A NEW APPROACH TO THE INITIAL VALUE PROBLEM

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Abstract: The initial condition problem for fractional linear system initialisation is studied in this paper. A new approach that involves functions belonging to the space of Laplace transformable distributions is presented. It is based on the generalised initial value theorem and on reinterpretations of the most common differintegration definitions.

Keywords: initial value problem, fractional differintegration

1. INTRODUCTION

The increase in the number of physical and engineering processes that are found to be best described by fractional differential equations has been motivation for the study and application of fractional calculus. The effective application of the fractional calculus to science and engineering problems needs a coherent fractional systems theory. In previous papers (Ortigueira 2000a and 2000b) we tried to do some contribution to that goal. However, a problem that seemed to be already solved originated an interesting discussion (Lorenzo and Hartley;Podlubny): the initialisation problem. The reason is in two facts:

a) Two different solutions are known.
b) They both seem to be unsatisfactory.

Lorenzo and Hartley (1998, 2000, 2001) showed that the proper initialisations of fractional differintegrals are non-constant functions, generalising the integer order case. They have treated the issue of initialisation in several papers where they formulated the problem correctly, analysed the effect of a wrong initialisation and proposed a solution.
In this paper, we will approach the problem from a different point of view having in mind two aspects:

a) the way how the initial values appear
b) the class of distributions having Laplace Transform (LT)

The paper proceeds as follows:
In section 2, we present the differintegrations we assume valid and equivalent from the Laplace Transform (LT) point of view. Taking the Leibniz rule as base, we will study the behaviour of the differintegrated near the origin in section 3. The initial value problem is treated in section 4 by:

a) enouncing the initial value theorem;
b) the statement of the problem;
c) studying the usual approaches
d) presenting the proposed solution.
e) giving two examples.

At last we will present some conclusions.

2. ON THE DIFFERINTEGRATION

In the following we will consider three formulations of the differintegration based on the general double convolution:

$$D^{\alpha}(t) = x(t) * \delta^{(n)}(t) * \delta^{(-\nu)}(t) \qquad (1)$$

where D means derivative ($\alpha>0$) or integral ($\alpha<0$), $n\in Z$, $0\leq\nu<1$, $\alpha=n-\nu$,

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$$\delta_{\pm}^{(n)}(t) = \begin{cases} D^{(n)}\delta(t) & n\geq 0 \\ \pm\frac{t^{(-n-1)}}{(-n-1)!}u(\pm t) & n<0 \end{cases} \tag{2}$$

with δ(t) as the impulse Dirac distribution, and

$$\delta_{\pm}^{(-\nu)}(t) = \begin{cases} \pm\frac{t^{\nu-1}}{\Gamma(\nu)}u(\pm t) & 0<\nu<1 \\ \delta(t) & \nu=0 \end{cases} \tag{3}$$

where + stand for forward and – for backward differintegrations. In the following we shall be concerned with the forward case. As it is clear, we have several possibilities in the computation of the differintegration according to how we use the associative property of the convolution. We have successively:

$$x^{(\alpha)}(t) = \delta^{(n)}(t)*\left\{x(t)*\delta^{(-\nu)}(t)\right\} \tag{4}$$

the Riemann-Liouville differintegration,

$$x^{(\alpha)}(t) = \left\{x(t)*\delta^{(n)}(t)\right\}*\delta^{(-\nu)}(t) \tag{5}$$

the Caputo differintegration, and

$$x^{(\alpha)}(t) = x(t)*\left\{\delta^{(n)}(t)*\delta^{(-\nu)}(t)\right\} \tag{6}$$

the Generalised Functions differintegration - also called Cauchy differintegration. It is interesting to remark that:

a) In the Riemann-Liouville differintegration we proceed sequentially by the computation of a ν order integration, followed by n derivative computations.
b) In the Caputo differintegration we invert the process, beginning by n derivative computations followed by a ν order integration.
c) In the Cauchy differintegration, the computation in done directly.
d) It is not hard to see that, if we retain the finite part, we can write:

$$\delta^{(\alpha)}(t) = \left\{\delta^{(n)}(t)*\delta^{(-\nu)}(t)\right\} = \begin{cases} \pm\frac{t^{-\alpha-1}}{\Gamma(-\alpha)}u(\pm t) & \alpha\neq 0 \\ \delta(t) & \alpha=0 \end{cases} \tag{7}$$

e) It is clear that, at least conceptually, we can admit mixed situations, because, with the above definitions, we treat the integration as the inverse operation of differentiation and vice-versa.

