Fractional Signal Processing: Scale Conversion

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***Abstract* - Scale conversion of discrete-time signals are studied taking as base the fractional discrete-time system theory. Some simulation results to illustrate the behaviour of the algorithms will be presented. A new algorithm for performing the zoom transform is also described.**

1 Introduction

In [1,2], the fractional discrete-time linear systems were introduced together with a fractional linear prediction that we used to interpolate discrete-time signals [3]. Here, we will address the problem of scale conversion that has connection with the rate conversion. We will present general scale conversion formulae that may be useful in the case of pulses, since the presence of a sync function makes the convergence very slow. As alternative, we propose the use of the fractional linear prediction.

The basic idea underlying the algorithm is the development of a system able to linear predicting the signal over instant times between the current ones, without converting the signal to the continuous-time domain. The new samples fit in between the original samples. The algorithm uses the Maximum Entropy Method to obtain the spectrum of the original integer domain signal. Using this spectrum estimate, we derived the coefficients of the fractional predictor. Here, we are going to present that algorithm together with some simulation results illustrating its behaviour. This algorithm is suitable for the interpolation of stationary stochastic processes. This means that when dealing with pulses the algorithm does not provide accurate samples.

The previous results suggested us to look for similar results in frequency domain. This is usually called zoom transform. There are two algorithms for performing the zoom of the DFT, but are somehow involved [4,5]. We are going to present a very simple algorithm that has two steps: a) computation of a matrix and b) multiplication of a vector by that matrix.

In section 2, we present general formulae for scale conversion and, in section 3, the fractional linear prediction is described together with corresponding simulation results. In section 4, we present the spectral zoom and some illustrating examples. At last some conclusions are outlined.

2 Scale conversion

Let consider a signal x_n, with Fourier Transform $X(e^{j\omega})$, and a real constant α such that $0<\alpha<1$. Define a new function $X_\alpha(e^{j\omega})$ by:

$$X_\alpha(e^{j\omega})=\begin{cases} X(e^{j\omega}) & \text{if } |\omega|\le\pi \\ 0 & \text{if } \pi<|\omega|<\dfrac{\pi}{\alpha} \end{cases} \qquad (1)$$

and repeat it with period $\dfrac{2\pi}{\alpha}$. Letting the coefficients of the corresponding Fourier Series be represented by c_n and putting $x_{\alpha n}= c_n/\alpha_,$ we obtain:

$$x_{\alpha n} = \sum_{k=-\infty}^{+\infty} x_k \frac{\sin[\pi(n\alpha-k)]}{\pi(n\alpha-k)} \qquad (2)$$

So, we can conclude that:

$$FT[x_{k\alpha}]=\frac{1}{\alpha}X(e^{j\omega/\alpha}) \qquad (3)$$

Consider another real constant $\beta\neq\alpha$, satisfying also $0<\beta<1$. It is not hard to show that:

$$x_{\alpha n} = \beta\sum_{k=-\infty}^{+\infty} x_{\beta k} \frac{\sin[\pi(n\alpha-\beta k)]}{\pi(n\alpha-\beta k)} \qquad (4)$$

If $\alpha>1$, the same procedure leads to:

$$x_{\alpha n} = \sum_{k=-\infty}^{+\infty} x_k \frac{\sin[\pi(n-k/\alpha)]}{\pi(n-k/\alpha)} \qquad (5)$$

that corresponds to an ideal lowpass filtering followed by a downsampling. Using (2) with β in the place of α, we obtain:

$$\frac{\sin[\pi(n\alpha-k)]}{\pi(n\alpha-k)}=\beta\sum_{m=-\infty}^{+\infty}\frac{\sin[\pi(\beta m-k)]}{\pi(\beta m-k)}\,\frac{\sin[\pi(n\alpha-\beta m)]}{\pi(n\alpha-\beta m)} \qquad (6)$$

that is an interesting relation involving sync functions. As seen, we can use (2) – or (4) – to perform a scale conversion. However, its usefulness is very limited since it

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cannot be used to perform a rate conversion as it is usually intended, due to the non-causality of the sync and the slow converging series.

