An Introduction to the Fractional Continuous-Time Linear Systems:
The 21st Century Systems

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Abstract

A brief introduction to the fractional continuous-time linear systems is presented. It will be done without needing a deep study of the fractional derivatives. We will show that the computation of the impulse and step responses is very similar to the classic. The main difference lies in the substitution of the exponential by the Mittag-Leffler function. We will present also the main formulae defining the fractional derivatives.

I. Introduction

The Fractional Calculus (FC) is a generalization of the traditional calculus that leads to similar concepts and tools, but with wider generality and applicability. By allowing derivative and integral operations of arbitrary real or complex order, it is to traditional calculus what the real or complex numbers are to the integers.

For almost 300 years fractional derivative was seen as an interesting, but abstract, mathematical concept. The development of the fractional calculus was mainly in the hands of mathematicians. This led to a number of competing definitions of the derivative and integral operators, originating a somewhat chaotic situation as people tried to extend the specific definitions of the traditional integer order to the more general arbitrary order context. Since the early 1990's more practically oriented scientists and engineers have been working with these various forms and felt the need for converging on physically meaningful formalisms.

We believe that the fractional calculus is ready for use in all aspects of Signals and Systems. What is necessary for researchers is to have access to the important tools of the theory. This is the main objective of this article: to introduce the fractional linear systems and offer some insights into how the involved mathematics is applied to very practical problems.

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It is already known that the non-integer order systems can describe dynamical behavior of materials and processes over vast time and frequency scales with very concise and computable models [4], [8], [12], [13], [16], [26], [29], [47]–[50], [53]–[56]. Nowadays well known concepts are being extended to the development robust control systems [4], [22], [23], [32], [49]–[51], [62]–[64], as well as signal filtering methods [8], [12], [36], [47], [60]. Of particular interest is the fact that the fractional systems exhibits both short and long term memory. While the short term memory corresponds to the "distribution of time constants" associated with the distribution of isolated poles and zeroes in the complex plane, the long term memory corresponds to infinitely many interlaced close to each other poles and zeros that in the limit originate a branch cut line [3], [39], [41], [42], [50], [62]. This translates to a lack specific time scale and, therefore, no new resonance or other instability effects and incorporates the power law behavior found in natural systems that show the greatest robustness to variation of environmental parameters.

Another interesting property of fractional derivatives lies in the intrinsic causality characteristic it enjoys forcing us to include time ordering into the setup of differential equations. As known, in integer order linear systems a given transfer function defines several systems—causal, anti-causal, and several acausal systems—accordingly to the region of convergence that we attach to it. In the fractional case, we only have two: causal and anti-causal systems [36].

In this paper we intend to present a simple approach to the basic structure underlying almost all the practical applications: the fractional linear systems. This presentation follows a way similar to the classic as described in most textbooks. In particular, we will show how to compute the impulse and step responses.

There is also a class of discrete-time fractional linear systems. However they are based on fractional delay difference equations and seem not to have any relation to the fractional derivatives (see [21], [37], [38]). The space-time systems will not be studied either [14], [27], [33]. The derivatives relatively to the space are not causal neither anti-causal and are conveniently described by central fractional derivatives [40].
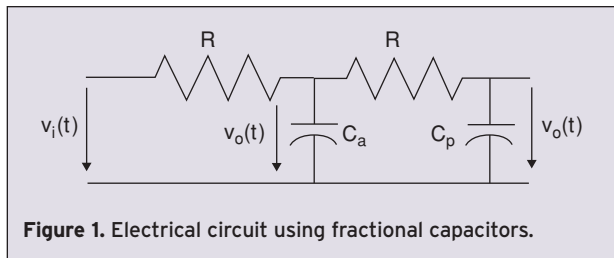

Figure 1. Electrical circuit using fractional capacitors.