These points are very important in the initial condition problem, as we will see later.

For later uses we are going to compute the α order derivative (α>0) of the power function $p(t)=t^{\beta}u(t)$, with β>-1. It is not hard to see that $p(t)=\Gamma(\beta+1).\delta^{(-\beta-1)}(t)$. So, independently of the used definition, we conclude that

$$D^{\alpha}p(t) = \Gamma(\beta+1).\delta^{(\alpha-\beta-1)}(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha+1}u(t) \tag{8}$$

that is a regular function if α<β and integrable if β-α>-1.

3. BEHAVIOUR NEAR THE ORIGIN

The initial condition problem has a very narrow connection with the behaviour of the differintegrated for positive near zero values. We will study it, now. Let us assume that we have a signal φ(t)=y(t).u(t). The differintegrated of φ(t) will be a causal signal. Leibniz rule states that

$$D^{\alpha}[y(t).u(t)]=\sum_{0}^{\infty}\binom{\alpha}{k}u^{(\alpha-k)}(t).y^{(k)}(t) = \sum_{0}^{\infty}\binom{\alpha}{k}\delta^{(\alpha-k-1)}(t).y^{(k)}(t) \tag{9}$$

As usually, put α=n-ν. Attending to (3):

$$\delta^{(\alpha-k-1)}(t) = \frac{t^{-n+k+\nu}}{\Gamma(\nu-n+k+1)}u(t) \tag{10}$$

that has the following behaviour near the origin:

$$\left.\frac{t^{-n+k+\nu}}{\Gamma(\nu-n+k+1)}\right|_{t=0} = \begin{cases} 0 & \text{if } k>n \\ \infty & \text{if } k\leq n \end{cases} \tag{11}$$

Then,

$$D^{\alpha}[y(t).u(t)] = \sum_{0}^{\infty}\binom{\alpha}{k}\frac{t^{-n+k+\nu}}{\Gamma(\nu-n+k+1)}.y^{(k)}(t)u(t) \tag{12}$$

But from (10) we have:

$$D^{\alpha}[y(t).u(t)] = \sum_{0}^{n}\binom{\alpha}{k}\frac{t^{-n+k+\nu}}{\Gamma(\nu-n+k+1)}.y^{(k)}(t)u(t) \tag{13}$$

To guarantee that each term of the summation is finite, we must assume that near the origin,

$$\lim_{t\to 0+}\frac{y^{(k)}(t)}{t^{n-k-\nu}} < \infty \tag{14}$$

So, let us assume that y(t) has the following format:

$$y(t) = t^{\beta}.g(t).u(t) \tag{15}$$

where g(t) and its derivatives $g^{(k)}(t)$, 1≤k≤n, must be finite as t→0+. It is not hard to conclude that, when t→0+, we have:

$$D^{\alpha}y(t)\Big|_{t=0+} = \begin{cases} 0 & \text{if } \beta>\alpha \\ \Gamma(\alpha+1)g(0+) & \text{if } \beta=\alpha \\ \infty & \text{if } \beta<\alpha \end{cases} \qquad (16)$$

4. THE INITIAL VALUE PROBLEM

1. STATEMENT OF THE PROBLEM

In problems with initial conditions it is a common practice to introduce unilateral transforms. However, there is no particular justification for such introduction. In fact, we intend to solve a given differential equation for values of t greater than a given initial instant, that, without loosing generality, we can assume to be the origin. To treat the question, it is enough to multiply both member of the equation by the unit step Heaviside function. Consider a given ordinary constant coefficient differential equation.

$$y^{(N)}(t) + a\, y(t) = x(t) \quad N\in Z_0^+ \qquad (17)$$

Assume that we want to solve it for t>0. The multiplication by u(t) leads to

$$y^{(N)}(t)u(t) + a\, y(t)u(t) = x(t)u(t) \qquad (18)$$

Thus, we have to relate $y^{(N)}(t)u(t)$ to $[y(t).u(t)]^{(N)}$. This can be done recursively provided that we account for the properties of the distribution δ and its derivatives. We obtain the well known result:

$$y^{(N)}(t).u(t)=[y(t).u(t)]^{(N)}-\sum_{i=0}^{N-1} y^{(N-1-i)}(0).\delta^{(i)}(t) \qquad (19)$$

that states that $y^{(N)}(t).u(t)=[y(t).u(t)]^{(N)}$ for t>0. They are different at t=0. This is the reason why we speak in initial values as being equivalent to initial conditions.

