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Discussion

Comments on "Modeling fractional stochastic systems as non-random fractional dynamics driven Brownian motions"

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ABSTRACT

Some results presented in the paper "Modeling fractional stochastic systems as non-random fractional dynamics driven Brownian motions" [I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999] are discussed in this paper. The slightly modified Grünwald-Letnikov derivative proposed there is used to deduce some interesting results that are in contradiction with those proposed in the referred paper.

The fractional calculus is a generalization of the traditional calculus that leads to similar concepts and tools, but with wider generality and applicability. By allowing derivative and integral operations of arbitrary real or complex order, it is to traditional calculus what the real or complex numbers are to the integers [1,2]. This means that we must recover the traditional calculus when the order is a positive integer number.

In "Modeling fractional stochastic systems as non-random fractional dynamics driven Brownian motions" [3] the introduction to fractional calculus presented there leads to several statements and results that deserve some comments, because they are in contradiction with the classic results and also with its own starting point (Eq. (1)).

Let us start as in [3] with the following definition of fractional derivative:

$$D_J^\alpha f(t) = \lim_{h\to 0+} \frac{\sum_{k=0}^{\infty}(-1)^k \binom{\alpha}{k} f[t+(\alpha-k)h]}{h^\alpha}. \tag{1}$$

From this definition we can obtain several interesting results as we will see next.

(1) The above defined derivative is equivalent to the Grünwald–Letnikov derivative. In fact and following the author's notation $FW(h)f(t) = f(t+h)$, we have:

$$(FW-1)^\alpha = FW^\alpha(1-FW^{-1})^\alpha.$$

and

$$\lim_{h\to 0+}\frac{(FW-1)^\alpha}{h^\alpha} = \lim_{h\to 0+} FW^\alpha \lim_{h\to 0+}\frac{(1-FW^{-1})^\alpha}{h^\alpha} \tag{2}$$

The first factor converges to 1 and the second leads to the forward Grünwald–Letnikov derivative. We could also conclude this directly from (1) by noting that $\alpha.h$ becomes negligible as h goes to zero.

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(2) The fractional derivative of a constant is zero. From (1), we have:

$$D_J^\alpha 1 = \lim_{h\to 0+} \frac{\sum_{k=0}^\infty (-1)^k \binom{\alpha}{k}}{h^\alpha} = \lim_{h\to 0+} \frac{(1-1)^\alpha}{h^\alpha} = \begin{cases} 0, & \alpha > 0 \\ \infty, & \alpha < 0. \end{cases} \tag{3}$$

We proved also that there is no fractional primitive of a constant. When we say that the Riemann–Liouville fractional derivative of a constant is not zero, it is a wrong statement, because we are effectively computing the derivative of a Heaviside step function.

(3) The fractional derivative of the exponential is

$$D_J^\alpha e^{st} = e^{st} \lim_{h\to 0+} \frac{\sum_{k=0}^\infty (-1)^k \binom{\alpha}{k} e^{s(\alpha-k)h}}{h^\alpha} = e^{st} \lim_{h\to 0+} \frac{e^{s\alpha h}(1-e^{-sh})^\alpha}{h^\alpha} = e^{st} \lim_{h\to 0+} \frac{(e^{sh}-1)^\alpha}{h^\alpha} = s^\alpha e^{st} \quad \text{if } \mathrm{Re}(s) > 0. \tag{4}$$

With this result and the use of the two-sided Laplace transform we obtain:

$$\mathscr{L}[D_J^\alpha f(t)] = s^\alpha F(s) \quad \text{for } \mathrm{Re}(s) > 0 \tag{5}$$

generalizing a well-known result. This means that there is a fractional linear system (differintegrator) with transfer function equal to

$$H(s) = s^\alpha \text{ for } \mathrm{Re}(s) > 0. \tag{6}$$

The corresponding impulse response is given by [1,4]:

$$h(t) = \delta^{(\alpha)}(t) = \frac{t^{-\alpha-1} u(t)}{\Gamma(-\alpha)}, \tag{7}$$

where $u(t)$ is the Heaviside unit step. Using the convolution property of the Laplace transform, we obtain from (5) and (7)

$$D_J^\alpha f(t) = \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^t f(\tau)(t-\tau)^{-\alpha-1} d\tau \tag{8}$$

that is the Liouville forward derivative. We obtained an integral formulation for the derivative without the drawbacks of the Riemann–Liouville derivative. However, we must refer that (1) has a wider applicability. The formula (2.3) in [3] can be obtained from (8) by multiplying $f(t)$ by $u(t)$.

