $P_\varepsilon(s)h = \gamma_{\varepsilon_{|\Gamma_s}}$, where $\gamma_\varepsilon \in \widehat{X}_s$ is the solution of (3.2) (that is, we consider a solution of (3.2) verifying also $\frac{\partial \gamma_\varepsilon}{\partial \rho}|_{\Gamma_s} + \alpha\, \gamma_{\varepsilon_{|\Gamma_s}}$ constant in θ). Considering $\delta(\rho)$ the solution of the linear two points boundary value problem, $\delta''(\rho) + \frac{1}{\rho}\delta'(\rho) = 0$, $2\pi\delta'(\varepsilon) = 0$, $\delta'(s) + \alpha\delta(s) = h$ (in fact, it is easy to prove that $\delta(\rho) = \frac{h}{\alpha}$) then, by uniqueness, $\gamma_\varepsilon(\rho, \theta) = \delta(\rho)$ is the solution of the previous problem.

Then, we can conclude that considering $\frac{\partial \gamma_\varepsilon}{\partial \rho}|_{\Gamma_s} + \alpha\gamma_{\varepsilon_{|\Gamma_s}} = h$ constant in θ, we also have $\gamma_\varepsilon(\rho, \theta)$ constant in θ (in fact, in this case, it is also constant in ρ) and therefore $\gamma_{\varepsilon_{|\Gamma_s}}$ has the same property. Consequently, $P_\varepsilon(s)h = \gamma_{\varepsilon_{|\Gamma_s}}$ is constant in θ and $P_\varepsilon : N \to N$.

Now, for each $s \in [\varepsilon, a)$ and $h \in M$, we define $P_\varepsilon(s)h = \gamma_{\varepsilon_{|\Gamma_s}}$, where $\gamma_\varepsilon \in \widehat{X}_s$ is the solution of (3.2) (that is, we consider a solution of (3.2) verifying also $\int_0^{2\pi} \frac{\partial \gamma_\varepsilon}{\partial \rho}|_{\Gamma_s} + \alpha\, \gamma_{\varepsilon_{|\Gamma_s}}\, \mathrm{d}\theta = 0$).

We have

$$
\begin{aligned}
& -\frac{\partial^2 \gamma_\varepsilon}{\partial \rho^2}(\rho,\theta) - \frac{1}{\rho^2}\frac{\partial^2 \gamma_\varepsilon}{\partial \theta^2}(\rho,\theta) - \frac{1}{\rho}\frac{\partial \gamma_\varepsilon}{\partial \rho}(\rho,\theta) = 0 \\
\Rightarrow\ & -\int_0^{2\pi} \frac{\partial^2 \gamma_\varepsilon}{\partial \rho^2}(\rho,\theta)\,\mathrm{d}\theta - \int_0^{2\pi} \frac{1}{\rho^2}\frac{\partial^2 \gamma_\varepsilon}{\partial \theta^2}(\rho,\theta)\,\mathrm{d}\theta - \int_0^{2\pi} \frac{1}{\rho}\frac{\partial \gamma_\varepsilon}{\partial \rho}(\rho,\theta)\,\mathrm{d}\theta = 0 \\
\Rightarrow\ & -\frac{\partial^2}{\partial \rho^2}\int_0^{2\pi} \gamma_\varepsilon(\rho,\theta)\,\mathrm{d}\theta - \frac{1}{\rho^2}\left.\frac{\partial \gamma_\varepsilon}{\partial \theta}(\rho,\theta)\right]_0^{2\pi} - \frac{1}{\rho}\frac{\partial}{\partial \rho}\int_0^{2\pi} \gamma_\varepsilon(\rho,\theta)\,\mathrm{d}\theta = 0 \\
\Rightarrow\ & -\frac{\partial^2}{\partial \rho^2}\int_0^{2\pi} \gamma_\varepsilon(\rho,\theta)\,\mathrm{d}\theta - \frac{1}{\rho}\frac{\partial}{\partial \rho}\int_0^{2\pi} \gamma_\varepsilon(\rho,\theta)\,\mathrm{d}\theta = 0.
\end{aligned}
$$

Considering $\delta(\rho) = \int_0^{2\pi} \gamma_\varepsilon(\rho,\theta)\,\mathrm{d}\theta$, since $\int_0^{2\pi} \frac{\partial \gamma_\varepsilon}{\partial \rho}|_{\Gamma_\varepsilon} = 0 \Rightarrow \frac{\partial}{\partial \rho}\int_0^{2\pi} \gamma_{\varepsilon_{|\Gamma_\varepsilon}}\,\mathrm{d}\theta = 0$ and also $\int_0^{2\pi} \frac{\partial \gamma_\varepsilon}{\partial \rho}|_{\Gamma_s} + \alpha\gamma_{\varepsilon_{|\Gamma_s}}\,\mathrm{d}\theta = 0 \Rightarrow \frac{\partial}{\partial \rho}\int_0^{2\pi} \gamma_{\varepsilon_{|\Gamma_s}}\,\mathrm{d}\theta + \alpha\int_0^{2\pi} \gamma_{\varepsilon_{|\Gamma_s}}\,\mathrm{d}\theta = 0$, we obtain the two points boundary value problem, $\delta''(\rho) + \frac{1}{\rho}\delta'(\rho) = 0$, $\delta'(\varepsilon) = 0$, $\delta'(s) + \alpha\delta(s) = 0$, which has the zero solution.

Then, we can conclude that considering $\int_0^{2\pi} \frac{\partial \gamma_\varepsilon}{\partial \rho}|_{\Gamma_s} + \alpha\gamma_{\varepsilon_{|\Gamma_s}}\,\mathrm{d}\theta = 0$, we also have $\int_0^{2\pi} \gamma_\varepsilon(\rho,\theta)\,\mathrm{d}\theta = 0$ for each ρ, and therefore $\int_0^{2\pi} \gamma_{\varepsilon_{|\Gamma_s}}\,\mathrm{d}\theta$ has the same property. Consequently, $P_\varepsilon(s)h = \gamma_{\varepsilon_{|\Gamma_s}}$ has null mean and $P_\varepsilon : M \to M$. ■

From equation (3.5), taking the derivative in a formal way, with respect to ρ, we obtain

$$
\begin{aligned}
&\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon \\
= \ & \frac{\partial P_\varepsilon}{\partial \rho}\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) + P_\varepsilon\left(\frac{\partial^2 \hat{u}_\varepsilon}{\partial \rho^2} + \alpha \frac{\partial \hat{u}_\varepsilon}{\partial \rho}\right) + \frac{\partial r_\varepsilon}{\partial \rho} + \alpha P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) + \alpha r_\varepsilon \\
= \ & \frac{\partial P_\varepsilon}{\partial \rho}\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) + P_\varepsilon\left(-f - \frac{1}{\rho}\frac{\partial \hat{u}_\varepsilon}{\partial \rho} - \frac{1}{\rho^2}\frac{\partial^2 \hat{u}_\varepsilon}{\partial \theta^2} + \alpha \frac{\partial \hat{u}_\varepsilon}{\partial \rho}\right) + \frac{\partial r_\varepsilon}{\partial \rho} \\
& + \alpha P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) + \alpha r_\varepsilon \\
= \ & \frac{\partial P_\varepsilon}{\partial \rho}\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) - P_\varepsilon f - \frac{1}{\rho}P_\varepsilon\frac{\partial \hat{u}_\varepsilon}{\partial \rho} - \frac{1}{\rho^2}P_\varepsilon\frac{\partial^2 \hat{u}_\varepsilon}{\partial \theta^2} + \alpha P_\varepsilon \frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \frac{\partial r_\varepsilon}{\partial \rho} \\
& + \alpha P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) + \alpha r_\varepsilon \\
= \ & \frac{\partial P_\varepsilon}{\partial \rho}\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) - P_\varepsilon f - \frac{1}{\rho}P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon - \alpha \hat{u}_\varepsilon\right) \\
& - \frac{1}{\rho^2}P_\varepsilon\frac{\partial^2}{\partial \theta^2}\left(P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) + r_\varepsilon\right) + \alpha P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon - \alpha \hat{u}_\varepsilon\right) + \frac{\partial r_\varepsilon}{\partial \rho} \\
& + \alpha P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) + \alpha r_\varepsilon \\
= \ & \frac{\partial P_\varepsilon}{\partial \rho}\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) - P_\varepsilon f - \frac{1}{\rho}P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) + \frac{1}{\rho}P_\varepsilon \alpha \hat{u}_\varepsilon \\
& - \frac{1}{\rho^2}P_\varepsilon\frac{\partial^2}{\partial \theta^2}P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) - \frac{1}{\rho^2}P_\varepsilon\frac{\partial^2 r_\varepsilon}{\partial \theta^2} + \alpha P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) - \alpha P_\varepsilon \alpha \hat{u}_\varepsilon + \frac{\partial r_\varepsilon}{\partial \rho} \\
& + \alpha P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) + \alpha r_\varepsilon \\
= \ & \frac{\partial P_\varepsilon}{\partial \rho}\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) - P_\varepsilon f - \frac{1}{\rho}P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) + \frac{1}{\rho}P_\varepsilon\left(\alpha P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) + \alpha r_\varepsilon\right) \\
& - \frac{1}{\rho^2}P_\varepsilon\frac{\partial^2}{\partial \theta^2}P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) - \frac{1}{\rho^2}P_\varepsilon\frac{\partial^2 r_\varepsilon}{\partial \theta^2} + \alpha P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) \\
& - \alpha P_\varepsilon\left(\alpha P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) + \alpha r_\varepsilon\right) + \frac{\partial r_\varepsilon}{\partial \rho} + \alpha P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) + \alpha r_\varepsilon \\
= \ & \frac{\partial P_\varepsilon}{\partial \rho}\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) - P_\varepsilon f - \frac{1}{\rho}P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) + \frac{1}{\rho}P_\varepsilon \alpha P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) \\
& + \frac{1}{\rho}P_\varepsilon \alpha r_\varepsilon - \frac{1}{\rho^2}P_\varepsilon\frac{\partial^2}{\partial \theta^2}P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) - \frac{1}{\rho^2}P_\varepsilon\frac{\partial^2 r_\varepsilon}{\partial \theta^2} + \alpha P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) \\
& - (\alpha P_\varepsilon)^2\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) - \alpha P_\varepsilon \alpha r_\varepsilon + \frac{\partial r_\varepsilon}{\partial \rho} + \alpha P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) + \alpha r_\varepsilon.
\end{aligned}
$$

Then, since $\dfrac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon$ is arbitrary (the prove is similar to the one of Remark 2.4.1), we obtain the following system

$$
\left\{
\begin{aligned}
& \frac{\partial P_\varepsilon}{\partial \rho} - \frac{P_\varepsilon}{\rho} + \frac{1}{\rho}\alpha P_\varepsilon^2 - \frac{1}{\rho^2}P_\varepsilon\frac{\partial^2}{\partial \theta^2}P_\varepsilon + 2\alpha P_\varepsilon - (\alpha P_\varepsilon)^2 = I \\
& -P_\varepsilon f + \frac{1}{\rho}P_\varepsilon \alpha r_\varepsilon - \frac{1}{\rho^2}P_\varepsilon\frac{\partial^2 r_\varepsilon}{\partial \theta^2} - \alpha^2 P_\varepsilon r_\varepsilon + \frac{\partial r_\varepsilon}{\partial \rho} + \alpha r_\varepsilon = 0 \\
& \hat{u}_\varepsilon = P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) + r_\varepsilon.
\end{aligned}
\right.
$$

Considering again M and N as in Lemma 2.2.4 and Lemma 2.2.5, respectively, from (3.4), we obtain

$$\begin{aligned} & \hat{u}_\varepsilon(\varepsilon) = P_\varepsilon(\varepsilon)\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho}(\varepsilon) + \alpha \hat{u}_\varepsilon(\varepsilon)\right) + r_\varepsilon(\varepsilon) \\ \Rightarrow\ & \hat{u}_\varepsilon(\varepsilon)_{|M} = P_\varepsilon(\varepsilon)\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho}(\varepsilon)|_M + \alpha \hat{u}_\varepsilon(\varepsilon)_{|M}\right) + r_\varepsilon(\varepsilon)_{|M} \\ \Rightarrow\ & 0 = P_\varepsilon(\varepsilon)\frac{\partial \hat{u}_\varepsilon}{\partial \rho}(\varepsilon)|_M + r_\varepsilon(\varepsilon)_{|M} \\ \Rightarrow\ & P_\varepsilon(\varepsilon)_{|M} = 0 \wedge r_\varepsilon(\varepsilon)_{|M} = 0, \end{aligned} \tag{3.8}$$

$$\begin{aligned} & \hat{u}_\varepsilon(\varepsilon) = P_\varepsilon(\varepsilon)\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho}(\varepsilon) + \alpha \hat{u}_\varepsilon(\varepsilon)\right) + r_\varepsilon(\varepsilon) \\ \Rightarrow\ & \hat{u}_\varepsilon(\varepsilon)_{|N} = P_\varepsilon(\varepsilon)\left(\frac{\partial \hat{u}_\varepsilon}{\partial \rho}(\varepsilon)|_N + \alpha \hat{u}_\varepsilon(\varepsilon)_{|N}\right) + r_\varepsilon(\varepsilon)_{|N} \\ \Rightarrow\ & \hat{u}_\varepsilon(\varepsilon) = P_\varepsilon(\varepsilon)\left(0 + \alpha \hat{u}_\varepsilon(\varepsilon)\right) + r_\varepsilon(\varepsilon)_{|N} = \alpha P_\varepsilon(\varepsilon)\hat{u}_\varepsilon(\varepsilon) + r_\varepsilon(\varepsilon)_{|N} \\ \Rightarrow\ & P_\varepsilon(\varepsilon)_{|N} = \frac{I}{\alpha} \wedge r_\varepsilon(\varepsilon)_{|N} = 0. \end{aligned} \tag{3.9}$$

From (3.8) and (3.9) we obtain $r_\varepsilon(\varepsilon) = r_\varepsilon(\varepsilon)_{|M} + r_\varepsilon(\varepsilon)_{|N} = 0$. In the same way, since

$$\begin{aligned} P_\varepsilon(\varepsilon)h &= P_\varepsilon(\varepsilon)\left(h_{|M} + h_{|N}\right) \\ &= P_\varepsilon(\varepsilon)h_{|M} + P_\varepsilon(\varepsilon)h_{|N} \\ &= \frac{I}{\alpha}h_{|N} \end{aligned}$$

we obtain $P_\varepsilon(\varepsilon) = \dfrac{proj_{|N}}{\alpha}$, denoting by $proj_{|N}$ the projection operator over the set N. Therefore, we have found the following system:

$$\begin{cases} \dfrac{\partial P_\varepsilon}{\partial \rho} - \dfrac{P_\varepsilon}{\rho} + \dfrac{1}{\rho}\alpha P_\varepsilon^2 - \dfrac{1}{\rho^2}P_\varepsilon\dfrac{\partial^2}{\partial\theta^2}P_\varepsilon + 2\alpha P_\varepsilon - (\alpha P_\varepsilon)^2 = I,\ P_\varepsilon(\varepsilon) = \dfrac{proj_{|N}}{\alpha} \\ -P_\varepsilon f + \dfrac{1}{\rho}P_\varepsilon \alpha r_\varepsilon - \dfrac{1}{\rho^2}P_\varepsilon\dfrac{\partial^2 r_\varepsilon}{\partial\theta^2} - \alpha^2 P_\varepsilon r_\varepsilon + \dfrac{\partial r_\varepsilon}{\partial \rho} + \alpha r_\varepsilon = 0,\ r_\varepsilon(\varepsilon) = 0 \\ \hat{u}_\varepsilon = P_\varepsilon\left(\dfrac{\partial \hat{u}_\varepsilon}{\partial \rho} + \alpha \hat{u}_\varepsilon\right) + r_\varepsilon,\ \hat{u}_\varepsilon(a) = 0. \end{cases}$$

3.2. Semi discretization and restriction to finite dimension

As in Section 2.3., every $\hat{v} \in \widehat{H}_s$ can be written in the form

$$\hat{v}(\rho,\theta) = \sum_{i=1}^{\infty} v_i(\rho) w_i(\theta), \tag{3.10}$$

where $w_i(\theta)$ are the elements of an Hilbert basis of $L^2(0,2\pi)$ formed by the eigenfunctions of the problem $-\dfrac{d^2 w_i}{d\theta^2} = \lambda_i w_i$, with periodic boundary conditions on 0 and 2π.

Using (3.10) and the definition of the norms referred in the previous section, we obtain, for all $s \in (\varepsilon, a]$:

$$\|\hat{v}(\rho)\|^2_{L^2(0,2\pi)} = \sum_{i=1}^{\infty} v_i^2, \quad \|\hat{v}(\rho)\|^2_{H^1_{\rho,P}(0,2\pi)} = v_1^2 + \sum_{i=2}^{\infty} \frac{\lambda_i}{\rho^2} v_i^2, \quad \|\hat{v}\|^2_{L^2_\rho(\varepsilon,s;L^2(0,2\pi))} = \int_\varepsilon^s \sum_1^\infty v_i^2 \rho \,\mathrm{d}\rho$$

$$|||\hat{v}|||^2_{\widehat{H}_s} = \alpha s \sum_1^\infty v_i^2(s) + \int_\varepsilon^s \sum_2^\infty \frac{\lambda_i}{\rho} v_i^2 \,\mathrm{d}\rho + \int_\varepsilon^s \sum_1^\infty \rho \left(\frac{\partial v_i}{\partial \rho}\right)^2 \mathrm{d}\rho.$$

By interpolation, we also have

$$\|\hat{v}(\rho)\|^2_{H^{1/2}_{\rho,P}(0,2\pi)} = v_1^2 + \sum_{i=2}^{\infty} \frac{\sqrt{\lambda_i}}{\rho} v_i^2, \quad \|\hat{v}(\rho)\|^2_{H^{3/2}_{\rho,P}(0,2\pi)} = v_1^2 + \sum_{i=2}^{\infty} \frac{{\lambda_i}^{3/2}}{\rho^3} v_i^2$$

and we define

$$\|\hat{v}(\rho)\|^2_{H^{1/2}_{\rho,P}(0,2\pi)}{}' = v_1^2 + \sum_{i=2}^{\infty} \frac{\rho}{\sqrt{\lambda_i}} v_i^2.$$

Obviously, all of these norms can be extended to the interval $(0,a]$.

Once again we embed the approximated problem (2.28) in a family of problems depending on h and s. For all $s \in (\varepsilon, a]$ we consider the finite dimension approximation defined on $\widehat{\Omega}_s \setminus \widehat{\Omega}_\varepsilon = (\varepsilon, s) \times (0, 2\pi)$ and, for each problem, we impose the boundary condition $\dfrac{\partial \hat{u}_\varepsilon}{\partial \rho}|_{\Gamma_s} + \alpha\, \hat{u}_{\varepsilon|_{\Gamma_s}} = h$. Considering $V^m = \langle w_1, \ldots, w_n \rangle$, we define $\widehat{H}^m_s = H^1_\rho(\varepsilon, s; V^m)$ and $\widehat{U}^m_s = \Big\{ v \in H^1_\rho(\varepsilon, s; V^m) : v_{|_{\Gamma_\varepsilon}} \text{ is constant} \Big\}$. Then, the approximation $\hat{u}^m_\varepsilon \in \widehat{U}^m_s$ of $\hat{u}_\varepsilon$ is the solution of

$$\begin{cases} \hat{u}^m_\varepsilon \in \widehat{U}^m_s \\ \displaystyle\int_0^{2\pi} \alpha \hat{u}^m_\varepsilon(s) \hat{v}^m_\varepsilon(s) s \,\mathrm{d}\theta + \int_\varepsilon^s \int_0^{2\pi} \frac{\partial \hat{u}^m_\varepsilon}{\partial \rho} \frac{\partial \hat{v}^m_\varepsilon}{\partial \rho} \rho + \frac{1}{\rho} \frac{\partial \hat{u}^m_\varepsilon}{\partial \theta} \frac{\partial \hat{v}^m_\varepsilon}{\partial \theta} \,\mathrm{d}\theta \,\mathrm{d}\rho \\ \displaystyle = \int_0^{2\pi} h^m \hat{v}^m_\varepsilon(s) s \,\mathrm{d}\theta + \int_\varepsilon^s \int_0^{2\pi} f^m \hat{v}^m_\varepsilon \rho \,\mathrm{d}\theta \,\mathrm{d}\rho, \ \forall \hat{v}^m_\varepsilon \in \widehat{U}^m_s. \end{cases} \tag{3.11}$$

We denote by $\beta_\varepsilon^m, \gamma_\varepsilon^m \in \widehat{U}_s^m$, respectively, the part of $\hat{u}_\varepsilon^m$ independent on h^m and linearly dependent on h^m, which means that, as in Section 2.4., we define the finite dimension operator $P_\varepsilon^m(s)$ by $\gamma_\varepsilon^m(s) = P_\varepsilon^m(s)h^m$ and $\beta_\varepsilon^m(s) = r_\varepsilon^m(s)$.

As in the previous chapter, for every $s \in (\varepsilon, a]$, $P_\varepsilon^m(s)$ is a linear operator and $P_\varepsilon^m(s) : V^m \to V^m$ (on which we consider in the first set the norm of $\left(H^{1/2}_{\rho,P}(0,2\pi)\right)'$, and in the second one the norm of $H^{1/2}_{\rho,P}(0,2\pi)$) and $r_\varepsilon^m(s) \in V^m$. Then we have

$$\hat{u}^m_{\varepsilon_{|\Gamma_s}} = P_\varepsilon^m(s)\left(\frac{\partial \hat{u}_\varepsilon^m}{\partial \rho}|_{\Gamma_s} + \alpha\, \hat{u}^m_{\varepsilon_{|\Gamma_s}}\right) + r_\varepsilon^m(s), \forall s \in [\varepsilon, a]. \tag{3.12}$$

Furthermore, the solution $\hat{u}_\varepsilon^m$ of (2.28) is given by

$$\hat{u}_\varepsilon^m(\rho,\theta) = \left(P_\varepsilon^m(\rho)\left(\frac{\partial \hat{u}_\varepsilon^m}{\partial \rho}|_{\Gamma_\rho} + \alpha\, \hat{u}^m_{\varepsilon_{|\Gamma_\rho}}\right)\right)(\theta) + (r_\varepsilon^m(\rho))(\theta). \tag{3.13}$$

From the last equality we can easily derive the following system:

$$\begin{cases} \dfrac{\partial P_\varepsilon^m}{\partial \rho} - \dfrac{P_\varepsilon^m}{\rho} + \dfrac{1}{\rho}\alpha (P_\varepsilon^m)^2 - \dfrac{1}{\rho^2} P_\varepsilon^m \dfrac{\partial^2}{\partial \theta^2} P_\varepsilon^m + 2\alpha P_\varepsilon^m - (\alpha P_\varepsilon^m)^2 = I,\ P_\varepsilon^m(\varepsilon) = \dfrac{proj^m_{|N}}{\alpha} \\ -P_\varepsilon^m f^m + \dfrac{1}{\rho} P_\varepsilon^m \alpha r_\varepsilon^m - \dfrac{1}{\rho^2} P_\varepsilon^m \dfrac{\partial^2 r_\varepsilon^m}{\partial \theta^2} - \alpha^2 P_\varepsilon^m r_\varepsilon^m + \dfrac{\partial r_\varepsilon^m}{\partial \rho} + \alpha r_\varepsilon^m = 0,\ r_\varepsilon^m(\varepsilon) = 0 \\ \hat{u}_\varepsilon^m = P_\varepsilon^m \left(\dfrac{\partial \hat{u}_\varepsilon^m}{\partial \rho} + \alpha \hat{u}_\varepsilon^m\right) + r_\varepsilon^m,\ \hat{u}_\varepsilon^m(a) = 0. \end{cases} \tag{3.14}$$

The proof of the next proposition is similar to the one of Proposition 2.4.2:

Proposition 3.2.1. *There exists a unique local solution to the system (3.14).*

By definition, we can write all $\hat{u}_\varepsilon^m \in \widehat{H}_s^m$ in the form

$$\hat{u}_\varepsilon^m(\rho,\theta) = \sum_1^m u_i(\rho) w_i(\theta). \tag{3.15}$$

Then, the coordinates $\{u_i(\rho)\}_{i=1}^m$ of $\hat{u}_\varepsilon^m$ must verify the following system (see Section 2.4. for the justification of the boundary conditions on ε):

$$\begin{cases} -\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial u_i}{\partial\rho}(\rho)\right)+\dfrac{\lambda_i}{\rho^2}u_i(\rho) \\ =\displaystyle\int_0^{2\pi}\hat{f}w_i(\theta)\,\mathrm{d}\theta=\hat{f}_i(\rho),\quad \varepsilon<\rho<s,\ i=1,\dots,m \\ u_i(\varepsilon)=0,\quad i=2,\dots,m \\ \dfrac{\partial u_1}{\partial\rho}(\varepsilon)=0 \\ \dfrac{\partial u_i}{\partial\rho}(s)+\alpha u_i(s)=h_i,\quad i=1,\dots,m. \end{cases} \tag{3.16}$$

As a consequence, the coordinates of γ_ε^m verify, for $\varepsilon<\rho<s$ and $i=1,\dots,m$

$$\begin{cases} -\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial\gamma_i}{\partial\rho}(\rho)\right)+\dfrac{\lambda_i}{\rho^2}\gamma_i(\rho)=0 \\ \gamma_i(\varepsilon)=0,\quad i=2,\dots,m \\ \dfrac{\partial\gamma_1}{\partial\rho}(\varepsilon)=0 \\ \dfrac{\partial\gamma_i}{\partial\rho}(s)+\alpha\gamma_i(s)=h_i,\quad i=1,\dots,m \end{cases} \tag{3.17}$$

and, for the coordinates of β_ε^m, we have

$$\begin{cases} -\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial\beta_i}{\partial\rho}(\rho)\right)+\dfrac{\lambda_i}{\rho^2}\beta_i(\rho) \\ =\hat{f}_i(\rho),\quad \varepsilon<\rho<s,\ i=1,\dots,m \\ \beta_i(\varepsilon)=0,\quad i=2,\dots,m \\ \dfrac{\partial\beta_1}{\partial\rho}(\varepsilon)=0 \\ \dfrac{\partial\beta_i}{\partial\rho}(s)+\alpha\beta_i(s)=0,\quad i=1,\dots,m. \end{cases} \tag{3.18}$$

For $i=1$, we know that the equation (3.17) has a solution of the form $\gamma_1(\rho)=c_1+c_2\log\rho$. So, determining the constants c_1 and c_2, we find that this solution is

$$\gamma_1(\rho)=\frac{1}{\alpha}h_1. \tag{3.19}$$

Similarly, for $i\geq 2$, we know that the equation has a solution of the form $\gamma_i(\rho)=c_1\rho^{\sqrt{\lambda_i}}+c_2\rho^{-\sqrt{\lambda_i}}$ and we find that this solution is

$$\gamma_i(\rho)=s\frac{\left(\frac{\rho}{\varepsilon}\right)^{\sqrt{\lambda_i}}-\left(\frac{\varepsilon}{\rho}\right)^{\sqrt{\lambda_i}}}{\sqrt{\lambda_i}\left(\left(\frac{s}{\varepsilon}\right)^{\sqrt{\lambda_i}}+\left(\frac{\varepsilon}{s}\right)^{\sqrt{\lambda_i}}\right)+\alpha\, s\left(\left(\frac{s}{\varepsilon}\right)^{\sqrt{\lambda_i}}-\left(\frac{\varepsilon}{s}\right)^{\sqrt{\lambda_i}}\right)}h_i, \tag{3.20}$$

on determining the constants.

In order to obtain an explicit formula for the coordinates of P_ε^m, we can also use the property exhibed in Remark 3.1.5. It's easy to see that without considering a particular value for the initial constant we obtain, using the same method as in Proposition 2.8.8, the general solution $p_i(\rho) = \frac{\rho}{\sqrt{\lambda_i}} \frac{c\,\rho^{2\sqrt{\lambda_i}} - 1}{c\,\rho^{2\sqrt{\lambda_i}} + 1}$, $i \geq 2$, for the equation (2.37). Therefore, as a consequence of Remark 3.1.5, denoting by $\bar{p}_i$ the coordinates of P_2, we have

$$(1 - \alpha \bar{p}_i) p_i = \bar{p}_i \;\Rightarrow\; (1 - \alpha \bar{p}_i) \frac{\rho}{\sqrt{\lambda_i}} \frac{c\,\rho^{2\sqrt{\lambda_i}} - 1}{c\,\rho^{2\sqrt{\lambda_i}} + 1} = \bar{p}_i$$
$$\Rightarrow\quad \bar{p}_i(\rho) = \frac{\rho(c\,\rho^{2\sqrt{\lambda_i}} - 1)}{\sqrt{\lambda_i}(c\,\rho^{2\sqrt{\lambda_i}} + 1) + \alpha\rho(c\,\rho^{2\sqrt{\lambda_i}} - 1)}.$$

From the initial condition $\bar{p}_i(\varepsilon) = 0$, $i \geq 2$, we can determine c as $\varepsilon^{-2\sqrt{\lambda_i}}$ and therefore we find, for the coordinates the P_ε^m, the explicit expression

$$\bar{p}_i(\rho) = \frac{\rho(\left(\frac{\rho}{\varepsilon}\right)^{2\sqrt{\lambda_i}} - 1)}{\sqrt{\lambda_i}(\left(\frac{\rho}{\varepsilon}\right)^{2\sqrt{\lambda_i}} + 1) + \alpha\rho(\left(\frac{\rho}{\varepsilon}\right)^{2\sqrt{\lambda_i}} - 1)},\ i \geq 2, \tag{3.21}$$

which corresponds to (3.20). The first coordinate of P_ε^m can not be achieved throught this process since the first component of $(I - \alpha P_2)$ is not invertible. In fact, from (3.19), we have $\bar{p}_1(\rho) = \frac{1}{\alpha}$.