In the above equation we have

$$[y(t).u(t)]^{(N)} +a[y(t).u(t)]=x(t)+\sum_{i=0}^{N-1} y^{(N-1-i)}(0).\delta^{(i)}(t) \qquad (20)$$

The initial conditions appear naturally, independently of using a transform. In fractional case, the problem is similar, but it is not so clear the introduction of the initial conditions. Consider the fractional analog of equations (17) and (18). We have:

$$y^{(\alpha)}(t) + a\, y(t) = x(t) \qquad (21)$$

and

$$y^{(\alpha)}(t)u(t) + a\, y(t)u(t) = x(t)u(t) \qquad (22)$$

Again our problem is to express $y^{(\alpha)}(t)u(t)$ in terms of $[y(t).u(t)]^{(\alpha)}$, or a function easily related to it.

2. THE INITIAL-VALUE THEOREM

The Abelian initial value theorem (Zemanian, 1965) is a very important result in dealing with the Laplace Transform. This theorem relates the asymptotic behaviour of φ(t) as t→0+ to the asymptotic behaviour of Φ(σ) =LT[φ(t)], as σ=Re(s) →+∞.

The initial-value theorem - Assume that φ(t) is a causal signal such that in some neighbourhood of the origin is a regular distribution corresponding to an integrable function. Also, assume that there is a real number β such that $\lim_{t\to 0+}\frac{\varphi(t)}{t^{\beta}}$ exists and is a finite complex value. Then

$$\lim_{t\to 0+}\frac{\varphi(t)}{t^{\beta}} = \lim_{\sigma\to\infty}\frac{\sigma^{\beta+1}\Phi(\sigma)}{\Gamma(\beta+1)} \qquad (23)$$

For proof see (Zemanian, 1965).

Let –1<α<β. Then

$$\lim_{t\to 0+}\frac{\varphi(t)}{t^{\alpha}} = \lim_{t\to 0+}\frac{\varphi(t)}{t^{\beta}}\frac{t^{\beta}}{t^{\alpha}} = 0$$

Similarly, if β<α,

$$\lim_{t\to 0+}\frac{\varphi(t)}{t^{\alpha}} = 0$$

Then, all the derivatives of order α<β have a zero initial value, while all the derivatives of order greater than β are infinite at t=0, in agreement with (16). The class of functions verifying (23) is very important and contains almost all the functions appearing when dealing with fractional linear systems. These functions verify the Watson-Doetsch Lemma (Henrici).

As $\lim_{t\to 0+}\frac{\varphi(t)}{t^{\beta}} = \lim_{t\to 0^+}\frac{\varphi(t)}{\Gamma(\beta+1)\delta^{(-\beta-1)}(t)}$ and $D^{\beta}\delta^{(-\beta-1)}(t) = u(t)$, we conclude that:

$$\lim_{t\to 0+}\frac{\varphi(t)}{\delta^{(-\beta-1)}(t)} = D^{\beta}\varphi(t)\Big|_{t=0+} \qquad (24)$$

So,

$$D^{\beta}\varphi(t)\Big|_{t=0+} = \lim_{\sigma\to\infty}\sigma^{\beta+1}\Phi(\sigma) \qquad (25)$$

that is a generalisation of the usual initial value theorem, obtained when β=0. It is interesting to remark that (23) is very similar to the usual l'Hôpital rule used to solve the 0/0 problems.

3. USUAL APPROACHES TO THE FRACTIONAL CASE

The initial value problem is solved traditionally by the use of the one-sided LT. A difficulty arises when it was found that the three fractional derivatives defined in section 2 lead to different solutions. Let y(t) be a function defined in R

In the Riemann-Liouville case, we obtain (Podlubny)

$$LT[D^{\alpha}_{rl}\, y(t)]=s^{\alpha}Y_L(s)-\sum_{i=0}^{n-1} s^{i}y^{(\alpha-1-i)}(0+) \qquad (26)$$

where n-1≤α<n and $Y_L(s)$ is the one-sided LT of y(t), while in Caputo case the result is:

$$LT[D^{\alpha}_{c}\, y(t)]=s^{\alpha}Y_L(s)-\sum_{i=0}^{n-1} s^{\alpha-i-1}y^{(i)}(0+) \qquad (27)$$