II. The Fractional Continuous-Time Linear Systems

A Simple Example

Consider the circuit in Figure 1. The capacitors in the circuit are fractors [4] with impedance equal to $\frac{K}{(j\omega\tau)^{1/2}}$ that we write as $\frac{1}{C_{1/2}(j\omega)^{1/2}}$ to have a format similar to the usual. Working in the frequency domain, we can write:

$$\frac{V_0(\omega)}{V_i(\omega)} = \frac{1}{R^2C_{1/2}^2(j\omega) + 3RC_{1/2}(j\omega)^{1/2} + 1}$$

Putting $s = j\omega$, as usually, we obtain the transfer function of the circuit:

$$H(s) = \frac{V_0(s)}{V_i(s)} = \frac{1}{R^2C_{1/2}^2 s + 3RC_{1/2}s^{1/2} + 1}$$

Let us assume that the transform of a derivative property of the Laplace transform remains valid:

$$TL[f^{(\alpha)}(t)] = s^{\alpha}F(s) \, Re(s) > 0$$

It is not hard to show that the input-output relation is given by:

$$a_3.Dv_o(t) + a_1.D^{1/2}v_o(t) + v_o(t) = v_i(t)$$

where D is the derivative operator and $a_1 = 3RC_{1/2}$ and $a_2 = R^2C_{1/2}^2$. We conclude that the above circuit is described by a fractional differential equation. We are going to show how to study this kind of systems.

General Description

Here, we will study the systems described by constant coefficient linear fractional differential equations: fractional linear time-invariant (FLTI) systems. They assume the general format

$$\sum_{n=0}^{N} a_n D^{\nu_n}y(t) = \sum_{m=0}^{M} b_m D^{\nu_m}x(t) \quad \nu_n < \nu_{n+1} \qquad (1)$$

where D means derivative and ν_n $n = 0, 1, 2, \ldots$ are derivative orders that we will assume to be positive real numbers. With this definition, we are in conditions to define and compute the Impulse Response and Transfer Function. Being a linear system the system described by (1) has the exponential as eigenfunction. Letting $x(t) = e^{st}$, where $s \in C$ and $t \in R$, we obtain $y(t) = H(s).e^{st}$, where $H(s)$ is the transfer function given by

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$$H(s) = \frac{\sum_{m=0}^{M} b_m s^{\nu_m}}{\sum_{n=0}^{N} a_n s^{\nu_n}} \qquad (2)$$

provided that $\mathrm{Re}(s) > 0$ or $\mathrm{Re}(s) < 0$. With $s = j\omega$, we obtain the Frequency Response, $H(j\omega)$, and can represent the Bode diagrams as in the usual systems. It is interesting to remark that the asymptotic amplitude Bode diagrams are constituted by straight lines with slopes that may assume any value, contrarily to the usual case where the slopes are multiples of 20 dB/decade.

The system represented by s^α is called differintegrator. However, we must be careful when dealing with s^α that is a multivalued expression defining an infinite number of Riemann surfaces. Each Riemann surface defines one function. Therefore, (1) can represent an infinite number of linear systems. However, only the principal Riemann surface may lead to a real system. Constraining s^α by imposing a region of convergence, we define a transfer function. Choosing the left half real axis as branch cut line we obtain the transfer function of the causal system. Its impulse response is given by

$$\delta_+^{(\alpha)}(t) = \frac{t^{-\alpha-1}u(t)}{\Gamma(-\alpha)} \qquad (3)$$

$\Gamma(.)$ is the Euler gamma function. In the anti-causal case we choose the right half real axis as branch cut line to obtain the impulse response

$$\delta_-^{(\alpha)}(t) = -\frac{t^{-\alpha-1}u(-t)}{\Gamma(-\alpha)} \qquad (4)$$

With $\alpha = -1$, we obtain the normal integrator impulse responses. With those impulse responses, we can obtain fractional the differintegrated of a given signal by the convolution. We are led to the forward

$$D_+^{(\alpha)}f(t) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty f(t-\tau).\tau^{-\alpha-1}d\tau \qquad (5)$$

and backward

$$D_-^{(\alpha)}f(t) = \frac{(-1)^{-\alpha}}{\Gamma(-\alpha)} \int_0^\infty f(t+\tau).\tau^{-\alpha-1}d\tau \qquad (6)$$

differintegrations. These formulae were obtained first by Liouville [9]. The second is also called Weyl differintegration [55].

