Introduction to fractional linear systems. Part 1: Continuous-time case

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Abstract: In the paper, the class of continuous-time linear systems is enlarged with the inclusion of fractional linear systems. These are systems described by fractional differential equations. It is shown how to compute the impulse, step, and frequency responses from the transfer function. The theory is supported by definitions of fractional derivative and integral, generalisations of the usual. An introduction to fractal signals as outputs of fractional differintegrators is presented. It is shown how to define a stationary fractal.

1 Introduction

Fractional calculus is nearly 300 years old. In fact, in a letter to Leibnitz, Bernoulli put him a question about the meaning of a non-integer derivative order. It was the beginning of a discussion on the theme that involved other mathematicians such as L' Hôpital, Euler and Fourier [1–3]. However we can trace the beginning of the fractional calculus in the works of Liouville and Abel. Abel solved an integral equation representing an operation of fractional integration. Liouville made several attempts and presented a formula for fractional integration

$$D^{-p}\varphi(t) = \frac{1}{(-1)^p\Gamma(p)}\int_0^\infty \varphi(t+\tau)\tau^{p-1}\,d\tau \qquad -\infty < t < \infty, p > 0 \tag{1}$$

where $\Gamma(p)$ is the gamma function. With the term $(-1)^p$ omitted, we call this the Liouville fractional integral. Liouville developed ideas on this theme and presented a generalisation of the notion of incremental ratio to define a fractional derivative. This idea was discussed again by Grünwald (1867) and Letnikov (1868). Riemann reached an expression similar to eqn. 1 for the fractional integral.

Holmgren (1865/66 and Letnikov 1868/74) discussed this problem when looking for the solution of differential equations, putting in a correct statement the fractional differentiation as inverse operation of the fractional integration. Hadamard proposed a method of fractional differentiation based on the differentiation of the Taylor's series associated with the function. Weyl (1917) defined a fractional integration suitable to periodic functions, and Marchaud (1927) presented a form of differentiation based on finite differences. More recently, the unified formulation of integration and differentiation (differintegration) based on Cauchy's integral [4–6] has gained great popularity.

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Applications to physics and engineering are not recent: application to viscosity dates back to the 1930s [7]. The work of Mandelbrot [8, 9] in the field of fractals had great influence and attracted attention to fractional calculus. During the last 20 years, application domains of fractional calculus have increased significantly: seismic analysis [7], dynamics of motor and premotor neurones of the oculomotor systems [10], viscous damping [11, 12], electric fractal networks [13], 1/f noise [14, 15], fractional order sinusoidal oscillators [16], and, more recently, control [17–19] and robotics [20]. However, there is no publication with a coherent presentation of fractional linear system theory. Most elementary books on signals and systems consider only the integer derivative order case and treat the corresponding systems, studying their impulse, step and frequency responses and their transfer function. It is not such a simple matter, if one substitutes fractional derivatives for the common derivatives.

The objective of this paper is to treat the fractional continuous-time linear system as is done with usual systems. Attempts have been made to create a formal framework for the study of fractional linear systems, but without the desired generality, coherence and usefulness of the final results [2, 11, 21–23]. To our knowledge, the approach we propose here is original, although it has an 'already seen' character. This is because we are dealing with very well known concepts. We merely generalise them to the fractional case.

We intend to make a first contribution for a correct understanding of some experimental results [14, 22, 23] and to create a new way into modelling, simulation, and estimation in real fractional systems.

To begin a study of fractional systems, we need to define fractional derivative. The most obvious approach to the fractional derivative is the Grünwald–Letnikov method, a generalisation of the usual definition based on the incremental ratio [1–3]. However, it is very hard to manipulate and obtain new results. This motivates us to adopt a definition based on the Laplace transforms.

Consider first the Laplace transform (LT) case. Essentially, we are looking for the Laplace inverse transform of s^α, $\delta^{(\alpha)}(t)$, with α being any positive real number. Generalising the well known property of the Laplace transform, the convolution of a Laplace transformable signal $x(t)$ with $\delta^{(\alpha)}(t)$ has $s^\alpha X(s)$ as the LT and is defined as the α order derivative of $x(t)$. If $\alpha < 0$, the definition remains valid and we are performing a (fractional) integration. For this reason, we assume α to be any real and speak of 'differintegration' [3, 5, 24]. As in the integer case, there are two functions with s^α as the LT, corresponding to the causal and anti-causal cases. The combination of these two allows us to obtain a definition of differintegration for bilateral signals with a Fourier transform (FT).

The results obtained with the LT are suitable for dealing with linear systems described by fractional linear differential equations. We define transfer function and impulse response in a very similar way to that normally used. We also make a brief study of the stability problem.

We present the definition of differintegration that we have adopted. Interesting topics are studied: differintegration of periodic functions, the unilateral Laplace transform and the initial conditions, and differintegration of the causal exponential. These results are fundamental to our study of the linear fractional continuous systems. We consider systems defined by fractional differential equations that are used to obtain the transfer function and the impulse response. This is obtained by partial fraction expansion. Examples are presented and the problem of the stability is treated briefly. We also consider the state variable formulation.

We study the stochastic processes output of the differintegrator when the input is white noise. We define a stationary fractional stochastic process, the hyperbolic noise. It is shown that this process belongs to the class of the so-called '1/f noise', and we obtain a generalisation of the Brownian motion. We also show why some attempts to define a stationary fractal have failed.

Note that in the following and otherwise stated, we assume to be in the context of the generalised functions (distributions). We always assume that they are either of exponential order or tempered distributions.

2 Fractional differintegration of continuous-time signals

2.1 Definition

The simplest way of defining fractional derivative is through the generalisation of the incremental ratio. It is called the Grünwald–Letnikov approach [3]. However, this approach is not suitable for the application we have in mind, since it is hard to manipulate. Therefore, we pursue a different method.

Instead of beginning by a definition of differintegration, let us invert the problem and take the LT as the starting point, not only for the differentiation, but also for integration – differintegration. Essentially, we intend to prolong the sequence

$$\ldots s^{-n} \ldots, s^{-2}, s^{-1}, 1, s^1, s^2, \ldots, s^n \ldots \qquad (2)$$

in order to include other kinds of exponents: rational or, generally, real (or even complex numbers). It is clear that there are two forms of obtaining the extension, depending on the choice for the region of convergence for the LT: the left and right half-planes. Let us begin by considering both cases with integer powers and obtaining the corresponding inverses. With these cases, we can form Table 1, where $\delta(t)$ is the Dirac impulse and $u(t)$ is the Heaviside unit step function.