With the Cauchy derivative, no initial conditions appear. Eq. (27) is very atractive, because it involves integer derivatives with simple physical interpretation. If we rewrite the previous equations in time, we have:

$$D^{\alpha}_{rl}\, y(t)=[y(t).u(t)]^{(\alpha)}-\sum_{i=0}^{n-1} y^{(\alpha-1-i)}(0+).\delta^{(i)}(t) \qquad (28)$$

and

$$D^{\alpha}_{c}\, y(t)=[y(t).u(t)]^{(\alpha)}-\sum_{i=0}^{n-1} y^{(i)}(0+).\delta^{(\alpha-i-1)}(t) \qquad (29)$$

respectively. We have three different results: (28), (29), and the Cauchy case that makes

$$y^{(\alpha)}(t).u(t)=[y(t).u(t)]^{(\alpha)} \qquad (30)$$

for any $t\geq 0$. We have to put the questions:

a) is there any correct result?
b) are all the three correct?

In the following section, we will approach the answers for these questions. Meanwhile, remark that in (28) we only affect the values at t=0: $D^{\alpha}_{rl}\, y(t)=y^{(\alpha)}(t).u(t)=[y(t).u(t)]^{(\alpha)}$ for t>0.

This is not the case of (28) where have initial functions, not initial values. We have: $D^{\alpha}_{c}\, y(t)\neq y^{(\alpha)}(t).u(t)\neq [y(t).u(t)]^{(\alpha)}$ for all $t\geq 0$.

This means that the first member in (29) in not $y^{(\alpha)}(t).u(t)$ but another function. We came to the conclusion that the correct solution for the initial value problem as it is normally interpreted is given by (28).

4. THE PROPOSED SOLUTION

It is interesting to see how the initial values appear and their meaning. Let y(t) be a given signal defined in R, continuous at t=0 and that $y(0)\neq 0$. Thus y(t)u(t) is not continuous at t=0. But $g_1(t) = y(t)u(t)-y(0+)u(t)$ is continuous and assumes the value zero. We subtracted the jump at t=0. The derivative of $g_1(t)$ is $g_1^{(1)}(t) =[y(t)u(t)]^{(1)}-y(0+)\delta(t)$. Again, $g_1^{(1)}(t)$ is not continuous but has a jump $g_1^{(1)}(0+)$. So,

$g_2(t) =[y(t)u(t)]^{(2)}- y(0+)\delta(t) - g_1^{(1)}(0+)\delta(t)$

is continuous and its derivative is not. The process continues and we obtain (19). Thus, the initial conditions appear from the need for becoming continuous the successive derivatives. In solving a differential equation, we only can accept that the highest derivative has a jump. Let us assume that y(t) has the format:

$$x(t)=t^N f(t).u(t) \qquad (31)$$

where f(t) is asumed to be analytic (only by simplicity) near t=0. It is not difficult to show that $D^i x(0+) = 0$ for i=0, …, N-1, $D^N x(0+) =N!f(0+)$ and $D^i x(0+) = \infty$ for i>N. This means that to obtain a function verifying (17) we only have to add several functions of the type (31):

$$y(t) = \sum_{i=0}^{N} t^i f_i(t)u(t) \qquad (32)$$

where the $f_i(t)$ are assumed to have regular derivatives near the origin, at least till the Nth derivative.

To get some insigt into the fractional case, remark that the function

$$g(t)=t^N f(t).u(t) - f(0+)\, t^N u(t) \qquad (33)$$

and all its derivatives $g^{(i)}(t)$ (i=1, 2, …, N) are continuous at t=0. Only the derivatives above N, are discontinuous.

In the fractional case we assume that we are dealing with a function of the form:

$$y(t) = \sum_{i=0}^{N} t^{\gamma_i} f_i(t)u(t) \qquad (34)$$

where the γ_i verify:

$$-1< \gamma_i < \gamma_{i+1} \qquad (35)$$

We introduce also a sequence β_n verifying:

$$\beta_n = \gamma_n - \sum_{k=0}^{n-1} \beta_k\, ,\ \ \beta_0=\gamma_0 \qquad (36)$$

According to what we said before, we can perform successive derivatives while $\beta_n > -1$.

