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A new symmetric fractional B-spline

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Abstract

A new definition of a symmetric fractional B-spline is presented. This generalises the usual integer order B-spline, that becomes a special case of the new one.

Keywords: B-spline; Fractional differintegration

1. Introduction

The signal processing with splines have been acquiring increasing interest due to its flexibility in interpolation, sampling and wavelet transform [3]. Recently, causal and symmetric fractional B-splines were proposed [4]. We have nothing to say relatively to the proposed causal splines. In fact they generalise the integer order B-splines in such a way that these are special cases of the fractional. However, this does not happen with the fractional ones. A closed look into the proposed B-spline definitions reveals that they are strange since the even integer order B-splines are not special cases of the fractional B-spline. Here, we will face the problem and propose new definitions that have the current integer order B-splines as particular cases.

2. Causal B-splines

Causal B-splines result from n-fold convolution of the rectangle function

$$\beta_+^0(t) = \begin{cases} 1 & 0 < t < 1 \\ \frac{1}{2} & t = 0, 1 \\ 0 & t < 0 \text{ or } t > 1. \end{cases} \tag{1}$$

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Its Laplace transform (LT) is an analytical function given by

$$B_+^n(s) = \left[\frac{1-e^{-s}}{s}\right]^{n+1} \tag{2}$$

that can be expressed as

$$B_+^n(s) = \frac{1}{s^{n+1}} \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k e^{-sk} \quad \text{for } \operatorname{Re}(s) \geqslant 0. \tag{3}$$

As

$$\mathrm{LT}^{-1}[s^{-n-1}] = \frac{t^n}{n!} u(t) = \frac{t_+^n}{n!} \tag{4}$$

we obtain, from (3):

$$\beta_+^n(t) = \frac{1}{n!} \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k (t-k)_+^n. \tag{5}$$

The situation is slightly different when we try to define a fractional spline. In this (causal) case, there is no problem. We only have to substitute α for n in (3) and (5) and summing up to ∞:

$$\beta_+^\alpha(t) = \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{+\infty} \binom{\alpha+1}{k} (-1)^k (t-k)_+^\alpha \tag{6}$$

It is important to remark that, while (3) represents a symmetric (relatively to $n/2$), for every positive integer, this does not happen with (6). This has implications in defining a symmetric fractional B-spline. The previous results are equal to those obtained in [4].

3. Symmetric B-splines

A nth degree B-spline, $\beta_0^n(t)$, is a symmetric function resulting from n-fold convolution of the rectangle function

$$\beta_0^0(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ \frac{1}{2} & |t| = \frac{1}{2} \\ 0 & |t| > \frac{1}{2}. \end{cases} \tag{7}$$

Its bilateral Laplace transform (BLT) is an analytic function given by

$$B_0^n(s) = \left[\frac{e^{s/2}-e^{-s/2}}{s}\right]^{n+1} = \frac{e^{s(n+1)/2}}{s^{n+1}} \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k e^{-sk}. \tag{8}$$

So, the corresponding FT exists and is given by

$$B_0^n(\omega) = \left[\frac{\sin(\omega/2)}{\omega/2}\right]^{n+1} = \frac{e^{j\omega(n+1)/2}}{(j\omega)^{n+1}} \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k e^{-j\omega k}. \tag{9}$$

From (8), we obtain:

$$\beta_0^n(t) = \frac{1}{n!}\sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k\left(t-k+\frac{n+1}{2}\right)_+^n. \tag{10}$$

On the other hand, introducing $t_0^n = t^n \mathrm{sgn}(t)$

$$\mathrm{FT}^{-1}[(\mathrm{j}\omega)^{-n-1}] = \frac{1}{2}\,\frac{t^n\mathrm{sgn}(t)}{n!} = \frac{t_0^n}{2n!} \tag{11}$$

we obtain, from (9):

$$\beta_0^n(t) = \frac{1}{2.n!}\sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k\left(t-k+\frac{n+1}{2}\right)_0^n \tag{12}$$

that seem to be different from (10), but due to the symmetry of the coefficients it represents the same function.

