# Fractional Discrete-Time Signal Processing: Scale Conversion and Linear Prediction[1]

by

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**Abstract**

A generalisation of the linear prediction for fractional steps is reviewed, widening well-known concepts and results. This prediction is used to derive a causal interpolation algorithm. A reconstruction algorithm for the situation where averages are observed is also presented. Scale conversion of discrete-time signals is studied taking as base the fractional discrete-time system theory. Some simulation results to illustrate the behaviour of the algorithms will be presented. A new algorithm for performing the zoom transform is also described.

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## 1. INTRODUCTION

In the following we shall be concerned with the study of the linear systems described by fractional difference equations [1,2]. We use the unit delay [3] with discrete-time Fourier Transform (FT) equal to $e^{j\omega\alpha}$, enabling to generalise, with the help of convolution, the usual translation property of the FT. With the Cauchy integrals we can treat the causal systems described by linear fractional difference equations, enabling us to define Transfer Function and Impulse Response [1]. Here we go further and enlarge the scale change property of the FT to the fractional case and its dual property that we use to perform a spectral zoom. This algorithm is simpler than the available in literature [4]. Essentially, it consists on a matrix computation and its multiplication by the vector of the DFT values.

The relevance of linear prediction in modern Signal Processing is a well-established fact. The one-step prediction has several practical applications, namely in Telecommunications and Speech Processing, for example, sampling rate conversion, equalization, and speech coding and recognition [5]. The d-step prediction (d positive integer) is useful in Geophysical Signal Processing and Economy. Here we generalise the concept for fractional steps. The basic idea underlying the proposed algorithm is to develop a system capable of linear predicting the signal over time instants, between the current ones, without converting the signal to the continuous-time domain. The new samples fit in between the original samples. This is intended to overcome the fact that the translation and scale change are done by non-causal and infinite duration operators. The practical algorithm we propose uses the Maximum Entropy Method to obtain the spectrum of the original integer sampled signal [6]. Using this spectrum estimate, it is possible to derive the coefficients of the fractional predictor [7]. The simulations present in this work will show that, from the fractional linear prediction method, it is possible to perform the interpolation of a given signal and also a scale change. Here, we are going to present that algorithm together with some simulation results illustrating its behaviour. This algorithm is suitable for the interpolation of stationary stochastic processes. This means that when dealing with pulses the algorithm does not provide accurate samples. The disaggregation of signals observed through MA systems is also studied. This corresponds to the case where the signal is not observed, but rather we have access to moving averages over the signal. These weighted averages may be done by non invertible and/or non causal operators.

In Section 2, a fractional delay and lead concepts review takes place. These concepts are the base for the theory of fractional linear prediction that is described in subsequent Sections. In Section 3 we generalise the scale change property and in Section 4 its dual is used to perform the spectral zoom. The fractional linear prediction is presented in Section 5. An algorithm for the computation of the optimum predictor coefficients is proposed. This predictor is used to produce a signal interpolation. In Section 6 we derive similar algorithm to disaggregate signals observed

through MA systems. At last, some examples to illustrate the behaviour of the algorithms are presented and some conclusion are outlined.

## 2. FRACTIONAL DISCRETE-TIME SYSTEMS

In practical applications of discrete systems, most of the times we deal with signals that are sampled versions of continuous-time band-limited signals. Normally, these signals are processed synchronously in the sense that the time domain is the set of integer numbers for all of them. However, there are applications where this does not happen. We can process signals obtained by sampling continuous-time signals with the same sampling interval but in different time instants, e.g. the so-called time-delayed processes [8]. On the other hand, in other applications, we may need to know the behaviour of a system between the sampling instants [9]. The current application of multi-rate techniques allows the conversion of a signal sampled with a given sampling interval to another one with a different sampling interval. These considerations motivate the generalisation of the notion to allow for fractional delay and lead. Basically we extend the definition of the usual Kronecker delta to give sense to $\delta_{n-\alpha}$. We shall be assuming $\alpha$ to be any non-integer real. If $\alpha<0$ we have a fractional lead; if $\alpha>0$ we have a fractional delay. When $\alpha$ is integer we fall into the usual scheme. As shown in [1],$\delta_{n-\alpha}$ is the inverse Fourier Transform of $e^{-j\omega\alpha}$ and is given by:

$$\delta_{n-\alpha} = \frac{\sin[\pi(n-\alpha)]}{\pi(n-\alpha)} = \frac{\sin(\pi\alpha)}{\pi\alpha}\cdot\frac{(-1)^n}{1-\frac{n}{\alpha}} \qquad n\in Z \tag{1}$$

and

$$x_{n-\alpha} = \sum_{m=-\infty}^{+\infty} x_m \cdot \frac{\sin[\pi(n-\alpha-m)]}{\pi[n-\alpha-m]} \qquad \alpha\in\,]0,1] \quad n\in Z \tag{2}$$

Defining a FT $X_\alpha(e^{j\omega})$ by:

$$X_\alpha(e^{j\omega}) = \sum_{-\infty}^{\infty} x_{n-\alpha}\, e^{-jn\omega} \tag{3}$$

we conclude that

$$X_\alpha(e^{j\omega}) = e^{-j\omega\alpha}\, X(e^{j\omega}) \tag{4}$$

It is a simple matter to show that the above relations remain valid for every $\alpha\in R$. With relations (2) to (4) we are in conditions to give immediately a meaning to fractional difference equations and to introduce a **Frequency Response** for the systems represented by such equations. Consider the Linear Time Invariant Systems characterised by a fractional difference equation:

$$\sum_{i=0}^{N_0} a_i\, y(n - \nu_i) = \sum_{j=0}^{M_0} b_j\, x(n - \nu_j) \qquad n \in Z \tag{5}$$

where the $\nu_n$ are the delays. The corresponding frequency response function is:

$$H(z) = \frac{\sum_{j=0}^{M_0} b_j z^{-\nu_j}}{\sum_{i=0}^{N_o} a_i z^{-\nu_i}} \qquad z = e^{j\omega} \tag{6}$$

As referred in [1], the corresponding **Transfer Function** is defined from Equation (6) through a Cauchy integral and, in general, it does not have a closed form. In all what follows, we will assume that the $\nu_n$ are either rational numbers or are multiple of a common real, $\nu$. Then, the Equations (5) and (6) adopt the format:

$$\sum_{i=0}^{N_0} a_i\, y(n - i\nu) = \sum_{j=0}^{M_0} b_j\, x(n - j\nu) \qquad n \in Z \tag{7}$$

and

$$H(z) = \frac{\sum_{j=0}^{M} b_j z^{-j\nu}}{\sum_{i=0}^{N} a_i z^{-i\nu}} \qquad z = e^{j\omega} \tag{8}$$

We will give the name “Fractional Autoregressive Moving Average (FARMA) Systems”.

## 3. SCALE CONVERSION

In Equation (2) we stated a way of relating two sequences defined in two instant sets $t_n = n$ and $\tau_n = n + \alpha$. This had as consequence a generalisation of the translation property of the Fourier Transform – Equation (4). Now, we are going to generalise another property: the scale conversion. By this we mean a conversion from one time grid $t_n = nT$ (T is assumed to be 1) to $\tau_n = n\alpha T$ ($0<\alpha<1$). This is equivalent to ideally make a D/A conversion followed by a sampling with $\alpha T$ as sampling interval. However, our procedure is valid for every discrete-time signal, without needing to assume it as being obtained by sampling a continuous-time signal. In spectral terms this conversion maintains the shape of the spectrum, but narrows it.