Let us see what happens. We are going to proceed step by step:

a) According to our assumptions β_0 is the least real for which $\lim_{t\to 0^+} \frac{y(t)}{t^{\beta_0}}$ is finite and nonzero. Let it be $y^{(\beta_0)}(0+)$.
b) Construct the function

$$f(t) = y(t).u(t) - y^{(\beta_0)}(0+)\, t^{\beta_0}u(t) \qquad (37)$$

c) All the derivatives $D^{\alpha}f(t)$ ($\alpha\leq\beta_0$) are continuous at t=0 and assume a zero value. Then

$$f^{(\beta_0)}(t) = [y(t).u(t)]^{(\beta_0)} - y^{(\beta_0)}(0+)\, u(t) \qquad (38)$$

is continuous at t=0.

d) Now, β_1 is the least real for which $\lim_{t\to 0^+} \frac{f^{(\beta_0)}(t)}{t^{\beta_1}}$ is finite and nonzero. Let it be $y^{(\beta_0+\beta_1)}(0+)$. Thus

$$f^{(\beta_0+\beta_1)}(t) =[y(t).u(t)]^{(\beta_0+\beta_1)} -y^{(\beta_0)}(0+)\, \delta^{(\beta_1)}(t) - y^{(\beta_0+\beta_1)}(0+)\, u(t) \qquad (39)$$

is again continuous at t=0.

e) Again β_2 is the least real for which $\lim_{t\to 0^+} \frac{f^{(\beta_0+\beta_1)}(t)}{t^{\beta_2}}$ is finite and nonzero. Let it be $y^{(\beta_0+\beta_1+\beta_2)}(0+)$. Thus

$$f^{(\beta_0+\beta_1+\beta_2)}(t) =[y(t).u(t)]^{(\beta_0+\beta_1+\beta_2)} -y^{(\beta_0)}(0+)\, \delta^{(\beta_1+\beta_2)}(t) -y^{(\beta_0+\beta_1)}(0+)\, \delta^{(\beta_2)}(t) - y^{(\beta_0+\beta_1+\beta_2)}(0+)u(t) \qquad (40)$$

Continuing with this procedure, we obtain a function:

$$f^{(\gamma_N)}(t) =[y(t).u(t)]^{(\gamma_N)} - \sum_{0}^{N-1} y^{(\gamma_m)}(0+)\delta^{(\gamma_{N-1}-\gamma_i)}(t) \qquad (41)$$

that is not continuous at t=0, but it can be made continuous if we subtract it $y^{(\gamma_N)}(0+)$. Equation (41) states the general formulation of the initial value problem solution.

To verify the coherence of the result, we are going to study some special cases:

a) $\gamma_i=i$, for i=0,1, …, N.

We have: $\beta_0=0$, $\beta_i=1$, for i=1, …, N-1 and:

$$f^{(N)}(t) =[y(t).u(t)]^{(N)} - \sum_0^{N-1} y^{(m)}(0+)\, \delta^{(N-1-m)}(t) \quad (42)$$

Applying the LT to both members we obtain:

$$s^N F(s) = s^N Y(s) - \sum_0^{N-1} y^{(m)}(0+)\, s^{(N-1-m)} \quad (43)$$

that is the usual formula for the initial value problem. It is clear that $f^{(N)}(t) =[y(t).u(t)]^{(N)}$ for t>0.

b) $\gamma_i=i\gamma$, for i=0,1, …, N.

We have: $\beta_0=0$, $\beta_i=\gamma$, for i=1, …, N-1. Then,

$$f^{(N\gamma)}(t) =[y(t).u(t)]^{(N\gamma)} - \sum_0^{N-1} y^{(m\gamma)}(0+)\delta^{((N-1-m)\gamma)}(t) \quad (44)$$

giving

$$s^{N\gamma}F(s) = s^{N\gamma}Y(s) - \sum_0^{N-1} y^{(m\gamma)}(0+)\, s^{(N-1-m)\gamma} \quad (45)$$

different from the results obtained with the one-sided LT and both Riemann-Liouville or Caputo differintegrations. This case is suitable for easy solution of equations of the type:

$$\sum_{n=0}^{N} a_n D^{n\nu}\, y(t) = \sum_{m=0}^{M} b_m D^{m\nu}\, x(t) \quad (46)$$

c) $\gamma_i=\gamma+i$, for i=0,1, …, N.