From the Transfer Function to the Impulse Response

The general case represented in (1) is not easy to solve, because, it is difficult to find the poles. For this reason, in the following, we shall be restricting our attention to the cases in which

1. the ν_n are irrational numbers but multiples of a given ν;
2. the ν_n are any rational numbers. In this case, write them in the format p_n/q_n.

Let ν be the greater common divider of the ν_n. Then $\nu_n = n.\nu$. We will assume that $\nu < 2$, for stability reasons. A differential equation with $\nu = 1/2$ is said semi-differential [5]. The coefficients and orders do not coincide necessarily with the previous ones, since some of the coefficients can be zero. For example, the equation $[aD^{1/3} + bD^{1/2}]y(t) = x(t)$ transforms into $[bD^{3.1/6} + aD^{2.1/6} + 0.D^{1/6}]y(t) = x(t)$.

With this formulation, the equations (1) and (2) assume the general formats

$$\sum_{n=0}^{N} a_n D^{n\nu} y(t) = \sum_{m=0}^{M} b_m D^{m\nu} x(t) \qquad (7)$$

and

$$H(s) = \frac{\sum_{m=0}^{M} b_m s^{m\nu}}{\sum_{n=0}^{N} a_n s^{n\nu}} \qquad (8)$$

With a Transfer Function as in (8) we can perform the inversion quite easily, by following the steps:

1. Transform $H(s)$ into $H(z)$, by substitution of s^ν for z. We are assuming that $H(z)$ is a proper fraction; otherwise, we have to decompose it in a sum of a polynomial (inverted separately) and a proper fraction.
2. The denominator polynomial in $H(z)$ is the indicial polynomial [5] or characteristic pseudo-polynomial [36]. Perform the expansion of $H(z)$ in partial fractions.
3. Substitute back s^ν for z, to obtain the partial fractions in the form

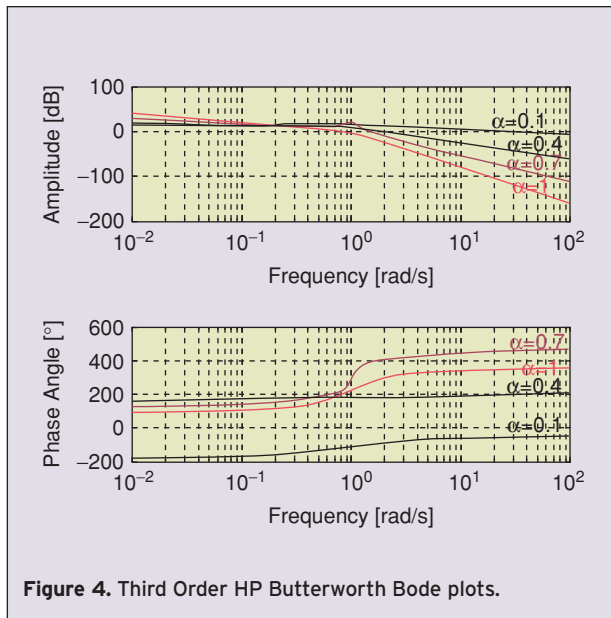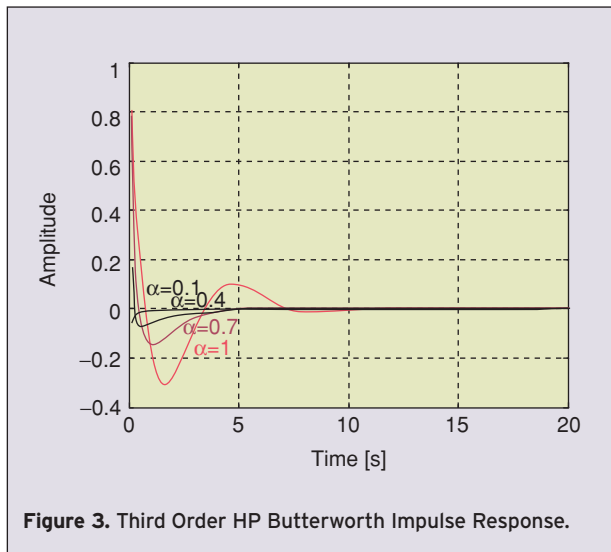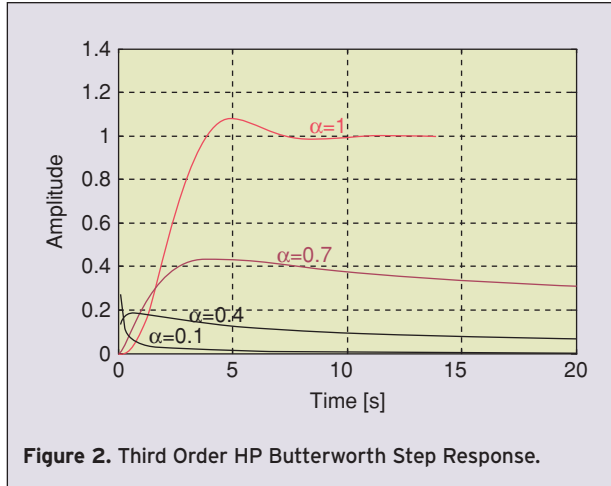$$F(s) = \frac{1}{(s^\nu - a)^k} \quad k = 1, 2, \dots \qquad (9)$$