To obtain a differintegration of a given function, we only have to convolve it with one of the functions presented in each row. Generalising the notation used for the usual derivatives, we denote by $\delta^{(-n)}(t)$ the nth order primitive of $\delta(t)$. This corresponds to saying that a differintegration of order n (integer) is given by

$$f^{(n)}(t) = \int_{-\infty}^{\infty} f(\tau)\delta^{(n)}(t-\tau)\mathrm{d}\tau \qquad (3)$$

where $\delta^{(n)}(t)$ is given in the causal case by

$$\delta^{(n)}(t) = \begin{cases} \text{the } n\text{th order derivative of } \delta(t) & \text{if } n > 0 \\ \dfrac{t^{-n-1}u(t)}{(-n-1)!} & \text{if } n < 0 \end{cases} \qquad (4)$$

To define a fractional differintegration, we have to generalise eqn. 2 to eqn. 4 for non-integer orders. Thus, we have to give a precise meaning to $\delta^{(\alpha)}(t) = LT^{-1}[s^\alpha]$. Note that, in the general case, $\alpha = \lfloor\alpha\rfloor + \nu$ $(0 \leq \nu < 1)$, and so we only have to consider the case $0 < |\alpha| < 1$, since $s^\alpha = s^{\lfloor\alpha\rfloor+\nu} = s^{\lfloor\alpha\rfloor}.s^\nu$, giving the following ($\delta^{(\alpha)}(t)$ always represents the α differintegrated of the Dirac's distribution $\delta(t)$):

$$\delta^{(\alpha)}(t) = \delta^{(\lfloor\alpha\rfloor)}(t)^{*}\delta^{(\nu)}(t) \qquad (5)$$

The solution for the problem is given by [2, 3]

$$\delta_{+}^{(\alpha)}(t) = \frac{t^{-\alpha-1}u(t)}{\Gamma(-\alpha)} \qquad (6)$$

for the causal case. It is a simple task to show that

$$\delta_{-}^{(\alpha)}(t) = -\frac{t^{-\alpha-1}u(-t)}{\Gamma(-\alpha)} \qquad (7)$$

is the solution for the anti-causal case. $\Gamma(t)$ is the Euler gamma function [3]. Returning to eqns. 6 and 7, if $\alpha > 0$, it is a derivation; if $\alpha < 0$, it is an integration. Essentially, eqns. 6 and 7 lead to the Riemann-Liouville and Weyl differintegration schemes, respectively [2, 3].

In the following, we consider the causal case and use the notation $D^\alpha x(t) = x^{(\alpha)}(t)$, where $x(t)$ is an exponential order signal with $X(s)$ as the LT. The differintegration enjoys some interesting properties [2–6], we present three of them.

2.1.1 P1 Semi-group property:

$$D^\alpha D^\beta x(t) = x(t)*\delta^{(\alpha)}(t)*\delta^{(\beta)}(t) = x(t)*\delta^{(\alpha+\beta)}(t) = D^{\alpha+\beta}x(t) \qquad (8)$$

If $\alpha = -\beta$, we conclude that the fractional differentiation and the fractional integration are inverse operations, as expected.

Table 1: Differintegration in context of Laplace transform

	...	s^{-n}	...	s^{-2}	s^{-1}	1	s	s^2	...	s^n	...
Causal Re(s) > 0	...	$t^{n-1}u(t)/(n-1)!$	...	$tu(t)$	$u(t)$	$\delta(t)$	$\delta'(t)$	$\delta''(t)$	...	$\delta^{(n)}(t)$	...
Anti-causal Re(s) < 0	...	$-t^{n-1}u(-t)/(n-1)!$	...	$-tu(-t)$	$-u(-t)$	$\delta(t)$	$\delta'(t)$	$\delta''(t)$	...	$\delta^{(n)}(t)$	...

2.1.2 Differintegration in the transform domain:

$$LT[t^{\alpha}x(t)u(t)] = (-1)^{\alpha}X^{(\alpha)}(s) \quad (9)$$

with

$$X^{(\alpha)}(s) = \frac{\Gamma(\alpha+1)}{2\pi j}\int_{a-j\infty}^{a+j\infty}(z-s)^{-\alpha-1}X(z)\mathrm{d}z \quad 0 < a < \mathrm{Re}(s) \quad (10)$$

This is similar to the differintegration formula used by Campos [4, 5] and Nishimoto [6], and is a generalisation of the well known Cauchy formula.

2.1.3 Differintegration of the product of two functions:

$$D^{\alpha}[x(t)\cdot y(t)] = \sum_{0}^{\infty}\binom{\alpha}{k}x^{(\alpha-k)}(t)y(t)^{(k)}(t) \quad (11)$$

This is the Leibnitz' formula for the differintegration of the product. The binomial coefficients (α/k) are given by

$$\binom{\alpha}{k} = \frac{(-1)^k(-\alpha)_k}{k!} \quad (12)$$

where $(a)_k = a(a+1)(a+2)\ \ldots\ (a+k-1)$ is the Pochhammer's symbol.

Although in the deductions of the previous results we used the LT, its validity is wider, allowing their use with other functions, as is the case of the exponential defined for all the time. Since we have in mind the study of causal linear systems, we are going to consider mainly the definition introduced in eqn. 6.

Till now, we considered a differintegration of signals with the LT that exclude, for example, the sinusoids defined for all the time and other similar signals with a Fourier transform, but not the Laplace transform (or its region of convergence is degenerate). This means that we look for a sequence like that presented in eqn. 2 but with $j\omega$ instead of s. We must begin by noting that the FT of the signum function, sgn(t), is given by $2/j\omega$. With this, we can present Table 2.

