

COGNITIVE MODELS FOR THE CONCEPT OF ANGLE

by

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Licenciatura, Faculdade de Ciências de Lisboa 1982

M. Ed., Boston University 1985

A Dissertation Submitted to the Graduate Faculty
of The University of Georgia in Partial Fulfillment

of the

Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

1999

Notary Public, Oconee County, Georgia
My Commission Expires August 14, 2002

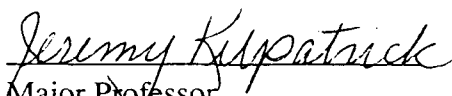
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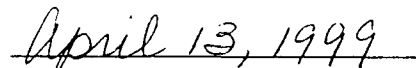
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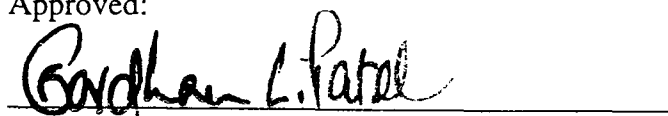
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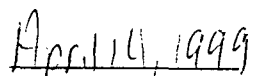
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The instructional models taught in class were similar to the students' models. The teachers addressed angle as a basic-level category, discussed its submodels, clarified the boundaries, and established cognitive reference points. They gradually increased the use of complex metaphors and of several models.

The study enriched the characterization of the first two levels of van Hiele theory and demonstrated the value of categorization theory in understanding how our comprehension of mathematics is rooted in basic human attributes pertaining to the material and social conditions of human life. The embodiment of mathematical ideas by the material world, including our bodies, needs greater emphasis in all facets of mathematics education.

INDEX WORDS: Angle, Cognitive models, Embodiment, Embodied cognition, Geometry, Lakoff, Language, Learning, Mark Johnson, Mathematical thinking, Metaphors, Prototypes, van Hiele theory

DEDICATION

To my mother
Maria Luiza

and

my daughter
Marta

ACKNOWLEDGEMENTS

There are many people that helped this dissertation throughout its long journey. First, I want to thank my major professor, Jeremy Kilpatrick, for his insights, his detailed comments, and his support. His persistence was a major force helping me to come to this point. He believed, perhaps more than I did, that my work could actually be finished. I don't think I will forget that, and I just wish that I am able to care about my students as he cared for me. I also thank him for his scholarly support. Today, it is an invisible component of my dissertation that only I can fully appreciate.

I also want to thank to all members of my committee: Jim Wilson, John Olive, Pat Wilson, and Jon Carlson. To all, I wish to tell that I thank them both for their keen critiques that invariably led to a deepening of the original ideas and for their support throughout these years.

I wish to extend my gratitude to the faculty, the students, and the staff of the Mathematics Education Department at the University of Georgia, where I found a rich environment that helped shape my thinking. I particularly remember vivid exchanges with Leslie Steffe, Julio Mosquera, Hiro Sekiguchi, Joe Zilliox, and Bob Moore which were an invaluable part of my education at UGA.

I wish to extend my thanks with gratitude to the fourth and fifth graders and their teachers who participated in the study. Their cooperation was vital to the study and was sincerely appreciated.

Many friends in Portugal helped this work. A word of appreciation must be added to the Secção de Ciências da Educação of the Faculdade de Ciências e Tecnologia. In particular I want to recognize Teresa Ambrósio for her enthusiasm and for providing the vital atmosphere necessary for the writing of several chapters. To Teresa Oliveira, I owe her support at several times and to João de Freitas the availability in setting up the technical details for my oral defense.

Many friends helped me through out the years, both in times of hope and in times of pessimism. I wish to express my appreciation to Ana, António Domingos, Belo, Darlinda, Henrique, João Pedro, Lurdes, and Manuel Saraiva for their belief in the viability of this work. Special gratitude must be expressed to Eduarda and Tó Zé for their support, without which the latest part of this work would have been much more difficult.

Finally, this work could not have been done without the support of the Portuguese Ministry of Education and the Fulbright Foundation.

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CHAPTER 1

ANGLES AS CONCEPT AND OBJECT

What is an angle? The search for an answer to this question motivated the present study. Angles seem to be complex geometrical objects. By comparison, take a line, for example. Although Heath (1956) discusses at length the different perspectives on lines taken throughout history, one can still *look* at a line (or a triangle, etc.). Lines seem to have a substance. But what is the substance of angles? An infinite area bounded by two rays, a rotation of a ray, or, as Euclid put it, the inclination of one line over another? What is the substance of *inclination*?

These initial considerations led me to search for answers from both a mathematical and a historical point of view. A survey of mathematical textbooks, from school mathematics, undergraduate courses, and at the graduate level, together with insights from research, evidenced several models of angle used in school mathematics. A review of angles in the history of mathematics (Matos, 1990, 1991b) revealed that (a) there is a vast array of nonisomorphic definitions of *angle* in contemporary mathematics, (b) there is disagreement over key elementary issues like whether a radian is a unit of measure, and (c) particular types of angles, like the angle of contingency, although discussed at length in the past because of its nonarchimedean properties, have been put aside by contemporary mathematics. Research on mathematics learning (Close, 1981) also showed that students display distinct conceptions of angles; namely, the static angle and the dynamic angle. Angles seemed, indeed, a complex issue.

Relevance of Angles for School Mathematics

But does the study of angles have any relevance for teaching and learning mathematics? From the structuralist perspective that swept school mathematics in the late 1950s and 1960s, it would certainly not be important. The determination with which Dieudonné banned triangles from school mathematics could very well be exerted also to angles. From the structuralist point of view, geometry was just a particular case of groups of transformations (Choquet, 1964; Dieudonné, 1964), and angles, in particular, were displaced by groups of matrices with specific characteristics. This is not the perspective from which I look at geometry. I prefer Freudenthal's (1973) phenomenological proposal:

[School] geometry is grasping space. ... It is grasping space in which the child lives, breathes and moves. The space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it. (p. 403)

Teaching geometry starts by letting students organize spatial phenomena and manipulating these means of organization, and not by acquainting children with broader mathematical structures, alien to their world of experience. Mathematics is seen as a topic of inquiry, rooted in our intellectual abilities, our social interactions, and our material world. My perspective on school geometry includes the study of triangles, circles, cubes, and similar entities, together with an awareness of the relationships among them and their components. For most students, the study of higher geometrical topics can only be successful if it is built upon this initial organization of spatial phenomena. In this approach to school geometry, angles do play a role, and they are a worthwhile topic for scientific research to scrutinize.

The Search for an Adequate Research Paradigm

The next question was: How should angles be studied? What would an adequate paradigm be that would produce credible results? We use *angles* to refer to a set of mathematical objects with specific characteristics. A way to study them would be to look at angles the same way that psychology and linguistics look at categories of things “in the world.”

New views have arisen about the ways in which we categorize common objects of the world (Armstrong, Gleitman, & Gleitman, 1983; Coleman & Kay, 1981; Pavitt & Haight, 1985; Rips, Shoben, & Smith, 1973; Rosch, 1973, 1978). These studies distinguish between the artificial categories used in former studies of categorization (red triangle) and the natural categories of “concepts designable by words in ‘natural languages’” (Rosch, 1973, p. 329). Natural categories are continuous and not discrete. They are not necessarily composed of combinations of simpler attributes, and some stimuli are considered better examples than others. The classical example is that of the category of birds. People have strong ideas of what a typical member of the class is and of the degree to which a given animal is or not a bird. More recently, categorization has become an integral part of cognitive scientists’ models of learning, helping to explain learning.

Psychologists have studied several mathematics categories. Armstrong et al. (1983) included in their research, together with other categories, the category of odd number. Participants were asked to rate the extent to which each given number represented their idea or image of the meaning of this category. Although odd number is clearly a definitional category—that is, there exists a unique criterion to determine membership in the category of odd numbers—participants considered, among others, 3 to be a better odd number than 501. The investigation showed that there is no difference in the extent in which the classification of well-defined categories differs from that of fuzzy categories (those without a clear definition). Moreover, participants seemed to agree extensively on

the classifications produced. The idea that categories, even of mathematical entities, are characterized solely by the properties shared by their members is questioned in the new views of categorization.

Producing a new account of the ways in which the human mind is able to categorize entities from its own experience was a task undertaken by Lakoff and Johnson (Johnson, 1987; Lakoff, 1987; Lakoff & Johnson, 1999). Basically, they argue that:

The mind is inherently embodied.

Thought is mostly unconscious.

Abstract concepts are largely metaphorical. (Lakoff & Johnson, 1999, p. 3)

They claim that their perspective has broad philosophical consequences. It affects the belief shared by many schools of thought that meaning is externally determined and is carried by signs and symbols. As a byproduct, this theory of cognitive science blends elegantly with a new attitude towards mathematical concepts proposed by some philosophers and by some researchers in the sociology and history of mathematics (Bloor, 1992; Lakatos, 1976). Mathematical definitions and proofs are seen as the product of the social negotiation of meanings. To study the concept of angle, under this approach, is to look for cognitive models that are embedded in this category, understand the ways in which they relate to each other, and, eventually, understand how they evolve over time, as students move from an elementary understanding of geometry to higher mathematical systems. The term *cognitive model* refers to a holistic structure governing our understandings of entities, events, and situations we encounter in everyday experience. In this study, I restrict the use of this term to the cognitive structures of the students. These models reflect *image schemas*, which are basic experiential structures that are a consequence of the nature of human biological capacities and the experience of functioning in a physical and social environment. The term *basic level* refers to a significant level of human interaction with the external environment, characterized by gestalt perception, mental imagery, and motor movements. Finally, I need to distinguish

between the models of the students (cognitive models) and the models occurring in educational contexts. The term *instructional model* is, then, applied to classroom exchanges or educational materials' content that mirrors a specific cognitive model.

Characterizing Complexity in Geometrical Reasoning

The search for the ways in which students move from an elementary understanding to higher, more abstract mathematical concepts has been pervasive in mathematics education research. In geometry, in particular, van Hiele theory may be credited with an elegant model for such abstracting processes. Its paradigm of recursive levels of abstraction appeals from both a mathematical and an educational point of view and has in broad outline been confirmed by research. As Schell (1998) puts it:

The van Hiele Model of Geometric Thought has been the pivot around which research in the learning of geometry has focused. (pp. 2-3)

The van Hieles have borrowed several psychological perspectives and incorporated them into the theory. The first version of the theory, developed in 1957, has a profound gestaltist flavor, whereas van Hiele's last book takes a more constructivist perspective. In the first version of the theory, abstraction was produced by progressive differentiation and restructuring of the field of perception (van Hiele-Geldof, 1984). In van Hiele's book *Structure and Insight* (1986), abstraction occurs at each level and is a recursive process of moving from signs to symbols through classroom interactions.

Lakoff and Johnson's perspectives on categorization stem mainly from linguistic studies. Although providing a thorough analysis of the structure of categories, they do not offer good descriptions of how categories are formed or how they are learned. The recognition of the importance of classroom interactions in the formation of geometric concepts makes van Hiele theory a useful complement to their work, providing a graded

perspective on abstraction and on the means by which generalized geometrical concepts are developed.

Research Questions

The purpose of the present study was to investigate the ways in which the geometrical concept of angle is understood by individual students, and to analyze the contexts involved in this understanding. This purpose was accomplished by addressing the following questions:

1. The first set of questions aim at understanding deeper cognitive structures that *underlie* the category of angle: What are the cognitive image schemas related to angles? How do they interact among themselves?

2. The second set of questions aim at understanding the *structure* of the category of angle: What are the cognitive models students have that constitute the category of angles for them? How do these models interact with each other? How do these models relate to image schemas?

3. The third set of questions aim at understanding how this structure connects with the *complexity* of students' geometric thinking: How are these cognitive models students have related to the van Hiele levels of geometric reasoning?

4. The fourth set of questions aim at understanding the processes under which these models are *taught*: What instructional models relating to the category of angle are used in classrooms? What classroom processes give rise to these models? What are the teachers' strategies for teaching these models? How do these strategies evolve through time?

5. The fifth set of questions aim at understanding the *background* against which the teaching and learning occur: What are the instructional models implicitly proposed to teachers and students by the materials used during the learning process? What happens to the models as the materials penetrate into more complex geometric topics?

Overview of the Dissertation

The next chapter (chapter 2) of this report begins with a review of the research related to the processes under which we categorize objects. The idea that categories of things are built from necessary and sufficient conditions is criticized, as it fails to explain a large body of research on this area. The key ideas of Lakoff and Johnson—like idealized cognitive models, embodied cognition, metaphoric and metonymic models, and image schemas—that largely support the theoretical background of this study are discussed here. This chapter ends with an initial discussion of the ways in which their ideas can be applied to results from research in mathematics education.

The third chapter bears on a different topic. For twenty years, van Hiele theory has been a reference for studies in geometry and in this chapter multiple aspects of the theory are discussed. The purpose is to understand how it can account for the study of complexity in geometrical teaching and learning.

The categorization studies discussed in chapter 2 and the van Hiele theory discussed in chapter 3 come from distinct research traditions. To use them together in this study, to apply them jointly to several research problems in mathematics education, required probing their compatibility and their power to shed light on those issues. That was accomplished in chapter 4, in which questions about learning geometry were jointly addressed by both theories, producing distinct interpretations.

Chapter 5 discusses the methodological issues of this research; namely, the context in which the study occurred, a description of the participants, and an account of procedures for data collection and for data analysis. The next three chapters present the results. In chapter 6, four image schemas related to the students' cognitive models of angles are presented. Chapter 7 discusses these models, together with the evidence for a basic level categorization of angles. This chapter ends by establishing a relationship between the cognitive models and the van Hiele levels. Chapter 8 analyzes lessons where angles were

taught and examines the materials used by the teachers and the students. The last chapter, chapter 9, presents the conclusions of this study.

CHAPTER 2

CATEGORIZATION OF CONCEPTS AND MATHEMATICAL OBJECTS

One of the concerns of psychologists and educators has been the ways in which students abstract ideas and concepts from their experience. These abstractions help us organize our experiential world. Gestalt theory conceived abstraction as a reorganization of the field of perception. Piaget viewed it as the formation of schemas. And cognitive scientists incorporated into it the mechanisms of generalization, differentiation, and pattern recognition.

Abstraction involves, among other things, the formation of categories of objects. Traditional psychological studies assumed that a small set of simple properties is necessary and sufficient to establish membership in a category. These categories have defining or critical attributes that determine which elements are or are not members of a category. Take, for example, the set of red flowers. Boundaries of this category are assumed to be sharp and not fuzzy; that is, given any flower, either it is red or not (Gardner, 1985). In Piaget's studies, for example, equivalence relationships determined membership in categories. Moreover, psychologists assumed that we use necessary and sufficient properties of each category when performing inferences, making deductions, and constructing a taxonomy of categories. In the last three decades, researchers have questioned each of these assumptions.

The classical view of categories was first challenged by Wittgenstein's work. He pointed out that the category of *games* does not have a set of common properties shared by all of its members. Some games share amusement, others luck, competition, skill, or a mix

of these. Wittgenstein (1953) conjectured that game was a cluster concept, held together by a variety of attributes but with no instance containing all the attributes. The category of games has a *family resemblance* structure. Like family members, games are similar to one another in some but not all ways. Some categories, like games or numbers, have no fixed boundaries and can be extended, depending on one's purposes. Some categories, like numbers or polyhedra, have central members. Any definition of numbers must include the integers, just as any definition of polyhedra must include the cube (Gardner, 1985; Lakoff, 1987).

The Study of Prototype Effects

Experiments by Rosch (1973, 1975a, 1975b, 1977, 1978; Rosch & Mervis, 1975; Rosch, Mervis, Gray, Johnson, & Boyes-Braem, 1976) and her colleagues empirically confirmed Wittgenstein's philosophical investigations. Their research proceeded in two directions: Horizontally, it looked for asymmetries among members of one category; vertically, it looked at asymmetries within nested categories (Rosch, 1978). Their work produced evidence of two phenomena: prototype effects and basic-level effects. Later, other researchers investigated prototype effects in inferences (Gelman & Markman, 1987; Gelman & O'Reilly, 1988; Rips, 1975), deductive reasoning (Cherniak, 1984), and other areas.

The term *prototype effects* refers to the experimental finding that in some categories not all members have equal status. Rosch's (1973) initial research with color showed that there are colors (focal colors) that have a special cognitive status. First, they were preferred by the participants as best examples. Second, participants learned them easier than the other colors. This finding ran contrary to the previous view that colors are arbitrarily named because colors, as part of one's conceptual system, are determined by language alone.

Rosch (1975a) called these focal colors *cognitive reference points* and *prototypes*. She later extended her research to other categories, usually of physical objects. In each case, she found prototype effects; that is, participants judged that certain members of each category were more representative of the category than others. She showed, for example, that her participants believed robins to be very typical birds, whereas chickens were less typical. Among sports, football was considered to be very typical, but weightlifting was not. To understand the manifestation of prototype effects under several conditions, Rosch and her coworkers used a wide variety of experimental devices. They asked participants to rate goodness of example and the sensibleness of sentences, measured participants' reaction time to declarations of membership, asked participants to produce examples of category members, and asked participants to represent proximity to the prototype spatially. Prototype effects were consistently shown under these several conditions but they were not found in every category. Rosch, for example, was unable to find prototype effects in categories of actions like walking or eating (D'Andrade, 1989).

Rosch also tried to find other consequences of these prototype effects. She showed that they predict performance on several tasks, focusing on the ways in which central members of a category are related to peripheral members and to the category itself. She found that: (a) less typical members of a category are less associated with that category; (b) typical members appear to have an advantage in perceptual recognition; and (c) when people think of a category member, they generally think of typical instances of that category. She also showed one asymmetry in the ways in which members of some categories are related to others: (d) people consider less typical members to be more similar or more representative examples than the converse. Finally, she showed that: (e) the categories studied had a structure of family resemblances (Anderson, 1980; Lakoff, 1987; Smith, 1989).

Basic-Level Effects

The ways in which people nest the classification of objects have been the object of research. Brown (1958) has been credited as the first to pose the problem:

We ordinarily speak of *the* name of a thing as if there were just one, but in fact, of course, every referent has many names. The dime in my pocket is not only a *dime*. It is also *money*, a *metal object*, a *thing*, and, moving to subordinates, it is a *1952 dime*, in fact a *particular 1952 dime* with a unique pattern of scratches, discolorations, and smooth places. (p. 14)

He pointed out that we have the feeling that some of these names are the “real names,” the others being achievements of the imagination. Although we know that Brown’s dime is a “coin” or a “thing,” we are compelled to think that its real name is “dime.” Moreover, these special names seem to be frequently linked with nonlinguistic actions. For example, we usually associate the word *flower* with the action of smelling one. This action is so important that mimicking the action of smelling a flower can stand for the class of flowers. Brown hypothesized that a child begins at a level of distinctive action that includes flowers, cats, and dimes. Later, she or he develops both to superordinate (plant and animal) and subordinate (rose and Siamese) categories (Lakoff, 1987).

Brown’s ideas prompted cognitive anthropologists to search for *folk taxonomies*—the ways in which cultures use the form “A is a kind of B.” Berlin and his coworkers (cited in Lakoff, 1987) examined folk classification of plants and animals of speakers of Tzeltal living in a region of Mexico. They found out that, although their informants could name animals and plants in a variety of ways, they tended to use a single level of classification. Berlin called this level of classification the *folk-generic level* (or *basic-level*). This level was in the middle of the folk classification hierarchy and had the following characteristics:

- People name things more readily at that level.
- Languages have simpler names for things at that level.
- Categories at that level have greater cultural significance.
- Things are remembered more readily at that level.
- At that level, things are perceived holistically, as a simple gestalt. while for identification at a lower level, specific details (called *distinctive features*) have to be picked out to distinguish, for example, among the kinds of oak. (Lakoff, 1987, p. 33)

Further research on the Tzeltal language discovered that most children initially learn names at this folk-generic level. Later they find out simultaneously how to differentiate and generalize these terms (Lakoff, 1987). Other examples of categorization at the basic-level are the distinction among the several pieces of furniture like tables, chairs, beds, but not including, for example, the Louis XV chair. For a person living at an urban site, the basic-level of plants may contain trees, flowers, grass, bushes, but not acacias, for example.

Rosch and her colleagues developed a series of experiments that confirmed most of Berlin's findings (Rosch et al., 1976). They found that the psychologically most basic-level was in the middle of taxonomic hierarchies. Basic-level categories are basic in perception, function, communication, and knowledge organization. Essentially Rosch et al. found that the basic-level is

- The highest level at which category members have similarly perceived overall shapes.
- The highest level at which a single mental image can reflect the entire category.
- The highest level at which a person uses similar motor actions for interacting with category members.
- The level at which subjects are faster at identifying category members.
- The level with the most commonly used labels for category members.
- The first level named and understood by children. ...
- The level at which terms are used in neutral contexts. ...
- The level at which most of our knowledge is organized. (Lakoff. 1987, p. 46)

Other researchers (D'Andrade, 1989) also found that, unlike scientific taxonomies, classification in folk taxonomies is not always exclusive; that is, instances can share several taxonomic levels. Moreover, the implicit categorization criteria may vary. Sometimes, for example, we use the function as the categorization criterion for a superordinate category (*clothing*). At other times, we use the unity of place (*furniture*). In other cases, we form categories (*groceries*) from a composition of criteria. There is also evidence that folk taxonomies are not very extensive (Randall, 1976).

Prototype Effects in Inferences

The existence of prototype effects in some categories has changed the ways in which researchers conceive the nature of human rules of inference. The consequence of finding that membership in a category is not sharply defined (from a cognitive standpoint) and that categories have cognitive reference points is that the inference of properties from one category member to the whole category may depend on the status of that member within the category.

Rips (1975) analyzed the ways in which category structure influences people's inferential judgments. The evidence suggests that people infer from the most representative members of categories in ways that they do not from less representative members. He found an asymmetry in the relationship between typical and peripheral members; namely, that new information about typical members is more likely to be generalized to the whole category than the reverse. For example, when told that robins on an island had a disease, participants were more likely to infer that ducks could also catch it than the reverse. In another investigation, Carey found that young children generalize asymmetrically in the domain of animals. When told that a human has a spleen, four year-olds assumed that any

animal, even a bee, has a spleen, but they did not generalize in the opposite direction (reported in Rosch, 1983).

Gelman and her colleagues, in a series of studies that investigated inferences in children, studied children's expectations about unknown properties of natural categories (Gelman & Markman, 1987). They compared the ways in which children infer to objects within a category and the ways in which they infer to objects of a different category. When instances had a similar appearance, 3- and 4-year-olds relied more on categories than on appearances for drawing inferences. In another study, Gelman and O'Reilly (1988) compared the ways in which preschool children and second graders were able to generalize from basic-level to superordinate categories. They found that children drew more inferences within basic-level categories (such as dog) than within superordinate categories (such as animal). Older children drew more inferences at the superordinate level than younger children.

Deduction often involves the restriction or the generalization of statements to subsets or supersets of a given category. If the category exhibits prototype effects, it is reasonable to expect that these will influence the outcome of deductive thinking. That is, deduction involving superordinates or subordinates is more difficult than deduction involving basic-level categories. Moreover, when a solver simplifies a problem by working on an example, that example should be "good"; that is, it should have the specific characteristics that are pertinent to the problem in question. Consequently, deduction using a prototype should be easier than using an example removed from the prototype.

These effects have not been studied extensively. For example, there has been almost no study of the effect of prototypical structures in mathematical proof. Cherniak (1984) appears to be the sole researcher who has studied this effect. His experiments seem to

indicate that, under certain conditions, people use a formally incorrect deductive reasoning heuristic that makes use of the prototypicality of categories.

Implications for Cognition

The identification of prototype and basic-level effects has destroyed the notion that concepts are organized by sets of necessary and sufficient conditions and has prompted the development of new cognitive models to accommodate the experimental findings. At the core of this theoretical effort is the notion of *mental representations*—“a set of constructs that can be invoked for the explanation of cognitive phenomena, ranging from visual perception to story comprehension” (Gardner, 1985, p. 383). This section examines the ways in which several psychological theories account for the effects described above.

The holistic perspective provides the simplest account for prototype effects. This theory maintains that, for example, a term like *dog* refers to the mental category dog, which is in itself an unanalyzable gestalt. It assumes that mental categories are composed of templates, usually imagistic, that are isomorphic to the object they represent, are unanalyzable, and implicitly show the relations between the several features or dimensions of the object. An object belongs to a certain class if it provides a holistic match to the template of the class. Computer scientists working in pattern recognition have been using this theory. The theory, however, seems to be limited to the categorization of concrete objects—it is difficult to talk about templates for categories like *furniture*, or for more abstract entities like *justice* (Smith & Medin, 1981).

Another theory, the featural perspective, rests on the idea that human minds use elementary categories and that only a few of the words we manipulate code unanalyzable concepts. Rather, most words are labels for mental categories that are themselves sets of simpler mental categories, usually called features, properties, or attributes. Each concept is

represented by groups of features that have a substantial probability of occurring in instances of the concept. Some proponents of this theory have developed the notion of a group of weighted features. An object is an instance of a category if the sum of its values for each feature is greater than a given threshold. Each member of a category is further removed from the prototype the more it differs in highly weighted features (Lakoff, 1987; Smith & Medin, 1981).

Some researchers (McDonald, 1989a, 1989b; Rips, Shoben, & Smith, 1973) have designed a similar theory using dimensions instead of features to represent concepts. This view departs from the featural perspective especially in the treatment of continuous dimensions like size. Specific instances depart from the prototype in continuous degrees. If the relevant dimensions of birds were thought to be animacy, size, and ferocity, for example, a robin might have a 1 in the feature of animacy, .7 in size, and .4 in ferocity. A key concept that is a consequence of this approach is the notion of *semantic metric spaces*, which are thought to be multidimensional Euclidean real spaces. The vector (1, .7, .4) in R^3 would represent a robin (Smith & Medin, 1981), providing a literal meaning to the notion of *semantic distance*, which is interpreted as the Euclidean distance in a semantic space. Typically, researchers using this theory attempt to determine relevant dimensions, discuss the meaning of clusters of concepts, or discuss meanings of translations in the semantic metric space. The semantic distance between an instance and the prototype corresponds to the degree of category membership.

Critics of these last two perspectives have pointed out that: (a) the knowledge represented in a concept includes more than a list of features, namely, the relationships among the features; and (b) the model does not provide for contextual, background, or cultural effects. The dimensional perspective also raises additional problems because of its use of semantic metric spaces. The requirement of orthogonal dimensions, the necessity of

the isotropy of the semantic space, and the possibility of coexistence of concepts and their members in the same space are examples of the difficulties of these theories (Lakoff, 1987; Smith & Medin, 1981).

Global theories of cognition have provided ways to accommodate prototype effects. Rumelhart (1980), for example, proposes that knowledge of each concept is represented by a *schema* (he uses *schemata* as the plural). A schema is “a data structure for representing the generic concepts stored in memory” (p. 34) and contains the relationships among the components of the concept in question. A very similar construct, a *frame*, is proposed by Minsky (1981). Rumelhart describes the features of schemas using four analogies.

First, each schema is a type of informal, private, unarticulated theory about the nature of events, objects, or situations that we face. The total set of our schemas constitutes our private theory about the nature of reality: “Schemata *are* our knowledge” (Rumelhart, 1980, p. 41). This means that we are constantly testing our theory, and that we use it to make predictions about unobserved events. According to Minsky (1981), this process is accomplished by an *information retrieval network*. Rumelhart also claims that schemas represent knowledge at all levels of abstraction.

Second, schemas are active processes, like procedures or computer programs. As such, they are able to determine the extent to which they account for the pattern of observations and are capable of invoking other subprocedures (or other *subframes*).

Third, schemas are like parsers that work with conceptual elements. On the one hand, we are able to find and verify the appropriate schemas. On the other hand, schemas enable us to find constituents and subconstituents in our observations.

Finally, the internal structures of schemas are like the scripts of plays that can be performed with different actors. The scripts correspond to prototypes of the concepts. They have several *variables* that can be “associated with (bound to) different aspects of the

environment on different instantiations of the schema” (Rumelhart, 1980, p. 35). Minsky (1981) talks about *terminals*, which are “slots that must be filled by specific instances or data” (p. 96). We are aware of the typical values of these variables and of their interrelationships.

Schemas and frames provide similar explanations of prototype effects. Rumelhart claims that the meaning of a concept is encoded in terms of the prototypical situations or events that instantiate that concept, and Minsky defines a frame as the representation of a stereotyped situation. In more specific explanations, both researchers stipulate that each schema’s variables have *default values* that are responsible for our expectations and other kinds of presumptions. These default values are “attached loosely to their terminals” (Minsky, 1981, p. 97), which allow their replacement by new items that better fit our experience.

Cultural Models

The previous models still do not account for contextual or background effects, nor do they provide any explanation for basic-level effects. From an educational point of view, contextual effects are very important for understanding children’s enculturation into school mathematics. It is a plausible conjecture that children’s conceptualizations depend heavily on the social and cognitive context in which learning takes place. Students and their teachers live in a school culture, and one can expect that they intersubjectively share mathematical concepts to some degree. Both students and teachers are also members of larger social groups, and one can expect that they bring the mathematical knowledge of such groups into the school.

A broader approach to mental representations is needed, one that takes into account the role played by the community of human minds acting on the individual. To describe this

common knowledge, cognitive anthropologists and linguists have developed the notion of a *cultural model*—“a cognitive schema that is intersubjectively shared by a social group” (D’Andrade, 1989, p. 809)—used by American researchers. A similar notion is that of *social representations*. These are systems of values, ideas, and practices with a twofold function: first, to establish an order that will enable individuals to orient themselves in their material and social world and to master it; and second, to enable communication to take place among the members of a community by providing them with a code for social exchange and a code for naming and classifying unambiguously the various aspects of their world and their individual and group history (Duveen & Lloyd, 1990; Moscovici, 1989). The construct of social representations was proposed by Durkheim and is used by investigators working in the European tradition.

A special case of cultural models is the notion of *scripts*, developed by Schank and Abelson (1976) in the context of text comprehension. Scripts are cultural models adapted to the study of events. A script is “a coherent sequence of events expected by the individual, involving him either as a participant or as an observer” (Abelson, 1976, p. 33), and it can be interpreted as an extension of schemas to dynamic episodes (Anderson, 1980; Gardner, 1985). A script may be thought of metaphorically as a cartoon strip. Forgas (1981) prefers the term *social episodes*, which he defines as “internal, cognitive representations about common, recurring interaction routines within a defined subcultural milieu” (p. 166). The departure of an instantiation of events from the expected prototypical episode gives rise to prototype effects. People, for example, have a “restaurant script” that is composed of a stereotyped set of events that they expect to happen under certain circumstances. The restaurant script depends on a sociocultural institution; that is, the existence of a place that serves and sells food (D’Andrade, 1989).

Idealized Cognitive Models

Recently, linguists and anthropologists have been converging in their study of cultural models. An example is Lakoff and Johnson's work on cognitive models. Lakoff and Johnson (Johnson, 1987, 1997; Lakoff 1987; Lakoff & Johnson, 1999) develop an idea of mental representations that borrows some of the features of Rumelhart's schemas, Minsky's frames, and Abelson's scripts, and adds linguistic and cultural components. They propose that we organize our knowledge by means of *idealized cognitive models*, and category structures and prototype effects are byproducts of that organization. As Johnson (1997) puts it:

Human beings understand their world by means of idealized cognitive models for the kinds of entities, events, and situations we encounter in everyday experience. Recent empirical studies in lexical semantics have shown that words do not map directly onto states of affairs in the world, but rather are defined by their roles in idealized models of situations, which are holistic structures called "frames". Words get their meaning by the role they play in frames. ... Frames are *imaginative*, not only because they are idealized models that do not exist objectively "in the world," but also because they are defined partly by image schemas and experiential metaphors. (p. 155)

Lakoff (1987) disagrees with two assumptions shared by the featural and the dimensional approaches. The first is that goodness of example is a direct reflection of degree of category membership; that is, people's willingness to say that a chicken is not a good bird implies that chickens do not have a high degree of membership in the category of birds. Although the construct of a graded membership in mental categories can explain some prototype effects, it cannot explain others. A classical example of a category (first presented by Fillmore) that does not have a graded membership is the category of "bachelor," for which there are clear conditions for membership. Nevertheless, persons like the Pope or Tarzan

do not have a clear status of membership in this category. Lakoff proposes that prototype effects in this category are produced not because the category is graded but because people have an idealized cognitive model of bachelor based on the context of a human society in which there are certain expectations about marriage and marriageable age. The worse the fit between that idealized cognitive model and our knowledge of the background conditions, the less appropriate we feel it is that the concept be used (Lakoff, 1987; Quinn & Holland, 1987). Later in this chapter, I discuss the category of even number, which is not graded either.

The second assumption shared by the featural and the dimensional perspectives with which Lakoff disagrees is that prototype effects mirror mental representations of categories; that is, categories are represented in the mind in terms of prototypes, and degrees of category membership are determined by their degree of similarity to the prototype. This assumption does not capture the complexity of some categories. For example, the concept of mother is based on a complex aggregate of several models:

- The birth model: The person who gives birth is the mother. ...
- The genetic model: The female who contributes the genetic material is the mother.
- The nurturance model: The female adult who nurtures and raises the child is the mother of that child.
- The marital model: The wife of the father is the mother.
- The genealogical model: The closest female ancestor is the mother. (Lakoff, 1987, p. 74)

The concept of mother is not defined by necessary and sufficient conditions, and all the models converge in a prototypical ideal case. Prototype effects can be explained by tensions between these models in some situations (stepmother, surrogate mother, foster mother, etc.).

Lakoff (1987) claims that a major source of prototype effects is associated with our use of *metonymy*—“a situation in which some subcategory or member or submodel is used ...

to comprehend the category as a whole” (p. 79). Social stereotypes, where a subcategory has a socially recognized status as standing for the category as a whole, are examples of our use of metonymy. For example, in the United States, the category “working mother” is not a mother who happens to be working. Rather it is defined in contrast to the social stereotype of a “housewife mother” that is defined by the nurturance model. Prototype effects in the case of a working mother arise from its comparison with only one of the models in the cluster and not against the whole category. Put another way, the “housewife mother” usually stands for the whole category of mothers:

Consider an unwed mother who gives up her child for adoption and then goes out and gets a job. She is still a mother, by virtue of the birth model, and she is working—but she is not a *working mother*!