Let us consider a signal $x_n$, with Fourier Transform $X(e^{j\omega})$, and a real constant $\alpha$ such that $0<\alpha<1$. Define a new function $X_\alpha(e^{j\omega})$ by:

$$X_\alpha(e^{j\omega})=\begin{cases} X(e^{j\omega}) & \text{if } |\omega| \le \pi \\ 0 & \text{if } \pi<|\omega|<\dfrac{\pi}{\alpha} \end{cases} \tag{9}$$

and repeat it with period $\frac{2\pi}{\alpha}$. Letting the coefficients of the corresponding Fourier Series be represented by $c_n$ and given by:

$$c_n = \frac{\alpha}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \tag{10}$$

with

$$X(e^{j\omega}) = \sum_{m=-\infty}^{+\infty} x_n\, e^{-j\omega n} \tag{11}$$

that inserted into Equation (10) and putting $x_{\alpha n}= c_n/\alpha$ allows us to obtain:

$$x_{\alpha n} = \sum_{k=-\infty}^{+\infty} x_k \frac{\sin[\pi(n\alpha-k)]}{\pi(n\alpha-k)} \tag{12}$$

Equation (9) means that,

$$\sum_{m=-\infty}^{+\infty} x_n\, e^{-j\omega n} = \alpha \sum_{m=-\infty}^{+\infty} x_{\alpha n}\, e^{-j\omega\alpha n} \qquad |\,\omega\,| < \pi \tag{13}$$

From this relation, can easily show that:

$$FT[x_{k\alpha}]=\frac{1}{\alpha}X(e^{j\omega/\alpha}) \tag{14}$$

Consider another real constant $\beta\neq\alpha$, satisfying also $0<\beta<1$. It is immediate to show that:

$$x_{\alpha n} = \beta \sum_{k=-\infty}^{+\infty} x_{\beta k}\frac{\sin[\pi(n\alpha-\beta k)]}{\pi(n\alpha-\beta k)} \tag{15}$$

Using (4) with $\beta$ in the place of $\alpha$, we obtain:

$$\frac{\sin[\pi(n\alpha-k)]}{\pi(n\alpha-k)} =\beta \sum_{m=-\infty}^{+\infty}\frac{\sin[\pi(\beta m-k)]}{\pi(\beta m-k)}\,\frac{\sin[\pi(n\alpha-\beta m)]}{\pi(n\alpha-\beta m)} \tag{16}$$

that is an interesting relation involving sinc functions. As seen, we can use Equation (12) – or Equation (16) – to perform a scale conversion. However, its usefulness is very limited since it cannot be used to perform a rate conversion as it is usually intended, due to the non-causality of the sinc and the slow converging series. On the other hand, the scale converted of a finite duration pulse does not have a finite duration.

If $\alpha>1$, the same procedure leads to:

$$x_{\alpha n} = \sum_{k=-\infty}^{+\infty} x_k \frac{\sin[\pi(n-k/\alpha)]}{\pi(n-k/\alpha)} \qquad (17)$$

that corresponds to an ideal lowpass filtering followed by a downsampling. In this case, the spectrum of the new signal has a different shape.

## 4. ZOOM TRANSFORM

The dual of the results obtained in the previous section have an interesting practical application: the zoom transform. Let us consider an L point sequence, $x_n$ n=0, …, L-1. Every N≥L point DFT sequence represent samples of the Discrete-Time Fourier Transform (DTFT). This sampling may hide some characteristics of the spectrum in a given particular band of interest. To avoid this problem two different methods of interpolation have been proposed [4,5] and usually referred as the zoom transform. Here, we propose an alternative approach. Let the DTFT of $x_n$ be $X(e^{j\omega})$. The DFT corresponds to sample $X(e^{j\omega})$:

$$DFT[x_n] = X(e^{j\frac{2\pi}{N}k}) \quad k=0, \ldots, N-1, N\geq L \qquad (18)$$

Denote this DFT by $X_N(k)$. Its inverse, $(DFT^{-1})$ is a N-period signal. If we take one period of this signal, add zeros and repeat the obtained sequence with a period $M=\alpha N$ ($\alpha>1$), we are sampling $X(e^{j\omega})$ at M uniformly spaced points, obtaining $X_M(k)$, k=0, …,M-1. Then, we have:

$$X_M(k) = \sum_{k=0}^{L-1} x_n e^{-j\frac{2\pi}{M}kn} \qquad k=0, ..., M-1 \qquad (19)$$

and

$$x_n = \frac{1}{N}\cdot \sum_{k=0}^{N-1} X_N(k).e^{j\frac{2\pi}{N}kn} \qquad n=0, ..., N-1 \qquad (20)$$

Inserting (20) into (19) we obtain:

$$X_M(k) = \frac{1}{N}\cdot \sum_{k=0}^{N-1} X_N(l).G(k,l) \quad k=0, ..., M-1 \qquad (21)$$

where

$$G(k,l) = \frac{1 - e^{j\frac{2\pi}{N}(l-k/\alpha)L}}{1 - e^{j\frac{2\pi}{N}(l-k/\alpha)}} \qquad (22)$$

for 0≤l<N and 0≤k<M. It is not hard to show that:

$$G(k,l) = L.\frac{\operatorname{sinc}\left[\frac{(l-k/\alpha)L}{N}\right]}{\operatorname{sinc}\left[\frac{l-k/\alpha}{N}\right]} e^{j\frac{\pi}{N}(l-k/\alpha)(L-1)} \quad (23)$$

Of course, we are not interested in zooming the whole spectrum, but a given band, corresponding to values of $k=m_1, \ldots, m_2$ with $m_1$ and $m_2$ as described below. Assume that we want to zoom the band $[f_1;f_2]$, with $0 \le f_1 < f_2 \le 1/2$. Let K be the number of points we want to compute. Then

$$\alpha = \frac{1}{(f_2-f_1)} \cdot \frac{K}{N} \quad (24)$$

and

$$m_i = \left\lfloor \frac{\alpha N}{2\pi} . f_i \right\rfloor \quad i=1,2 \quad (25)$$

where $\lfloor x \rfloor$ means the integer part of x. In the following Figure, we illustrate the results obtained by the application of the algorithm for zooming 2 regions of the spectrum shown in the upper strip of the Figure 1.

Figure 1

## 5. FRACTIONAL LINEAR PREDICTION

In Section 3 we showed how to change the scale of a signal, from a theoretical point of view. Here we will present an algorithm that can be used to do it. Essentially, we perform an interpolation taking as base the Fractional Linear Prediction [1,7]. This is a generalisation of the notion of linear prediction for any fractional d-step prediction ($d \in R$) and was proposed without details in [1]. We will now go into the details of this topic. We shall be working in the context of a stationary real stochastic process. Let x(n), $n \in Z$, be such a process and let $R_x(k)$ be its autocorrelation function.

**Definition 5.1**

Let x(n) be a real stationary stochastic process, observed from $-\infty$ to n-1. We define the Nth order d-step prediction at n-1+d by:

$$\hat{x}(n-1+d) = -\sum_{i=1}^{N} a_i \, x(n-i) \quad (26)$$

where $a_i$ (i=1, ..., N) are the coefficients of the d-step predictor (d=1, corresponds to the usual one-step prediction).

The predictor coefficients will be chosen in order to minimise the prediction error power:

$$P_d = E[\,(x(n-1+d) - \hat{x}(n-1+d))^2] \quad (27)$$

where x(n-1+d) can be defined by Equation (26).