The coordinates of P_ε^m verify $\bar{p}_i(\rho) \geq 0$, $\forall i \geq 1$, and since $\left(\frac{\rho}{\varepsilon}\right)^{2\sqrt{\lambda_i}} + 1 > 0$, from (3.21) we deduce also that $\bar{p}_i(\rho) < \dfrac{1}{\alpha}$, for $i \geq 2$. Then,

$$\bar{p}_i(\rho) \leq \frac{1}{\alpha}, i \geq 1. \tag{3.22}$$

3.3. Estimates on P_ε^m and r_ε^m

We begin this section with the usual "trace theorem", valid both for functions γ_ε^m and β_ε^m:

Proposition 3.3.1. *For all $\rho \in (\varepsilon, s]$ $(s \in [\varepsilon, a])$, there exists $k > 0$ (independent of ρ) such that*

$$\sqrt{\rho} \|\xi^m(\rho)\|_{H^{1/2}_{\rho,P}(0,2\pi)} \leq k |||\xi^m|||_{\widehat{H}_s},$$

for all $\xi^m \in \widehat{H}_s$, verifying $\frac{\partial \xi_1}{\partial \rho}(\varepsilon) = 0$ and $\xi_i(\varepsilon) = 0$, for $i \geq 2$.

Proof. Since, for $i \geq 2$,

$$\xi_i^2(\rho) = 2\int_\varepsilon^\rho \xi_i(t)\frac{\partial \xi_i}{\partial t}(t)\,\mathrm{d}t$$

we have

$$\begin{aligned}\sqrt{\lambda_i}\xi_i^2(\rho) \;&\leq\; 2\sqrt{\lambda_i}\left|\int_\varepsilon^\rho \xi_i(t)\frac{\partial \xi_i}{\partial t}(t)\,\mathrm{d}t\right| \leq 2\left(\int_\varepsilon^\rho \frac{\lambda_i}{t}\,\xi_i^2\,\mathrm{d}t\right)^{1/2}\left(\int_\varepsilon^\rho t\left(\frac{\partial \xi_i}{\partial t}\right)^2\mathrm{d}t\right)^{1/2} \\ &\leq\; \int_\varepsilon^\rho \frac{\lambda_i}{t}\,\xi_i^2\,\mathrm{d}t + \int_\varepsilon^\rho t\left(\frac{\partial \xi_i}{\partial t}\right)^2\mathrm{d}t.\end{aligned} \tag{3.23}$$

Summing up from 2 to m, we obtain:

$$\sum_2^m \sqrt{\lambda_i}\xi_i^2(\rho) \leq \sum_2^m \int_\varepsilon^\rho \frac{\lambda_i}{t}\,\xi_i^2\,\mathrm{d}t + \sum_2^m \int_\varepsilon^\rho t\left(\frac{\partial \xi_i}{\partial t}\right)^2\mathrm{d}t. \tag{3.24}$$

On the other hand, as in Proposition 3.1.1, we have

$$\int_\rho^s t\,\xi_1(t)\frac{\partial \xi_1}{\partial t}(t)\,\mathrm{d}t = \frac{1}{2}s\,\xi_1^2(s) - \frac{1}{2}\rho\,\xi_1^2(\rho) - \frac{1}{2}\int_\rho^s \xi_1^2(t)\,\mathrm{d}t.$$

Then,

$$\begin{aligned}&\rho\,\xi_1^2(\rho) - s\,\xi_1^2(s) + \int_\rho^s \xi_1^2(t)\,\mathrm{d}t = -2\int_\rho^s t\,\xi_1(t)\frac{\partial \xi_1}{\partial t}(t)\,\mathrm{d}t \\ \leq\;& 2\int_\rho^s \left|t\xi_1(t)\frac{\partial \xi_1}{\partial t}(t)\,\mathrm{d}t\right| \leq 2\left(\int_\rho^s \xi_1^2(t)\,\mathrm{d}t\right)^{1/2}\left(\int_\rho^s t^2\left(\frac{\partial \xi_1}{\partial t}\right)^2\mathrm{d}t\right)^{1/2} \\ \leq\;& \int_\rho^s \xi_1^2(t)\,\mathrm{d}t + \int_\rho^s t^2\left(\frac{\partial \xi_1}{\partial t}\right)^2\mathrm{d}t.\end{aligned}$$

Consequently,

$$\rho\,\xi_1^2(\rho) - s\,\xi_1^2(s) \leq \int_\rho^s t^2\left(\frac{\partial \xi_1}{\partial t}\right)^2\mathrm{d}t \leq s\int_\rho^s t\left(\frac{\partial \xi_1}{\partial t}\right)^2\mathrm{d}t \leq s\int_\varepsilon^s t\left(\frac{\partial \xi_1}{\partial t}\right)^2\mathrm{d}t.$$

So,

$$\begin{aligned}&\rho\,\xi_1^2(\rho) + \sum_2^m \sqrt{\lambda_i}\xi_i^2(\rho) \leq \rho\,\xi_1^2(\rho) + \sum_2^m \int_\varepsilon^\rho \frac{\lambda_i}{t}\,\xi_i^2\,\mathrm{d}t + \sum_2^m \int_\varepsilon^\rho t\left(\frac{\partial \xi_i}{\partial t}\right)^2\mathrm{d}t \\ \leq\;& \rho\,\xi_1^2(\rho) + \sum_2^m \int_\varepsilon^s \frac{\lambda_i}{t}\,\xi_i^2\,\mathrm{d}t + \sum_2^m \int_\varepsilon^s t\left(\frac{\partial \xi_i}{\partial t}\right)^2\mathrm{d}t \\ \leq\;& s\,\xi_1^2(s) + s\int_\varepsilon^s t\left(\frac{\partial \xi_1}{\partial t}\right)^2\mathrm{d}t + \sum_2^m \int_\varepsilon^s \frac{\lambda_i}{t}\,\xi_i^2\,\mathrm{d}t + \sum_2^m \int_\varepsilon^s t\left(\frac{\partial \xi_i}{\partial t}\right)^2\mathrm{d}t \\ \leq\;& \frac{1}{\alpha}\alpha\,s\xi_1^2(s) + \sum_2^m \int_\varepsilon^s \frac{\lambda_i}{t}\,\xi_i^2\,\mathrm{d}t + (s+1)\sum_1^m \int_\varepsilon^s t\left(\frac{\partial \xi_i}{\partial t}\right)^2\mathrm{d}t \\ \leq\;& \frac{1}{\alpha}\alpha\,s\sum_1^m \xi_i^2(s) + \sum_2^m \int_\varepsilon^s \frac{\lambda_i}{t}\,\xi_i^2\,\mathrm{d}t + (s+1)\sum_1^m \int_\varepsilon^s t\left(\frac{\partial \xi_i}{\partial t}\right)^2\mathrm{d}t \\ \leq\;& \max\left\{1, \frac{1}{\alpha}, s+1\right\} |||\xi_\varepsilon^m|||_{\widehat{H}_s} \leq \max\left\{\frac{1}{\alpha}, a+1\right\} |||\xi_\varepsilon^m|||_{\widehat{H}_s}.\end{aligned}$$

∎

Proposition 3.3.2. *For γ_ε^m solution of (3.17), we have*

$$|||\gamma_\varepsilon^m|||^2_{\widehat{H}_s} \leq s \, \|\gamma_\varepsilon^m(s)\|_{H^{1/2}_{s,P}(0,2\pi)} \, \|h^m\|_{H^{1/2}_{s,P}(0,2\pi)}{}',$$

for all $s \in [\varepsilon, a]$.

Proof. From (3.17) we obtain,

$$\begin{aligned}
& -\int_\varepsilon^s \frac{\partial^2 \gamma_i}{\partial \rho^2} \gamma_i \rho \, \mathrm{d}\rho + \int_\varepsilon^s \frac{1}{\rho^2} \lambda_i \gamma_i^2 \rho \, \mathrm{d}\rho - \int_\varepsilon^s \frac{1}{\rho} \frac{\partial \gamma_i}{\partial \rho} \gamma_i \rho \, \mathrm{d}\rho = 0 \\
\Rightarrow & -\frac{\partial \gamma_i}{\partial \rho} \, \gamma_i \, \rho \Big]_\varepsilon^s + \int_\varepsilon^s \frac{\partial \gamma_i}{\partial \rho} \left(\frac{\partial \gamma_i}{\partial \rho} \rho + \gamma_i \right) \mathrm{d}\rho + \int_\varepsilon^s \frac{1}{\rho} \lambda_i \gamma_i^2 \, \mathrm{d}\rho - \int_\varepsilon^s \frac{\partial \gamma_i}{\partial \rho} \gamma_i \, \mathrm{d}\rho = 0 \\
\Rightarrow & \quad -\frac{\partial \gamma_i}{\partial \rho}(s) \, \gamma_i(s) \, s + \int_\varepsilon^s \left(\frac{\partial \gamma_i}{\partial \rho} \right)^2 \rho \, \mathrm{d}\rho + \int_\varepsilon^s \frac{1}{\rho} \lambda_i \gamma_i^2 \, \mathrm{d}\rho = 0 \\
\Rightarrow & \quad -(h_i - \alpha \gamma_i(s)) \, \gamma_i(s) \, s + \int_\varepsilon^s \left(\frac{\partial \gamma_i}{\partial \rho} \right)^2 \rho \, \mathrm{d}\rho + \int_\varepsilon^s \frac{1}{\rho} \lambda_i \gamma_i^2 \, \mathrm{d}\rho = 0 \\
\Rightarrow & \sum_1^m h_i \, \gamma_i(s) \, s = \alpha \, s \sum_1^m \gamma_i^2(s) + \sum_1^m \int_\varepsilon^s \left(\frac{\partial \gamma_i}{\partial \rho} \right)^2 \rho \, \mathrm{d}\rho + \sum_1^m \int_\varepsilon^s \frac{1}{\rho} \lambda_i \gamma_i^2 \, \mathrm{d}\rho \\
\Rightarrow & \quad \sum_1^m h_i \, \gamma_i(s) \, s = |||\gamma_\varepsilon^m|||^2_{\widehat{H}_s}.
\end{aligned} \tag{3.25}$$

On the other hand,

$$\begin{aligned}
\sum_1^m h_i \gamma_i(s) &= h_1 \gamma_1(s) + \sum_2^m \frac{\sqrt{s}}{\sqrt[4]{\lambda_i}} \, h_i \, \frac{\sqrt[4]{\lambda_i}}{\sqrt{s}} \gamma_i(s) \\
&\leq \left(h_1^2 + \sum_2^m \frac{s}{\sqrt{\lambda_i}} \, h_i^2 \right)^{1/2} \left(\gamma_1^2(s) + \sum_2^m \frac{\sqrt{\lambda_i}}{s} \gamma_i^2(s) \right)^{1/2} \\
&= \|h^m\|_{H^{1/2}_{s,P}(0,2\pi)}{}' \, \|\gamma_\varepsilon^m(s)\|_{H^{1/2}_{s,P}(0,2\pi)}.
\end{aligned}$$

Consequently,

$$|||\gamma_\varepsilon^m|||^2_{\widehat{H}_s} \leq s \, \|h^m\|_{H^{1/2}_{s,P}(0,2\pi)}{}' \, \|\gamma_\varepsilon^m(s)\|_{H^{1/2}_{s,P}(0,2\pi)}.$$

■

As a direct consequence of Theorem 3.3.1 and Proposition 3.3.2, we have the following theorem:

Theorem 3.3.3. *There exists* $k = \left(\max\left\{\frac{1}{\alpha}, a+1\right\}\right)^2 > 0$ *(independent of* s *and* ε*) such that*

$$\|\gamma_\varepsilon^m(s)\|_{H^{1/2}_{s,P}(0,2\pi)} \le k \, \|h^m\|_{\, H^{1/2}_{s,P}(0,2\pi)\;'}. \tag{3.26}$$

The above theorem tell us that the operator P_ε^m is continuous and

$$\|P_\varepsilon^m\|_{\mathcal{L}\left(\; H^{1/2}_{s,P}(0,2\pi)\;', H^{1/2}_{s,P}(0,2\pi)\right)} \le k,$$

where k is a constant that does not depende both on ε and s.

Theorem 3.3.4. *There exists* $k = \max\left\{1, \dfrac{1}{\alpha}\right\} > 0$ *(independent of* s *and* ε*) such that*

$$\|\gamma_\varepsilon^m(s)\|_{H^{3/2}_{s,P}(0,2\pi)} \le k \, \|h^m\|_{H^{1/2}_{s,P}(0,2\pi)}. \tag{3.27}$$

Proof. Multiplying (3.23), for the particular case of $\rho = s$, by λ_i, we obtain

$$\lambda_i^{3/2}\gamma_i^2(s) \le \int_\varepsilon^s \frac{1}{\rho}\lambda_i^2\gamma_i^2(\rho)\,\mathrm{d}\rho + \int_\varepsilon^s \rho\lambda_i\left(\frac{\partial\gamma_i}{\partial\rho}(\rho)\right)^2 \mathrm{d}\rho$$

and summing up from 2 to m,

$$\sum_2^m \lambda_i^{3/2}\gamma_i^2(s) \le \sum_2^m \int_\varepsilon^s \frac{1}{\rho}\lambda_i^2\gamma_i^2(\rho)\,\mathrm{d}\rho + \sum_2^m \int_\varepsilon^s \rho\lambda_i\left(\frac{\partial\gamma_i}{\partial\rho}(\rho)\right)^2 \mathrm{d}\rho. \tag{3.28}$$

On the other hand, on (3.25), considering $i = 1$, we have

$$\begin{aligned} &\alpha\, s\gamma_1^2(s) = h_1\gamma_1(s)s - \int_\varepsilon^s \left(\frac{\partial\gamma_1}{\partial\rho}\right)^2 \rho\,\mathrm{d}\rho \le h_1\gamma_1(s)s \\ \Rightarrow\quad &s^3\gamma_1^2(s) \le \frac{s^3}{\alpha}h_1\gamma_1(s) \end{aligned}$$

and, considering $i \ge 2$ and multiplying by λ_i, we get

$$\begin{aligned} &\int_\varepsilon^s \lambda_i\left(\frac{\partial\gamma_i}{\partial\rho}\right)^2 \rho\,\mathrm{d}\rho + \int_\varepsilon^s \frac{1}{\rho}\lambda_i^2\gamma_i^2\,\mathrm{d}\rho \\ &= \lambda_i h_i\;\gamma_i(s)\;s - \lambda_i\alpha\,s\gamma_i^2(s) \le \lambda_i h_i\;\gamma_i(s)\;s \\ &\Rightarrow \sum_2^m \int_\varepsilon^s \lambda_i\left(\frac{\partial\gamma_i}{\partial\rho}\right)^2 \rho\,\mathrm{d}\rho + \sum_2^m \int_\varepsilon^s \frac{1}{\rho}\lambda_i^2\gamma_i^2\,\mathrm{d}\rho \le \sum_2^m \lambda_i h_i\;\gamma_i(s)\;s. \end{aligned}$$

Consequently,

$$\begin{aligned}
& s^3\|\gamma_\varepsilon^m(s)\|^2_{H^{3/2}_{s,P}(0,2\pi)} \\
= \;& s^3\gamma_1^2(s)+\sum_2^m \lambda_i^{3/2}\gamma_i^2(s) \\
\le \;& s^3\gamma_1^2(s)+\sum_2^m \int_\varepsilon^s \frac{1}{\rho}\lambda_i^2\gamma_i^2(\rho)\,\mathrm{d}\rho+\sum_2^m \int_\varepsilon^s \rho\lambda_i\left(\frac{\partial\gamma_i}{\partial\rho}(\rho)\right)^2\mathrm{d}\rho \\
\le \;& \frac{s^3}{\alpha}h_1\gamma_1(s)+\sum_2^m \lambda_i h_i\;\gamma_i(s)\;s \\
\le \;& \max\left\{1,\frac{1}{\alpha}\right\}s^3\left(h_1\;\gamma_1(s)+\sum_2^m \lambda_i\,h_i\;\gamma_i(s)\;\frac{1}{s^2}\right) \\
\le \;& \max\left\{1,\frac{1}{\alpha}\right\}s^3\left(h_1\;\gamma_1(s)+\sum_2^m \lambda_i^{3/4}\lambda_i^{1/4}\,h_i\;\gamma_i(s)\;\frac{1}{s^{3/2}}\frac{1}{s^{1/2}}\right) \\
\le \;& \max\left\{1,\frac{1}{\alpha}\right\}s^3\left(h_1^2+\sum_2^m \frac{\lambda_i^{1/2}}{s}h_i^2\right)^{1/2}\left(\gamma_1^2(s)+\sum_2^m \frac{\lambda_i^{3/2}}{s^3}\gamma_i^2(s)\right)^{1/2} \\
= \;& \max\left\{1,\frac{1}{\alpha}\right\}s^3\,\|h^m\|_{H^{1/2}_{s,P}(0,2\pi)}\|\gamma_\varepsilon^m(s)\|_{H^{3/2}_{s,P}(0,2\pi)}.
\end{aligned}$$

Then,

$$\|\gamma_\varepsilon^m(s)\|_{H^{3/2}_{s,P}(0,2\pi)} \le \max\left\{1,\frac{1}{\alpha}\right\}\|h^m\|_{H^{1/2}_{s,P}(0,2\pi)}.$$

∎

By interpolation, we achieve the following corollary, which is a direct consequence of Propositions 3.3.3 and 3.3.4:

Corollary 3.3.5. *There exists $k>0$ (independent of s and ε) such that*

$$\|\gamma_\varepsilon^m(s)\|_{H^1_{s,P}(0,2\pi)} \le k\,\|h^m\|_{L^2(0,2\pi)}. \tag{3.29}$$

With respect to the function β_ε^m, solution of (3.18), we have the following estimations:

Proposition 3.3.6. *For all $\rho\in(\varepsilon,s]$, there exists $c>0$ (independent of ρ) such that*

$$\|\beta_\varepsilon^m\|_{L^2_\rho(\varepsilon,s;H^{1/2}_{\rho,P}(0,2\pi))} \le c.$$

Proof. Since the coordinates of β_ε^m, for $\varepsilon < \rho \le s$ and $i = 1, \dots, m$, verify (3.18), we have

$$
\begin{aligned}
& -\int_\varepsilon^s \frac{\partial^2 \beta_i}{\partial \rho^2}\beta_i \rho \,\mathrm{d}\rho - \int_\varepsilon^s \frac{\partial \beta_i}{\partial \rho}\beta_i \,\mathrm{d}\rho + \int_\varepsilon^s \frac{\lambda_i}{\rho}\,\beta_i^2 \,\mathrm{d}\rho = \int_\varepsilon^s \left(\int_0^{2\pi} \hat{f} w_i(\theta)\,\mathrm{d}\theta\right)\beta_i\rho\,\mathrm{d}\rho \\
\Rightarrow \quad & -\frac{\partial \beta_i}{\partial \rho}\beta_i\rho\Big]_\varepsilon^s + \int_\varepsilon^s \frac{\partial \beta_i}{\partial \rho}\left(\frac{\partial \beta_i}{\partial \rho}\rho + \beta_i\right)\mathrm{d}\rho - \int_\varepsilon^s \frac{\partial \beta_i}{\partial \rho}\beta_i\,\mathrm{d}\rho + \int_\varepsilon^s \frac{\lambda_i}{\rho}\,\beta_i^2\,\mathrm{d}\rho \\
& = \int_\varepsilon^s \left(\int_0^{2\pi} \hat{f} w_i(\theta)\,\mathrm{d}\theta\right)\beta_i\rho\,\mathrm{d}\rho \\
\Rightarrow \quad & \beta_i^2(s)\alpha\, s + \int_\varepsilon^s \left(\frac{\partial \beta_i}{\partial \rho}\right)^2 \rho\,\mathrm{d}\rho + \int_\varepsilon^s \frac{\lambda_i}{\rho}\,\beta_i^2\,\mathrm{d}\rho = \int_\varepsilon^s \left(\int_0^{2\pi} \hat{f} w_i(\theta)\,\mathrm{d}\theta\right)\beta_i\rho\,\mathrm{d}\rho \\
\Rightarrow \quad & \sum_1^m \left(\beta_i^2(s)\alpha\, s + \int_\varepsilon^s \left(\frac{\partial \beta_i}{\partial \rho}\right)^2 \rho\,\mathrm{d}\rho + \int_\varepsilon^s \frac{\lambda_i}{\rho}\,\beta_i^2\,\mathrm{d}\rho\right) = \sum_1^m \int_\varepsilon^s \left(\int_0^{2\pi} \hat{f} w_i(\theta)\,\mathrm{d}\theta\right)\beta_i\rho\,\mathrm{d}\rho \\
\Rightarrow \quad & |||\beta_\varepsilon^m|||_{\widehat{H}_s}^2 = \int_\varepsilon^s \int_0^{2\pi} \hat{f}\left(\sum_1^m \beta_i w_i(\theta)\right)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta = \int_\varepsilon^s \int_0^{2\pi} \hat{f}\beta_\varepsilon^m \rho\,\mathrm{d}\rho\,\mathrm{d}\theta.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\int_\varepsilon^s \int_0^{2\pi} \hat{f}\beta_\varepsilon^m \rho\,\mathrm{d}\rho\,\mathrm{d}\theta & \le \|\hat{f}\|_{L^2_\rho(\varepsilon,s;L^2(0,2\pi))}\|\beta_\varepsilon^m\|_{L^2_\rho(\varepsilon,s;L^2(0,2\pi))} \\
& \le \|\hat{f}\|_{L^2_\rho(0,a;L^2(0,2\pi))}\|\beta_\varepsilon^m\|_{L^2_\rho(\varepsilon,s;H^{1/2}_{\rho,P}(0,2\pi))}.
\end{aligned}
$$

Therefore,

$$
|||\beta_\varepsilon^m|||_{\widehat{H}_s}^2 \le \|\hat{f}\|_{L^2_\rho(0,a;L^2(0,2\pi))}\|\beta_\varepsilon^m\|_{L^2_\rho(\varepsilon,s;H^{1/2}_{\rho,P}(0,2\pi))} \tag{3.30}
$$

and from Proposition 3.3.1, we obtain

$$
\begin{aligned}
& \rho\|\beta_\varepsilon^m(\rho)\|^2_{H^{1/2}_{\rho,P}(0,2\pi)} \le k\|\hat{f}\|_{L^2_\rho(0,a;L^2(0,2\pi))}\|\beta_\varepsilon^m\|_{L^2_\rho(\varepsilon,s;H^{1/2}_{\rho,P}(0,2\pi))} \\
\Rightarrow \quad & \int_\varepsilon^s \rho\|\beta_\varepsilon^m(\rho)\|^2_{H^{1/2}_{\rho,P}(0,2\pi)}\,\mathrm{d}\rho \le \int_\varepsilon^s k\|\hat{f}\|_{L^2_\rho(0,a;L^2(0,2\pi))}\|\beta_\varepsilon^m\|_{L^2_\rho(\varepsilon,s;H^{1/2}_{\rho,P}(0,2\pi))}\,\mathrm{d}\rho \\
\Rightarrow \quad & \|\beta_\varepsilon^m\|^2_{L^2_\rho(\varepsilon,s;H^{1/2}_{\rho,P}(0,2\pi))} \le k(s-\varepsilon)\|\hat{f}\|_{L^2_\rho(0,a;L^2(0,2\pi))}\|\beta_\varepsilon^m\|_{L^2_\rho(\varepsilon,s;H^{1/2}_{\rho,P}(0,2\pi))} \\
\Rightarrow \quad & \|\beta_\varepsilon^m\|_{L^2_\rho(\varepsilon,s;H^{1/2}_{\rho,P}(0,2\pi))} \le k\,s\|\hat{f}\|_{L^2_\rho(0,a;L^2(0,2\pi))} \le k\,a\|\hat{f}\|_{L^2_\rho(0,a;L^2(0,2\pi))} = c.
\end{aligned}
$$

■

Proposition 3.3.7. *For all $\rho \in (\varepsilon, s]$, there exists $k > 0$ (independent of ρ) such that*

$$
\|\beta_\varepsilon^m(\rho)\|_{H^{1/2}_{\rho,P}(0,2\pi)} \le k.
$$

Proof. Using (3.24), we have

$$\sum_{2}^{m} \sqrt{\lambda_i}\beta_i^2(\rho) \leq \sum_{2}^{m} \int_{\varepsilon}^{\rho} \frac{\lambda_i}{t}\, \beta_i^2 \,\mathrm{d}t + \sum_{2}^{m} \int_{\varepsilon}^{\rho} t \left(\frac{\partial \beta_i}{\partial t}\right)^2 \mathrm{d}t.$$

Then,

$$\begin{aligned}
&\rho\beta_1^2(\rho) + \sum_{2}^{m} \sqrt{\lambda_i}\beta_i^2(\rho) \leq \rho\beta_1^2(\rho) + \sum_{2}^{m} \int_{\varepsilon}^{\rho} \frac{\lambda_i}{t}\, \beta_i^2 \,\mathrm{d}t + \sum_{2}^{m} \int_{\varepsilon}^{\rho} t \left(\frac{\partial \beta_i}{\partial t}\right)^2 \mathrm{d}t \\
\leq\ & \frac{1}{\alpha}\sum_{1}^{m} \alpha\rho\beta_i^2(\rho) + \sum_{2}^{m} \int_{\varepsilon}^{\rho} \frac{\lambda_i}{t}\, \beta_i^2 \,\mathrm{d}t + \sum_{1}^{m} \int_{\varepsilon}^{\rho} t \left(\frac{\partial \beta_i}{\partial t}\right)^2 \mathrm{d}t \\
\leq\ & \max\left\{1, \frac{1}{\alpha}\right\} |||\beta_\varepsilon^m|||^2_{\widehat{H}_\rho},
\end{aligned}$$

that is,

$$\rho\|\beta_\varepsilon^m\|^2_{H^{1/2}_{\rho,P}(0,2\pi)} \leq c_1 |||\beta_\varepsilon^m|||^2_{\widehat{H}_\rho}. \tag{3.31}$$

From (3.30) we have, for all $t \in (\varepsilon, \rho)$,

$$|||\beta_\varepsilon^m|||^2_{\widehat{H}_\rho} \leq \|\hat{f}\|_{L^2_t(0,a;L^2(0,2\pi))} \|\beta_\varepsilon^m\|_{L^2_t(\varepsilon,\rho;H^{1/2}_{t,P}(0,2\pi))} \leq c_2 \|\beta_\varepsilon^m\|_{L^2_t(\varepsilon,\rho;H^{1/2}_{t,P}(0,2\pi))}. \tag{3.32}$$

From Proposition 3.3.1, for all $t \in (\varepsilon, \rho)$, $\exists c_3 > 0$ (independent of t) such that

$$\sqrt{t}\|\beta_\varepsilon^m(t)\|_{H^{1/2}_{t,P}(0,2\pi)} \leq c_3 |||\beta_\varepsilon^m|||_{\widehat{H}_\rho}. \tag{3.33}$$

Then, from (3.33) and (3.32), we obtain

$$\begin{aligned}
& t\|\beta_\varepsilon^m(t)\|^2_{H^{1/2}_{t,P}(0,2\pi)} \leq c_2 c_3^2 \|\beta_\varepsilon^m\|_{L^2_\tau(\varepsilon,\rho;H^{1/2}_{\tau,P}(0,2\pi))} \\
\Rightarrow\ & \int_{\varepsilon}^{\rho} t\|\beta_\varepsilon^m(t)\|^2_{H^{1/2}_{t,P}(0,2\pi)} \,\mathrm{d}t \leq c_2 c_3^2 \int_{\varepsilon}^{\rho} \|\beta_\varepsilon^m\|_{L^2_\tau(\varepsilon,\rho;H^{1/2}_{\tau,P}(0,2\pi))} \,\mathrm{d}t \\
\Rightarrow\ & \|\beta_\varepsilon^m\|^2_{L^2_t(\varepsilon,\rho;H^{1/2}_{t,P}(0,2\pi))} \leq c_2 c_3^2 (\rho - \varepsilon) \|\beta_\varepsilon^m\|_{L^2_t(\varepsilon,\rho;H^{1/2}_{t,P}(0,2\pi))} \\
\Rightarrow\ & \|\beta_\varepsilon^m\|_{L^2_t(\varepsilon,\rho;H^{1/2}_{t,P}(0,2\pi))} \leq c_2 c_3^2 \rho.
\end{aligned}$$

Again from (3.32) we get

$$|||\beta_\varepsilon^m|||^2_{\widehat{H}_\rho} \leq c_2^2 c_3^2 \rho$$

and back to (3.31) we obtain

$$\rho\|\beta_\varepsilon^m\|^2_{H^{1/2}_{\rho,P}(0,2\pi)} \leq c_1 c_2^2 c_3^2 \rho,$$

as desired. ■

In the sequence of Proposition 3.2.1, the following proposition is a direct consequence of Propositions 3.3.3, 3.3.4, 3.3.6, 3.3.7 and Corollary 3.3.5:

Proposition 3.3.8. *P_ε^m is a global solution of (3.14) and is C^1 from $[\varepsilon, a]$ with values in $\mathcal{L}(V^m, V^m)$; consequently, r_ε^m is a global solution of (3.14) and $r_\varepsilon^m \in H^1(\varepsilon, a; V^m)$.*

3.4. Passing to the limit

First, we are going to pass to the limit when $m \to \infty$. In this passage we use the same arguments of Chapter 2. In fact, applying again Propositions 2.8.1 and 2.8.2, we can prove the following result, following the same steps of Corollary 2.8.3:

Corollary 3.4.1. *For all $s \in (\varepsilon, a)$, $r_\varepsilon^m(s) \to r_\varepsilon(s)$ strongly in $H^{1/2}_{\rho,P}(0, 2\pi)$, when $m \to \infty$. Also, for all $s \in (\varepsilon, a)$ and for a fixed h, $P_\varepsilon^m(s)h \to P_\varepsilon(s)h$, strongly in $H^{1/2}_{\rho,P}(0, 2\pi)$, weakly in $H^{3/2}_{\rho,P}(0, 2\pi)$ and strongly in $H^1_{\rho,P}(0, 2\pi)$, when $m \to \infty$.*

Now, from (3.14), we obtain

Proposition 3.4.2. *For every $h, \bar{h}$ in $L^2(0, 2\pi)$, the operator $P_\varepsilon \in L^\infty\big((\varepsilon, a); \mathcal{L}\big(L^2(0, 2\pi), H^1_{\rho,P}(0, 2\pi)\big)\big)$ satisfies the following equation*

$$\left(\frac{\partial P_\varepsilon}{\partial \rho} h, \bar{h}\right)_{L^2(0,2\pi)} - \left(\frac{1}{\rho} h, P_\varepsilon \bar{h}\right)_{L^2(0,2\pi)} + \left(\frac{1}{\rho^2}\frac{\partial}{\partial \theta} P_\varepsilon h, \frac{\partial}{\partial \theta} P_\varepsilon \bar{h}\right)_{L^2(0,2\pi)}$$
$$+\frac{1}{\rho}\alpha \left(P_\varepsilon h, P_\varepsilon \bar{h}\right)_{L^2(0,2\pi)} + 2\alpha \left(P_\varepsilon h, \bar{h}\right)_{L^2(0,2\pi)} - \alpha^2 \big(P_\varepsilon h, P_\varepsilon \bar{h}\big)_{L^2(0,2\pi)} = \big(h, \bar{h}\big)_{L^2(0,2\pi)},$$

in $\mathcal{D}'(\varepsilon, a)$, and $P_\varepsilon(\varepsilon) = \dfrac{proj_{|N}}{\alpha}$.