The reason is that it is the nurturance model, not the birth model, that is relevant. Thus, a biological mother who is not responsible for nurturance cannot be a working mother, though an adoptive mother, of course, can be one. (Lakoff, 1987, p. 80)

Other kinds of metonymic models include: typical examples, ideals, paragons, salient examples. Neither the featural and dimensional perspectives, Rumelhart's schemas, Minsky's frames, nor Schank and Abelson's scripts account for prototype effects that result from metonymy.

The category of mother combines two different models: One is a cluster of models, and the other is a stereotypic model. The category of mother has a structure with a composite prototype: “The best example of a mother is a biological mother who is a housewife principally concerned with nurturance, not working at a paid position, and married to the child's father” (Lakoff, 1987, p. 82). Although mothers in other cultures and in other historical times differ in several aspects from this description, it is likely that such cultures or epochs have their prototypes for mothers. Prototype effects arise when a given

individual is compared with the prototype as a result of a particular *representativeness structure*. Representativeness structures exist in other categories and are the ones detected by the research on prototype effects. As these structures are linear—they focus on closeness to the prototype—they hide much of the complexity of the category.

Another important source in the construction of all kinds of models is the use of *metaphor*. Each metaphor is based on a similarity between a source and a target domain, together with a source-to-target mapping (Lakoff, 1987). Metaphors allow us to extend the similarities between the domains beyond their initial state, and they structure most of our conceptual system. We use them to map structures usually in the physical world and eventually in the mental world into other domains through imaginative processes (Lakoff & Johnson, 1980):

The result of any such mapping, from physical experience in the source domain to social or psychological experience in the target domain, is that elements, properties, and relations that could not be conceptualized in image-schematic form without the metaphor can now be so expressed in the terms provided by the metaphor. (Quinn & Holland, 1987, p. 28)

Johnson (1987) elaborated this notion and proposed the construct of *kinesthetic image schemas* (1987) or *schematic mental images* or *image schemas* (1997)—basic or primitive experiential structures that are a consequence of the nature of human biological capacities and the experience of functioning in a physical and social environment. Reason, for Johnson, is no longer detached from human beings as functioning organisms. Image schemas significantly structure our experience prior to, and independent of, any concepts and are responsible for many of the metaphors we use in abstract domains. Examples of these schemas include: the container schema that consists of a boundary distinguishing an interior from an exterior; the part-whole schema that involves the whole, the parts, and a

configuration; the link schema, where there are two entities and a link connecting them; the center-periphery schema, where a central element is thought to be more important than the periphery; the source-path-goal schema that includes a source, a destination, a path, and a direction; the up-down schema; the front-back schema; and the linear order schema (Johnson, 1987, 1997; Lakoff, 1987). Quinn and Holland (1987) argued that these imagetic schemas, from which metaphors are based, are not only predicated on our bodily experiences but may also be built on elements shared by the cultural group.

In summary, Lakoff (1987) and Johnson (1987) propose that the structure of thought in general, and the categorization in natural languages in particular, is characterized by cognitive models that fall into four types:

1. *Propositional models* that specify elements, their properties, and the relations holding among them.
2. *Image-schematic models* that specify schematic images.
3. *Metaphoric models* that are mappings from one of the above models in one domain to a corresponding structure in another domain.
4. *Metonymic models* that make use of the previous models and map one element of the model to another.

By distinguishing among these types of cognitive models, Lakoff was able to propose a process of creating complex cognitive models that can “characterize the overall category structure, indicate what the central members are, and characterize the links in the internal chains” (p. 114). He argued that there is a “significant level of human interaction with the external environment (the basic-level), characterized by gestalt perception, mental imagery, and motor movements” (p. 269). This is the level at which people function most efficiently and successfully using basic-level and image-schematic concepts as proposed by Johnson (1987). Gelman’s finding mentioned previously that older children generalize more easily

to superordinate categories than do younger children supports this proposal (Gelman & O'Reilly, 1988).

But how do categories gain their structure? In what ways do metaphors relating distinct domains of experience come about? Why this metaphor and not another? How does each individual relate to particular metaphors? How do we learn metaphors? These are crucial questions for educators. Grady and colleagues (Grady, 1996; Grady, Taub, & Morgan, 1996) distinguish between *primary* (or *primitive*) and *complex* (or *compound*) metaphors. The former pair subjective experience and judgment with sensorimotor experience and the later are formed from the primary ones through fitting together small metaphorical "pieces" into larger wholes. *Conflation*, a stage at which the two domains that will later be related metaphorically are combined, plays a key role in the formation of primary metaphors (Lakoff & Johnson, 1999). Metaphors have also been shown as being the sources for innovative solutions to problems. Indurkha (1994) denominated them *similarity-creating metaphors* and Carroll (1994) *visual metaphors*. But how does this process occur? And how do teachers create contexts to facilitate the creation of rich mathematical metaphors? Are there specific pitfalls to avoid?

Categorization of Mathematical Objects

Mathematical categories have been the object of research by psychologists, linguists, and mathematics educators. For some of these researchers, mathematical knowledge is a field of certainty, bound by the laws of logic and a clear example of analytic truth. Armstrong, Gleitman, and Gleitman's (1983) investigation on prototype effects provides a typical example. In an attempt to prove that prototype effects are unrelated to the ways in which we categorize, they compared the categorization of mathematical entities with the categorization of real world objects. The rationale for this approach was that if prototype

effects could be found in mathematical categories, like “even number,” then Rosch’s prototype theory would be wrong, because these prototype effects are unrelated to membership gradience—the category of even numbers has a clear membership rule. Their implicit assumption was that mathematical entities have a clear declarative membership rule, and that their participants were applying it.

This conception of mathematics runs contrary to developments in the philosophy, history, and sociology of mathematics (Bloor, 1991; Lakatos, 1976; Restivo, Bendegem, & Fischer, 1993). As a result of investigating the roots of mathematical knowledge, researchers in these fields have been proposing that mathematics knowledge is generated by social interactions and that mathematical truth is intersubjectively shared by the community of mathematicians. Lakatos’s work, in particular, shows how mathematicians themselves may not be in agreement over the meanings of mathematical entities, even when such meanings are provided by definitions. Although Lakatos’s field was the history and philosophy of mathematics, he provided evidence of prototype effects in the category of polyhedra. In his historical account of the discussions over a precise definition of the concept of polyhedra, Lakatos showed how some mathematicians came up with counterexamples of polyhedra that did not verify Euler’s formula and how other mathematicians claimed that they were presenting “monsters” and using “wrong” definitions of polyhedra. Moreover, there are central examples of polyhedra—all mathematicians would agree that any definition of polyhedra should include prototypes like the five platonic solids.

Mathematics educators (Matos, 1991a; Sfard, 1994) have pointed out how Lakoff and Johnson’s proposals may entail a change in the view about the nature of mathematical knowledge. They reflect a shift from an objectivist viewpoint, in which mathematics categories are a reflection of the structure of an external reality, to a subjective perspective

where these categories are mental products shaped by three ingredients: an embodied cognition linking our inner mental workings to the particularities of our physical body, social interactions that rely on language, and an external reality that, although inaccessible, conditions all our mental world. The affective dimensions of rationality may, however, still be missing from this picture (Damásio, 1994; Lakoff, 1997).

A second point can be made about Armstrong et al's (1983) investigation. Even if we accept that mathematical categories are classical, we would still have to show that they were viewed as such by the participants. As Gardner pointed out (1985), the study may very well show that even mathematical categories display a structure similar to other categories. Later in this chapter, I discuss further this research.

A strong case for the subjectivity inherent in mathematical entities was put forth by Fischbein (1987). In his review of the role of intuition in thought, he gave examples of what he termed *analogic* and *paradigmatic models* in mathematics and physics. His definition of *analogic models* is similar to the previous notion of metaphors—"two objects or two systems are said to be analogical if, on the basis of a certain partial similarity, one feels entitled to assume that the respective entities are similar in other respects as well" (p. 127). A *paradigm*, in Fischbein's terminology, is an instance of a category that is used to represent the whole category and is thought to be a particularly good example of the category. This definition shares both the characteristics of a prototype and a metonymic model. Fischbein also agreed that mathematical categories may not be classical. He proposed that when we define a concept we never do it as a pure logical construct:

The meaning subjectively attributed to [the concept], its potential associations, implications and various usages are tacitly inspired and manipulated by some particular *exemplar*, accepted as a representative for the whole class. (p. 143)

Fischbein's point is as much about students as about mathematicians themselves. He compared these reasoning processes with Kuhn's paradigms in scientific thought and called this phenomenon "the paradigmatic nature of intuitive judgment" (p. 143).

To adjust Lakoff's theory to mathematics learning, Dörfler (1991, 1995) proposes three changes. First, he emphasizes the need to stress a social perspective on the genesis of knowledge, which, in a way, is already implicit in Lakoff's account. Second, he interprets image schematic models as condensations of actions, much as Piaget would. Third, in order to adapt the theory to mathematics, Dörfler proposes an image schema to be a perceptive or cognitive interaction with an object-like model—a material model, a drawing, or just an imagined model—together with the manipulation of this model. He calls this material model the *concrete carrier* for the image schema.

Evidence of prototype effects has been empirically found by mathematics educators. McDonald (1989b) studied the relationship between high school students' level of cognitive development and the way in which they structure geometrical content. This investigation was followed up a year later to study changes in the structure geometrical content (McDonald, 1989a). Both studies used ratings of the similarity of two mathematical terms (*equilateral triangle, right triangle, parallel, corresponding sides, ratio, and others*) to produce a map of these concepts into a two-dimensional semantic mental space. The studies also compared students' maps with experts' maps. McDonald's research showed that the semantic map of students at the formal level was significantly closer to the experts' map than was that of the students at the concrete level. Moreover, after one year, the formal group had more stable structures than the concrete group. Although these studies exhibit differences in the geometric concepts of the students, it is difficult to claim that they allow us to model students' mental structures. As shown above, the notion of semantic mental spaces has severe limitations in the study of mental representations of concepts.

Presmeg (1992, 1997a, 1997b) focused on prototypical mental images and the use of imagery in metaphoric and metonymic ways in mathematics. She was especially concerned with the role played by these mental images in more advanced thought processes in mathematics and found that imagery is central in the development of mathematical reasoning, even in areas that have been traditionally considered akin to algebraic, nonvisual explorations.

Concept Images and Concept Definitions

The construct of prototypes has also been used in another area of research in geometry learning. Some researchers have reported that students' choice of examples of geometric concepts and their definitions of the same concepts do not match (Burger, 1985b; Fuys, Geddes, & Tischler, 1985; Mason, 1989; Wilson, 1988). The construct of *concept images* has been proposed by Vinner and his colleagues (Tall & Vinner, 1981; Vinner, 1983; Vinner & Hershkowitz, 1983) as an explanation for these findings.

A concept image is "the total cognitive structure that is associated with a given concept" (Tall & Vinner, 1981, p. 152) and is composed of the images associated with that concept together with a set of properties and processes. For example, the concept image of a function may include a picture of the graph and a picture of the algebraic expression that defines the function, together with the students' definition of function. *Concept image* is contrasted with *concept definition*, which is a verbal definition that accurately explains the concept (Vinner, 1983) and may differ from the mathematical definition (Vinner & Hershkowitz, 1983). Vinner distinguished between formal and informal learning and claimed that in the latter people need a concept image and not a concept definition. Concept definitions introduced by means of a definition remain inactive or are eventually forgotten. In a specific intellectual task, only portions of the concept image are actually evoked

(*temporary or evoked concept image*). These portions might be contradictory and produce conflict in one person's mind when these opposing portions of the concept image are used simultaneously (Tall & Vinner, 1981; Vinner, 1983).

Vinner and his colleagues have used this construct to interpret the finding that often visual identifications and drawings made by students do not match their definitions (Hershkowitz, Bruckheimer, & Vinner, 1987; Vinner & Dreyfus, 1989; Vinner & Hershkowitz, 1980; Vinner & Hershkowitz, 1983). Although these researchers did not perform their investigations within the framework of categorization theory, Hershkowitz (1989b) attempted a reinterpretation of their findings that is consonant with categorization theory and van Hiele theory. I discuss her proposals in chapter 4.

Applying the construct of concept image to the problem of the mismatch between the students' choice of examples and their definitions, one could say that their concept images and concept definitions did not align and that the concept image took precedence in identification or production tasks. However, one would still fail to explain the incompleteness of the definitions, the absence of a distinction between necessary and sufficient conditions, the ambiguity of the terms, and how a contradiction between an imaged and a propositional representation of concepts could occur in students' minds.

There are other ways in which the construct of concept images is not open to research. Although concept images have a domain similar to that of Lakoff's image-schematic models and although concept definitions are close to his propositional models, Hershkowitz (1987) claimed that prototypes are mainly a visual phenomenon and provided no explanation for prototype effects derived from propositional models that do not involve mental images. To conclude this chapter, I present examples of a metonymic model and a metaphoric model not encompassed by the notion of concept images.

An Example: The Concept of Number

Rosch (1975a) studied examples of cognitive reference points. They included vertical and horizontal lines and numbers that are powers of 10. As part of her research, Rosch looked for prototype effects using linguistic *hedges*—“terms referring to types of metaphorical distance” (p. 533)—like *almost*, *virtually*, *essentially*, or *loosely speaking*. She made use, for example of such stimuli as “103 is essentially 100.” She found that in the decimal system, multiples of ten constitute reference points. Both 97 and 102 were judged essentially 100, but not vice versa, and both were considered closer to 100 than 100 was close to them. As a byproduct, these asymmetries called into question the isotropy of semantic spaces.

Analyzing these results from a linguistic perspective, Lakoff (1987) added that the natural numbers, for most people, are characterized by the words for the integers between zero and nine, plus addition and multiplication tables and rules of arithmetic. These digits are the central members of the category of natural numbers from which the other members are generated. Any natural number can be written as a sequence of digits, the properties of large numbers are understood in terms of the properties of the single-digit numbers, and computations with large numbers are understood in terms of computation with single-digit numbers. Each of the single digits generates subcategories of its own when multiplied by 10, 100, and so on. These results were actually predicted by Wertheimer (1938). He may be credited as being the first to draw attention to the special place multiples of ten have in our vocabulary: “He is a man in his thirties,” or “X died in the twenties of the last century.”

Natural numbers are an example of a category composed of some central members and some rules for generating the other members. Lakoff (1987) claimed this category is a metonymic model in which the single-digit numbers stand for the whole category. He also claimed that the category of natural numbers itself is a central category in more general

categories of numbers. For example, rational numbers are understood as quotients of natural numbers, real numbers as infinite sequences of single digits, and so on. These other categories of numbers are understood metonymically in terms of the natural numbers. Every axiomatic system involving numbers must include the natural numbers, so their centrality is reflected even in the work of mathematicians. Data from researchers in mathematics education confirm this “dissolution of hierarchies” (Fischbein, 1987, p. 147), at least in older students. Tall and Vinner (1981) report that often students did not regard $\sqrt{2}$ as a complex number although some of them defined real numbers as “complex numbers with imaginary part zero” (p. 154). The structure of the category of numbers in younger children has yet to be understood.

The effects found by Rosch are explained because we use the powers of ten as a submodel to comprehend the relative size of the numbers, especially in the context of approximations and estimations. We also use other models to comprehend numbers. For example, in the context of body temperature, 98.6° Fahrenheit (or 37° Celsius) is a cognitive reference point where fever is involved, and as far as American money is concerned, a cognitive model often includes multiples of five (nickels, dimes, quarters). Each of these models produces prototype effects.

These prototype effects are not equivalent to graded category membership. In fact, participants in Armstrong et al.’s (1983) investigation agreed that the categories of even and odd numbers are well defined. Nevertheless, the researchers found prototype effects using reaction time and ratings. Lakoff (1987) claimed that these effects are the result of the superposition of the other models over the even-odd structure of the natural numbers.

Another Example: Preferred Triangles

Geometry, in particular, relies heavily on metaphors. Examples are the terms *altitude*, *height*, *base*, *length*, and *width*. We talk about “the altitude of a triangle,” “the altitude of a trapezoid,” “the altitude of a parallelogram,” but very rarely about “the altitude of a rectangle” (*length* and *width* are used instead), or the “altitude of a square” (we use *side*), and never about “the altitude of a rhombus.” The same can be said about the term *base*. There is “the base of a triangle” but not “the base of a rhombus.” *Base* and *altitude* are also used with solids in the same way. Virtually every textbook will say that to calculate the area of a rectangle multiply the *length* and the *width*, whereas the area of triangle is computed using the *base* and the *height*.

The phrase “the altitude of” is used in ordinary English mainly in relation to mountains. We use the phrases “height of a building” or “height of a person” but not “the altitude of a house.” Although we would say that “the plane is *at* an altitude of 9 kilometers,” we are less likely to say “the altitude *of* the plane is 9 kilometers.” The underlying message is that we are asking students to imagine triangles as being like mountains, whereas rectangles are thought to be like rooms or football fields (rhombuses apparently are thought to be diamonds or kites). We are, in fact, using a “mountain metaphor” to work with triangles and a “football field” or a “room metaphor” to compute the areas of rectangles.

Although such worldly terminology would be condemned by Hilbertian formalists, its use helps students attribute meaning to their actions on mathematical objects. But mathematicians themselves also make an extensive use of metaphors. As Thom (1973) puts it, “the mathematician gives a meaning to every proposition” (p. 202). The terms *manifold*, *fiber bundle*, *curvature*, *projection*, *kernel*, *closure*, and many others are all evidence of mathematicians’ concern for meaning. Lakoff and Nuñez (1997) provide many such examples that show the pervasiveness of metaphors in mathematics and distinguish

between *grounding metaphors* and *linking metaphors*, the former ground mathematical ideas in everyday experience and the later link one branch of mathematics to another.

The mountain metaphor may only be part of the picture. Several researchers (Burger & Shaughnessy, 1986; Fisher, 1978; Fuys et al., 1985; Presmeg, 1992; Vinner, 1983; Wilson, 1986) have reported students' preference for the upright/horizontal position of geometric figures, and in one case (Burger & Shaughnessy, 1986), one of the informants (Bud) distinguished among several triangles by the directions in which they were pointing. Some researchers (Fuys et al., 1985) have interpreted these phenomena as "perceptual difficulties" (p. 137), but this description does not provide specific information. I claim instead that it is a cognitive, not perceptual, problem produced by the interaction of several cognitive models. The up-down schema in Johnson's terminology (Lakoff, 1987) may account for the preferred orientation, and a metaphor mapping the human act of pointing to some of Bud's triangles may help to explain his answers. I discuss these points in chapter 4.

The use of such processes is a necessary and unavoidable characteristic of thought. It facilitates the identification of relevant elements and their relationships and permits their integration with previous knowledge (Petrie, 1979). There is, however, an unwanted side effect. It is hard to imagine the direction in which an obtuse triangle points. In the culture of school mathematics, moreover, it is irrelevant where triangles are pointing, and students tend to have problems when they attempt to apply the mountain metaphor to triangles that are not in the "mountain position" (no side horizontal) or that do not look like mountains (obtuse triangles with a horizontal side other than the larger side). A non-obtuse triangle in a "mountain position" seems to be the cognitive reference point (Lakoff, 1987) used by students. Prototype effects are likely to occur with different triangles, as shown by the investigations of Vinner and Hershkowitz (1983) and Wilson (1986), in which the concept

of the altitude of a triangle was included. Comparable effects were detected by Mariotti (1995) in polyhedra.

Conclusion

At the beginning of this chapter I started by discussing the role of abstraction in the formation of mathematical concepts. I showed how developments in categorization are questioning the very notion of a *concept*: namely, the conviction that propositional statements suffice to sharply define the boundaries of mental categories, in particular mathematical categories. I propose instead that the ways in which we organize our concepts (as individuals or as a group) assumes complex forms involving mental images, propositional statements, and metaphoric and metonymic projections. Attempts have been made to incorporate this perspective into mathematics education. Sfard (1994, 1997), in particular, reflected upon the relationship between the formation of metaphors and the process under which meaning of abstract mathematics objects is created.

CHAPTER 3

THE VAN HIELE THEORY

Dina van Hiele-Geldof and Pierre van Hiele developed their theory in Holland, when, in the middle of the 1950s, they wrote their doctoral dissertations under the direction of Hans Freudenthal. Pierre was essentially concerned with the study of geometrical insight, and Dina was developing a didactical approach to geometry for 12- and 13-year-olds.

They produced their research amid the dawning of tremendous changes in the field of mathematics education. Cuisenaire rods and the geoboard had just emerged, and there was a strong movement in Britain towards the creation of what later became the Association of Teachers of Mathematics. The Royaumont meeting that played a significant role shaping the modern math movement in Europe was still in the future, as was the launching of Sputnik. In Holland, in particular, discussions of the teaching of geometry were popular, and the van Hieles played a very important role in them. So, although mathematics curricula had not yet been changed, the international community was discussing new methods, new purposes, and new curricular content (Matos, 1985).

The van Hieles's work reflects this duality of influence. On the one hand, they developed their work in the context of an Euclidean geometry curriculum (now virtually gone in most of the world) that viewed geometry as an instrument for exercising the mind's logical abilities. On the other hand, their pedagogical standpoint embodied a very contemporary approach. This approach is visible in Pierre's concern for insight and Dina's emphasis on the manipulation of shapes, the use of geoboards, and drawing using rulers

and compasses. Her students were drawing, folding, arguing, comparing, and observing, and such activities are at the core of today's recommendations for geometrical activities.

The research of the van Hieles is based on three elements. There was, on the one hand, a strong structuralist basis for their work. They were not siding with Bourbakian structuralism that proposed the early study of broad mathematical structures. They were advocating instead that mathematics education should take advantage of structures that are "out there" in the world. Structures, in fact, permeated their view of the world, of the organization of cognition, and of mathematics teaching and learning. On the other hand, the influence of gestalt psychology provided a framework for the analysis of the perception and interpretation of these structures. Finally, the van Hieles were concerned with the didactics of mathematics, especially the development of insight in the classroom. Their research developed out of their previous work as teachers and maintained direct connections with mathematics classrooms. Dina van Hiele-Geldof's work consisted of the development of new teaching methods, and Pierre van Hiele incorporated into the theory the interactions that occur in a classroom setting.

A Gestalt View of Cognition

The van Hieles were essentially concerned with the actual teaching of mathematics and did not provide a detailed psychological account of mathematics learning. Nevertheless, some of their proposals had a psychological base that I analyze in the following sections.

Cognition, for Pierre van Hiele, proceeded recursively from the construction of a global perception, to the formation of a mental structure, its progressive differentiation, and its final restructuration into a new mental structure. For the van Hieles, as for gestalt psychology, there were no isolated objects nor concepts per se, but all entities existed in a context, a *structure* in Pierre van Hiele's terms.

Van Hiele did not provide a definition of *structures*. Instead, he explained some of their properties, described kinds of structures, and gave some examples. Following Popper's distinction among three worlds, van Hiele proposed that there are several kinds of structures: (a) the structures in the world we live in (World 1); (b) the structures in our mind (World 2); and (c) structures in the world of common human knowledge (World 3). Van Hiele insisted that, in cognition, it is very important that a structure can be seen as a totality because a structure is more than the sum of its elements.

Pierre van Hiele borrowed from gestalt psychology four properties of structures:

- 1) Structures can be extended;
- 2) A structure may be seen as part of a finer structure;
- 3) A structure may be seen as a part of a more inclusive structure; and
- 4) A given structure may be isomorphic with another structure.

The first and fourth properties are obvious because they involve activities that are innate in human thinking. The other two properties assert that, in line with the gestalt tradition, structures have other structures embedded in them or are part of larger structures. Van Hiele claimed that these two last properties should be taught.

Mental structures exist in World 2 and are built upon structures in World 1. Examples of mental structures include the following:

- Action structures, which are automatic motor actions that we cannot make explicit, like the movement of the fingers of a pianist playing the piano or of a driver of a car who can react directly to road signs. These structures are very close to stimulus-response patterns, and van Hiele questioned whether these can be considered mental structures at all.

- Visual structures, which are constructed in our mind reacting directly to structures in World 1.

- A global structure of acting, which is van Hiele's terminology for imitation.

The formation of mental structures demands rapid switches between receptive and active moments. Receptive moments permit “the absorption of the ‘spontaneous’ structures emanating from the materials” (van Hiele, 1984a, p. 237). During the active moments the individual concentrates “on the analysis and modification of these structures” (p. 237).

Learning, for the van Hieles, is a progressive differentiation and restructuring of fields that produce new and more complex mental structures. Higher levels are attained if the rules governing lower structures “have been made explicit and studied” (van Hiele, 1986, p. 6) leading to the development of more complex mental structures. Mental development proceeds as the students gradually transform their structures (*transstructuring*) or actually substitute one structure for another (*restructuring*). Transstructuring occurs, for example, when, in van Hiele theory, the original visual structures are gradually transformed into abstract structures. Instances in which restructuring occurs are, among others: (a) a restructuring of the field of observation leading to the integration of several structures that have been developed independently and (b) the solution of a problem for which we have to try several structures.

Pierre van Hiele departed from other structuralist approaches to mathematics education of his time. For example, Piagetian educators (and also Dienes and Papy) attempted to use in classrooms the “three mother structures of the Bourbaki mathematicians” (Piaget, 1970, p. 23): the algebraic, the order, and the topological structures. Van Hiele’s structures are all based on World 1 structures that can be immediately perceived as a gestalt.

Insight was, for van Hiele (1986), a key mechanism that allows students to visualize different fields (*structures* in his terminology) that permit them to build more complex concepts. He used the gestalt idea that “insight might be understood as the result of perception of a structure” (p. 5), and proposed that it is characterized by the following properties:

1. Insight requires *adequacy* either in a new situation or within an established structure. This adequacy demands a social mechanism for establishing criteria for objectivity, which I discuss later.

2. Insight also requires *intention* which means that the person will act according to the perceived structure and not at random.

3. Insight cannot be planned. (pp. 24, 154)

Fostering insight must focus on the development of students' ability to see structures as part of finer structures, or as part of more inclusive structures.

The creation of mental structures has two distinct "acts of thought" (van Hiele, 1984a, p. 238). First, there is an undifferentiated identification of the structure under observation. At the beginning of geometric thought, for example,

the presentation of concrete (study) material evokes visual undifferentiated structures.

Children become familiar with these structures fairly early in life, long before they reach the level of secondary education. (p. 238)

These undifferentiated structures are not truly mathematical, nor can they produce a truly mathematical insight. After this first identification, the analysis of the object enables us to abstract and eliminate a certain number of its qualities, which leads to new forms of identification and thus to new mental structures.

A second act of thought proposed by van Hiele is the classification of interrelated structures. When we have several principles of classification, the principles of classification themselves are a new undifferentiated structure. Then the process starts again, recursively, resulting in a new structure with classifying principles of the classifying principles themselves. Van Hiele (1986) called this new process of thought a "higher level of thinking" (p. 238), and maintained that "it takes place under the influence of a teaching-learning program" (p. 50).

Students' Learning

As noted in a previous section, van Hiele proposed that learning is a process recursively progressing through discrete levels of thinking—"jumps in the learning curve" (van Hiele & van Hiele-Geldof, 1958, p. 75)—that can be enhanced by an adequate didactical procedure. He assumed that there are several levels of geometric learning and that the passage from one level to the next must occur through a sequence of phases of instruction. The van Hieles characterized the levels as follows:

Level 1 (Visualization) - Figures are judged by their appearance

Level 2 (Descriptive) - Figures are bearers of their properties

Level 3 (Theoretical) - Properties are logically ordered

Level 4 (Formal logic) - Geometry is understood as an axiomatic system. (van Hiele, 1986; van Hiele & van Hiele-Geldof, 1958)

In some of their works, the van Hieles also proposed the existence of a fifth level (the nature of formal logic, in which axiomatic systems are studied), or even higher levels (van Hiele, 1984b, 1986; van Hiele-Geldof, 1984a). Pierre van Hiele warned, however, that these levels have very little importance in the teaching of geometry and urged researchers to concentrate on the first three levels.

According to van Hiele (1986), students, for example, understand an isosceles triangle differently at each level. At Level 1, the student develops visual structures of isosceles triangles based on previous undifferentiated structures and is able to recognize isosceles triangles among other triangles. At this level, World 1 and World 2 are the main actors. At Level 2, with the mediation of language, the visual shape loses its importance, and the student recognizes an isosceles triangle by the totality of its geometrical properties. The student knows, for example, that an isosceles triangle has two equal sides and two equal angles, that the altitude bisects the side common to the equal angles, and that it has one axis of symmetry. As the visual aspect of the figures loses its importance, imperfect drawings

are no longer a problem. At Level 3, the object of study is the nature of the relations among the theorems, and the student is able to differentiate the field of geometrical properties, distinguishing between definitions and propositions. The student understands that the fact that an isosceles triangle has two equal sides implies that it must have two equal angles. These are, however, local relationships, and only at Level 4 does the student feel the need for the establishment of a global relationship among the properties. Only then can he or she relate the properties of an isosceles triangle to the axioms of Euclidean geometry. At Level 5, a student is able to discuss whether an axiom that postulates the existence of isosceles triangles is independent of a subset of the Hilbert axioms.

This description of the levels has an interesting characteristic. Consider an analogy. In the beginning the carpenter has certain tools with which he or she can manufacture new tools. Once produced, these new tools become available to construct even more complex tools. Now apply this metaphor to the van Hiele levels. One can see in the isosceles triangle example that some elements are mentally constructed at each level; that is, they are new concepts that the student forms based on the observation of structures. Other elements are mentally manipulated; they are mental tools from whose manipulation the constructed elements are produced. For example, at Level 1 the student constructs an image of a figure by observing it. At the end of Level 2, after this image has been manipulated, a new structure will emerge, and he or she will be able to observe the properties of the figure in that new structure. At Level 3, the student will reflect upon the properties and will eventually order them logically. This order will become the basis for an axiomatic system at Level 4. At Level 5, the student will reflect on the axiomatic systems and will understand formal logic. This relationship between observed and manipulated objects can be summarized in the following table:

Level	Manipulated Objects	Observed Objects
1		Figures
2	Figures	Properties
3	Properties	Ordering of properties
4	Ordering of properties	Axiomatic system
5	Axiomatic system	Logic

Students notice the observed objects as they become familiar with the structure of the manipulated objects. This relationship between manipulated and observed objects has several implications. On the one hand, what was intrinsic at one level becomes extrinsic at the next level. This means that the relationship between levels is one not of subsumption but of “aboutness,” making it impossible to skip levels or pass through them in a different order (Olson, Kieren, & Ludwig, 1987). On the other hand, each level has its own linguistic symbols, its own language, which makes the communication difficult between persons operating at different levels. In fact, if the teacher is talking about properties, trying to show their logical relationships (Level 3), but the language that students possess only allows them to understand the manipulation of figures (Level 2), van Hiele claimed that communication is impossible. He thought that this problem explains geometry students’ frequent complaint that they do not understand what the teacher is talking about.

A Didactical View of Cognition

Van Hiele (1986) proposed that there is an additional property that the theory possesses, namely, that “the transition from one level to the following is not a natural process; it takes place under the influence of a teaching-learning program. The transition is not possible without the learning of a new language” (p. 50). This teaching-learning program comprises a precise didactical sequence of five *phases* (or *stages*) of learning.

During the first phase (*information*), the teacher holds a conversation with the students in order for them to get acquainted with the working domain. The teacher, for example, shows triangles and informs students that they are called “triangles.”

During the second phase (*guided orientation*), students are guided by tasks that they establish themselves or that are given by the teacher, to find networks of relations between the objects they are manipulating. The purpose is to guide students through the differentiation of new structures from the ones observed in the first phase.

The third phase (*explicitation*) is based upon class discussions in which the students give their opinions about the regularities they have found, become conscious about the relations, and express them in words. In other words, this phase involves making the new structures observed in the previous phase explicit through language. The class discussions permit students to learn the necessary language to express what they have discovered. The teacher now introduces all the technical language. For the van Hiele, true understanding requires the successful accomplishment of this phase.

In the fourth phase (*free orientation*), the teacher gives students general tasks, and they have the opportunity to get acquainted with the topic from all directions. During the fifth phase (*integration*), the teacher does not present anything new. The students build a “global survey” of what was learned before. New discoveries are integrated into the existing structures, thus promoting the process of restructuring. The teacher’s role is to help students see how it all fits together.

Suppose, for example, that a teacher is preparing a group of lessons to lead his or her students from Level 1 to Level 2 on the topic “rhombus” (van Hiele, 1986, p. 54). The teacher will show his or her students several rhombuses and will ask if other figures are rhombuses or not. This will constitute the first phase of Level 1. At this point, it would be meaningless to discuss logical reasons why a figure is a rhombus because at Level 1 figures are visually perceived. Although the students can distinguish and name rhombuses,

they perform these actions on the basis of a global visual recognition. During the second phase, other kinds of activities will be performed on the rhombus. For example, the rhombus will be folded on its axes of symmetry, and the angles and the sides will be measured. These will be followed, in the third phase, by a discussion among the students about what they have discovered. For the next phase, the teacher will pose the problem of drawing a rhombus given some of its sides and vertices. Finally, in the last phase, the properties will be summarized and memorized.

A Linguistic Perspective

Pierre van Hiele often claimed that the movement to a higher level is accompanied by the learning of a new language. His main point can be appreciated if one notes that, recalling the example of the last section, the properties of the rhombus code relations among its components. In this last example, students move from the knowledge of the properties of a rhombus (Level 2) to the knowledge of the logical relations among the properties (Level 3). Yet, the discussions that students can have about the properties are very different from an exchange of ideas about the relations among the properties themselves. For example, in the first case, students may discuss whether any rhombus has a symmetry of 90° . In the second, they may discuss whether the equality of the sides of the rhombus implies that some sides are parallel.

Consider in more detail how this linguistic shift occurs during the phases of learning. At the beginning of the first phase, students possess several signficatory items (*symbols*, in van Hiele's terminology). In the case of Level 1, these can be images of figures, or, in the case of other levels, they can be properties, relations among properties, or axiomatic systems. If during the first phase, the teacher were to send a message that used those signficatory items as objects of reflection, the message would not be easily decoded by the student. At the beginning of the first phase, he or she is still using these signficatory items

mainly as *instruments of action*, and is just starting to use them as *instruments of reflection*, following von Glasersfeld's (1974) terminology.