4. Invert each partial fraction.
5. Add the different partial Impulse Responses.

Partial Fraction Inversion

We are going to see how to invert $F(s) = \frac{1}{s^\nu - a}$. Using the properties of the geometric series, it is a simple task to obtain:

$$F(s) = s^{-\nu} \sum_{n=0}^{\infty} a^n s^{-n\nu} \qquad (10)$$

with $\mathrm{Re}(s) > |a|^{1/\nu}$ defining the region of convergence. However, all the terms of the series are analytic for $\mathrm{Re}(s) > 0$. For this reason, we can invert this series term by term, to obtain:

Figure 2. Third Order HP Butterworth Step Response.

Figure 3. Third Order HP Butterworth Impulse Response.

Figure 4. Third Order HP Butterworth Bode plots.

$$f(t) = t^{\nu-1} \sum_{n=0}^{\infty} \frac{a^n t^{n\nu}}{\Gamma(n\nu + \nu)} . u(t) \qquad (11)$$

that is a special case of the two parameter Mittag-Leffler function that is a generalization of the exponential to what it reduces when $\nu = 1$. This function is well studied {see [1], [7], [19], [51], [54]}. An interesting implementation was done by Prof. Podlubny and can be found at the site of MatLab. It is an implementation of the two parameter generalized Mittag-Leffler function with precision control—usage: mlf(alfa, beta, z, p). Equation (11) suggests us to work with the step response instead of the impulse response to avoid derivatives or working with non-regular functions near the origin.

The $k > 1$ case in (9) does not present great difficulties except some additional work. It can be obtained from the $k = 1$ case by repeated convolution or by differentiation. For example:

$$\frac{1}{(s^\nu - a)^2} = -\frac{1}{\nu s^{1-\nu}} \frac{d}{ds}\left[\frac{1}{s^\nu - a}\right] \qquad (12)$$

$$\frac{1}{(s^\nu - a)^2} = s^{-2\nu} \sum_{n=0}^{\infty} (n + 1) a^n S^{-n\nu} \qquad (13)$$

We do not go further, since this example shows how we can proceed in the general case. We can use formula (13) to obtain the corresponding formula.

Example—Third Order LP Butterworth filter

The Stability Problem

The impulse and step responses of this filter are shown in Figures 2 and 3, respectively. In Figure 4, we present the corresponding Bode plots. The study of the stability of the FLTI systems we are going to do is based on the BIBO stability criterion that implies stability when the impulse response is absolutely integrable.