**Theorem 5.1 -** *According to the previous definition and assuming that the correlation matrix of x(n) has, at least rank N, the optimum d-step predictor is given by the solution of the following set of Normal Equations[1,7]:*

$$\sum_{i=1}^{N} a_i.R_x(k-i) = -R_x(-k-d-1) \quad k=1,2,\ldots, N \tag{28}$$

In a matrix format, Equation (28) can be written:

$$\mathbf{R_x}.a=-\mathbf{r}_d \tag{29}$$

where it is evident the meaning of the vectors and matrix. The corresponding minimum error power is easily obtained by inserting Equation (28) in Equation (27) and it is given by:

$$P_{dmin} = R(0) + \sum_{i=1}^{N} a_i.R(-i-d+1) \tag{30}$$

Now, let $p_i^n$ (i=0,1, ..., n), with $p_0^n = 1$, be the nth order one-step linear predictor coefficients and $P^n(z)$ the corresponding Z Transform. As known, the predictors allow us to obtain the Cholesky factorisation for the inverse of the Toeplitz matrix, $\mathbf{R}_N$, in (29) [5,6]:

$$\mathbf{R}_N^{-1} = \mathbf{P}.\mathbf{D}.\mathbf{P}^H \quad ([1]) \tag{31}$$

where **P** is a lower triangular matrix having the one-step predictors as columns, **D** is a diagonal matrix with the inverses of the one-step prediction error powers.

The substitution of Equation (31) into Equation (29) allows us to express the a's in terms of the one-step predictor coefficients. It is not hard to show that:

$$\mathbf{a} = \mathbf{P}.\mathbf{v} \tag{32}$$

with **v** given by:

$$\mathbf{v} = \mathbf{D}.\mathbf{P}^H.\mathbf{r}_d \tag{33}$$

where $\mathbf{r}_d$ is the vector in the right hand side in Equation (29). To compute this vector we can use Equation (2):

$$R(-k-d+1) = R(k-1+d) = \frac{\sin(\pi d)}{\pi d}.\sum_{-\infty}^{\infty} R(n)\frac{(-1)^{k-1-n}}{\alpha+k-1-n} \tag{34}$$

But, as the autocorrelation function is an even function, we can transform the previous equation into:

$$R(-k-d+1) = R(k-1+d) = \frac{(-1)^{k-1}\sin(\pi d)}{\pi(\alpha+k-1)}.\left[R(0)+ 2\sum_{n=1}^{\infty}\frac{(-1)^n R(n)}{1-\left(\frac{n}{\alpha+k-1}\right)^2}\right] \tag{35}$$

Since the coefficients go to zero, at least quadratically, the series converges quickly, allowing its truncation.

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[1] H means conjugate transpose.

So, with Equations (29) and (35) we can compute the coefficients of the fractional predictor, provided that we use a suitable autocorrelation function estimate.

The use of the Z Transform in Equation (32) lead to an interesting result:

$$A(z) = \sum_{i=1}^{N} v_i . z^{-i} P^{N-i}(z) \qquad (36)$$

where $P^k(z)$ is the kth order prediction error filter transfer function and the $v_i$ are the components of the vector **v** defined in Equation (33). The result stated in Equation (36) is important, since it shows that the predicted value is a linear combination of all the forward prediction error signals { Figure 1} ([1]).

Figure 2

Assume now that x(n) is an AR(N-1) stationary stochastic process. Then the longest (with greater order) optimum fractional d-step predictor has order N [5,6].

This allows us to devise a better way to compute R(k+d). As the process is AR(N-1), the (N-1)th one-step predictor defines, together with the prediction error, $P_{N-1}$, the spectrum of the process [5,6]:

$$S_x(\omega) = \frac{P_{N-1}}{\left| \sum_{n=0}^{N-1} P_i^{N-1} \cdot e^{-jwn} \right|^2} \qquad (37)$$

that can be used to obtain R(k+d):

$$R(k+d) = FT^{-1}\left[ e^{j\omega d} S_x(\omega) \right] \qquad (38)$$

With these results we can take advantage of the well-known linear prediction methods (e. g. modified covariance or Burg algorithms) [6]. The proposed algorithm has the following steps:

1 - Compute the N-1 linear predictors using a suitable algorithm.

2 - Use the (N-1)th linear predictor to estimate the spectrum, $S_x(\omega)$, and the corresponding autocorrelation, of the signal.

3 - Multiply $S_x(\omega)$ by $e^{jd\omega}$ and compute the inverse Fourier Transform to obtain the vector $\mathbf{r}_{d.}$

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[1] Instead of the Cholesky factorization we could use the Gohberg-Semencul formula [6]. In this case, A(z) would be expressed in terms of the (N-1)th order forward and backward predictors, only, but the "coefficients" of the linear combination would be polynomials in z. A direct application of the Levinson algorithm would allow to obtain A(z) recursively (this will be done in a future work). The approach used here has some advantages that will be clear in the following.

4 - Use Equation (29) to obtain the coefficients of the fractional predictor.

This algorithm is simple and computationally efficient. Although obtained under the hypothesis that the signal is AR(N-1), it will be useful in other situations, namely in the ARMA case.

## 6. RECONSTRUCTION FROM MA MEASUREMENTS

We are going to generalise the previous results and propose a solution for an interesting problem that appears, for example, in Economy. Let us assume that instead of observing the stochastic process x(n) for $n \in Z$, we observe an aggregate time series obtained as an MA process ([1]):

$$y(n) = \sum_{i=0}^{M} b_i \, x(n-i\alpha) \tag{39}$$

The problem we want to solve is: Can we "recover" the unobserved values x(n-iα) for i=0, …,M?

The answer is positive. Let us see how we obtain the referred values. The procedure is similar to the one followed in Section 5.

***Definition 6.1***

Let y(n) be a real stationary stochastic process, observed from -∞ to n-1 and satisfying (1). We define the Nth order d-step prediction of x(n) from y(n) at n+lα (0<α<1 and l=0, …,M) by:

$$\hat{x}(n+l\alpha) = - \sum_{i=1}^{N} a_i \, y(n-i) \tag{40}$$

where $a_i$ (i=1, ..., N) are the coefficients of the lα-step predictor.

As seen, this is a generalisation of the problem solved in Section 5. If we put M=0, y(n)=x(n) and we return back to the normal d-step prediction. Again, the predictor coefficients will be chosen in order to minimise the prediction error power:

$$P_d = E[ \left(x(n-1+d) - \hat{x}(n-1+d)\right)^2] \tag{41}$$

**Theorem 6.1 -** *According to the previous definition the optimum lα-step predictor is given by the solution of the following set of Normal Equations:*

$$\sum_{i=1}^{N} a_i . R_y(k-i) = -R_{xy}(-k-l\alpha) \qquad k=1,2,\ldots,N \tag{42}$$

The minimisation of the prediction error power - Equation (41) - is easily performed by differentiation of its right hand side in order to all the $a_i$ (i=1, ..., N) and leads to the normal Equations (42). As y(n) is obtained through an MA

operation, the correlation matrix of y(n) has surely great than N rank. To compute the cross-correlation in Equation (42), we define $S_y(e^{j\omega})$ as the spectrum of y(n), $S_{xy}(e^{j\omega})$ as the cross spectrum and $B(e^{j\omega})$ the frequency response of the MA filter in Equation (39):

$$B(e^{j\omega\alpha}) = \sum_{i=0}^{M} b_k e^{-jk\omega\alpha} \quad (43)$$

We have too:

$$S_y(e^{j\omega}) = B(e^{j\omega\alpha}).\ S_{xy}(e^{j\omega}) \quad (44)$$

Thus

$$R_{xy}(-k-l\alpha) = FT^{-1}\left[e^{-j\omega l\alpha} S_{yx}(e^{j\omega})\right] \quad (45)$$

where $S_{yx}(e^{j\omega}) = S^{*}_{yx}(e^{j\omega})$ . We must be careful when implementing Equation (45), since the factors on the right have different periods. If we use a FFT with length L in the computation of $S_y$, we must use $L/\alpha$ when computing $B(e^{j\omega\alpha})$, though only L points are used.