In the rhombus example, when students finish Level 1 they are able to discuss whether a rhombus possesses the property that "its diagonals are perpendicular." At the end of Level 1, they are capable of using the signifiatory items *components of rhombuses* as instruments of reflection. They know what a *rhombus* is, what the *diagonals* are, and the meaning of saying that "the diagonals are perpendicular." They also know how to condense the relations between these terms when they state properties of the components of rhombuses. Yet at Level 2, the signifiatory items as instruments of action become the properties. The teacher is sending messages about relations among properties and not about relations among the figures and their components. The properties become the instruments of action, and it is only at the end of the second phase of Level 2 that the student will start to understand the properties as instruments of reflection. This means precisely that their symbols become signals, in Pierre van Hiele's terminology. At the end of the second phase, students begin to reflect on the symbols they had at the beginning, and the symbols become signals.

During the third phase of learning, the student will make "explicit the structures" (van Hiele, 1986, p. 97) that were previously only known implicitly. Before the third phase, symbols are controlled implicitly ("one has understood the structure and knows how to work with it," p. 79). The third phase aims at allowing students to talk about the new things they have discovered during the second phase, in order to make the symbols explicit. Only then does "it [become] possible to talk with other people about it" (p. 79). The analogy between a passive language (the portion of the language that a person understands) and an active language (the portion of the language a person speaks) is apparent. The purpose of the third phase is to force the transformation of a passive language acquired during the first two phases into an active language. It is also during this

phase that the teacher introduces the technical terminology—technical words will not add new symbols, they will only allow precision (shortcuts) in the discussion.

Now that students possess the symbols and know how to operate with them in the context of a given level, the fourth phase will allow further explorations of the topic, and “in this way signals, precursors of [new] symbols, are developed” (van Hiele, 1986, p. 97). The free orientation will provide students with the basis for the construction of new relations among the old symbols that will constitute the initial symbols of the next level.

Clearly, each level has its own linguistic symbols and its own system of relations that connect these symbols. Symbols and relations in the van Hiele theory are dependent upon the context in which they are produced, and the theory produces a hierarchy of levels that contains a hierarchy of symbols. This construction permits van Hiele to answer his initial problem of the apparent differences between the languages of the teacher and the students. It also allows him to state one of his major claims: Two persons who reason at two different levels cannot understand each other.

Another important consequence that derives from this didactical process is van Hiele’s implicit explanation of the stability of concepts. Piaget, for example, proposed the mechanisms of accommodation and equilibration to account for the change and stability of our concepts. Van Hiele took a different path. In the van Hiele model, change and stability of concepts are achieved by the social interactions among students and teacher that take place mainly during explicitation, free orientation, and integration. This approach guarantees that the geometrical concepts will be intersubjectively shared, and therefore objective, in van Hiele’s sense—truth shared by a social group having an adequate language.

One area to which van Hiele (1986) applied the theory was the problem of the reason why some students resort to rote learning. For example, he analyzed arithmetic teaching in the first grade. If the teacher teaches arithmetic at the second level to students who are still

at the first level, rote learning will most likely occur. From the teacher's point of view, the mathematical knowledge he or she has explained to the students is an action structure, because it is so common to him or her as to be automatic. From the point of view of the students, since they are still at the first level, they are not capable of deciphering the teacher's justifications but can only hope to imitate the teacher's actions. In the long run, with enough persistence, the students are able to calculate as the teacher does, but their knowledge is only an imitation and is incapable of adjusting to new situations.

A Gestalt Approach in Practice

The depth to which gestalt principles influence the theory can be appreciated if we take a closer look at Dina van Hiele-Geldof's work (1984b) developed in her 1957 dissertation. She starts the teaching of geometry with an observation of cubes. After having counted and observed the faces, vertices, and edges, she asks students to build a cube. Then activities of the same type are performed with other regular polyhedra—for example, the octahedron and the tetrahedron—and some terms are introduced.

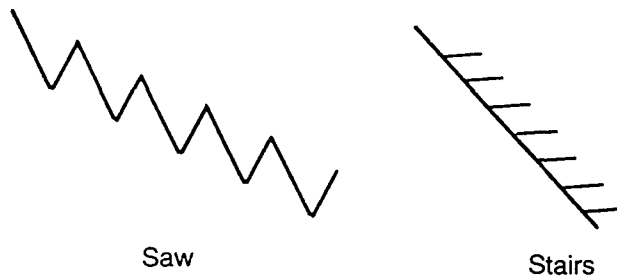
During this first part, van Hiele-Geldof's emphasis is on what students can immediately see; that is, on what van Hiele calls the "'spontaneous' structures of the material" (1984a, p. 237). An assumption of objectivity is implicit in this approach, as everyone is assumed to observe the same structure and interpret it similarly. There is no need for the negotiation of meaning at this point, and language is used only to present some new terms.

Van Hiele-Geldof then moves to a unit based on tessellations and begins by asking students to look for a way to tessellate a sidewalk with squares. After they have completed this tessellation, she leads the discussion to what they see in that tessellation. It is possible to observe straight lines, groups of equidistant straight lines, distinct groups of parallel straight lines, right angles, squares, and so forth. She then asks students to find other means of paving the floor, and for each of the tessellations she leads discussions about what

can be seen in these structures. Depending on the particular tessellations, other organizations of geometric objects may be perceived.

Van Hiele-Geldof then leads students to observe the tessellation (the perceptual field, the structure) from more than one perspective. She wants students to restructure their perceptual field in such a way that they can see another type of organization. She continues to guide students through the spontaneous structures, but now language must play the role of showing other structures embedding or encompassing the original structure perceived by the students. These are the two properties of structures that Pierre van Hiele claimed should be taught.

In the process of observation and discussion of these tessellations, properties like “the angles of triangles sum to 180° ” and the ideas of *saw* (a broken line with two directions) and *stairs* (a line segment intersected on one side by parallel line segments) are used to represent the properties of the angles between groups of parallel line segments.



The equality of opposite angles of a parallelogram can be established using saws and stairs. Now, Dina prepares the shift from Level 2 to Level 3, making use of genealogical trees of properties to facilitate students' reflection on the logical structure of the properties of figures.

A Critique of the Theory

Van Hiele theory provides a fresh perspective on mathematics teaching and learning in general, and on geometric learning in particular. Its impact on mathematics education can be compared to a paradigm shift during which

scientists adopt new instruments and look in new places. Even more important, during revolutions scientists see new and different things when looking with familiar instruments in places they have looked before. (Kuhn, 1970, p. 111)

This passage portrays what has happened to mathematics educators who have used the van Hiele theory to study the teaching and learning of geometry. Most of its importance comes from the way in which the levels are defined and how they are articulated.

Nevertheless, the theory leaves broad areas for which it provides no explanation. In this section, I discuss some of its limitations in the areas of cognitive development, the goals of geometric learning, the importance of individual differences, and students' autonomy in the learning process.

Van Hiele's perspective on cognition is not (nor does it intend to be) a psychological theory on its own terms. On the one hand, it relies very heavily sometimes on gestalt psychology and at others times on a form of constructivism. On the other hand, it leaves many areas uncovered. Imagery is one such area. In the last two decades there has been an appraisal of the role imagetic thinking plays in mathematics learning and in scientific work (Bishop, 1980; Clements & Battista, 1992; English, 1997; Fischbein, 1987; Hershkowitz, 1989a; Hershkowitz, Parzysz, & van Dormolen, 1996; Sutherland & Mason, 1995). The idea that "at the third level, it is no longer possible to use visual structures to clarify ideas" (van Hiele, 1986, p. 141) denies the role that mental images play in higher order thinking. Other areas at the heart of the learning of geometry, like visualization, spatial orientation, and the two-dimensional representation of three-dimensional objects, are also absent.

In fact, the theory implicitly assumes that the teaching and learning of geometry should aim at developing a deductive approach. The theory does not explicitly contemplate areas such as spatial orientation and representation, measurement, trigonometry, or analytic geometry, which are very important in contemporary approaches to geometry. As was noted at the beginning of this chapter, this omission has its roots in the pre-new-math environment in which the theory was created. Today, the goals for geometric learning have changed. The first Standards document of the National Council of Teachers of Mathematics (NCTM, 1989), for example, included, among some goals for geometry learning consonant with the van Hiele theory, others such as the development of a spatial sense, the visualization and representation of geometric figures, the representation and solution of problems using geometric models, the development of an appreciation of geometry, and proficiency in analytic geometry, none of which the theory directly addresses.

The van Hieles' narrow interpretation of the goals of the geometry curriculum has impacted most of the investigations about the theory. In a review of research on geometry learning (Hershkowitz, 1989a), it was apparent that the research community has perceived van Hiele-based research as an area distinct from, for example, visualization, spatial orientation, higher level concepts (like transformation geometry or proof), measurement, and geometric problem solving. The only exception to this trend is research that attempted an extension of the theory to the Logo environment (Olive, Lankenau, & Scally, 1986; Olson et al., 1987).

Among the areas in which the theory does not provide satisfactory explanations is that of individual differences. In van Hiele theory students are always treated as a homogeneous group, and there are no such entities as individual students with different cognitive styles and different learning preferences. The purpose of the van Hieles' didactical approach is to socialize (normalize) students into the mathematical knowledge agreed upon by the

“committee of experts” (van Hiele, 1986, p. 217) constituted by the community of mathematicians.

The theory does not admit the possibility that students might develop autonomous mathematical knowledge. One reason is that, as mentioned above, Level 1 is based on the perception of the “‘spontaneous’ structures of the material,” which are objective in the sense that many people agree on their content. Consequently, all students are assumed to perceive these structures similarly. Another feature that hinders students’ autonomic processes is the proposed role of the teacher. Throughout the van Hieles’ discussion of the phases of learning, the teacher is assumed to be the source of knowledge in the classroom. The teacher is to hold conversations, help students trace relations between the language symbols, and guide the explicitation phase, providing the technical terms. The teacher assumes the role of the enculturator (some would say the enforcer) of the students into the accepted school mathematics culture. Throughout this process, students are never expected to contribute their own knowledge or experiences, nor are they expected to produce alternative mathematical productions. This presupposition of the objectivity of mathematics and the nullification of the role of students’ knowledge in school mathematics makes it extremely difficult to explain why some students seem to build unconventional mathematical concepts (Vinner & Hershkowitz, 1983; Wilson, 1986).

In spite of such limitations, the theory has replaced previous paradigms of geometric learning proposed by Piaget, Bruner, and the behaviorists. Many mathematics educators seem to believe that the theory can be amended to fit more contemporary views and have used it in some areas of research. Later in this chapter, I note some attempts that have been made to articulate the theory with alternative geometric perspectives.

Van Hiele and Mathematics

In this section, I discuss two issues that reveal some of the positions of van Hiele in relation to mathematics knowledge: the nature of mathematical truth and the nature of mathematical knowledge.

Van Hiele's philosophical position about truth is what Lakatos (1970) calls *truth by consensus*; that is, van Hiele suggested that objectivity is a social artifact, never absolute and always a matter of degree. In fact, he claimed that there are two ways in which objectivity is achieved: by relying on the opinion of a group of experts and by establishing operational norms that reflect a consensus within a group. In either case, the development of an adequate language is paramount. Mathematical propositions satisfy his criterion for objective judgments because they are discussible, testable, and shared by a sufficient number of people. Moreover, mathematical language permits very little variation of interpretation, which means that propositions have a high potential for being accepted. Still, the issue of their certainty is not resolved externally to the group. To obtain a higher degree of certainty, van Hiele (1986) proposed that mathematics is based on a "sufficiently ordered field of nonmathematical, or perhaps not yet mathematical, knowledge" (p. 219) from which mathematical elements and their relationships originate. In this field, the relationship between these elements is sufficiently known and accepted to make the elements of the field an objective base for mathematics. Van Hiele did not provide examples that would help characterize this field of nonmathematical knowledge, and one can only infer that he was trying to base mathematics both on the knowledge we share through our culture and on the "'spontaneous' structures of the material" (van Hiele, 1984a, p. 237) on which geometric knowledge is based.

This position with respect to mathematical knowledge explains why van Hiele considered mathematical structures, together with Islamic architecture or a winding staircase, examples of *rigid* (or *strong*) structures. Rigid structures are those that can be

uniquely extended. The style of a painter, a man driving in heavy traffic, or fallen leaves are examples of *feeble* structures whose process of extension may vary. If a structure is strong, it will usually be possible to superpose a mathematical structure onto it. Van Hiele, however, proposed that such distinction is not absolute but a matter of degree. In particular, "mathematical structures are very rigid if the rule of the structure is given. But they lose their rigidity if the rule is not given" (van Hiele, 1986, p. 20), as when one has the first few terms of a sequence and can use several mathematical rules to continue it.

Research on the van Hiele Levels

Van Hiele theory did not produce immediate effects on the field of mathematics education. Soviet researchers used the theory in the late 1960s as a base for the development of a new geometry curriculum (Wirszup, 1976). Only after Freudenthal (1973) and Wirszup (1976) presented brief discussions of the theory did other researchers start to focus on it. Investigators' interest in the theory has prompted research in three areas: (a) validating the theory, (b) extending it, and (c) using it as a framework for research. These categories are not exclusive, and in fact many researchers investigated more than one of them. I briefly analyze the first two categories.

When the theory was discovered by mathematics educators, there was virtually no empirical research, besides the van Hieles' studies, to support the theory. Since it departed radically from the tradition of research at the time, the objective of the first studies was to validate the theory. These studies attempted to confirm the hierarchical nature of the levels, their sequence, their discreteness, and their appropriateness to describe events in geometry classrooms. The van Hiele theory, however, is general in nature and lacks operational definitions. Consequently, each of these researchers faced the very difficult task of operationalizing the levels by designing his or her own instruments and procedures. Different methodologies were produced, some using tests and others interviews, and there

is disagreement among researchers on their validity (Crowley, 1990; Gutiérrez, Jaime, 1998; Gutiérrez, Jaime, & Fortuny, 1991; Usiskin & Senk, 1990; Wilson, 1990).

Mayberry (1981) conducted the first North American investigation on the van Hiele theory after the van Hieles themselves. A key concern in her research was “to investigate whether the levels form a hierarchy and whether the levels were topic-free; that is, that the student performed at the same level on different geometric concepts” (Mayberry, 1981, p. 9). She identified several topics in geometry and developed a sequence of tasks that would assign a level to each participants for each of the topics. She showed that the sequence of the lower levels seemed to hold but found that most of her participants were at different levels for different topics. Other researchers obtained similar results using the same instruments (e.g., Mason, 1989).

The sequence of the levels was also evaluated in a project at the University of Chicago (Usiskin, 1982). After testing a large sample of high school students in the United States, the researchers confirmed the sequence of the first four levels and raised doubts about the testability of Level 5. Gutiérrez and Jaime (1987) designed a research study with objectives similar to Mayberry’s, but used a much larger sample and a different methodology, using tests instead of interviews. They designed three tests based on different topics (polygons, measurement, and solids) and found that although the global hierarchy of the first four levels held, the results were inconclusive on the issue of the globality of the levels across geometric topics.

The results obtained by Mayberry and confirmed by Gutiérrez and Jaime apparently call into question the possibility of assigning a global van Hiele level to each student. Could, for example, a given student be at Level 1 on triangles and at Level 3 for circles? Apparently not, according to the original theory, but the globality of the levels could be reinterpreted by assuming that the levels have a fractal-like structure in which a global level of thinking could be assigned to each student, influencing, but not determining, the level of

thinking for each conceivable mathematical topic. In an interview reported by Mayberry (1981), Pierre van Hiele indicated that students might be at different levels on different concepts. He proposed that higher levels in one topic might facilitate learning in other topics. However, no investigation has yet tackled the problems raised by this issue; namely, how far apart can knowledge of mathematical topics be? Clearly it seems possible that a student may be well advanced in one topic but ignorant of everything in another. But what are the consequences of this situation for the learning of the second topic? To what extent does advanced knowledge in one topic allow students to move to a high level in another topic? If they can move in this way, are there some topics that would be better suited to facilitate the subsequent learning of other topics?

The project directed by Burger (Burger & Shaughnessy, 1986) aimed at determining whether the van Hiele levels can serve as a model of students' development in geometry. The researchers designed eight tasks and videotaped students' responses. In a procedure similar to that used by Mayberry (1981) and by Gutiérrez and Jaime (1987), they attributed a van Hiele level to each subject for each of the eight tasks. They also assigned an overall van Hiele level to each student. Finally, they developed a list of level indicators aiming at a better characterization of the levels. Their results support the hierarchical nature of the levels. They designed their investigation so that it permitted the detection of sources of confounding effects in assigning levels to students. A rater of the videotapes would assign a predominant level on a given task to a student, adding conflicting evidence that would not support this assignment. For example, one rater might assign Level 1 to a task with conflicting evidence for Level 2, and another might do the reverse. No global quantitative results about these discrepancies are provided in the literature cited here, but as far as the van Hiele theory is concerned, this research seems to help characterize the levels qualitatively. Contrary to van Hiele, it rejects the idea that the levels are discontinuous. In fact, Burger and Shaughnessy believed that many of the discrepancies in the assignment of

a level to some students may have been due to the possibility that they were in a transition process from one level to the next. Other researchers have also felt the need to classify students in a transitional stage (Fuys, Geddes, & Tischler, 1985; Olive et al., 1988). Burger and Shaughnessy also found evidence that some students oscillate between two levels, sometimes during the same task. Fuys, Geddes, and Tischler also confirmed this phenomenon. Other students seemed to regress to lower levels after having successfully completed geometry courses. These facts allowed Burger and Shaughnessy (1986) to conclude that

the levels appear to be dynamic rather than static and of a more continuous nature than their descriptions would lead one to believe. Students may move back and forth between levels quite a few times while they are in transition from one level to the next (p. 45)

It is, however, far from clear that this oscillation between levels is necessarily caused by a transitional state between one level and the next. An alternative explanation could be provided if the nature of the levels is assumed to be such that every student is capable of jumping to lower levels if he or she feels that these are appropriate to the solution of the problem in question. In other words, one can assume that human cognition allows for changes in level depending on the context in which the person is operating. The oscillation would not then be caused by a transition, an abnormal state, but would be inherent in the levels themselves.

Fuys et al. (1985) developed a working model of the levels with detailed level descriptors. They then elaborated three teaching modules, each correlated with specific level descriptors that included instructional activities and assessment tasks. They conducted clinical interviews based on the modules for sixth and ninth graders. These interviews allowed them to characterize the geometrical knowledge of the students.

The investigations just cited started from the theory and developed instruments that would operationalize it. Villiers and Njisane (1987) took a different approach. They developed their own geometry test with questions on several “geometric thought categories” (p. 117) and used it with a large population of 9th- through 12th-grade South African students. They concluded that: (1) reading and interpreting definitions did not correlate with other aspects of geometric learning, (2) the hierarchy of geometric abilities was very close to van Hiele’s proposals, and (3) the hierarchical classification of geometric figures was much more difficult than longer deduction. Their results are discussed in chapter 4.

Despite the number and variety of the investigations devoted to validating the theory, there are some areas that have received no attention from researchers. For example, no investigations have used the theory to describe geometric learning in regular classrooms. Most investigations either used tests (Gutiérrez & Jaime, 1987; Usiskin, 1982; Villiers & Njisane, 1987), interviewed students in clinical settings (Burger & Shaughnessy, 1986; Fuys et al., 1985; Mayberry, 1981), or conducted actual teaching in clinical settings (Fuys et al., 1985). None observed interactions in regular geometry classrooms. Although some curriculum development projects have been influenced by the van Hieles’ ideas (Maxwell & Schell, 1998; Treffers, 1987; Wirszup, 1976), no recent investigation has attempted to change the didactical process of geometric learning as the van Hieles proposed and to evaluate the effects on the quality of the geometric learning. Consequently, there is no information about the validity of the didactical process—the phases of learning—proposed by the van Hieles for moving from one level to the next. On the contrary, the very fact that researchers have been able to study reliably the levels in the absence of this didactical process suggests that the transition from one level to the next is produced under much broader teaching conditions than those anticipated by the van Hieles.

Virtually every researcher who interviewed students has presented strong evidence that the distinction between the levels is accompanied by the linguistic distinctions predicted by the theory. There is, however, no information about the linguistic details that accompany the progression from one level to the next.

A second category of investigations attempted to “articulate the paradigm” (Kuhn, 1970, p. 29) either by building upon the theory or by establishing links between the van Hiele theory and other theories. In an early theoretical work, Hoffer (1981) provided a description of the levels in terms of other “skills” that play an important role in geometric learning: visual, verbal, drawing, logical, and applied skills (pp. 15-17). He provided examples of each of these skills in each of the levels. In his article he calls the first level “*recognition*,” and presents examples of visual skills for each of the levels.

Lunkenbein came closer to establishing a representation-based model compatible with the van Hiele theory. In an initial didactical experiment (Lunkenbein, 1983b), he confirmed the explanatory power of the first two levels. In a second approach (1983a), he hinted at a cognitive interpretation of the theory in terms of mental images and attempted to develop links with Piaget’s theory of groupings:

A grouping consists of:

- (1) “static” elements, called states or objects and which represent instances of a given conceptual context or problem situation;
- (2) “dynamic” elements, called operations or transformations that act on the states or objects and relate them to each other;
- (3) properties of or relations between states or objects brought out as the effect of the operations on the states. (p. 256)

Lunkenbein defined three types of groupings: (1) the *infralogical groupings* relate spatial figures to each other on a concrete basis and are the result of activities involving concrete materials and actions; (2) the *groupings of the classes of a partition* relate those

figures on the grounds of verification or not of specific properties; and (3) the *groupings of logical inclusion* are based on class-inclusion relationships. Lunkenbein claimed that these three types of groupings correspond to the first three levels of the van Hiele theory.

Although Lunkenbein believed that class inclusion is an attribute of the third level, Villiers and Njisane (1987) concluded that it should precede Level 3. Kay (1986) took a different point of view toward characterizing this class-inclusion problem. She developed a teaching experiment that started with quadrilaterals, proceeded to rectangles, and ended with squares. She used names for these shapes that emphasized the class-inclusion relationships, and at the end of the year a majority of her first graders understood the hierarchical relationships among these figures. She speculated that van Hiele theory may be well adapted to instruction that proceeds from specific to general, but not the reverse. In chapter 4, I discuss this issue and propose a different interpretation.

In an investigation aimed at a description of the impact that Logo has on students' mathematical thinking, Olive, Lankenau, and Scally (reported in Olive, Scally, & Skaftadottir, 1987) conducted two case studies that suggest links between the van Hiele theory and the Solo Taxonomy. One of their subjects who was at the lowest van Hiele level tended to execute tasks at the unistructural and multistructural Solo levels. The other subject, who was at a higher van Hiele level, performed mainly at the relational and even extended abstract Solo levels. Other investigations (Clements & Battista, 1988; Olson et al., 1987) proposed the development of hierarchical levels in the Logo environment that were similar to van Hiele's levels.

Apparently, no investigations have attempted to link the van Hiele theory with recent research on cognitive science. This study attempted to establish such links to the process of categorization.

CHAPTER 4
CATEGORIZATION OF MATHEMATICAL OBJECTS
AND THE VAN HIELE THEORY

In the last two chapters, I present an overview of two theories relevant for the study of the concept of angle. Categorization studies, and especially the work of Johnson and Lakoff in the context of cognitive semantics, provide a framework within which the category of angles can be understood. Van Hiele theory was born out of the mathematical education community, and so it comes from a different scientific tradition. The present chapter should be understood as an effort to examine the viability of the simultaneous use of both theories in the study of geometric concepts and to explore the ways in which they jointly help to shed light on research problems in mathematics education. In other words, as the purpose of this study is to investigate the concept of angle by making use of two distinct theories, it is useful to understand the ways in which they are potentially useful to, or may actually hinder, research efforts.

As angles are at the center of this study, I choose to analyze in this chapter five areas that have been the focus of research on geometric learning. In each of them, researchers have detected evidence of students' nonstandard mathematical knowledge. None of these cases seems to be partially or completely explained by van Hiele theory. I attempt to look at each area in light of the discussion about the categorization of mathematical objects that was developed in chapter 2 and provide an explanation of the sources of that knowledge. This analysis reveals prototype effects with roots in distinct cognitive models. Later, I propose

an outline of the ways in which van Hiele theory should be amended to account for those effects.

Preliminary Questions

In terms of categories, the van Hiele theory explains how geometrical objects move from a category composed of concrete geometrical objects to increasingly abstract categories. These categories are characterized first by properties, then by clusters of properties, and finally by minimal sets of properties. Van Hiele proposes that the objects to be categorized vary depending on the level of thinking the person is using. Categories also encompass a hierarchy of abstractions: There is a basic-level and image-schematic models on which other models are built, but there is not a precise organization as in van Hiele theory.

Van Hiele theory also claims that each level is characterized by a specific language, with its own specific symbols and relations, and, as shown in chapter 3 (p. 55), researchers have confirmed this theoretical prediction. Researchers investigating the properties of categories seem not to have focused on this characteristic.

Van Hiele theory is based on classic categories. For example, one key characteristic of this theory is the notion that when there is a shift to an upper level, the observed objects become manipulated objects. This move involves whole categories of objects, and it may be the case that not all members of these categories have the same status. The van Hiele theory does not make any such distinction.

An Attempt to Explain Some Prototype Effects in the Framework of van Hiele Theory

Hershkowitz (1989a) attempted to accommodate some prototype effects in the structure of the van Hiele levels. Her description is closely influenced by Vinner's concept images described in chapter 2 (p. 30).

According to Hershkowitz (1989), for each concept, students form one or more prototypical examples composed of the critical attributes of the concept together with specific noncritical attributes that have strong, salient, visual characteristics. Students use these prototypical examples in the first two van Hiele levels "as a model in their judgment of other instances" (p. 83), in what she terms *prototypical* judgment. This kind of judgment is contrasted with *analytical* judgment, which is a correct judgment based on the concept's critical attributes. At the first two van Hiele levels, students make use of the prototype in distinct ways. *Type 1 prototypical judgment* occurs at Level 1 when students use the prototype as a frame of reference to perform visually comparisons. At Level 2, students make use of the prototype's attributes instead of the prototype's image and try to impose them on every instance. This is called *Type 2 prototypical judgment*. Level 3 is, in contrast, based upon analytical (or *Type 3*) judgment. Hershkowitz found that as students move through the levels, Type 1 judgment decreases but never disappears completely, whereas Type 2 completely disappears.

The featural and dimensional perspectives discussed in chapter 2 (pp. 16-17) seem to have been the main influence on Hershkowitz's (1989a) approach. Consequently, by assuming that prototype effects derive only from an overevaluation of specific features of the geometric objects, her model shares many of its limitations: the impossibility of explaining basic-level effects, especially those derived from nonvisual sources, or metaphoric models, among others. In particular, her definition of prototypes seems to be

narrow. She explicitly states that “the prototype phenomenon and prototypical judgment seem to be mostly a product of visual processes” (p. 83). In fact, her proposals only explain phenomena that involve visual prototypes. In general, although Hershkowitz’s proposals adequately characterize some research findings at the first van Hiele levels, they must be enlarged to incorporate others.

The Influence of Visual Prototypes and Metaphoric Models

Prototype effects caused by image-schematic models, characterized by a gestalt of a geometrical figure, are well known by researchers (Burger, 1985a; Clements, Sarama, & Battista, 1998; Fuys et al., 1985; Hasegawa, 1997; Hershkowitz et al., 1987; Junqueira 1995; Mariotti, 1995; Mason, 1989; Presmeg, 1992; Scally, 1987; Shaughnessy & Burger, 1985; Vinner & Hershkowitz, 1983; Wilson, 1986b). They are the most simple cases of prototype effects. The characteristics of these prototypes can be summarized as follows:

1. A preferred position; namely, triangles, squares, rectangles, and parallelograms must have a horizontal base (Fuys et al., 1985; Hasegawa, 1997; Mason, 1989; Presmeg, 1992; Scally, 1987; Vinner & Hershkowitz, 1983; Wilson, 1986b);
2. Symmetry; for example, obtuse triangles with their smallest side as the base are not recognized, or a right triangle is thought of as a half triangle (Burger, 1985a; Fuys et al., 1985; Hasegawa, 1997; Vinner & Hershkowitz, 1983); and
3. An overall balanced shape; namely students do not recognize “skinny” triangles, “pointy” triangles, or extremely small squares (Burger, 1985a; Fuys et al., 1985; Hershkowitz et al., 1987).

These characteristics provide a good description of prototypical geometrical figures. Moreover, the gestalts do not require some characteristics that are significant from a standard mathematical point of view. For example, sides may be curved or “crooked”

(Burger, 1985a; Fuys et al., 1985; Scally, 1987). There is also some evidence showing that students form such image-schematic models even when only a verbal definition is given (Vinner & Hershkowitz, 1983). Students also show substantial agreement about these models.

In fact, it may happen that in some cases these image-schematic models are intermixed with some metonymic models. In the case of triangles, Vinner and Hershkowitz (1983) present evidence suggesting that “overall balanced” isosceles triangles are taken metonymically as best examples of the whole category of triangles. This idea can be extended to other categories. It is reasonable to conjecture that a whole set of “overall balanced” rectangles may stand for the whole category of rectangles. In this particular case, there is historical evidence that the search for overall balanced rectangles gave rise to the propositional model of *golden rectangles* built out of the metonymic model of *balanced rectangles*. The quest for the ultimate perfect rectangle in Greece, embodying beauty, universal order, unity, among others, produced a well determined (propositional) procedure to obtain one.

There is a substantial agreement about the sources of these models. Upon entering school, children are able to identify balls, cans, boxes, and other shapes. In school they are taught the names of two-dimensional geometric figures. The former objects are real-world objects. Children learn about them by manipulation or observation, and they are capable of identifying them regardless of their position. The latter objects, geometric figures, tend to be learned and used mainly in school. Research has shown (Fuys et al., 1985) that geometric figures are usually presented in pictures that match the three characteristics of students’ mental images described above: preferred position, symmetry, and balanced shape. In summary, children’s prototypes are heavily influenced by the best exemplars shown to them in the school environment.

Image-schematic models, however, do not present a complete picture of children's learning. There is evidence that there are also perceptual problems involved in the identification of geometrical figures, namely, in the perception of right angles. Vinner and Hershkowitz (1983) provided some evidence suggesting that isolated right angles or right angles included in right triangles are more difficult to identify when none of the sides is horizontal. Some of their participants used the strategy of turning the figure so that they could achieve a better identification.

Other models may also be involved in the identification of geometrical figures. In chapter 2, I mentioned a metaphoric model used (or implicitly used) by the mathematical community, the "mountain metaphor." Here I analyze two examples of students' metaphoric models. Burger (1985) reports that Bud (one of his participants) explained that some of his triangles were different from others because they were "pointing that way [to the right, or down]" (p. 52). The idea that triangles point in one direction is what Johnson (1987) would call a metaphoric model based on our kinesthetic image schema of pointing. Triangles, for Bud, are embodied; that is, some of their properties "are a consequence of the nature of human biological capacities and of the experience of functioning in a physical and social environment" (Lakoff, 1987, p. 12). This way to think about triangles is not Bud's particular model. Rather it is a social model because we are all able to understand Bud's point. In some contexts, we ourselves would be willing to say that a triangle is pointing in one direction.

Fuys et al. (1985) report an example of another metaphoric model. One of their participants (Gene), when asked if a square was a rectangle, answered "Na, that's a box" (p. 83). Of course, Gene knew that, literally, a square is not a box. He was using a box as a metaphoric model of a square. Gene also thought that "the sides of a rectangle" referred to the vertical sides, whereas the horizontal sides were not "sides" but "top" and "bottom."

Again, he was using the metaphoric model of a rectangle thought of as a box. When we use English words to speak about objects in the world that look like rectangles, like boxes, we may make this linguistic distinction among sides, bottom, and top. Both these metaphoric models were based on Gene's experiences with objects in his environment.

Prototype effects produced by image-schematic and metaphoric models are reported to occur with students at the first two van Hiele levels. At the first level, students make extensive use of them, and at the second level, students may describe properties of the geometric figures that are heavily influenced by them. Hershkowitz (1989a) has detected these effects even in subjects at Level 3. There is, however, evidence suggesting students' ability to overcome these effects after Level 1 (Junqueira, 1995).

The role played by these models at higher levels is unclear. There are strong arguments to support the idea that mental images play a powerful role in higher order thinking in general, and scientific thought in particular (Clements, 1981). However, it is uncertain whether these are still image-schematic models as I have defined them here, or if they have different characteristics.

The Classification Issue

In this section, I discuss van Hiele's predictions about the hierarchical classification of quadrilaterals, analyze the mathematical basis for this classification, describe research findings that challenge the predictions of the theory, and propose an alternative explanation.

The classification of quadrilaterals is an area in which van Hiele theory makes predictions that challenge the usual notion of what students are able to learn. According to the dominant interpretation of the theory, at the end of the first level, students are able to identify rectangles, squares, rhombuses, and other figures. At the end of the second level, they are able to enumerate several properties that each of these figures has. Only at the third

level do students agree with the usual hierarchical classification of quadrilaterals, namely, that a square is a special type of rectangle and that both are special parallelograms. The reason for this change is that at the third level students are able to understand locally logical connections between properties, and consequently are capable of accepting the logical consequences of a definition (van Hiele, 1984a).

Occasionally, the van Hieles defended the position that class inclusion of quadrilaterals could be understood at Level 2. Dina van Hiele-Geldof (1984a) said it explicitly: "At level 0 [Level 1 in my terminology] a square is not perceived as a rhombus; at the first level of thinking [Level 2] it is self-evident that a square is a rhombus" (p. 222). Pierre van Hiele said the same thing in his 1973 book *Begrip en Inzicht*. He proposed that class inclusion may occur at Level 2 since a child may realize that a square is a rhombus because it has all its properties (cited in Villiers & Njisane, 1987). This interpretation has not been used by the research community. Perhaps it is not known, or perhaps it runs contrary to the way in which Level 3 is conceived; namely, the acceptance of squares as particular rectangles is thought to be the consequence of understanding the role of definitions, which occur at Level 3, or perhaps it has not been used because van Hiele himself proposed the opposite position elsewhere (van Hiele, 1984a).