3 Scale conversion by fractional prediction

In the following we will present an algorithm for scale conversion based on the fractional prediction. This is based on the theory of the fractional linear systems [1]. The starting point is the definition of fractional delay and lead:

$$x_{n+\alpha}=\sum_{m=-\infty}^{+\infty} x_m \frac{\sin\left[p(a+n-m)\right]}{p\left[a+n-m\right]} \qquad (7)$$

where $\alpha\in R$ and $n\in Z$. The relation (2.1) is a convolution of x_n and a $\delta_{n+\alpha}$ given by:

$$\delta_{n+a}=\frac{\sin[\pi(\alpha+n)]}{[\pi(\alpha+n)]}=\frac{\sin(\pi\alpha)}{\pi\alpha}\frac{(-1)^n}{1+\frac{n}{\alpha}} \qquad (8)$$

which can be considered as the impulse response of a reconstruction filter, δ_{n+a}, such that

$$x_{n+\alpha} = x_n * \delta_{n+a} \qquad (9)$$

or applying the Fourier Transform (FT):

$$X_\alpha(e^{j\omega}) = e^{j\omega\alpha} X(e^{j\omega}) \qquad (10)$$

with $X_\alpha(e^{j\omega}) = FT[x_{n+\alpha}]$,thus generalising a well-known result.

The previous relations are the bases for the d-step prediction we will present. We shall be working in the context of a stationary real stochastic process.

Let x(n) be a real stationary stochastic process, observed from $-\infty$ to n-1 and let $R_x(k)$ be its autocorrelation function.

We define the Nth order d-step prediction at the instant n-1+d ($0<d\leq1$) by:

$$\hat{x}(n-1+d) = -\sum_{i=1}^{N} a_i\, x(n-i) \qquad (11)$$

where a_i (i=1, ..., N) are the coefficients of the d-step predictor (d=1, corresponds to the usual one-step prediction).

The predictor coefficients are chosen in order to minimise the prediction error power:

$$P_d = E[\left(x(n-1+d) - \hat{x}(n-1+d)\right)^2] \qquad (12)$$

Assuming that the correlation matrix of x(n) has, at least rank N, the optimum d-step predictor is given by the solution of the following set of normal equations [1,3]:

$$\sum_{i=1}^{N} a_i.R_x(k-i) = -R_x(-k-d-1) \quad k=1,2,\ldots, N \qquad (13)$$

that can be written in a matrix format as:

$$\mathbf{R}_x.a=-\mathbf{r}_d \qquad (14)$$

To compute this vector $\mathbf{r}_d$ we can use [3]:

$$R(k-1+d)=$$

$$\frac{(-1)^{k-1}\sin(\pi d)}{\pi(d+k-1)}.\left[R(0)+2\sum_{n=1}^{+\infty}\frac{(-1)^n R(n)}{1-\left(\frac{n}{d+k-1}\right)^2}\right] \qquad (15)$$

So, with equations (8) and (9) we can compute the coefficients of the fractional predictor, provided that we use a suitable autocorrelation function estimate. If x(n) is an AR(N-1) stationary stochastic process, the longest (with greater order) optimum fractional d-step predictor has order N [3]. In the non-AR case, we are expecting that the predictor although theoretically not finite may be truncated. This allows us to devise a better way to compute R(k+d). Assuming a AR(N-1) process, the (N-1)th one-step predictor defines, together with the prediction error, P_{N-1} , the spectrum of the signal [3,5]:

$$S_x(\omega) = \frac{P_{N-1}}{\left|\sum_{n=0}^{N-1} p_i^{N-1}\cdot e^{-jwn}\right|^2} \qquad (16)$$

that can be used to obtain:

$$R(k) = FT^{-1}\left[S_x(\omega)\right] \qquad (17)$$

and

$$R(k+d) = FT^{-1}\left[e^{j\omega d}S_x(\omega)\right] \qquad (18)$$

With these results we can take advantage of the well-known linear prediction methods (e. g. modified covariance or Burg algorithms) [3,5]. The proposed algorithm has the following steps:

1 - Compute the N-1 linear predictor using a suitable algorithm.

2 - Use the (N-1)th linear predictor to estimate the spectrum, $S_x(\omega)$, and the corresponding autocorrelation, of the signal.

3 - Multiply $S_x(\omega)$ by $e^{jd\omega}$ and compute the inverse Fourier Transform to obtain the vector $\mathbf{r}_{d.}$

4 - Use (17), (18) and (14) to obtain the coefficients of the fractional predictor.

This algorithm is simple and computationally efficient. Although obtained under the hypothesis that the signal is AR(N-1), it will be useful in other situations, namely in the ARMA case.