Proof. For a fixed m_0, let $h, \bar{h} \in V^{m_0}$. Then, from (3.14), we obtain, for $m \geq m_0$

$$\left(\frac{\partial P_\varepsilon^m}{\partial \rho} h, \bar{h}\right)_{L^2(0,2\pi)} - \left(\frac{P_\varepsilon^m}{\rho} h, \bar{h}\right)_{L^2(0,2\pi)} + \left(\frac{1}{\rho}\alpha (P_\varepsilon^m)^2 h, \bar{h}\right)_{L^2(0,2\pi)}$$
$$- \left(\frac{1}{\rho^2} P_\varepsilon^m \frac{\partial^2}{\partial \theta^2} P_\varepsilon^m h, \bar{h}\right)_{L^2(0,2\pi)} + \big(2\alpha P_\varepsilon^m h, \bar{h}\big)_{L^2(0,2\pi)} - \big((\alpha P_\varepsilon^m)^2 h, \bar{h}\big)_{L^2(0,2\pi)} = \big(h, \bar{h}\big)_{L^2(0,2\pi)}.$$

Considering $\phi \in \mathcal{C}_0^1[\varepsilon, a)$ (that is, $\phi(a) = 0$ and we can have $\phi(\varepsilon) \neq 0$), we have:

$$\int_\varepsilon^a \left(\frac{\partial P_\varepsilon^m}{\partial \rho} h, \bar{h}\right)_{L^2(0,2\pi)} \phi\rho \, \mathrm{d}\rho - \int_\varepsilon^a \left(\frac{P_\varepsilon^m}{\rho} h, \bar{h}\right)_{L^2(0,2\pi)} \phi\rho \, \mathrm{d}\rho$$
$$+ \int_\varepsilon^a \left(\frac{1}{\rho}\alpha (P_\varepsilon^m)^2 h, \bar{h}\right)_{L^2(0,2\pi)} \phi\rho \, \mathrm{d}\rho - \int_\varepsilon^a \left(\frac{1}{\rho^2} P_\varepsilon^m \frac{\partial^2}{\partial \theta^2} P_\varepsilon^m h, \bar{h}\right)_{L^2(0,2\pi)} \phi\rho \, \mathrm{d}\rho$$
$$+ \int_\varepsilon^a \big(2\alpha P_\varepsilon^m h, \bar{h}\big)_{L^2(0,2\pi)} \phi\rho \, \mathrm{d}\rho - \int_\varepsilon^a \big((\alpha P_\varepsilon^m)^2 h, \bar{h}\big)_{L^2(0,2\pi)} \phi\rho \, \mathrm{d}\rho$$
$$= \int_\varepsilon^a \big(h, \bar{h}\big)_{L^2(0,2\pi)} \phi\rho \, \mathrm{d}\rho.$$

Integrating by parts the first term, since $P_\varepsilon^m(\varepsilon)h = \dfrac{h_{|N}}{\alpha}$ and $\phi(a) = 0$, we have

$$
\begin{aligned}
&-(\frac{h_{|N}}{\alpha},\bar{h})_{L^2(0,2\pi)}\phi(\varepsilon)\varepsilon - \int_\varepsilon^a \big(P_\varepsilon^m h,\bar{h}\big)_{L^2(0,2\pi)}\,\phi'\rho\,\mathrm{d}\rho - 2\int_\varepsilon^a \left(P_\varepsilon^m \frac{1}{\rho}h,\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho \\
&+\int_\varepsilon^a \left(\frac{1}{\rho}\alpha(P_\varepsilon^m)^2 h,\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho - \int_\varepsilon^a \left(\frac{1}{\rho^2}P_\varepsilon^m \frac{\partial^2}{\partial\theta^2}P_\varepsilon^m h,\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho \\
&+\int_\varepsilon^a \big(2\alpha P_\varepsilon^m h,\bar{h}\big)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho - \int_\varepsilon^a \big((\alpha P_\varepsilon^m)^2 h,\bar{h}\big)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho \\
&=\int_\varepsilon^a \big(h,\bar{h}\big)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho.
\end{aligned}
$$

Now, integrating by parts the fifth term, and taking into account the periodic boundary conditions, we obtain

$$
\begin{aligned}
&-(\frac{h_{|N}}{\alpha},\bar{h})_{L^2(0,2\pi)}\phi(\varepsilon)\varepsilon - \int_\varepsilon^a \big(P_\varepsilon^m h,\bar{h}\big)_{L^2(0,2\pi)}\,\phi'\rho\,\mathrm{d}\rho - 2\int_\varepsilon^a \left(\frac{1}{\rho}h,P_\varepsilon^m\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho \\
&+\int_\varepsilon^a \frac{1}{\rho}\alpha\big(P_\varepsilon^m h,P_\varepsilon^m\bar{h}\big)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho + \int_\varepsilon^a \left(\frac{1}{\rho^2}\frac{\partial}{\partial\theta}P_\varepsilon^m h,\frac{\partial}{\partial\theta}P_\varepsilon^m\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho \\
&+\int_\varepsilon^a 2\alpha\big(P_\varepsilon^m h,\bar{h}\big)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho - \alpha^2\int_\varepsilon^a \big(P_\varepsilon^m h,P_\varepsilon^m\bar{h}\big)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho \\
&=\int_\varepsilon^a \big(h,\bar{h}\big)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho.
\end{aligned}
$$

In the previous equality all the integrands are bounded, as a consequence of Corollary 3.3.5. In fact, as in Chapter 2, for $h \in L^2(0,2\pi)$ we have $\|P_\varepsilon^m h\|_{H^1_{\rho,P}(0,2\pi)}$ bounded and consequently both $\|P_\varepsilon^m h\|_{L^2(0,2\pi)}$ and $\left\|\dfrac{1}{\rho}\dfrac{\partial}{\partial\theta}(P_\varepsilon^m h)\right\|_{L^2(0,2\pi)}$ are bounded (notice that we have, for instance, $(P_\varepsilon^m h,\bar{h})_{L^2(0,2\pi)} \le \|P_\varepsilon^m h\|_{L^2(0,2\pi)}\|\bar{h}\|_{L^2(0,2\pi)}$). Then, we can use Lebesgue's theorem and according to Corollary 3.4.1, we can pass to the limit and obtain

$$
\begin{aligned}
&-(\frac{h_{|N}}{\alpha},\bar{h})_{L^2(0,2\pi)}\phi(\varepsilon)\varepsilon - \int_\varepsilon^a \big(P_\varepsilon h,\bar{h}\big)_{L^2(0,2\pi)}\phi'\rho\,\mathrm{d}\rho \\
&-2\int_\varepsilon^a \left(\frac{1}{\rho}h,P_\varepsilon\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho + \int_\varepsilon^a \frac{1}{\rho}\alpha\big(P_\varepsilon h,P_\varepsilon\bar{h}\big)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho \\
&+\int_\varepsilon^a \left(\frac{1}{\rho^2}\frac{\partial}{\partial\theta}P_\varepsilon h,\frac{\partial}{\partial\theta}P_\varepsilon\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho + \int_\varepsilon^a 2\alpha\big(P_\varepsilon h,\bar{h}\big)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho \\
&-\alpha^2\int_\varepsilon^a \big(P_\varepsilon h,P_\varepsilon\bar{h}\big)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho = \int_\varepsilon^a \big(h,\bar{h}\big)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho.
\end{aligned}
\tag{3.34}
$$

In fact, since $P_\varepsilon^m h \to P_\varepsilon h$ strongly in $H^1_{\rho,P}(0,2\pi)$, then both $P_\varepsilon^m h \to P_\varepsilon h$ and $\dfrac{\partial}{\partial\theta}P_\varepsilon^m h \to \dfrac{\partial}{\partial\theta}P_\varepsilon h$ strongly in $L^2(0,2\pi)$.

Now, since $\mathcal{D}(\varepsilon,a) \subset \mathcal{C}_0^1[\varepsilon,a)$, we can take $\phi \in \mathcal{D}(\varepsilon,a)$ in the previous equality and integrate backwards the second term, obtaining

$$\begin{aligned}
&\int_\varepsilon^a \left(\frac{\partial P_\varepsilon}{\partial\rho}h,\bar h\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho-\int_\varepsilon^a\left(\frac{1}{\rho}h,P_\varepsilon\bar h\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho\\
&+\int_\varepsilon^a\frac{1}{\rho}\alpha\left(P_\varepsilon h,P_\varepsilon\bar h\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho+\int_\varepsilon^a\left(\frac{1}{\rho^2}\frac{\partial}{\partial\theta}P_\varepsilon h,\frac{\partial}{\partial\theta}P_\varepsilon\bar h\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho\\
&+\int_\varepsilon^a 2\alpha\left(P_\varepsilon h,\bar h\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho-\alpha^2\int_\varepsilon^a\left(P_\varepsilon h,P_\varepsilon\bar h\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho\\
&=\int_\varepsilon^a\left(h,\bar h\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho,
\end{aligned}\tag{3.35}$$

for $h,\bar h\in V^{m_0}$. Therefore, by density, when $m_0\to\infty$, we obtain (3.35), for $h,\bar h\in L^2(0,2\pi)$. Then, from the equality in $\mathcal{D}'(\varepsilon,a)$

$$\begin{aligned}
\left(\frac{\partial P_\varepsilon}{\partial\rho}h,\bar h\right)_{L^2(0,2\pi)}&=\left(\frac{1}{\rho}h,P_\varepsilon\bar h\right)_{L^2(0,2\pi)}-\frac{1}{\rho}\alpha\left(P_\varepsilon h,P_\varepsilon\bar h\right)_{L^2(0,2\pi)}-\left(\frac{1}{\rho^2}\frac{\partial}{\partial\theta}P_\varepsilon h,\frac{\partial}{\partial\theta}P_\varepsilon\bar h\right)_{L^2(0,2\pi)}\\
&-2\alpha\left(P_\varepsilon h,\bar h\right)_{L^2(0,2\pi)}+\alpha^2\left(P_\varepsilon h,P_\varepsilon\bar h\right)_{L^2(0,2\pi)}+\left(h,\bar h\right)_{L^2(0,2\pi)},
\end{aligned}$$

and using again Corollary 3.3.5 (notice that the result is independent of m), we see that $\left(\frac{\partial P_\varepsilon}{\partial\rho}h,\bar h\right)_{L^2(0,2\pi)}\in L^\infty(\varepsilon,a)$. Using the same reasoning as in the proof of Proposition 2.8.4, from $\left(\frac{\partial P_\varepsilon}{\partial\rho}h,\bar h\right)_{L^2(0,2\pi)}\in L^2_\rho(\varepsilon,a)$ and $\left(P_\varepsilon(\rho)h,\bar h\right)_{L^2(0,2\pi)}\in L^2_\rho(\varepsilon,a)$, we deduce that $\left(P_\varepsilon(\rho)h,\bar h\right)_{L^2(0,2\pi)}$ is continuous in ρ. Consequently, for $\phi\in\mathcal{C}_0^1[\varepsilon,a)$ we can integrate (3.34) backwards to obtain $P_\varepsilon(\varepsilon)h=\frac{h_{|N}}{\alpha}$. ■

With respect to the equation on r_ε^m, we obtain the following result:

Proposition 3.4.3. *The function r_ε belongs to $\mathcal{C}\left(\varepsilon,a,L^2(0,2\pi)\right)$, satisfies $r_\varepsilon(\varepsilon)=0$, and for every h in $H^{1/2}_{\rho,P}(0,2\pi)$ verifies the following equation*

$$\begin{aligned}
&\left(\frac{1}{\rho}\alpha r_\varepsilon,P_\varepsilon h\right)_{L^2(0,2\pi)}+\left\langle\frac{1}{\rho^2}\frac{\partial r_\varepsilon}{\partial\theta},\frac{\partial}{\partial\theta}P_\varepsilon h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)}\\
&-\alpha^2\left(r_\varepsilon,P_\varepsilon h\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho+\left(\frac{\partial r_\varepsilon}{\partial\rho},h\right)_{L^2(0,2\pi)}+\alpha\left(r_\varepsilon,h\right)_{L^2(0,2\pi)}=\left(f,P_\varepsilon h\right)_{L^2(0,2\pi)},
\end{aligned}$$

in $\mathcal{D}'(\varepsilon,a)$.

Proof. For a fixed m_0, let $h\in V^{m_0}$. Then, from (3.14), we obtain, for $m\geq m_0$

$$\begin{aligned}
&\left(-P_\varepsilon^m f^m,h\right)_{L^2(0,2\pi)}+\left\langle P_\varepsilon^m\frac{1}{\rho}\alpha r_\varepsilon^m,h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)}\\
&-\left\langle P_\varepsilon^m\frac{1}{\rho^2}\frac{\partial^2 r_\varepsilon^m}{\partial\theta^2},h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)}-\alpha^2\left\langle P_\varepsilon^m r_\varepsilon^m,h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)}\\
&+\left\langle\frac{\partial r_\varepsilon^m}{\partial\rho},h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)}+\alpha\left(r_\varepsilon^m,h\right)_{L^2(0,2\pi)}=0.
\end{aligned}$$

Considering $\phi \in \mathcal{C}_0^1[\varepsilon, a)$, we have:

$$\begin{aligned}
&\int_\varepsilon^a (-P_\varepsilon^m f^m, h)_{L^2(0,2\pi)}\, \phi\rho \,\mathrm{d}\rho + \int_\varepsilon^a \left\langle P_\varepsilon^m \frac{1}{\rho}\alpha r_\varepsilon^m, h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)} \phi\rho \,\mathrm{d}\rho \\
&- \int_\varepsilon^a \left\langle P_\varepsilon^m \frac{1}{\rho^2}\frac{\partial^2 r_\varepsilon^m}{\partial\theta^2}, h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)} \phi\rho \,\mathrm{d}\rho \\
&-\alpha^2 \int_\varepsilon^a \langle P_\varepsilon^m r_\varepsilon^m, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)} \phi\rho \,\mathrm{d}\rho \\
&+ \int_\varepsilon^a \left\langle \frac{\partial r_\varepsilon^m}{\partial\rho}, h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)} \phi\rho \,\mathrm{d}\rho + \alpha \int_\varepsilon^a (r_\varepsilon^m, h)_{L^2(0,2\pi)}\, \phi\rho \,\mathrm{d}\rho = 0.
\end{aligned}$$

Integrating by parts the fifth term, since $r_\varepsilon^m(\varepsilon) = 0$ (and $\phi(a) = 0$), we obtain

$$\begin{aligned}
&- \int_\varepsilon^a (f^m, P_\varepsilon^m h)_{L^2(0,2\pi)}\, \phi\rho \,\mathrm{d}\rho + \int_\varepsilon^a \left(\frac{1}{\rho}\alpha r_\varepsilon^m, P_\varepsilon^m h\right)_{L^2(0,2\pi)} \phi\rho \,\mathrm{d}\rho \\
&- \int_\varepsilon^a \left\langle \frac{1}{\rho^2}\frac{\partial^2 r_\varepsilon^m}{\partial\theta^2}, P_\varepsilon^m h\right\rangle_{H^{3/2}_{\rho,P}(0,2\pi)\ ',H^{3/2}_{\rho,P}(0,2\pi)} \phi\rho \,\mathrm{d}\rho \\
&-\alpha^2 \int_\varepsilon^a (r_\varepsilon^m, P_\varepsilon^m h)_{L^2(0,2\pi)}\, \phi\rho \,\mathrm{d}\rho - \int_\varepsilon^a (r_\varepsilon^m, h)_{L^2(0,2\pi)}\, \phi'\rho \,\mathrm{d}\rho \\
&- \int_\varepsilon^a \left(\frac{1}{\rho} r_\varepsilon^m, h\right)_{L^2(0,2\pi)} \phi\rho \,\mathrm{d}\rho + \alpha \int_\varepsilon^a (r_\varepsilon^m, h)_{L^2(0,2\pi)}\, \phi\rho \,\mathrm{d}\rho = 0.
\end{aligned}$$

Integrating by parts the third term and according to the periodic boundary conditions, we have

$$\begin{aligned}
&- \int_\varepsilon^a (f^m, P_\varepsilon^m h)_{L^2(0,2\pi)}\, \phi\rho \,\mathrm{d}\rho + \int_\varepsilon^a \left(\frac{1}{\rho}\alpha r_\varepsilon^m, P_\varepsilon^m h\right)_{L^2(0,2\pi)} \phi\rho \,\mathrm{d}\rho \\
&+ \int_\varepsilon^a \left\langle \frac{1}{\rho^2}\frac{\partial r_\varepsilon^m}{\partial\theta}, \frac{1}{\partial\theta} P_\varepsilon^m h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)} \phi\rho \,\mathrm{d}\rho \\
&-\alpha^2 \int_\varepsilon^a (r_\varepsilon^m, P_\varepsilon^m h)_{L^2(0,2\pi)}\, \phi\rho \,\mathrm{d}\rho - \int_\varepsilon^a (r_\varepsilon^m, h)_{L^2(0,2\pi)}\, \phi'\rho \,\mathrm{d}\rho \\
&- \int_\varepsilon^a \left(\frac{1}{\rho} r_\varepsilon^m, h\right)_{L^2(0,2\pi)} \phi\rho \,\mathrm{d}\rho + \alpha \int_\varepsilon^a (r_\varepsilon^m, h)_{L^2(0,2\pi)}\, \phi\rho \,\mathrm{d}\rho = 0.
\end{aligned}$$

From Corollary 3.4.1 and Lebesgue's theorem (again all the integrands are bounded as a consequence of Proposition 3.3.7 and Corollary 3.3.5), we can pass to the limit in the previous equality. Then,

$$\begin{aligned}
&- \int_\varepsilon^a (f, P_\varepsilon h)_{L^2(0,2\pi)}\, \phi\rho \,\mathrm{d}\rho + \int_\varepsilon^a \left(\frac{1}{\rho}\alpha r_\varepsilon, P_\varepsilon h\right)_{L^2(0,2\pi)} \phi\rho \,\mathrm{d}\rho \\
&+ \int_\varepsilon^a \left\langle \frac{1}{\rho^2}\frac{\partial r_\varepsilon}{\partial\theta}, \frac{1}{\partial\theta} P_\varepsilon h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)} \phi\rho \,\mathrm{d}\rho \\
&-\alpha^2 \int_\varepsilon^a (r_\varepsilon, P_\varepsilon h)_{L^2(0,2\pi)}\, \phi\rho \,\mathrm{d}\rho - \int_\varepsilon^a (r_\varepsilon, h)_{L^2(0,2\pi)}\, \phi'\rho \,\mathrm{d}\rho \\
&- \int_\varepsilon^a \left(\frac{1}{\rho} r_\varepsilon, h\right)_{L^2(0,2\pi)} \phi\rho \,\mathrm{d}\rho + \alpha \int_\varepsilon^a (r_\varepsilon, h)_{L^2(0,2\pi)}\, \phi\rho \,\mathrm{d}\rho = 0.
\end{aligned} \tag{3.36}$$

In fact, in addition to the converge properties on P_ε^m exhibited in the previous proof, we have $\dfrac{\partial}{\partial\theta} r_\varepsilon^m \to \dfrac{\partial}{\partial\theta} r_\varepsilon$ strongly in $\left(H^{1/2}_{\rho,P}(0,2\pi)\right)'$, since $r_\varepsilon^m \to r_\varepsilon$ strongly in $H^{1/2}_{\rho,P}(0,2\pi)$, and $\frac{\partial}{\partial\theta}P_\varepsilon^m h \to \frac{\partial}{\partial\theta}P_\varepsilon h$ weakly in $H^{1/2}_{\rho,P}(0,2\pi)$. Now, since $\mathcal{D}(\varepsilon,a) \subset \mathcal{C}^1_0[\varepsilon,a)$, we can take $\phi \in \mathcal{D}(\varepsilon,a)$ in the previous equality and integrate backwards the fifth term, obtaining

$$\begin{aligned}
&-\int_\varepsilon^a (f,P_\varepsilon h)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho + \int_\varepsilon^a \left(\frac{1}{\rho}\alpha r_\varepsilon, P_\varepsilon h\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho\\
&+\int_\varepsilon^a \left\langle \frac{1}{\rho^2}\frac{\partial r_\varepsilon}{\partial\theta}, \frac{1}{\partial\theta}P_\varepsilon h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)} \phi\rho\,\mathrm{d}\rho\\
&-\alpha^2\int_\varepsilon^a (r_\varepsilon,P_\varepsilon h)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho + \int_\varepsilon^a \left(\frac{\partial r_\varepsilon}{\partial\rho}, h\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho\\
&+\alpha\int_\varepsilon^a (r_\varepsilon,h)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho = 0
\end{aligned} \tag{3.37}$$

for $h \in V^{m_0}$. Then, by density, when $m_0 \to \infty$, we have (3.37) for $h \in H^{1/2}_{\rho,P}(0,2\pi)$ (notice that with this choice for h, the third term is well defined).

Again by Proposition 3.3.6 and Corollary 3.3.5 (the result is independent of m), from the equality

$$\begin{aligned}
\left(\frac{\partial r_\varepsilon}{\partial\rho}, h\right)_{L^2(0,2\pi)} &= -\left(\frac{1}{\rho}\alpha r_\varepsilon, P_\varepsilon h\right)_{L^2(0,2\pi)} - \left\langle \frac{1}{\rho^2}\frac{\partial r_\varepsilon}{\partial\theta}, \frac{1}{\partial\theta}P_\varepsilon h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)}\\
&+\alpha^2 (r_\varepsilon,P_\varepsilon h)_{L^2(0,2\pi)} - \alpha (r_\varepsilon,h)_{L^2(0,2\pi)} + (f,P_\varepsilon h)_{L^2(0,2\pi)},
\end{aligned}$$

in $\mathcal{D}'(\varepsilon,a)$, it's easy to see that $\frac{\partial r_\varepsilon}{\partial\rho} \in L^\infty\left(\varepsilon,a,\left(H^{1/2}_{\rho,P}(0,2\pi)\right)'\right)$. Analogously to the proof of Proposition 2.8.5, from $\frac{\partial r_\varepsilon}{\partial\rho} \in L^2_\rho\left(\varepsilon,a,\left(H^{1/2}_{\rho,P}(0,2\pi)\right)'\right)$ and $r_\varepsilon \in L^2_\rho\left(\varepsilon,a,H^{1/2}_{\rho,P}(0,2\pi)\right)$, we deduce that $r_\varepsilon \in \mathcal{C}\left(\varepsilon,a,L^2(0,2\pi)\right)$. Consequently, for $\phi \in \mathcal{C}^1_0[\varepsilon,a)$ we can integrate (3.36) backwards to obtain $r_\varepsilon(\varepsilon) = 0$. ∎

Regarding the equation on $\hat{u}^m_\varepsilon$, we now have:

Proposition 3.4.4. *For every h in $\left(H^{1/2}_{\rho,P}(0,2\pi)\right)'$, $\hat{u}_\varepsilon$ satisfies the following equation*

$$\begin{aligned}
&\langle \hat{u}_\varepsilon, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '}\\
&= \left\langle P_\varepsilon\left(\frac{\partial \hat{u}_\varepsilon}{\partial\rho} + \alpha\hat{u}_\varepsilon\right), h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} + \langle r_\varepsilon, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '}
\end{aligned}$$

in $\mathcal{D}'(\varepsilon,a)$, with $\hat{u}_\varepsilon(a) = 0$.