A review of the research does not provide conclusive evidence to accept or reject either of the previous points of view regarding class inclusion. Many researchers have accepted as a crucial characteristic of Level 3 the ability to correctly classify quadrilaterals and have incorporated it into their criteria for level classification. To decide the position of class inclusion in the hierarchy, one would need to correlate students' ability to understand class inclusion with their levels, which would need to have been determined without recourse to class inclusion itself. That was the approach followed by Villiers and Njisane (1987). They identified eight geometric thought categories (GTCs):

1. recognition and representation of figures

2. visual recognition of properties
3. use and understanding of terminology
4. verbal description of properties of a figure or its recognition from a verbal description
5. one-step deduction
6. longer deduction
7. hierarchical classification (inclusion of classes)
8. reading and interpretation of given definitions

A test that included questions related to these GTCs was given to a large number of 9th-through 12th-grade South African students. Villiers and Njisane grouped the results for each category and performed a statistical analysis looking for ordered relationships. They found that Category 8 (reading and interpretation of definitions) did not correlate with the other categories. That is hardly surprising given the close relations of this category with linguistic rather than geometric abilities. The other seven categories did correlate with each other and formed a hierarchy, the first being the easiest and the seventh the hardest. Only then did they assign a level to each of the seven categories. They concluded that the levels match the hierarchy, which confirmed the predicted sequence of the levels.

Villiers and Njisane research provides strong evidence that the seventh category (hierarchical classification) is either at Level 3 or above. In fact, in contrast with all the other seven categories, only 5% of the students correctly answered half the test items that related to this category. Moreover, there was almost no change in the percentage of correct answers across the grades. Also, category 7 was the most difficult of all, far more difficult than deductions involving several steps. These results seem to suggest that only students at the final stages of Level 3 or even at the beginning of Level 4 could understand hierarchical classification well. As Villiers (1994) puts it:

children's difficulty with hierarchical class inclusion (especially older children) does not necessarily lie with the logic of the inclusion as such, but often with the *meaning*

of the activity, both linguistic and functional: linguistic in the sense of correctly interpreting the language used for class inclusions, and functional in the sense of understanding why it is more useful than a partition classification. (p. 17)

Other investigators have reported the extreme difficulty of hierarchical classification (Burger & Shaughnessy, 1986; Fuys et al., 1985; Mason, 1989; Usiskin, 1982). One researcher reported that “to many children, squares are not rectangles and rectangles are not parallelograms, even though they share many properties. Such beliefs persist among many high school students, even those who memorize correct definitions of the shapes” (Burger, 1985a, p. 53).

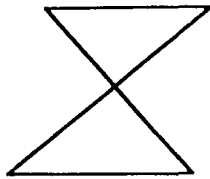
There have been some attempts to explain the source of such difficulty. Fuys et al. (1985) call it “interference of prior learning” (p. 137), which is not very specific. Others have suggested that textbooks and instructional practices may be responsible (Burger, 1985a) and still others have contended that teaching the hierarchical classification of quadrilaterals runs against a constructivist perspective that values students’ active participation in the learning process (Villiers, 1994). However, the question still remains of what is specific to the classification of quadrilaterals that makes it more difficult to understand than, for example, the classification of figures into quadrilaterals, triangles, circles, and others.

Kay (1986) provides a more conclusive explanation. She showed how the use of a different naming procedure facilitated first graders’ learning of quadrilaterals’ inclusion. She began instruction with the category of quadrilaterals and proceeded to the subordinate categories of rectangle-quadrilateral and square-rectangle. Apparently by avoiding naming problems in the class inclusion hierarchy of quadrilaterals, she was able to teach successfully to the majority of her first graders the hierarchical relationships among classes

of quadrilaterals. Kay's dissertation shows that language can help students distinguish among the several types of quadrilaterals.

First graders are capable of understanding class-inclusion relationships, namely, that dogs and cats are kinds of animals, or that triangles and squares are kinds of shapes. Therefore Kay's finding that some first graders can understand certain objects as kinds of others should not come as a surprise. It is, however, a surprise from the perspective of van Hiele theory, because her first graders were hardly at Level 3—the theory's necessary and sufficient condition to understand the hierarchical classification of quadrilaterals. Kay concluded that the theory may describe the development of concepts within a hierarchy when instruction proceeds from specific-to-general but not the reverse. In what follows, I show that the classification of quadrilaterals, as understood in school mathematics, is not simply a move from general-to-specific, and I propose a different interpretation.

I first argue that there is no absolute classification of quadrilaterals and in fact that the very notion of quadrilateral is based on a particular notion of geometry. At first sight the way in which we classify quadrilaterals seems to be the correct way. The classification is based on a Euclidean approach to geometry, and it is hard to imagine other ways to organize our mathematical knowledge that would be scientifically adequate. However, there is no complete agreement about the classification of quadrilaterals in the scientific community. In fact, there is no such thing as a standard definition of *quadrilateral*. Some mathematicians include the requirement that a quadrilateral be a simple curve, whereas others do not. Consequently, the drawing below, where the intersection of the two lines is not a vertex, is a quadrilateral for some (Vinner & Hershkowitz, 1983) and not even a geometric figure for others.



There is a similar disagreement about the definition of *trapezoid*. For some a parallelogram is a special kind of trapezoid (see Burger, 1985a, p. 55; or European textbooks), but not for others.

One can extend this relativistic point of view further. The class of quadrilaterals is significant in the context of Euclidean geometry but is irrelevant in other approaches, like analytic geometry or linear algebra (Dieudonné, 1964). The notion of a trapezoid has no importance in Logo, but spirolaterals, immaterial to Euclidean geometry, do. The traditional hierarchical classification of quadrilaterals is seriously challenged by a Kleinian approach to geometry, where the relevant classification procedures involve invariants under groups of transformations.

Mathematicians are familiar with this lack of agreement and have mastered the art of living with it. Usually the issue is resolved by negotiating a definition of the terms and accepting its consequences. In more extreme cases, this disagreement may be at the very source of the production of mathematical knowledge (Lakatos, 1976). Mathematicians have learned to accept that there is no “natural” (scientifically accepted) way of defining mathematical objects, just as there is no “natural” way to categorize objects in the real world.

Van Hiele himself (1986) seems to agree that the classification of quadrilaterals is socially determined:

An intelligent person need not conclude that every square is a rhombus; this is only submission to a traditional choice. In some Greek philosophies, a square could not be a rhombus, for it had some properties a rhombus could not have. Even now there are

people arguing that a parallelogram is not a trapezoid because a trapezoid can have only one pair of parallel sides. (p. 50)

One can say the same about the classification of triangles.

As I have shown previously (chapter 2), investigations of the ways in which we form our mental representations indicate that we start categorizing at what is called the basic-level. Categories at this level share many of the characteristics of the Level 1 of van Hiele theory; namely, category members are globally perceived, they have a similar overall shape, and there is a mental image associated with the category. This is the first level named and understood by children, who are later able to differentiate subordinate and superordinate categories predicated on these basic categories (Lakoff, 1987; Rosch et al., 1976).

It is possible to use this basic-level to interpret the ways in which we categorize geometric figures. Typically, by kindergarten or first grade, children already know how to group and name different shapes of objects, namely, balls, boxes, stars, coins, footballs, and cones, among others. Schooling usually means that they will learn to name additional groups of shapes like squares, triangles, rectangles, and circles. Sometimes parallelograms, diamonds, or kites are also included. These groups have what Fischbein calls “the same level of generality” (1987, p. 147); that is, each of these groups of shapes is assimilated into a basic-level category: Each group of shapes has a common name and shares a common mental image. There is one basic-level for squares, another for rectangles, still another for balls, and so on. These basic-levels of the categories used in early grades are sometimes called *folk-generic levels* because they reflect the levels that are most commonly used by the society to communicate. Even when one mathematics educator asks another to draw a rectangle, he or she usually means a folk-generic level rectangle and

not a square. At the end of van Hiele's Level 1, children have learned to identify and name these distinct basic-level categories.

Later the basic-level categories may be grouped informally into two superordinate categories, the three-dimensional figures and the two-dimensional figures. This distinction between these two superordinate categories is not sharp, and it is unclear whether it exists for many children, or whether it has dimensional connotations. In fact, many children experience only textbook representations of mathematical solids, and others experience squares and triangles as flat prisms. Moreover, two-dimensional figures are metaphorically thought of by children as three-dimensional shapes. For example, a circle is often called a ball, and a rectangle is a door. (Lakoff would say that the ball is a metaphor for a circle and the door a metaphor for a rectangle.)

At some point in their instruction, students are asked to switch the status of the majority of the categories of quadrilaterals from basic-level to superordinates. Squares remain a basic-level category, but rectangles and rhombus are now superordinates that contain both the old categories and the category of squares. The category of parallelograms is now a superordinate category containing most of the others. The category of trapezoids may either maintain its status as a basic-level category or shift to a superordinate category that includes all the other basic-level categories, depending on the definition used. The understanding of a hierarchical classification of quadrilaterals involves the destruction of an organization of the category of quadrilaterals as a collection of basic-level categories, and its reconstruction into a category in which basic-levels exist only for a small portion of its members.

Research has shown that this cognitive reorganization is very difficult for students to accept. Burger (1985b) reports the case of a good ninth-grade student who in the last month of his geometry course would not admit that rectangles were parallelograms. His definition of parallelograms was "a quadrilateral with opposite sides parallel and has no

right angles” (p. 12). This student knew that this was a “bad definition” because it was not the textbook definition, but it was his own definition and he was able to be mathematically consistent with it. Apparently, this student was carrying his basic-level concept of parallelogram up to Level 3.

There seems to be no other domain of students’ experience outside mathematics where such a global reorganization takes place. In their taxonomic experience, the only major shifts were probably associated with learning facts such as that “dolphins are not fish.” In this case, children have to shift the basic-level dolphins from the superordinate “fish” to the superordinate “mammals,” but “dolphins” remains a basic-level category. There is evidence that students accept the global reorganization of geometrical categories as a peculiarity of school mathematics and are ready to forget it later, reverting to the earlier basic-level categories (Burger & Shaughnessy, 1986; Shaughnessy & Burger, 1985). Kay (1986) successfully circumvented the class-inclusion problem by teaching the social entities rectangle-quadrilateral and square-rectangle when children had not yet had any contact with school geometry.

There are other examples of this type of interference between mathematical hierarchies and basic-level effects. In geometry, the hierarchical classification of triangles seems to present similar problems (Burger & Shaughnessy, 1986). In arithmetic, as I mentioned in chapter 2, students are reluctant to consider specific integers or real numbers as examples of complex numbers, even when they can quote the accepted definitions (Fischbein, 1987; Tall & Vinner, 1981). A close look at Lakatos’s (1976) case study on the historical development of the concept of polyhedra shows there is evidence of such effects in the community of mathematicians.

The difficulty students experience in understanding the classification of quadrilaterals is, therefore, an example of basic-level effects. These seem to explain why the hierarchical

classification of quadrilaterals is so hard for students, whereas the understanding of the category of quadrilaterals, pentagons, or hexagons is not. It is possible to conclude that the classification of quadrilaterals is much more than a class-inclusion problem. It does not involve exclusively the learning of new hierarchical relation: rather it involves also learning that in the school mathematics culture some basic-level categories change their hierarchical status.

This interpretation has some consequences for the van Hiele theory. On the one hand, the characteristics of the first van Hiele level seem to be compatible with the interpretation and are confirmed by the anthropological and psychological research on basic-levels. On the other hand, van Hiele theory underestimates the psychological restructuring necessary to accept this classification or, in other words, overestimates the power of deduction at Level 3. It may be the case that students are only able to accept fully the consequences of definitions, especially when they run contrary to a basic-level classification, at Level 4.

Prototypical Actions (Scripts) in Geometry

Investigations focusing on students' difficulties in drawing elements in a figure have also found prototype effects. Previous explanations have interpreted these phenomena as prototype effects produced by comparisons with a prototypical image of the expected drawing. In this section, I argue for a more dynamic interpretation that takes into account the actions that students are expecting to perform when asked to draw elements of geometric figures. I focus on two important cases: drawing the altitude of a triangle and drawing the diagonals of a polygon.

This interpretation is compatible with Dörfler's (1989) account of the learning of mathematics. He proposes that learning is a cognitive reconstruction of the mathematical content involving the accomplishment of relevant actions. The main tool allowing a

reflection of and about action are the *protocols for action* (p. 214). These protocols try to denote and describe the characteristic and relevant stages, the steps, and the outcomes of the actions, through the use of perceptive objects, like written signs. Protocols for action are the cognitive reconstruction of the concept (p. 215).

Research on students' drawing of the altitude of triangles has focused on determining the characteristics of the triangle with which students experience the most problems. Vinner and Hershkowitz (1983) asked students to draw an altitude in 14 triangles. The triangles varied in their orientation, their type (isosceles, right, or obtuse), and whether the altitude to be drawn was inside or outside the triangle. The results showed that the orientation had almost no effect on the students' ability to draw the altitude. However, altitudes that fell on a side or outside the triangle and triangles that deviated from isosceles triangle had a negative impact on students' performance. As a consequence of their research, Vinner and Hershkowitz were able to produce a sequence of triangles of increasing and statistically significantly different difficulty in which to draw an altitude. From the easiest to the most difficult, the sequence is isosceles triangle (non-equilateral) with altitude to the side of different length, scalene triangle with altitude falling inside the triangle, obtuse triangle with altitude falling outside the triangle, and right triangle.

These researchers conducted a similar investigation using the diagonals of a polygon (Hershkowitz et al., 1987; Hershkowitz & Vinner, 1984). It showed that in the case of concave polygons, only those diagonals inside the polygon and that did not contain any side were drawn.

The models involved in these investigations are distinct from the ones previously described because they involve action. The participants were not using image-schematic models exclusively. They were expected to perform a sequence of actions familiar in a

certain context. This sequence of familiar actions fits exactly the definition of *script* mentioned previously (Abelson, 1976).

An interpretation of Vinner and Hershkowitz's research using scripts may say that the typical script for drawing the altitude of a triangle occurs in the context of isosceles (non-equilateral) triangles. Students then seem to attempt to adapt this script to the other cases. When given an isosceles triangle, the student draws the altitude of the triangle so that it falls perpendicularly on the middle of the side that has different length (74% of the students were able to do it). When the triangle is quasi-isosceles, this script is still maintained, but it breaks down for many students, producing prototype effects when the triangle is considerably nonsymmetric (only 40% of the students answered correctly). In the case of the isosceles triangle, the altitude coincides with the median and with the perpendicular bisector. This is no longer the case when the triangle does not resemble an isosceles triangle. The original script is changed by the students into two incompatible scripts. A considerable number of students choose to draw the median (20%), whereas a smaller number (7%) draw a perpendicular bisector to the base. The original script breaks down for an even greater number of students in the last two cases (only 32% and 30.5% of the students answered correctly in the last two cases, respectively), and again some students choose the median (21% and 20%), others the perpendicular bisector (7% and 9%). A similar interpretation can be produced in the case of the diagonals of polygons.

The research performed in these areas has not attempted any connections with the van Hiele levels. Nor did van Hiele give examples of the different types of actions that students would be able to perform at each of the levels. Apparently the only theoretical attempt in this direction was made by Hoffer (1981). Junqueira's (1995) study established some connections between students actions and verbalizations. She studied ninth graders'

justifications for their geometric constructions using the Cabri-Géomètre software. She found that some of these justifications could be linked to van Hiele levels:

Types of justification	van Hiele levels
Based on visual appearance	Level 1
Description of the process of construction together with the observation of invariant relationships	Level 2
Deduced in one or two steps showing logical orderings of the properties of figures	Level 3
Mixed	Level transition

She argued that participants developed specific scripts using Cabri-Géomètre to solve geometric problems, and their justifications are a hint of the geometric reasoning involved in these scripts.

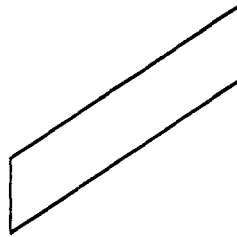
The Mismatch Between Visual and Verbal Representations

In this section I apply some of the constructs outlined in the previous sections to one issue that has been discussed among researchers on geometric learning. One important consequence of prototype effects due to image-schematic and metaphoric models and basic-level effects is the apparent mismatch between students' visual productions and their definitions. In this section I argue that those differences are the result of expecting a mathematical type of reasoning when students are not using it.

Wilson (1986a, 1986b, 1988) investigated the relationship between students' definitions and their choices of examples. On the one hand, she found that students' choice of examples was based on a limited image-schematic model that became increasingly

complex as students' knowledge expanded. On the other hand, their definitions departed to some extent from the classroom definitions. Often students' definitions were incomplete, with ambiguous terms, and did not distinguish between necessary and sufficient conditions. Although from the observer's point of view, the students were apparently not applying the definitions they wrote when choosing examples, they said they were using definitions to help them make decisions and used reasoning that made sense to them, rarely applying formal logic.

Other researchers have reported similar findings. Mason (1989), for example, reported the case of three students who identified the figure below as a rectangle and included "has four right angles" in their definition of rectangle. Later, in interviews, Mason found that these students were aware of the right angle characteristic of rectangles, but they decided that the attribute "has two long sides and two short sides" took precedence.



Fischbein (1987) offers an explanation of this apparent conflict. His point of view is that students have to learn the role of explicitly defining concepts as a necessary condition of avoiding errors in using the terminology. For him, the students have to learn the following:

When you affirm that *a parallelogram is a quadrilateral* the opposite sides of which are parallel (or with opposite equal angles) this is *exactly* what is meant by a parallelogram. Nothing is said about adjacent angles or sides. They may be equal or not. (p. 153)

It is this literal meaning, characteristic of mathematical definitions, that children fail to use, and that may explain students' ambiguity of language and their lack of distinction between necessary and sufficient conditions. After all, contrary to mathematics, everyday language draws much of its power from its equivocal nature.

Consequently, when students at the first two van Hiele levels are defining a geometric term, they are not *defining* but *describing* the term. At the first level, they are describing their image-schematic models of geometric objects in global terms, and at the second level, they are also describing them but referring to their properties. That explains the ambiguity of students' language uncovered by Wilson and the preference for some relevant characteristics over others mentioned by Mason. Learning to define in mathematical terms involves using the opposite procedure. Students have to learn to accept that the images are produced by the definitions. In fact, they have to transform their image-schematic and metaphoric models into propositional models.

There is also a second distinction not made by the van Hiele theory: the distinction between a set of necessary and sufficient conditions and a *minimal* set of necessary and sufficient conditions. A student at Level 3 may formulate a set of conditions like "A rectangle is a quadrilateral with four right angles and two pairs of parallel sides." Although this is not a minimal set, it uniquely defines rectangles. The student may very well understand that this set of conditions determines the stipulations under which a quadrilateral is a rectangle, but still not understand that the set is not minimal and therefore not a definition. In fact, it is only possible to be certain that specific sets are minimal when one has a relatively complete perspective on the axiomatic system. Moreover, even being concerned with minimal sets makes sense only when one is considering the organization of an axiomatic system. Consequently, a full understanding of definitions requires Level 4 thinking.

In the analysis above I focused on how children verbalized the particularities of a given class of geometric shapes. That process is only part of the picture since there is the complementary process: namely, how children find shapes given their verbal description. Shaughnessy and Burger's (1985) research included a task of the type "What's my shape?" This task asked students to guess the researcher's shape given several statements. The statements were presented one at a time, and students could make a guess at any point. Most of the students did not take the statements as mathematical conditions, nor did they distinguish between necessary and sufficient conditions or take the set of all statements as a mathematical conjunction. For example, after the first two propositions ("It is a closed figure with four sides" and "It has two long sides and two short sides"), many students guessed that the shape was a rectangle. The next three statements seemed to confirm that the mystery shape was indeed a rectangle, but when the sixth statement came ("The two long sides are not the same length"), their guesses collapsed.

It seems that the students were using the kind of thought processes that work in everyday life; namely, that the large majority of the shapes with the first two characteristics they had previously found in school mathematics were indeed rectangles. Among the entities we call shapes, rectangles were salient. Apparently, for each clue they would make one best guess based on a prototypical type of reasoning. The processes used by students in these two complementary areas (producing a definition given an image and producing an image given a definition) seem to depart extensively from standard mathematical processes, and it is difficult to use the usual characterization of the van Hiele levels to explain them.

Conclusion

This chapter intends to understand the ways in which two distinct paradigms, research on categorization and van Hiele theory, can illuminate studies in mathematics education. The interplay of these two theories clarifies five research issues and shows that:

1. There is evidence that image-schematic models play a very important role at the first van Hiele levels and have a decreasing role after Level 3 (Hershkowitz, 1989). It is possible to conjecture that images play a role in higher geometric thinking.

2. There is evidence of an extensive use of other imagetic processes like metaphoric models by virtually every participant in school and university mathematics from children at the first level (Gene's box metaphor) to textbooks (the mountain metaphor) to mathematicians (Lakatos, 1976).

3. There is evidence of basic-level effects that may begin at the first level and continue their influence well into Level 3 (Villiers & Njisane, 1987; the classification of quadrilaterals) and may even affect mathematical research.

4. There is evidence of scripts that seem to affect the ways in which students perform geometrical constructions (Vinner & Hershkowitz, 1983; Junqueira, 1995), but their relationship to the van Hiele levels still needs to be clarified by research.

As noted previously in chapter 3, the van Hiele theory considers the mathematical structures as prime examples of very rigid structures, which students are assumed to understand by a mixed process involving observation, teaching, and explicitation (van Hiele, 1986). The theory, as stated before, provides very useful insights into the didactics of geometry and has been the source of very revealing research and development in the field. But changes in the theory, as it was originally put forward by the van Hieles, must be made to accommodate the explanation of the phenomena discussed in this chapter prompted by the work in categorization.

In the remainder of this section, I discuss changes in the theory that may allow the previous discussion to be understood within it. These changes will also be incorporated in the ways in which the present study interprets the theory.

There are two types of changes that need to be addressed: changes in the implicit cognitive theory and changes in the characterization of the levels. I do not address changes in the phases of learning because of the lack of investigation in this area.

A first change in the implicit cognitive theory has to do with the assumption about the “spontaneous’ structures of the material” (van Hiele, 1984a, p. 237). The structuralist premise that mathematical structures are somehow objectively embedded in the material and that our role is to observe them poses tremendous obstacles to understanding both mathematics from a historical and cultural perspective and the process of children’s production of mathematical ideas. As shown during the discussion of the categorization of mathematical objects, and later in the discussion of the classification of quadrilaterals, there are no such things as “natural” (objective) mathematical objects, nor “natural” (objective) ways to classify them. Consequently, a first change is to drop this requirement of an external spontaneous organization waiting to be objectively perceived by the human mind.

A second change, which is a natural consequence of the first, is to accept that the process under which we shape our mathematical knowledge is constructive. By this, I mean the assumption that students do not acquire mathematical knowledge by the observation of external structures, nor is their mathematical knowledge a mere extension of the teacher’s. Rather, mathematical structures are built by the students themselves. This change permits the existence of autonomous student productions that depart from standard mathematical knowledge, and it accepts that students will generate definitions, images, metaphors, and convincing arguments that are significant for them. The role of the teacher

could, however, be maintained as an active participant in the formation of these constructions.

A third change has to do with discreteness in the learning curve. Research on the levels has shown no such jumps, nor has it shown sharp separations between the levels. The levels should be understood rather as a complex progression through geometric knowledge (Burger & Shaughnessy, 1986).

A fourth change, resulting from the previous one, involves the characterization of Levels 3 and 4. Research shows that students have major difficulties in the use of definitions. I mentioned how Villiers and Njisane (1987) found that the category of definitions did not correlate with other geometric abilities, how basic-level effects produced major prototype effects in the classification of quadrilaterals even when students knew the definitions, and how an understanding of minimal sets of conditions required Level 4 thinking. I also showed how students made use of thinking strategies relying heavily on prototypes in tasks like “What’s my shape?” A full comprehension of the consequences of using mathematical definitions seems to be acquired by students only at the very end of Level 3, and most likely requires an understanding at Level 4.

The simultaneous use of the work of Lakoff and Johnson and the van Hiele theory proved productive. It provided new perspectives on some research issues on geometric learning, and prompted the need for several changes in the van Hiele theory. These changes are incorporated into the background of the empirical study, whose methodology is the topic of the next chapter.

CHAPTER 5

METHODOLOGY

To achieve the main goal of this study, an initial survey of the types of angle used in school mathematics was performed (Appendix A). Then, the teaching, learning, and understanding of the concept of angle were investigated with fourth- and fifth-grade teachers and students. The participating teachers had been working with the Department of Mathematics Education of the University of Georgia in the Geometry and Measurement Project (McKillip & Wilson, 1990) to revise the elementary school geometry and measurement curriculum.

The first three questions of the present study dealt with students' cognitive models and were addressed through a test with geometric tasks related to the concept of angle taken by all the students in the participating classes and an interview conducted with some selected students from those classes. The fourth question was related to teaching events and was addressed through the observation of lessons in which the concept of angle was used. These observations were focused on episodes involving instructional models. The fifth question was addressed by the analysis of mathematics textbooks and other materials used by the participating teachers and students.

Participants

The study was conducted in May and June 1990 with two fourth-grade and two fifth-grade classes of an elementary school in rural north Georgia that had been participating in a curriculum development project.

The Geometry and Measurement Project

The Geometry and Measurement Project (McKillip & Wilson, 1990) aimed at revising the geometry and measurement strands of the elementary school mathematics curriculum for kindergarten through Grade 6. It worked for four years, from 1986 to 1990, in one Georgia school and one South Carolina school and had already finished when the data for the present study were collected. The project produced 158 geometry and measurement lessons, each including activities for students and instructional procedures for the teachers. These lessons were developed by the project staff, reviewed by some of the project's participating teachers, and first tried out in 1988 by the teachers in their classrooms. Lessons were then reviewed by some participating teachers and revised by the project staff. A second field trial was conducted during the school year 1988-1989, after which a final revision was performed. During the second and the third year, as the teachers were experimenting with the lessons, the project staff helped them prepare to teach individual lessons and attended the classes where the lessons were being taught to help teachers and students.

During the second and the third year of the project, inservice workshops were conducted with the participating teachers. These workshops aimed at introducing the teachers to ideas of content and method that formed the basis for the project and to get them involved in the actual process of curriculum development. Inservice workshops were conducted during the school year 1988-1989 as the teachers were trying out the lessons with their classes and focused, among other things, on the lessons the teachers would use.

I was a member of the project staff for the whole project and participated in the writing and revision process of the lessons, especially those involving solid geometry and measurement of length, area, and volume. During the school years 1987-1988 and 1988-1989, I attended second-, fourth-, and fifth-grade classes where the project lessons were being tried out. I also performed the analysis of tests given to students following the field trials of the second and third years.

The Participating Teachers

The teachers participating in the present study had more than ten years of experience teaching mathematics and were involved with the project. Both fourth-grade teachers participated in the project from the beginning. They both attended the inservice workshops, developed prototype lessons, and tested the materials on their classes. During the second year of the project I attended classes of these teachers. These teachers continued to use the project lessons in the school year 1989-1990, after the field trials were over and the final version of the lessons was produced.

The two fifth-grade participating teachers worked with the project from the second year on. They attended the workshops and tried out lessons in their classes. In the second year I attended two classes of Teacher D. This teacher continued to make extensive use of the project lessons during the school year 1989-1990.

The Participating Students

There were 33 fourth graders and 24 fifth graders in all, from two fourth-grade and two fifth-grade classes. A total of 16 students were interviewed. Each of the fifth graders had been in the same school the previous year. Moreover, most of the fifth graders had been in classes taught by the fourth-grade teachers, and I had observed them during the previous year.

This study proposed to identify distinct cognitive models of angle and also to examine relations between these models and the van Hiele levels. Therefore there was a need to select students having distinct background geometrical knowledge. Having explained this necessity to the teachers, an *extreme cases sampling method* was used (Patton, 1986). A test that included geometric tasks related to the concept of angle was designed for each grade level (Appendix B) and was given to each class by its respective teacher. Results on these tests enabled me, together with each teacher, to select students below the average and students above the average representing a diversity of familiarity with geometry. In addition, the test responses provided data on specific tasks and a basis for questioning during the interview.

Four students, two above average and two below average, were selected from the class of each fourth-grade teacher, Teachers H and A. Tapes from one below the average fourth grader's interview were lost due to an equipment malfunction. Fifth-grade Teacher E only selected three students from her class. The other fifth-grade teacher, Teacher D, devoted more attention to the teaching of geometry, specifically of concepts relating to angles. Hoping that her students would provide a broader perspective on cognitive models of the concept of angle, the teacher and I chose six students from her class, three above average and three below average. Table 1 lists the pseudonyms of the students grouped by their teachers. The names of the participants in this study (students, teachers) have been changed to maintain confidentiality.

Table 1. Participating Students by Grade and Teacher

Grade	Teacher	Students
4	H	Beth, Mally, Rick, Susan
	A	James, Laurie, Louise
5	Z	Julie, Linda, Mike
	D	Alice, Angela, Bob, Hill, Jessie, Marie

Instruments

Tests Used to Select Students for the Interview

Two tests were developed, one for each grade level (Appendix B). Each test had two parts, which were administered in succession at times that were at the discretion of the classroom teacher. Students usually took about 20 minutes to complete both parts.

The two tests were similar in most of the questions. On both tests, there was a set of four questions that asked students:

1. to draw an angle,
2. to draw another angle that was different in some way from the first angle,
3. to explain how the angles were different, and
4. to explain how the angles were alike.

This set of questions was later used in the interview in Task B4 (Appendix C). A fifth question probed students' understanding of the infinity of the set of angles. A last common question asked students to identify angles in figures. The potential angles provided variation along the following attributes: straightness of the sides, concavity/convexity of the figure, size of the angle, orientation of the angle, and angle of tangency.

Fourth-grade students were also asked to identify points inside an angle (Questions 7 through 9). Fifth-grade students were asked to distinguish between the measure of an obtuse and an acute angle (Question 10) and to answer a question that required them to use facts related to the internal angles of a triangle (Question 11).

Development of the Tasks Used in the Interviews

Interview tasks (Appendix C) were designed to elicit attributes significant to the students' characterization of angles and to assess their van Hiele levels. The tasks attempted to judge students' van Hiele levels across a broad range of competencies (Hoffer, 1981). Special techniques were used so that tasks could elicit students' meanings. An overall

concern about *neutrality* was incorporated into the interviews. Neutrality means that the person being interviewed can tell the interviewer anything without engendering the interviewer's favor or disfavor with regard to the content of the answer (Patton, 1986). This was accomplished by an initial conversation with the student, explaining that the interview had the purpose of obtaining his or her opinions about angles, without any evaluative purpose. Occasionally a variation of the *illustrative examples* format was used in several questions (Patton, 1986). This is a special way of achieving neutrality. It has the purpose of letting the interviewee know that the interviewer has had contact with a broad variety of answers. In the format used in this research, the student was asked questions in the form: "Some students told me that.... What do you think?"

Other tasks included questions that used *simulation* techniques (Patton, 1986). The effect of these questions was to provide a context for what would otherwise be difficult questions. An example can be found in Task V1 (Appendix C), where the student is asked to imagine a phone conversation with a friend.

Tasks Used in the Interviews

The interview tasks covered five areas of geometrical competencies: verbal, drawing, visual, applied, and logical (Hoffer, 1981). The interview protocols are reproduced in Appendix C.

Students' ability to describe geometrical objects was one of the competencies investigated in this study. There were two tasks whose main purpose was the investigation of students' ability to describe and compare angles and turns: Tasks V1 and V2. The first required students to describe an angle "over the phone," and the second asked for the description of a turn. As the emphasis of these tasks was on the verbalization of attributes of angles, these were the first tasks of the interview, when no material representations of angles had yet been presented to the student.

Students' ability to make angles and turns was evaluated by tasks that required them to perform turns with an arrow, draw these turns, and draw angles: Tasks B3 and B4. In the first task, I asked students to draw a 90° turn that I performed. The second task explored drawings of angles made by the students during the test. Meanings attached to these drawings were also explored verbally during the interviews.

The ability to visually identify and compare angles and turns was included in Tasks I4, I2, I5, and C4. The first task had two versions. The short version was used with students who had rejected angles with curved sides on the identification question of the test. This version consisted of a verbal exploration of students' answers to the identification question. Students were essentially asked why they had chosen to identify or not identify an angle at a particular vertex. Special attention was paid to vertices with curved segments and to concave figures. The longer version of the first task was used with students who accepted angles with curved sides. This version had three parts: (a) the identification of angles in a series of drawings that provided variation along the attributes of straightness of the sides, presence or absence of a vertex, number of angles shown, size of the angle, orientation of the angle, and angle of tangency; (b) a comparison between pairs of angles that varied along the attributes of size of the angle, size of the sides, orientation of the angle, and orientation of the sides; and (c) a part similar to the shorter version. Task I2 investigated the attribute of flatness in students' prototypes of angle by asking students to identify angles in solids. Task I5 investigated the components of angles; namely, the vertex, the infinity of the sides, and, with fifth-grade students, the interior of an angle. Task C4 prompted students to compare turns. Pairs of representations of turns were presented to the students, and I asked which turned more. The comparisons varied on the attributes of: angle of turn, radius of turn, direction of turn, congruency modulo one turn, and speed of turn.

Students were asked to apply their knowledge of angles mainly in two tasks. One task, Task D2, asked students to partition an angle in a scalene triangle into two equal angles.

Another task, Task P4, asked students to solve a problem involving the sum of the internal angles of a quadrilateral. This last task was used with fifth-grade students only. Other tasks, like Tasks V1, I2, I5, and D2, included specific questions that required students to apply their knowledge of angles.

Logical competencies were mainly explored in Task P3, which was used with fifth-grade students only. In this task, I gave some information about an angle or a class of angles, and the student had to identify which angle or class of angles was involved. Qualifications like *all*, *some*, *may*, and *should* were used. Logical competencies were also present in all the tasks fourth graders were given except Task I5.

Procedure

This section describes the procedures of the study. It includes a description of the student interviews, the class observations, and the textbook analysis.