Table 2 suggests the introduction of the following definition of differintegration of $\delta(t)$:

$$\delta_0^{(\alpha)}(t) = \frac{t^{-\alpha-1}\mathrm{sgn}(t)}{2\cdot\Gamma(-\alpha)} \quad (13)$$

This can be considered as the arithmetic mean of causal and anti-causal differintegrations, if we extend their regions of convergence to include the imaginary axis. This is the Rieman–Hilbert problem: to define a function on the boundary of the analyticity regions of two functions. We have

$$\begin{aligned} 2\cdot\Delta(\omega) &= 2\cdot FT[\delta_0^{(\alpha)}(t)] \\ &= FT[\delta_+^{(\alpha)}(t)] + FT[\delta_-^{(\alpha)}(t)] \\ &= \lim_{\mathrm{S}\in C^r\to \mathrm{j}\omega} s^{\alpha} + \lim_{s\in C^l\to \mathrm{j}\omega} s^{\alpha} \end{aligned} \quad (14)$$

where C^r and C^l represent the right and left half complex planes. This immediately gives $\Delta(\omega) = (j\omega)^{\alpha}$.

Before finishing here, we must consider the application fields of the second approach. It is not hard to see that if the function has the LT convergent in a strip that includes the imaginary axis, both definitions give the same result. If the strip degenerates in the imaginary axis, as is the case of many bilateral signals, the function has the FT (at least in the generalised sense) and we must use eqn. 12. Of course, in applications to the study of causal systems, we must use the approach based on the LT.

2.2 Differintegration of periodic functions

Now we have the problem of the differintegration of a periodic function, $x_p(t)$. These functions do not have the LT, but they have the FT. A given periodic function, with period T, can be considered as a sum of delayed versions of a given basic wavelet. Mathematically, we have

$$x_p(t) = \sum_{-\infty}^{\infty}x(t-nT) = x(t)*\sum_{-\infty}^{\infty}\delta(t-nT) \quad (15)$$

leading to

$$D^{\alpha}x_p(t) = \sum_{-\infty}^{\infty}x_0^{(\alpha)}(t-nT) = x(t)*\sum_{-\infty}^{\infty}\delta_0^{(\alpha)}(t-nT) \quad (16)$$

This shows that the differintegrator of a periodic function can be obtained by convolving the wavelet with the differintegrator of the comb signal. To obtain the corresponding Fourier series, we must find the differintegration of the exponential $e^{j\omega_0 t}$ for all $t\in R$. The FT of this function is $2\pi\delta(\omega-\omega_0)$. This leads us to conclude that its α order fractional differintegrator is the inverse FT of $2\pi\cdot(j\omega)^{\alpha}\cdot\delta(\omega-\omega_0) = 2\pi\cdot(j\omega_0)^{\alpha}\cdot\delta(\omega-\omega_0)$. The inverse of this transform is

$$[e^{j\omega_0 t}]^{(\alpha)} = (j\omega_0)^{\alpha}e^{j\omega_0 t} \quad (17)$$

Using the FT in eqn. 16, we have

$$\begin{aligned} (j\omega)^{\alpha}X_p(\omega) &= \mathrm{X}(\omega)\cdot(j\omega)^{\alpha}\sum_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{j}\omega nT} \\ &= \frac{2\pi}{T}\cdot\sum_{-\infty}^{\infty}X(\omega)\cdot(j\omega)^{\alpha}\delta\left(\omega-n\frac{2\pi}{T}\right) \\ &= \frac{2\pi}{T}\cdot\sum_{-\infty}^{\infty}X\left(n\frac{2\pi}{T}\right)\cdot\left(jn\frac{2\pi}{T}\right)^{\alpha}\delta\left(\omega-n\frac{2\pi}{T}\right) \end{aligned} \quad (18)$$

If $\alpha<0$, we must assume a zero mean value periodic function $X(0)=0$. Inverting the FT and putting $\omega_0 = 2\pi/T$, we obtain

$$D^{(\alpha)}x_p(t) = \sum_{-\infty}^{+\infty}(\mathrm{j}n\omega_0)^{\alpha}\cdot C_n\mathrm{e}^{\mathrm{j}n\omega_0 t} \quad (19)$$

with evident meaning for C_n, thus generalising a well known result of the ordinary Fourier series.

Table 2: Differintegration in context of Fourier transform

...	$(j\omega)^{-n}$	...	$(j\omega)^{-3}$	$(j\omega)^{-2}$	$(j\omega)^{-1}$	1	$(j\omega)$	$(j\omega)^2$	...	$(j\omega)^n$	...
...	$(1/2)[t^{n-1}\mathrm{sgn}(t)]/(n-1)!$	...	$(1/2)(t^2/2)\mathrm{sgn}(t)$	$(1/2)t\cdot\mathrm{sgn}(t)$	$(1/2)\mathrm{sgn}(t)$	$\delta(t)$	$\delta'(t)$	$\delta''(t)$	...	$\delta^{(n)}(t)$	...

2.3 Unilateral Laplace transform and initial conditions

The solution of fractional differential equations can be done using the unilateral LT. As in the ordinary equations, this transform introduces the initial conditions naturally. For this, we need a generalisation of the usual theorem of the LT of the derivative. Let $x(t)$ be a signal and $X(s)$ its unilateral LT. It is known that

$$LT[D^n x(t)] = s^n X(s) - \sum_{i=0}^{n-1} s^{n-1-i} x^{(i)}(0) \qquad (20)$$

To generalise this result, for any order $\alpha > 0$, let m be the least natural number greater than or equal to α: $m \geq \alpha$. $D^{-(m-\alpha)}$ then corresponds to integration and multiplication by $s^{-(m-\alpha)}$ in the transformed domain. We have successively [3]

$$\begin{aligned} LT[D^\alpha x(t)] &= LT[D^m[D^{-(m-\alpha)}x(t)]] = s^m LT[D^{-(m-\alpha)}x(t)] \\ &\quad - \sum_{i=0}^{m-1} s^{m-i-1} D^i[D^{-(m-\alpha)}x(t)]_{t=0} \\ &= s^m[s^{-(m-\alpha)}X(s)] - \sum_{i=0}^{m-1} s^{m-i-1} D^{i-(m-\alpha)}x(0) \\ &= s^\alpha X(s) - \sum_{i=0}^{m-1} s^{m-i-1} D^{i-m+\alpha}x(0) \qquad (21) \end{aligned}$$

which is the desired result (this can also be obtained from eqn. 11, by putting $y(t) = u(t)$.) With this result, we can deduce generalisations of the initial and final value theorems. For the former, we obtain

$$x^{(\alpha-1)}(0) = \lim_{s\to\infty} s^\alpha X(s) \quad \text{Re}(s) > 0 \qquad (22)$$

For the final value, we have

$$x^{(\alpha-1)}(\infty) = \lim_{s\to 0} s^\alpha X(s) \quad \text{Re}(s) > 0 \qquad (23)$$

The proofs are very similar to the usual, and so we do not present them.