(4)

$$D_J^\alpha \left[D_J^\beta f(t) \right] = D_J^\beta \left[D_J^\alpha f(t) \right] = D_J^{\alpha+\beta} f(t).$$

To prove this statement we start again from (1). We write

$$\begin{aligned} D_J^\alpha \left[D_J^\beta f(t) \right] &= \lim_{h\to 0+} \frac{\sum_{k=0}^\infty \binom{\alpha}{k} (-1)^k \left[\sum_{n=0}^\infty \binom{\beta}{n} (-1)^n f[t + (\alpha+\beta)h - (k+n)h] \right]}{h^\alpha h^\beta} \\ &= \lim_{h\to 0+} \frac{\sum_{n=0}^\infty \binom{\beta}{n} (-1)^n \left[\sum_{k=0}^\infty \binom{\alpha}{k} (-1)^k f[t + (\alpha+\beta)h - (k+n)h] \right]}{h^\alpha h^\beta} \end{aligned} \tag{9}$$

for any $\alpha, \beta \in \mathbf{R}$, or even $\in \mathbf{C}$. With a change in the summation, we obtain

$$D_J^\alpha [D_J^\beta f(t)] = \lim_{h\to 0+} \frac{\sum_{m=0}^\infty \left[\sum_{n=0}^m \binom{\alpha}{m-n} \binom{\beta}{n} \right] (-1)^m f[t + (\alpha+\beta)h - mh]}{h^{\alpha+\beta}}$$

As $\sum_0^m \binom{\beta}{m-n} \binom{\beta}{n} = \binom{\alpha+\beta}{m}$

$$D_J^\alpha [D_J^\beta f(t)] = \lim_{h\to 0+} \frac{\sum_{m=0}^\infty \binom{\alpha+\beta}{m} (-1)^m f[t + (\alpha+\beta)h - mh]}{h^{\alpha+\beta}} = D_J^{\alpha+\beta} f(t).$$

This general result contradicts those presented in [3] (Section 3.4). We may ask where is the reason for the noncommutativity of the derivative proposed in [3]. If we compare the above Liouville derivative (8) with formulae (3.6)–(3.8) in [3], we conclude that there

(a) We are computing the derivative of the product $f(t).u(t)$.
(b) The derivative has several steps that introduce "initial conditions".

These facts together with the following result explain the noncommutativity of the derivative introduced in [3].

(5) To explain the above referred behaviour, we are going to compute the derivative of the product of two functions: $f(t) = \varphi(t)\cdot\psi(t)$. Assume that one of them has a Taylor expansion:

$$\varphi[t+(\alpha-k)h] = \sum_{n=0}^{\infty}\varphi^{(n)}(t)(\alpha-k)^n h^n$$

and the other has Laplace transform, $\Psi(s)$. Then

$$D_f^{\alpha} f(t) = \sum_{n=0}^{\infty}\varphi^{(n)}(t)\lim_{h\to 0+}\frac{\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}(\alpha-k)^n h^n\psi[t+(\alpha-k)h]}{h^{\alpha}} \tag{10}$$

and

$$\mathscr{L}[(\alpha-k)^n f[t+(\alpha-k)h]) = (\alpha-k)^n h^n \mathrm{e}^{s(\alpha-k)h}\psi(s) = \psi(s).\frac{d^n \mathrm{e}^{s(\alpha-k)h}}{\mathrm{d}s^n} \tag{11}$$

that leads to

$$\mathscr{L}\left[\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}(\alpha-k)^n h^n\psi[t+(\alpha-k)h]\right] = \psi(s).\frac{d^n(\mathrm{e}^{sh}-1)^{\alpha}}{\mathrm{d}s^n}.$$