Student Interviews

Interviews with students provided the core of the research data used in this study. At the end of the school year, students' understanding of angles was assessed by a videotaped interview composed of several tasks and problem-solving situations that essentially followed a *standardized open-ended format* (Patton, 1986). This format is used when it is the intent of the researcher to minimize interviewer effects. In this format, every participant is asked exactly the same questions, and probing questions are placed in the interview protocol at appropriate places. In this study, I followed the protocol for each task, but occasionally, when a student's answers did not seem clear, I pursued a line of questioning that departed from the outlined sequence. Moreover, the fifth-grade students were given two extra tasks based on the curriculum at that grade level. All interviews took place in a room at the students' school.

Data analysis of the interviews occurred in two distinct phases: First the interviews were vertically analyzed: that is, the interviews were sequentially analyzed, one after the other, and categories relating to the purpose of the study were developed. Then a horizontal analysis was performed to examine consistency within each category: that is, each category generated in the previous phase was checked for internal consistency, and a comparison among the categories was performed, to seek duplicated categories or omissions in the categorization of data. As one of the objectives of this work was the development of cognitive models of the students' categorization of angles, *nuclear categories* relating to these cognitive models were identified. These nuclear categories were triangulated with data produced by a systematic analysis of key words that could be linked to each model. Several models were developed by identifying subcategories for each category that specified the elements, their relationships, the type of cognitive model they were related to, prototype effects, and metaphorical projections. Finally, image schemas proposed by Johnson (1987) that were related simultaneously to several models of angles were identified.

A second phase of data analysis operationalized a van Hiele criterion developed for the analysis of angles. Specific tasks from the interviews were analyzed horizontally across the interviews, and each student was classified according to the van Hiele levels using an adaptation of the descriptors provided by Fuys, Geddes, and Tischler (1988) (Appendix D).

Memos (Strauss, 1988) were written for each student. These included a description of the student's performance on each task, comments on the van Hiele levels of specific behaviors, and any additional comments I could produce.

Class Observations

I observed lessons in the targeted classes where angles were being taught. These observations included lessons whose main focus was on the concept of angle itself (identifying, drawing, defining, classifying, measuring) as well as lessons that made use of

the concept of angle (side/angle relationships in triangles.) Table 2 shows the distribution of the 16 observations.

Table 2. Number of Lessons Observed by Grade and Teacher

Grade	Teacher	Lessons
4	H	1
	A	1
5	Z	5
	D	9

Note. See Appendix E for a list of the topics of the lessons.

I assumed the role of a non-participant observer (Goetz & LeCompte, 1984), although on some occasions when students or the teacher requested help, I shifted to the role of a participant. Field notes (Patton, 1986) were recorded on an observation grid to categorize observed phenomena into verbalizations, which included remarks addressed to the whole class by the teacher, exchanges between teacher and students, and exchanges among students; and actions, which included drawings, movements, and nonverbal messages produced by the participants. My comments and impressions were also recorded, together with some general information about the class. Artifacts (Goetz & LeCompte, 1984), which included students' work and other material produced or used by the teachers, were also collected.

Materials Analysis

The four teachers participating in this research used extensively lessons produced by the Geometry and Measurement Project (McKillip & Wilson, 1990). There were four basic ideas underlying the lessons: (a) children can learn much more geometry and measurement than they learn in school, (b) children should learn geometry and measurement in the context of their personal activity, (c) children should talk about the mathematical knowledge

they are developing, and (d) teachers should be treated as professionals able to make their own curriculum decisions. Teachers using the lessons were encouraged to choose from among the lessons those that would be suitable for the needs of their classes and to use them in the order they felt was appropriate.

Teachers occasionally used the regular textbook (Thoburn, Forbes, & Bechtel, 1982a, 1982b) together with other materials. Copies of all these materials relating to the observed lessons used by each teacher were collected. These artifacts were then analyzed by a procedure similar to the one used in the analysis of student interviews.

Summary

A survey, focusing on the types of angle used by school mathematics, was performed initially. Data collection in this study was accomplished in the following sequence. Between April and May of 1990, fourth-grade and fifth-grade classes making use of the concept of angle were observed and the materials were collected. Tests were given to these classes in the week between May 21 and May 25, 1990. By the end of this week students to be interviewed were chosen by me and the teacher. Between the last week of May and the first week of June 1990 the students were interviewed.

Data analysis proceeded by the development of cognitive models of angles through the analysis of student interviews, and these are presented in the first two sections of chapter 7. The understanding of these models as metaphors of image schemas was the next step. Literature characterizing image schemas was reviewed again, especially Johnson (1987), and image schemas relevant for this study were identified and the associated subschemas were developed. These are presented in chapter 6. Only then were cognitive models related to van Hiele levels, and these connections are presented in the third section of chapter 7. Instructional models associated with students' cognitive models observed in the classrooms and in the materials were then analyzed and are presented in chapter 8.

CHAPTER 6

COGNITIVE IMAGE SCHEMAS RELATED TO ANGLE

In the course of the development of cognitive models of angle, four image schemas not directly related to angles were found to be incorporated into them: the *container*, the *turn*, the *path*, and the *link* schemas. Most of them were analyzed by Johnson (1987). Instances of these schemas (Lakoff, 1987), labeled *subschemas* in the present study, that relate to the category of angles are presented. A metaphoric schema, *far is up, near is down*, was pervasively used during the interviews and is also described. In this chapter, these schemas are presented. Diagrammatic representations of the schemas are provided when appropriate. These representations do not attempt to portray images associated with the models. Rather, they illustrate the elements and their relationships by means of a graphical image.

The Container Schema

Our understanding of geometric concepts relies heavily on our awareness of spatial relationships. The ways in which we make these spatial relationships meaningful to us characterize spatial forms in general, and geometric forms in particular. Johnson (1987) proposes that our pervasive physical experience of containment and boundedness produces an image-schematic structure that partially accounts for the ways in which we understand spatial relationships. “We are intimately aware of our bodies as three-dimensional containers into which we put certain things (food, water, air) and out of which other things emerge (food and water wastes, air, blood, etc.)” (p. 21). We also have experiences of

physical containment as we move into and out of many distinct objects and as we put physical entities into and take them out of containers. Johnson proposes that these experiences occur in a repeatable spatial and temporal organization, producing typical schemas for physical containment. These experiences of physical containment are metaphorically projected into nonphysical entities. They also show up in our expressions about geometric entities. Even with one-dimensional objects, we say that points lie *in* a circle or *in* a line segment.

Lakoff and Johnson (1999) propose that this schema (which they call the *container* schema) has the following structure: a boundary, an inside, and an outside. It is a gestalt structure because the parts make no sense without the whole. These are related because the boundary separates the inside from the outside. Figure 1 is a diagrammatic representation of the *container* schema.

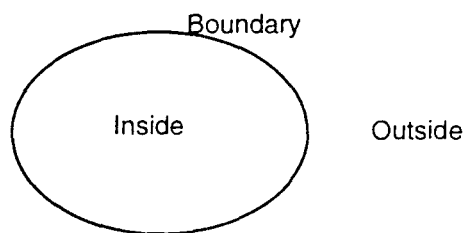


Figure 1. Diagram of the *container* schema.

In this study, the *container* schema was metaphorically related to various models. Lakoff and Johnson (1980) call these metaphoric projections *ontological metaphors* because they are “ways of viewing events, activities, emotions, ideas, etc., as entities and substances” (p. 25). For example, experiences of perceiving containers from the inside contributed to the *angles are interior corners* model. Experiences with containers taken from the outside also showed up in several models and schemas. In this case, these containers had specific attributes: namely, they had protruding points (in the case of the *angles are points* model), they were the locus from which substances emanated (*angle are sources*), or

they opened or closed (*angles open* model). Each of these models of angles relies on a metaphoric projection of a particular subschema of the *container* schema adapted to the idiosyncrasies of the container in question or to the transformations that the container will undergo. I briefly describe some of these subschemas relevant to this study: the *pointed object*, the *interior corner*, the *source*, and the *open* subschemas.

In our lives we experience pointed objects: a pen point, a dagger, a needle, a thorn, and so on. There are some attributes that we associate with pointed objects: a sharp or tapering end, a projecting part. The *pointed object* subschema is a kind of container characterized by four elements: a boundary, an inside, an outside, and a point (Figure 2).

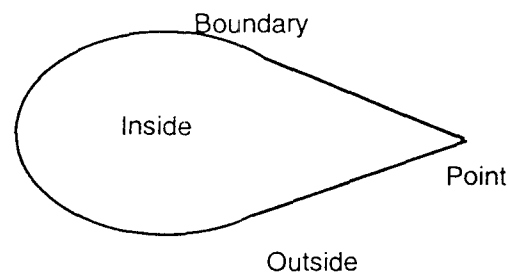


Figure 2. Diagram of the *pointed object* subschema.

The container has a special kind of boundary from which a point protrudes. This schema may be thought of as a special kind of *container* schema from which a point “comes out.” This endows the pointed object with a force that projects part of the container to the outside. In other words, pointed objects are understood using a *part-whole* schema (Lakoff, 1987), and a special part is the point.

The *interior corner* subschema is associated with our experiences of being inside a special kind of container, one that has a corner. It has four elements: a boundary, an inside, an outside, and a force. The force is a movement from the inside to the outside that forces the boundary to protrude. This subschema is experienced from inside the container (Figure 3).

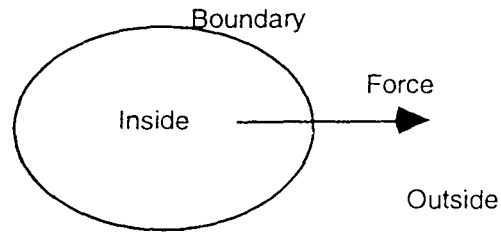


Figure 3. Diagram of the *interior corner* subschema.

Another subschema of the *container* schema is the *source* subschema. It is related to our experiences with containers from which specific entities (liquids, light, air, etc.) emerge. One model of angles the students evidenced, the *angles are sources* model, was metaphorically projected from this schema. This subschema is composed of six elements: a boundary, an inside, an outside, an origin, an entity projected from it, and a trajectory that starts at the origin and is followed by the entity (Figure 4).

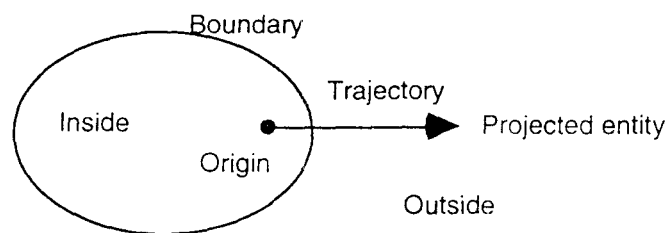


Figure 4. Diagram of the *source* subschema.

Containers may undergo specific transformations. One transformation is that they may open. In other words, we have the *open* subschema of the *container* schema, in which a container is in one state and undergoes a transformation that changes it into a second state. In State 1 the interior of the container is not available. But in State 2 the interior of the container becomes accessible to the outside (Figure 5).

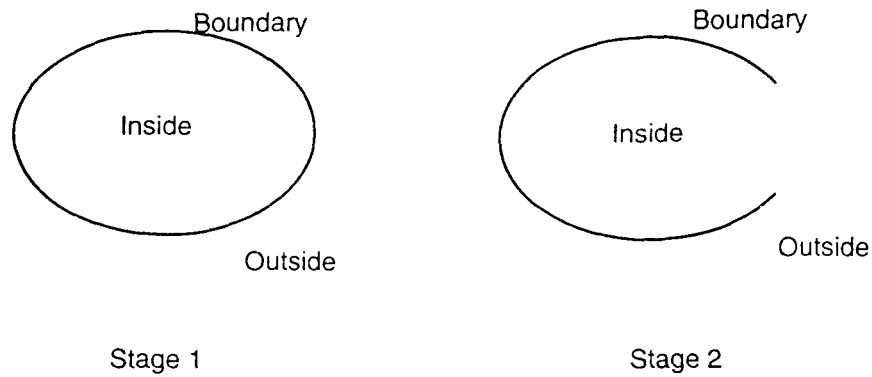


Figure 5. Diagram of the *open* subschema.

This is a graded transformation because the container can open more or less. Containers open in dissimilar ways. Boxes, doors, flowers, eyes, newspapers, and wounds open in distinct manners, which means that there are distinct transformations that fit this subschema. The way in which a door or a box opens suggests that a special part of the boundary actually changes its position relative to the container by a transformation related to the *turn* schema discussed in the next section.

The Turn Schema

This is the only schema not referred to by Johnson. It is associated with our motor experiences of rotating following a trajectory with a starting and a stopping position, as in *turn around*, *turn left*, *turn right*, or the act of turning so as to face a different direction. It is distinct from the action of following a path and turning a corner, which are classified as a different schema. The *turn* schema is an image schema composed of a body with an orientation and a trajectory that has a starting and a stopping point (Figure 6). A possible attribute of the trajectory is the speed with which the movement is performed. Another one is the extent of the turn.

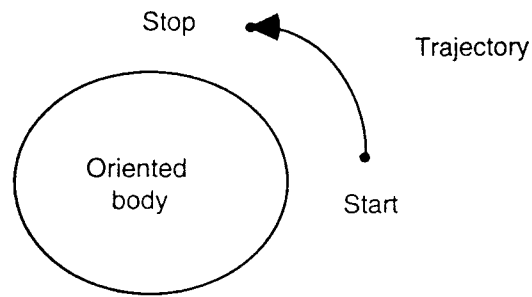


Figure 6. Diagram of the *turn* schema.

There are certain kinds of turns that we perform more often than others, like *turn around*, *turn left*, *turn right*, or *turn all the way around*. In fact, translating to ordinary words what in mathematical language would be, for example, a 45° *turn to your left*, yields some sentences rarely used in everyday life. The four kinds of turns listed are those used most frequently and constitute cognitive reference points for this schema. This schema is also used when saying “she is turned that way,” meaning “she is facing that direction.” In this case, the emphasis is put on the stopping point of the trajectory. This schema is also metaphorically used in expressions like “let’s turn our attention to the matter at hand.”

The *open* subschema I discuss in the previous section is related, at least in the way it showed up in this study, to both the *container* and the *turn* schemas because the change from Stage 1 to Stage 2 occurs as part of the boundary performs a turn.

The Path Schema

Our common experiences involve paths that we encounter either as actual routes we walk, as trajectories we observe, or as paths we imagine. In all of these cases there is a recurring image-schematic pattern composed of an origin or a starting point, a goal or endpoint, and a sequence of contiguous locations connecting the origin to the goal. “Paths are thus routes for moving from one point to another” (Johnson, 1987, p. 113). Johnson proposes that there are certain consequences of assuming a *path* schema: (a) a path

presupposes that to move from the starting point to the endpoint one must go over all the intermediate points, (b) paths may have directionality, and (c) there is a temporal dimension associated with the *path* schema, which provides an important way to understand temporality. Johnson proposes several metaphorical models that are projected from the *path* schema: for example, *purposes are physical goals*.

The *path* schema showed up in the present study in several variations. A special kind of path is used by the *angles are sources* model and another by the *angles are contours* model. In these models the endpoint is irrelevant. An event associated with paths is implicit in the *angles are two connecting lines* model, but here the endpoint is a special landmark, in this case the *path* schema has an “end-point focus” (Lakoff, 1987, p. 423).

The *contour* subschema is a special kind of *path* schema composed of two trajectories, a landmark, and an entity that proceeds through the two trajectories. The landmark is the endpoint of one trajectory and the starting point of another. This is a dynamic model, and the switch from one trajectory to the other is performed instantly. It also presupposes that the two trajectories do not have the same direction (Figure 7).

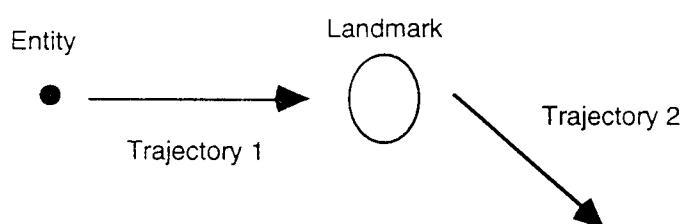


Figure 7. Diagram of the *contour* subschema.

The Link Schema

We experience the binding of physical objects that involves a spatial contiguity and whose closeness relates the connected objects through the link. We also learn temporal connections as we experience events that are temporally related. Sometimes we may have

the feeling of observing causal connections. There are several types of causal connections. One is the genetic connection in which one or more entities are related to a common source. Another causal connection is the functional linking of entities. This links objects that may be intrinsically unrelated but that are linked by virtue of their relation within a functional unity. An example of a functional linking is solubility or compressibility (Johnson, 1987).

The simplest structure of the *link* schema consists of two entities connected by a bonding structure. Typically these entities are spatially contiguous within our perceptual field (Johnson, 1987). If the entities are interpreted abstractly, we may have, for example, logical entities connected by logical connectives. This means that “the metaphorical elaboration of the link schema is one of the primary ways in which we are able to establish connectedness in our understanding” (p. 119). Lakoff (1987) argues that this *link* schema is at the root of our understanding of relational structures. A version of this schema is implicit in the *angles are two connecting lines* model.

One of the ways we understand linking is the action (the event) of meeting people. In this sense it has strong social and cultural dimensions. This *meeting* subschema, at least in the way it showed up in this study, has several elements: two entities that follow trajectories (paths) that end at the same landmark (the meeting point). Figure 8 shows this subschema.

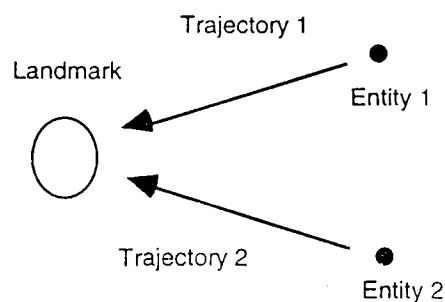


Figure 8. Diagram of the *meeting* subschema.

This subschema involves an action, namely, that the two entities follow different paths and at some point in time are at the same position, where an event, a meeting, occurs. The *meeting* subschema is related to both the *link* schema (the two entities became linked) and the *path* schema (both trajectories are paths and there is a focus on the endpoint). The subschema shows up in the *angles are two connecting lines* model.

Metaphoric Projection of the Up-Down Schema

Geometric figures have an orientation in relation to a person. Some features are closer than others. We tend to express this idea by a metaphor. We usually replace the *close-far* feature by the *up-down* orientational schema. This is seen in our terminology for some geometrical properties (base, height). Some schemas, although they are not models of angles, are constantly used by students. One is an *up-down* orientation, and another is the *far is up, near is down* metaphoric schema.

When drawing geometrical figures, we tend to take as a cognitive reference the horizontal. The dichotomy horizontal/vertical provides a very useful orientation in our daily lives. We eat or write on tables, sleep in beds, and watch the sea, all of which are more or less horizontal. Horizontal paths tend to be easier to transverse. This daily (bodily) experience leads to the development of a *horizontal* schema. This schema naturally transforms the horizontal into a cognitive reference (Rosch, 1975a). I showed in chapter 2 (p. 33) how a preference for horizontality can explain several naming procedures that geometry uses together with some nonstandard uses of geometric terms by students. This schema is closely associated with the *vertical* schema described by Johnson (1987).

Another schema used by students is what may be called a *far is up, near is down* metaphoric schema. In their experiences in classrooms, students often asked to look at diagrams (including words and other entities) that are drawn on the board. Many times teachers ask them to copy these diagrams. As they do this, students naturally perform a

projection of what was drawn on the board, having top-to-bottom, left-to-right dimensions, to a drawing on their desk or tables that has far-to-near, left-to-right dimensions. The top-to-bottom dimension is projected into a dimension of more or less close to the student in such a way that what was a horizontal line on the board becomes a line that is in front of the student. This conversion leaves out the original names, and so students still use “up” and “down” to describe elements that are more or less further away from them. The chart below shows the correspondence between the *up/down* and the *far/near* schemas:

<i>up/down</i> schema	<i>far/near</i> schema
up	far
down	near
higher	farther
lower	nearer

This metaphor is not restricted to schools. In fact, it is used very early in the life of children when they start to draw pictures. In the present study, the students made extensive use of this schema without any conflict.

CHAPTER 7

STRUCTURE OF THE CATEGORY OF ANGLE

This chapter is devoted to the description and analysis of the basic-level category for angles and the students' cognitive models of angles. Attempts were made to identify prototype effects, cognitive reference points, and metaphoric and metonymic projections. Seven cognitive models were identified: *angles are points*, *angles are interior corners*, *angles are sources*, *angles open*, *angles turn*, *angles are contours*, and *angles are two connecting lines*.

Basic-Level Categorization of Angles

Students learning angles started by forming basic-level categories of this new geometric entity. At this level, category members are globally perceived, they have an overall similar shape, and there is a mental image associated with the category. These basic-level categories share most of the characteristics of van Hiele Level 1. In this section I characterize prototypical mental images of angles, some of which are associated with re-enactment of classroom actions.

Basic-level elements of the category of angles were composed of acute angles, right angles, and obtuse angles. Very rarely students mentioned other angles. Table 3 shows the number of answers of students to one test question asking them to draw an angle, together with the answers to a second question that asked them to draw another angle different in some way from the first.

Table 3

Frequency of Choices of Basic-Level Elements by Grade Level

Grade	Basic-level elements of the category of angle				
	Acute	Right	Obtuse	180° angle	360° angle
4	25	20	5	2	0
5	20	16	7	0	1

Note. $n = 52$ at Grade 4; $n = 44$ at Grade 5.

Both acute angles and right angles stand out in Table 3 as central elements in the category of angles. Obtuse angles, 180° angles and 360° angles are very rarely chosen, and angles between 180° and 360° were not mentioned. Moreover, the 180° angle and the 360° angle were only included in answers to the second question. The acute angles drawn by the students measured between 30° and 60°. Throughout the interviews this pattern was repeated. It was as if all the category of angles was metonymically projected into the acute angles between 30° and 60° or the right angles.

Task V2 asked students to perform turns and so it is possible to have an indication of the structure of the category of turns. Table 4 shows the kinds of turns students chose to make. In this task, full turns stand out as central in the category of turns, and no student performed a quarter turn or a turn less than 90°. During the interviews students were aware of, and, in fact, used other kinds of turns, but these were not central in the category of angles. Students may have seen them as important as long as angles were being learned at school. All students that performed turns different from a full turn fall into two very distinct groups: four students, mainly fourth graders, that just flicked the spinner so that it would spin randomly, and four fifth graders having a good performance in geometry, that executed their turns carefully so as to show exactly the kind of turn they intended to show. These latter students were making use of a scholarly category of turns, well blended with the category of angles, whereas the former group of students was making use of a “worldly” category of turns.

Table 4

Frequency of Turns Performed by Students by Grade Level

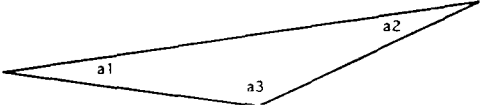
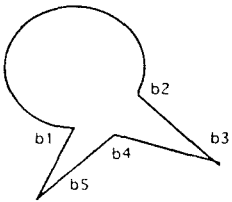
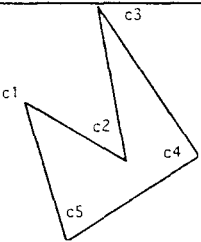
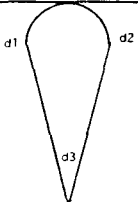
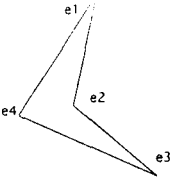
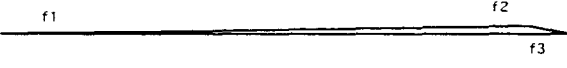
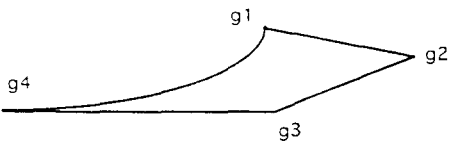
Grade	Full turn	Half turn	2 full turns	540°+ turn	270°+ turn	180°- turn	Other
4	3	1	0	0	0	0	3
5	3	1	1	1	1	1	1

Note. $n = 7$ at Grade 4; $n = 9$ at Grade 5.

I now look into the image schemas the students had for the category of angles. It is plausible to assume that almost all the participating students used image-schematic models composed of rich mental images, as all basic-level categories do (Lakoff, 1987). A first grasp of these models may be obtained by analyzing the students' responses both in the interviews and on the test. Table 5 shows the number of students that, when asked on the test to identify angles in several geometric figures, identified angles in several configurations. In general, students in both grades recognized convex angles (angles at a convex vertex of a configuration) much more often than concave angles. Also, convex vertices of configurations with curved sides (g1, g4) were seen as angles, even by fifth graders, as long as their appearance did not depart significantly from that of an acute angle: b1, b2, d1, and d2 were rarely chosen. These preferences were confirmed during the interviews. Obtuse angles (a3, f2, g3) were less likely to be seen as angles than acute angles were, confirming the centrality of acute angles and the periphery of obtuse angles. Many times these images were associated with a preferred orientation of the angles. Most angles drawn by the students had one side horizontal. For example, a non-horizontal side was considered "slanted" by Beth.

Table 5.

Number of Students That Identified Angles in Each Configuration

Configuration Put an A at the point of each angle in the figures below:	Angle	Grade	
		4	5
	a1	32	22
	a2	32	22
	a3	29	23
	b1	6	4
	b2	6	4
	b3	32	21
	b4	9	13
	b5	32	21
	c1	31	22
	c2	14	16
	c3	31	22
	c4	16	20
	c5	16	20
	d1	5	3
	d2	5	3
	d3	32	24
	e1	32	21
	e2	14	12
	e3	32	21
	e4	24	23
	f1	23	21
	f2	16	16
	f3	27	23
	g1	30	17
	g2	31	24
	g3	23	19
	g4	25	19

Note. $n = 33$ at Grade 4; $n = 24$ at Grade 5.

Occasionally students used common symbols from their environment to explain what angles are. For example, Beth said that to make different angles you could “turn it around and make it like a L, or you can make it a different way.... You can put it a different way and then make a [*sic*] angle.” Louise said that to make an obtuse angle a friend of hers would have to “kind of try to make an A, but not just like an A. ... [It’s an A] but further apart.” And a right angle “is kind of like half of a square.” Jessie said that “an angle is just ... a sort of a V shape.” In these two sequences angles are metaphorically thought of as letters. It is as if angles have a shape that resembles other shapes known to students. The metaphor is used only as shapes are concerned.

Image-schematic models are not only composed of rich mental images. It is possible to observe that they are also composed of re-enactments of actions associated with the ways in which angles were introduced. In all classes, angles had been taught in association with dynamic “protractors” made from two sticks, usually unequal in length, connected by means of a flexible drinking straw (Figure 9). One stick could be moved about the other in a circular motion. The trajectory was the movement produced by one stick as it rotated about the connection point.

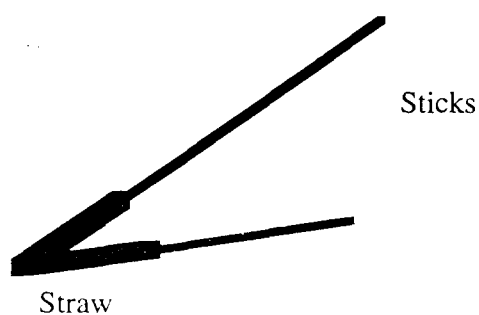
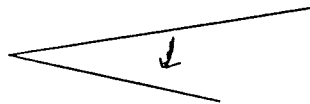


Figure 9. Dynamic protractor used to introduce angles.

Two fourth-grade students, Beth and Mally, exhibited an image-schematic model of angles that very closely resembled a re-enactment of the actions they experienced in class.

In this model, an angle is composed of two unequal sticks and a trajectory. Beth's characterization of angles, for example, reflected the use of this model:

Beth: A [*sic*] angle is what you.... [It has] a long part [seems to be holding an imaginary line between her hands] and then a short part [moves her left hand so that the imaginary drawing is like the figure below].



You move it like.... You move the top part, and then you say you have half. And then you move it around. And then you move around again, and you have a whole [completes a full turn with her left hand].

Beth was remembering what angles had been like in the class when she talked about them. She mentioned a “long part” and a “short part.” Similarly, Mally explained that an angle “has two long pieces or two short pieces.” Both students seemed to refer to the protractor made with two unequal straws that had been used in class. Asked to draw angles that were different, Beth drew one right angle and one acute angle. When she was asked to explain her drawings, she resorted to the *two connecting lines* model and said that the second angle “moved the top down some,” by which she meant that the oblique side on the top had been rotated so that the angle was smaller. These two students exhibited what Grady and colleagues (Grady, 1996; Grady, Taub, & Morgan, 1996) would call a *primary* metaphor for the category of angles, with a strong *conflation* (Lakoff & Johnson, 1999) between angles and the protractor.

Cognitive Models of Angle

In this section I characterize seven cognitive models of angle that the students used. These models are metaphors for the image schemas described in the beginning of this chapter, and the correspondences in each metaphoric projection are outlined.

Angles Are Points

Seven students characterized angles as similar to points:

Angles are kind of a point.

Angles have a point.

Angles have a sharp end.

Angles come out as a point.

This angle is sharper than that one.

Angles are like thorns on a rosebush.

Angles are like the tip of a triangle.

All these expressions, together with some others, reveal a cognitive model of angles thought of as points or pointed objects. This is a metaphorical model projected from the schema *pointed object* mentioned previously. The correspondence between the two can be characterized as follows:

<i>pointed object</i> subschema	<i>angles are points</i> model
container	angle
point	vertex
edge (point)	vertex
edges (boundary)	sides
end	vertex
inside	inside of angle
relations	
sharp end	vertex of an acute angle
protruding	angles come out as a point

This model has an implicit “viewpoint” (Johnson, 1987, p. 36) because the movement is observed from the outside, so one can see the *sharp ends* of angles. This model was extensively used by almost all fourth graders (James, Laurie, Mally, Rick, and Susan), and some fifth graders (Jessie and Mike), who thought that *angles have a point*. Other students used it occasionally.

[Angles] must have a point. (Laurie)

[Angles] have ... a point in their end. (Rick)

[An angle] has a point in it. (Mally)

[Angles] would always have a point. (Alice)

[Angles have a] sharp end. (Rick)

Mally indicated that the similarities between angles and points may constitute a learning problem because as “triangles all have points on it, [some students] might think that it is a angle.” Most students believed that a point is something that angles *have*. Two students (James and Jessie) also claimed occasionally that angles *are* points. James said that an angle is “kind of a point” but distinguished between triangles and angles. The former *have* three points, whereas the latter *are* actually points. There seemed to be two similar models at work here. For most of the students, angles were an entity with a part that is a point. In Lakoff and Johnson’s terms, one would say that these students understood angles using the *part-whole* schema, one of the parts being the vertex, which was understood in terms of the pointed object subschema. James and Jessie, however, were exhibiting a conflation between the domain *angle* and *point* characteristic of a primary metaphor (Grady et al., 1996; Lakoff & Johnson, 1999).

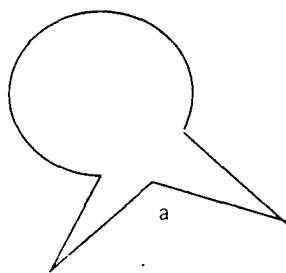
This schema had consequences for the meaning attributed to the action of measuring angles. Jessie explained how to measure an angle:

Jessie: [An angle is] a line with a point at the end of them, the lines with a point set at the end of them, and you measure them by degrees. And I would tell you [how to] measure the points. When you measure them, you measure the side of it.

I noted above how the *pointed object* schema may involve a protruding force that is associated with movement. Its movement was instantiated by Rick using a mixture of words and gestures. He said that angles “got pointing ends at the end, they got pointing ends when they come up,” and he placed his hands vertically in an inverted V. Rick repeated this last gesture several times. Students expressed other ways in which the movement produced by this force could function. Angles “come out as a point” (Rick) or angles point “across” (Mike), or “you look at the point where [the angle] ends” (Alice). This model does not exclude angles with curved sides. In fact, some students (e.g., Jessie) accepted angles with curved sides but rejected drawings where the lines did not connect.

If angles are thought of as pointed objects, it makes sense to use language with them that is usually associated with pointed objects. Some angles can be *sharper* than others (Laurie, Susan), and Mally showed an angle that “doesn’t really have a pointy end.” Susan said that an “angle is straight at the edges.”

This model can be seen as being applied in another context. Mike only recognized angles in convex configurations. He first called the vertex *a* in the figure below “an arrow” and said it was pointing up and should point down.



Later, with other figures, he distinguished between “in” and “out.” If the point is *out* there is an angle, if it is *in* there is not. Geometric figures contain something (they are

containers), and angles are protruding points that come out. Mike's approach had difficulty dealing with obtuse angles and certainly excluded angles larger than 180° . So, for Mike, angles in figures could only be conceived of in the interior of the figures.

The notion of sharpness helps to reveal further the structure of this model. Mally, Susan, and Laurie, for example, recognized obtuse angles in several contexts. However, they mostly used acute angles and right angles as examples. Although they made use of other types of angles, right angles and acute angles seemed to stand for the whole set of angles. Their model of angles had a metonymical structure in which right angles and acute angles could stand for all angles. This metonymical structure seems to be a source of prototype effects because an obtuse angle "doesn't really have a pointed end" (Mally), or it is not as sharp as an acute angle (Susan, Laurie). This metaphorical model can give rise to other metaphorical models. Laurie said that an angle is like "the edge of a pencil, the point of a pencil." Others remarked that an angle "looks like,... sort [of] like a thorn on a rosebush" (James), like "the tip of a triangle" (Susan), or like "the pointing top of a triangle" (Mike), and Jessie, for example, associated angles having small amplitudes with "little" points.

Angles Are Interior Corners

Many students associated angles with corners. In the previous model, the angle (the points) were perceived from outside. But here students used a modification of the *container* schema, relating angles to a corner as perceived from the inside. In other words, they used a metaphoric model *angles are interior corners* projected from the *interior corner* subschema of the *container* schema.

Angela used this model, for example, in the following excerpt in which she had been asked to explain to a friend what an angle is:

Angela: I guess like in a corner of a wall, in the very corner, that's the angle, I guess. That goes across from wall to wall.

José: Goes across?

A: You have a corner [puts her left hand on the table and opens two fingers like a V], and [it] goes across [points at the “vertex” formed by her fingers] like that [she gestures]:



J: So you tell her to think of a corner of a room.