The simplest FLTI system is the system with transfer function $H(s) = s^\nu$ with s belonging to the principal Riemann surface. If $\nu > 0$, the system is definitely unstable, since the impulse response is not absolutely integrable, even in a finite interval. If $-1 < \nu < 0$, the impulse response remains a limited function when t increases indefinitely and it is absolutely integrable in every finite interval. Therefore, we will say that the system is wide sense stable. This case is interesting to the study of the fractional stochastic processes. If $\nu = -1$, the normal integrator, the system is wide sense stable. The case $\nu < -1$ corresponds to an unstable system, since the impulse response is not a limited function when t goes to $+\infty$.

Consider the LTI systems with transfer function H(s) a quotient of two polynomials in s^ν. The transformation $w = z^q$, transforms the sector $0 \le \theta \le 2\pi/q\{\theta = \arg(z)\}$

into the entire complex plane. So, the sector $\frac{\pi}{2q} \leq \theta \leq \frac{\pi}{2q} + \frac{\pi}{q}$ is transformed in the left half plane. Consider the first Riemann surface of $z = s^\nu$ defined by $\theta = \arg(s) \in (-\pi; \pi]$. This domain is transformed into $\varphi = \arg(z) \in (-\pi\alpha; \pi\alpha]$. However the poles leading to instability must be inside the sector $(-\pi\alpha/2; \pi\alpha/2)$. We have two situations leading to stability:

1. There are no poles inside the sector $(-\pi\alpha; \pi\alpha]$.
2. There are poles but they are in the sectors: $(-\pi\alpha; -\pi\alpha/2)$ and $(\pi\alpha/2; \pi\alpha)$.

The poles with argument equal to $\pm\pi\alpha/2$ may lead to wide sense stable systems as in the usual systems. These conclusions come from properties of the Mittag-Leffler function. To give a simple example, consider the transfer function $H(s) = \frac{1}{s^\alpha + 1}$, with $0 < \alpha < 2$. It is easy to see that there is no pole in the principal Riemann surface. So, it represents a stable system.

The Applications

The applications to physics and engineering are not recent; the application to viscosity dates back to the thirties in the last century. The works of Mandelbrot in the field of fractals had great impact and attracted the attention to fractional calculus. During the last 20 years the application domains of fractional calculus increased significantly. Here is a brief list of some of the reasons why the fractional calculus is catching on:

- There is evidence that several biological and man made signals have spectra that do not increase or decrease by multiples of 20 dB. This happens, for example, with ECG, speech, music, etc [13], [28], [29], [58], [59], [65]–[68]. The electric power line is a channel with such characteristics.
- The long range processes (1/f noises)—the fractional Brownian motion (fBm) is the most famous—have been attracting the attention because their importance in many practical systems [18]–[20], [30], [31]. Although there are several methods for analysis and synthesis of such signals, for example, using wavelets; modelling with fractional derivatives has proven to be more efficient and natural [43], [44].
- The famous Curie law stating that the current in an insulator increases proportionally to a negative power of the time leads to the known "supercapacitors" that have impedance proportional to $1/s^\alpha$, with $0 < \alpha < 1$ [67]. Electrochemists have used the Constant Phase Elements (CPE) description for over 60 years. The new terminology is "fractance" to indicate an impedance with fractional order response. As these devices become available commercially, we will be rewriting many of the rules for design of filters and controllers [4], [6], [26], [57], [61].

This explains why we can find fractional calculus in

- Materials Theory [13], [16], [53], [59]
- Control [4], [23], [32], [47]–[50], [62]–[64]
- Viscoelasticity [28]
- Electromagnetism [10], [11]
- Statistical Mechanics [58]
- Diffusion Theory [14], [15], [27], [33]
- Internet Traffic [68]
- Bioengineering [25]

and in other areas [1], [24], [25], [45], [46], [54]. Very interesting are the engineering applications of A. Oustaloup and his group [22], [48]–[50], [53], Vinagre et al. [32], [56], [62]–[64], Machado et al. [2], [3], [23], and the applications in Physics by Agrawal, Baleanu, and Nigmatullin (see the papers in [1], [24], [45], [46], [54]).