## 7. SIMULATION RESULTS

Let us start by exemplifying the simplest case corresponding to a fractional prediction with step d=0.5 of sinusoidal data. Considering the signal $x(n)=\sin(2\pi n/8)+\cos(2\pi n/12)$ in noise (20dB) we computed an $8^{th}$ order fractional predictor as pointed before. The results are displayed together with the original values in Figure 3. The presence of noise makes the signal behave like an ARMA signal. This means that we must increment the predictor order to obtain a better prediction.

Figure 3

prediction. This can be seen in Figure 4 where we present the result obtained with a $16^{th}$ order predictor. As seen, the result is clearly better, since the estimate of the spectrum is more reliable.

In the same line of thoughts we substituted the sinusoidal data by a sum of sinc functions, a band limited signal but not AR, not even ARMA. The results are shown in Figure 5. We see that even with a low predictor order the result is still good.

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[1] It is not necessary to be causal. Here, we use a causal MA, by simplicity. Else, it is not necessary to assume invertibility.

Figure 4

Figure 5

If we had added noise, we would need to increment the predictor order to obtain a similar result. Obviously, we are not restricted to d=0.5. Consider that *d* assumes 3 values, d=0.25, d=0.5, d=0.75, and keep the predictor of order 4. We insert 3 values between each set of two original values. The results obtained are displayed in Figure 6. As it is easy to conclude, we were making a rate increase by integer values. Of course, we can obtain a fractional rate increase (or decrease) by decimation.

Figure 6

In the presented algorithm we computed the spectrum of the signal. In a real time computation it may happen that we cannot compute a spectrum estimate. We propose to avoid this by assuming the knowledge of the bandwidth and substituting the signal spectrum by the absolute square of the Frequency Response of the filter designed to pass the signal. In Figure 7, we present preliminary results showing promising performances.

Figure 7

To illustrate the performances of the method, we present some numerical results. We proceed in the following way:

a) Generate a signal with L points and a given signal to noise ratio;
b) Down-sample it by 1/2 factor;
c) Use the previous algorithm to estimate the removed values.

For each simulation we computed the error between each original and estimated value and the corresponding error power. In Figure 8 we present the result of a simulation using as original signal a sum of sinc functions.

Figure 8

To study the influence of the predictor length we made several simulations in the referred conditions and computed the average error power over 10 realizations of each of the referred signals. The results are presented in the following pictures.

Figure 9

Figure 10

Figure 11

We can conclude that even with low predictor orders (lower than 10) we can interpolate quite well non-AR signals. As seen, we really made a rate conversion by a factor equal to 2. The generalization for other factors is not difficult to implement and will be subject of a future publication.

In the following Figures, we illustrate the application of the disaggregation algorithm. The signal used for the prediction was obtained from the original signal by substituting each pair of consecutive points by their average. We present the results for sinusoidal data (Figure 12) and sinc data (Figure 13).

Figure 12

Figure 13

In the next two figures, we repeat the previous simulations but in a situation where we substituted every 3 points by its average.

Figure 14

Figure 15

These results show that these algorithms can be useful. However, some work need to be done about the preditor length that is an important parameter.

## 8. CONCLUSIONS

In this paper we proposed a generalisation of the usual linear prediction to fractional step linear prediction. This allows us to predict the value of a signal defined at a uniform time grid to any point between any two-grid points. We presented some illustrating examples showing the use of the algorithm to perform a signal interpolation. In a future publication we will present quantitative results illustrating the performance of the algorithm. From the examples, we can confirm the

algorithm ability to perform a rate conversion. Besides its performance the algorithm can be implement in a recursive way. On the other hand, giving a lattice form to the predictor turns out to be a simple task.

We presented new algorithms for interpolation and scale conversion of discrete-time signals based on the theory of fractional discrete-time systems. We presented some simulation results to illustrate the behaviour of the algorithms when applied in a rate increase by a factor 2 for different sets of signals. We concluded that even with low order predictors we can perform a rate increase. Based in the results of Section 1 we also derived a very simple but efficient algorithm for the zoom transform.

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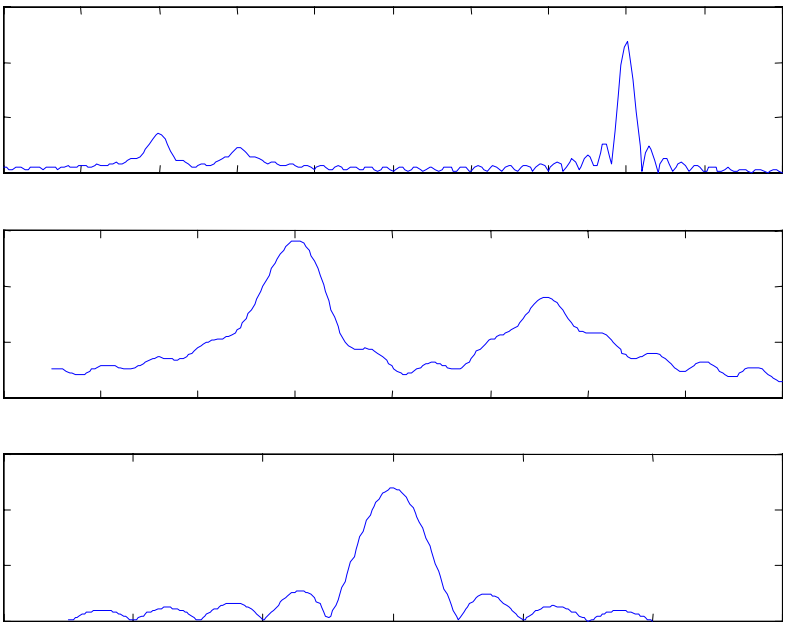