A: [Nods.]

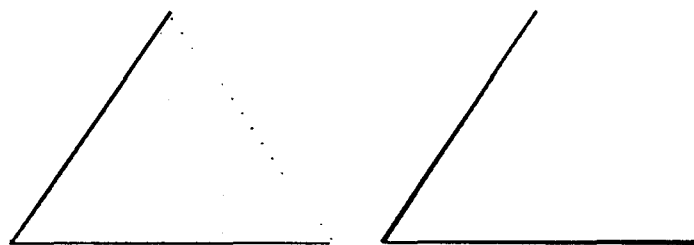
J: And think about....

A: The two walls that are right next to it [the corner].

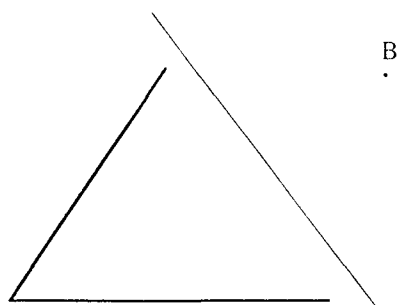
In this excerpt, Angela was implicitly using the following correspondences:

<i>interior corner</i> subschema	<i>angles are interior corners</i> model
container	angle
goes across from wall to wall (inside)	interior
walls (boundary)	sides
corner of the room (force)	vertex

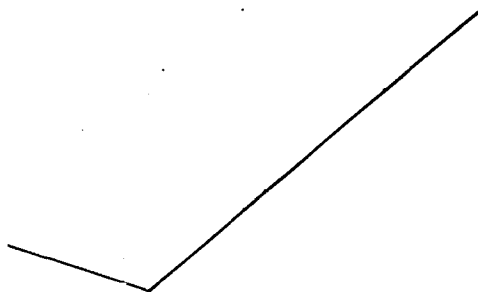
In this model, there are two versions of what constitutes the inside. Many students believed that the angle had a limited interior. For example, Jessie, Julie, and Linda indicated that the interior of the angle in Question 7 of the test is the drawing on the left below, whereas Beth indicated that it is the drawing on the right:



Susan also believed that the interior of this angle was like the drawing on the right and explained that point B was not inside the angle because “it’s way out there” and traced a line as indicated in the figure below:

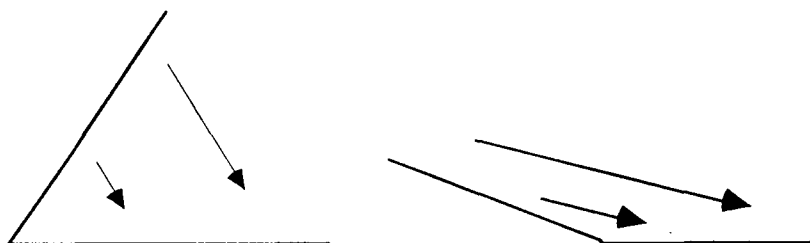


The interior of the angle in Question 9 was similarly expressed. Susan, for example, indicated the interior to be as follows



Mally had a similar idea both in Question 7 and in Question 8 of the test, but resorted to gestures to express it:

Mally: See, right down here, I have just in between these two lines right here. [She gestures as indicated below.]

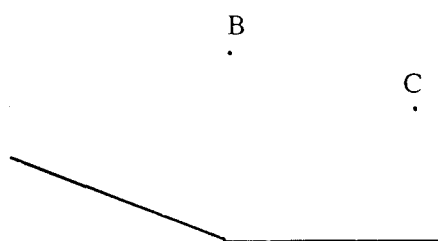


Beth and Linda, although indicating that angles have limited interiors, believed that there not a clear boundary. When asked, in Question 8, if she could draw a line showing the separation between the interior and the exterior of the angle, Linda said:

Linda: No. See, you could draw one like that, but that wouldn't be it.

However, later, in Question 9, she drew such a line non-verbally, expressing that it was an approximation. Beth shared a similar opinion. In Question 8, she said that

Beth: [the interior of the angle] would probably go up like this. [Starts at the rightmost endpoint of the horizontal side and goes near the point C.] Or it probably wouldn't make the C. ... [C is] in between.



Beth also believed that point B was "there too with the C."

A second version of this model conceived the interior to be like an infinite surface. Hill and Marie expressed this second opinion. When asked what points were inside the angles (Question 7 of the test) they both answered that it depended on whether a concave or a convex angle was intended, and proceeded to give the adequate answers for both cases.

Hill carefully extended the sides of the angles so as to be sure whether specific points were or not in the inside. Marie also expressed the idea that the line segments could be extended.

In this task, Bob and Mike, after giving answers according to the limited interior version of the *angles are interior corner* model, asked whether the lines could be extended. After obtaining the answer that the sides of the angles are infinite, they proceeded to adjust their answers to the second version of the model.

The second version of this model, conceiving of angle as a corner with an “infinite area,” showed up more clearly in two students with better knowledge of geometry than most students that participated in this study. The first version of this model was shared by many students, most of whom also exhibited the *angles are points* model. It can be conjectured that as students learn more geometry both the model *angles are points* and the limited interior version of the *angles are interior corners* model converge into the infinite surface version of the second model, which resembles closely the *portion of the plane* kind of angle used in school mathematics (Appendix A).

Angles Are Sources

Some students thought of angles as sources of trajectories. In this model, an angle is a projection of two rays from the vertex. It establishes the following correspondences:

<i>Source</i> subschema	<i>Angle is a source</i> model
container	angle
projected entity (two rays)	sides
trajectories	straight lines
starting point	vertex
inside	interior
Relations	
lines that can go on forever	infinite length of sides

Alice and Hill made use of this model. When asked why she thought there was an angle in a triangle, Alice answered:

Alice: Really it starts at a point [points to the vertex of the angle]. I didn't do it because, just because it starts at a point. It starts at a point, and it has a certain way that it goes [traces the sides of the angle].

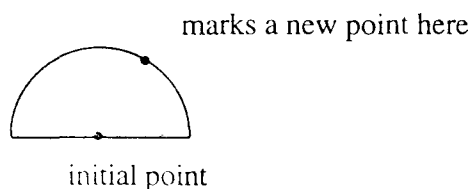
Alice was identifying angles with a certain kind of path, for which she indicated both the origin and the trajectory. The trajectory was also special: it went “a certain way,” in this case, straight paths emanating from the vertex. Hill expressed the same idea:

Hill: [Angles] have a starting point like a ray [traces one side of an angle], and then they go on and on. You can make the angle like a ray.

This model was also visible in the way in which Alice interpreted the action of drawing an angle:

Alice: You always make a point and then you can draw where you want it to be.

They always start at some point. [She draws the figure below.]



Because they can be ... anywhere, but they have a starting point.

Sometimes, Alice and Hill referred only to some elements of the model. The idea that the vertex is the source of something can be found in Alice's comments. On several occasions, she mentioned that angles “start at a point,” or “have a starting point.” Hill described the trajectory as the extension of two lines. Two other students (Bob and Mally) hinted at the use of this model by using gestures indicating the two trajectories.

There are certain metaphors the students used that were related to this model. Hill, for example, believed a ray is similar to an angle:

Hill: Because if you have a ray [draws a ray], you have a point [points to the endpoint of the ray] that goes on and on. But they might get it confused for an angle because it has a point like an angle, a vertex. Not a vertex, because a vertex has... it's where two lines meet. But a point to where a line goes on and on, never stops, does not go away, as a, sort of like a flashlight.

Several times he referred to this metaphor, but Alice proposed a different metaphor. She says that an angle "is like, if you're walking straight, if you start from a point on the playground and now you can keep walking for a long time." It is a dynamic model, because an angle is thought of as the source of a dispersing movement of two lines. Students that made use of this model saw it as independent of the length and the straightness of the sides. This model has close resemblances both with the *portion of the plane* and the *set of two rays* types of angles in school mathematics.

Angles Open

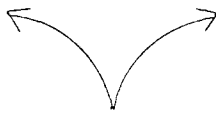
Angles are thought of in the *angles open* model as geometric objects that have the property of opening. It is a metaphoric projection from the *open* subschema of the *container* schema.

Two students (Bob, Marie) mentioned that angles "open up," and one student (Alice) compared angles using "the opening of the angle." Laurie, for example, said that an angle is "like when a door opens." Bob used this model several times. In his use of the model, angles opened much like a lid or a door, in which one side would rotate from a "starting point." For example, Bob drew a right angle and an obtuse angle each having a horizontal side, and he used this model to explain the similarities between them. He said

Bob: They both start at the same point, you see. They open up at the same point [indicates the horizontal lines in both angles]. They both start right here. One of them [right angle] goes right here. One of them [obtuse angle] opens a little bit there.

Bob also used this model to explain why he identified an angle in a triangle. To do that, he traced the “starting point” (the beginning position of one of the sides), and he showed where the action of opening (a rotation) took place by saying, “[It] opens up right there,” drawing a small arc inside the angle near the vertex. The *angles open* model that Bob was using here is related both to the *container* and the *turn* schemas.

Marie also used the *angles open* model. However, she used a different kind of transformation unrelated to the *turn* schema. Marie showed that angles open by making an unfolding gesture with her hands, like the blossoming of flowers:



This transformation is different from Bob’s turn and is closer to the way in which newspapers or flowers open. Each time Marie used this model, she continued the discussion by using the *angles turn* model.

Alice, Jessie, and Mally used the *angles open* model to compare angles. Alice compared angles according to “the opening of the angle,” which she associated with a gesture encompassing the interior of the angle:



Jessie explained that two angles are different because one of them “is a new open kind,” and Mally characterized congruent angles by saying that “they both open about the same length” and made a small V with her fingers.

In summary, these are correspondences that characterize the *angles open* model as a metaphoric projection of the *open* subschema:

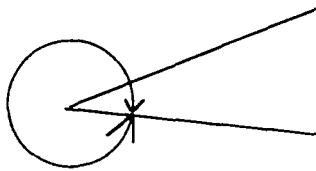
<i>open</i> subschema	<i>angles open</i> model
container	angle
transformation (stage 1 to stage 2)	open up
transformation (stage 1 to stage 2)	turn
transformation (stage 1 to stage 2)	unfolding gesture
small transformation	opens a little bit
degree of openness	opening of the angle
degree of openness	angles open the same length
degree of openness	angle opens a little bit (small angle)
Relations	
part of the border changes position	one side of the angle turns

Angles Turn

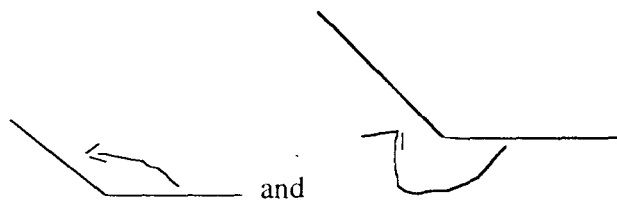
Students used several ways to refer to the relation between angles and turns: “angles turn,” “angles have turns,” “angles are turns,” “angles and turns are the same.” Some students explicitly identified angles with a rotating line. I propose that most of these usages of the notion of turn represent variations of the model *angles turn*. More specifically, the image schema model *turn* described above is metaphorically projected onto the model of angles as turning bodies.

Ten students (Alice, Angela, Beth, Bob, Hill, Jessie, Julie, Louise, Mally, Marie) said that angles turn. Sometimes they identified angles with turns. Bob, for example, saw “no difference” between angles and turns. Jessie said that “a turn is the same as [an] angle,” and Angela stated that turns and angles “sort of are alike, because an angle, it can turn.” Louise said that angles can “turn bigger or wider.” Alice said that “as long as it starts at a point and turns about wider, it [is] a angle.” James several times compared angles by saying that one turned more than the other. Several of these students explained that certain figures were angles because there was a turn involved. Linda, however, rejected this model saying that an angle “stays in one place.” Only three students explicitly said that a line (Alice, Hill) or a ray (Julie) was turning.

Some students added some gestures to the use of this model. Marie, for example, when asked to show points in the interior of an acute angle, asked: “Which way does it turn? This way or this way?” She traced the two choices:



The use of the *angles turn* model enable her to distinguish between the two angles. This distinction is not so easily noted using other models. Later she traced two other angles like this:



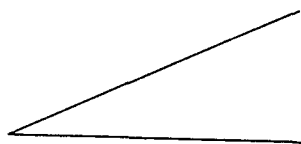
During this task, Marie constantly referred to angles as turning objects. Bob also made similar gestures. Alice and Julie used this model in a more elaborate way. They both referred to angles as a line (or a ray) that turns.

The image schema of a turn endows the metaphoric model *angles turn* with a metonymic structure. Louise, for example, described the special angle she called “turn around,” which meant a 90° angle, and Mally referred at several points to angles that went “all the way around.” In another example, Jessie, when referring to an obtuse angle, said that “[the obtuse angle] is almost a finished turn.” Previously she had said that the same obtuse angle “is almost half of a whole turn.” The “finished turn” here seemed to be a 180° turn, and the obtuse angle was compared with it. Bob referred to the “full turn” as a means to compare angles. The “finished turn,” the “whole turn,” and the “full turn” were used as cognitive reference points.

The image-schematic nature of the projected model could be observed in Alice's use of incorporation. Asked about the differences between an acute and an obtuse angle, she asked me to become the angle and "go like that" or "turn all the way around." She was asking me to make use of my bodily experiences associated with *going* or *turning around* to make sense of what happens with angles.

Alice pointed out a problem in establishing turns as a metaphor for angles:

Alice: [A turn is different from an angle] because angles, they have ... they always have a point. I mean, like this right here [points to the center of a spinner] is a angle. The corner right here, that is a turn, it is not really a angle. [It] is a angle but... Okay, most angles they always start at a starting point. ... It is like most angles, they always have a point. [Draws the figure below.]



Like, they always start, most of them. But some angles like that [refers to angles made by turning the spinner], they can curve around like that [draws]:



making a square, something like that. But mostly, when you turn, like when someone tell you to turn around, there would be more than half of a circle [draws]:



Alice was essentially saying that the vertex, a key element in an angle, is not easily apparent in a turn.

In summary, there is a correspondence between the *turns* schema and the *angles turn* model:

<i>turn</i> schema	<i>angles turn</i> model
body	line or ray
trajectory	turn
trajectory	go around
turned this way	direction of the angle
full turn	360° angle
turn around	180° angle
turn around	90° angle (Louise)
turn all the way around	360° angle
stop and turn	acute angle
almost a finished turn	obtuse angle

Some students associated angles and turns with a circle. Apparently, one of their models for circle was a metaphoric model projected from the same model that the metaphoric model *angles turn* used. In other words, they were using a model *circles turn* together with *angles turn* and concluded that, “in a way,” both entities, angles and circles, can be equated.

Alice, for example, was able to provide an extensive explanation of this association:

Alice: The only thing I could think of is a circle that's bigger than all of the angles because there is a full turn [traces a circle on the table, then makes a half turn with the pencil]. A full A circle is really an angle because if it's like you started [at] one point [points to the edge of the pencil]. This is the point you started [points to the other endpoint of the pencil], and You started here [points to the edge of the pencil]. It is like you can turn the pencil all the way around [rotates the pencil 180° to the left]. Then you can start there, there

would be a angle. Then you keep turning. That would be a angle, that would be a angle [indicates successive angles as she rotates the pencil around the endpoint opposite to the edge beginning at the horizontal]. And that would be the end where you started at [ends the rotation of the pencil by moving it to its initial position]. And that would be a circle.

Alice developed this idea on other occasions. Different students also developed similar ideas. Jessie, for example, said that “the turn and [the] angle, they both can ... they both can wind a circle.” Alice went one step further. She also developed a similar metaphoric model for a sphere and compared the sphere to an angle:

Alice: In a way it’s a angle, because it starts and it turns around [traces a great circle of the sphere]. It starts as an angle, but then it goes around. So in a way I would say this is a angle.

She also used the same method to compare cylinders to angles.

Students made use of this model when they wanted to convey a dynamic interpretation of angles. This model, however, is not appropriate for referring to geometric entities like acute angles, obtuse angles, vertices, or sides, because they do not correspond to attributes of the original *turn* schema. Hill used this model to reject the possibility of curved sides in an angle, saying that the drawing was “not a line that turns.” In fact, in this model the sides of an angle are conserved by a rotation, and so they cannot become curved. No student made use of this model to explain his or her acceptance of angles with curved sides.

The direction of turning is one attribute of the *turn* schema that was not projected to the *angles turn* model. The velocity of turning is another attribute that should not have been imported. There is, however, some indirect evidence that velocity may have been associated with angles by some students. When asked to compare turns, some students indicated that turns made faster produced larger turns (James, Jessie, Laurie, Linda, Mike):

Jessie: Because the faster it goes the more it has inside the angle, the more space.

Other believe that the slow turns turn more (Beth, Julie, Louise, Susan).

Louise: [The slow turn] probably rotated more, because it used more tie, it took longer.

It is reasonable to suppose that this property of the *turn* schema may be transferred to the *angles turn* model, causing some learning problems.

The idea that angles turn was also found when I investigated the kinds of angles used in school mathematics (Appendix A). Two kinds of angles involved rotations depending whether angles differing by multiples of turns were considered equivalent. The cognitive model *angles turn* shows strong similarities with the type *modular external rotation*, whereas no student gave indication of conceiving turns above one full turn, as the *continuous external rotation* type of angle does.

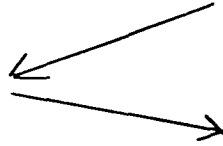
Angles Are Contours

Alice used the *path* schema extensively as a metaphor for angles, corners, turns, and triangles. All of these figures can be described by specific paths. In particular, she used the model *angles are contours*, which is a metaphoric model from the *contour* subschema described previously. Some correspondences are as follows:

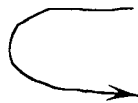
<i>Contour</i> subschema	<i>Angles are contours</i> model
trajectories	sides
landmark	vertex
entity	lines
Relations	
entity goes through the trajectories	lines curve
landmark is a stopping point	
landmark is a curve	

A good example is provided by Alice as she used the *contour* subschema to distinguish between angles and corners:

Alice: [An angle] would always have a point. [She gestures.]



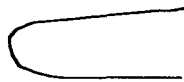
Alice: You always.... It would always have a point, and.... A corner, it doesn't always.... Sometimes it can curve. [She gestures.]



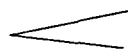
This set of gestures is associated with the model of angles as contours. As Alice put it the easiest way to draw an angle is “just look at how does the lines curve.” At the same time, she made a gesture similar to the figure above. She was talking about a continuous path that moves along a line, at a certain point makes a sudden “curve,” and continues along another line. This curve is so sudden that it produces a point. At a later time. Alice even stressed this instantaneous change of direction by drawing a small curve around the vertex of an angle.

Alice made repeated use of the *angles are contours* model. Later she used it to distinguish between angles and turns:

Alice: If you're gonna turn a corner, sometimes they're round. [Traces the figure below.]



And sometimes they're not. [Traces the figure below.]



Sometimes they are just like angles. Like, you can walk and then you have to turn.

[Traces the figure below.]



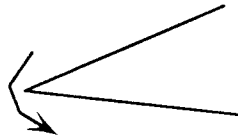
Other [times] you can just, you can just curve. [Traces the following figure.]



In other words, for Alice, an angle was a special path composed of two trajectories with a special relation to each other, whereas a turn has a special path that follows a curve. In the excerpt above, one can see that the proposed angle is composed of a “line” that “goes straight” but that “curves” at some point. In another segment, Alice explained the relationship among the trajectories and the landmark:

Alice: [The angle] kind of comes down and then ... comes down and curves like that

[and she gestures]:



Alice used the attributes of the trajectories to explain why the sides of the angles are straight:

Alice: [It is not an angle] because this curves [traces one “curved” side back and forth], and angles mostly don’t curve. It’s like one line goes straight, and then it curves like that [draws a semicircle]. But a angle doesn’t curve when it goes [draws a line segment].

Hill also used the *angles are contours* model as a basis to develop a metaphor for angles. An angle can be seen as a path that people follow. They go along a line, stop at a corner (the vertex), and then turn and continue in another direction:

Hill: A right angle [traces one side of a right angle], when it comes as a vertex toward sort of like a stopping point ... it makes a turn go up, just like a sidewalk on a corner. People come this way, and then they stop and come this way, and then they have sort of a stopping point [points to the vertex of a right angle] before they turn, or they can come this way [traces the other side of the right angle away from the vertex].

In the case of Alice, the use of the *angles are contours* model seemed to be a part of a more general model. Several times she used a *path* schema to refer to turns.

Straightness of sides seems to be an attribute associated with the *angles are contours* model. There may be, however, some learning difficulties associated with this model. It may be difficult for students to distinguish between angles that add to 360° . Alice also showed some difficulties understanding the relation between angles and turns, namely, that there can be an angle associated with each turn.

The notion that angles are related to changes in direction of paths, as in the model *angles are contours*, was found in the *intrinsic rotation* type of angle discussed in Appendix A.

Angles Are Two Connecting Lines

In the metaphoric model of *angles are two connecting lines*, angles are thought of as projections from the *meeting* subschema. Hill, Jessie, Mally, and Susan made use of a model of an angle as a meeting point of two lines. This model involves “two lines that come together,” as Hill and Susan put it, and angles are “where two lines meet” or where they are “together,” where there is a “cross” between two lines, as Susan said, or where “two lines connect,” as Jessie and Mally said. The vertex is “where two lines meet,” as Hill

put it. An angle is “where two lines meet,” and to find angles one would have to look for “where two lines come together,” said Susan.

Jessie used the *angles are two connecting lines* model extensively as a means to justify the existence of angles. From her statements, one can observe that this model was associated with a visual configuration of two intersecting lines and the action of two lines attempting to connect to each other. When asked why there were angles in a triangle, Jessie replied.

Jessie: Right here is connecting the lines, is connecting right there. And they're having a point right here. there, and there [points to the vertices].

Later when explaining why she did not identify angles in other triangles, she said that “I must have overlooked ... the lines connecting this and that [points to the vertices].” In many other instances, she discussed angles in terms of connecting lines.

Jessie seemed to endow the connecting lines with a will to connect. For example, when explaining what was most difficult to learn about angles she said,

Jessie: One other thing that is difficult to learn about angles. the most difficult thing, is when you try to have. you know. parallel... parallel lines [draws two parallel lines with a curved portion on one end so that the lines intersect] ... and get them confused. When, you know. the lines that have a line, you know, the lines that keep going, and then the lines that connect when they keep going so they are curved, it connects. I think it's the most difficult thing about learning angles.

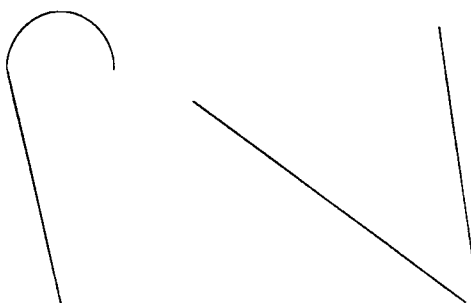
She later repeated the same drawing:

Jessie: [I am talking about] the ones that connects [sic] when they're still going [draws again two parallel lines with a curved portion on one end so that the lines intersect], like when they start going like that [emphasizes the small curved piece she drew].

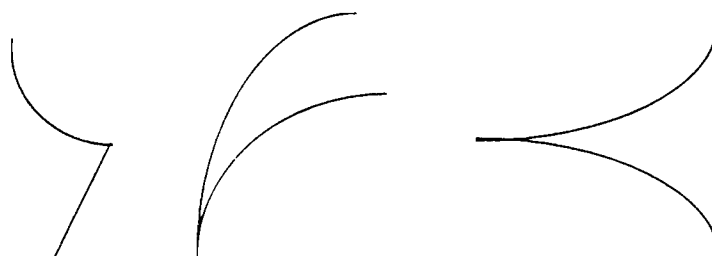
This model involves several correspondences:

<i>Meeting subschema</i>	<i>Angles are two connecting lines model</i>
entities	lines
trajectories	go a certain way
landmark (meeting point)	vertex
relations	
trajectories meet	lines intersect
trajectories meet	lines connect

This model excludes configurations in which lines do not intersect visibly or that do not contain two distinguishable lines. Jessie did not identify angles in the following configurations:



She did identify the following as angles:



Direct connections between this last model and the *set of two rays* kind of angle used in school mathematics (Appendix A) can be established.

Summary

This section shows seven cognitive models of angles displayed by the participating students. These models are metaphoric projections of subschemas discussed in the previous section. Figure 10 describes the relationships among the several image schemas together with their subschemas and their metaphoric projections into students' cognitive models. Associations between cognitive models and types of angles used in school mathematics (Appendix A) are also summarized.

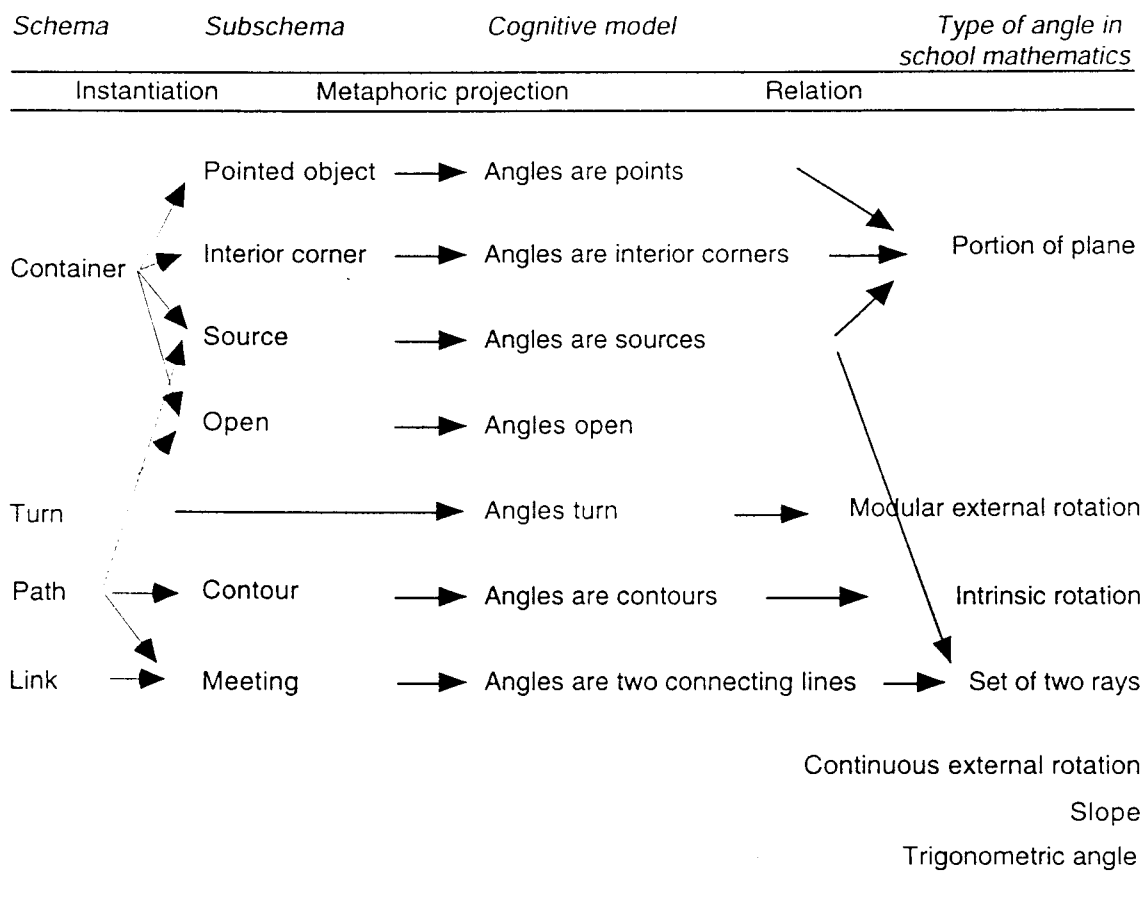


Figure 10. Relationship among schemas, subschemas, cognitive models and types of angles in school mathematics in the category of angles.

For example, the *path* schema has as instantiations the *source*, the *contour*, and the *meeting* subschemas. The *angles are sources* model is a metaphoric projection of the *source* subschema. This model was found to bear a relationship both with the *portion of the plane* and the *set of two rays* types of angles used in school mathematics. No relation was found among fourth and fifth graders' models and the types *continuous external rotation*, the *slope*, and the *trigonometric angle*, which show up much later in the curriculum.

Learning the Concept of Angle

This section addresses the second question of this study, which sought to understand how the category of angles changes its complexity and how is it related to students' van Hiele levels for the concept of angle.

Learning the Basic-Level Concept of Angle

All the fourth graders were starting to learn about angles; that is, they were learning the initial details of a new kind of school entity that their teachers were calling an *angle*. Two cases are typical of the ways in which they were conceiving angles, the cases of Louise and Rick.

Louise was able to draw angles carefully and recognized most of the angles in the test. She recognized the angles with a very small amplitude but missed all the angles at concave vertices. Later, during the interview, she identified two of them. She identified as angles some configurations with curved sides and was not sure about others: "I don't know, because it is straight right here [points to the straight side], and it curves right here [points to the curved side]." She believed that if the sides of an angle were extended, that would make the angle "longer."

Louise thought that angles have different shapes and different sizes. When asked what she meant by "shape," she answered "like this [points at a right angle] is kind of like half of a square. And this [points to an obtuse angle] is like the shape of a wide, wide A." During

the interview, she also clarified the difference between “shape” and “size.” When I drew a right angle with sides longer than the right angle she had drawn previously and asked her if the two were different, she said that the two angles were not different and added that “they have the same shapes even though they are not the same sizes.” Then, I drew an angle congruent to her right angle but with shorter sides and flipped vertically. I asked her which one was bigger. She said that “it’s the same, just upside down.”

Louise’s descriptions of angles relied heavily either on images or on actions: “Angles have one side. I mean, if they were trying to make a triangle, and they would only have to cut in half to make a angle.” All the angles “use letters to show their points.” To explain to a friend how to do a 90° turn, she said: “I would tell her just to draw a circle. I could tell her ‘it’s like a clock and go from number nine down to number six.’” She summarized what angles are by saying:

Louise: There is [sic] different kinds of angles, and one of them is turn around. You can just take it [out of] two parts of a square. It is just one line, and the other two connect it. And you can fit a corner of a paper into it. That’s how you turn around. There is [also an] obtuse angle. It’s kind of like an A but not exactly.

José: And how would you tell [a] friend how to make an angle?

L: Turn around, it is just the corner of a piece of paper. And for an obtuse angle, she can kind of try to make as A, but not just like an A.

J: How is it different from an A?

L: [It’s an A] but further apart.

Louise’s last description of angles show how she was making sense of this new category of things by relating it, simultaneously, to several other entities she knew: turns, squares, lines that connect, letters, and corners. It is possible to recognize several cognitive models described above: *angles turn*, *angles are two connecting lines*, and *angles are interior corners*.

Rick had much in common with Louise. He drew striking angles with thick sides. Apparently, he needed to keep a close relationship between angles and the physical means by which they were represented in class. In other terms, he was functioning within primary metaphors for angles. He chose to draw angles in geometric figures: in a triangle or a square. He identified the vertices of these angles with one of the vertices of each figure. Rick recognized most, but not all, the angles in the test. When asked during the interview why he skipped some angles, he said that he was not sure about some of them because they “would be like a square.” He identified angles that had a small amplitude. During the interview, he expressed doubts as to whether angles in concave configurations were angles.

For Rick, some angles were alike because they had “a point in their end.” When asked if an angle with shorter sides was different from a congruent angle with longer sides, he said they were different, because “that one is just a little bigger than this one” although they were congruent. When asked if an acute angle was different from an obtuse angle, he said that the two angles were different because we could rotate one of the sides of the acute angle to make it like the obtuse. When asked if two congruent angles facing different directions were different, he answered, “They’re different. They’re the same but they’re different. One is turned that way, and the other is turned that way.”

Up to this point, it is possible to see similarities between Rick’s and Louise’s concepts. Rick, however, brought up something extra. As it was difficult for him to express all the complexities of angles by means of speech, many times he resorted to gestures. For example, when asked to explain to a friend over the phone what an angle is, he said (and enacted) the following:

Rick: Triangle, square, and circle, and sharp ends. [Pauses] They have [pauses] a [point] on the end [makes a V with his left hand], a pointing end [gestures upward with his hands as if modeling a pyramid]. They have a angle [make a

kite with his hands using his forefingers and his thumbs]. They come up

[repeats the modeling of a pyramid].

To explain the difference between angles and corners, Rick made use of the model *angles are points*:

Rick: A corner is like a sharp end, and a angle is something, whatever it is, that comes up like that [makes a gesture with both palms of his hands that looks like a tent or the roof of a house] or something like that, like it comes up like that [repeats the gesture]. like...

[...]

José: How are they the same?

R: They got pointing ends at the end. They got pointing ends when they come up [repeats the gesture].

Rick begins by using the *angles are interior corners* model and shifts after to the model *angles are points*.

Both Louise and Rick understood angles in terms of other categories of objects. Rick, in particular, needed to bring up examples from the real world to explain what angles were like. His striking and abundant gestures showed how these categories in the world, and consequently his angles, had rich images associated with them and were full of movement. As Lakoff and Johnson (1999) would put it, the metaphors used by these two students were conflating the two domains: school geometry and the real world; that is, angles were for them a primary metaphor.

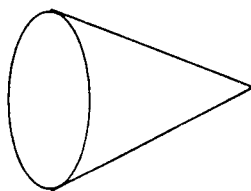
These two students above were using metaphors of image schemas to understand a new category, angles, that had been taught to them recently. Alice, however, although using rich images, was taking advantage of metaphors in a different sense. Two excerpts show how she was using similarity-creating metaphors (Indurkha, 1994), with which she could use metaphors as a strategy to imagine geometric entities in a new way. -

At the beginning of the interview, Alice did not interpret turns as angles. When shown a rotating arrow, she did not initially believe it to have any relationship to angles. Only later did she say that “in a way” the rotating arrow was an angle. She saw this only after having drawn a 45° turn of the arrow. After drawing a turn of 90° , she said, spontaneously:

Alice: Funny. In a way it is like a angle [looks at her drawing]. The way I see it [it is an angle]. Because you kind of turn in there [traces the 90° turn]. ... It is like this pencil. If I had it, and I turn it that way [rotates the pencil 90° on the top of the table having the middle of the pencil as the center of rotation], it will end right there because of the point of the pencil. In a way it is like a angle and in a way it is like a triangle, because at this end ... at both ends they'd curve right there [Alice refers to the two curves sketched by the pencil, one at the edge and the other at the other endpoint]. So in a way it is like a circle.