2.4 Differintegration of the causal exponential

This case is exceptionally important, since it appears in the computation of the impulse and step responses of linear causal systems. We do not consider the case of integer differintegration. Here, we have

$$\begin{aligned} D^{(\alpha)}[e^{at} \cdot u(t)] &= \frac{1}{\Gamma(-\alpha)} \int_0^t (t-\tau)^{-\alpha-1} e^{a\tau} d\tau \\ &= e^{at} \frac{1}{\Gamma(-\alpha)} \int_0^t \tau^{-\alpha-1} e^{-a\tau} d\tau \quad t \geq 0 \qquad (24) \end{aligned}$$

We represent this function by $E_\alpha(t,a)$, with $E_0(t,a) = e^{at}u(t)$ and $E_\alpha(t,0) = \delta^{(\alpha-1)}(t)$. The study of this function is normally done in terms of the incomplete gamma function [1, 2]. Here, we use the power series approach.

The generalised Leibnitz' rule allows us to obtain directly a power series for $E_\alpha(t,a)$. We only have to make $x(t) = u(t)$ and $y(t) = e^{at}$ in eqn. 11. As $u^{(\alpha)}(t) = \delta^{(\alpha-1)}(t) = t^{-\alpha}/\Gamma(1-\alpha)$, (another power series for this function is presented elsewhere [2]) then,

$$E_\alpha(t,a) = e^{at} \sum_0^\infty \binom{\alpha}{k} \frac{t^{-\alpha+k}}{\Gamma(-\alpha+k+1)} a^k \qquad (25)$$

This gives, after some manipulation

$$E_\alpha(t,a) = t^{-\alpha} e^{at} \frac{1}{\Gamma(-\alpha)} \sum_0^\infty \frac{(-1)^k (at)^k}{k! k - \alpha} u(t) \qquad (26)$$

This equation shows that all the non-integer derivatives of the causal exponential are not continuous at the origin and go to infinity. This has important implications in applications to linear systems. The LT of this function is easily obtained, following the theory developed in Section 2.2, and is

$$LT[E_\alpha(t,a)] = \frac{s^\alpha}{s-a} \quad \text{Re}(s) > \text{Re}(a) \qquad (27)$$

As $\text{Re}(s) > \text{Re}(a)$, we can always work in the region $|s| > |a|$, and so eqn. 27 can be written

$$LT[E_\alpha(t,a)] = \sum_0^{+\infty} a^n s^{\alpha-n-1} \quad |s| > |a| \qquad (28)$$

This expresses a special case of the more general result known as Hardy's theorem [25]:

Let the series

$$F(s) = \sum_0^{+\infty} a_n \Gamma(\alpha+n+1), s^{-\alpha-n-1} \qquad (29)$$

be convergent for some $\text{Re}(s) > s_0 > 0$ and $\alpha > -1$. The series

$$f(t) = \sum_0^{+\infty} a_n t^{\alpha+n} \qquad (30)$$

converges for all $t > 0$ and $F(s) = LT[f(t)]$.

3 Fractional continuous-time linear systems

3.1 Description

The most common and useful continuous-time linear systems are the lumped parameter systems, described by linear differential equations. The simplest of these systems are the integrators, differentiators and constant multipliers (amplifiers/attenuators). The referred lumped parameter linear systems are associations (cascade, parallel or feedback) of those simple systems. Here, we study the systems that result from the use of fractional differentiators or integrators, and that are described by linear fractional differential equations. We assume that the coefficients of the equation are constant, and so the corresponding system is a fractional linear time-invariant (FLTI) system. With this definition, we are ready to define and compute the impulse response and transfer function.

We therefore consider FLTI systems described by a differential equation with the general format

$$\sum_{n=0}^{N} a_n D^{\nu_n} y(t) = \sum_{m=0}^{M} b_m D^{\nu_m} x(t) \qquad (31)$$

where D is the derivation operator and ν_n are the orders of differintegration that, in the general case, are complex numbers. Here, we assume they are positive real numbers. As usual, we apply the LT to eqn. 31, obtaining easily

$$H(s) = \frac{\sum_{m=0}^{M} b_m s^{\nu_m}}{\sum_{n=0}^{N} a_n s^{\nu_n}} \qquad (32)$$

which is the transfer function, provided that $\mathrm{Re}(s) > 0$ or $\mathrm{Re}(s) < 0$. If we use the FT instead of the LT, we would obtain the frequency response, $H(j\omega)$, and could represent the Bode diagrams as in usual systems. It is interesting to note that the asymptotic amplitude Bode diagrams constitute straight lines with slopes that, at least in principle, may assume any value; this is in contrast to the usual case, where the slopes are multiples of 20 dB/decade. To obtain the frequency response directly from eqn. 32, we must proceed as in Section 2.1. The result would be that achieved by letting s go to $j\omega$ in eqn. 32.

3.2 From the transfer function to the impulse response

To obtain the impulse response from the transfer function, we proceed almost as usual. However, we must be careful. Let us begin by considering the simple case of a differintegrator:

$$H(s) = s^{\alpha} \quad \alpha \neq 0 \tag{33}$$

s^{α} is a multi-valued expression defining an infinite number of Riemann surfaces. Each Riemann surface defines one function. Therefore, eqn. 33 can represent an infinite number of linear systems. However, only the principal Riemann surface $\{z: -\pi \leq \arg(z) < \pi\}$ may lead to a real system. Constraining this function by imposing a region of convergence, we define a transfer function. Eqn. 6 or 7 gives the corresponding impulse response.