*Figure 1 – Zoom Transform.*

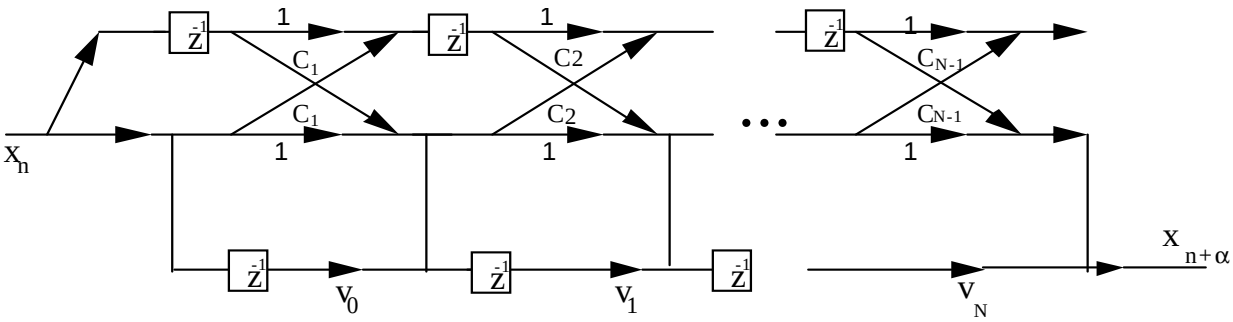


*Figure 2 - Lattice form for the fractional predictor*

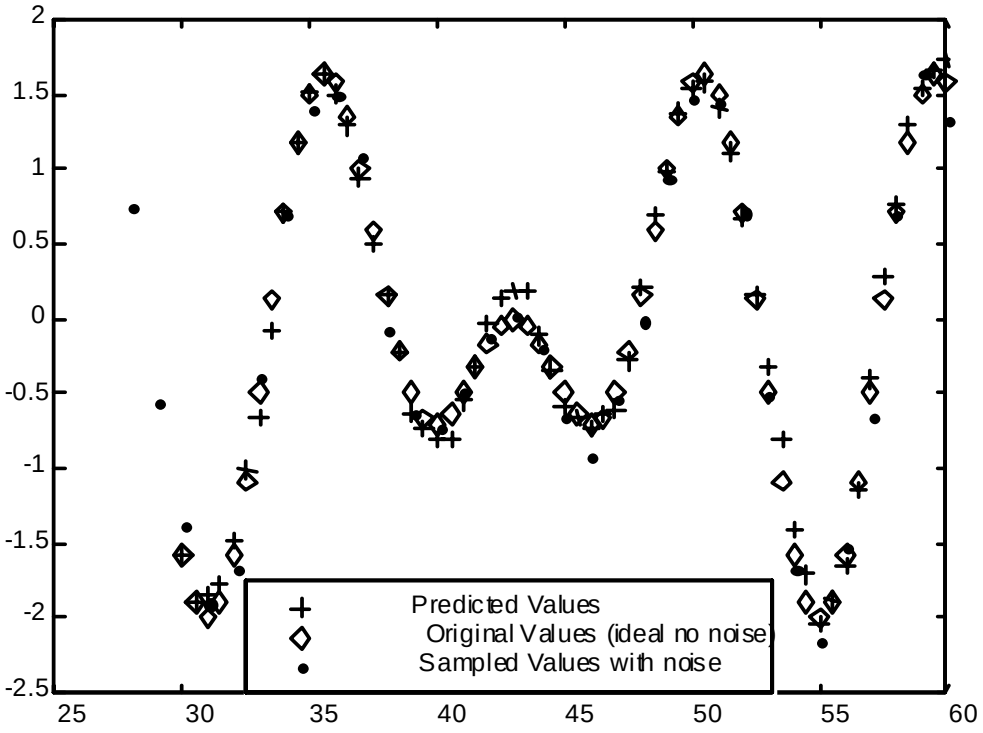


*Figure 3 - prediction of noisy (20dB) sinusoidal data with a predictor of order 8.*

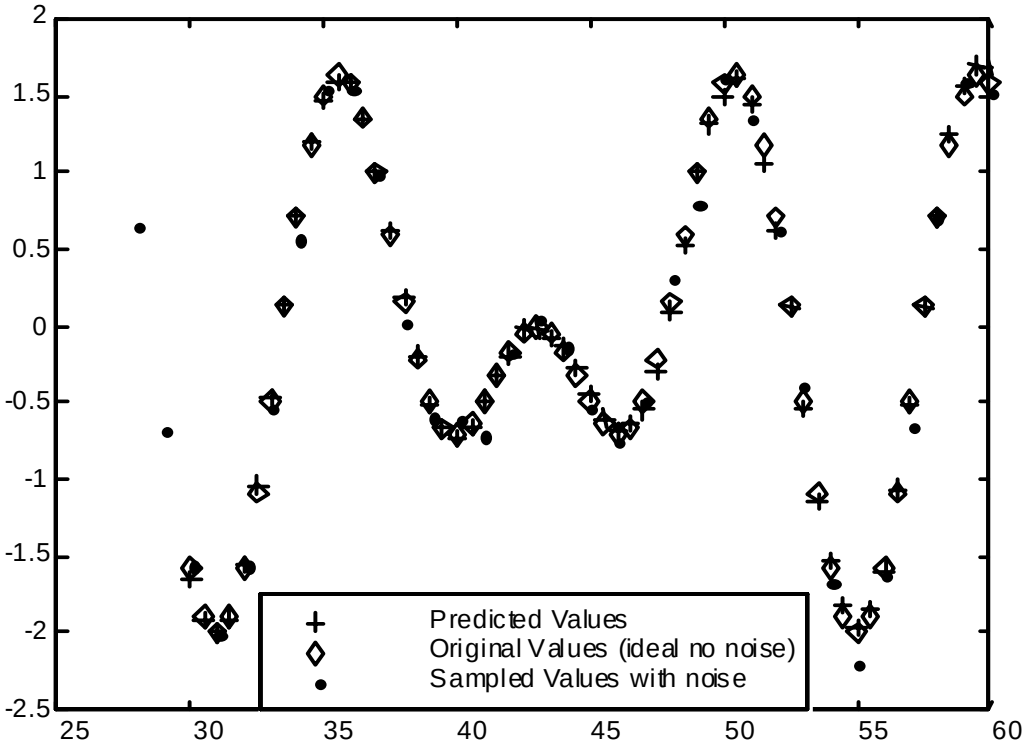


*Figure 4 - prediction of noisy (20dB) sinusoidal data with a predictor of order 16.*

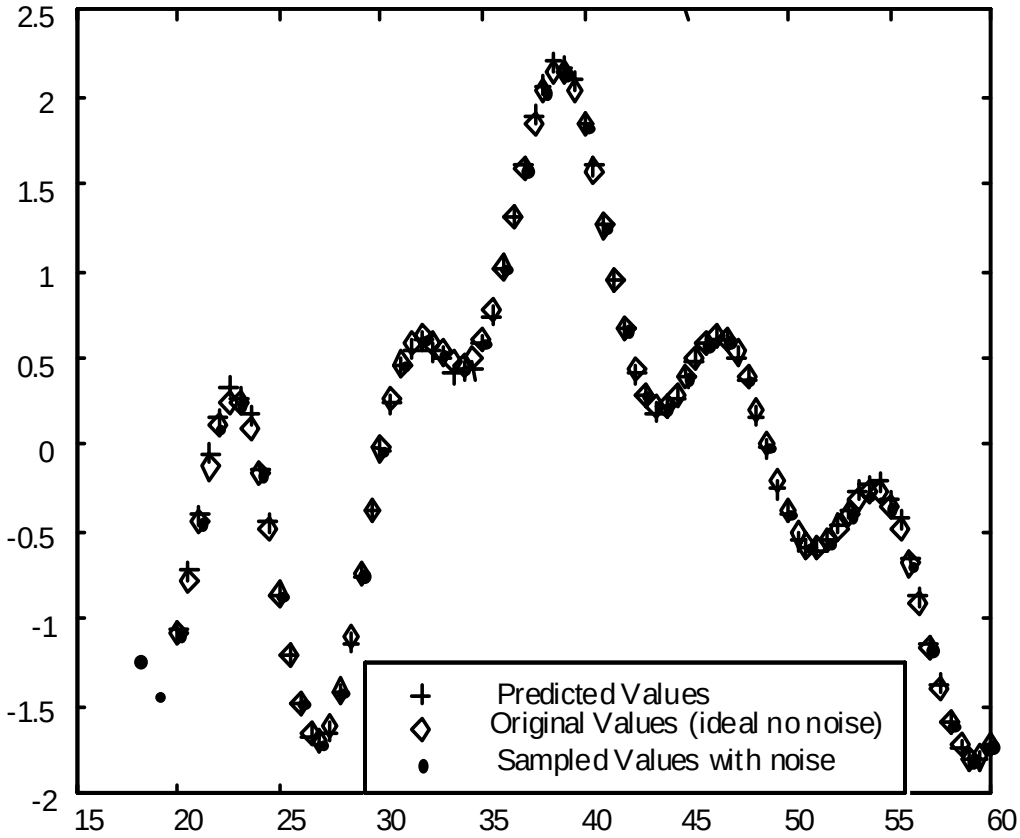


*Figure 5 – prediction of a sum of sinc (100dB) functions with predictor order 4.*