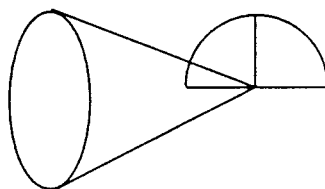
She is using a similarity-creating metaphor, and that is “funny.” She interpreted a turn as an angle using an image schema that incorporates a generalized picture of a right angle (and a right triangle). In another instance, she tried to think of a cone as a metaphor for an angle, and, reflecting on the consequences, came up with very creative geometric thinking:

Alice: I was thinking about [the relationship between angles and spheres], see. In a way [the cone] seems like it is a angle because of the way it is. If you looked at it this way, [positions the cone as in the figure below]:



In a way, it would be a angle because it has this starting point [points to the vertex], and it goes from there to there [traces two edges of the cone]. So in a way it would be a angle. If you looked at it from there [places the cone with the

base horizontal], it doesn't really look like an angle. But if you held it [as in the beginning position], it would look like a angle, mostly because it's like.... If you start at the bottom. [Puts the protractor over the cone with her finger, holding them together.]



You start at that point [means the vertex], it would be a angle. You can't really tell, but in a way it looks like a angle. In a way, it looks like a angle.

José: Could you measure it?

A: I don't know. It would be hard to measure this, because it rounds at the point like this [joins the protractor and the cone in the same position]. So, in a way you wouldn't know where to start. So, if I try to do like this. I don't know.... I would say about 15, from right here. In a way it looks like it. I can't really tell.

Alice came up with an estimation of the “angle measure” of the cone, although she was aware of the uncertainties of her procedure. Through out the interview, Alice used many times expressions like “in a way it is,” or “it is a kind of,” or “it seems like” as she was verbalizing metaphors and establishing similarities. Her metaphors were, however, complex metaphors (Lakoff & Johnson, 1999) as she could limit their scope by using the term “really.”

The episodes above show ways in which some students were learning the composition of the basic-level entities of a new geometrical category. This perspective has a strong similarity with van Hiele Level 1, and all interactions reported above are at this level.

Structuring the Category of Angle

Few students went beyond the understanding of basic-level categories of angles; that is, few responded at a van Hiele level above the first. Some students, however, developed a deeper understanding of the concept. A first example can be taken from Marie's long list of attributes related to angles, among them some properties. When asked how she would describe an angle to a friend over the phone, she said:

Marie: If you look at ... the corners on a square ... [you can see angles]. Or [if] you trace the square on another square and then you look at the corners, the lines are angles. If you erase everything except the corners and everything in the middle of a square, you have a angle. Angles can be bigger than that. You have angles on a triangle and you erase all the triangle except where are the corners, you have a angle. Angles can go bigger, or [pauses]. Angles can go all the way around to it. Angles can go to a straight line or all the way around. If you had two sticks and you put them together, they are straight. And you can put [them] right on top of each other. If you do that, and open up the sticks, a little bit [you can see an angle] If you have a string and putting together holding those [points], and open them up, your top end will be the vertex.

Up to this point in this transcript, Marie's verbalizations refer to global descriptions of angles and can be classified at van Hiele Level 1. She handles a sequence of several models of angles (*angles are interiorcorners*, *angles turn*, and *angles open*) very well. Differently from the primary metaphors seen in other students, she uses what Grady et al. (1996) would classify as complex metaphors; that is, her models are metaphoric projections but she very clearly distinguishes the attributes to which the metaphor is applicable from those that it is not. For example, although her initial angles are expressed as being at the corners of figures, she immediately makes clear that she conceives angle as being "greater than that" and exemplifies with 180° and 360° angles.

Without interruption, she continues her description of angles as follows:

Marie: All angles have vertices, and all angles have two sides. And no matter how far you open your angle, [you can] make it come back around again, so that they are lined up again. That's still a angle. If you make them straight out, that's an angle. Everything between those [are] angles too.

Marie is now using a different kind of discourse, one that describes *properties* of angles. In this second excerpt, she displays a typical van Hiele level 2 behavior. She continues to use models (*angles turn*, and *angles open*) but is relating them to propositions about the generality of angles. In Lakoff's (1987) terminology, she is displaying a propositional model for angles; that is, one that specifies "elements, their properties, and the relations holding among them" (p. 113). Prototype effects were nevertheless present: if the two sides of the angle "come back round again ... that's *still* a angle" (my emphasis). Although she knew that a 360° angle is still an angle, she felt the need to emphasize it. Other students, although displaying behaviors at van Hiele Level 2, manifested prototype effects. Hill and Marie, for example, identified g1 and g4 as angles in the figure below from the test.



When questioned during the interview about his identification of g1 in the test, Hill said the following:

Hill: That shouldn't 've been there, I don't know why I put it there. It's not an angle because a line doesn't... It's a curve.

José: And what about this one [g4]?

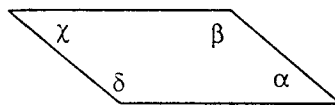
H: I would say that wouldn't be there either. Because it curves up as it goes [traces g4-g1]. A straight line goes straight.

J: So you say [that] this one is not an angle either. Why did you put them?

H: At that time I was thinking that this line [traces g1-g4] wasn't... Since it was connected that it eventually... be a line somehow, because it curved in and then it was straighten out.

In a similar vein, Marie argued that near the vertices the lines were straight, so her answers were correct. In the test, both Hill and Marie were misled by prototype effects into identifying as angles configurations with curved sides. But, during the interview, both were able to either correct this mistake or produce a reason for their answers that would not conflict with the notion that angles have straight sides.

In the last section I mentioned how Alice was able to use metaphors as a means to think about the global properties of angles. She was also capable, in the context of another task, of revealing a deeper understanding of angles. In Task P4, Alice was trying to find the sum of the four angles of a parallelogram. She first measured using the protractor, obtaining an angle of 65 degrees, another of 68 degrees, and two of 117 degrees.



As she was measuring the last angle, δ , she said:

Alice: [The angle δ] is about 117 too. And that one [χ] will probably be the same as that one [α], because these [traces the parallel lines $\beta-\alpha$ and $\chi-\delta$] go the same way. Because that [$\beta-\alpha$] is the same length as that [$\chi-\delta$], and that [$\beta-\chi$] is the same length as that [$\alpha-\delta$]. Then if that [$\beta-\delta$] is the same, then that [$\alpha-\chi$] should be the same.

In this excerpt, she is not relying on images nor actions alone. Having obtained different numbers when measuring χ and α , she is able to state that the amplitudes should be equal. She explains this by informally referring to a property associated with systems of

two pairs of parallel lines. Alice expresses this property using the metaphor *parallellines are lines that go the same way*. She concludes her reasoning by asserting, “Then if that [$\beta-\delta$] is the same then that [$\alpha-\chi$] should be the same.” The figure shows two diagonals of distinct length, but she concludes, however, that they are equal. Although her conclusion is false, she is performing an inference, obviously not relying on the figure.

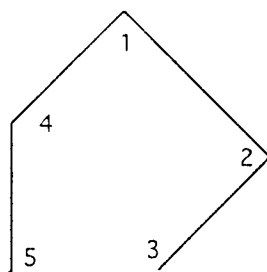
Marie could use the complex set of characteristics of angles she was aware of to solve geometric problems. In the task involving the calculation of the internal angles of a parallelogram, after I divided the parallelogram into two triangles, Marie said that its internal angles add to 360° “because two triangles.... If you add up all the angles of a triangle, you would get 180 degrees. And 180 degrees times two is 360.”

This task was developed further as I asked Marie if she thought that the same kind of argument could be used with quadrilaterals other than parallelograms. At first, she said no:

Marie: You can only do it for ... parallelograms. Because if you divide this in two triangles, the triangles won't be equal. If you divide with that [quadrilateral], ... I don't think the triangles will be equal.

She later corrected herself: “That would be, [360] Because all triangles add up [to 180], no matter what shape they are. Doesn't matter. ... They always are. The angles always add up to 180 degrees.”

When asked whether the same rule would apply to a pentagon, she did not find a way to split the pentagon into triangles. She did not draw diagonals that would help solve the problem, so tried to estimate the angles in the pentagon.



She identified Angles 1 and 2 as right angles, argued that Angles 3 and 4 are equal, proceeded to measure one of them and obtained 50° because of a measuring error. She decided to move in a different direction and claimed that Sides 1-4 and 2-3 were equal. Sides 5-4 and 3-5 were also equal. She then compared Angle 5 with a right angle and concluded it was a right angle. At this point, she checked that she had 270° "so far." I then suggested that she divide the two remaining angles in some way. She then drew Diagonal 4-3 and concluded immediately "This right here is a [tri]angle, that's 180 degrees. This right here [the remaining quadrilateral] is 360 degrees. Would it be ... 540 degrees?" She argued later that this procedure would work with any pentagon.

Structuring the category of angles involved the awareness of several properties associated with angles and the ability to use multiple models of angles as a means to solve geometrical problems. It bears a strong resemblance to van Hiele Level 2. Marie's case represents the most complex instance of this structurization found among the participating students, but two other students (Angela and Hill) were also able to exercise an approximate behavior in specific instances. In all of these cases, prototype effects were also detected.

Conclusion

The first section of this chapter describes the basic-level of the category of angle. This level is composed of elements like *acute angles*, *right angles*, *obtuse angles*, and also *quarterturns*, *half turns*, and, occasionally, *full turns*. For the students beginning their study of angles, these are barely differentiated wholes, relying heavily on images and actions. Prototype effects can be detected, affecting features like the length and the straightness of the sides, the amplitude of the angle, or the preference for a horizontal position for one side. Metonymic effects can also be detected, as students are usually

producing either acute angles or right angles as standing for the category of all angles. These effects still show up for students with a more sophisticated knowledge of angles.

The next section analyzes seven cognitive models of angles by showing how they are metaphoric projections of the schemas discussed in chapter 6. Relations between these models of angle and bodily experiences of containment, motor actions like turning or walking, and social events like coming into contact with somebody are established. Relations between these models and types of angles used in school mathematics are also highlighted.

The last section discusses the ways in which these models relate to the complexity in geometric thinking. Initial metaphoric models, closely linking the source and the target domain of the metaphor, are found to change to complex metaphors, as these domains drift apart. The basic-level of the category, closely linked to van Hiele Level 1, loses its prominence, as students move to Level 2. At this level, the category of angles has the structure of a cluster of distinct models, which students use as they see fit for the situation at hand. Propositional models emerge, but prototype effects continue to show. Imagination, that is, the ability to reason using rich images, was found to be a powerful tool for the mathematical explorations of some students.

CHAPTER 8

MODELS OF ANGLES IN TEACHING AND IN MATERIALS

Chapter 7 contains an account of several models of angles exhibited by students, together with an identification of the ways in which these models were related to broader schemas. In this chapter, I move from the individual sphere into the social context in which these models came about and investigate the models of angle taught in the participants' classes. I looked for traces of these (or other) models both in the classroom discourse and the educational materials that dealt with the concept of angle.

As stated in chapter 5, all classes chosen to participate in the study were also taking part in a research project on curriculum development in geometry (McKillip & Wilson, 1990). This project proposed that angles should be introduced by making extensive use of the *angles turn* metaphor. Angles should be explored so that students would understand that angles cannot be measured by a ruler, but that the amount of turn was important in comparing angles. To make sense of the action of measuring angles, students should initially be introduced to informal units of measurement, after which the formal unit *degree* was introduced. Properties involving angles were to come later by the observation of relationships and the invention of rules (Wilson & Adams, 1992).

I begin by showing how the teachers chose to teach this specific model of angle. Next I show how the structure of this model was shaped through specific classroom strategies. I also show the ways in which links between this model, other models of angle, and other models of mathematical entities were formed. Other models of angle were also present either as a major teaching topic or as a brief reference. I show how these models were

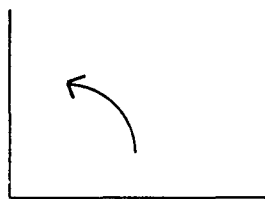
used to investigate specific geometric topics, and I finish by analyzing the ways in which these models were expressed in the materials used by the teachers.

Teaching a Model of Angle

Activities aimed at the introduction of angles were observed in all classes participating in the present study. In these lessons, *angles turn* was the model proposed to students.

In the fourth-grade classes, both teachers decided to spend only one hour introducing angles, moving thereafter to other geometric topics not related to angles. The fifth-grade Teacher D spent one hour on the introduction of angles and continued by exploring topics related to angles for several subsequent lessons. The methodology used by these three teachers was very similar. They started by dividing the class into groups. The students in each group were asked to construct a “dynamic protractor” made from two different-sized D-Stix and a flexible drinking straw. Tape was used to keep the D-Stix in place. During the construction of the protractor, all teachers pointed out to the students that it did not matter if the two sides were different lengths. Then, using the protractor, the teachers discussed the relationship between angles and the turning sticks. An example taken from Teacher D’s fifth-grade class shows a typical sequence:

Teacher D enacts the following motion with the D-Stix:

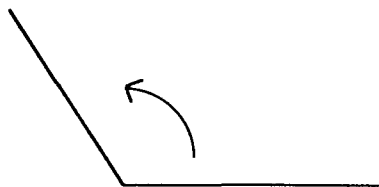


Teacher D: This is a quarter turn. A quarter angle [sic] looks like a right angle.

Student A: Yeah, a 90° angle. When two lines form a 90° angle.

Teacher: You can rotate it counterclockwise [and she explains the meaning of this word].

The teacher enacts the following:



Teacher: When you have it like this, what would you call it?

Student B: A left angle.

Teacher: No, it is more than one fourth of a turn and less than half a turn.

The teacher then continues by asking students to show angles relative to one fourth turn and then to one half turn.

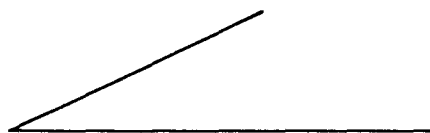
One can see in this sequence that the teacher was showing the students a state of affairs and was talking about it. Really, she was presenting two joined sticks and saying that that artifact was an angle. Most students were aware that literally that device was not an angle. For they implicitly knew that the teacher was actually aiming at introducing some new instructional entity and was using the two sticks as an educational strategy. The two sticks were perceived as a metaphor for angles. Moreover, at this stage, it was a primary metaphor, because the students were not yet able to distinguish clearly between the two metaphoric domains. As she was presenting a dynamic model, she needed to show the action needed to produce a specific angle and to speak the words that went along with this new entity. In Vygotskian terms, she was making the protractor a mediating instrument. The students, on the other hand, had two different reactions. Student A recognized the configuration the teacher was showing and was implicitly saying, "I know that, it's a right angle." He proceeded by stating what these angles were

for him: “When two lines form a 90° angle.” He was recalling a different cognitive model, *angles are meeting points*, and was relating it to the new perspective the teacher was proposing by means of a proposition. Student B was a different matter, and one can imagine what happened. From the point of view of the student, the oblique side was bending toward the left. He had already heard about a “right” angle, so it would be plausible to imagine that this would be a “left” angle. The teacher’s reaction to this assertion was to implicitly say that that was not an adequate (socially acceptable) interpretation, and she proceeded to express an adequate one.

A similar sequence but with a different focus can be observed in fourth-grade Teacher A’s class:

Teacher A. [Look at] what’s happening between these two sticks. [Look at how far they are] apart.

She holds the protractor in her hands like the following:



She rotates the oblique stick around the horizontal stick.

Teacher A: As I swing this around what is happening to the angle?

Student: It is wider.

As she joins the two sticks, she notes the following:

Teacher: As the sticks rotate, the angle disappears. It is like a clock. ... Look at what happens now. First around. It is smaller. Now it’s really big.



She enacts the configuration on the left. She pauses, and then she ends her movement with an angle larger than a 180° angle. The use of the instructional model *angles turn* enables her to go beyond the types of angles usually taught at this grade level and meaningfully speak about angles greater than 180° .

Teacher A also made the protractor a mediated instrument, but she performed a deeper analysis of the kinds of appropriate actions than Teacher D did. She probably knew that her fourth graders had never met angles before, and so from the beginning, she carefully warned her students as to where the focus of their attention should be: “[Look at] what’s happening between these two sticks. [Look at how far they are] apart.” This was a different model of angle (*angles are meeting points*) that the teacher used only on this occasion. She then proceeded by dynamically exploring the whole range of possibilities and by relating the whole experience to an object from the outside world: “It is like a clock.” On a different occasion, she asked students to identify the vertex. Although one student indicated one of the endpoints of the D-Stix, whereas another one wondered, “Do we have a vertex on one half turn?” many students were able to point out the vertex.

Occasionally, the students played a more prominent role. Below is an episode taken from fifth-grade Teacher Z that illustrates how students’ models were socialized. She chose to begin a series of lessons involving angles by introducing the concept. She did not follow the project’s proposed sequence, deciding instead to start by using the textbook (Thoburn, Forbes, & Bechtel, 1982b), where angles were presented as two rays sharing an endpoint. Only after this introduction did she distribute protractors made of D-Stix and proceed as the other teachers did. She had previously explained that acute angles turn less than a quarter turn, whereas obtuse angles turn more. Her definition considered as obtuse those angles that turn more than one half turn, which is not the standard mathematical terminology. But she was consistent with her terminology. Then she asked

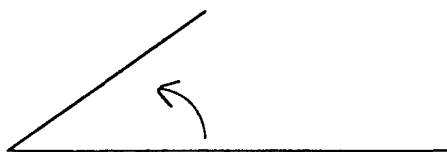
students to show her acute angles with their D-Stix protractors. One student showed the following:



Teacher Z: What did you do? You did it like this?



Teacher Z: That would be an obtuse angle. You need to do like this:



Teacher Z was setting standards for the adequacy of students' mathematical actions: "You need to do like this." Teacher D proposed that her students distinguish between these two angles by drawing a small arc "inside" the intended angle.

Fourth-grade Teacher H discussed with her students what was the largest angle and the smallest angle. She also asked students to develop and expand their model by drawing an angle and then drawing another one that was very different. One student drew a right angle with one side horizontal and then another angle with a very small amplitude. Another student drew a right angle with sides of different lengths and then another right angle with both sides at a small tilt and with congruent sides. Teacher H explained to this student that "what matters is the amount of turn."

In summary, this section has shown how teachers made the protractor a mediating instrument for the metaphoric understanding of angle as a turn. They showed the appropriate physical actions, tried to attribute adequate language to them, and enriched the actions by showing multiple kinds of turns, comparing the action with the movement of the hands of a clock or with doors opening and closing, or using terminology connected to turns (“all the way around,” “swing”); that is, established a primary metaphor. Occasional connections to other models were spontaneously introduced by the students and referred to by the teachers. Students were frequently asked to expand this new model and were sometimes corrected so that their actions and utterances would fit the teacher’s expectations.

It is also important to note two aspects of angle that were not explicitly addressed in the lessons. A first aspect has to do with the ways in which the dynamic protractor was used. Almost all angles shown by the teachers had one side horizontal. A preferred position for one side of the angles was therefore implicitly being created. This practice also established a distinction between the two sides of the protractor (one stood still and the other moved). A second aspect, related to the formation of images, played a more prominent role. It is an intrinsic characteristic of school geometry that all its objects can be “seen.” In the lessons one can see that image-schematic models of angles were constantly present or were constantly in the process of being shaped.

These lessons went well. The model, however, differed from teachers’ usual practices at teaching angle, and some teachers occasionally got confused over the appropriate way of using the protractor. The students were, nevertheless, able to complete the tasks proposed by the teachers, and, from their answers to the teachers’ questions, one can infer that, in general, they understood the lessons.

Teaching the Structure of a Model

After having acquainted their students with the model *angles turn*, the teachers looked deeper into the structure of the model, essentially by characterizing some of its submodels (right, acute, and obtuse angles) and identifying some cognitive reference points. All the teachers showed “full turns,” “complete turns,” or “whole turns” (which meant a 360° turn of one stick around the other); “half turns” (180° turns); and “half of half turns” or “quarter turns” (90° turns). All of these turns constituted reference points in the model, and the other angles were named in relation to them. These points also provided the model with an ordered structure; that is, it is possible to say which angles are larger and which are smaller. The diagram in Figure 11 captures the structure of the model, identifying its submodels.

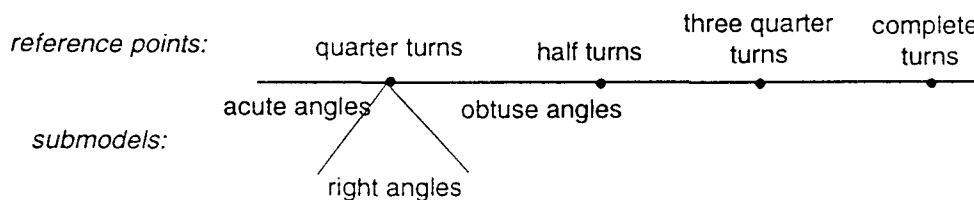


Figure 11. Structure of the *angles turn* model.

All teachers started by looking into the *quarter turns* submodel. Teacher Z did this explicitly by asking questions like, “What things do you have on your desk that show quarter turns?” Or she said, “You can check if it is a quarter turn by using a piece of paper or a book.” Teacher D used a similar strategy:

Teacher D: Give me examples of a quarter-turn angle. ... Look for the corners.

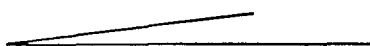
Student (aside): It's got to be a square to be a corner.

Both examples show the teachers' concern with establishing links between a specific submodel and other elements in the students' previous experience. The second also shows how a student was understanding and fulfilling the teacher's expectations.

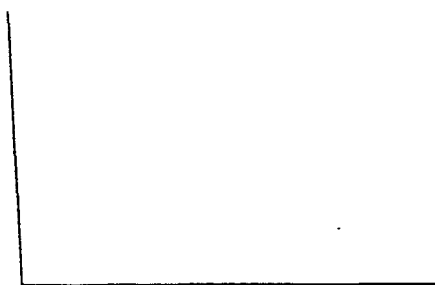
An example of the teaching of the boundaries of the submodels is provided by the following sequence. Teacher Z faces the class with the dynamic protractor in her hands and says the following:

Teacher Z: Now I am going to turn [to get an obtuse angle], and you will tell me when to stop. [You will tell me to stop] when I have an obtuse angle.

She starts with the configuration below:



She then rotates the upper stick counterclockwise. When she arrives at the configuration below, she stops:

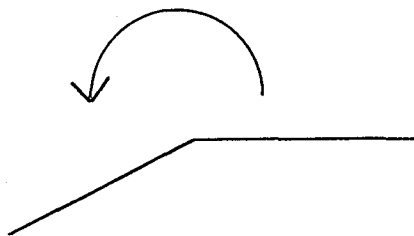


She asks, "Is this an obtuse angle?" The students are undecided, but eventually they come to agree that it is obtuse.

The students' indecision apparently came from the fact that the configuration was visually much closer to a right angle than to an obtuse angle. The teacher chose to put this case before the class precisely because of the potential for confusion between an image-schematic model of a right angle and the standard submodel of an obtuse angle. She was making clearer the boundaries between the submodels for right angles and for obtuse angles.

In the following example, fifth-grade Teacher D employs several reference points, using them to name other angles:

Teacher D: Now show me [angles that are] less than one half and more than one fourth of a turn. ... Now in this case? How much is it? [She enacts the motion below.]



Student A: More than one half by more than one third of one fourth.

Student B: More than one half by more than one fourth of one fourth.

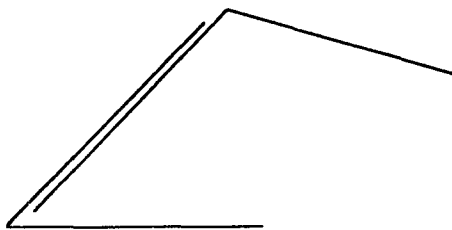
There is silence. The teacher is nonverbally signaling that they are giving wrong answers.

Teacher: Remember your fractions. Where is three fourths?

It was hard for the students to find the correct name for the angles. The teacher then decided to resort to a model that had been used in another mathematical topic. Her last remark is very interesting and is discussed in the next section.

Using a Different Cognitive Model of Angle

Occasionally there was a need to compare angles, which proved very difficult to do in the absence of a measuring system. In both fourth-grade classes, several students found it very hard to compare angles by superimposing their drawings. They did not understand that vertices should be made to coincide and that it might be helpful to overlap one side. One boy overlapped two angles like this:



One girl in Teacher H's class even proposed that to compare angles one could "measure the distance between the sides," hinting at a different model.

To overcome those problems, the project proposed the use of a unit called a "wedge" (Wilson, 1990). After the introduction of angles, both fifth-grade teachers decided to teach ways to measure angles by using wedges. In these classes, a "wedge" was an actual physical object made from a paper circle by folding and cutting a circular sector with a 30° amplitude. This was, in fact, a new metaphoric model, which I call the *sector* model, that did not show up in the students' interviews. It is characterized by the following elements: a body with a special configuration, two different kinds of borders, and a special point (Figure 12).

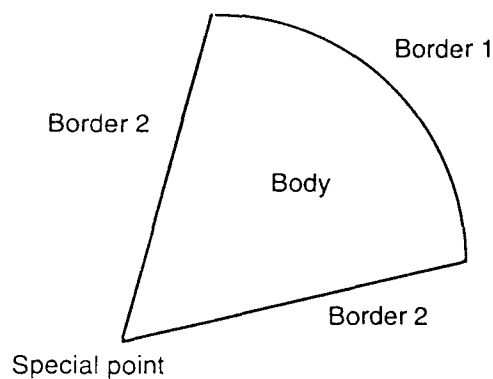


Figure 12. The *sector* model.

One difference between this model and the models presented in chapter 7 is that inherent in this case is a joining operation between several bodies to produce a new instance of the model (Figure 13).

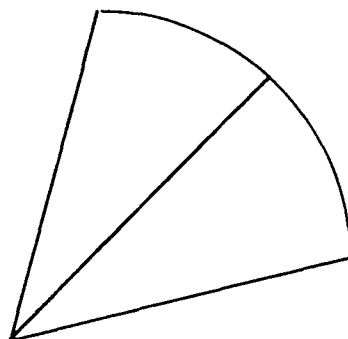


Figure 13. The joining operation on the *sector* model.

Figure 13 shows that joining two bodies produces a new body having the same elements. One can also see that the common border disappears. Angles are here seen as metaphoric projections of this model in the following way:

<i>Sector model</i>	angle
body	interior of angle
border 1	
border 2	side
special point	vertex
<i>relations</i>	
joining	adding

The model *angles turn* was not the focus of these lessons. It was superseded by the *sector* model, and both fifth-grade teachers took care to establish links between this new model and the previous model. This was done during the initial phase of the construction of a wedge. Here is what happened in Teacher Z's fifth-grade class:

Teacher Z: We are going to use wedges to measure angles.

She asks students working individually to cut out a circle (15 cm in diameter) from a worksheet.

Teacher: First thing I want you to do is hold [the circle] like this [with the dynamic protractor on top of it] and make a full turn to see it going all the way around.

She then gives directions for constructing and cutting out a semicircle.

Teacher: [Holds the semicircle.] How much of a turn would you make to [complete this semicircle]?

The teacher now tells students how to cut the semicircle into three equal parts, which are then cut in half. She tells them the following:

Teacher: This is going to be your measurement unit. I will call it *wedges*.

In this episode, one can see how the teacher was constantly recalling the old model as she was directing students in the construction of the new one. Instructions like “make a full turn to see it going all the way around” were not needed for constructing the wedges. They were, however, vital to the establishment of a relationship between the partition of a semicircle and the partition of a turn and to allow the actions that were going to be performed with wedges to become a mediating tool for the understanding of angles.

Occasional relationships between the *sector* model and other models were established. For example, a link with the *angles are interior corners* model was established when the teacher asked students to “see how many you can fit into that angle.” It was as if Border 2 was not a real border. As a means to help students complete one worksheet, she also said:

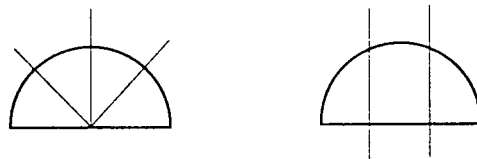
Teacher: You have to place wedges. Sometimes you have half a wedge. It is like a pie.

Sometimes you may have like half of a piece of pie.

I return later to this reference to a piece of pie.

Some students had difficulty choosing the proper position for the vertex in the *sector* model. Teacher D went through a very similar sequence, but she decided to ask students to find ways of dividing the semicircle into three equal parts by themselves. This proved

to be a challenging task for the students. Although many students divided the semicircles shown in figure on the left below, at least two did the division on the right.



Although many students understood the joining relationship, some did not. In Teacher D's class, one student "added" the wedges as shown in the figure below:



But overall, in Teacher D's class, students were able to use this model to measure and draw angles. The answers to three worksheets given by Teacher D in the third lesson observed by me, show that a large majority of students was able to accurately measure and draw the angles. Only in the cases of angles involving fractions of wedges, like $1\frac{1}{2}$ and $1\frac{1}{3}$, could some students not provide adequate answers.

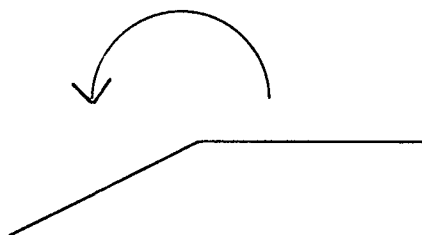
In summary, this section has shown how a new model was introduced and how the teachers carefully established links to previous models of angles as a means to ensure that this new device would become a tool in the proper understanding of angles.

Relations to Other Models

Another way to give structure to a model is to make explicit its relationship to other models. Both students and teachers did that spontaneously from the beginning. Fifth-grade Teacher Z, for example, started one lesson by using the *angles are meeting points* model and later changed to the *angles turn* model. On several occasions, students brought other models to the forefront. I mentioned earlier that one student stated that the angle

becomes “wider.” At other times, the word *corner* showed up. The term *distance* between the two sides of an angle was used on at least two occasions, once by a student and once by Teacher D.

I have cited two references to fractions and pies. The first instance was when Teacher D was asking students to name the angle below:



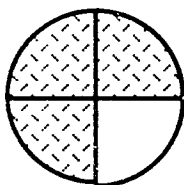
The teacher gave the hint “Remember your fractions.” The second reference was at the end of the previous section (p. 161). These references provide an example of a connection between this methodology for teaching angles and another mathematical topic. Through the similarity between the names and the images, the *sector* model was very rapidly associated by both teacher and students with the *pizza* model used in the study of fractions.

After her hint “remember your fractions,” the following dialogue occurred:

Teacher D: Where is three fourths?

Student: Three fourths is seventy five.

Teacher: Very good. Come to the board, please. Where is three fourths? Draw a circle like an apple.



The teacher shows how much three fourths of a turn is. She also shows that the unshaded part can also be one quarter turn. She concludes by saying that it is important to know how the turn was performed in order to relate angles to turns.

The teacher was using the students' previous knowledge of fractions to lead them to understand angles. The *pizza* model for fractions became a metaphoric projection of the *sector* model. This projection was accomplished essentially by a visual association between the two models. The teacher was not, however, completely successful, and some students still showed some confusion, especially between one fourth and one third.

Using Models to Investigate Geometrical Knowledge

The exploration of a model can lead to some unexpected mathematical consequences. In Teacher H's class, understanding full turns was easily accomplished, and several students were able to draw a full turn. One boy even proposed that there is an angle when the two sides are joined together. He drew two line segments:

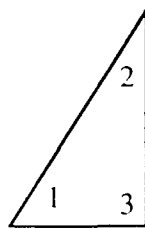


Then he said that when "we have just one side, this is a turn" also. He drew the following:

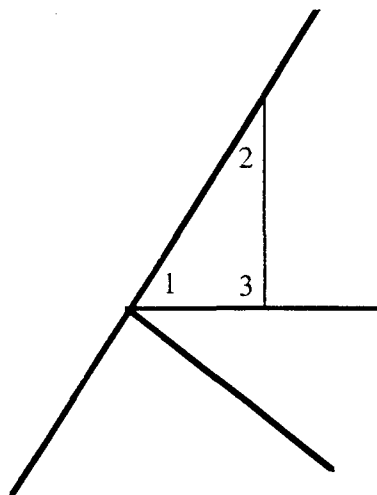


By this drawing, he was pointing out that a ray is the same as a 360° angle, which is a consequence of the model *angles turn* and is a remarkable inference for a fourth grader.

There were other ways in which different models were used to investigate geometrical knowledge. After spending several lessons on teaching some geometric concepts, Teacher D discussed angle sums in triangles and quadrilaterals. She started by showing the following triangle on the overhead projector:



With the help of a student, she used the D-Stix protractor to copy the angles. Her final drawing looked like the figure below:



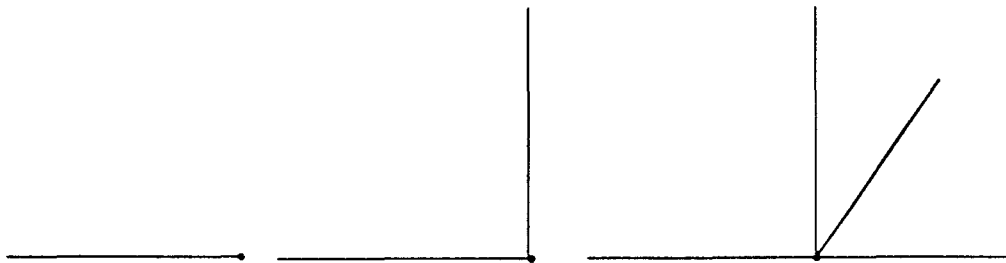
After making this drawing, she separated the students into groups of two and distributed two triangles and one quadrilateral to each group.

Teacher D: When you get your triangles, please number them [the angles] 1, 2, 3. The first thing [is to] tear up Angle 1, then Angle 2, and then Angle 3. I want to line them up so that we can find out how much of a turn all those turns would make. ... [Please remember] sides on sides. vertex on vertex.