In [4], a symmetric fractional B-spline is defined as inverse FT of the function:

$$B_0^\alpha(\omega) = \left|\frac{\sin(\omega/2)}{\omega/2}\right|^{\alpha+1}. \tag{13}$$

However, this definition has the disadvantage of giving a strange spline, when α is an even positive integer. To avoid this, we are going to present a centred fractional spline that does not have this drawback. Let $\alpha = n + v$, where n is a positive integer and $0 < v < 1$. We define a fractional α-order B-spline as the function that has

$$B_0^\alpha(\omega) = \left[\frac{\sin(\omega/2)}{\omega/2}\right]^{n+1}\left|\frac{\sin(\omega/2)}{\omega/2}\right|^{v} \tag{14}$$

as FT. When α is an integer, $v = 0$ and we obtain the normal n-order B-spline, while when $v \neq 0$, we obtain a fractional centred and symmetric B-spline that is the convolution of two even functions. The reason of proposing such definition in the fact that the inverse FT of $[\sin(\omega/2)/\omega/2]^v$ is a complex function.

For computing $\beta_0^{v+n}(t)$, we can proceed recursively by successive convolutions with $\beta_0^0(t)$.

$$\beta_0^{v+n}(t) = \beta_0^{v+n-1}(t) * \beta_0^0(t) \tag{15}$$

From (15) and noting that $\beta_0^0(t) = u(t+\frac{1}{2}) - u(t-\frac{1}{2})$ we obtain:

$$\beta_0^\alpha(t) = \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \beta_0^{\alpha-1}(\tau)\,\mathrm{d}\tau \tag{16}$$

and

$$\frac{\mathrm{d}\beta_0^\alpha(t)}{\mathrm{d}t} = \beta_0^{\alpha-1}(t+\tfrac{1}{2}) - \beta_0^{\alpha-1}(t-\tfrac{1}{2}). \tag{17}$$

On the other hand, if $\alpha_1 = n + v$ and $\alpha_2 = m + \mu$, with $v + \mu < 1$,

$$\beta_0^{\alpha_1}(t) * \beta_0^{\alpha_2}(t) = \beta_0^{\alpha_1+\alpha_2+1}(t). \tag{18}$$

If $v + \mu \geqslant 1$, we can obtain a similar relation if we define a modified B-spline by allowing v to be negative in equation (14). We put, for example, $\alpha_2 = m + 1 - \varepsilon$. Now, we are going to compute $\beta_0^{v+n}(t)$ directly.

As shown in Appendix A, we have:

$$|2\sin(\omega/2)|^{\nu}[2\sin(\omega/2)]^{n+1} = \frac{\mathrm{e}^{\mathrm{j}\omega(n+1)/2}}{(\mathrm{j})^{n+1}} \sum_{k=-\infty}^{+\infty} a(n,k)\,\mathrm{e}^{-\mathrm{j}\omega k} \tag{19}$$

with

$$a(n,k) = \frac{\Gamma(\nu+n+2)}{\Gamma(\nu/2+1)\Gamma(\nu/2+n+2)} \begin{cases} 1 & k=0 \\ \left[\dfrac{-\nu/2-n-1}{\nu/2+1}\right]_k & k>0 \\ \left[\dfrac{-\nu/2}{\nu/2+n+2}\right]_{|k|} & k<0 \end{cases}. \tag{20}$$

We are in conditions to express (14) in the Fourier series format:

$$B_0^{\alpha}(\omega) = \frac{1}{|\omega|^{\nu}(\mathrm{j}\omega)^{n+1}} \sum_{k=-\infty}^{+\infty} a(n,k)\,\mathrm{e}^{-\mathrm{j}\omega(k-(n+1)/2)}. \tag{21}$$

The inverse Fourier transform of $1/|\omega|^{\nu}(\mathrm{j}\omega)^{n+1}$ is computed in Appendix B and is given by

$$\mathrm{FT}^{-1}\left[\frac{1}{|\omega|^{\nu}(\mathrm{j}\omega)^{n+1}}\right] = \frac{1}{2\Gamma(\nu+n+1)\cos(\nu\pi/2)}|t|^{\nu+n}\mathrm{sgn}^{n+1}(t). \tag{22}$$

Inserting (22) into (21) and using (20) we obtain:

$$\beta_0^{\nu+n}(t) = \frac{(\nu+n+1)}{2\cos(\nu\pi/2)\Gamma(\nu/2+1)\Gamma(\nu/2+n+2)} \sum_{k=-\infty}^{+\infty} b(n,k)\left|t-k+\frac{n+1}{2}\right|^{\nu+n} \times\mathrm{sgn}^{n+1}\left(t-k+\frac{n+1}{2}\right) \tag{23}$$

that is the expression of the $\alpha = \nu + n$ order B-spline. The coefficients $b(n,k)$ are given by

$$b(n,k) = \begin{cases} 1 & k=0 \\ \left[\dfrac{-\nu/2-n-1}{\nu/2+1}\right]_k & k>0 \\ \left[\dfrac{-\nu/2}{\nu/2+n+2}\right]_{|k|} & k<0 \end{cases} \tag{24}$$

that have a form suitable for recursive computation.