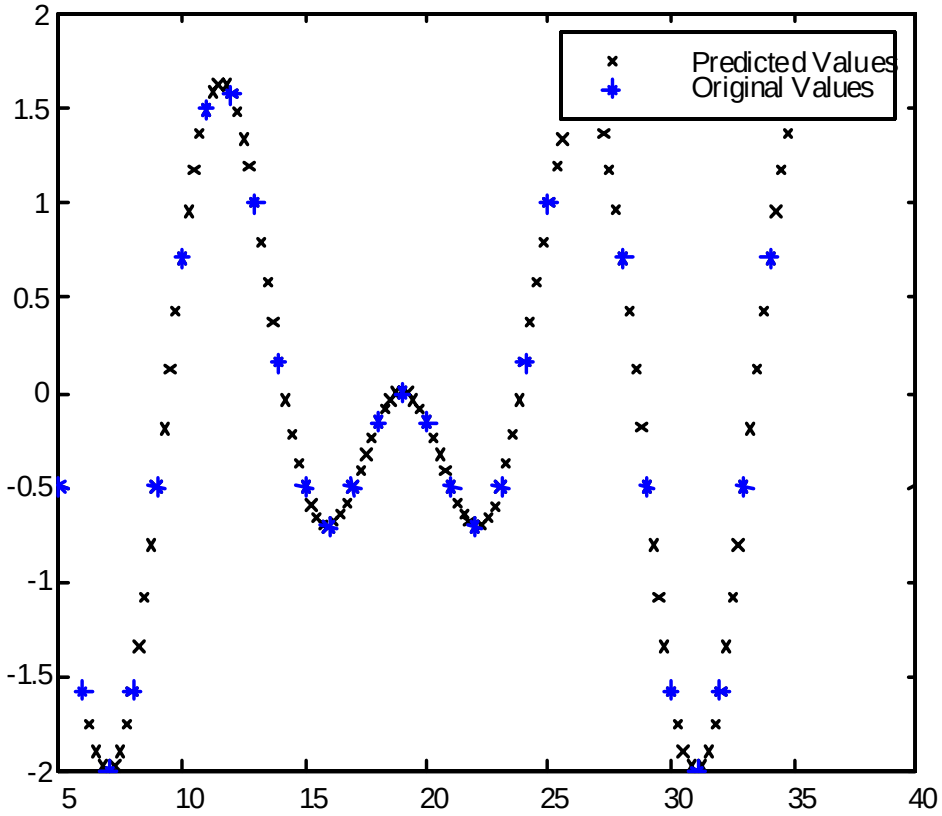


*Figure 6 – interpolation using fractional prediction with steps 0.25, 0.5 , and 0.75.*