Modeling, Identification, and Implementation

As in the usual systems, modeling, identification, and implementation are very interesting tasks. In the fractional case, they are slightly more difficult due to the fact of having, at least, one extra degree of freedom: the fractional order. However, this difficult increments the possibilities of obtaining more reliable and robust systems. This is challenging and people working in the area has been giving different interesting answers. We can refer the following approaches:

1. Circuit implementations with fractional elements—It consists of using the classic circuit theory, but with fractional capacitors [4], [26], [61] and coils [57].
2. Trans-finite circuits—The infinite transmission lines are circuits with fractional behavior [7], [17], [51], [52], but there are other interesting circuits with similar characteristics like the tree fractance (a tree of RC circuits) and chain fractance (a series of parallel RC) circuits.
3. Band-limited approximations—It is an engineer approach. There are several ways of doing the design and implementation we can refer a) the CRONE that uses the Bode diagrams [47]–[50], [56] band b) the continued fraction approaches [62]–[64]. Both construct pole-zero systems with interlaced poles and zeros.
4. Identification from frequency data—It consists on a least-squares approach in the frequency domain. The more interesting algorithm uses a generalized Levy method [54], [60].
5. Discrete-time implementations—there are several algorithms that start from an s to z conversion and design an ARMA model [2], [3], [6], [23], [41], [42].

Although there is no consensus neither standard design rules, there are several interesting applications where the implemented systems proved to be better than the corresponding integer order systems.

III. On the Derivative Definitions

Some Considerations

In the previous sections the fractional linear systems were studied in a formal way very close to the ordinary. Strangely the notion of fractional derivative was introduced but a formal defintion was not presented. This means that the formalism just develop avoids the need for a concrete fractional derivative definition. However, there are applications where we need to use the definition of fractional derivative. One of these cases is the discrete-time realization of fractional linear systems considered above. However, the derivative definition will be presented together with a sequence of integral representations. There are several texts on fractional calculus interesting from a mathematical point of view [19], [34], [35], [55]. However, they may not be interesting from an engineering approach at least when starting. The reason lies in the introduction of fractional derivative definitions that are not useful for engineering applications.

From Differences to Derivatives

Let $f(z)$ be a complex variable function and introduce Δ_d and Δ_r as finite "direct" and "reverse" [39] order one differences defined by:

$$\Delta_d f(z) = f(z) - f(z - h) \tag{14}$$

and

$$\Delta_r f(z) = f(z + h) - f(z) \tag{15}$$

with $h \in C$ and, as before, we assume that $Re(h) > 0$. For any order (including the negative integer case) we have [39], [55]:

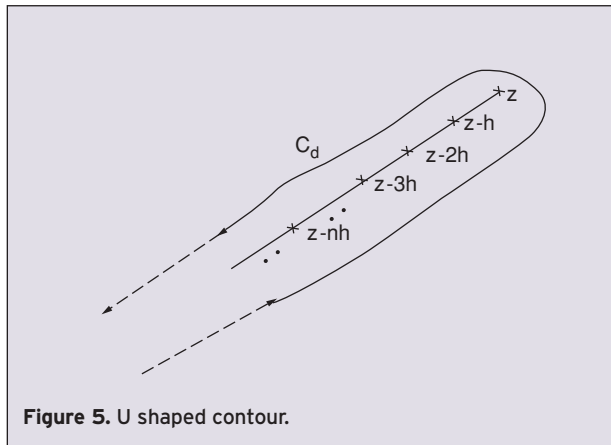$$\Delta_d^\alpha f(z) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z - kh) \tag{16}$$

and

Figure 5. U shaped contour.

$$\Delta_r^\alpha f(z) = (-1)^\alpha \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z + kh) \tag{17}$$

where $\binom{\alpha}{k}$ are the binomial coefficients.