In Figs. 1–4 we present some splines for values of $n = 0$, 1, 2, and 3 and $\nu = 0.2$, 0.4, 0.6, 0.8, and 0.9. To make a fair comparison, in Fig. 5, we present in the same time scale all the α order splines with $\alpha = 0.2k$, $k = 0, 1, \ldots, 19$. As it can be seen, the fractional B-splines interpolate the integer ones and become smooth as the order increases.

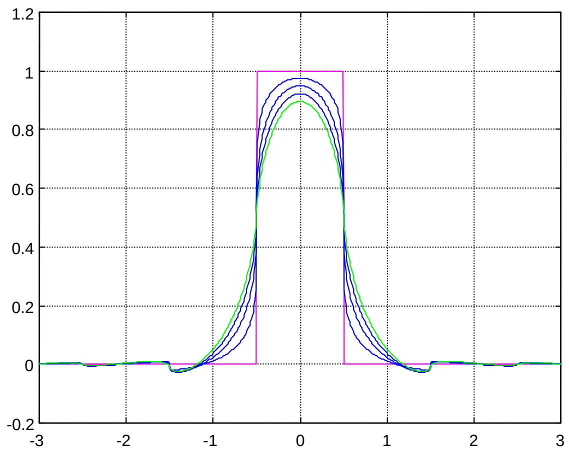

Fig. 1. Splines of orders 0, 0.2, 0.4, 0.6, and 0.8.

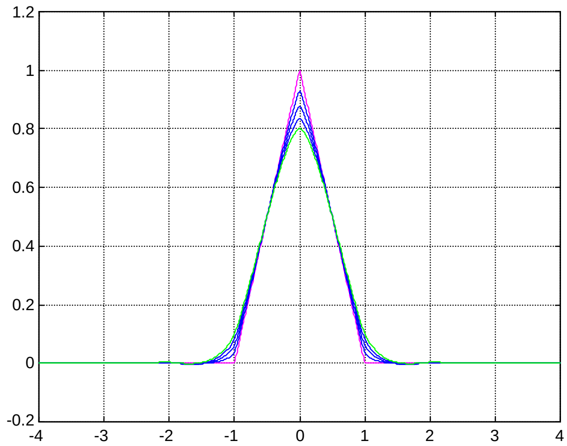

Fig. 2. Splines of orders 1, 1.2, 1.4, 1.6, and 1.8.

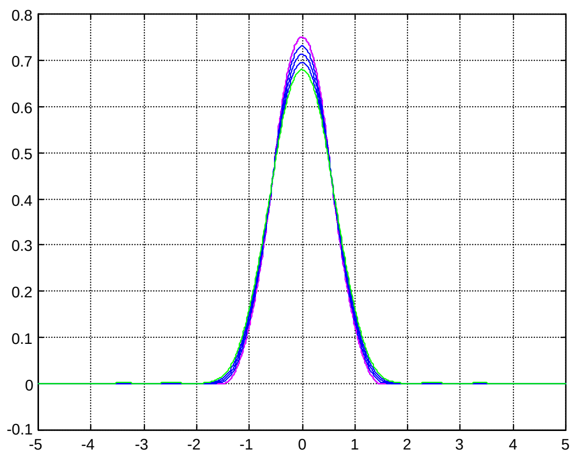

Fig. 3. Splines of orders 2, 2.2, 2.4, 2.6, and 2.8.

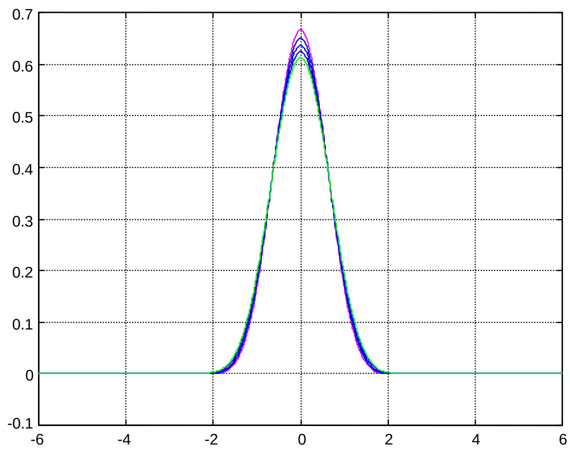

Fig. 4. Splines of orders 3, 3.2, 3.4, 3.6, and 3.8.