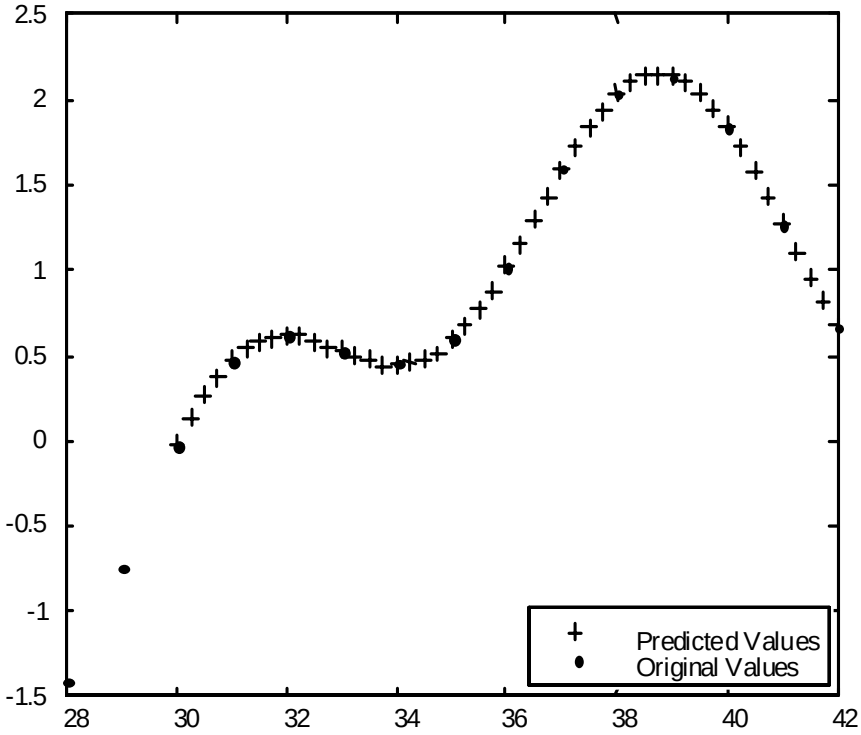


*Figure 7- prediction without knowing the signal spectrum.*

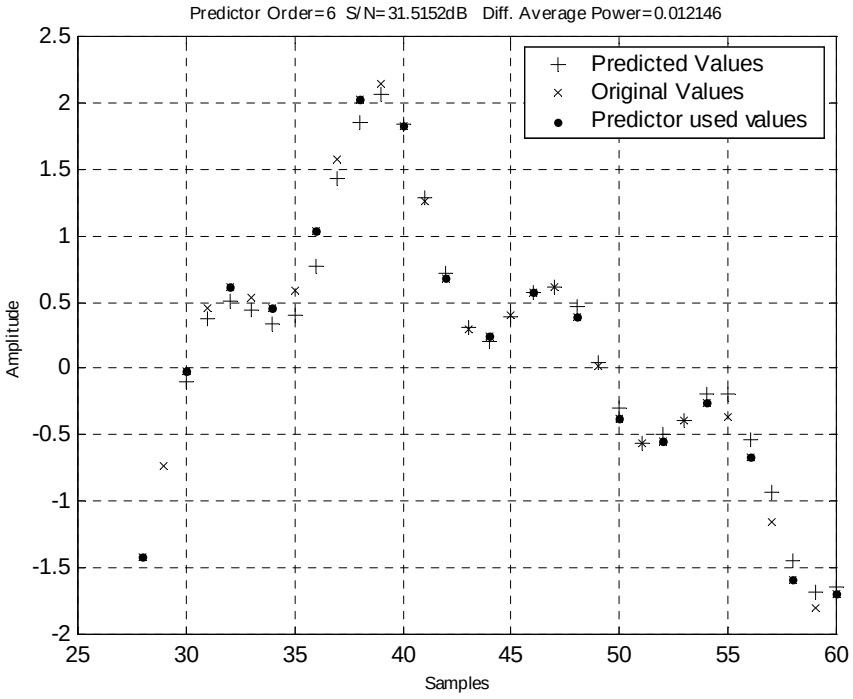


*Figure 8 – Fractional prediction of a sum of sinc functions.*

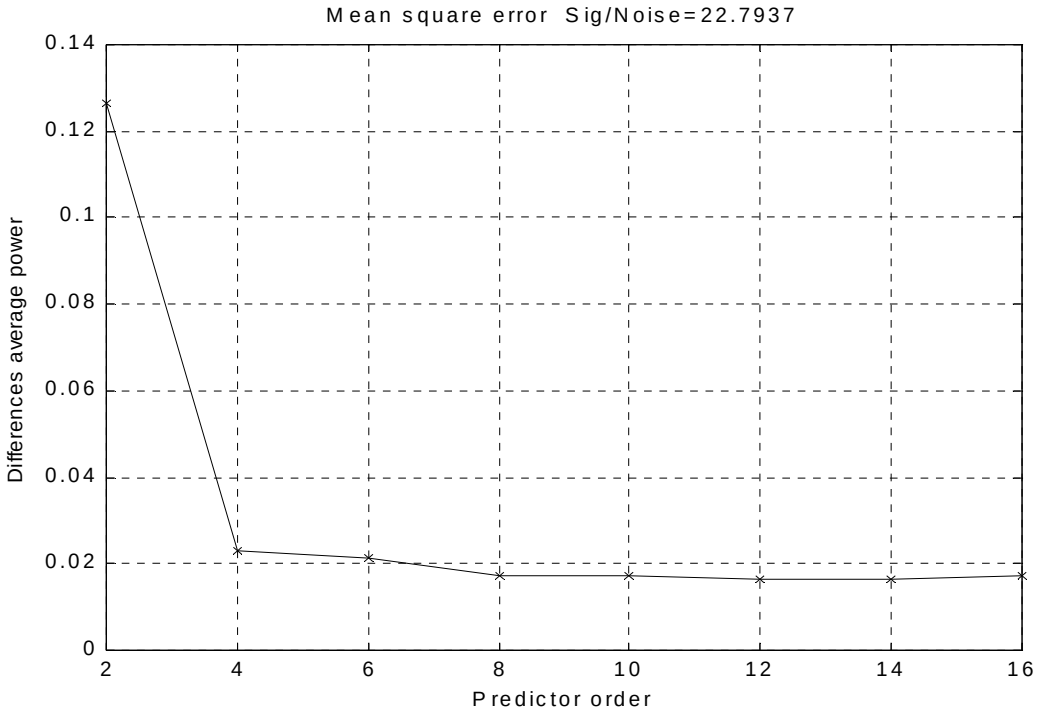


*Figure 9 – mean error power for 10 realizations of one sinusoid as function of predictor length.*

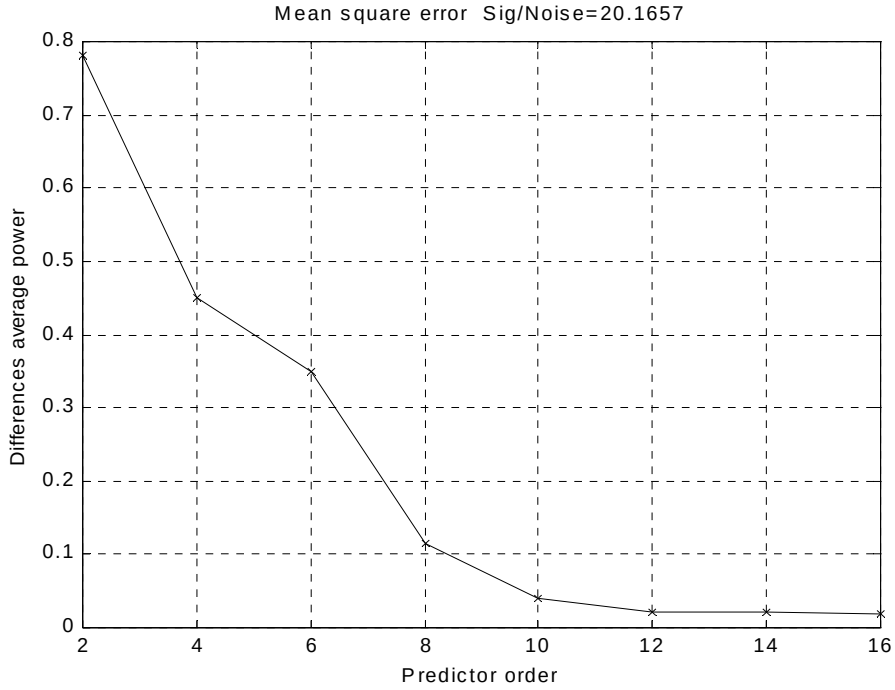


*Figure 10 – mean error power for 10 realizations of 2 sinusoids as function of predictor length.*

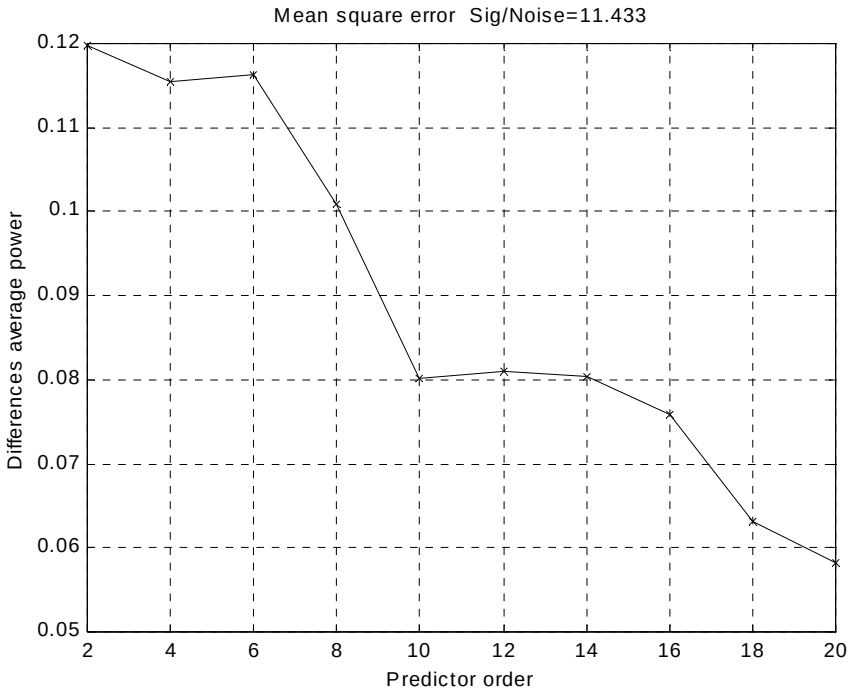


Figure 11 – mean error power for 10 realizations of several sincs as function of predictor length.