Now, go a step further and consider the simple case corresponding to the fraction

$$H(s) = \frac{1}{(s^{v} - a)} \quad v > 0 \tag{34}$$

where v is a real number. The equation $s^{v} = a$ has infinite solutions on a circle of radius $|a|^{1/v}$. However, in the general case, we cannot ensure the existence of one pole in the principal Riemann surface. This is not the case for $0 < v \leq 1$. In this case, we may have one pole on that branch. For this reason, we focus on the following cases:

(a) v_n are rational numbers that we write in the form p_n/q_n. Let q be the least common multiple of the q_n; then $v_n = n/q$, where n and q are positive integer numbers. So, $v_n = n \cdot v$, with $v = 1/q$ (a differential equation with $v = 1/2$ is semi-differential [28]). The coefficients and orders do not coincide necessarily with the previous ones, since some of the coefficients can be zero: for example

$$[aD^{1/3} + bD^{1/2}]y(t) = x(t)$$

transforms into:

$$[bD^{3.1/6} + aD^{2.1/6} + 0 \cdot D^{1/6}]y(t) = x(t) \tag{35}$$

(b) v_n are irrational numbers but multiples of a $(0 < v < 1)$

Eqns. 31 and 32 then assume the general forms

$$\sum_{n=0}^{N} a_n D^{nv} y(t) = \sum_{m=0}^{M} b_m D^{mv} x(t) \tag{36}$$

and

$$H(s) = \frac{\sum_{m=0}^{M} b_m s^{mv}}{\sum_{n=0}^{N} a_n s^{nv}} \tag{37}$$

With a transfer function as in eqn. 37 we can perform the inversion quite easily, by following the steps below:

(i) transform $H(s)$ into $H(z)$, by substitution of s^{v} for z [we are assuming that $H(z)$ is a proper fraction; otherwise, we have to decompose it in a sum of a polynomial (inverted separately) and a proper fraction.]
(ii) the denominator polynomial in $H(z)$ is the indicial polynomial [3] or characteristic pseudo-polynomial [18]; perform the expansion of $H(z)$ in partial fractions (we must use only the zeros of the indicial polynomials that are really in the principal Riemann surface.)
(iii) substitute back s^{v} for z, to obtain the partial fractions in the form

$$F(s) = \frac{1}{(s^{v} - a)^{k}} \quad k = 1, 2, \ldots \tag{38}$$

(iv) invert each partial fraction.
(v) add the different partial impulse responses.

3.3 Partial fraction inversion

3.3.1 Rational case: We proceed to the inversion of the partial fraction (eqn. 38), considering first the $k = 1$ and $v = 1/q$ case. Using the well known result referring the sum of the first q terms of a geometric sequence, we obtain

$$F(s) = \frac{1}{(s^{v} - a)} = \frac{\sum_{j=1}^{q} a^{j-1} s^{1-jv}}{s - a^{q}} \tag{39}$$

because $r = b/x$, we obtain

$$\sum_{j=0}^{q-1} r^{j} = \frac{1 - r^{q}}{1 - r} \Rightarrow \sum_{j=0}^{q-1} b^{j} \cdot x^{-j} = \frac{1 - b^{q} \cdot x^{-q}}{1 - b/x} \text{ or } x^{q} - b^{q}$$
$$= (x - b) \cdot \sum_{j=1}^{q} b^{j-1} \cdot x^{q-j}$$
$$\text{from where } \frac{1}{x - b} = \frac{\sum_{j=1}^{q} b^{j-1} \cdot x^{q-j}}{x^{q} - b^{q}} \tag{40}$$

We conclude that the LT inverse of a partial fraction as $F(s) = 1/s^{1/q} - a$ is a linear combination of q fractional derivatives of $E_0(t, a^{q}) = e^{a^{q} \cdot t} \cdot u(t)$:

$$f(t) = \frac{1}{a} \cdot \sum_{j=1}^{q} a_j \cdot E_{1-jv}(t, a^{q}) \tag{41}$$

The $k > 1$ case in eqn. 38 does not present great difficulties, except some additional work. We can use eqn. 39 repeatedly and the convolution to solve the problem. Alternatively, we can differentiate. For example

$$\frac{1}{(s^{v} - a)^{2}} = \frac{1}{v} s^{1-v} \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{s^{v} - a}\right] \tag{42}$$

$$LT^{-1}\left[\frac{1}{(s^{v} - a)^{2}}\right] = \frac{1}{v} D^{1-v}[t^{v} \cdot f(t)] \tag{43}$$

with $f(t)$ given by eqn. 41. We do not proceed further, since this example shows how we can proceed in the general case.

3.3.2 Irrational case: Consider now the case corresponding to v an irrational number, $0 < v < 1$. To invert eqn. 38 when v is irrational, we can use a known result [4]: the LT of the Mittag–Leffler function

$$\psi(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(nv + l)} \cdot u(t) \qquad (44)$$

is given by

$$\Psi(s) = \frac{s^{v-1}}{s^v - 1}, \quad \mathrm{Re}(s) > 1 \qquad (45)$$

Therefore, in terms of this function, it is not hard to conclude that F(s) is the LT of the $(1 - v)$th order derivative of the function $\psi(\mathrm{at}^v)$

$$f(\mathrm{t}) = \mathrm{D}^{1-v}[\psi(at^v)] \qquad (46)$$

The Mittag-Leffler function can be considered as a generalisation of the exponential with which it coincides when $v = 1$. Eqn. 41 (and eqn. 45) suggests that we work with the step response instead of the impulse response, to avoid derivatives or working with non-regular functions near the origin.

3.4 Stability of FLTI continuous-time systems

Our study of the stability of the FLTI systems is based on the BIBO (bound input, bound output) stability criterion, which implies stability when the impulse response is absolutely integrable.

The simplest FLTI system is the system with transfer function $H(s) = s^v$, where s belongs to the principal Riemann surface. If $v > 0$, the system is definitely unstable, since the impulse response is not absolutely integrable, even in a finite interval. If $-1 < v < 0$, the impulse response remains a limited function when t increases indefinitely and it is absolutely integrable in every finite interval. Therefore, we say that the system is wide-sense stable. This case is interesting to the study of the fractional stochastic processes. If $v = -1$, the normal integrator, the system is wide-sense stable. The case $v < -1$ corresponds to an unstable system, since the impulse response is not a limited function when t goes to $+\infty$.