We obtain: $\beta_0=\gamma$, $\beta_i=1$, for i=1, …, N-1. Then,

$$f^{(N+\gamma)}(t) =[y(t).u(t)]^{(N+\gamma)} - \sum_0^{N-1} y^{(m+\gamma)}(0+)\delta^{(N-1-m)}(t) \quad (47)$$

and

$$s^{N+\gamma}F(s) = s^{N+\gamma}Y(s) - \sum_0^{N-1} y^{(m+\gamma)}(0+)\, s^{N-1-m} \quad (48)$$

with $\gamma=0$, we obtain (19) again. With $\alpha=N+\gamma$, equation (48) can be rewritten as:

$$s^{N+\gamma}F(s) = s^{N+\gamma}Y(s) - \sum_0^{N-1} y^{(\alpha-1-i)}(0+)\, s^{i} \quad (49)$$

that is the Riemann-Liouville solution.

d) Putting $\gamma_i = i$, i=0, …,N, $\gamma_N=-\varepsilon$, $0<\varepsilon<1$, and $\alpha=N-\varepsilon$, we obtain:

$$f^{(\alpha)}(t) =[y(t).u(t)]^{(\alpha)} - \sum_0^{N} y^{(m)}(0+)\delta^{(\alpha-m)}(t) \quad (50)$$

that is the Caputo solution.

The choice of a given set of initial conditions will be dictated by additional considerations, like:

a) physical interpretation
b) availability
c) facility of computation

5. Examples

1 - Consider the system described by the equation (21) with $\alpha=3/2$. As in the equation we only have two terms we are not constraint and can choose any “way” to go from 0 to α. We are going to consider 3 cases:

a) $\gamma_i = 1/2.i$ (i=0,1,2,3) or $\beta_0=0$ and $\beta_i=1/2$ (i=1,2,3). We have now:

$$s^{3/2}F(s) = s^{3/2}Y(s) - \sum_0^{2} y^{(m/2)}(0+)\, s^{(2-m)/2} \quad (51)$$

with

$$F_f(s)= \frac{\sum_0^{2} y^{(m/2)}(0+)\, s^{(2-m)/2}}{s^{3/2}+a} \quad (52)$$

As the corresponding free term.

b) $\gamma_i = 1/2+i$ (i=0,1) or $\beta_0=1/2$ and $\beta_1=3/2$.giving the Riemann-Liouville solution:

$$s^{3/2}F(s) = s^{3/2}Y(s) – y^{(1/2)}(0+) \quad (53)$$

The same solution can be obtained with $\gamma_I = -1/2+i$ (i=0,1,2). Now, the free term is given by:

$$F_f(s)=y^{(1/2)}(0+).\frac{1}{s^{3/2}+a} \quad (54)$$

c) $\gamma_i = i$ (i=0,1, 2) and $\gamma_3 = 3/2$ or $\beta_0=0$, $\beta_i=1$ (i=1,2), and $\beta_3=-1/2$. It comes:

$$s^{3/2}F(s) = s^{3/2}Y(s) - \sum_0^{1} y^{(m)}(0+)\, s^{3/2-m} \quad (55)$$

giving the free term:

$$F_f(s)= \frac{\sum_0^{2} y^{(m)}(0+)\, s^{(3/2-m)}}{s^{3/2}+a} \quad (56)$$

Of course, there are other solutions. This last solution seems to be the more appealing, because it involves integer order derivatives that are easily obtained and have simple physical interpretation. However, the corresponding time response has a $\delta(t)$.

2 - The situation is somehow different if we have an intermediary term as it is the case of the equation:

$$y^{(\alpha)}(t) + a\, y^{(1)}(t) + by(t)= x(t) \quad (57)$$

Now, when going from $\gamma=0$ to $\gamma=3/2$, we have to “pass” by $\gamma=1$. So, the above solution b) is not suitable to deal with this case. However, the other two serve in this case.

For example, for the c) case, we have:

$$F_f(s)=\frac{ay(0+) + \sum_{0}^{2} y^{(m)}(0+)\, s^{(3/2-m)}}{s^{3/2}+as + b} \qquad (58)$$

5. CONCLUSIONS

We approached the initial conditions problem from a sequential point of view. The proposed solution showed that, in general, we must speak in initial functions instead of initial values, in the sense that we can modify the functions, not only at t=0, but at all t≥0. With this point of view, we obtained a broad set of initial conditions that we can choose according to our interests or facility in solving a specific problem. We worked in the context of the Laplace transformable distributions class that cover most of functions we are interested in.

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