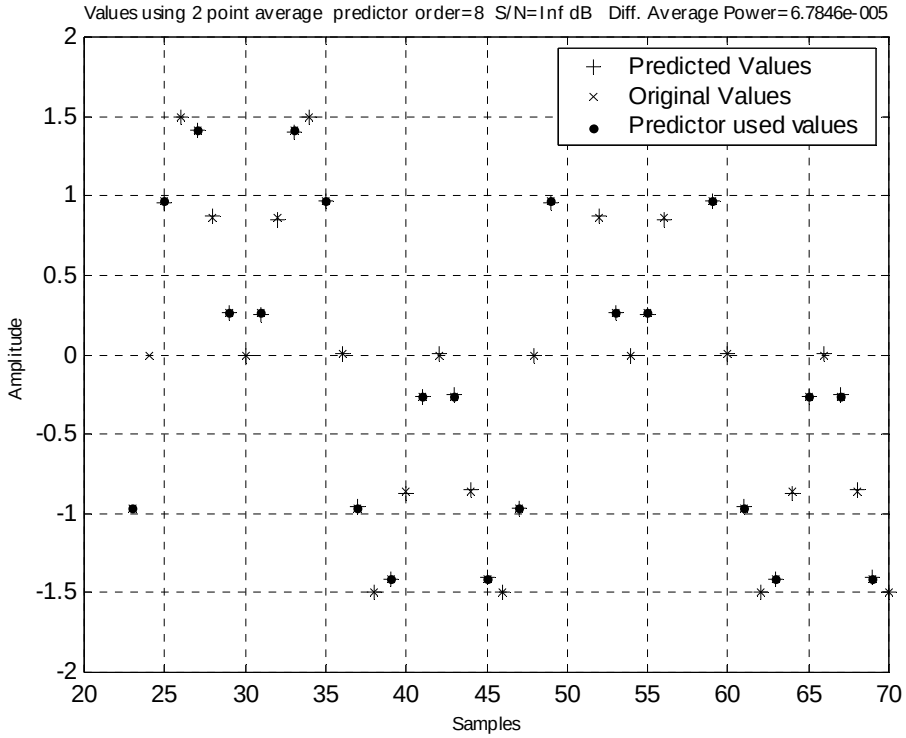


*Figure 12 – Fractional reconstruction of a sum of two sinusoids.*

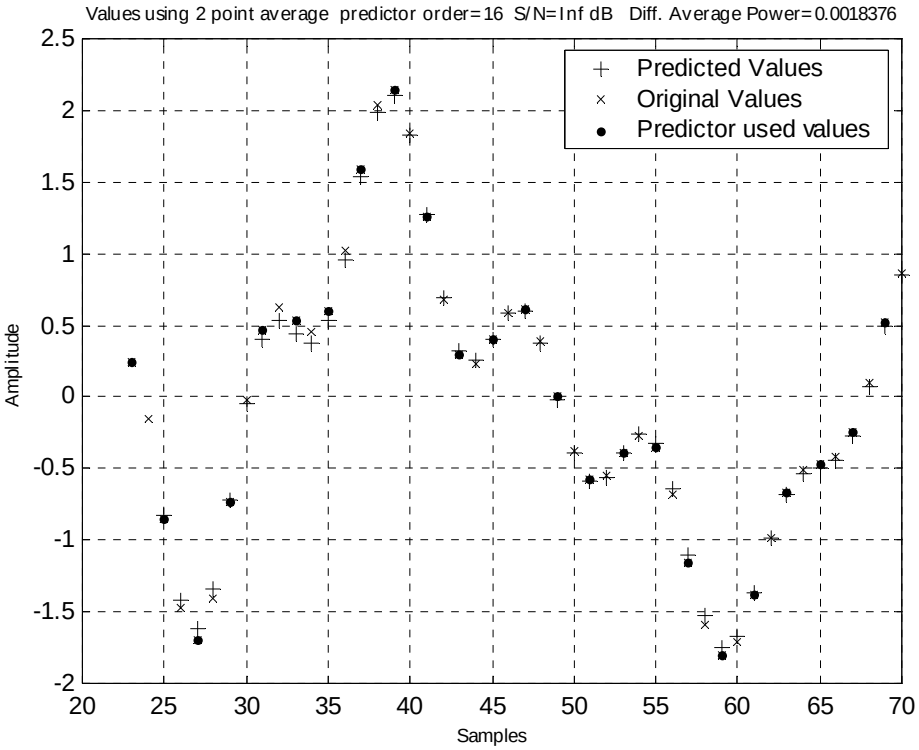


*Figure 13 – Fractional reconstruction of a sum of four sinc signals.*

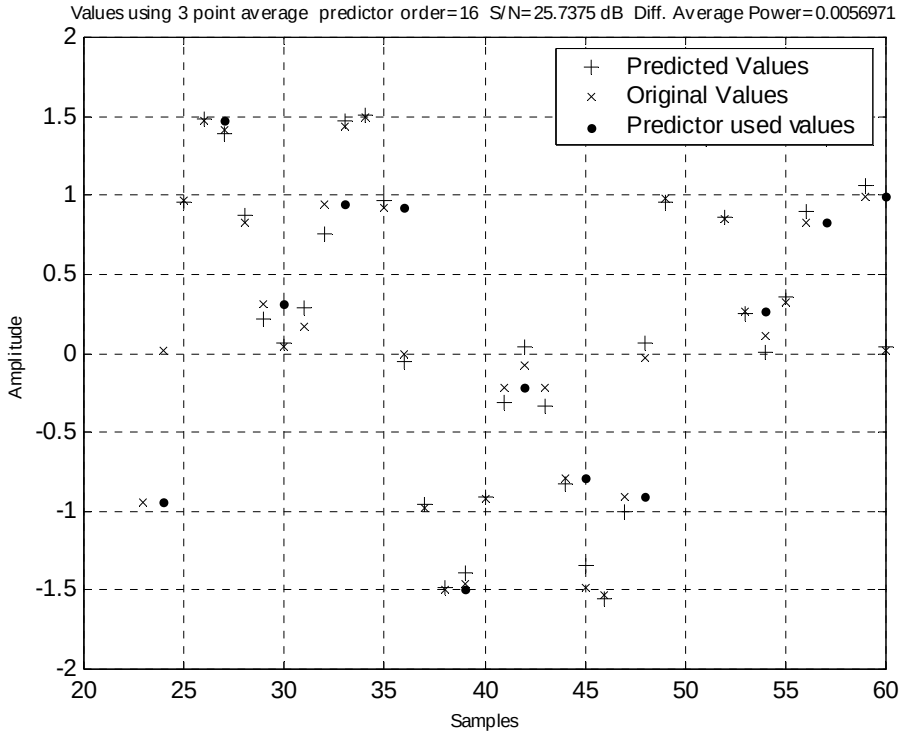


*Figure 14 – Fractional reconstruction of a sum of two sinusoids.*

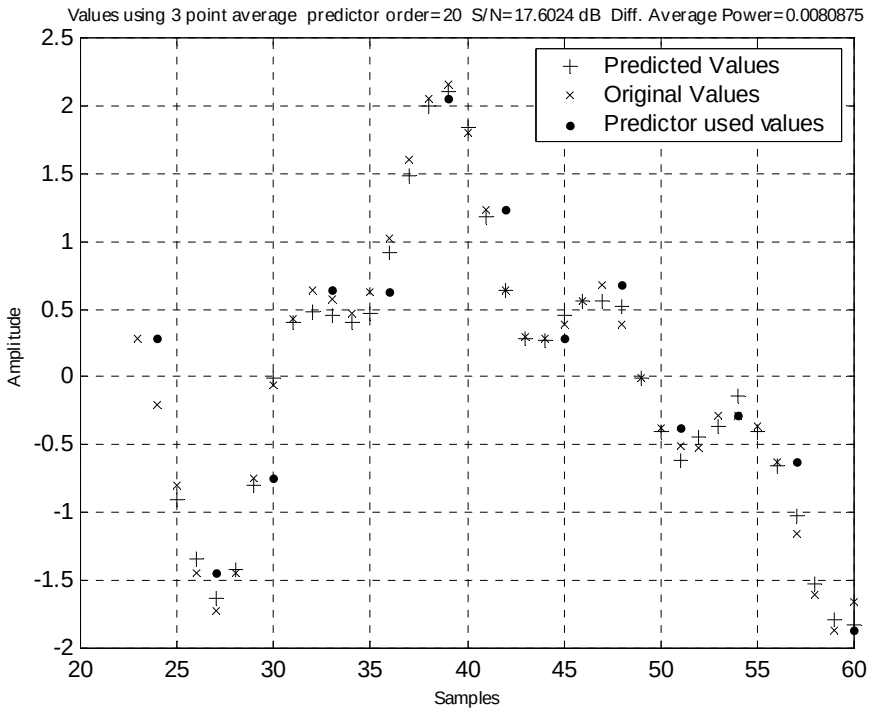


*Figure 15 – Fractional reconstruction of a sum of four sinc signals.*