Proof. For a fixed m_0 let $h \in V^{m_0}$. Then, from (3.14), we obtain, for $m \geq m_0$

$$\begin{aligned} & \left\langle P_\varepsilon^m \left(\frac{\partial \hat{u}_\varepsilon^m}{\partial \rho} + \alpha \hat{u}_\varepsilon^m \right), h \right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' - \langle \hat{u}_\varepsilon^m, h \rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' \\ = \ & \langle -r_\varepsilon^m, h \rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}'. \end{aligned}$$

Considering $\phi \in \mathcal{D}(\varepsilon, a)$, we have:

$$\begin{aligned} & \int_\varepsilon^a \left\langle P_\varepsilon^m \left(\frac{\partial \hat{u}_\varepsilon^m}{\partial \rho} + \alpha \hat{u}_\varepsilon^m \right), h \right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' \phi \rho \, \mathrm{d}\rho \\ & - \int_\varepsilon^a \langle \hat{u}_\varepsilon^m, h \rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' \phi \rho \, \mathrm{d}\rho = \int_\varepsilon^a \langle -r_\varepsilon^m, h \rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' \phi \rho \, \mathrm{d}\rho \\ \Rightarrow \ & \int_\varepsilon^a \left\langle \frac{\partial \hat{u}_\varepsilon^m}{\partial \rho}, P_\varepsilon^m h \right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' \phi \rho \, \mathrm{d}\rho \\ & + \int_\varepsilon^a \alpha \langle \hat{u}_\varepsilon^m, P_\varepsilon^m h \rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' \phi \rho \, \mathrm{d}\rho \\ & - \int_\varepsilon^a \langle \hat{u}_\varepsilon^m, h \rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' \phi \rho \, \mathrm{d}\rho = \int_\varepsilon^a \langle -r_\varepsilon^m, h \rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' \phi \rho \, \mathrm{d}\rho. \end{aligned}$$

Then, by Proposition 2.8.2, Corollary 3.4.1 and Lebesgue's theorem, we can pass to the limit and obtain

$$\begin{aligned} & \int_\varepsilon^a \left\langle \frac{\partial \hat{u}_\varepsilon}{\partial \rho}, P_\varepsilon h \right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' \phi \rho \, \mathrm{d}\rho \\ & + \int_\varepsilon^a \alpha \langle \hat{u}_\varepsilon, P_\varepsilon h \rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' \phi \rho \, \mathrm{d}\rho \\ & - \int_\varepsilon^a \langle \hat{u}_\varepsilon, h \rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' \phi \rho \, \mathrm{d}\rho = \int_\varepsilon^a \langle -r_\varepsilon, h \rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' \phi \rho \end{aligned} \tag{3.38}$$

for $h \in V^{m_0}$. Then, by density we have (3.38) for $h \in \left(H^{1/2}_{\rho,P}(0, 2\pi) \right)'$. ∎

At this point, we want to pass to the limit when $\varepsilon \to 0$. For this, we will use the same arguments as in Chapter 2, that is, Lemma 2.8.14 and Proposition 2.8.15. In fact, the following Corollary is a direct consequence of Proposition 2.8.15:

Corollary 3.4.5. *$\hat{u}_\varepsilon(\rho) \to \hat{u}(\rho)$, when $\varepsilon \to 0$, strongly in $H^{1/2}_{\rho,P}(0, 2\pi)$, for all $\rho \in (0, a)$, where $\hat{u}_\varepsilon$ and $\hat{u}$ are the solutions of (2.2) and (2.1), respectively.*

Therefore, considering $\hat{u}(\rho) = P(\rho)h + r(\rho)$, we obtain:

Corollary 3.4.6. *For all $\rho \in (0,a)$, $r_\varepsilon(\rho) \to r(\rho)$ strongly in $H^{1/2}_{\rho,P}(0,2\pi)$, when $\varepsilon \to 0$. Also, for all $\rho \in (0,a)$ and for a fixed h, $P_\varepsilon(\rho)h \to P(\rho)h$, strongly in $H^{1/2}_{\rho,P}(0,2\pi)$, weakly in $H^{3/2}_{\rho,P}(0,2\pi)$ and strongly in $H^1_{\rho,P}(0,2\pi)$, when $\varepsilon \to 0$.*

Proof. Applying Corollary 3.4.5, for all $\rho \in (0,a)$, we obtain $P_\varepsilon(\rho)h + r_\varepsilon(\rho) \to P(\rho)h + r(\rho)$, strongly in $H^{1/2}_{\rho,P}(0,2\pi)$. Taking $h = 0$, we obtain $r_\varepsilon(\rho) \to r(\rho)$ and consequently $P_\varepsilon(\rho)h \to P(\rho)h$, strongly in $H^{1/2}_{\rho,P}(0,2\pi)$. Now, from Proposition 3.3.4, $P_\varepsilon(\rho)$ is bounded in $H^{3/2}_{\rho,P}(0,2\pi)$ (the result is independent of ε and m) and consequently we can extract a subsequence converging weakly. By density (since $P_\varepsilon(\rho)h \to P(\rho)h$, strongly in $H^{1/2}_{\rho,P}(0,2\pi)$, for all $\rho \in (0,a)$) that subsequence converges also to $P(\rho)h$. Since $H^{3/2}_{\rho,P}(0,2\pi) \subset H^1_{\rho,P}(0,2\pi)$, with $H^{3/2}_{\rho,P}(0,2\pi)$ dense in $H^{1/2}_{\rho,P}(0,2\pi)$, then $P_\varepsilon(\rho)h \to P(\rho)h$ strongly in $H^1_{\rho,P}(0,2\pi)$, for all $\rho \in (0,a)$. ∎

Now we can pass to the limit, when $\varepsilon \to 0$, successively on P_ε, r_ε and $\hat{u}_\varepsilon$:

Proposition 3.4.7. *For every $h, \bar{h}$ in $L^2(0,2\pi)$, the operator P satisfies the following equation*

$$\left(\frac{\partial P}{\partial \rho}h, \bar{h}\right)_{L^2(0,2\pi)} - \left(\frac{1}{\rho}h, P\bar{h}\right)_{L^2(0,2\pi)} + \frac{1}{\rho}\alpha\left(Ph, P\bar{h}\right)_{L^2(0,2\pi)} + \left(\frac{1}{\rho^2}\frac{\partial}{\partial\theta}Ph, \frac{\partial}{\partial\theta}P\bar{h}\right)_{L^2(0,2\pi)}$$
$$+2\alpha\left(Ph, \bar{h}\right)_{L^2(0,2\pi)} - \alpha^2\left(Ph, P\bar{h}\right)_{L^2(0,2\pi)} = \left(h, \bar{h}\right)_{L^2(0,2\pi)},$$

in $\mathcal{D}'(0,a)$.

Proof. We consider equation (3.35), for $\phi \in C^1_0(0,a)$ and $h, \bar{h} \in L^2(0,2\pi)$. Integrating by parts its first term, we obtain

$$-\left(\frac{h_{|N}}{\alpha}, \bar{h}\right)_{L^2(0,2\pi)}\phi(\varepsilon)\varepsilon - \int_\varepsilon^a \left(P_\varepsilon h, \bar{h}\right)_{L^2(0,2\pi)} \phi'\rho\,\mathrm{d}\rho - 2\int_\varepsilon^a \left(\frac{1}{\rho}h, P_\varepsilon\bar{h}\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho$$
$$+\int_\varepsilon^a \frac{1}{\rho}\alpha\left(P_\varepsilon h, P_\varepsilon\bar{h}\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho + \int_\varepsilon^a \left(\frac{1}{\rho^2}\frac{\partial}{\partial\theta}P_\varepsilon h, \frac{\partial}{\partial\theta}P_\varepsilon\bar{h}\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho$$
$$+\int_\varepsilon^a 2\alpha\left(P_\varepsilon h, \bar{h}\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho - \alpha^2\int_\varepsilon^a \left(P_\varepsilon h, P_\varepsilon\bar{h}\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho$$
$$= \int_\varepsilon^a \left(h, \bar{h}\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho.$$

Then, considering $P_\varepsilon h = 0$, for all $\rho \in [0,\varepsilon)$, gives

$$
\begin{aligned}
&-(\frac{h_{|_N}}{\alpha},\bar{h})_{L^2(0,2\pi)}\phi(\varepsilon)\varepsilon-\int_0^a\big(P_\varepsilon h,\bar{h}\big)_{L^2(0,2\pi)}\phi'\rho\,\mathrm{d}\rho-2\int_0^a\left(\frac{1}{\rho}h,P_\varepsilon\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho\\
&+\int_0^a\frac{1}{\rho}\alpha\left(P_\varepsilon h,P_\varepsilon\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho+\int_0^a\left(\frac{1}{\rho^2}\frac{\partial}{\partial\theta}P_\varepsilon h,\frac{\partial}{\partial\theta}P_\varepsilon\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho\\
&+\int_0^a 2\alpha\left(P_\varepsilon h,\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho-\alpha^2\int_0^a\left(P_\varepsilon h,P_\varepsilon\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho\\
&=\int_0^a\left(h,\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho-\int_0^\varepsilon\left(h,\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho.
\end{aligned}
\tag{3.39}
$$

When $\varepsilon \to 0$, we have

$$
\begin{aligned}
&-\int_0^a\left(Ph,\bar{h}\right)_{L^2(0,2\pi)}\phi'\rho\,\mathrm{d}\rho-2\int_0^a\left(\frac{1}{\rho}h,P\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho\\
&+\int_0^a\frac{1}{\rho}\alpha\left(Ph,P\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho+\int_0^a\left(\frac{1}{\rho^2}\frac{\partial}{\partial\theta}Ph,\frac{\partial}{\partial\theta}P\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho\\
&+\int_0^a 2\alpha\left(Ph,\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho-\alpha^2\int_0^a\left(Ph,P\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho\\
&=\int_0^a\left(h,\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho.
\end{aligned}
\tag{3.40}
$$

In fact, the terms $(\frac{h_{|_N}}{\alpha},\bar{h})_{L^2(0,2\pi)}\phi(\varepsilon)$ and $\big(h,\bar{h}\big)_{L^2(0,2\pi)}\phi\rho$ are obviously bounded, since $h,\bar{h}\in L^2(0,2\pi)$, $\phi\in C_0^1(0,a)$ and $\rho<a$. Then, when $\varepsilon\to 0$, we have $(\frac{h_{|_N}}{\alpha},\bar{h})_{L^2(0,2\pi)}\phi(\varepsilon)\varepsilon\to 0$ and $\int_0^\varepsilon\big(h,\bar{h}\big)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho\to 0$. Further, due to Corollary 3.4.6, we have $P_\varepsilon h\to Ph$ strongly in $H^1_{\rho,P}(0,2\pi)$, which implies that both $P_\varepsilon h\to Ph$ and $\frac{\partial}{\partial\theta}P_\varepsilon h\to\frac{\partial}{\partial\theta}Ph$ strongly in $L^2(0,2\pi)$, for all $\rho\in[0,a]$. In order to use Lebesgue's theorem, we need also to have all the integrands in (3.39) bounded, for all $\rho\in[0,a]$. But, from $\big(P_\varepsilon h,\bar{h}\big)_{L^2(0,2\pi)}\phi'\rho\le\|P_\varepsilon h\|_{L^2(0,2\pi)}\,\|\bar{h}\|_{L^2(0,2\pi)}\phi'\rho$, this term is bounded for all $\rho\in[\varepsilon,a]$, since $\|P_\varepsilon h\|_{H^1_{\rho,P}(0,2\pi)}$ is bounded for $\rho\in[\varepsilon,a]$ (notice that the result of Corollary 3.3.5 is independent of m and ε), $\bar{h}\in L^2(0,2\pi)$, $\phi'\in C_0(0,a)$ and $\rho<a$. Then, since we have considered $P_\varepsilon h=0$, for all $\rho\in[0,\varepsilon)$, we also have $\|P_\varepsilon h\|_{H^1_{\rho,P}(0,2\pi)}$ bounded for all $\rho\in[0,a]$, and consequently, $\big(P_\varepsilon h,\bar{h}\big)_{L^2(0,2\pi)}\phi'\rho$ is bounded, for all $\rho\in[0,a]$. All the other terms, that is, $2\alpha\left(P_\varepsilon h,\bar{h}\right)_{L^2(0,2\pi)}\phi\rho$, $\alpha^2\left(P_\varepsilon h,P_\varepsilon\bar{h}\right)_{L^2(0,2\pi)}\phi\rho$, $\alpha\left(P_\varepsilon h,P_\varepsilon\bar{h}\right)_{L^2(0,2\pi)}\phi$ and $2\left(\frac{1}{\rho}h,P_\varepsilon\bar{h}\right)_{L^2(0,2\pi)}\phi\rho\ \Big(=2\left(h,P_\varepsilon\bar{h}\right)_{L^2(0,2\pi)}\phi\Big)$ are bounded for the same reasons, on $[0,a]$, and $\left(\frac{1}{\rho}\frac{\partial}{\partial\theta}P_\varepsilon h,\frac{1}{\rho}\frac{\partial}{\partial\theta}P_\varepsilon\bar{h}\right)_{L^2(0,2\pi)}\phi\rho$ is also bounded as a consequence of the boundeness of $\|P_\varepsilon h\|_{H^1_{\rho,P}(0,2\pi)}$ on $[0,a]$.

Now, since $\mathcal{D}(0,a)\subset\mathcal{C}_0^1(0,a)$, we can take $\phi\in\mathcal{D}(0,a)$ in the previous equality and integrating backwards the first term, we get

$$\int_0^a \left(\frac{\partial P}{\partial \rho}h,\bar{h}\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho - \int_0^a \left(\frac{1}{\rho}h, P\bar{h}\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho$$
$$+\int_0^a \frac{1}{\rho}\alpha\left(Ph, P\bar{h}\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho + \int_0^a \left(\frac{1}{\rho^2}\frac{\partial}{\partial\theta}Ph, \frac{\partial}{\partial\theta}P\bar{h}\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho$$
$$+\int_0^a 2\alpha\left(Ph,\bar{h}\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho - \alpha^2\int_0^a \left(Ph, P\bar{h}\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho$$
$$= \int_0^a \left(h,\bar{h}\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho,$$

and consequently, we find the equality in $\mathcal{D}'(0,a)$:

$$\left(\frac{\partial P}{\partial \rho}h,\bar{h}\right)_{L^2(0,2\pi)} = \left(\frac{1}{\rho}h, P\bar{h}\right)_{L^2(0,2\pi)} - \frac{1}{\rho}\alpha\left(Ph, P\bar{h}\right)_{L^2(0,2\pi)} + \left(h,\bar{h}\right)_{L^2(0,2\pi)}$$
$$-\left(\frac{1}{\rho^2}\frac{\partial}{\partial\theta}Ph, \frac{\partial}{\partial\theta}P\bar{h}\right)_{L^2(0,2\pi)} - 2\alpha\left(Ph,\bar{h}\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho + \alpha^2\left(Ph, P\bar{h}\right)_{L^2(0,2\pi)}.$$

■

Proposition 3.4.8. *For every h in $H^{1/2}_{\rho,P}(0,2\pi)$ the function r verifies the following equation*

$$\left(\frac{1}{\rho}\alpha r, Ph\right)_{L^2(0,2\pi)} + \left\langle \frac{1}{\rho^2}\frac{\partial r}{\partial\theta}, \frac{1}{\partial\theta}Ph\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)} - \alpha^2\left(r, Ph\right)_{L^2(0,2\pi)}$$
$$+\left(\frac{\partial r}{\partial\rho}, h\right)_{L^2(0,2\pi)} + \alpha\left(r,h\right)_{L^2(0,2\pi)} = \left(f, Ph\right)_{L^2(0,2\pi)},$$

in $\mathcal{D}'(0,a)$.

Proof. We consider (3.37), for $\phi \in \mathcal{C}^1_0(0,a)$ and $h \in H^{1/2}_{\rho,P}(0,2\pi)$. Integrating by parts its fifth term, we obtain

$$-\int_\varepsilon^a \left(f, P_\varepsilon h\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho + \int_\varepsilon^a \left(\frac{1}{\rho}\alpha r_\varepsilon, P_\varepsilon h\right)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho$$
$$+\int_\varepsilon^a \left\langle \frac{1}{\rho^2}\frac{\partial r_\varepsilon}{\partial\theta}, \frac{1}{\partial\theta}P_\varepsilon h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)} \phi\rho\,\mathrm{d}\rho$$
$$-\alpha^2\int_\varepsilon^a (r_\varepsilon, P_\varepsilon h)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho - (r_\varepsilon(\varepsilon), h)_{L^2(0,2\pi)}\phi(\varepsilon)\varepsilon - \int_\varepsilon^a (r_\varepsilon, h)_{L^2(0,2\pi)} \phi'\rho\,\mathrm{d}\rho$$
$$-\int_\varepsilon^a (\frac{1}{\rho}r_\varepsilon, h)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho + \alpha\int_\varepsilon^a (r_\varepsilon, h)_{L^2(0,2\pi)} \phi\rho\,\mathrm{d}\rho = 0.$$

Then, considering $r_\varepsilon = 0$ and $P_\varepsilon h = 0$, for all $\rho \in [0,\varepsilon)$, gives

$$\begin{aligned}
&-\int_0^a (f,P_\varepsilon h)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho + \int_0^a \left(\frac{1}{\rho}\alpha r_\varepsilon, P_\varepsilon h\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho\\
&+\int_0^a \left\langle \frac{1}{\rho^2}\frac{\partial r_\varepsilon}{\partial\theta}, \frac{1}{\partial\theta}P_\varepsilon h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)}\phi\rho\,\mathrm{d}\rho\\
&-\alpha^2\int_0^a (r_\varepsilon,P_\varepsilon h)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho - (r_\varepsilon(\varepsilon),h)_{L^2(0,2\pi)}\phi(\varepsilon)\varepsilon - \int_0^a (r_\varepsilon,h)_{L^2(0,2\pi)}\,\phi'\rho\,\mathrm{d}\rho\\
&-\int_0^a (\frac{1}{\rho}r_\varepsilon,h)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho + \alpha\int_0^a (r_\varepsilon,h)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho = 0.
\end{aligned} \tag{3.41}$$

When $\varepsilon \to 0$, we have

$$\begin{aligned}
&-\int_0^a (f,Ph)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho + \int_0^a \left(\frac{1}{\rho}\alpha r, Ph\right)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho\\
&+\int_0^a \left\langle \frac{1}{\rho^2}\frac{\partial r}{\partial\theta}, \frac{1}{\partial\theta}Ph\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)}\phi\rho\,\mathrm{d}\rho\\
&-\alpha^2\int_0^a (r,Ph)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho - \int_0^a (r,h)_{L^2(0,2\pi)}\,\phi'\rho\,\mathrm{d}\rho\\
&-\int_0^a (\frac{1}{\rho}r,h)_{L^2(0,2\pi)}\phi\rho\,\mathrm{d}\rho + \alpha\int_0^a (r,h)_{L^2(0,2\pi)}\,\phi\rho\,\mathrm{d}\rho = 0.
\end{aligned} \tag{3.42}$$

In fact, due to Corollary 3.4.6, we have $P_\varepsilon h \to Ph$ and $r_\varepsilon \to r$ strongly in $H^{1/2}_{\rho,P}(0,2\pi)$, for all $\rho \in [0,a]$. Then, $\dfrac{\partial r_\varepsilon}{\partial\theta} \to \dfrac{\partial r}{\partial\theta}$ strongly in $\left(H^{1/2}_{\rho,P}(0,2\pi)\right)'$ and, in addition, we have $\dfrac{\partial}{\partial\theta}P_\varepsilon h \to \dfrac{\partial}{\partial\theta}Ph$ weakly in $H^{1/2}_{\rho,P}(0,2\pi)$. In order to use Lebesgue's theorem, we need also to have all the integrands in (3.41) bounded, for all $\rho\in[0,a]$. But, we have seen in the proof of Proposition 3.4.7 that $\|P_\varepsilon h\|_{H^1_{\rho,P}(0,2\pi)}$ is bounded for all $\rho\in[0,a]$. In the same way, since we have considered $r_\varepsilon = 0$, for all $\rho\in[0,\varepsilon)$, we also have, as a consequence of Proposition 3.3.7, that $\|r_\varepsilon\|_{H^{1/2}_{\rho,P}(0,2\pi)}$ is bounded for all $\rho\in[0,a]$ (notice that the result of Proposition 3.3.7 is independent of ε and m). Then, from $(f,P_\varepsilon h)_{L^2(0,2\pi)}\,\phi\rho \le \|f\|_{L^2(0,2\pi)}\|P_\varepsilon h\|_{L^2(0,2\pi)}\phi\rho$ this term is bounded on $[0,a]$ since $\|P_\varepsilon h\|_{H^1_{\rho,P}(0,2\pi)}$ is bounded on $[0,a]$, $f\in L^2_\rho(0,a;L^2(0,2\pi))$ (for all $\rho\ge 0$), $\phi\in\mathcal{C}^1_0(0,a)$ and $\rho<a$. The term $\left(\dfrac{1}{\rho}\alpha r_\varepsilon, P_\varepsilon h\right)_{L^2(0,2\pi)}\phi\rho\ \left(=(\alpha r_\varepsilon,P_\varepsilon h)_{L^2(0,2\pi)}\phi\right)$ is also bounded on $[0,a]$, since $\|r_\varepsilon\|_{H^{1/2}_{\rho,P}(0,2\pi)}$ and $\|P_\varepsilon h\|_{H^1_{\rho,P}(0,2\pi)}$ are bounded on $[0,a]$ and we have $(\alpha r_\varepsilon,P_\varepsilon h)_{L^2(0,2\pi)} \le \alpha\|r_\varepsilon\|_{L^2(0,2\pi)}\|P_\varepsilon h\|_{L^2(0,2\pi)} \le \alpha\|r_\varepsilon\|_{H^{1/2}_{\rho,P}(0,2\pi)}\|P_\varepsilon h\|_{L^2(0,2\pi)}$. All the other terms - $(r_\varepsilon,P_\varepsilon h)_{L^2(0,2\pi)}\,\phi\rho$, $(\dfrac{1}{\rho}r_\varepsilon,h)_{L^2(0,2\pi)}\phi\rho\ \left(=(r_\varepsilon,h)_{L^2(0,2\pi)}\phi\right)$ and $(r_\varepsilon,h)_{L^2(0,2\pi)}\,\phi'\rho$ ($\phi'\in\mathcal{C}_0(0,a)$) - are bounded on $[0,a]$ for the same reasons. Also, is bounded on $[0,a]$ the term $\left\langle\dfrac{\partial r_\varepsilon}{\partial\theta},\dfrac{1}{\rho}\dfrac{\partial}{\partial\theta}P_\varepsilon h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)}\phi$, again since $\|r_\varepsilon\|_{H^{1/2}_{\rho,P}(0,2\pi)}$ and $\|P_\varepsilon h\|_{H^{3/2}_{\rho,P}(0,2\pi)}$ are bounded on $[0,a]$. Furthermore, since $|(r_\varepsilon(\varepsilon),h)_{L^2(0,2\pi)}| \le \|r_\varepsilon(\varepsilon)\|_{L^2(0,2\pi)}\|h\|_{L^2(0,2\pi)} \le \|r_\varepsilon(\varepsilon)\|_{H^{1/2}(0,2\pi)}\|h\|_{H^{1/2}(0,2\pi)}$, $\phi\in\mathcal{C}^1_0(0,a)$ and $h\in L^2(0,2\pi)$, the term $(r_\varepsilon(\varepsilon),h)_{L^2(0,2\pi)}\phi(\varepsilon)$ is bounded and therefore $(r_\varepsilon(\varepsilon),h)_{L^2(0,2\pi)}\phi(\varepsilon)\varepsilon\to 0$, when $\varepsilon\to 0$.

Now, since $\mathcal{D}(0,a)\subset\mathcal{C}^1_0(0,a)$, we can take $\phi\in\mathcal{D}(0,a)$ in the previous equality and integrating backwards the fifth term, we get

$$-\int_0^a (f, Ph)_{L^2(0,2\pi)} \phi\rho \,\mathrm{d}\rho + \int_0^a \left(\frac{1}{\rho}\alpha r, Ph\right)_{L^2(0,2\pi)} \phi\rho \,\mathrm{d}\rho$$
$$+\int_0^a \left\langle \frac{1}{\rho^2}\frac{\partial r}{\partial \theta}, \frac{1}{\partial \theta}Ph\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ', H^{1/2}_{\rho,P}(0,2\pi)} \phi\rho \,\mathrm{d}\rho$$
$$-\alpha^2 \int_0^a (r, Ph)_{L^2(0,2\pi)} \phi\rho \,\mathrm{d}\rho + \int_0^a \left(\frac{\partial r}{\partial \rho}, h\right)_{L^2(0,2\pi)} \phi\rho \,\mathrm{d}\rho$$
$$+\alpha \int_0^a (r, h)_{L^2(0,2\pi)} \phi\rho \,\mathrm{d}\rho = 0,$$

and consequently, in $\mathcal{D}'(0,a)$, we have

$$\left(\frac{\partial r}{\partial \rho}, h\right)_{L^2(0,2\pi)} = (f, Ph)_{L^2(0,2\pi)} - \left(\frac{1}{\rho}\alpha r, Ph\right)_{L^2(0,2\pi)}$$
$$-\left\langle \frac{1}{\rho^2}\frac{\partial r}{\partial \theta}, \frac{1}{\partial \theta}Ph\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ', H^{1/2}_{\rho,P}(0,2\pi)} + \alpha^2 (r, Ph)_{L^2(0,2\pi)} - \alpha (r, h)_{L^2(0,2\pi)}.$$

■

Proposition 3.4.9. *For every h in $\left(H^{1/2}_{\rho,P}(0,2\pi)\right)'$, $\hat{u}$ satisfies the following equation*

$$\langle \hat{u}, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '}$$
$$= \left\langle P\left(\frac{\partial \hat{u}}{\partial \rho} + \alpha \hat{u}\right), h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} + \langle r, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '}$$

in $\mathcal{D}'(0,a)$.

Proof. Let $\phi \in \mathcal{C}^1_0(0,a)$. Since $\phi(0) = 0$, in a neighborhood of the origin, and $\dfrac{\partial \tilde{\hat{u}}_\varepsilon}{\partial \rho} = 0$, for $\rho \in (0,\varepsilon)$, from (3.38) we obtain

$$\int_0^a \left\langle \frac{\partial \tilde{\hat{u}}_\varepsilon}{\partial \rho}, P_\varepsilon h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} \phi\rho \,\mathrm{d}\rho$$
$$+\int_0^a \left\langle \alpha\tilde{\hat{u}}_\varepsilon, P_\varepsilon h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} \phi\rho \,\mathrm{d}\rho - \int_0^\varepsilon \left\langle \alpha\tilde{\hat{u}}_\varepsilon, P_\varepsilon h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} \phi\rho \,\mathrm{d}\rho$$
$$-\int_0^a \left\langle \tilde{\hat{u}}_\varepsilon, h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} \phi\rho \,\mathrm{d}\rho + \int_0^\varepsilon \left\langle \tilde{\hat{u}}_\varepsilon, h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} \phi\rho \,\mathrm{d}\rho$$
$$= \int_0^a \langle -r_\varepsilon, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} \phi\rho \,\mathrm{d}\rho - \int_0^\varepsilon \langle -r_\varepsilon, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} \phi\rho \,\mathrm{d}\rho.$$

Since the terms $\left(\tilde{\hat{u}}_\varepsilon, h\right)_{L^2(0,2\pi)} \phi\rho$, $(-r_\varepsilon, h)_{L^2(0,2\pi)} \phi\rho$ and $\left\langle \alpha\tilde{\hat{u}}_\varepsilon, P_\varepsilon h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} \phi\rho$ are bounded in $[0,\varepsilon)$ by a constant not depending on ε, for ε arbitrarily small (we remind that the result for $\tilde{\hat{u}}_\varepsilon$ is due to Lemma 2.5.2 and the results for P_ε and r_ε are a consequence of considering $r_\varepsilon = 0, P_\varepsilon h = 0, \forall \rho \in [0,\varepsilon)$), we have $\int_0^\varepsilon \left\langle \tilde{\hat{u}}_\varepsilon, h \right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' \phi\rho\,\mathrm{d}\rho \to 0$, $\int_0^\varepsilon \langle r_\varepsilon, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' \phi\rho\,\mathrm{d}\rho \to 0$ and $\int_0^\varepsilon \left\langle \alpha\tilde{\hat{u}}_\varepsilon, P_\varepsilon h \right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' \phi\rho\,\mathrm{d}\rho \to 0$, as $\varepsilon \to 0$. This way, using the results of Corollary 3.4.5 and Corollary 3.4.6 and passing to the limit when $\varepsilon \to 0$ through Lebesgue's theorem, we obtain

$$\begin{aligned}&\int_0^a \left\langle \frac{\partial \hat{u}}{\partial \rho}, Ph \right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' \phi\rho\,\mathrm{d}\rho + \int_0^a \langle \alpha\hat{u}, Ph\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' \phi\rho\,\mathrm{d}\rho\\ &- \int_0^a \langle \hat{u}, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' \phi\rho\,\mathrm{d}\rho = \int_0^a \langle -r, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' \phi\rho\,\mathrm{d}\rho,\end{aligned}$$

and consequently since $\mathcal{D}(0,a) \subset \mathcal{C}^1_0(0,a)$, we have

$$\begin{aligned}&\left\langle \frac{\partial \hat{u}}{\partial \rho}, Ph \right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' + \langle \alpha P\hat{u}, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}'\\ &- \langle \hat{u}, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}' = \langle -r, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)}{}'\end{aligned}$$

in $\mathcal{D}'(0,a)$. ■

As seen in section 2.5., using the appropriate conditions of regularity around the origin (that is, $f \in \mathcal{C}^{0,\alpha}(\Omega)$), we can define the value of $\hat{u}(0)$ (as a constant), and consequently we have $\hat{u}(0) \in N$, with N defined in Lemma 2.2.5. Also, since $\dfrac{\partial \hat{u}}{\partial \rho} = \dfrac{\partial u}{\partial x}\cos(\theta) + \dfrac{\partial u}{\partial y}\sin(\theta)$ (see proof of Proposition 2.5.4) and we have assumed enough regularity around the origin, we have $\int_0^{2\pi} \dfrac{\partial \hat{u}}{\partial \rho}(0)\,\mathrm{d}\theta = \int_0^{2\pi} c_1\cos(\theta) + c_2\sin(\theta)\,\mathrm{d}\theta = 0$, from which we conclude that $\dfrac{\partial \hat{u}}{\partial \rho}(0) \in M$, with M defined in Lemma 2.2.4.

Therefore, from $\hat{u}_{|\Gamma_s} = P(s)\left(\dfrac{\partial \hat{u}}{\partial \rho}|_{\Gamma_s} + \alpha\,\hat{u}_{|\Gamma_s}\right) + r(s), \forall s \in [0,a]$ we obtain

$$\begin{aligned}&\hat{u}(0) = P(0)\left(\frac{\partial \hat{u}}{\partial \rho}(0) + \alpha\hat{u}(0)\right) + r(0)\\ \Rightarrow\ &\hat{u}(0)_{|M} = P(0)\left(\frac{\partial \hat{u}}{\partial \rho}(0)|_M + \alpha\hat{u}(0)_{|M}\right) + r(0)_{|M}\\ \Rightarrow\ &0 = P(0)\left(\frac{\partial \hat{u}}{\partial \rho}(0)|_M + 0\right) + r(0)_{|M} \qquad (3.43)\\ \Rightarrow\ &0 = P(0)\frac{\partial \hat{u}}{\partial \rho}(0)|_M + r(0)_{|M}\\ \Rightarrow\ &P(0)_{|M} = 0,\ r(0)_{|M} = 0,\end{aligned}$$

$$\begin{aligned}
&\hat{u}(0) = P(0)\left(\frac{\partial \hat{u}}{\partial \rho}(0) + \alpha\hat{u}(0)\right) + r(0)\\
\Rightarrow\ &\hat{u}(0)_{|_N} = P(0)\left(\frac{\partial \hat{u}}{\partial \rho}(0)|_N + \alpha\hat{u}(0)_{|_N}\right) + r(0)_{|_N}\\
\Rightarrow\ &\hat{u}(0) = P(0)\left(0 + \alpha\hat{u}(0)\right) + r(0)_{|_N}\\
\Rightarrow\ &\hat{u}(0) = \alpha P(0)\hat{u}(0) + r(0)_{|_N}\\
\Rightarrow\ &P(0)_{|_N} = \frac{I}{\alpha},\ r(0)_{|_N} = 0.
\end{aligned} \tag{3.44}$$

From (3.43) and (3.44) we obtain $r(0) = r(0)_{|_M} + r(0)_{|_N} = 0$. In the same way, since

$$\begin{aligned}
P(0)h\ &= P(0)\left(h_{|_M} + h_{|_N}\right)\\
&= P(0)h_{|_M} + P(0)h_{|_N}\\
&= \frac{I}{\alpha}h_{|_N}
\end{aligned}$$

we obtain $P(0) = \dfrac{proj_{|_N}}{\alpha}$.