Divide (16) by h^α to obtain the fractional incremental ratio. Performing the computation of its limit as $h \to 0^+$, we obtain the direct Grünwald-Letnikov derivative given by:

$$D_d^\alpha f(z) = \lim_{h \to 0+} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z - kh)}{h^\alpha} \tag{18}$$

where h is any complex number in the right hand complex plane. Making a substitution $h \to -h$ {or using (17)}, we obtain the reverse Grünwald-Letnikov derivative.

$$D_r^\alpha f(z) = \lim_{h \to 0+} (-1)^\alpha \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z + kh)}{h^\alpha} \tag{19}$$

Expression (18) corresponds to the forward Grünwald-Letnikov fractional derivative [39], [51], [55] while (19) is the backward Grünwald-Letnikov fractional derivative [39]. It is interesting to remark that these definitions were proposed first by Liouville [9].

Although we are not concerned here with existence problems, we must refer that in general we can have the direct derivative without existence of the reverse one and *vice-versa*. For example, let us apply both definitions to the function $f(z) = e^{az}$. If $Re(a) > 0$, expression (18) converges to $D_d^\alpha f(z) = a^\alpha e^{az}$, while (19) diverges. On the other hand, if $f(z) = e^{-az}$ equation (18) diverges while (19) converges to $D_r^\alpha f(z) = (-a)^\alpha e^{-az}$. It is interesting to remark that, if z and h are real, in (18) we are using the current and past values of the function: it is a *causal derivative*; On the other hand in (19) we use the current and future values: it is an *anti-causal derivative*.

Integral Representations of Derivatives

Let $f(z)$ be a complex variable function analytic in a region that includes a half straight line starting at z and defined by $z - nh$, with $n \in Z$; h is any complex in the right hand d'Argand plane. Consider the U shaped contour represented in Figure 5.

Assume that this line is inside the analyticity region.

With the above definitions and conditions we can state the following result [34], [39]

$$D_+^\alpha f(z) = \frac{\Gamma(\alpha + 1)}{2\pi i} \oint_C f(w) \frac{1}{(w - z)^{\alpha+1}} dw \tag{20}$$

The right hand side is the generalized Cauchy derivative. C is the U-shaped path in Figure 5 lying in the left half plane defined by the straight line passing over z. Making

a substitution $h \to -h$ we obtain a generalized Cauchy with a branch cut line on the right half plane

$$D_-^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \oint_C f(w) \frac{1}{(w-z)^{\alpha+1}} dw \qquad (21)$$

We must remark that the right hand side remains the same excepting the integration path. Now, it lies in the right hand complex plane defined by a vertical straight line passing over z.

We can go a bit further by deforming the contour used in (20) and (21) in order to transform it in the Hankel path [39], [51]. We obtain:

$$D_\pm^\alpha f(z) = \frac{e^{j(\pi-\theta)\alpha}}{\Gamma(-\alpha)}$$
$$\times \int_0^\infty \frac{\left[f(x.e^{j\theta}+z) - \sum_0^N \frac{f^{(n)}(z)}{n!} e^{jn\theta} x^n \right]}{x^{\alpha+1}} dx$$

$$(22)$$

where θ is the angle between the positive real axis and the branch cut line. This is a regularized integral "a la Hadamard", but, contrarily to the usual, obtained without rejecting any infinite part. If $\theta = \pi(+)$, we have the forward derivative, while with $\theta = 0(-)$, we obtain the backward one.

For functions with LT, we obtain (5) and (6) [39].

IV. Conclusions

We made a brief introduction to the fractional linear systems. We did it without needing a deep study of the fractional derivatives. We showed that the computation of the impulse and step responses is very similar to the classic. The main difference lies in the substitution of the exponential by the Mittag-Leffler function. We presented also the main formulae defining the fractional derivatives.

Some applications and implementations were considered. From them we can predict a great future for fractional systems. We agree with Prof. Nishimoto when he that says that fractional calculus is the 21st century calculus and say that fractional systems will be the 21st century systems. The number of published papers in different areas has been increasing and will continue to grow up in parallel with the diffusion of the theory.

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