These considerations tell us that a system with polynomial transfer function $P_K(s^v) = \Sigma_{i=0}^{K} a_i s^{vi}$ is stable if $|K \cdot v| < 1$. Thus, in the case of a pole-zero system, with N poles, the number of zeros must be less than $N + 1/v$. Consider the LTI systems with transfer function $H(s)$ a quotient of two polynomials in s^v. Let us go back to the partial fraction (eqn. 38) with $v = 1/q$ and $k = 1$, which was transformed into eqn. 39. The poles are therefore integer powers of complex numbers. The transformation $w = z^q$ transforms the sector $0 \leq \theta \leq 2\pi/q\{\theta = \arg(z)\}$ into the entire complex plane. The sector $\pi/2q \leq \theta \leq \pi/2q + \pi/q$ is then transformed in the left half-plane [36]. It is not difficult to show that the same happens to the sectors obtained by rotating the previous one by an angle $i \cdot 2\pi/q$ $(i = 1, \ldots, q-1)$. Then, if we let θ be the argument of a in eqn. 39. we have stability if the poles are in one of the sectors $\pi/2q + 2\pi/q\, i \leq \theta \leq \pi/2q + \pi/q + (2\pi/q)\, i$ $(i = 0, \ldots, q-1)$. Thus, the impulse responses of these kind of linear systems are linear combinations of functions of the type represented in eqn. 41. These functions are not regular at the origin, where they are proportional to $t^{-1/q}$, but the impulse responses are absolutely integrable functions, leading to regular step response functions. Thus, these systems are stable in the usual meaning. If v is not rational, the situation is similar, provided that $0 < v < 1$. It is not hard to see that the region of stability is defined by $(\pi/2) \cdot v + 2\pi v i \leq \theta \leq \pi/2v + \pi v + 2\pi v i$ $(i = 0, \ldots, N-1)$, with $N = \lceil 1/v \rceil$.

3.5 Initial conditions and free response

In the previous Section, we have described the procedure to find the impulse response of a linear fractional system. Here we consider the response of the system corresponding to a given set of initial conditions. Apply the unilateral LT to both members of eqn. 37. With the help of eqn. 20 and after rearranging the terms, we obtain as the LT of the free response

$$Y_f(s) = \frac{C(s)}{A(s^v)} \qquad (47)$$

where $A(s^v)$ is the polynomial in the denominator of eqn. 37, and $C(s) = \Sigma_{i=0}^{I} c_i s^i$, with I as the smallest integer greater than or equal to $\max(N, M)v - 1$; the coefficients c_i $(i = 0, 1, \ldots, I)$ are linear combinations of the fractional derivatives of the input and output at $t = 0$. To have an idea about the behaviour of $y_f(t)$, let $h_a(t)$ be the inverse LT of $1/A(s^v)$. From eqn. 47 we conclude that $y_f(t)$ is a linear combination of $h_a(t)$ and its I integer derivatives $D^i[h_a(t)]$. Using the initial value theorem eqn. 20, we conclude that $y_f(t)$ goes to zero as $t \to \infty$. In general, $y_f(t)$ is irregular at $t = 0$, because the numerator in eqn. 47 has a higher degree than the denominator.

3.6 Practical example

The 'single-degree-of-freedom fractional oscillator' consists of a mass and a fractional Kelvin element, and it is applied in viscoelasticity theory. The equation of motion is [20]

$$mD^2x(t) + cD^\alpha x(t) + kx(t) = f(t) \qquad (48)$$

where m is the mass, c the damping constant, k the stiffness, x the displacement and f the forcing function. Fenander [21] includes another fractional term in eqn. 48, leading to a different transfer function. Eqn. 48 has been studied [7, 11, 23], but its solution was not found explicitly. The impulse response has been given implicitly as an integral [11]; approximate solutions have been proposed [3], allowing the computation of the most important parameters: damping factor and damped natural frequency. To solve eqn. 48 Koh and Kelly [7] use numerical schemes. However, none of these methods solves effectively eqn. 48. We are going to do this. Let us introduce the parameters $\omega_0 = \sqrt{(k/m)}$ as the undamped natural frequency of the system and $\zeta = c/2m\omega_0^{2-\alpha}$.

Following the work of Koh and Kelly [7], we rewrite eqn. 48 in the form

$$D^2x(t) + 2\omega_0^{2-\alpha}\zeta D^\alpha x(t) + \omega_0^2 x(t) = f(t) \qquad (49)$$

Let $\alpha = 1/2$, and apply the LT. The transfer function is

$$H(s) = \frac{1}{s^2 + 2\omega_0^{2-\alpha}\zeta s^\alpha + \omega_0^2} \qquad (50)$$

with indicial polynomial $s^4 + 2\omega_0^{3/2}\zeta s + \omega_0^2$. Its roots can be found by a standard procedure, but it is rather difficult to reach useful conclusions. However, as the coefficients in s^3 and s^2 are zero, we can conclude that four roots are on two vertical straight lines with symmetric abscissas, but only two belong to the first Riemann surface. For example, with $\omega_0 = 1$ rad/s and $\zeta = 0.05$, the roots are $s_1 = 0.7073 + j0.7319$, $s_2 = 0.7073 - j0.7319$, $s_3 = -0.7073 + j0.6819$, and $s_4 = -0.7073 - j0.6819$, but only s_1 and s_2 belong to the first Riemann surface. Using the results in Section 3.3.1, we obtain the impulse response as

$$h(t) = \mathrm{Re}\{r \cdot s_1 \cdot e^{(-0.0354+j1.0353)t}u(t) + D^{1/2}[r \cdot e^{(-0.0354+j1.0353)t}u(t)]\} \quad (50)$$

where r is the residue at s_1. Fig. 1 presents the results of solving eqn. 49 for $\alpha = 1$, 1/2, and 2/3, with $\omega_0 = 1$ rad/s and $\zeta = 0.05$. However, Fig. 1 does not show, for $\alpha \neq 1$, $h(t)$ is not regular for $t = 0$.