Proposition 3.4.10. *For all $\rho \in (0, a)$ there is a unique solution $\hat{u}(\rho)$ for the boundary value problem $\hat{u}(\rho) = P(\rho)\left(\frac{\partial \hat{u}}{\partial \rho}(\rho) + \alpha\,\hat{u}(\rho)\right) + r(\rho)$, $\hat{u}(a) = 0$.*

Proof. Using the notations of Remark 3.1.5, we have seen that $(I - \alpha P_2)P_1 = P_2$, for the non constant part of the operators, that is, we consider the projection of the previous equality on the set M. Then, supposing that the problem $\hat{u}(\rho) = P_2(\rho)\left(\frac{\partial \hat{u}}{\partial \rho}(\rho) + \alpha\,\hat{u}(\rho)\right) + r(\rho)$, $\hat{u}(a) = 0$, has two solutions $\hat{u}^1(\rho)$ and $\hat{u}^2(\rho)$, we know that $w(\rho) = \hat{u}^1(\rho) - \hat{u}^2(\rho)$ satisfies the boundary value problem $w(\rho) = P_2(\rho)\left(\frac{\partial w}{\partial \rho}(\rho) + \alpha\, w(\rho)\right)$, $w(a) = 0$. So, we have $(\alpha\, P_1 + I)\, w(\rho) = (\alpha\, P_1 + I)\, P_2(\rho)\left(\frac{\partial w}{\partial \rho}(\rho) + \alpha\, w(\rho)\right) = P_1\left(\frac{\partial w}{\partial \rho}(\rho) + \alpha\, w(\rho)\right)$ and, consequently, $P_1\frac{\partial w}{\partial \rho}(\rho) - w(\rho) = 0$. Also, since $\hat{u}^1(\rho)$ and $\hat{u}^2(\rho)$ are continuous (again by Lemma 2.5.2), $w(\rho)$ is continuous. Thus, taking the inner product with $\frac{\partial w}{\partial \rho}(\rho)$, in the duality $H^{1/2}_{\rho,P}(0, 2\pi)$, $\left(H^{1/2}_{\rho,P}(0, 2\pi)\right)'$, we obtain:

$$\left(P_1(\rho)\frac{\partial w}{\partial \rho}(\rho), \frac{\partial w}{\partial \rho}(\rho)\right) - \left(w(\rho), \frac{\partial w}{\partial \rho}(\rho)\right) = 0.$$

Then, we find, as in Proposition 2.8.19,

$$\int_\rho^a \left(P_1(\varrho)\frac{\partial w}{\partial \varrho}(\varrho), \frac{\partial w}{\partial \varrho}(\varrho)\right)\, \mathrm{d}\varrho + \frac{1}{2}\|w(\rho)\|^2_{L^2(0,2\pi)} = 0,$$

since $w(a) = 0$ and w is continuous on $[\rho, a]$. Since we are summing, in the previous equation, two positive quantities (notice that P_1 is a positive operator because both P_2 and $I - \alpha P_2$ are positive), we must have $\|w(\rho)\|_{L^2(0,2\pi)} = 0$. According to the continuity previously established, we conclude that $\hat{u}^1(\rho) = \hat{u}^2(\rho)$.

With respect to the constant part of the operator P_2, that is, its projection on the set N, we consider now that the equation $\hat{u}(\rho) = \frac{1}{\alpha}\left(\frac{\partial \hat{u}}{\partial \rho}(\rho) + \alpha\, \hat{u}(\rho)\right) + r(\rho)$, $\hat{u}(a) = 0$, has two solutions $\hat{u}^1(\rho)$ and $\hat{u}^2(\rho)$. Then, $w(\rho) = \hat{u}^1(\rho) - \hat{u}^2(\rho)$ is the solution of $\frac{\partial w}{\partial \rho}(\rho) = 0$, $w(a) = 0$. Obviously, $w(\rho) = 0$ a.e. on $(0, a)$. ■

As a consequence of Propositions 3.4.7, 3.4.8 and 3.4.9 and using the initial conditions computed above, we finally achieve the following result:

Theorem 3.4.11. *Considering $\phi \in \mathcal{D}(0, a)$ we obtain:*

1. *for every $h, \bar{h}$ in $L^2(0, 2\pi)$, the operator P satisfies the equation*

$$\left(\frac{\partial P}{\partial \rho} h, \bar{h}\right)_{L^2(0,2\pi)} - \left(\frac{1}{\rho} h, P\bar{h}\right)_{L^2(0,2\pi)} + \frac{1}{\rho}\alpha \left(Ph, P\bar{h}\right)_{L^2(0,2\pi)} + 2\alpha \left(Ph, \bar{h}\right)_{L^2(0,2\pi)}$$
$$+ \left(\frac{1}{\rho^2}\frac{\partial}{\partial \theta} Ph, \frac{\partial}{\partial \theta} P\bar{h}\right)_{L^2(0,2\pi)} - \alpha^2 \left(Ph, P\bar{h}\right)_{L^2(0,2\pi)} = \left(h, \bar{h}\right)_{L^2(0,2\pi)}$$

   in $\mathcal{D}'(0, a)$, with the initial condition $P(0) = \frac{proj_{|N}}{\alpha}$;

2. *for every h in $H^{1/2}_{\rho,P}(0, 2\pi)$, the function r satisfies the equation*

$$\left(\frac{1}{\rho}\alpha r, Ph\right)_{L^2(0,2\pi)} + \left\langle \frac{1}{\rho^2}\frac{\partial r}{\partial \theta}, \frac{1}{\partial \theta} Ph\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi)\ ',H^{1/2}_{\rho,P}(0,2\pi)} - \alpha^2 \left(r, Ph\right)_{L^2(0,2\pi)}$$
$$+ \left(\frac{\partial r}{\partial \rho}, h\right)_{L^2(0,2\pi)} + \alpha \left(r, h\right)_{L^2(0,2\pi)} = \left(f, Ph\right)_{L^2(0,2\pi)}$$

   in $\mathcal{D}'(0, a)$, with the initial condition $r(0) = 0$;

3. *for every h in $\left(H^{1/2}_{\rho,P}(0, 2\pi)\right)'$, $\hat{u}$ satisfies the equation*

$$\langle \hat{u}, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '}$$
$$= \left\langle P\left(\frac{\partial \hat{u}}{\partial \rho} + \alpha \hat{u}\right), h\right\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '} + \langle r, h\rangle_{H^{1/2}_{\rho,P}(0,2\pi),\ H^{1/2}_{\rho,P}(0,2\pi)\ '}$$

   in $\mathcal{D}'(0, a)$, with the initial condition $\hat{u}(a) = 0$.

Chapter 4

The factorization method in a general star shaped domain

In order to generalize the invariant embedding method to more general geometries, in this chapter we apply it to the case of a star shaped domain. Here, the family of curves which limits the sub-domains has no invariant geometry but, as in the precedent cases, are homothetic to one another and homothetic to a point. We study the case where the moving boundary starts on the outside boundary of the domain and shrinks to that point.

4.1. Statement of the problem

Let Ω be an open set containing the origin O, star-shaped with respect to O, with boundary $\Gamma = \partial\Omega$. As in the two previous chapters, we consider the problem (1.3), with $f \in L^2(\Omega)$. We also consider that the domain $\widehat{\Omega}$ is now defined in polar coordinates by $x = \rho\cos(\theta), y = \rho\sin(\theta)$, $0 < \rho \leq \varphi(\theta)$, where $\varphi(\theta) \in \mathcal{C}^1([0, 2\pi])$ is such that $\varphi(2\pi) = \varphi(0)$, $\varphi'(2\pi) = \varphi'(0)$ and $\varphi(\theta) < k$, for a strictly positive constant k. Using the transformation $\tau = \rho/\varphi(\theta)$ we obtain $x = \tau\varphi(\theta)\cos(\theta), y = \tau\varphi(\theta)\sin(\theta), \tau \leq 1$. In the new system of coordinates (τ, θ), the Laplace equation becomes

$$\begin{aligned}&\left(\frac{1}{\varphi^2(\theta)} + \frac{(\varphi'(\theta))^2}{\varphi^4(\theta)}\right)\frac{\partial^2 u}{\partial\tau^2} + \left(-2\frac{\varphi'(\theta)}{\varphi^3(\theta)}\right)\frac{1}{\tau}\frac{\partial^2 u}{\partial\tau\partial\theta}\\&+\left(-\frac{\varphi''(\theta)}{\varphi^3(\theta)} + 2\frac{(\varphi'(\theta))^2}{\varphi^4(\theta)} + \frac{1}{\varphi^2(\theta)}\right)\frac{1}{\tau}\frac{\partial u}{\partial\tau} + \frac{1}{\varphi^2(\theta)}\frac{1}{\tau^2}\frac{\partial^2 u}{\partial\theta^2} = -f.\end{aligned} \tag{4.1}$$

The Laplace equation can also be written in the form

$$\frac{\sqrt{(\varphi'(\theta))^2+\varphi^2(\theta)}}{\tau\varphi^2(\theta)}\mathcal{D}_v\left(\frac{\tau\varphi^2(\theta)}{\sqrt{(\varphi'(\theta))^2+\varphi^2(\theta)}}A\,\mathcal{G}_d(u)\right)=-f,$$

where

$$\mathcal{G}_d(u)=\left(\frac{1}{\varphi(\theta)}\frac{\partial u}{\partial\tau},\frac{1}{\tau\sqrt{(\varphi'(\theta))^2+\varphi^2(\theta)}}\frac{\partial u}{\partial\theta}\right),$$

$$\mathcal{D}_v(u_1,u_2)=\left(\frac{1}{\varphi(\theta)}\frac{\partial u_1}{\partial\tau}+\frac{1}{\tau\sqrt{(\varphi'(\theta))^2+\varphi^2(\theta)}}\frac{\partial u_2}{\partial\theta}\right)$$

and A is the symmetrical matrix

$$A=\begin{bmatrix} 1+\dfrac{(\varphi'(\theta))^2}{\varphi^2(\theta)} & -\dfrac{\varphi'(\theta)}{\varphi^2(\theta)}\sqrt{(\varphi'(\theta))^2+\varphi^2(\theta)} \\ -\dfrac{\varphi'(\theta)}{\varphi^2(\theta)}\sqrt{(\varphi'(\theta))^2+\varphi^2(\theta)} & 1+\dfrac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\end{bmatrix}.$$

Considering $\widehat{\Omega}\setminus\widehat{\Omega}_\varepsilon$ the domain defined between $\rho=\varepsilon\varphi(\theta)$ and $\rho=\varphi(\theta)$, for $\varepsilon<1$, the transformation $\tau=\frac{\rho}{\varphi(\theta)}$ leads to the domain $\check{\Omega}\setminus\check{\Omega}_\varepsilon$, which is now the rectangle $(\varepsilon,1)\times(0,2\pi)$. Consequently, the equivalent of problem (2.2), in this new system of coordinates, is

$$\begin{cases} -\dfrac{\sqrt{(\varphi'(\theta))^2+\varphi^2(\theta)}}{\tau\varphi^2(\theta)}\mathcal{D}_v\left(\dfrac{\tau\varphi^2(\theta)}{\sqrt{(\varphi'(\theta))^2+\varphi^2(\theta)}}A\,\mathcal{G}_d(\check{u}_\varepsilon)\right)=f, & \text{in } \check{\Omega}\setminus\check{\Omega}_\varepsilon \\ \check{u}_{\varepsilon_{|\Gamma_1}}=0 \\ \check{u}_{\varepsilon_{|\Gamma_\varepsilon}} \text{ constant} \\ \displaystyle\int_{\Gamma_\varepsilon}\frac{\partial\check{u}_\varepsilon}{\partial n_A}\,\mathrm{d}\Gamma=0 \\ \check{u}_{\varepsilon_{|\theta=0}}=\check{u}_{\varepsilon_{|\theta=2\pi}} \\ \left(\dfrac{\partial\check{u}_\varepsilon}{\partial\theta}\right)_{|\theta=0}=\left(\dfrac{\partial\check{u}_\varepsilon}{\partial\theta}\right)_{|\theta=2\pi}, \end{cases} \tag{4.2}$$

where $\dfrac{\partial\check{u}_\varepsilon}{\partial n_A}=\vec{n}.A\,\mathcal{G}_d(\check{u}_\varepsilon)$. Since

$$\begin{aligned}\int_{\Gamma_\varepsilon}\frac{\partial\check{u}_\varepsilon}{\partial n_A}\,\mathrm{d}\Gamma &=\int_0^{2\pi}\left(\left(\frac{1}{\varphi(\theta)}+\frac{(\varphi'(\theta))^2}{\varphi^3(\theta)}\right)\frac{\partial\check{u}_\varepsilon}{\partial\tau}-\frac{\varphi'(\theta)}{\varphi^2(\theta)}\frac{1}{\varepsilon}\frac{\partial\check{u}_\varepsilon}{\partial\theta}\right)\varepsilon\varphi(\theta)\,\mathrm{d}\theta\\ &=\int_0^{2\pi}\left(\varepsilon\left(1+\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\frac{\partial\check{u}_\varepsilon}{\partial\tau}-\frac{\varphi'(\theta)}{\varphi(\theta)}\frac{\partial\check{u}_\varepsilon}{\partial\theta}\right)\mathrm{d}\theta,\end{aligned}$$

defining

$$\delta\check{u}_\varepsilon=\tau\left(1+\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\frac{\partial\check{u}_\varepsilon}{\partial\tau}-\frac{\varphi'(\theta)}{\varphi(\theta)}\frac{\partial\check{u}_\varepsilon}{\partial\theta}, \tag{4.3}$$

we have

$$\int_{\Gamma_\varepsilon} \frac{\partial \check{u}_\varepsilon}{\partial n_A}\, \mathrm{d}\Gamma = \int_0^{2\pi} \delta \check{u}_{\varepsilon_{|\Gamma_\varepsilon}}\, \mathrm{d}\theta.$$

Since $\int_{\Omega\setminus\Omega_\varepsilon} |v(x,y)|^2\, \mathrm{d}x\, \mathrm{d}y = \int_{\widehat{\Omega}\setminus\widehat{\Omega}_\varepsilon} |\hat{v}(\rho,\theta)|^2 \rho\, \mathrm{d}\rho\, \mathrm{d}\theta = \int_0^{2\pi}\int_\varepsilon^1 |\check{v}(\tau,\theta)|^2 \tau\varphi^2(\theta)\, \mathrm{d}\tau\, \mathrm{d}\theta$, to the space $L^2(\Omega\setminus\Omega_\varepsilon)$ corresponds the space $L^2_\tau((\varepsilon,1)\times(0,2\pi))$ (considering $\|\check{v}\|^2_{L^2_\tau((\varepsilon,1)\times(0,2\pi))} = \int_0^{2\pi}\int_\varepsilon^1 |\check{v}(\tau,\theta)|^2\tau\varphi^2(\theta)\, \mathrm{d}\tau\, \mathrm{d}\theta$).

Furthermore, we denote by $L^2_\tau(\varepsilon,1)$ the L^2-space of functions with the measure $\tau\varphi^2(\theta)\, \mathrm{d}\tau$, by $H^1_\tau(\varepsilon,1)$ the space of functions $\check{v}$ such that $\check{v}\in L^2_\tau(\varepsilon,1)$ and $\frac{1}{\varphi}\frac{\partial\check{v}}{\partial\tau} \in L^2_\tau(\varepsilon,1)$ and by $H^1_{\tau,p}(0,2\pi)$ the space of functions $\check{v}$ verifying $\check{v}\in L^2(0,2\pi)$, $\frac{1}{\tau\varphi(\theta)}\frac{\partial v}{\partial\theta} \in L^2(0,2\pi)$ and such that $\check{v}$ has periodic boundary conditions $\check{v}(0)=\check{v}(2\pi)$. Then, we consider

$$\|\check{v}(\theta)\|^2_{L^2_\tau(\varepsilon,1)} = \int_\varepsilon^1 |\check{v}|^2\tau\varphi^2(\theta)\, \mathrm{d}\tau; \quad \|\check{v}(\theta)\|^2_{H^1_\tau(\varepsilon,1)} = \int_\varepsilon^1 |\check{v}|^2\tau\varphi^2(\theta) + \left(\frac{\partial\check{v}}{\partial\tau}\right)^2 \tau\, \mathrm{d}\tau;$$

$$\|\check{v}(\tau)\|^2_{L^2(0,2\pi)} = \int_0^{2\pi} |\check{v}|^2\, \mathrm{d}\theta; \quad \|\check{v}(\tau)\|^2_{H^1_{\tau,p}(0,2\pi)} = \int_0^{2\pi} |\check{v}|^2 + \frac{1}{\tau^2\varphi^2(\theta)}\left(\frac{\partial\check{v}}{\partial\theta}\right)^2 \mathrm{d}\theta$$

and

$$\begin{aligned}\|\check{v}\|^2_{\check{H}_\varepsilon} &= \int_0^{2\pi}\int_\varepsilon^1 \left(|\check{v}|^2 + \frac{1}{\tau^2\varphi^2(\theta)}\left(-\frac{\varphi'(\theta)}{\varphi(\theta)}\tau\frac{\partial\check{v}}{\partial\tau} + \frac{\partial\check{v}}{\partial\theta}\right)^2\right)\tau\varphi^2(\theta)\, \mathrm{d}\tau\, \mathrm{d}\theta \\ &+ \int_0^{2\pi}\int_\varepsilon^1 \left(\frac{\partial\check{v}}{\partial\tau}\right)^2 \tau\, \mathrm{d}\tau\, \mathrm{d}\theta \\ &= \int_0^{2\pi}\int_\varepsilon^1 |\check{v}|^2\tau\varphi^2(\theta) + \left(\frac{1}{\sqrt{\tau}}\frac{\partial\check{v}}{\partial\theta} - \sqrt{\tau}\frac{\varphi'(\theta)}{\varphi(\theta)}\frac{\partial\check{v}}{\partial\tau}\right)^2 + \tau\left(\frac{\partial\check{v}}{\partial\tau}\right)^2 \mathrm{d}\tau\, \mathrm{d}\theta,\end{aligned}$$

where $\check{H}_\varepsilon$ is the equivalent, in this system of coordinates, to the Hilbert space $H^1(\Omega\setminus\Omega_\varepsilon)$.

Proposition 4.1.1. *The $\check{H}_\varepsilon$ operator*

$$\begin{aligned}\tau\varphi^2(\theta)\check{\Delta} = &\left(1+\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\tau\frac{\partial^2}{\partial\tau^2} + \left(-2\frac{\varphi'(\theta)}{\varphi(\theta)}\right)\frac{\partial^2}{\partial\tau\partial\theta} \\ &+ \left(-\frac{\varphi''(\theta)}{\varphi(\theta)} + 2\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)} + 1\right)\frac{\partial}{\partial\tau} + \frac{1}{\tau}\frac{\partial^2}{\partial\theta^2}\end{aligned}$$

is self-adjoint.

Proof. Considering $u, v \in \check{H}_\varepsilon$, we obtain

$$
\begin{aligned}
\left(\tau\varphi^2(\theta)\check{\Delta}u, v\right) &= \left(\left(1+\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\tau\frac{\partial^2 u}{\partial\tau^2}, v\right) + \left(\left(-2\frac{\varphi'(\theta)}{\varphi(\theta)}\right)\frac{\partial^2 u}{\partial\tau\partial\theta}, v\right) \\
&+ \left(\left(-\frac{\varphi''(\theta)}{\varphi(\theta)} + 2\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)} + 1\right)\frac{\partial u}{\partial\tau}, v\right) + \left(\frac{1}{\tau}\frac{\partial^2 u}{\partial\theta^2}, v\right) \\
&= \left(\frac{\partial^2 u}{\partial\tau^2}, \left(1+\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\tau v\right) + \left(\frac{\partial^2 u}{\partial\tau\partial\theta}, -2\frac{\varphi'(\theta)}{\varphi(\theta)}v\right) \\
&+ \left(\frac{\partial u}{\partial\tau}, \left(-\frac{\varphi''(\theta)}{\varphi(\theta)} + 2\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)} + 1\right)v\right) + \left(\frac{\partial^2 u}{\partial\theta^2}, \frac{1}{\tau}v\right) \\
&= \left(u, 2\left(1+\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\frac{\partial v}{\partial\tau} + \left(1+\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\tau\frac{\partial^2 v}{\partial\tau^2}\right) \\
&+ \left(u, -2\frac{\varphi''(\theta)\varphi(\theta) - (\varphi'(\theta))^2}{\varphi^2(\theta)}\frac{\partial v}{\partial\tau} - 2\frac{\varphi'(\theta)}{\varphi(\theta)}\frac{\partial^2 v}{\partial\tau\partial\theta}\right) \\
&- \left(u, \left(-\frac{\varphi''(\theta)}{\varphi(\theta)} + 2\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)} + 1\right)\frac{\partial v}{\partial\tau}\right) + \left(u, \frac{1}{\tau}\frac{\partial^2 v}{\partial\theta^2}\right) \\
&= \left(u, \left(1+\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\tau\frac{\partial^2 v}{\partial\tau^2}\right) + \left(u, \left(-2\frac{\varphi'(\theta)}{\varphi(\theta)}\right)\frac{\partial^2 v}{\partial\tau\partial\theta}\right) \\
&+ \left(u, \left(-\frac{\varphi''(\theta)}{\varphi(\theta)} + 2\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)} + 1\right)\frac{\partial v}{\partial\tau}\right) + \left(u, \frac{1}{\tau}\frac{\partial^2 v}{\partial\theta^2}\right) = \left(u, \tau\varphi^2(\theta)\check{\Delta}v\right).
\end{aligned}
$$

■

Proposition 4.1.2. *Let $\check{U}_\varepsilon = \{\check{u}_\varepsilon \in \check{H}_\varepsilon \,:\, \check{u}_{\varepsilon_{|\Gamma_1}} = 0 \ \wedge\ \check{u}_{\varepsilon_{|\Gamma_\varepsilon}}$ is constant$\}$. $\check{U}_\varepsilon$ is an Hilbert space and the variational formulation of problem (4.2) is*

$$
\begin{cases}
\check{u}_\varepsilon \in \check{U}_\varepsilon \\
\displaystyle\int_0^{2\pi}\int_\varepsilon^1 A\,\mathcal{G}_d(\check{u}_\varepsilon)\mathcal{G}_d(\check{v}_\varepsilon)\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta = \int_0^{2\pi}\int_\varepsilon^1 f\check{v}_\varepsilon\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta \\
\forall \check{v}_\varepsilon \in \check{U}_\varepsilon.
\end{cases}
\tag{4.4}
$$

Proof. Multiplying (4.1) by $\check{v}_\varepsilon \in \check{U}_\varepsilon$, we get

$$
\begin{aligned}
&\int_0^{2\pi}\int_\varepsilon^1 \left(1+\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\tau\frac{\partial^2\check{u}_\varepsilon}{\partial\tau^2}\check{v}_\varepsilon\,\mathrm{d}\tau\,\mathrm{d}\theta + \int_0^{2\pi}\int_\varepsilon^1 \left(-2\frac{\varphi'(\theta)}{\varphi(\theta)}\right)\frac{\partial^2\check{u}_\varepsilon}{\partial\tau\partial\theta}\check{v}_\varepsilon\,\mathrm{d}\tau\,\mathrm{d}\theta \\
&+ \int_0^{2\pi}\int_\varepsilon^1 \left(-\frac{\varphi''(\theta)}{\varphi(\theta)} + 2\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)} + 1\right)\frac{\partial\check{u}_\varepsilon}{\partial\tau}\check{v}_\varepsilon\,\mathrm{d}\tau\,\mathrm{d}\theta \\
&+ \int_0^{2\pi}\int_\varepsilon^1 \frac{1}{\tau}\frac{\partial^2\check{u}_\varepsilon}{\partial\theta^2}\check{v}_\varepsilon\,\mathrm{d}\tau\,\mathrm{d}\theta = -\int_0^{2\pi}\int_\varepsilon^1 f\check{v}_\varepsilon\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\ & \int_0^{2\pi}\left[\tau\frac{\partial\check{u}_\varepsilon}{\partial\tau}\check{v}_\varepsilon\left(1+\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\right]_\varepsilon^1 \mathrm{d}\theta \\
& -\int_0^{2\pi}\int_\varepsilon^1 \frac{\partial\check{u}_\varepsilon}{\partial\tau}\left(1+\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\left(\check{v}_\varepsilon+\tau\frac{\partial\check{v}_\varepsilon}{\partial\tau}\right)\mathrm{d}\tau\,\mathrm{d}\theta \\
& -\int_0^{2\pi}\left[\frac{\varphi'(\theta)}{\varphi(\theta)}\frac{\partial\check{u}_\varepsilon}{\partial\theta}\check{v}_\varepsilon\right]_\varepsilon^1 \mathrm{d}\theta+\int_0^{2\pi}\int_\varepsilon^1\frac{\varphi'(\theta)}{\varphi(\theta)}\frac{\partial\check{u}_\varepsilon}{\partial\theta}\frac{\partial\check{v}_\varepsilon}{\partial\tau}\mathrm{d}\tau\,\mathrm{d}\theta \\
& -\int_\varepsilon^1\left[\frac{\partial\check{u}_\varepsilon}{\partial\tau}\frac{\varphi'(\theta)}{\varphi(\theta)}\check{v}_\varepsilon\right]_0^{2\pi}\mathrm{d}\tau \\
& +\int_0^{2\pi}\int_\varepsilon^1\frac{\partial\check{u}_\varepsilon}{\partial\tau}\left(\left(\frac{\varphi''(\theta)\varphi(\theta)-(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\check{v}_\varepsilon+\frac{\varphi'(\theta)}{\varphi(\theta)}\frac{\partial\check{v}_\varepsilon}{\partial\theta}\right)\mathrm{d}\tau\,\mathrm{d}\theta \\
& +\int_0^{2\pi}\int_\varepsilon^1\left(-\frac{\varphi''(\theta)}{\varphi(\theta)}+2\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}+1\right)\frac{\partial\check{u}_\varepsilon}{\partial\tau}\check{v}_\varepsilon\,\mathrm{d}\tau\,\mathrm{d}\theta+\int_\varepsilon^1\left[\frac{\partial\check{u}_\varepsilon}{\partial\theta}\frac{1}{\tau}\check{v}_\varepsilon\right]_0^{2\pi}\mathrm{d}\tau \\
& -\int_0^{2\pi}\int_\varepsilon^1\frac{\partial\check{u}_\varepsilon}{\partial\theta}\frac{1}{\tau}\frac{\partial\check{v}_\varepsilon}{\partial\theta}\mathrm{d}\tau\,\mathrm{d}\theta=-\int_0^{2\pi}\int_\varepsilon^1 f\check{v}_\varepsilon\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta \\
\Rightarrow\ & -\check{v}_\varepsilon\int_0^{2\pi}\left(1+\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\varepsilon\frac{\partial\check{u}_\varepsilon}{\partial\tau}(\varepsilon)-\frac{\varphi'(\theta)}{\varphi(\theta)}\frac{\partial\check{u}_\varepsilon}{\partial\theta}\mathrm{d}\theta \\
& +\int_\varepsilon^1\frac{1}{\tau}\check{v}_\varepsilon(2\pi)\left(\frac{\partial\check{u}_\varepsilon}{\partial\theta}(2\pi)-\tau\frac{\varphi'(2\pi)}{\varphi(2\pi)}\frac{\partial\check{u}_\varepsilon}{\partial\tau}(2\pi)\right)\mathrm{d}\tau \\
& -\int_\varepsilon^1\frac{1}{\tau}\check{v}_\varepsilon(0)\left(\frac{\partial\check{u}_\varepsilon}{\partial\theta}(0)-\tau\frac{\varphi'(0)}{\varphi(0)}\frac{\partial\check{u}_\varepsilon}{\partial\tau}(0)\right)\mathrm{d}\tau \\
& -\int_0^{2\pi}\int_\varepsilon^1\left(1+\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\tau\frac{\partial\check{u}_\varepsilon}{\partial\tau}\frac{\partial\check{v}_\varepsilon}{\partial\tau}\mathrm{d}\tau\,\mathrm{d}\theta \\
& +\int_0^{2\pi}\int_\varepsilon^1\frac{\varphi'(\theta)}{\varphi(\theta)}\left(\frac{\partial\check{u}_\varepsilon}{\partial\theta}\frac{\partial\check{v}_\varepsilon}{\partial\tau}+\frac{\partial\check{u}_\varepsilon}{\partial\tau}\frac{\partial\check{v}_\varepsilon}{\partial\theta}\right)\mathrm{d}\tau\,\mathrm{d}\theta \\
& -\int_0^{2\pi}\int_\varepsilon^1\frac{1}{\tau}\frac{\partial\check{u}_\varepsilon}{\partial\theta}\frac{\partial\check{v}_\varepsilon}{\partial\theta}\mathrm{d}\tau\,\mathrm{d}\theta=-\int_0^{2\pi}\int_\varepsilon^1 f\check{v}_\varepsilon\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta \\
\Rightarrow\ & \int_0^{2\pi}\int_\varepsilon^1\left(1+\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\tau\frac{\partial\check{u}_\varepsilon}{\partial\tau}\frac{\partial\check{v}_\varepsilon}{\partial\tau}\mathrm{d}\tau\,\mathrm{d}\theta \\
& -\int_0^{2\pi}\int_\varepsilon^1\frac{\varphi'(\theta)}{\varphi(\theta)}\left(\frac{\partial\check{u}_\varepsilon}{\partial\theta}\frac{\partial\check{v}_\varepsilon}{\partial\tau}+\frac{\partial\check{u}_\varepsilon}{\partial\tau}\frac{\partial\check{v}_\varepsilon}{\partial\theta}\right)\mathrm{d}\tau\,\mathrm{d}\theta \\
& +\int_0^{2\pi}\int_\varepsilon^1\frac{1}{\tau}\frac{\partial\check{u}_\varepsilon}{\partial\theta}\frac{\partial\check{v}_\varepsilon}{\partial\theta}\mathrm{d}\tau\,\mathrm{d}\theta=\int_0^{2\pi}\int_\varepsilon^1 f\check{v}_\varepsilon\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta.
\end{aligned}
$$

■

We can now establish the uniqueness of solution for the problem (4.2):

Proposition 4.1.3. *For each $\varepsilon > 0$, problem (4.4) has a unique solution.*

Proof. We consider $a(\check{u}_\varepsilon,\check{v}_\varepsilon) = \displaystyle\int_0^{2\pi}\int_\varepsilon^1 A\,\mathcal{G}_d(\check{u}_\varepsilon)\mathcal{G}_d(\check{v}_\varepsilon)\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta$ and $(f,\check{v}_\varepsilon) = \displaystyle\int_0^{2\pi}\int_\varepsilon^1 f\check{v}_\varepsilon\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta$, for $\check{u}_\varepsilon,\check{v}_\varepsilon\in\check{H}_\varepsilon$ and $f\in L^2_\tau((\varepsilon,1)\times(0,2\pi))$. Then,

$$\begin{aligned}
&a(\check{u}_\varepsilon,\check{v}_\varepsilon)\\
\le\;& \left(\int_0^{2\pi}\!\!\int_\varepsilon^1 A\mathcal{G}_d(\check{u}_\varepsilon).\mathcal{G}_d(\check{u}_\varepsilon)\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta\right)^{1/2}\left(\int_0^{2\pi}\!\!\int_\varepsilon^1 A\mathcal{G}_d(\check{v}_\varepsilon).\mathcal{G}_d(\check{v}_\varepsilon)\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta\right)^{1/2}\\
=\;& \left(\int_0^{2\pi}\int_\varepsilon^1\left(\frac{1}{\sqrt{\tau}}\frac{\partial\check{u}_\varepsilon}{\partial\theta}-\sqrt{\tau}\frac{\varphi'(\theta)}{\varphi(\theta)}\frac{\partial\check{u}_\varepsilon}{\partial\tau}\right)^2+\tau\left(\frac{\partial\check{u}_\varepsilon}{\partial\tau}\right)^2\mathrm{d}\tau\,\mathrm{d}\theta\right)^{1/2}\\
&\left(\int_0^{2\pi}\int_\varepsilon^1\left(\frac{1}{\sqrt{\tau}}\frac{\partial\check{v}_\varepsilon}{\partial\theta}-\sqrt{\tau}\frac{\varphi'(\theta)}{\varphi(\theta)}\frac{\partial\check{v}_\varepsilon}{\partial\tau}\right)^2+\tau\left(\frac{\partial\check{v}_\varepsilon}{\partial\tau}\right)^2\mathrm{d}\tau\,\mathrm{d}\theta\right)^{1/2}\\
\le\;& \|\check{u}_\varepsilon\|_{\check{H}_\varepsilon}\|\check{v}_\varepsilon\|_{\check{H}_\varepsilon}
\end{aligned}$$

and consequently a is continuous.