4. Conclusions

In this paper we proposed a new fractional order B-spline generalising the usual integer order B-spline that is a special case of the new one. Examples illustrate this fact and show that the fractional order interpolate the integer ones. The main drawback is in the discontinuity relatively to the order when we approach each integer from the left.

Appendix A. The Fourier series of $|2\sin(\omega/2)|^\nu\ [2\sin(\omega/2)]^{n+1}$

We begin by noting that

$$|2\sin(\omega/2)|^\nu = \lim_{s\to j\omega}[1-e^{-s}]^{\nu/2}[1-e^{s}]^{\nu/2}. \tag{A.1}$$

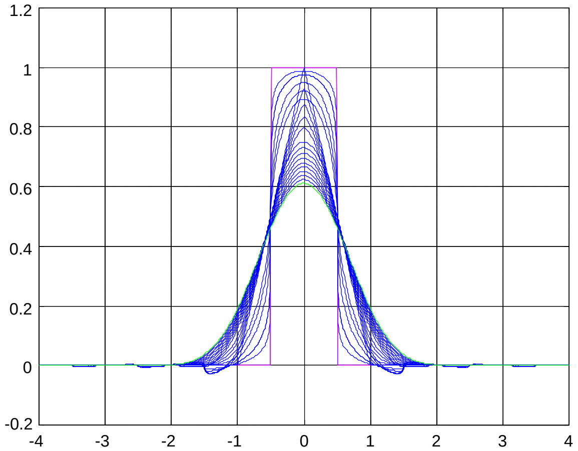

Fig. 5. Splines of orders $n+v$ with $n=0,1,2,3$ and $v=0,0.2,0.4,0.6$ and 0.8.

Expanding each term on the right using the binomial series and computing the autocorrelation of the coefficients we obtain the Fourier series associated to the function on the left. To do it we remark that for every $\theta \in \mathrm{R}$, but non-even integer [1].

$$\sum_{k=0}^{\infty} \binom{\theta}{k} \binom{\theta}{k+n} = \frac{\Gamma(1+2\theta)}{\Gamma(\theta+n+1)\Gamma(\theta-n+1)} \tag{A.2}$$

leading to

$$|2\sin(\omega/2)|^{v} = \Gamma(v+1) \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{\Gamma(v/2+k+1)\Gamma(v/2-k+1)} \mathrm{e}^{-\mathrm{j}\omega k}. \tag{A.3}$$

On the other hand,

$$[2\sin(\omega/2)]^{n+1} = \frac{\mathrm{e}^{\mathrm{j}\omega(n+1)/2}}{(\mathrm{j})^{n+1}} \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k \mathrm{e}^{-\mathrm{j}\omega k}. \tag{A.4}$$

The product of (A.3) and (A.4) has the Fourier series:

$$|2\sin(\omega/2)|^{v}[2\sin(\omega/2)]^{n+1} = \frac{\mathrm{e}^{\mathrm{j}\omega(n+1)/2}}{(\mathrm{j})^{n+1}} \sum_{k=-\infty}^{+\infty} a(n,k)\,\mathrm{e}^{-\mathrm{j}\omega k} \tag{A.5}$$

with

$$a(n,k) = \Gamma(v+1)(-1)^k \sum_{m=0}^{n+1} \frac{\binom{n+1}{m}}{\Gamma(v/2+k-m+1)\Gamma(v/2-k+m+1)}. \tag{A.6}$$

It is not hard to show that (A.6) can be written in terms of the Gauss hypergeometric function:

$$a(n,k) = \Gamma(v+1) \frac{(-1)^k}{\Gamma(v/2+k+1)\Gamma(v/2-k+1)} \sum_{m=0}^{n+1} \frac{(-n-1)_m(-v/2-k)_m}{(v/2-k+1)_m m!}$$

$$= \Gamma(v+1)\frac{(-1)^k}{\Gamma(v/2+k+1)\Gamma(v/2-k+1)^2}F_1(-n-1,-v/2-k;v/2-k+1;1). \quad (A.7)$$

Using the properties of the hypergeometric function, we obtain:

$$a(n,k) = (-1)^k \frac{\Gamma(v+n+2)}{\Gamma(v/2+k+1)\Gamma(v/2-k+n+2)} = (-1)^k \binom{v+n+1}{v/2+k}. \quad (A.8)$$