3.7 State-space formulation

In some applications, e.g. control, the state-space formulation is very important [18, 27]. It is not hard to obtain it from eqn. 36. It can be written for the time-variant case as $\boldsymbol{s}^{(\nu)}(t) = \boldsymbol{A}(t)$, $\boldsymbol{s}(t) + \boldsymbol{B}(t)$, $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t) = \boldsymbol{C}(t)$, $\boldsymbol{s}(t) + \boldsymbol{D}(t)$, $\boldsymbol{x}(t)$. To solve the dynamic equation, it is necessary to introduce the fractional state transition operator $\Phi(t, \tau)$, which is a generalisation of the usual state transition operator. Heuristically, we could conclude that the required operator can be represented by the usual Peano–Baker series with a substitution of a ν-order integration for the usual one. In the time-invariant case, this operator is related to the Mittag–Leffler function. However, it is very difficult to manipulate. In addition, it does not enjoy all the features of the ordinary one, i.e. the semi-group property even in the time-invariant case, in fact

$$\begin{aligned} f(0, t) &= \int_0^t (t-\tau)^{-\alpha-1} d\tau \\ &= \int_0^{t1} (t-\tau)^{-\alpha-1} d\tau + \int_{t1}^t (t-\tau)^{-\alpha-1} d\tau \\ &\neq f(0, t_1) + f(t_1, t) \Rightarrow \Phi(0, t) \neq \Phi(0, t_1) \cdot \Phi(t_1, t) \end{aligned} \quad (51)$$

This has a very important consequence: the operator $\Phi(\tau, t)$ is not the inverse operator of $\Phi(t, \tau)$. Such an inverse operator might be obtained with the help of the anti-causal differintegration operator: this must be a subject for further research.

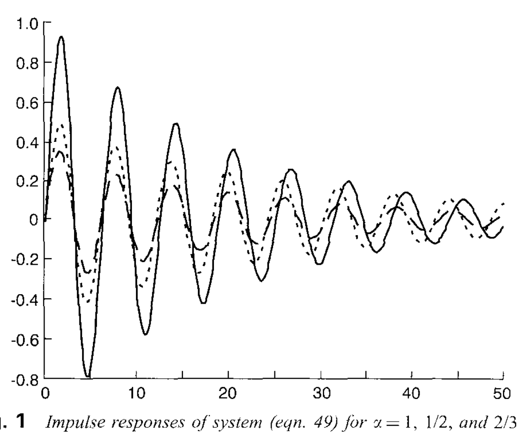

Fig. 1 *Impulse responses of system (eqn. 49) for $\alpha = 1$, 1/2, and 2/3*
—— $\alpha = 1$
····· $\alpha = 1/2$
– – – $\alpha = 2/3$

4 Fractional integrators and fractal signals

In the previous Section we have studied fractional linear systems described by fractional differential equations (eqn. 36) and having transfer functions with expansions in partial fractions with the general format (eqn. 38).

Now we generalise these results by introducing the notions of fractional order pole and similarly fractional order zero. In terms of systems, we consider systems with transfer function

$$H(s) = (s - a)^{\nu} \quad (52)$$

corresponding to a ν-order zero if $\nu > 0$, and to a ν-order pole if $\nu < 0$. Assuming the interest in the causal case $\mathrm{Re}(s) > 0$, the corresponding impulse response is given by

$$h(t) = e^{-at} \cdot \delta^{(\nu)}(t) \quad (53)$$

where $\delta^{(\nu)}(t)$ is the ν-order differintegrator of $\delta(t)$. The special cases obtained by putting $a = 0$ are very important: it is a differintegrator (fractional differentiator if $\nu > 0$, and integrator if $\nu < 0$). They have been used in fractal modelling [21, 22] and 1/f noise [27, 28], and can be represented by one of the fractional differential equations

$$\frac{d^{-\nu} y(t)}{dt^{-\nu}} = x(t) \text{ or } y(t) = \frac{d^{\nu} x(t)}{d^{\nu}} \quad (54)$$

According to the usual BIBO stability criterion, both systems are always unstable. However, if $-1 \leq \nu < 0$, we say that the corresponding fractional integrator is a wide-sense stable system, as seen in Section 3.4.

We define fractional stochastic process as the output of a fractional system. Let $h(t)$ be the impulse response of the system. The signal

$$x(t) = h(t) * w(t) \quad (55)$$

is a fractional stochastic process, where $w(t)$ is a stationary white noise. Among these kinds of signal, the 1/f noise are of special importance. Keshner [14] refers to several examples of 1/f noise. From his considerations, we may conclude that those signals seem to belong to one of two types: those with the spectrum of the form $1/f^{\gamma}$ for every f, and those with that form only for f above a given value. This means that the former can be considered as the output of a differintegrator, and the second may be the output of a low-pass fractional system with a pole near the origin. This case is similar to the ordinary one. In the following, we deal with the first case, with the system defined by the transfer function $H(s) = s^{\nu}$ (with s belonging to the principal Riemann surface) and its impulse response

$$h(t) = \delta^{(\nu)}(t) = \frac{t^{-\nu-1}}{\Gamma(-\nu)} \cdot u(t) \text{ with } -1 \leq \nu < 0 \quad (56)$$

to have a stable system. Let $w(t)$ be a continuous-time stationary white noise with variance σ^2. We call α-order hyperbolic noise, $r_{\alpha}(t)$, the output of an α-order integrator, $-1/2 < \alpha < 0$, when the input is white noise, $w(t)$

$$r_{\alpha}(t) = \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^{t} w(\tau) \cdot (t-\tau)^{-\alpha-1} d\tau \quad (57)$$

where we put $r_0(t) = w(t)$. The name hyperbolic comes from the fact that $r_{\alpha}(t)$ is a wide-sense stationary stochastic process with a hyperbolic autocorrelation function (although the signal has an infinite power, its mean is constant and the autocorrelation function $E\{r_\alpha(t+\tau)\cdot r_\alpha(t)\}$ depends only on τ, not on t:

$$R(\tau) = \sigma^2 \frac{|\tau|^{-2\alpha-1}}{2\cdot\Gamma(-2\alpha)\cdot\cos\alpha\pi} \tag{58}$$

To obtain this function, we must compute the convolution

$$\begin{aligned} R(\tau) &= \sigma^2 h(\tau) * h(-\tau) \\ &= \sigma^2 \frac{1}{\Gamma^2(-\alpha)} \int_0^\infty t^{-\alpha-1}(t+\tau)^{-\alpha-1} u(t+\tau)\,dt \end{aligned} \tag{59}$$

This is readily reduced to

$$R(\tau) = \frac{\sigma^2}{\Gamma^2(-\alpha)} \begin{cases} \int_0^\infty t^{-\alpha-1}(\tau+t)^{-\alpha-1}\,dt & \text{if } \tau \geq 0 \\ \int_{|\tau|}^\infty t^{-\alpha-1}(t-|\tau|)^{-\alpha-1}\,dt & \text{if } \tau < 0 \end{cases} \tag{60}$$