Also, taking $B=\begin{bmatrix} b_{1,1} & b_{1,2}\\ b_{2,1} & b_{2,2}\end{bmatrix}$, with $b_{1,2}=b_{2,1}=\dfrac{\sqrt{\frac{\varphi^2(\theta)+(\varphi'(\theta))^2-\varphi(\theta)\sqrt{\varphi^2(\theta)+(\varphi'(\theta))^2}}{\varphi^2(\theta)}}}{\sqrt{2}}$ and $b_{1,1}=b_{2,2}=-\dfrac{\frac{\varphi(\theta)+\sqrt{\varphi^2(\theta)+(\varphi'(\theta))^2}}{}\sqrt{\frac{\varphi^2(\theta)+(\varphi'(\theta))^2-\varphi(\theta)\sqrt{\varphi^2(\theta)+(\varphi'(\theta))^2}}{\varphi^2(\theta)}}}{\sqrt{2}\varphi'(\theta)}$, we have

$$a(\check{u}_\varepsilon,\check{u}_\varepsilon)=\int_0^{2\pi}\int_\varepsilon^1 A\,\mathcal{G}_d(\check{u}_\varepsilon).\mathcal{G}_d(\check{u}_\varepsilon)\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta=\|B\,\mathcal{G}_d(\check{u}_\varepsilon)\|^2_{L^2_\tau((\varepsilon,1)\times(0,2\pi))}.$$

Since, through Poincaré's theorem, there exists $c>0$ such that

$$\|\check{u}_\varepsilon\|^2_{L^2_\tau((\varepsilon,1)\times(0,2\pi))}\le c\,\|B\,\mathcal{G}_d(\check{u}_\varepsilon)\|^2_{L^2_\tau((\varepsilon,1)\times(0,2\pi))},\tag{4.5}$$

we have

$$\begin{aligned}
&\|\check{u}_\varepsilon\|^2_{L^2_\tau((\varepsilon,1)\times(0,2\pi))}+\|B\,\mathcal{G}_d(\check{u}_\varepsilon)\|^2_{L^2_\tau((\varepsilon,1)\times(0,2\pi))}\le(c+1)\|B\,\mathcal{G}_d(\check{u}_\varepsilon)\|^2_{L^2_\tau((\varepsilon,1)\times(0,2\pi))}\\
\Rightarrow\;&\|\check{u}_\varepsilon\|^2_{\check{H}_\varepsilon}\le(c+1)\|B\,\mathcal{G}_d(\check{u}_\varepsilon)\|^2_{L^2_\tau((\varepsilon,1)\times(0,2\pi))}.
\end{aligned}$$

Then,

$$a(\check{u}_\varepsilon,\check{u}_\varepsilon)=\|B\,\mathcal{G}_d(\check{u}_\varepsilon)\|^2_{L^2_\tau((\varepsilon,1)\times(0,2\pi))}\ge\frac{1}{c+1}\|\check{u}_\varepsilon\|^2_{\check{H}_\varepsilon}\tag{4.6}$$

and a is coercive.

Also, since $f\in L^2_\tau((\varepsilon,1)\times(0,2\pi))$,

$$\begin{aligned}
(f,\check{v}_\varepsilon)\;&=\int_0^{2\pi}\int_\varepsilon^1 f\check{v}_\varepsilon\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta\le\|f\|^2_{L^2_\tau((\varepsilon,1)\times(0,2\pi))}\|\check{v}_\varepsilon\|^2_{L^2_\tau((\varepsilon,1)\times(0,2\pi))}\\
&\le c\,\|\check{v}_\varepsilon\|^2_{\check{H}_\varepsilon}
\end{aligned}$$

and the linear form is also continuous. Then, the result is a direct application of the Lax-Milgram's theorem, since all the other hypotheses are easily verified.

■

4.2. Invariant embedding

Similarly to the two previous chapters, we consider $H^{1/2}_{\tau,p}(0,2\pi)$ to be the 1/2 interpolate between $H^1_{\tau,p}(0,2\pi)$ and $L^2(0,2\pi)$, and $\left(H^{1/2}_{\tau,p}(0,2\pi)\right)'$ as the 1/2 interpolate between $\left(H^1_{\tau,p}(0,2\pi)\right)'$ and $L^2(0,2\pi)$. Using the technique of invariant embedding, we embed problem (4.2) in a family of similar problems defined on $[s,1]\times[0,2\pi]$, for $s\in[\varepsilon,1)$. For each problem we impose the boundary condition $\delta\check{u}_{s|\Gamma_s}=h$, where $\delta\check{u}_{s|\Gamma_s}=\left(1+\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)s\frac{\partial\check{u}_s}{\partial\tau}|_{\Gamma_s}-\frac{\varphi'(\theta)}{\varphi(\theta)}\frac{\partial\check{u}_s}{\partial\theta}|_{\Gamma_s}$. Thus,

$$\begin{cases}-\dfrac{\sqrt{(\varphi'(\theta))^2+\varphi^2(\theta)}}{\tau\varphi^2(\theta)}\mathcal{D}_v\left(\dfrac{\tau\varphi^2(\theta)}{\sqrt{(\varphi'(\theta))^2+\varphi^2(\theta)}}A\,\mathcal{G}_d(\check{u}_s)\right)=f, \text{ in } \check{\Omega}\setminus\check{\Omega}_s\\ \check{u}_{s_{|\Gamma_1}}=0\\ \delta\check{u}_{s_{|\Gamma_s}}=h\\ \check{u}_{s_{|\theta=0}}=\check{u}_{s_{|\theta=2\pi}}\\ \left(\dfrac{\partial\check{u}_s}{\partial\theta}\right)_{|\theta=0}=\left(\dfrac{\partial\check{u}_s}{\partial\theta}\right)_{|\theta=2\pi}.\end{cases}\tag{4.7}$$

In (4.7) we take $h\in\left(H^{1/2}_{\tau,p}(0,2\pi)\right)'$.

In order to decouple this problem, we define:

Definition 4.2.1. For every $s\in[\varepsilon,1)$ and $h\in\left(H^{1/2}_{\tau,p}(0,2\pi)\right)'$ we define $\mathbf{P}(s)h=\gamma_{s_{|\Gamma_s}}$, where $\gamma_s\in\{\check{v}\in\check{H}_s:\check{v}_{|\Gamma_1}=0\}$ is the solution of

$$\begin{cases}-\dfrac{\sqrt{(\varphi'(\theta))^2+\varphi^2(\theta)}}{\tau\varphi^2(\theta)}\mathcal{D}_v\left(\dfrac{\tau\varphi^2(\theta)}{\sqrt{(\varphi'(\theta))^2+\varphi^2(\theta)}}A\,\mathcal{G}_d(\gamma_s)\right)=0, \text{ in } \check{\Omega}\setminus\check{\Omega}_s\\ \gamma_{s_{|\Gamma_1}}=0\\ \delta\gamma_{s_{|\Gamma_s}}=h\\ \gamma_{s_{|\theta=0}}=\gamma_{s_{|\theta=2\pi}}\\ \left(\dfrac{\partial\gamma_s}{\partial\theta}\right)_{|\theta=0}=\left(\dfrac{\partial\gamma_s}{\partial\theta}\right)_{|\theta=2\pi}\end{cases}\tag{4.8}$$

and $r(s) = \beta_{s_{|\Gamma_s}}$ where $\beta_s \in \{\check{v} \in \check{H}_s : \check{v}_{|\Gamma_1} = 0\}$ is the solution of

$$\begin{cases} -\dfrac{\sqrt{(\varphi'(\theta))^2+\varphi^2(\theta)}}{\tau\varphi^2(\theta)}\mathcal{D}_v\left(\dfrac{\tau\varphi^2(\theta)}{\sqrt{(\varphi'(\theta))^2+\varphi^2(\theta)}}A\,\mathcal{G}_d(\beta_s)\right)=f, \quad \text{in } \check{\Omega}\setminus\check{\Omega}_s \\ \beta_{s_{|\Gamma_1}} = 0 \\ \delta\beta_{s_{|\Gamma_s}} = 0 \\ \beta_{s_{|\theta=0}} = \beta_{s_{|\theta=2\pi}} \\ \left(\dfrac{\partial\beta_s}{\partial\theta}\right)_{|\theta=0} = \left(\dfrac{\partial\beta_s}{\partial\theta}\right)_{|\theta=2\pi}. \end{cases} \tag{4.9}$$

By linearity of (4.7) we have

$$\check{u}_{s_{|\Gamma_s}} = \mathbf{P}(s)\delta u_{s_{|\Gamma_s}} + r(s), \forall s \in [\varepsilon, 1]. \tag{4.10}$$

Furthermore, the solution $\check{u}_\varepsilon$ of (4.2) is given by

$$\check{u}_\varepsilon(\tau,\theta) = (\mathbf{P}(\tau)\delta\check{u}_{\varepsilon_{|\Gamma_\tau}})(\theta) + (r(\tau))(\theta). \tag{4.11}$$

Proposition 4.2.1. *Considering the Hilbert space $\check{U}_s = \{\check{u}_s \in \check{H}_s : \check{u}_{s_{|\Gamma_1}} = 0\}$, the variational formulation of problem (4.7) is*

$$\begin{cases} \check{u}_s \in \check{U}_s \\ \displaystyle\int_0^{2\pi}\!\!\int_s^1 A\,\mathcal{G}_d(\check{u}_s)\mathcal{G}_d(\check{v}_s)\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta = -\int_0^{2\pi}\check{v}_{s_{|\Gamma_s}}h\,\mathrm{d}\theta + \int_0^{2\pi}\!\!\int_s^1 f\check{v}_s\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta \\ \forall \check{v}_s \in \check{U}_s. \end{cases} \tag{4.12}$$

Proof. Using the computations of Proposition 4.1.2, we find

$$\begin{aligned} &\int_0^{2\pi}\frac{\partial\check{u}_s}{\partial\tau}|_{\Gamma_1}\check{v}_{s_{|\Gamma_1}}\left(1+\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\mathrm{d}\theta - \int_0^{2\pi}s\frac{\partial\check{u}_s}{\partial\tau}|_{\Gamma_s}\check{v}_{s_{|\Gamma_s}}\left(1+\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\mathrm{d}\theta \\ &-\int_0^{2\pi}\int_s^1\frac{\partial\check{u}_s}{\partial\tau}\left(1+\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\left(\check{v}_s+\tau\frac{\partial\check{v}_s}{\partial\tau}\right)\mathrm{d}\tau\,\mathrm{d}\theta \\ &-\int_0^{2\pi}\frac{\varphi'(\theta)}{\varphi(\theta)}\frac{\partial\check{u}_s}{\partial\theta}|_{\Gamma_1}\check{v}_{s_{|\Gamma_1}}\mathrm{d}\theta + \int_0^{2\pi}\frac{\varphi'(\theta)}{\varphi(\theta)}\frac{\partial\check{u}_s}{\partial\theta}|_{\Gamma_s}\check{v}_{s_{|\Gamma_s}}\mathrm{d}\theta + \int_0^{2\pi}\!\!\int_s^1\frac{\varphi'(\theta)}{\varphi(\theta)}\frac{\partial\check{u}_s}{\partial\theta}\frac{\partial\check{v}_s}{\partial\tau}\mathrm{d}\tau\,\mathrm{d}\theta \\ &-\int_s^1\left(\frac{\partial\check{u}_s}{\partial\tau}\frac{\varphi'(\theta)}{\varphi(\theta)}\check{v}_s\right)_{|\theta=2\pi}\mathrm{d}\tau + \int_s^1\left(\frac{\partial\check{u}_s}{\partial\tau}\frac{\varphi'(\theta)}{\varphi(\theta)}\check{v}_s\right)_{|\theta=0}\mathrm{d}\tau \\ &+\int_0^{2\pi}\int_s^1\frac{\partial\check{u}_s}{\partial\tau}\left(\left(\frac{\varphi''(\theta)\varphi(\theta)-(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\check{v}_s + \frac{\varphi'(\theta)}{\varphi(\theta)}\frac{\partial\check{v}_s}{\partial\theta}\right)\mathrm{d}\tau\,\mathrm{d}\theta \end{aligned}$$

$$
\begin{aligned}
&+\int_0^{2\pi}\int_s^1\left(-\frac{\varphi''(\theta)}{\varphi(\theta)}+2\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}+1\right)\frac{\partial\check{u}_s}{\partial\tau}\check{v}_s\,\mathrm{d}\tau\,\mathrm{d}\theta\\
&+\int_s^1\left(\frac{\partial\check{u}_s}{\partial\theta}\frac{1}{\tau}\check{v}_s\right)_{|\theta=2\pi}\mathrm{d}\tau-\int_s^1\left(\frac{\partial\check{u}_s}{\partial\theta}\frac{1}{\tau}\check{v}_s\right)_{|\theta=0}\mathrm{d}\tau\\
&-\int_0^{2\pi}\int_s^1\frac{\partial\check{u}_s}{\partial\theta}\frac{1}{\tau}\frac{\partial\check{v}_s}{\partial\theta}\,\mathrm{d}\tau\,\mathrm{d}\theta=-\int_0^{2\pi}\int_s^1 f\check{v}_s\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta\\
\Rightarrow\quad&-\int_0^{2\pi}\int_s^1\frac{\partial\check{u}_s}{\partial\tau}\left(1+\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\left(\check{v}_s+\tau\frac{\partial\check{v}_s}{\partial\tau}\right)\mathrm{d}\tau\,\mathrm{d}\theta\\
&+\int_0^{2\pi}\int_s^1\frac{\varphi'(\theta)}{\varphi(\theta)}\frac{\partial\check{u}_s}{\partial\theta}\frac{\partial\check{v}_s}{\partial\tau}\,\mathrm{d}\tau\,\mathrm{d}\theta\\
&+\int_0^{2\pi}\int_s^1\frac{\partial\check{u}_s}{\partial\tau}\left(\left(\frac{\varphi''(\theta)\varphi(\theta)-(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\check{v}_s+\frac{\varphi'(\theta)}{\varphi(\theta)}\frac{\partial\check{v}_s}{\partial\theta}\right)\mathrm{d}\tau\,\mathrm{d}\theta\\
&+\int_0^{2\pi}\int_s^1\left(-\frac{\varphi''(\theta)}{\varphi(\theta)}+2\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}+1\right)\frac{\partial\check{u}_s}{\partial\tau}\check{v}_s\,\mathrm{d}\tau\,\mathrm{d}\theta\\
&-\int_0^{2\pi}\int_s^1\frac{\partial\check{u}_s}{\partial\theta}\frac{1}{\tau}\frac{\partial\check{v}_s}{\partial\theta}\,\mathrm{d}\tau\,\mathrm{d}\theta=\int_0^{2\pi}\delta\check{u}_{s_{|\Gamma_s}}\check{v}_{s_{|\Gamma_s}}\,\mathrm{d}\theta-\int_0^{2\pi}\int_s^1 f\check{v}_s\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta\\
\Rightarrow\quad&\int_0^{2\pi}\int_s^1\left(1+\frac{(\varphi'(\theta))^2}{\varphi^2(\theta)}\right)\tau\frac{\partial\check{u}_s}{\partial\tau}\frac{\partial\check{v}_s}{\partial\tau}\,\mathrm{d}\tau\,\mathrm{d}\theta\\
&-\int_0^{2\pi}\int_s^1\frac{\varphi'(\theta)}{\varphi(\theta)}\left(\frac{\partial\check{u}_s}{\partial\theta}\frac{\partial\check{v}_s}{\partial\tau}+\frac{\partial\check{u}_s}{\partial\tau}\frac{\partial\check{v}_s}{\partial\theta}\right)\mathrm{d}\tau\,\mathrm{d}\theta\\
&+\int_0^{2\pi}\int_s^1\frac{1}{\tau}\frac{\partial\check{u}_s}{\partial\theta}\frac{\partial\check{v}_s}{\partial\theta}\,\mathrm{d}\tau\,\mathrm{d}\theta=-\int_0^{2\pi}\check{v}_{s_{|\Gamma_s}}\delta\check{u}_{s_{|\Gamma_s}}\,\mathrm{d}\theta+\int_0^{2\pi}\int_s^1 f\check{v}_s\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta.
\end{aligned}
$$

∎

Using the variational formulation (4.12) and Lax-Milgram theorem it can be easily proved that problem (4.7) is well posed.

As a direct consequence of the computations exhibited in Proposition 4.2.1 (taking $f = 0$ and $h = 0$, respectively), we can prove the following corollary:

Corollary 4.2.2. *The variational formulation of problem (4.8) is*

$$
\begin{cases}
\gamma_s\in\check{U}_s\\
\displaystyle\int_0^{2\pi}\int_s^1 A\,\mathcal{G}_d(\gamma_s)\mathcal{G}_d(\overline{\gamma}_s)\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta=-\int_0^{2\pi}\overline{\gamma}_{s_{|\Gamma_s}}h\,\mathrm{d}\theta\\
\forall\overline{\gamma}_s\in\check{U}_s
\end{cases}
$$

and the variational formulation of problem (4.9) is

$$\begin{cases} \beta_s \in \check{U}_s \\ \displaystyle\int_0^{2\pi}\!\!\int_s^1 A\,\mathcal{G}_d(\beta_s)\mathcal{G}_d(\overline{\beta}_s)\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta = \int_0^{2\pi}\!\!\int_s^1 f\overline{\beta}_s\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta \\ \forall\overline{\beta}_s \in \check{U}_s. \end{cases}$$

In the next proposition, we collect some properties of the operator $\mathbf{P}$, which are similar to the ones found on Chapter 2.

Proposition 4.2.3. *The linear operator* $\mathbf{P}(s) : \left(H^{1/2}_{\tau,p}(0,2\pi)\right)' \to H^{1/2}_{\tau,p}(0,2\pi)$ *is continuous, self-adjoint and negative definite, for all* $s \in [\varepsilon, 1)$*.*

Proof. As in Proposition 2.2.3, the operator $\mathbf{P}(s)$ is continuous since it is the composition of continuous operators: $h \to \gamma_s \to \gamma_{s_{|\Gamma_s}}$, defined, respectively, in the spaces $\left(H^{1/2}_{\tau,p}(0,2\pi)\right)'$, $\check{H}_s$ and $H^{1/2}_{\tau,p}(0,2\pi)$. Considering γ_s and $\overline{\gamma}_s$ two solutions of (4.8), with $\delta\gamma_{s_{|\Gamma_s}} = h$ and $\delta\overline{\gamma}_{s_{|\Gamma_s}} = \overline{h}$, respectively, using (4.8) and Corollary 4.2.2, we have:

$$\begin{aligned} &\int_0^{2\pi}\!\!\int_s^1 A\,\mathcal{G}_d(\gamma_s)\mathcal{G}_d(\overline{\gamma}_s)\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta = -\int_0^{2\pi}\delta\gamma_{s_{|\Gamma_s}}\overline{\gamma}_{s_{|\Gamma_s}}\,\mathrm{d}\theta \\ \Rightarrow\quad -&\int_0^{2\pi}\!\!\int_s^1 A\,\mathcal{G}_d(\gamma_s)\mathcal{G}_d(\overline{\gamma}_s)\tau\varphi^2(\theta)\,\mathrm{d}\tau\,\mathrm{d}\theta = \left\langle h, \mathbf{P}(s)\overline{h}\right\rangle_{H^{1/2}_{\tau,p}(0,2\pi)\;',\,H^{1/2}_{\tau,p}(0,2\pi)}. \end{aligned}$$

Since

$$\left\langle h, \mathbf{P}(s)\overline{h}\right\rangle_{H^{1/2}_{\tau,p}(0,2\pi)\;',\,H^{1/2}_{\tau,p}(0,2\pi)} = \left\langle \overline{h}, \mathbf{P}(s)h\right\rangle_{H^{1/2}_{\tau,p}(0,2\pi)\;',\,H^{1/2}_{\tau,p}(0,2\pi)}$$

we conclude that $\mathbf{P}(s)$ is a self-adjoint operator.

Taking $\gamma_s = \overline{\gamma}_s$ we have

$$\langle h, \mathbf{P}(s)h\rangle_{H^{1/2}_{\tau,p}(0,2\pi)\;',\,H^{1/2}_{\tau,p}(0,2\pi)} = -\|B\,\mathcal{G}_d(\gamma_s)\|^2_{L^2_\tau((s,1)\times(0,2\pi))} \tag{4.13}$$

and consequently $\mathbf{P}(s)$ is a negative operator. Then, using (4.6), we have

$$\langle h, \mathbf{P}(s)h\rangle_{H^{1/2}_{\tau,p}(0,2\pi)\;',\,H^{1/2}_{\tau,p}(0,2\pi)} \leq -c_1\|\gamma_s\|^2_{\check{H}_s},$$

with $c_1 = \frac{1}{c+1}$.

Again as in Proposition 2.2.3, since $\Delta\gamma_s = 0$, there exists $k_s > 0$ (the constant should depend on s, due to the utilization of polar coordinates) such that

$$\langle h, \mathbf{P}(s)h \rangle_{H^{1/2}_{\tau,p}(0,2\pi)'\!,\,H^{1/2}_{\tau,p}(0,2\pi)} \leq -c_1\|\gamma_s\|^2_{\check{H}_s} = -c_1\|\gamma_s\|^2_{H(\Delta,\check{\Omega}\backslash\check{\Omega}_s)} \leq -c_2\left\|\delta\gamma_{s_{|\Gamma_s}}\right\|^2_{H^{1/2}_{\tau,p}(0,2\pi)'},$$

with $c_2 = \frac{c_1}{k_s^2}$. which proves that $\mathbf{P}(s)$ is a negative definite operator.

Furthermore, from (4.13), Poincaré's inequality and Holder's inequality, we have

$$c_1\|\gamma_s\|^2_{\check{H}_s} \leq \|B\mathcal{G}_d(\gamma_s)\|^2_{L^2_\tau((s,1)\times(0,2\pi))} \leq \|h\|_{H^{1/2}_{\tau,p}(0,2\pi)'}\|\gamma_s(s)\|_{H^{1/2}_{\tau,p}(0,2\pi)},$$

and, on the other hand, due to trace theorem, there exists $c_s > 0$ (again, c_s should depend on s) such that

$$\|\gamma_s(s)\|_{H^{1/2}_{\tau,p}(0,2\pi)} \leq c_s\|\gamma_s\|_{\check{H}_s}.$$

Then,

$$\begin{aligned} &\frac{c_1}{c_s^2}\|\gamma_s(s)\|^2_{H^{1/2}_{\tau,p}(0,2\pi)} \leq c_1\|\gamma_s\|^2_{\check{H}_s} \leq \|h\|_{H^{1/2}_{\tau,p}(0,2\pi)'}\|\gamma_s(s)\|_{H^{1/2}_{\tau,p}(0,2\pi)} \\ \Rightarrow\ &\|\gamma_s(s)\|_{H^{1/2}_{\tau,p}(0,2\pi)} \leq \frac{c_s^2}{c_1}\|h\|_{H^{1/2}_{\tau,p}(0,2\pi)'}. \end{aligned}$$

■

Considering $\check{N} = \left\{u \in H^{1/2}_{\tau,p}(0,2\pi) : u \text{ is constant}\right\}$ and $\check{M} = \left\{v \in \left(H^{1/2}_{\tau,p}(0,2\pi)\right)' : \int_0^{2\pi} v\,\mathrm{d}\theta = 0\right\}$, the proof of the next proposition is similar to the one of Proposition 2.2.7:

Proposition 4.2.4. *For any $\check{u} \in N$ there exists a unique solution $\check{v} \in M$ for the equation $\check{u} = \mathbf{P}(\varepsilon)\,\check{v} + r(\varepsilon)$, for given $r(\varepsilon)$ and $\mathbf{P}(\varepsilon)$.*

Since $\check{u} \in \check{N}$, when we multiply the equality $\check{u} = \mathbf{P}(\varepsilon)\,\check{v} + r(\varepsilon)$ by $\check{w} \in \check{M}$ and integrate on $[0,2\pi]$, we find $\int_0^{2\pi} \mathbf{P}(\varepsilon)\check{v}\,\check{w}\,\mathrm{d}\theta = -\int_0^{2\pi} r(\varepsilon)\,\check{w}\,\mathrm{d}\theta$, that is, $(\mathbf{P}(\varepsilon)\check{v} + r(\varepsilon))_{|M} = 0$. The constant $\check{u}$ is given by $\int_0^{2\pi} \frac{\mathbf{P}(\varepsilon)\check{v} + r(\varepsilon)}{2\pi}\,\mathrm{d}\theta$.