Letting $q(n+1,k)$ be given by

$$q(n,k) = \frac{(-1)^k}{\Gamma(v/2+k+1)\Gamma(v/2-k+n+2)} \quad (A.9)$$

we can express it as

$$q(n,k) = \frac{1}{\Gamma(v/2+1)\Gamma(v/2+n+2)} \begin{cases} 1 & k=0 \\ \left[\dfrac{-v/2-n-1}{v/2+1}\right]_k & k>0 \\ \left[\dfrac{-v/2}{v/2+n+2}\right]_{|k|} & k<0 \end{cases} \quad (A.10)$$

where we represented $(a)_k/(b)_k$ by $[a/b]_k$ for simplification. Thus, $a(n,k)$ is given by

$$a(n,k) = \frac{\Gamma(v+n+2)}{\Gamma(v/2+1)\Gamma(v/2+n+2)} \begin{cases} 1 & k=0 \\ \left[\dfrac{-v/2-n-1}{v/2+1}\right]_k & k>0 \\ \left[\dfrac{-v/2}{v/2+n+2}\right]_{|k|} & k<0 \end{cases}. \quad (A.11)$$

It is interesting to remark that, if $v=0$, $a(n,k)=0$ for $k<0$ and $a(n,k)=(-1)^k\binom{n+1}{m}$ for $k=0,1,\ldots,n+1$, leading to (12).

Appendix B. Inverse transform of $|\omega|^{-v}(\mathrm{j}\omega)^{-n-1}$

To compute the inverse Fourier transform of $G(\omega)=|\omega|^{-v}(\mathrm{j}\omega)^{-n-1}$ we write it as

$$G(\omega) = |\omega|^{-v}(\mathrm{j}\omega)^{-n-1} = \frac{(-\mathrm{j})^{n+1}}{|\omega|^{v+n+1}} \begin{cases} 1 & \omega>0 \\ (-1)^{n+1} & \omega<0 \end{cases}. \quad (B.1)$$

If n is odd, we obtain:

$$G(\omega) = \frac{(-1)^{n+1/2}}{|\omega|^{v+n+1}}. \quad (B.2)$$

while, if n is even,

$$G(\omega) = \frac{(-\mathrm{j})(-1)^{n/2}}{|\omega|^{v+n+1}}\,\mathrm{sgn}(\omega). \quad (B.3)$$

If α is a noninteger real, the inverse Fourier transform of $|\omega|^{-\alpha}$ is given by [2]:

$$\mathrm{FT}^{-1}[|\omega|^{-\alpha}] = \frac{\Gamma(1-\alpha)\sin(\alpha\pi/2)}{\pi}\,|t|^{\alpha-1} \tag{B.4}$$

or

$$\mathrm{FT}^{-1}[|\omega|^{-\alpha}] = \frac{1}{2\Gamma(\alpha)\cos(\alpha\pi/2)}\,|t|^{\alpha-1}. \tag{B.5}$$

We obtain immediately to (B.2):

$$\mathrm{FT}^{-1}[G(\omega)] = \frac{1}{2\Gamma(v+n+1)\cos(v\pi/2)}\,|t|^{v+n}. \tag{B.6}$$

To treat the n even case, we only have to remark that

$$\frac{1}{|\omega|^{v+n+1}}\,\mathrm{sgn}(\omega) = -\frac{\mathrm{d}}{\mathrm{d}\omega}\left[\frac{1}{(v+n)|\omega|^{v+n}}\right]$$

and use the property: $\mathrm{FT}[t\,f(t)] = \mathrm{j}F'(\omega)$. We obtain, then:

$$\mathrm{FT}^{-1}[G(\omega)] = \frac{1}{2\Gamma(v+n+1)\cos(v\pi/2)}\,t\,|t|^{v+n-1} = \frac{1}{2\Gamma(v+n+1)\cos(v\pi/2)}\,|t|^{v+n}\mathrm{sgn}(t). \tag{B.7}$$

We can join (B.5) and (B.6) in the form:

$$\mathrm{FT}^{-1}\left[\frac{1}{|\omega|^{v}(\mathrm{j}\omega)^{n+1}}\right] = \frac{1}{2\Gamma(v+n+1)\cos(v\pi/2)}|t|^{v+n}\mathrm{sgn}^{n+1}(t) \tag{B.8}$$

or

$$\mathrm{FT}^{-1}\left[\frac{1}{|\omega|^{v}(\mathrm{j}\omega)^{n+1}}\right] = \frac{1}{2\Gamma(v+n+1)\cos(v\pi/2)}|t|^{v}t^{n}\mathrm{sgn}(t) \tag{B.9}$$

that gives (11) when $v = 0$.

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