To illustrate the application of the method, we present some simulation results. We proceed in the following way:

[1] Generate a signal with L points and a given signal to noise ratio;

[2] down-sample it by 1/2 factor;

[3] Use the previous algorithm to estimate the removed values.

For each simulation we computed the error between each original and estimated value and the corresponding error power. In Figure 1 we present the result of a simulation using as original signal a sum of sync functions.

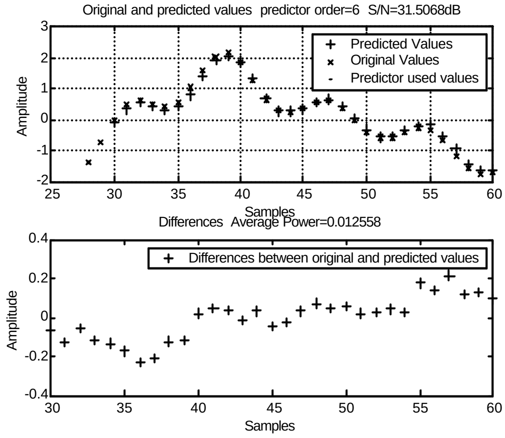

Figure 1 – Fractional prediction of a sum of sync functions

Obviously, we are not restricted to d=0.5. Consider that *d* assumes 3 values, d=0.25, d=0.5, d=0.75, and keep the predictor of order 4. We insert 3 values between each set of two original values. The results obtained are displayed in figure 2. As it is easy to conclude, we were making a rate increase by integer values. Of course, we can obtain a fractional rate increase (or decrease) by decimation.

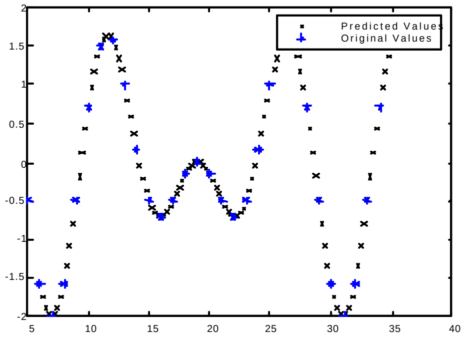

Figure 2 – interpolation using fractional prediction with steps 0.25, 0.5 , and 0.75.

To study the influence of the predictor length we made several simulations in the referred conditions and computed the average error power over 10 realizations of each of the referred signals. The results are presented in the following pictures.

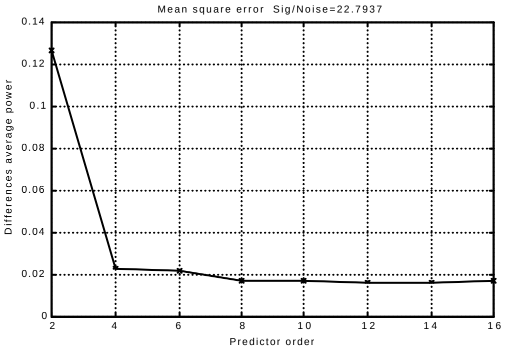

Figure 3 – mean error power for 10 realizations of one sinusoid as function of predictor length.

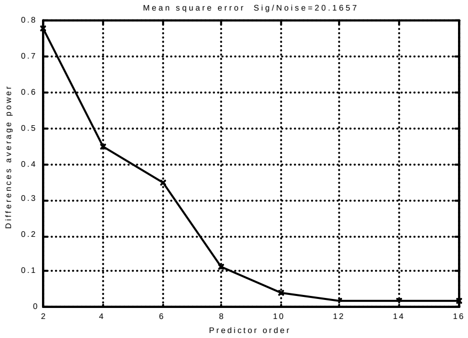

Figure 4 – mean error power for 10 realizations of 2 sinusoids as function of predictor length

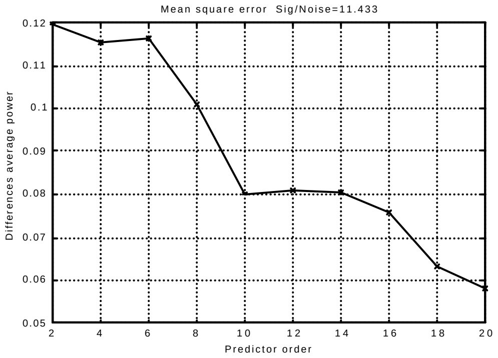

Figure 5 – mean error power for 10 realizations of several syncs as function of predictor length.