Remark 4.2.5. *Particularizing $\check{v} = \delta\check{u}_\varepsilon(\varepsilon)$, defined by (4.3), on the equality of Proposition 4.2.4, we conclude that $\check{u}_\varepsilon(\varepsilon)$, the initial condition of problem 4.2, is uniquely determined through the relation $\check{u}_\varepsilon(\varepsilon) = \mathbf{P}(\varepsilon)\delta\check{u}_\varepsilon(\varepsilon) + r(\varepsilon)$ and is, as we have seen, a constant. In the sequel we are going to denote this constant by $\check{u}_\varepsilon(\varepsilon) = \Upsilon(r(\varepsilon), \mathbf{P}(\varepsilon))$.*

4.3. Formal calculations

Let $b = \frac{\varphi'(\theta)}{\varphi(\theta)}$. With this notation the Laplace equation can be written in the form

$$(1+b^2)\tau\frac{\partial^2 \breve{u}}{\partial\tau^2} - b\frac{\partial^2 \breve{u}}{\partial\tau\partial\theta} = b\frac{\partial^2 \breve{u}}{\partial\theta\partial\tau} + \left(\frac{\partial b}{\partial\theta} - (1+b^2)\right)\frac{\partial \breve{u}}{\partial\tau} - \frac{1}{\tau}\frac{\partial^2 \breve{u}}{\partial\theta^2} - \tau\varphi^2(\theta)f \tag{4.14}$$

and further,

$$\delta\breve{u} = (1+b^2)\tau\frac{\partial \breve{u}}{\partial\tau} - b\frac{\partial \breve{u}}{\partial\theta}. \tag{4.15}$$

From (4.11), taking the derivative in a formal way, with to respect to τ and using (4.15), we obtain

$$\begin{aligned}
\frac{\partial \breve{u}}{\partial\tau} &= \frac{\partial \mathbf{P}}{\partial\tau}\delta\breve{u} + \mathbf{P}\frac{\partial}{\partial\tau}\delta\breve{u} + \frac{\partial r}{\partial\tau}\\
&= \frac{\partial \mathbf{P}}{\partial\tau}\delta\breve{u} + \mathbf{P}\frac{\partial}{\partial\tau}\left((1+b^2)\tau\frac{\partial \breve{u}}{\partial\tau} - b\frac{\partial \breve{u}}{\partial\theta}\right) + \frac{\partial r}{\partial\tau}\\
&= \frac{\partial \mathbf{P}}{\partial\tau}\delta\breve{u} + \mathbf{P}\left((1+b^2)\tau\frac{\partial^2 \breve{u}}{\partial\tau^2} - b\frac{\partial^2 \breve{u}}{\partial\tau\partial\theta}\right) + \mathbf{P}(1+b^2)\frac{\partial \breve{u}}{\partial\tau} + \frac{\partial r}{\partial\tau}.
\end{aligned}$$

Using (4.14), we find

$$\frac{\partial \breve{u}}{\partial\tau} = \frac{\partial \mathbf{P}}{\partial\tau}\delta\breve{u} + \mathbf{P}b\frac{\partial}{\partial\theta}\left(\frac{\partial \breve{u}}{\partial\tau}\right) + \mathbf{P}\frac{\partial b}{\partial\theta}\frac{\partial \breve{u}}{\partial\tau} - \mathbf{P}\frac{1}{\tau}\frac{\partial^2 \breve{u}}{\partial\theta^2} - \mathbf{P}\tau\varphi^2(\theta)f + \frac{\partial r}{\partial\tau}.$$

Then, from (4.15), we compute

$$\begin{aligned}
\delta\breve{u} = \;& (1+b^2)\tau\frac{\partial \mathbf{P}}{\partial\tau}\delta\breve{u} + (1+b^2)\tau\mathbf{P}b\frac{\partial}{\partial\theta}\left(\frac{\partial \breve{u}}{\partial\tau}\right) + (1+b^2)\tau\mathbf{P}\frac{\partial b}{\partial\theta}\frac{\partial \breve{u}}{\partial\tau} - (1+b^2)\mathbf{P}\frac{\partial^2 \breve{u}}{\partial\theta^2}\\
& -(1+b^2)\mathbf{P}\tau^2\varphi^2(\theta)f + (1+b^2)\tau\frac{\partial r}{\partial\tau} - b\frac{\partial \breve{u}}{\partial\theta}\\
= \;& (1+b^2)\tau\frac{\partial \mathbf{P}}{\partial\tau}\delta\breve{u} + (1+b^2)\tau\mathbf{P}b\frac{\partial}{\partial\theta}\left(\frac{1}{(1+b^2)\tau}\delta\breve{u} + \frac{b}{(1+b^2)\tau}\frac{\partial \breve{u}}{\partial\theta}\right) + (1+b^2)\tau\mathbf{P}\frac{\partial b}{\partial\theta}\\
& \left(\frac{1}{(1+b^2)\tau}\delta\breve{u} + \frac{b}{(1+b^2)\tau}\frac{\partial \breve{u}}{\partial\theta}\right) - (1+b^2)\mathbf{P}\frac{\partial^2 \breve{u}}{\partial\theta^2} - (1+b^2)\mathbf{P}\tau^2\varphi^2(\theta)f\\
& +(1+b^2)\tau\frac{\partial r}{\partial\tau} - b\frac{\partial \breve{u}}{\partial\theta}\\
= \;& (1+b^2)\tau\frac{\partial \mathbf{P}}{\partial\tau}\delta\breve{u} + (1+b^2)\mathbf{P}b\frac{\partial\left(\frac{1}{1+b^2}\right)}{\partial\theta}\delta\breve{u} + (1+b^2)\mathbf{P}\frac{b}{1+b^2}\frac{\partial}{\partial\theta}\delta\breve{u} + (1+b^2)\mathbf{P}b\\
& \frac{\partial\left(\frac{b}{1+b^2}\right)}{\partial\theta}\frac{\partial \breve{u}}{\partial\theta} + (1+b^2)\mathbf{P}\frac{b^2}{1+b^2}\frac{\partial^2 \breve{u}}{\partial\theta^2} + (1+b^2)\mathbf{P}\frac{\partial b}{\partial\theta}\frac{1}{1+b^2}\delta\breve{u} + (1+b^2)\mathbf{P}\frac{\partial b}{\partial\theta}\frac{b}{1+b^2}\\
& \frac{\partial \breve{u}}{\partial\theta} - (1+b^2)\mathbf{P}\frac{\partial^2 \breve{u}}{\partial\theta^2} - (1+b^2)\mathbf{P}\tau^2\varphi^2(\theta)f + (1+b^2)\tau\frac{\partial r}{\partial\tau} - b\frac{\partial \breve{u}}{\partial\theta}.
\end{aligned}$$

Using again (4.11),

$$\begin{aligned}
\delta\breve{u} = \;& (1+b^2)\tau\frac{\partial \mathbf{P}}{\partial\tau}\delta\breve{u} + (1+b^2)\mathbf{P}b\frac{\partial\left(\frac{1}{1+b^2}\right)}{\partial\theta}\delta\breve{u} + (1+b^2)\mathbf{P}\frac{b}{1+b^2}\frac{\partial}{\partial\theta}\delta\breve{u} + (1+b^2)\mathbf{P}b \\
& \frac{\partial\left(\frac{b}{1+b^2}\right)}{\partial\theta}\frac{\partial}{\partial\theta}\left(\mathbf{P}\delta\breve{u}+r\right) + (1+b^2)\mathbf{P}\frac{b^2}{1+b^2}\frac{\partial^2}{\partial\theta^2}(\mathbf{P}\delta\breve{u}+r) + (1+b^2)\mathbf{P}\frac{\partial b}{\partial\theta}\frac{1}{1+b^2}\delta\breve{u} \\
& +(1+b^2)\mathbf{P}\frac{\partial b}{\partial\theta}\frac{b}{1+b^2}\frac{\partial}{\partial\theta}\left(\mathbf{P}\delta\breve{u}+r\right) - (1+b^2)\mathbf{P}\frac{\partial^2}{\partial\theta^2}\left(\mathbf{P}\delta\breve{u}+r\right) - (1+b^2)\mathbf{P}\tau^2\varphi^2(\theta)f \\
& +(1+b^2)\tau\frac{\partial r}{\partial\tau} - b\frac{\partial}{\partial\theta}\left(\mathbf{P}\delta\breve{u}+r\right) \\
= \;& (1+b^2)\tau\frac{\partial \mathbf{P}}{\partial\tau}\delta\breve{u} + (1+b^2)\mathbf{P}b\frac{\partial\left(\frac{1}{1+b^2}\right)}{\partial\theta}\delta\breve{u} + (1+b^2)\mathbf{P}\frac{b}{1+b^2}\frac{\partial}{\partial\theta}\delta\breve{u} + (1+b^2)\mathbf{P}b \\
& \frac{\partial\left(\frac{b}{1+b^2}\right)}{\partial\theta}\left(\frac{\partial}{\partial\theta}\mathbf{P}\right)\delta\breve{u} + (1+b^2)\mathbf{P}b\frac{\partial\left(\frac{b}{1+b^2}\right)}{\partial\theta}\frac{\partial r}{\partial\theta} + (1+b^2)\mathbf{P}\frac{b^2}{1+b^2}\left(\frac{\partial^2}{\partial\theta^2}\mathbf{P}\right)\delta\breve{u} \\
& +(1+b^2)\mathbf{P}\frac{b^2}{1+b^2}\frac{\partial^2 r}{\partial\theta^2} + (1+b^2)\mathbf{P}\frac{\partial b}{\partial\theta}\frac{1}{1+b^2}\delta\breve{u} + (1+b^2)\mathbf{P}\frac{\partial b}{\partial\theta}\frac{b}{1+b^2}\left(\frac{\partial}{\partial\theta}\mathbf{P}\right)\delta\breve{u} \\
& +(1+b^2)\mathbf{P}\frac{\partial b}{\partial\theta}\frac{b}{1+b^2}\frac{\partial r}{\partial\theta} - (1+b^2)\mathbf{P}\left(\frac{\partial^2}{\partial\theta^2}\mathbf{P}\right)\delta\breve{u} - (1+b^2)\mathbf{P}\frac{\partial^2 r}{\partial\theta^2} \\
& -(1+b^2)\mathbf{P}\tau^2\varphi^2(\theta)f + (1+b^2)\tau\frac{\partial r}{\partial\tau} - b\left(\frac{\partial}{\partial\theta}\mathbf{P}\right)\delta\breve{u} - b\frac{\partial r}{\partial\theta}.
\end{aligned}$$

Considering $\delta\breve{u}$ arbitrary we find

$$\begin{aligned}
& \tau\frac{\partial \mathbf{P}}{\partial\tau} + \mathbf{P}b\frac{\partial\left(\frac{1}{1+b^2}\right)}{\partial\theta} + \mathbf{P}\frac{b}{1+b^2}\frac{\partial}{\partial\theta} + \mathbf{P}b\frac{\partial\left(\frac{b}{1+b^2}\right)}{\partial\theta}\frac{\partial}{\partial\theta}\mathbf{P} + \mathbf{P}\frac{b^2}{1+b^2}\frac{\partial^2}{\partial\theta^2}\mathbf{P} + \mathbf{P}\frac{\partial b}{\partial\theta}\frac{1}{1+b^2} \\
& +\mathbf{P}\frac{\partial b}{\partial\theta}\frac{b}{1+b^2}\frac{\partial}{\partial\theta}\mathbf{P} - \mathbf{P}\frac{\partial^2}{\partial\theta^2}\mathbf{P} - \frac{b}{1+b^2}\frac{\partial}{\partial\theta}\mathbf{P} - \frac{I}{1+b^2} = 0
\end{aligned}$$

and

$$\mathbf{P}b\frac{\partial\left(\frac{b}{1+b^2}\right)}{\partial\theta}\frac{\partial r}{\partial\theta} + \mathbf{P}\frac{b^2}{1+b^2}\frac{\partial^2 r}{\partial\theta^2} + \mathbf{P}\frac{\partial b}{\partial\theta}\frac{b}{1+b^2}\frac{\partial r}{\partial\theta} - \mathbf{P}\frac{\partial^2 r}{\partial\theta^2} - \mathbf{P}\tau^2\varphi^2(\theta)f + \tau\frac{\partial r}{\partial\tau} - \frac{b}{1+b^2}\frac{\partial r}{\partial\theta} = 0,$$

that is,

$$\tau\frac{\partial \mathbf{P}}{\partial\tau} + \mathbf{P}\frac{\partial}{\partial\theta}\left(\frac{b}{1+b^2}\right) - \mathbf{P}\frac{\partial}{\partial\theta}\left(\frac{1}{1+b^2}\frac{\partial}{\partial\theta}\right)\mathbf{P} - \frac{b}{1+b^2}\frac{\partial}{\partial\theta}\mathbf{P} - \frac{I}{1+b^2} = 0 \tag{4.16}$$

and

$$-\mathbf{P}\frac{\partial}{\partial\theta}\left(\frac{1}{1+b^2}\frac{\partial r}{\partial\theta}\right) - \mathbf{P}\tau^2\varphi^2(\theta)f + \tau\frac{\partial r}{\partial\tau} - \frac{b}{1+b^2}\frac{\partial r}{\partial\theta} = 0. \tag{4.17}$$

Again from (4.11) and considering the initial conditions on Γ_1 on (4.2) we obtain

$$\mathbf{P}(1) = 0 \text{ and } r(1) = 0.$$

Further, we can determine the unknown constant of Proposition 4.2.4, and we find

$$\begin{cases} \tau\dfrac{\partial \mathbf{P}}{\partial \tau} + \mathbf{P}\dfrac{\partial}{\partial \theta}\left(\dfrac{b}{1+b^2}\right) - \mathbf{P}\dfrac{\partial}{\partial \theta}\left(\dfrac{1}{1+b^2}\dfrac{\partial}{\partial \theta}\right)\mathbf{P} - \dfrac{b}{1+b^2}\dfrac{\partial}{\partial \theta}\mathbf{P} - \dfrac{I}{1+b^2} = 0, \\ \mathbf{P}(1) = 0 \\ -\mathbf{P}\dfrac{\partial}{\partial \theta}\left(\dfrac{1}{1+b^2}\dfrac{\partial r}{\partial \theta}\right) - \mathbf{P}\tau^2\varphi^2(\theta)f + \tau\dfrac{\partial r}{\partial \tau} - \dfrac{b}{1+b^2}\dfrac{\partial r}{\partial \theta} = 0, \quad r(1) = 0 \\ \check{u}_\varepsilon = \mathbf{P}\delta\check{u}_\varepsilon + r, \quad \check{u}_\varepsilon(\varepsilon) = \Upsilon(r(\varepsilon), \mathbf{P}(\varepsilon)) \ \ \text{(see Remark 4.2.5).} \end{cases} \tag{4.18}$$

It is now easy to see that, for the particular case of $\varphi(\theta) = a$ (a constant), in which case we are back to the circular domain of radius a, this system reduces to the system (2.17). In fact, (4.18) takes the form

$$\begin{cases} \tau\dfrac{\partial \mathbf{P}}{\partial \tau} - \mathbf{P}\dfrac{\partial^2}{\partial \theta^2}\mathbf{P} - I = 0, \quad \mathbf{P}(a) = 0 \\ -\mathbf{P}\dfrac{\partial^2 r}{\partial \theta^2} - \mathbf{P}\tau^2 a^2 f + \tau\dfrac{\partial r}{\partial \tau} = 0, \quad r(a) = 0 \\ \check{u}_\varepsilon = \mathbf{P}\left(\tau\dfrac{\partial \check{u}}{\partial \tau}\right) + r \end{cases}$$

and substituting $\mathbf{P}\tau a = P$, we obtain

$$\begin{cases} \tau\dfrac{\partial\left(\dfrac{P}{\tau a}\right)}{\partial \tau} - \dfrac{P}{\tau a}\dfrac{\partial^2}{\partial \theta^2}\dfrac{P}{\tau a} - I = \dfrac{1}{a}\dfrac{\partial P}{\partial \tau} - \dfrac{1}{a}\dfrac{P}{\tau} - \dfrac{P}{\tau^2 a^2}\dfrac{\partial^2}{\partial \theta^2}P - I = 0, \quad P(a) = 0 \\ -\dfrac{P}{a\tau^2}\dfrac{\partial^2 r}{\partial \theta^2} - Paf + \dfrac{\partial r}{\partial \tau} = 0, \quad r(a) = 0 \\ \check{u}_\varepsilon = P\dfrac{1}{a}\dfrac{\partial \check{u}}{\partial \tau} + r, \end{cases}$$

which corresponds to (2.17), on substituting reversely $\tau = \frac{\rho}{a}$.

4.4. Another formulation

In this section we are going to obtain a second formulation for the decoupled system, which is intrinsic with the problem. Let α be the angle $(\vec{OM}, \vec{n})$ where M is a point on Γ and $\vec{n}$ is the outward normal to Γ at M. We assume that $-\pi/2 < \alpha_0 \leq \alpha \leq \alpha_1 < \pi/2$ and that the equation of Γ in polar coordinates is given by $\rho = \varphi(\theta)$. We consider the homothety of center O and ratio $0 < \tau < 1$, which transforms Ω to Ω_τ with boundary Γ_τ, and the following system of curvilinear coordinates: for $M \in \Omega$, (τ, t) are such that M', the image of M by a $1/\tau$ homothety, belongs to Γ and $t, 0 \leq t < t_0$, is the curvilinear abscissa of M' on Γ (t_0 is the length of Γ); $\breve{u}(\tau,t) = u(x_1,x_2)$. This new system of coordinates and the one defined on Section 4.1. are related by the equalities $\cos(\alpha)\,\mathrm{d}t = \tau\varphi\,\mathrm{d}\theta$ and $\tan(\alpha) = \frac{\varphi'}{\varphi}$. In this coordinates, the exterior normal to Γ_τ can be written in the form $\dfrac{\partial}{\partial n} = \dfrac{1}{\varphi\cos(\alpha)}\dfrac{\partial}{\partial\tau} - \tan(\alpha)\dfrac{\partial}{\partial t}$.

Considering now that $\breve{v}_\varepsilon$ is solution of the homogeneous equation $\Delta\breve{v}_\varepsilon = 0$ and using the computations exhibit on Proposition 4.2.1, we obtain

$$\begin{aligned}&\int_\varepsilon^1\int_0^{2\pi}\left(1+\frac{(\varphi')^2}{\varphi^2}\right)\tau\frac{\partial\breve{v}_\varepsilon}{\partial\tau}\frac{\partial\breve{u}_\varepsilon}{\partial\tau} - \frac{\varphi'}{\varphi}\left(\frac{\partial\breve{v}_\varepsilon}{\partial\theta}\frac{\partial\breve{u}_\varepsilon}{\partial\tau} + \frac{\partial\breve{v}_\varepsilon}{\partial\tau}\frac{\partial\breve{u}_\varepsilon}{\partial\theta}\right) + \frac{1}{\tau}\frac{\partial\breve{v}_\varepsilon}{\partial\theta}\frac{\partial\breve{u}_\varepsilon}{\partial\theta}\,\mathrm{d}\theta\,\mathrm{d}\tau\\ = &-\int_0^{2\pi}\delta\breve{v}_\varepsilon\,\breve{u}_\varepsilon\,\mathrm{d}\theta = -\int_0^{2\pi}\left(\left(1+\frac{(\varphi')^2}{\varphi^2}\right)\varepsilon\frac{\partial\breve{v}_\varepsilon}{\partial\tau} - \frac{\varphi'}{\varphi}\frac{\partial\breve{v}_\varepsilon}{\partial\theta}\right)\breve{u}_\varepsilon\,\mathrm{d}\theta,\end{aligned}$$

and making the referred change of coordinates, we get the following equality

$$\begin{aligned}&\int_\varepsilon^1\int_0^{\tau t_0}\frac{1}{\varphi\cos(\alpha)}\frac{\partial\breve{v}_\varepsilon}{\partial\tau}\frac{\partial\breve{u}_\varepsilon}{\partial\tau} - \tan(\alpha)\left(\frac{\partial\breve{v}_\varepsilon}{\partial t}\frac{\partial\breve{u}_\varepsilon}{\partial\tau} + \frac{\partial\breve{v}_\varepsilon}{\partial\tau}\frac{\partial\breve{u}_\varepsilon}{\partial t}\right) + \frac{\varphi}{\cos(\alpha)}\frac{\partial\breve{v}_\varepsilon}{\partial t}\frac{\partial\breve{u}_\varepsilon}{\partial t}\,\mathrm{d}t\,\mathrm{d}\tau\\ = &-\int_0^{\varepsilon t_0}\left(\frac{1}{\varphi\cos(\alpha)}\frac{\partial\breve{v}_\varepsilon}{\partial\tau} - \tan(\alpha)\frac{\partial\breve{v}_\varepsilon}{\partial t}\right)\breve{u}_\varepsilon\,\mathrm{d}t.\end{aligned}$$

Now we derive, for $\breve{\Omega}\setminus\breve{\Omega}_\tau = \{(s,t)\in]0,1[\times]0,t_0[:\ \tau < s < 1 \wedge 0 < t < st_0\}$,

$$\begin{aligned}&\int_\tau^1\int_0^{st_0}\frac{1}{\varphi\cos(\alpha)}\frac{\partial\breve{v}_\tau}{\partial\tau}\frac{\partial\breve{u}_\tau}{\partial\tau} - \tan(\alpha)\left(\frac{\partial\breve{v}_\tau}{\partial t}\frac{\partial\breve{u}_\tau}{\partial\tau} + \frac{\partial\breve{v}_\tau}{\partial\tau}\frac{\partial\breve{u}_\tau}{\partial t}\right) + \frac{\varphi}{\cos(\alpha)}\frac{\partial\breve{v}_\tau}{\partial t}\frac{\partial\breve{u}_\tau}{\partial t}\,\mathrm{d}t\,\mathrm{d}s\\ = &-\int_0^{\tau t_0}\left(\frac{1}{\varphi\cos(\alpha)}\frac{\partial\breve{v}_\tau}{\partial\tau} - \tan(\alpha)\frac{\partial\breve{v}_\tau}{\partial t}\right)\breve{u}_\tau\,\mathrm{d}t.\end{aligned}\tag{4.19}$$

Then, applying the change of coordinates $t' = \dfrac{t}{\tau}$ to the right hand side of (4.19), deriving the resulting equality with respect to the variable τ, and then applying the same change of coordinates to the left hand side, we obtain, successively,

$$\begin{aligned}&\frac{\partial}{\partial\tau}\left(\int_\tau^1\int_0^{st_0}\frac{1}{\varphi\cos(\alpha)}\frac{\partial\breve{v}_\tau}{\partial\tau}\frac{\partial\breve{u}_\tau}{\partial\tau} - \tan(\alpha)\left(\frac{\partial\breve{v}_\tau}{\partial t}\frac{\partial\breve{u}_\tau}{\partial\tau} + \frac{\partial\breve{v}_\tau}{\partial\tau}\frac{\partial\breve{u}_\tau}{\partial t}\right) + \frac{\varphi}{\cos(\alpha)}\frac{\partial\breve{v}_\tau}{\partial t}\frac{\partial\breve{u}_\tau}{\partial t}\,\mathrm{d}t\,\mathrm{d}s\right)\\ = &-\frac{\partial}{\partial\tau}\left(\int_0^{t_0}\left(\frac{\tau}{\varphi\cos(\alpha)}\frac{\partial\breve{v}_\tau}{\partial\tau} - \tan(\alpha)\frac{\partial\breve{v}_\tau}{\partial t'}\right)\breve{u}_\tau\,\mathrm{d}t'\right)\\ \Rightarrow &-\int_0^{\tau t_0}\frac{1}{\varphi\cos(\alpha)}\frac{\partial\breve{v}_\tau}{\partial\tau}\frac{\partial\breve{u}_\tau}{\partial\tau} - \tan(\alpha)\left(\frac{\partial\breve{v}_\tau}{\partial t}\frac{\partial\breve{u}_\tau}{\partial\tau} + \frac{\partial\breve{v}_\tau}{\partial\tau}\frac{\partial\breve{u}_\tau}{\partial t}\right) + \frac{\varphi}{\cos(\alpha)}\frac{\partial\breve{v}_\tau}{\partial t}\frac{\partial\breve{u}_\tau}{\partial t}\,\mathrm{d}t\\ &= -\int_0^{t_0}\frac{\partial}{\partial\tau}\left(\left(\frac{\tau}{\varphi\cos(\alpha)}\frac{\partial\breve{v}_\tau}{\partial\tau} - \tan(\alpha)\frac{\partial\breve{v}_\tau}{\partial t'}\right)\breve{u}_\tau\right)\mathrm{d}t'\\ \Rightarrow &\int_0^{t_0}\frac{\tau}{\varphi\cos(\alpha)}\frac{\partial\breve{v}_\tau}{\partial\tau}\frac{\partial\breve{u}_\tau}{\partial\tau} - \tan(\alpha)\left(\frac{\partial\breve{v}_\tau}{\partial t'}\frac{\partial\breve{u}_\tau}{\partial\tau} + \frac{\partial\breve{v}_\tau}{\partial\tau}\frac{\partial\breve{u}_\tau}{\partial t'}\right) + \frac{\varphi}{\tau\cos(\alpha)}\frac{\partial\breve{v}_\tau}{\partial t'}\frac{\partial\breve{u}_\tau}{\partial t'}\,\mathrm{d}t'\\ &= \int_0^{t_0}\frac{\partial}{\partial\tau}\left(\left(\frac{\tau}{\varphi\cos(\alpha)}\frac{\partial\breve{v}_\tau}{\partial\tau} - \tan(\alpha)\frac{\partial\breve{v}_\tau}{\partial t'}\right)\breve{u}_\tau\right)\mathrm{d}t'.\end{aligned}$$

As before, we consider the family of problems $\mathcal{P}_{\tau,h}$, each one defined on $\breve{\Omega}\setminus\breve{\Omega}_\tau$, adding a Neumann boundary condition $\frac{\partial \breve{u}_\tau}{\partial n}|_{\breve{\Gamma}_\tau} = h$. By linearity, there exist $P(\tau)$ and $r(\tau)$ such that $\breve{u}_\tau(\tau) = P(\tau)h + r(\tau)$. In the new variables (τ, t'), the normal derivative $\frac{\partial}{\partial n}$ becomes $\frac{\partial}{\partial n_\tau} = \frac{1}{\varphi\cos(\alpha)}\frac{\partial}{\partial \tau} - \frac{\tan(\alpha)}{\tau}\frac{\partial}{\partial t'}$. So, we can write the last equality in the form

$$\begin{aligned}
&\int_0^{t_0} \frac{\tau}{\varphi\cos(\alpha)}\frac{\partial \breve{v}_\tau}{\partial \tau}\frac{\partial \breve{u}_\tau}{\partial \tau} - \tan(\alpha)\left(\frac{\partial \breve{v}_\tau}{\partial t}\frac{\partial \breve{u}_\tau}{\partial \tau} + \frac{\partial \breve{v}_\tau}{\partial \tau}\frac{\partial \breve{u}_\tau}{\partial t}\right) + \frac{\varphi}{\tau\cos(\alpha)}\frac{\partial \breve{v}_\tau}{\partial t}\frac{\partial \breve{u}_\tau}{\partial t}\,\mathrm{d}t \\
&= \int_0^{t_0} \frac{\partial}{\partial \tau}\left(\tau\frac{\partial \breve{v}_\tau}{\partial n_\tau}\breve{u}_\tau\right)\mathrm{d}t,
\end{aligned}$$

dropping the prime on t', in order to simplify the notation. Then, using the Laplace equation in the form $-\frac{\partial}{\partial \tau}\left(\frac{\tau}{\varphi\cos(\alpha)}\frac{\partial \breve{u}_\tau}{\partial \tau} - \tan(\alpha)\frac{\partial \breve{u}_\tau}{\partial t}\right) - \frac{\partial}{\partial t}\left(-\tan(\alpha)\frac{\partial \breve{u}_\tau}{\partial \tau} + \frac{\varphi}{\tau\cos(\alpha)}\frac{\partial \breve{u}_\tau}{\partial t}\right) = f\tau\varphi\cos(\alpha)$ (in the variables (τ, t')), we have successively,