As $\frac{(\mathrm{e}^{sh}-1)^{\alpha}}{h^{\alpha}}$ tends to s^{α} as h decreases to 0, and the nth derivative of s^{α} is equal to $\frac{(-\alpha)_n(-1)^n}{n!}s^{\alpha-n} = \binom{\alpha}{n}s^{\alpha-n}$ we obtain

$$\mathscr{L}\left[\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}(\alpha-k)^n h^n\psi[t+(\alpha-k)h]\right] = \binom{\alpha}{n}s^{\alpha-n}$$

and finally

$$\lim_{h\to 0+}\frac{\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}(\alpha-k)^n h^n\psi[t+(\alpha-k)h]}{h^{\alpha}} = \binom{\alpha}{n}\psi^{(\alpha-n)}(t). \tag{12}$$

Inserting (12) into (10) we obtain the derivative of the product:

$$D_f^{\alpha}[\varphi(t)\psi(t)] = \sum_{n=0}^{\infty}\binom{\alpha}{n}\varphi^{(n)}(t)\psi^{(\alpha-n)}(t). \tag{13}$$

This is the generalized Leibniz rule. The deduction presented here is different from others presented in literature [1,2] because it is based on the Grünwald–Letnikov derivative. As we can see it is non-commutative in agreement with our above affirmation. Eq. (13) states a result that conflicts with equation (4.12) in [3].

(6) The definition of fractional derivative stated in formulae (3.6)–(3.8) in [3] is of limited interest since it cannot be applied to important functions like the negative power $t^{-\alpha}u(t)$, $\alpha > 0$, or $t^{\nu-1}\sum_{n=0}^{\infty}\frac{a^n t^{n\gamma}}{\Gamma(n\nu+\nu)}\cdot u(t)$ that is the inverse Laplace transform of $\frac{1}{s^{\nu}-a}$. This is a very important function in fractional linear systems theory.

(7) The fractional Taylor series presented in Section 4.1 in [3] is also of limited interest. To see why, let us apply it to the exponential e^{at}, $a > 0$. We have:

$$\mathrm{e}^{a(t+h)} = \sum_{0}^{\infty}\frac{h^{\alpha k}}{\Gamma(\alpha k+1)}a^{\alpha k}\mathrm{e}^{at}$$

or

$$\mathrm{e}^{ah} = \sum_{0}^{\infty}\frac{(ah)^{\alpha k}}{\Gamma(\alpha k+1)}$$

that is an incorrect relation if $\alpha\neq 1$.

(8) The fractional Brownian motion (fBm) is currently defined by [5–7]:

$$B_H(t) - B_H(0) = \frac{1}{\Gamma(H+1/2)}\left\{\int_{-\infty}^{0}\left[(t-\tau)^{H-1/2} - (-\tau)^{H-1/2}\right]\mathrm{d}B(\tau) + \int_0^t (t-\tau)^{H-1/2}\mathrm{d}B(\tau)\right\} \tag{14}$$

and enjoys several interesting properties. Its autocorrelation is given by [6,7]:

$$R_H(t,\tau) = \frac{V_H}{2}\left[|t|^{2H} + |\tau|^{2H} - |t-\tau|^{2H}\right] \tag{15}$$

with

$$V_H = \frac{\sigma^2}{\Gamma(2H+1)\sin H\pi} \tag{16}$$

In [3] the author defines the fBm by:

$$B_H(t) - B_H(0) = \frac{1}{\Gamma(H+1/2)}\int_0^t (t-\tau)^{H-1/2} w(\tau)\mathrm{d}\tau \tag{17}$$

This formula was rejected by Mandelbrot and van Ness [6] because it "puts too great importance on the origin for many applications". Besides (17) cannot lead to (15) as it is easy to verify by direct computation of the autocorrelation. In fact, the fBm can be defined in terms of the integral of a fractional noise that is the Liouville derivative, (8), of white noise [8]. Recently, it was proved that the backward and central derivatives can also be used [9].

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