We can conclude that even with low predictor orders (lower than 10) we can interpolate quite well non-AR signals.

4 Zoom Transform

The results obtained in section 1 can be extrapolated to an interesting practical application: the zoom transform. Let us consider a L point sequence, x_n n=0, …, L-1. Every N≥L point DFT sequence represent samples of the Discrete-Time Fourier Transform (DTFT). This sampling may become unapparent some characteristics of the spectrum in a given particular band of interest. To avoid this problem two different methods of interpolation have been proposed [4,6] and usually referred as the zoom transform. Here, we propose an alternative approach. The DTFT of x_n is given $X(e^{j\omega})$. The DFT corresponds to sample $X(e^{j\omega})$:

$$\mathrm{DFT}[x_n] = X(e^{j\frac{2\pi}{N}k}) \quad k=0, \ldots, N-1, N\geq L \qquad (19)$$

Denote this DFT by $X_N(k)$. Its inverse, (DFT^{-1}) is a N-period signal. If we take one period of this signal, add zeros and repeat the obtained sequence with a period $M=\alpha N$ $(\alpha>1)$, we are sampling $X(e^{j\omega})$ in M uniformly spaced points, obtaining $X_M(k)$, $k=0, \ldots, M-1$. Then, we have:

$$X_M(k) = \sum_{k=0}^{L-1} x_n\, e^{-j\frac{2\pi}{M}kn} \qquad k=0, ..., M-1 \qquad (20)$$

and

$$x_n = \frac{1}{N}.\sum_{k=0}^{N-1} X_N(k).e^{j\frac{2\pi}{N}kn} \qquad n=0, ..., N-1 \qquad (21)$$

Inserting (21) into (20) we obtain:

$$X_M(k) = \frac{1}{N}.\sum_{k=0}^{N-1} X_N(l).G(k,l) \quad k=0, ..., M-1 \qquad (22)$$

where

$$G(k,l) = \frac{1 - e^{j\frac{2\pi}{N}(l-k/\alpha)L}}{1 - e^{j\frac{2\pi}{N}(l-k/\alpha)}} \qquad (23)$$

for $0\leq l<N$ and $0\leq k<M$. It is not hard to show that:

$$G(k,l) = L.\frac{\mathrm{sync}\left[\frac{(l-k/\alpha)L}{N}\right]}{\mathrm{sync}\left[\frac{l-k/\alpha}{N}\right]}\, e^{j\frac{\pi}{N}(l-k/\alpha)(L-1)} \qquad (24)$$

Of course, we are not interested in zooming the whole spectrum, but a given band, corresponding to values of $k=m_1, \ldots, m_2$ with m_1 and m_2 as described below. Assume that we want to zoom the band $[f_1;f_2]$, with $0\leq f_1<f_2\leq\pi$. Let K be the number of points we want to compute. Then

$$\alpha = \frac{1}{(f_2-f_1)}.\frac{K}{N} \qquad (25)$$

and

$$m_i = \left\lfloor \frac{\alpha N}{2\pi}.f_i \right\rfloor \quad i=1,2 \qquad (26)$$

where $\lfloor x \rfloor$ means the integer part of x. In the following figure, we illustrate the application of the algorithm for zooming 2 regions of the spectrum shown in the upper strip of the figure 6.

5 Conclusions

In this paper we presented new algorithms for interpolation and scale conversion of discrete-time signals based on the theory of fractional discrete-time systems. We presented some simulation results to illustrate the behaviour of the algorithms when applied in a rate increase by a factor 2 for different sets of signals. We concluded that even with low order predictors we can perform a rate increase. We presented also an illustration of the linear prediction with several fractional steps. Based in the results of section 1 we also derived a very simple but efficient algorithm for the zoom transform.

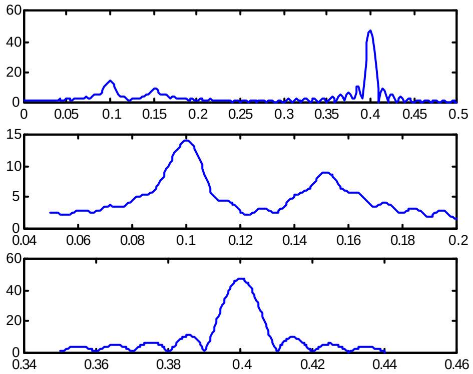

Figure 6 – zoom transform

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