$$\begin{aligned}
&\int_0^{t_0} \frac{\tau}{\varphi\cos(\alpha)}\frac{\partial \breve{v}_\tau}{\partial \tau}\frac{\partial \breve{u}_\tau}{\partial \tau} - \tan(\alpha)\left(\frac{\partial \breve{v}_\tau}{\partial t}\frac{\partial \breve{u}_\tau}{\partial \tau} + \frac{\partial \breve{v}_\tau}{\partial \tau}\frac{\partial \breve{u}_\tau}{\partial t}\right) + \frac{\varphi}{\tau\cos(\alpha)}\frac{\partial \breve{v}_\tau}{\partial t}\frac{\partial \breve{u}_\tau}{\partial t}\,\mathrm{d}t \\
= \;& \int_0^{t_0} \frac{\partial}{\partial \tau}\left(\tau\frac{\partial \breve{v}_\tau}{\partial n_\tau}\left(P\frac{\partial \breve{u}_\tau}{\partial n_\tau} + r\right)\right)\mathrm{d}t = \int_0^{t_0} \frac{\partial}{\partial \tau}\left(\tau\frac{\partial \breve{v}_\tau}{\partial n_\tau}P\frac{\partial \breve{u}_\tau}{\partial n_\tau} + \tau\frac{\partial \breve{v}_\tau}{\partial n_\tau}r\right)\mathrm{d}t \\
= \;& \int_0^{t_0} \frac{\partial}{\partial \tau}\left(\tau\frac{\partial \breve{v}_\tau}{\partial n_\tau}\right)P\frac{\partial \breve{u}_\tau}{\partial n_\tau} + \tau\frac{\partial \breve{v}_\tau}{\partial n_\tau}\frac{\partial P}{\partial \tau}\frac{\partial \breve{u}_\tau}{\partial n_\tau} + \tau\frac{\partial \breve{v}_\tau}{\partial n_\tau}P\frac{\partial}{\partial \tau}\left(\frac{\partial \breve{u}_\tau}{\partial n_\tau}\right) + \frac{\partial}{\partial \tau}\left(\tau\frac{\partial \breve{v}_\tau}{\partial n_\tau}\right)r \\
&+\tau\frac{\partial \breve{v}_\tau}{\partial n_\tau}\frac{\partial r}{\partial \tau}\,\mathrm{d}t \\
= \;& \int_0^{t_0} \frac{\partial}{\partial t}\left(\tan(\alpha)\frac{\partial \breve{v}_\tau}{\partial \tau} - \frac{\varphi}{\tau\cos(\alpha)}\frac{\partial \breve{v}_\tau}{\partial t}\right)P\frac{\partial \breve{u}_\tau}{\partial n_\tau} + \tau\frac{\partial \breve{v}_\tau}{\partial n_\tau}\frac{\partial P}{\partial \tau}\frac{\partial \breve{u}_\tau}{\partial n_\tau} - \frac{\partial \breve{v}_\tau}{\partial n_\tau}P\frac{\partial \breve{u}_\tau}{\partial n_\tau} \\
&+\frac{\partial \breve{v}_\tau}{\partial n_\tau}P\left[\frac{\partial}{\partial t}\left(\tan(\alpha)\frac{\partial \breve{u}_\tau}{\partial \tau} - \frac{\varphi}{\tau\cos(\alpha)}\frac{\partial \breve{u}_\tau}{\partial t}\right) - f\tau\varphi\cos(\alpha)\right] \\
&+\frac{\partial}{\partial t}\left(\tan(\alpha)\frac{\partial \breve{v}_\tau}{\partial \tau} - \frac{\varphi}{\tau\cos(\alpha)}\frac{\partial \breve{v}_\tau}{\partial t}\right)r + \tau\frac{\partial \breve{v}_\tau}{\partial n_\tau}\frac{\partial r}{\partial \tau}\,\mathrm{d}t \\
= \;& \int_0^{t_0} \frac{\partial}{\partial t}\left(\tan(\alpha)\frac{\partial \breve{v}_\tau}{\partial \tau} - \frac{\varphi}{\tau\cos(\alpha)}\frac{\partial \breve{v}_\tau}{\partial t}\right)\breve{u}_\tau + \tau\frac{\partial \breve{v}_\tau}{\partial n_\tau}\frac{\partial P}{\partial \tau}\frac{\partial \breve{u}_\tau}{\partial n_\tau} - \frac{\partial \breve{v}_\tau}{\partial n_\tau}P\frac{\partial \breve{u}_\tau}{\partial n_\tau} \\
&+\frac{\partial \breve{v}_\tau}{\partial n_\tau}P\frac{\partial}{\partial t}\left(\tan(\alpha)\frac{\partial \breve{u}_\tau}{\partial \tau} - \frac{\varphi}{\tau\cos(\alpha)}\frac{\partial \breve{u}_\tau}{\partial t}\right) - \frac{\partial \breve{v}_\tau}{\partial n_\tau}Pf\tau\varphi\cos(\alpha) + \tau\frac{\partial \breve{v}_\tau}{\partial n_\tau}\frac{\partial r}{\partial \tau}\,\mathrm{d}t \\
\Rightarrow \;& \int_0^{t_0} \frac{\tau}{\varphi\cos(\alpha)}\frac{\partial \breve{v}_\tau}{\partial \tau}\frac{\partial \breve{u}_\tau}{\partial \tau} - \tan(\alpha)\frac{\partial \breve{v}_\tau}{\partial t}\frac{\partial \breve{u}_\tau}{\partial \tau} - \tan(\alpha)\frac{\partial \breve{v}_\tau}{\partial \tau}\frac{\partial \breve{u}_\tau}{\partial t} + \frac{\varphi}{\tau\cos(\alpha)}\frac{\partial \breve{v}_\tau}{\partial t}\frac{\partial \breve{u}_\tau}{\partial t}\,\mathrm{d}t \\
= \;& \int_0^{t_0} -\tan(\alpha)\frac{\partial \breve{v}_\tau}{\partial \tau}\frac{\partial \breve{u}_\tau}{\partial t} + \frac{\varphi}{\tau\cos(\alpha)}\frac{\partial \breve{v}_\tau}{\partial t}\frac{\partial \breve{u}_\tau}{\partial t} + \tau\frac{\partial \breve{v}_\tau}{\partial n_\tau}\frac{\partial P}{\partial \tau}\frac{\partial \breve{u}_\tau}{\partial n_\tau} - \frac{\partial \breve{v}_\tau}{\partial n_\tau}P\frac{\partial \breve{u}_\tau}{\partial n_\tau} \\
&+P\frac{\partial \breve{v}_\tau}{\partial n_\tau}\frac{\partial}{\partial t}\left(\tan(\alpha)\frac{\partial \breve{u}_\tau}{\partial \tau} - \frac{\varphi}{\tau\cos(\alpha)}\frac{\partial \breve{u}_\tau}{\partial t}\right) - \frac{\partial \breve{v}_\tau}{\partial n_\tau}Pf\tau\varphi\cos(\alpha) + \tau\frac{\partial \breve{v}_\tau}{\partial n_\tau}\frac{\partial r}{\partial \tau}\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
= \quad & \int_0^{t_0} -\tan(\alpha)\frac{\partial \check{v}_\tau}{\partial \tau}\frac{\partial \check{u}_\tau}{\partial t} + \frac{\varphi}{\tau\cos(\alpha)}\frac{\partial \check{v}_\tau}{\partial t}\frac{\partial \check{u}_\tau}{\partial t} + \tau\frac{\partial \check{v}_\tau}{\partial n_\tau}\frac{\partial P}{\partial \tau}\frac{\partial \check{u}_\tau}{\partial n_\tau} - \frac{\partial \check{v}_\tau}{\partial n_\tau}P\frac{\partial \check{u}_\tau}{\partial n_\tau} \\
& -\frac{\partial \check{v}_\tau}{\partial t}\tan(\alpha)\frac{\partial \check{u}_\tau}{\partial \tau} + \frac{\partial \check{v}_\tau}{\partial t}\frac{\varphi}{\tau\cos(\alpha)}\frac{\partial \check{u}_\tau}{\partial t} - \frac{\partial \check{v}_\tau}{\partial n_\tau}Pf\tau\varphi\cos(\alpha) + \tau\frac{\partial \check{v}_\tau}{\partial n_\tau}\frac{\partial r}{\partial \tau}\,\mathrm{d}t \\
\Rightarrow \quad & \int_0^{t_0} \frac{\tau}{\varphi\cos(\alpha)}\frac{\partial \check{v}_\tau}{\partial \tau}\frac{\partial \check{u}_\tau}{\partial \tau}\,\mathrm{d}t = \int_0^{t_0} \tau\frac{\partial \check{v}_\tau}{\partial n_\tau}\frac{\partial P}{\partial \tau}\frac{\partial \check{u}_\tau}{\partial n_\tau} - \frac{\partial \check{v}_\tau}{\partial n_\tau}P\frac{\partial \check{u}_\tau}{\partial n_\tau} + \frac{\partial \check{v}_\tau}{\partial t}\frac{\varphi}{\tau\cos(\alpha)}\frac{\partial \check{u}_\tau}{\partial t} \\
& -\frac{\partial \check{v}_\tau}{\partial n_\tau}Pf\tau\varphi\cos(\alpha) + \tau\frac{\partial \check{v}_\tau}{\partial n_\tau}\frac{\partial r}{\partial \tau}\,\mathrm{d}t \\
\Rightarrow \quad & \int_0^{t_0} \left(\tau\frac{\partial \check{v}_\tau}{\partial n_\tau} + \tan(\alpha)\frac{\partial \check{v}_\tau}{\partial t}\right)\left(\varphi\cos(\alpha)\frac{\partial \check{u}_\tau}{\partial n_\tau} + \frac{\varphi\sin(\alpha)}{\tau}\frac{\partial \check{u}_\tau}{\partial t}\right)\mathrm{d}t = \int_0^{t_0} \tau\frac{\partial \check{v}_\tau}{\partial n_\tau}\frac{\partial P}{\partial \tau}\frac{\partial \check{u}_\tau}{\partial n_\tau} \\
& -\frac{\partial \check{v}_\tau}{\partial n_\tau}P\frac{\partial \check{u}_\tau}{\partial n_\tau} + \frac{\partial \check{v}_\tau}{\partial t}\frac{\varphi}{\tau\cos(\alpha)}\frac{\partial \check{u}_\tau}{\partial t} - \frac{\partial \check{v}_\tau}{\partial n_\tau}Pf\tau\varphi\cos(\alpha) + \tau\frac{\partial \check{v}_\tau}{\partial n_\tau}\frac{\partial r}{\partial \tau}\,\mathrm{d}t \\
\Rightarrow \quad & \int_0^{t_0} \tau\varphi\cos(\alpha)\frac{\partial \check{v}_\tau}{\partial n_\tau}\frac{\partial \check{u}_\tau}{\partial n_\tau} + \varphi\sin(\alpha)\frac{\partial \check{v}_\tau}{\partial n_\tau}\frac{\partial P}{\partial t}\frac{\partial \check{u}_\tau}{\partial n_\tau} + \varphi\sin(\alpha)\frac{\partial \check{v}_\tau}{\partial n_\tau}\frac{\partial r}{\partial t} \\
& +\varphi\sin(\alpha)\frac{\partial P}{\partial t}\frac{\partial \check{v}_\tau}{\partial n_\tau}\frac{\partial \check{u}_\tau}{\partial n_\tau}\,\mathrm{d}t = \int_0^{t_0} \tau\frac{\partial \check{v}_\tau}{\partial n_\tau}\frac{\partial P}{\partial \tau}\frac{\partial \check{u}_\tau}{\partial n_\tau} - \frac{\partial \check{v}_\tau}{\partial n_\tau}P\frac{\partial \check{u}_\tau}{\partial n_\tau} + \tau\frac{\partial \check{v}_\tau}{\partial n_\tau}\frac{\partial r}{\partial \tau} \\
& +\frac{\varphi\cos(\alpha)}{\tau}\frac{\partial P}{\partial t}\frac{\partial \check{v}_\tau}{\partial n_\tau}\frac{\partial P}{\partial t}\frac{\partial \check{u}_\tau}{\partial n_\tau} + \frac{\varphi\cos(\alpha)}{\tau}\frac{\partial P}{\partial t}\frac{\partial \check{v}_\tau}{\partial n_\tau}\frac{\partial r}{\partial t} - \frac{\partial \check{v}_\tau}{\partial n_\tau}Pf\tau\varphi\cos(\alpha)\,\mathrm{d}t.
\end{aligned}
$$

Formally, we obtain

$$
\begin{aligned}
& \left(\tau\varphi\cos(\alpha)\frac{\partial \check{v}_\tau}{\partial n_\tau}, \frac{\partial \check{u}_\tau}{\partial n_\tau}\right) + \left(\varphi\sin(\alpha)\frac{\partial P}{\partial t}\frac{\partial \check{v}_\tau}{\partial n_\tau}, \frac{\partial \check{u}_\tau}{\partial n_\tau}\right) + \left(\varphi\sin(\alpha)\frac{\partial \check{v}_\tau}{\partial n_\tau}, \frac{\partial P}{\partial t}\frac{\partial \check{u}_\tau}{\partial n_\tau}\right) \\
& = \left(\tau\frac{\partial \check{v}_\tau}{\partial n_\tau}, \frac{\partial P}{\partial \tau}\frac{\partial \check{u}_\tau}{\partial n_\tau}\right) - \left(\frac{\partial \check{v}_\tau}{\partial n_\tau}, P\frac{\partial \check{u}_\tau}{\partial n_\tau}\right) + \left(\frac{\varphi\cos(\alpha)}{\tau}\frac{\partial P}{\partial t}\frac{\partial \check{v}_\tau}{\partial n_\tau}, \frac{\partial P}{\partial t}\frac{\partial \check{u}_\tau}{\partial n_\tau}\right)
\end{aligned}
\tag{4.20}
$$

and

$$
\begin{aligned}
& \left(\varphi\sin(\alpha)\frac{\partial \check{v}_\tau}{\partial n_\tau}, \frac{\partial r}{\partial t}\right) = -\left(P\frac{\partial \check{v}_\tau}{\partial n_\tau}, f\tau\varphi\cos(\alpha)\right) + \left(\tau\frac{\partial \check{v}_\tau}{\partial n_\tau}, \frac{\partial r}{\partial \tau}\right) \\
& +\left(\frac{\varphi\cos(\alpha)}{\tau}\frac{\partial P}{\partial t}\frac{\partial \check{v}_\tau}{\partial n_\tau}, \frac{\partial r}{\partial t}\right).
\end{aligned}
\tag{4.21}
$$

We emphasize that this operator P, and the operator $\mathbf{P}$ of the precedent sections, are not the same. Nevertheless, $\dfrac{\partial}{\partial n}$ and δ verify the direct relation $\dfrac{\partial}{\partial n} = \left(\dfrac{\cos(\alpha)}{\tau\varphi}\right)\delta$. This means that we can prove the equivalence between the formulations (4.16) and (4.20) (and, similarly, between (4.17) and (4.21)). In fact, through the change of coordinates $\cos(\alpha)\,\mathrm{d}t = \varphi\,\mathrm{d}\theta$, we find $\dfrac{\partial}{\partial n_\tau} = \left(\dfrac{\cos(\alpha)}{\tau\varphi}\right)\delta$ and, consequently, from (4.16), we obtain successively

$$\tau\frac{\partial \mathbf{P}}{\partial \tau}+\mathbf{P}\frac{\partial}{\partial\theta}\left(\frac{b}{1+b^2}\right)-\mathbf{P}\frac{\partial}{\partial\theta}\left(\frac{1}{1+b^2}\frac{\partial}{\partial\theta}\right)\mathbf{P}-\frac{b}{1+b^2}\frac{\partial}{\partial\theta}\mathbf{P}-\frac{I}{1+b^2}=0$$

$$\Rightarrow\quad \tau\frac{\partial \mathbf{P}}{\partial \tau}+\mathbf{P}\frac{\varphi}{\cos(\alpha)}\frac{\partial}{\partial t}\left(\frac{b}{1+b^2}\right)-\mathbf{P}\frac{\varphi}{\cos(\alpha)}\frac{\partial}{\partial t}\left(\frac{1}{1+b^2}\frac{\varphi}{\cos(\alpha)}\frac{\partial}{\partial t}\right)\mathbf{P}-\frac{b}{1+b^2}\frac{\varphi}{\cos(\alpha)}\frac{\partial}{\partial t}\mathbf{P}$$
$$-\frac{I}{1+b^2}=0$$

$$\Rightarrow\quad \tau\frac{\partial \mathbf{P}}{\partial \tau}+\mathbf{P}\frac{\varphi}{\cos(\alpha)}\frac{\partial}{\partial t}\big(\sin(\alpha)\cos(\alpha)\big)-\mathbf{P}\frac{\varphi}{\cos(\alpha)}\frac{\partial}{\partial t}\left(\cos^2(\alpha)\frac{\varphi}{\cos(\alpha)}\frac{\partial}{\partial t}\right)\mathbf{P}-\sin(\alpha)\cos(\alpha)$$
$$\frac{\varphi}{\cos(\alpha)}\frac{\partial}{\partial t}\mathbf{P}-I\cos^2(\alpha)=0$$

$$\Rightarrow\quad \tau\frac{\partial \mathbf{P}}{\partial \tau}+\mathbf{P}\frac{\varphi}{\cos(\alpha)}\frac{\partial}{\partial t}\big(\sin(\alpha)\cos(\alpha)\big)-\mathbf{P}\frac{\varphi}{\cos(\alpha)}\frac{\partial}{\partial t}\left(\varphi\cos(\alpha)\frac{\partial}{\partial t}\right)\mathbf{P}-\varphi\sin(\alpha)\frac{\partial}{\partial t}\mathbf{P}$$
$$-I\cos^2(\alpha)=0$$

$$\Rightarrow\quad \tau\frac{\partial}{\partial\tau}\left(P\frac{\cos(\alpha)}{\tau\varphi}\right)+P\frac{\cos(\alpha)}{\tau\varphi}\frac{\varphi}{\cos(\alpha)}\frac{\partial}{\partial t}\big(\sin(\alpha)\cos(\alpha)\big)-P\frac{\cos(\alpha)}{\tau\varphi}\frac{\varphi}{\cos(\alpha)}\frac{\partial}{\partial t}\left(\varphi\cos(\alpha)\frac{\partial}{\partial t}\right)$$
$$P\frac{\cos(\alpha)}{\tau\varphi}-\varphi\sin(\alpha)\frac{\partial}{\partial t}P\frac{\cos(\alpha)}{\tau\varphi}-I\cos^2(\alpha)=0$$

$$\Rightarrow\quad \left(\frac{\frac{\partial P}{\partial\tau}\tau-P}{\tau^2}\right)\frac{\tau\cos(\alpha)}{\varphi}+\frac{P}{\tau}\frac{\partial}{\partial t}\big(\sin(\alpha)\cos(\alpha)\big)-\frac{P}{\tau}\frac{\partial}{\partial t}\left(\varphi\cos(\alpha)\frac{\partial}{\partial t}\right)P\frac{\cos(\alpha)}{\tau\varphi}-\varphi\sin(\alpha)$$
$$\frac{\partial}{\partial t}P\frac{\cos(\alpha)}{\tau\varphi}-I\cos^2(\alpha)=0$$

$$\Rightarrow\quad \frac{\partial P}{\partial\tau}\frac{\cos(\alpha)}{\varphi}-P\frac{\cos(\alpha)}{\tau\varphi}-\frac{1}{\tau}\frac{\partial}{\partial t}P\sin(\alpha)\cos(\alpha)-\frac{P}{\tau}\frac{\partial}{\partial t}\left(\varphi\cos(\alpha)\frac{\partial}{\partial t}\right)P\frac{\cos(\alpha)}{\tau\varphi}-\varphi\sin(\alpha)$$
$$\frac{\partial}{\partial t}P\frac{\cos(\alpha)}{\tau\varphi}-I\cos^2(\alpha)=0$$

$$\Rightarrow\quad \tau\frac{\partial P}{\partial\tau}-P-\frac{\partial}{\partial t}P\varphi\sin(\alpha)-\frac{P}{\tau}\frac{\partial}{\partial t}\left(\varphi\cos(\alpha)\frac{\partial}{\partial t}\right)P-\varphi\sin(\alpha)\frac{\partial}{\partial t}P-\tau\varphi\cos(\alpha)I=0$$

$$\Rightarrow\quad \tau\frac{\partial P}{\partial\tau}-P+P\frac{\partial}{\partial t}\big(\varphi\sin(\alpha)\big)-\frac{P}{\tau}\frac{\partial}{\partial t}\left(\varphi\cos(\alpha)\frac{\partial}{\partial t}P\right)-\varphi\sin(\alpha)\frac{\partial}{\partial t}P-\tau\varphi\cos(\alpha)I=0$$

which corresponds formally to (4.20).

4.5. Defining $u(0)$

Proposition 4.5.1. *Considering u_ε the solution of problem (1.4), $u_{\varepsilon|_{\Gamma_\varepsilon}}$ is bounded by a constant not depending on ε.*

Proof. The first part of the proof consists on showing that we have

$$\inf_{\Gamma_\varepsilon} w_\varepsilon \leq u_{\varepsilon|_{\Gamma_\varepsilon}} \leq \sup_{\Gamma_\varepsilon} w_\varepsilon, \tag{4.22}$$

where $w_\varepsilon \in H_0^1(\Omega)$ is the solution of the problem $-\Delta w_\varepsilon = \tilde{f}_\varepsilon = \begin{cases} f, & \Omega \setminus \Omega_\varepsilon \\ 0, & \Omega_\varepsilon. \end{cases}$

From $-\Delta w_\varepsilon = \tilde{f}_\varepsilon$, in $H_0^1(\Omega)$, we find $\int_\Omega -\Delta w_\varepsilon = \int_\Omega \tilde{f}_\varepsilon = \int_{\Omega\setminus\Omega_\varepsilon} f = -\int_\Gamma \frac{\partial w_\varepsilon}{\partial n}$. On the other hand, from the formulation of problem (1.4) and choosing a test function equal to one, we find $\int_{\Omega\setminus\Omega_\varepsilon} -\Delta u_\varepsilon = \int_{\Omega\setminus\Omega_\varepsilon} f = -\int_{\Gamma_\varepsilon} \frac{\partial u_\varepsilon}{\partial n} - \int_\Gamma \frac{\partial u_\varepsilon}{\partial n} = -\int_\Gamma \frac{\partial u_\varepsilon}{\partial n}$. Therefore, we have the equality $\int_\Gamma \frac{\partial u_\varepsilon}{\partial n} = \int_\Gamma \frac{\partial w_\varepsilon}{\partial n}$.

Let us now suppose that $u_{\varepsilon|_{\Gamma_\varepsilon}} = c_\varepsilon < \inf_{\Gamma_\varepsilon} w_\varepsilon$. Then, $u_\varepsilon - w_\varepsilon$ satisfies:

$$\begin{cases} -\Delta(u_\varepsilon - w_\varepsilon) = 0, \text{ in } \Omega \setminus \Omega_\varepsilon \\ (u_\varepsilon - w_\varepsilon)_{|_\Gamma} = 0 \\ (u_\varepsilon - w_\varepsilon)_{|_{\Gamma_\varepsilon}} < 0. \end{cases} \tag{4.23}$$

From (4.23) and using the maximum principle we can also conclude that $u_\varepsilon - w_\varepsilon \leq 0$, in $\overline{\Omega \setminus \Omega_\varepsilon}$ and, in fact, $u_\varepsilon - w_\varepsilon < 0$, in $\Omega \setminus \Omega_\varepsilon$. As a consequence, using the definition of directional derivative, we find that $\frac{\partial u_\varepsilon}{\partial n}|_\Gamma \geq \frac{\partial w_\varepsilon}{\partial n}|_\Gamma$.

From $\frac{\partial(u_\varepsilon - w_\varepsilon)}{\partial n}|_\Gamma \geq 0$ and $\int_\Gamma \frac{\partial(u_\varepsilon - w_\varepsilon)}{\partial n} = 0$ we conclude that $\frac{\partial(u_\varepsilon - w_\varepsilon)}{\partial n}|_\Gamma = 0$.

Therefore, we have $u_\varepsilon - w_\varepsilon < 0$, in $\Omega \setminus \Omega_\varepsilon$, and $(u_\varepsilon - w_\varepsilon) = 0$, in Γ. Using Lemma 3.4 of [14], for each point of Γ, we find $\frac{\partial(u_\varepsilon - w_\varepsilon)}{\partial n} > 0$ a.e. on Γ and we reach a contradiction. So, we must have $\inf_{\Gamma_\varepsilon} w_\varepsilon \leq c_\varepsilon$.

Analogously, one can show that $c_\varepsilon \leq \sup_{\Gamma_\varepsilon} w_\varepsilon$.

For the second part of the proof, using [14] (Theorem 8.15, page 189, with $q = 4$), we can show that $\|w_\varepsilon\|_{L^\infty(\Omega)}$ is bounded by a constant not depending on ε (only depends on constants concerning $\|f\|_{L^2(\Omega)}$ and the size of Ω) and the result follows. ■

Now we are able to establish the value of u on the origin. Obviously, this general method could also be applied on Chapter 2, instead of the direct computations presented there.

Proposition 4.5.2. *Let $f \in \mathcal{C}^{0,\alpha}(\Omega)$ Then, when ε converges to 0, $u_{\varepsilon_{|\Gamma_\varepsilon}}$ converges to $u(0)$.*

Proof. Considering u the solution of problem (1.3), since $f \in \mathcal{C}^{0,\alpha}(\Omega)$, we have $u \in \mathcal{C}^{2,\alpha}(\Omega)$. Let, as previously, $-\Delta w_\varepsilon = \tilde{f}_\varepsilon$, $w_\varepsilon \in H_0^1(\Omega)$. Therefore, $v_\varepsilon = w_\varepsilon - u$ satisfies $-\Delta(v_\varepsilon) = \tilde{g_\varepsilon}$, where $\tilde{g_\varepsilon} = \begin{cases} -f, & \Omega_\varepsilon \\ 0, & \Omega \setminus \Omega_\varepsilon. \end{cases}$ Using again [14] we can show that $\|v_\varepsilon\|_{L^\infty(\Omega)} \leq k(\|v_\varepsilon\|_{L^2(\Omega)} + \|\tilde{g_\varepsilon}\|_{L^2(\Omega)})$, where k is a constant not depending on ε. When $\varepsilon \to 0$ we have $\|v_\varepsilon\|_{L^2(\Omega)} \to 0$ and $\|\tilde{g_\varepsilon}\|_{L^2(\Omega)} \to 0$, then $\|v_\varepsilon\|_{L^\infty(\Omega)} \to 0$. So, for $\delta > 0$ there exists $\varepsilon > 0$ such that $|v_\varepsilon(x)| \leq \frac{\delta}{2}$ and $|u(x) - u(0)| \leq \frac{\delta}{2}, \forall x \in \Omega_\varepsilon \cup \Gamma_\varepsilon$. Then, for $x \in \Gamma_\varepsilon$, $|w_\varepsilon(x) - u(0)| = |v_\varepsilon(x) + u(x) - u(0)| \leq \delta$ and consequently, $-\delta \leq \inf_{\Gamma_\varepsilon}(w_\varepsilon(x)) - u(0) = \inf_{\Gamma_\varepsilon}(w_\varepsilon(x) - u(0)) \leq \sup_{\Gamma_\varepsilon}(w_\varepsilon(x) - u(0)) = \sup_{\Gamma_\varepsilon}(w_\varepsilon(x)) - u(0) \leq \delta$. Using (4.22), we find $-\delta \leq \inf_{\Gamma_\varepsilon} w_\varepsilon - u(0) \leq u_{\varepsilon_{|\Gamma_\varepsilon}} - u(0) \leq \sup_{\Gamma_\varepsilon} w_\varepsilon - u(0) \leq \delta$, which implies that $u_{\varepsilon_{|\Gamma_\varepsilon}} \to u(0)$, when $\varepsilon \to 0$. ∎

4.6. Conclusion

Using the Galerkin method and the adequate properties on the operator P and function r, we hope to justify the preceding formal calculations, following the same steps of Chapter 2. We expect to obtain, after passing to the limit when the dimension tends to infinity, the following result:

Claim 4.6.1. *Denoting by $\mathcal{I}$ the interval $(0, t_0)$, by $(.,.)$ the scalar product in $L^2(\mathcal{I})$, and considering $\phi \in \mathcal{D}(0,1)$, then P, r and $\breve{u}_\tau$ satisfy:*

1. *the negative self-adjoint operator $P \in \mathcal{L}\left(L^2(\mathcal{I}), H^1_{\tau,p}(\mathcal{I})\right) \cap \mathcal{L}\left(\left(H^{1/2}_{\tau,p}(\mathcal{I})\right)', H^{1/2}_{\tau,p}(\mathcal{I})\right) \cap \mathcal{L}\left(\left(H^1_{\tau,p}(\mathcal{I})\right)', L^2(\mathcal{I})\right)$, bounded as a function of τ, satisfies, for every $h, \bar{h}$ in $L^2(\mathcal{I})$, the Riccati equation*

$$\left(\frac{dP}{d\tau}h, \bar{h}\right) - \left(\frac{\varphi \sin\alpha}{\tau}h, \frac{\partial}{\partial t} \circ P\bar{h}\right) - \left(\frac{\partial}{\partial t} \circ Ph, \frac{\varphi \sin\alpha}{\tau}\bar{h}\right) + \left(\frac{\varphi \cos\alpha}{\tau^2}\frac{\partial}{\partial t} \circ Ph, \frac{\partial}{\partial t} \circ P\bar{h}\right) - \left(\frac{1}{\tau}h, P\bar{h}\right) = \left(\varphi \cos\alpha\, h, \bar{h}\right) \tag{4.24}$$

in $\mathcal{D}'(0,1)$, with the initial condition $P(1) = 0$;

2. *for every h in $H^{1/2}_{\tau,p}(\mathcal{I})$, r satisfies the equation*

$$\left(\frac{\partial r}{\partial \tau}, h\right) - \left(\frac{\varphi \sin\alpha}{\tau}\frac{\partial r}{\partial t}, h\right) + \left(\frac{\varphi \cos\alpha}{\tau^2}\frac{\partial r}{\partial t}, \frac{\partial}{\partial t} \circ Ph\right) = \left(\varphi \cos\alpha \hat{f}, Ph\right) \tag{4.25}$$

in $\mathcal{D}'(0,1)$, with the initial condition $r(1) = 0$;

3. *for every h in $\left(H^{1/2}_{\tau,p}(\mathcal{I})\right)'$, $\breve{u}_\tau$ satisfies the equation*

$$\begin{aligned} &-\left(\frac{1}{\varphi \cos\alpha}\frac{\partial \breve{u}_\tau}{\partial \tau}, Ph\right) + \left(\frac{\tan\alpha}{\tau}\frac{\partial \breve{u}_\tau}{\partial t}, Ph\right) + \langle \breve{u}_\tau, h\rangle_{H^{1/2}_{\tau,p}(\mathcal{I}),\ H^{1/2}_{\tau,p}(\mathcal{I})\ '} \\ &= \langle r, h\rangle_{H^{1/2}_{\tau,p}(\mathcal{I}),\ H^{1/2}_{\tau,p}(\mathcal{I})\ '} \end{aligned} \tag{4.26}$$

in $\mathcal{D}'(0,1)$, with the initial condition $\breve{u}_\tau(0) = u(0)$ given by Proposition 4.5.2.

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Table of Notation