To perform the computation, we make the substitutions $t+\tau=\tau/\xi$ in the first integral and $t=|\tau|/\xi$ in the second to obtain

$$R(\tau) = \sigma^2 \frac{|\tau|^{-2\alpha-1} B(1+2\alpha, -\alpha)}{\Gamma^2(-\alpha)} \tag{61}$$

where $B(x,y)$ is the beta function. As $B(x,y)=\Gamma(x)\cdot\Gamma(y)/\Gamma(x+y)$ [4] and $\Gamma(z)\cdot\Gamma(1-z)=\pi/\sin(\pi z)$, we obtain eqn. 58. This relation shows that we only have a (wide-sense) stationary hyperbolic noise if $-1/2 \leq \alpha < 0$. The other cases do not lead to a valid autocorrelation function of a stationary stochastic process, since it does not have a maximum at the origin. For example, for $0<\alpha<1/2$ (differentiator) $\Gamma(-2\alpha)$ is negative, which means that only for $\alpha \in]-1/2, 0[$ we obtain a stationary (wide-sense) stochastic process. This explains the negative results in trying to define a stationary fractional Brownian motion starting from the definition presented by Mandelbrot and Van Ness [9]: there it is assumed $\alpha=-\mathrm{H}-1/2$ with $0<H<1$, and so $-3/2<\alpha<-1/2$, which is outside the stationarity range. The above process is a somehow 'strange' process with infinite power. However, the power inside any finite frequency band is always finite. Its spectrum is

$$S(\omega) = \sigma^2 |\omega|^{2\alpha} \tag{62}$$

justifying the name '1/f noise'.

In the following, we generalise the notion of the Wiener–Lévy process (Brownian motion). Let $r_\alpha(t)$ be a continuous-time stationary hyperbolic noise. Define a process $v_\alpha(t)$, $t \geq 0$, by

$$v_\alpha(t) = \int_0^t r_\alpha(\tau)\,d\tau \tag{63}$$

If $-1/2<\alpha<0$, we call this process a generalised Wiener-Lévy process. This is also a generalisation of the ordinary Brownian noise that is obtained with $\alpha=0$.

For the proof, note that $v_\alpha(0)=0$ and $E\{v_\alpha(t)\}=0$ for every $t \geq 0$ (if $w(t)$ is a Gaussian white noise, and so it is $r_\alpha(t)$ and $v_\alpha(t)$). We only have to show that the increments

$$\Delta v_\alpha(t,s) = v_\alpha(t) - v_\alpha(s) = \int_s^t r_\alpha(\tau)\,d\tau \tag{64}$$

are stationary. The computation of the variance is slightly involved. We have

$$\begin{aligned} \operatorname{Var}\{\Delta v_\alpha(t,s)\} &= E\left[\int_s^t r_\alpha(\tau)\,d\tau \int_s^t r_\alpha(\tau')\,d\tau'\right] \\ &= \int_s^t \int_s^t E[r_\alpha(\tau) r_\alpha(\tau')]\,d\tau\,d\tau' \end{aligned} \tag{65}$$

Using eqn. 58, we obtain

$$\operatorname{Var}\{\Delta v_\alpha(t,s)\} = \sigma^2 \frac{1}{2\cdot\Gamma(-2\alpha)\cdot\cos\alpha\pi} \int_s^t \cdot \int_s^t |\tau-\tau'|^{-2\alpha-1}\,d\tau\,d\tau' \tag{66}$$

As $t>s\geq 0$ and with a variable change, it is a simple matter to obtain

$$\operatorname{Var}\{\Delta v_\alpha(t,s)\} = \sigma^2 \frac{1}{2\cdot\Gamma(-2\alpha)\cdot\cos\alpha\pi} \frac{2}{(t-s)} \int_{-(t-s)}^{(t-s)} |u|^{-2\alpha-1}\left[1-\frac{|u|}{(t-s)}\right] du \tag{67}$$

But, as (α is negative)

$$\begin{aligned} &\frac{2}{(t-s)} \int_{-(t-s)}^{(t-s)} |u|^{-2\alpha-1}\left[1-\frac{|u|}{(t-s)}\right] du \\ &= \frac{2}{-2\alpha(-2\alpha+1)} (t-s)^{-2\alpha} \end{aligned} \tag{68}$$

and

$$\Gamma(-2\alpha)\cdot(-2\alpha)\cdot(-2\alpha+1) = \Gamma(-2\alpha+2)$$

we obtain

$$\operatorname{Var}\{\Delta v_\alpha(t,s)\} = \sigma^2 \frac{(t-s)^{-2\alpha+1}}{\Gamma(-2\alpha+2)\cdot\cos\alpha\pi} \tag{69}$$

This result confirms that the increments are stationary. In particular, with $t=s+T$, we obtain

$$\operatorname{Var}\{\Delta v_\alpha(s+T,s)\} = \sigma^2 \frac{T^{-2\alpha+1}}{\Gamma(-2\alpha+2)\cdot\cos\alpha\pi} \tag{70}$$

and

$$\operatorname{Var}\{\Delta v_\alpha(s+T,s)\} = \sigma^2 \frac{T^{2H}}{\Gamma(2H+1)\cdot\sin H\pi} \tag{71}$$

by putting $H=-\alpha+1/2$ [20]. As $\alpha \in]-1/2, 0[$, $H \in]1/2, 1[$, obtaining a similar formula. With these values for H and $\alpha=-H+1/2$, we are confirming the considerations of Mandelbrot and Van Ness [9] (with α and H as above, the stochastic process $B_H(t,\omega)$, defined in definition 2.1 [9], will be surely differentiable, in contradiction with proposition 4.2). These results are also in agreement with the theory and results developed by Reed *et al.* [26]. We note that, contrary to ordinary Brownian motion, $v_\alpha(t)$ does not have independent increments.

5 Conclusions

In this paper, we have presented a class of linear systems: the fractional continuous-time linear systems. For the definitions of these systems, we have introduced the definitions of fractional derivative. These definitions allowed us to present an approach very similar to that used in the study of ordinary linear systems; we were led to the notions of fractional impulse and frequency responses. We have shown how to compute them.

We have also studied briefly the stability of these systems, introduced the state-space representation, and introduced the fractal signals as outputs of fractional systems. In particular, we have studied the outputs of differintegrators, the fractals, and shown how to define a stationary fractal.

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