The *angles turn* model was not really helping the students understand the problem. The strategy intended for the students involved tearing up figures to allow manipulation of their angles. That is much closer to other models, like *angles are interior corners*, *angles are two connecting lines*, or the *sector* model, than to the *angles turn* model. The teacher became aware of this and decided to change her approach:

Teacher D: [There is] another way. Draw a dot and a line [and put together the three angles].

She showed the following on the overhead projector:



The teacher then reported to the class an event that seemed relevant to the solution of the problem:

Teacher D: K found a different way to do it. Please show how you are doing it [K is tracing the angles and does not need to tear them.]

Most students were solving the problem, but some continued to have difficulty deciding how to put the angles together. The teacher kept reminding the students “side on side, vertex on vertex” as she went from group to group. The next activity involved adding the angles of a quadrilateral, and the students were much more successful at that.

It is important now to look at a lesson that occurred after the students had been involved with several models of angles. Fifth-grade students of Teacher D have been studying angles for over a week and are about to explore special kinds of triangles and quadrilaterals. The teacher distributes D-Stix and connectors, and asks the students to work in pairs. She starts by asking students to construct a triangle with all sides different, one with three sides congruent, and one with two sides congruent. Then the following exchange occurs:

Teacher D: Now I have a real problem for you: Make a triangle with all angles equal.

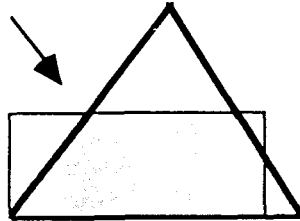
Immediately after the teacher’s challenge, one student asks for a protractor and is given one. Most of the students proceed to solve the problem by visually comparing the angles.

One student exhibits a triangle:

Student A: Do these look equal?

José: You could check with a piece of paper.

The student then uses a piece of paper to compare the angles. He puts the paper like this:



He holds his finger where the arrow is. He then compares the angle formed on the paper by the edge and the line from this finger in the corner with the other angles.

Student A: Yes, they are right.

The teacher is having a similar dialogue with another student.

Teacher: How do you know they are equal?

Student B: I stick this in and move it to the other. [He means that he uses a pair of Dividers and moves it from one angle to the other.]

As students are successfully coming up with triangles with equal angles, the teacher asks them “Do they have equal sides?”

The teacher now asks students to “create a triangle with two angles that are congruent.” Student C brings his completed triangle to show the teacher.

Teacher D: Which sides are equal?

Student C: [Pointing to the two equal sides] this and this.

Teacher: Which angles are equal?

The student hesitates. He points to the equal sides but, from the teacher's nonverbal language, sees that something is wrong. Another student corrects him and points to the interior of the angles.

Student C: Oh! You mean the corners [and then points correctly].

Other students have different perspectives:

Teacher D: How do you know they are equal?

Student E: By looking.

Student F: You could trace it.

Some students are also using a protractor to measure the angles. The teacher then attempts to relate the number of equal angles in a triangle to the number of equal sides. She challenges the students to find a triangle with two right angles. Marie very quickly says it is impossible. Hill argues that "if one angle is a right angle and the other is also [a right angle], then we have a square."

Throughout this lesson, the students were using several models of angles. In fact, students were at times interacting by using distinct models, but that did not hinder communication. Some students were apparently using a generalized model of angle that was able to integrate the other ways to "see" angles. There was no sign of the primary metaphors used earlier by some students during the introduction of angles.

By the end of the lesson, it appeared as though the successful students were the ones that could choose whatever model would be best adapted to the situation. The teacher also played a very important role in challenging students and producing provocative comments from time to time. From the perspective of van Hiele levels, the lesson entailed a constant movement back and forth between Level 2 and Level 3.

Models in the Materials

The four teachers participating in the present study had at their disposal various materials to assist in the preparation of their lessons. Most of these materials came from the Geometry and Measurement Project mentioned above (McKillip & Wilson, 1990), but sometimes the teachers would use a textbook (Thoburn, Forbes, & Bechtel, 1982a, 1982b) or other materials. In this section, I look at the models of angle incorporated in all these materials.

The materials proposed by the project took the form of lessons intended as a geometry course for the elementary school. These lessons included straightforward directions for activities to be carried out by the teacher, some worksheets, and teaching notes that included suggestions, comments, accounts of classes, and so on. The four participating teachers were observed as they were using lessons that involved directly or indirectly the concept of angle. There were two kinds of lessons: (a) lessons aimed at building an artifact and relating it to the introduction of a geometrical concept; and (b) lessons proposing explorations of specific geometrical properties.

Two lessons (numbered 2.06 and 2.07 in Appendix E) were of the first kind. They had the purpose of developing the “concept of angle” and proposed the construction of the dynamic protractor mentioned previously. In the text of both lessons, the model *angles turn* is proposed. In Lesson 2.06, the protractor is described. This description prepares the teacher for the way in which the protractor is to function: It has two arms, a bottom arm that stays horizontal and a top arm that always rotates on the desktop. There is a suggestion that the teacher have students make angles with their arms. A picture indicating the appropriate procedures for producing “a right angle or a 1/4 turn” is included. Teachers are advised to ask students to look for right angles on their desks or in the classroom. The lesson ends by proposing that teachers establish certain cognitive

reference points (1/4 turn, 1/2 turn, 3/4 turn) and that other angles be compared to them by using terms like “more than” or “less than.”

Lesson 2.07 expands the previous lesson and presents some important teaching directions:

The students must now develop three crucial ideas about angle:

1. the size of an angle is determined by the amount of turn, not by the length of the arms.
2. Neither arm of an angle needs to be horizontal.
3. In order to properly label an angle we need to know which arm turns and the direction it turns. The arc (or an arrow) in the drawing must be present or the student must be able to describe which arm turned and in what way. (McKillip & Wilson, 1990, Lesson 2.07, p. 1)

The first two recommendations aim directly at countering specific prototype effects common in students' misconceptions about angle. This concern is further explored in the worksheet for the lesson. The third recommendation is a way both to improve the efficacy of the tool by adopting a convention (the arc or the arrow) to eliminate an ambiguity and to make sure that students can carry out the actions.

In these lessons there are occasional references to other models. A drawing exemplifying the appropriate movement shows three positions: namely, the start position, the opening, and the right angle. The opening, which is the *angles open* model, is shown as an intermediate position. Corners are also briefly mentioned.

Lesson 2.10 is the second type of lesson. It aims at comparing the angle sums of triangles and quadrilaterals and determining that the former is a half turn and the second a whole turn. No new devices are introduced. The models are to be used to obtain new geometrical knowledge.

The lesson starts by suggesting that the teacher begin by showing a triangle on the overhead projector and using the dynamic protractor “to show the turning of each angle.” Then the teacher is to ask, “How much turning is there altogether?” This first part aims at enacting each angle as a turn and setting up the problem to be investigated in terms of this model.

The lesson continues by proposing that the teacher give students cutouts of triangles, have them tear off the corners, and reassemble them side by side “like the pieces of a puzzle.” Several steps are proposed to achieve this reassembly. The same kind of activity is then proposed for quadrilaterals. Throughout this part of the lesson, *angles are interior corners* is the prevailing model. Angles become physical objects that occupy space (a region) that can be moved from one place to the other, and, that can be put side by side to produce a new angle.

Lesson 2.29 taught by Teacher D also fits the second type of lesson. It aims at identifying special triangles and quadrilaterals using the sides and the angles. It proposed that students should be given D-Stix and asked to find figures that match specific descriptions like “a triangle that has all sides congruent.” In the case of quadrilaterals, the materials also propose, as a summary, a diagram showing the relationships among the several types of quadrilaterals. This lesson does not use a specific model of angle even though references to the *angles are interior corners* model can be found.

The teachers occasionally used the textbooks as a source for their lessons (Thoburn et al., 1982a, 1982b). Whereas the project materials were intended specifically for the teachers, the textbooks were intended for both teachers and students. The textbooks’ style was therefore closer to “teaching”; it addresses the student directly.

The fourth-grade textbook (Thoburn et al., 1982a) uses two instructional models of angle. Angles are presented on page 318. The reader learns that a ray is a line segment that “goes on forever.” The reader also learns that if two rays have the same endpoint

they form an angle and that their common endpoint is the vertex of the angle. Then a figure (Figure 14) is added.

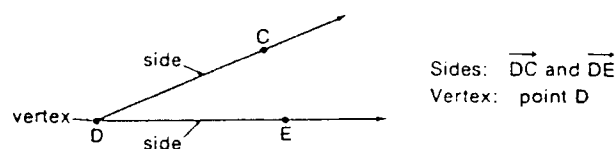


Figure 14. Figure displaying the model *angles are two connecting lines* (from Thoburn et al., 1982a, p. 318).

This is an instructional version of the model *angles are two connecting lines* and is identical to the type of angle called *set of two rays* in Appendix A.

On the bottom half of the same page, the authors propose that angles also form corners: “An angle that forms a square corner is a *right angle*” (p. 318, emphasis in original). A figure shows a rectangular drawing in the interior of the angle (Figure 15).

C. An angle that forms a square corner is a *right angle*.

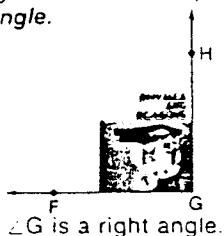


Figure 15. Picture suggesting the model *angles are interior corners* (from Thoburn et al., 1982a, p. 318).

This is an instructional version of a second model, *angles are interior corners*, in the limited interior version. It is somehow applied only to right angles. This page is making a simultaneous use of these two models, which continues on the following pages. Page 319 may implicitly add a third model. At the bottom of the page are printed two figures (Figure 16):

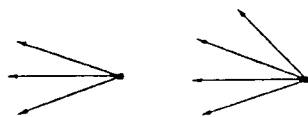


Figure 16. Figures suggesting the model *angles are sources*
(from Thoburn et al., 1982a, p. 319).

These figures may be visually associated with the model *angles are sources*. Although no words are given that would connect angles to sources, the two pictures may suggest the model.

The fifth-grade book (Thoburn et al., 1982b) starts by recalling the definition of angle as “formed by two rays with the same endpoint” (p. 284), again, literally, a textbook version of the *set of two lines* type of angle. A figure is presented on this page (Figure 17).

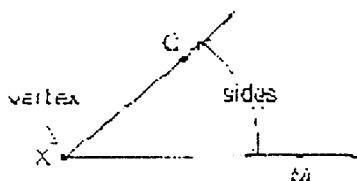


Figure 17. Figure displaying the model *angles are two connecting lines*
(from Thoburn et al., 1982b, p. 284).

In the figure, the sides are specified from the interior of the angle by an arc, which may suggest the *angles are interior corners* model. The book continues by explaining what measuring an angle means, which is to “find how many units [of measure] fill the inside of the angle” (p. 284). The idea that angles have an inside and that measuring is filling an angle is used extensively in the following pages. At the same time, all drawings of individual angles show them as two rays with a common endpoint; that is, through the model *angles are two connecting lines*. As in the book for fourth graders, the fifth-grade book contains on this page (p. 284) several figures that strongly suggest visually the model *angles are sources* (Figure 18).

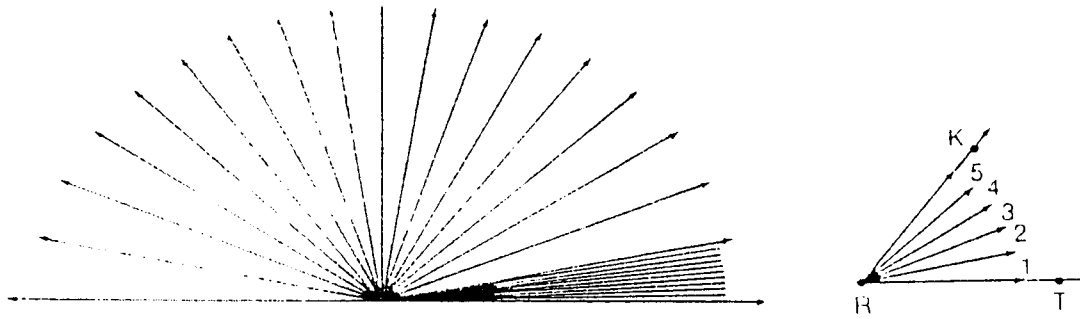


Figure 18. More figures suggesting the model *angles are sources*
(from Thoburn et al., 1982b, p. 284).

After this introduction to angle measurement, the fifth-grade book continues by explaining how to measure angles using a protractor, explaining how to classify angles, and discussing properties of figures.

After those pages in which the book explains how angles are measured by a protractor, the only actions with angles that are accounted for verbally are measuring and comparing. Angles are increasingly shown in the context of geometric figures. In the absence of linguistic references to angles, one must resort to an analysis of the images. In them, the interior of the angles is shown as having special importance: right angles systematically have a small square inside at the vertex, and other angles occasionally have small arcs or have their degree measure stated in the interior near the vertex. As individual angles disappear, the model *angles are two connecting lines* loses its importance, and the model *angles are interior corners* shows up extensively in the figures.

CHAPTER 9

CONCLUSIONS, IMPLICATIONS, AND RECOMMENDATIONS

The purpose of this study was to explore the ways in which the geometrical concept of angle is understood by individual students, together with an analysis of the contexts involved in this process. The study began with an analysis of the notion of mathematical concept. I reviewed several perspectives on categorization, especially George Lakoff and Mark Johnson's work, relating them to specific mathematical topics. Mathematical concepts, however, take distinct forms as students became acquainted with them at different grades. The work of Dina and Pierre van Hiele shows that these changes are also changes in the complexity of mathematical relationships. It thus was natural to look to the van Hiele Theory for insights that would provide a deeper understanding of the quality of students' mathematical thinking.

These theoretical explorations were accompanied by an empirical investigation focused on the learning of the concept of angle in Grades 4 and 5 at an elementary school in the Southern United States. From an analysis of responses to tasks posed on a written test and in an interview, students' cognitive models of angle were identified and categorized and were related to the students' van Hiele levels. The social context associated with these models was examined by observing the lessons in which the concept of angle was taught and by analyzing the materials used by the teachers.

Conclusions

Cognitive Structures Underlying the Concept of Angle

The first set of questions aimed at cognitive structures that *underlie* the category of angle. This category was found to be grounded in image schemas produced by our interactions with various environments. The schemas were the *container*, the *turn*, the *path*, and the *link* schemas. Intrinsic bodily experiences of containment, elementary motor actions like turning or walking, and basic social events like coming into contact with somebody are at the root of these schemas and are used idiosyncratically by individuals. Instances were noted of each of these schemas, subschemas, that were metaphorically projected onto the cognitive models of angle that students displayed. These instances resembled experiences with physical objects, like corners or points; or actions performed on objects, like opening or turning; actions performed by objects, like opening or pouring; or actions performed in relation to objects (or people), like going around. All of these were

recurring structures of our bodily interactions in the world, and they exist across *all* our perceptual modalities (visual, tactile, olfactory, aural, etc.). They are not fixed structures or images, but rather dynamic patterns of our interactions within various evolving environments. (Johnson, 1997, p. 156)

These environments are thought of as having physical, social, and cultural dimensions (Lakoff, 1987).

Structure of the Category of Angles

The second set of questions addressed by this study aimed at understanding the *structure* of the category of angle. For the participant students (and also for their teachers) angles were found to be a class of geometric objects much like many classes of objects that we find in our everyday lives. For most students, all the fourth graders and most of the fifth graders, angles—that is, the category of angles—were composed of

basic-level entities like right angles, acute angles, obtuse angles, half turns, and full turns. Each of these basic-level entities had rich mental images associated with it. Moreover, acute angles and right angles were central elements in the category of angles; that is, when students—and teachers—were referring to angles in general, the images that came to mind (or the drawings that were produced) were usually those of acute or of right angles. In other words, metonymic projections were produced, that yielded prototype effects: for example, obtuse angles were not as good exemplars of the category of angles as acute or right angles.

It is also possible to determine features possessed by the image schemas associated with this basic-level. Students preferred convex angles over nonconvex angles, and configurations with curved sides were recognized as angles, provided they did not look as though they had amplitudes larger than a right angle. Acute and right angles were taken metonymically to represent all angles. Students usually drew or enacted angles with one side horizontal or vertical, or in some cases, having a horizontal or a vertical line of symmetry. These preferences at the basic-level yielded prototype effects, with the angles having these features being used as better exemplars of angles than others. These prototype effects have their roots in models of angles built out of metaphoric projections.

This study revealed that angles are a cluster category composed of many different cognitive models. Seven models of angles were found—*angles are points*, *angles are interior corners*, *angles are sources*, *angles open*, *angles turn*, *angles are contours*, and *angles are two connecting lines*—and they shaped the ways in which students drew angles, enacted angles, gestured to illustrate the specificities of angles, or argued about the characteristics of angles. In brief, these models fashioned students' understanding of the category of angles. All these models were metaphoric projections of image schemas, and metonymic projections were found in some of them. All derived from basic understandings about the ways in which the students related to the material world, and all of them show how angles are shaped by our bodies and brains. In Lakoff and Nuñez's

(1997) terminology, they were grounding metaphors. The models were also a source of prototype effects. Pointed objects (*angles are points* model) do not produce good instances of obtuse angles, neither do they clarify the requirement of the straightness of sides, for example. The model *angles are two meeting lines* do not account easily for angles greater than 180° . It was also possible to highlight ways in which cognitive models similar to the ones found in this study may be at the very source of most of the mathematical models used by mathematicians in the definition of angle. These models were also ideosyncratic, as each student had a personal way of using the models. One fourth grader with a very limited knowledge of angles constantly resorted to the *angles are points* model which he accompanied with gestures of his own. A fifth grader with an extensive knowledge of geometry used mainly the *angles turn* model, again resorting to special gestures. Another fifth grader used, again in a very personal way, both this model and the *angles are two connecting lines* model. It is as if every student had to cognitively make his or her own model of angles, as he or she was re-enacting or re-presenting the actions and the images observed in class.

Complexity of Students' Geometric Thinking

The third set of questions aimed at understanding how this structure connects with the *complexity* of students' geometric thinking. Geometric thinking, as investigated in this study, is strongly connected to the development of a new category of entities of a geometric nature. Behaviors at van Hiele Level 1, Visualization, enacted by students were associated with images and motor actions, as was the case for two students. For one of them, his terminology was not always adequate to express his ideas, and he resorted heavily to gestures. This level was found to be strongly connected with the formation of a basic-level category of angles. Primary metaphors shown in statements like "angles *are* points" or "angles have two long pieces," expressed by four fourth graders and one fifth grader, were also used at this level.

Few students displayed actions at van Hiele Level 2, Descriptive. Only one student well exemplified the level, although three others occasionally exhibited behavior at this level. As van Hiele predicted, this level is characterized by the emergence of propositional models, and students' actions are based upon properties of geometric elements. These students were also using a concept of angle composed of complex metaphors. Occasionally, prototype effects were shown, but these students were able to correct them. All these students were able to understand and use several cognitive models simultaneously. They were also capable of entertaining interactions at Level 1. It was as if at this second level, the complexity of their category of angles changed both in *extension*, allowing them to use many models at the same time, and in *intention*, so that they could separate themselves from the source domains of metaphors and from the basic-level of angles. They were, however, able to show typical behavior at this level when appropriate.

An analysis based on van Hiele theory does not explain, however, all the complex mathematical activities that were detected. One fifth grader used metaphors—that is, similarity-creating metaphors (Indurkha, 1994)—as tools to imagine angles as other kinds of mathematical objects. She also claimed that this imagining activity was “funny.” Spontaneously, she compared angles to triangles, circles, cylinders, cones, discussing the ways in which angles “seem like” these other mathematical entities and how they “really” departed from them. In Lakoff and Nuñez's (1997) terminology, she was linking metaphors between two mathematical domains. Imagining how one mathematical object is like another, or can be thought of as another, is an important source of mathematical ideas. Thinking about geometry using algebra (Descartes), about calculus using arithmetic (Weierstrass), and about metamathematics using arithmetic (Gödel), for example, gave powerful insight. That is what this student was doing within the constraints of her fifth-grade mathematical knowledge.

Instructional Models for the Concept of Angle

This study looked for the ways in which the students acquired the cognitive models of angle. Class observation revealed similarities between students' cognitive models and the instructional model taught in class. The teachers began by addressing angles as a basic-level category. The model *angles turn* was extensively used, and initially the teachers created contexts for students to provide rich images and actions associated with this model. Occasionally, the model *angles are two meeting lines* was also used. In this initial phase, these instructional models were used as primary metaphors, with no clear distinction between the source domain and the target domain.

The teachers then went on to deepen the discussion. They discussed the submodels (acute, right, obtuse, quarter turn, and others) of the basic category of angles, clarified their boundaries, and established cognitive reference points. Other models of angle were occasionally used. They gradually moved students away from the initial "real world" models; that is, they moved away from the initial primary metaphors into complex metaphors. To teach angle measurement, the teachers introduced a new model, the *sector* model. Again, the teachers began by presenting this model as a primary metaphor. The *sector* model had strong resemblances with a similar model used with fractions. It was clear that the teachers and students connected the two models.

As the lessons moved on, it was possible to detect changes in the instructional models. First, the use of several models simultaneously became more common. Second, while remaining metaphors of deeper schemas, the models no longer had a direct connection to real world objects or situations. Third, prototype effects could still be detected.

The instructional models in the educational materials used in the lessons framed the last set of questions addressed by this study. The models in the lessons used by the teachers had very strong resemblances to the instructional sequence they followed. The models relied strongly on the metaphor *angles turn* and proposed that teachers make use

of a tool, a “dynamic protractor” to mediate it to the students. The use of this material enabled students to experience a context for actions involving a broad range of angles, from 0° to 360° . Comparison of the angles within this model was related to the amount of turn. Measurement using informal units of measurement of angle was possible by employing “wedges.” This was enabled by the introduction of a new model, the *sector* model. The textbooks were heavily based on the model *angles are two connecting lines*, but occasionally other models were shown, explicitly (as the *angles are interior corners* model) or implicitly (as the *angles are sources* model). Gradually, in both the lessons and the textbooks, the category of angle became more abstract, as less space was devoted to work with specific models and more was dedicated to the exploration of geometric properties involving angles.

Implications and Recommendations

This study began with the question “What is an angle?” Ideas originating in the study of the ways in which we categorize, especially those coming from the work of Johnson and Lakoff, proved to be of crucial importance in exploring that question. Van Hiele theory was also important as a means to reflect upon the growing complexity of students’ geometric reasoning. This study confirmed my initial idea that angles are a complex subject. Other mathematical topics may prove to be simpler. It is, however, my contention that its findings can have implications in broader areas, namely, characterizing cognition and revising van Hiele theory. Recommendations are also made concerning practice and research in mathematics education.

Implications for Characterizing Cognition

The structure of the category of angles was found to be very complex. There was a basic-level heavily grounded in perception, images, and motor actions. The seven models of angles were metaphoric projections of image schemas relating to fundamental ways in

which humans relate to their physical, social, and cultural environment. This organization could account for prototype effects and for the role of imagination in the formation of mathematical concepts. The picture obtained for the category of angle confirms that this category was not determined by necessary and sufficient conditions. Also the meaning of the word *angle* was not fixed by referring to an abstract disembodied entity apart from human experience. On the contrary, it was embodied.

It was possible to envision the ways in which this category changes under the influence of schooling. Students begin understanding angles by means of primary metaphors in which source domain and target domain are conflated. Gradually these primary metaphors are transformed into complex metaphors as students learn to separate (abstract) the two domains. The basic-level categorization of angles is, however, never forgotten. Its availability can account for many prototype effects found even in the more able students.

Abstraction was found to be not a departure from “real world” characterizations of angles into more intangible realms of knowledge, but the development of a categorization that made use of a multitude of cognitive models, a cluster of models, that competed with each other for primacy at solving a given task. The idea of the trading floor of a “stock exchange” (Kilpatrick, 1985, p. 13) of models, which is itself a metaphor, seems an apt description of the kind of reasoning used by some of the more advanced students.

This characterization of the category of angles also refutes the idea that its structure could be completely accounted for by describing the relevant social environment and its influence upon students’ concept of angle. Social interactions in the classrooms guided by the teachers played a fundamental part in shaping the students’ categorization of angles. But social influences alone cannot explain why an angle was thought to be like a point or a corner, for example. They cannot explain the determinant role of image schemas structuring the category of angles.

This study barely scratched the surface of the role played by social interactions in

shaping categories, and much research needs to be done in this area. This role is of special importance for research on improving teaching methodologies.

Implications for the van Hiele Theory

The van Hiele theory was found to be useful in describing complexity in geometric thinking, yielding valuable insights in interpreting students' verbalizations and actions. Although the van Hiele theory was not the focus of this study, several changes in the theory were needed to account for the results. Several changes are discussed in chapter 4 and were incorporated into the interpretation of the theory used in this study. Two types of changes were addressed there: changes in the implicit cognitive theory (that there are external spontaneous structures ready to be perceived by the learner and that learning proceeds uniformly) and changes in the characterization of the levels (that movement through the levels is continuous and that a full understanding of the classification of quadrilaterals requires Level 4). This study shows that further modifications should be made, all of them consequences of assuming that cognition is embodied.

The description of the first two levels can be enriched by the findings of this study. Level 1 is characterized by basic-level categorizations based on rich images and motor actions, as van Hiele anticipated. But, contrary to van Hiele's ideas, there are no spontaneous structures of the material on which our cognition is based. Instead, image schemas grounded in our material and social life yield idiosyncratic metaphoric models that are construed by each individual. Level 2, as van Hiele proposed, is characterized by the appearance of propositional models built upon those of the previous level. These new models imply, however, the disappearance of neither the basic-level of categorization nor the metaphoric projections. Neither do they imply that the language of previous levels is not understood at higher levels. Contrary to van Hiele theory, previous levels are not lost; they remain an important asset in a student's cognitive repertoire. They play a crucial role in imagination, understood as the formation of similarity-creating metaphors. The relation

between primary and complex metaphors and the van Hiele levels was not clear in the study. Although students with primary metaphors were at Level 1 and students with complex metaphors were at Level 2, there was no clear evidence for a sharp distinction. An alternative possibility would be that complex metaphors emerge as students progress through Level 1.

This study did not address thinking above Level 2. Therefore, there is no evidence as to the kind of models that characterize Level 3 and above. Since from van Hiele's perspective Level 3 is characterized by a local logical ordering of properties, it is only possible to conjecture that a *conduit* metaphor (Johnson, 1987) for this ordering might come into play at this level. Table 6 summarizes this discussion.

Table 6
Relation Between van Hiele Levels and Categorization

van Hiele level	Manipulated object	Observed object	Categorization
1		Figure	Basic-level, schemas, metaphors
2	Figure	Property	Propositional models
3	Property	Ordering of properties	Conduit metaphor?
4	Ordering of properties	Axiomatic system	?
5	Axiomatic system	Logic	?

Recommendations for Teaching

The ways in which students conceive of the category of angles suggest some recommendations for teaching strategies. The present study showed that a new mathematical concept is learned in the same way as other concepts are: by establishing a basic-level structure that relates the new concept metaphorically to previous models of other concepts. Abstraction is gradually built up by enriching the available models for thinking about an object and by separating the domains involved in the metaphor.

Complex mathematical reasoning is supported by the existence of a rich background of models. These developments require a departure from practices in which the initial statement of definitions is an essential teaching mechanism. Mathematical categories are not established cognitively by verbalizing necessary and sufficient conditions. In other words, for most students stating a definition does not help their learning.

Teachers need to pay special attention to prototype effects. There is no way to learn a category that is general enough to exclude prototype effects. Research shows the pervasiveness of prototype effects, as one of humanity's cognitive tools for relating to the environment. Mathematics is no different in this respect from other subjects, and it is impossible, even if it were desirable, to remove these effects from the teaching and learning process. But then, given that the propositional structure of mathematics attempts to counter those very effects, what should teachers do? In this study, the teachers moved gradually from an initial basic-level category of angles to the more complex categorization described above. Pitfalls originating in prototype effects were occasionally pointed out by the teachers so as to build a more robust category. The centrality of certain subcategories (acute angles and right angles) was not denied, but occasionally noncentral exemplars of the category of angles were discussed.

This study showed the role played by image-schematic models in promoting mathematical reasoning. One student's explorations of the similarities between angles and corners illustrates the sort of activities that should take place occasionally in mathematics classrooms. Comparing and contrasting mathematical entities informally may prove to be a powerful means of strengthening students' mathematical knowledge.

Finally, teachers should be aware that several models are available on which to build specific mathematical topics. Appendix A, for example, shows several options for the case of angles, and this study showed connections between those types of angle, the cognitive models of angles of students, and very basic image schemas. Not all models, however, have the same educational value. Lakoff and Nuñez (1997) discuss an example

of a metaphoric model for teaching negative numbers that proves to be inadequate for specific mathematical properties of these numbers. Although there is a good correspondence between the source domain and the target domain for adding and subtracting, the model fails in the case of multiplication.

Recommendations for Research

The perspective on cognition outlined above raises a number of research questions in mathematics education. A first area concerns enlarging the experiential basis for understanding the role played by metaphors and basic-level categorization in school mathematics. There is no reason to believe that this role is confined to teaching and learning angles, and, although there is mounting evidence from the history of mathematics of the ways in which mathematics uses metaphors, much research still needs to be done at the school level on how metaphors might be introduced and used. Longitudinal studies investigating the development of metaphors at several grade levels or linking the formation of primary metaphors to the formation of advanced mathematical thinking are needed. In what concerns the concept of angle, three types of angle used in school mathematics that were identified in Appendix A, the *continuous external rotation*, the *slope*, and the *trigonometric angle*, belong to more advanced mathematics curricula and were not found in this study. Research focusing on these types of angle may uncover relations that broaden the scope of the diagram in Figure 10 (p. 136). Although the *continuous external rotation* type of angle has, plausibly, roots in the *angles turn* model, the other two types of angle are apparently of a very distinct nature, and the metaphoric projections may prove to be very different. The *slope* type of angle may be related to schemas associated with actions of climbing (up or down), which are experientially very different from the four schemas identified in this study. There is historical support for this conjecture (Matos, 1990), as Chinese or Egyptian mathematics, for example, did not use angles as Greek mathematics did, but were very proficient in calculating slopes. The

trigonometric angle may be a cluster of several models relating to experiences involving vibrations and very complex metaphoric mathematical models such as the number line.

A second area refers to the consequences of the perspective for an epistemology of mathematics. Social influences on the development of mathematics have been pointed out for some time. The role of external social factors (Struik, 1942) and of internal social factors (Restivo, 1990; Wilder, 1981) in shaping mathematics has been acknowledged. The study of the influences of social factors on the development of mathematics has been moving from a *sociology of mathematics* to a *sociology of mathematical knowledge* (Bloor, 1991), the latter claiming that social factors play a key role in shaping mathematical knowledge itself. What would a comparable epistemology of mathematics look like that stemmed from an embodied approach to cognition? It would certainly depart from platonist approaches and share broad areas of agreement with the work of Lakatos (1976) and of Bloor (1991). This area needs further development. It is a crucial area for mathematics education, as epistemologies of mathematics are vital factors in learning and teaching school mathematics.

A third area is also an epistemological problem. It relates to the nature of mathematics thinking. Mathematical argumentation at higher levels carries a logical compulsion; that is, the truthfulness of a mathematical theorem appears inevitable to those who go through its proof. After understanding the proof, one sees the mathematical conclusion as following necessarily from the premises. Rejecting a platonist approach to mathematical ideas requires explaining such a necessity. Although some argue that it is an artifact of social environments (Bloor, 1991), the origins of this cognitive compulsion still need to be shown. Johnson and Lakoff (Johnson, 1987; Lakoff & Johnson, 1999) have argued that logic too is embodied, but better empirical evidence for this claim is needed. Mathematics education seems to be a good area in which the claim could be investigated.

A fourth area relates to mathematization. Mathematics can be thought of as composed of both concepts and processes, although the distinction is blurry. It appears that an embodied approach to cognition can successfully explain the formation of categories of objects in general, and of mathematical objects in particular. But it still has a long way to go before it can come to grips with the processes used in advanced mathematical thinking. How can it account for processes like generalizing, representing, visualizing, conjecturing, inducing, analyzing, synthesizing, abstracting, or formalizing (Dreyfus, 1989). What is the role of metaphors in these processes? Again, mathematics education may be of help by providing a fertile ground for research.

This study has augmented the van Hiele theory, by enriching the characterization of the first two levels, and has demonstrated the value of categorization theory in understanding how our comprehension of mathematics objects is rooted in basic human attributes pertaining to the material and social conditions of human life. The focus was on the category of angles, but this investigation has implications for research into and the teaching of mathematical objects across the school curriculum. The embodiment of mathematical ideas by the material world, including our bodies, needs greater emphasis in all facets of mathematics education.

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APPENDICES

Appendix A

Different Types of Angle Used in School Mathematics

Mathematics educators and mathematicians have characterized the types of angle that are used in mathematics and in school mathematics. At issue is the fact that curricula make use of distinct mathematical definitions of *angle*, defined over different domains, with different properties, and making use of different representations. At least three mathematics educators discuss different types of angles:

- Freudenthal (1973) characterizes four types of angles from a mathematical standpoint. The elementary geometry angle, which is the angle of a non-ordered pair of rays in the non-oriented plane, determined between 0° and 180° ; the goniometry angle, which is an ordered pair of rays in the oriented plane determined modulo 2π ; the analytic geometry angle, whose difference from the previous ones is that it involves lines instead of rays; and the space geometry angle, which is a non-ordered pair of lines in the non-oriented plane between 0° and 90° .

- Close (1982) divides mathematical angles into *static* and *dynamic*. A static angle is the portion of a plane included between two straight lines in the plane that meet in a point; a dynamic angle is a plane rotation necessary to bring one side to the position of the other.

- Saxon (1985, p. 51) seems to be the only high school book that contrasts four definitions of angle: (1) An angle can be defined to be the geometric figure formed by

two rays that have a common endpoint. The measure of the angle is the measure of the opening between the rays. (2) An angle is the region bounded by two radii and an arc of a circle. The measure of the angle is the ratio of the length of the arc to the length of the radius. (3) An angle is the difference in direction of two intersecting lines. (4) An angle is the rotation of a ray about its endpoint.

These classifications do not exhaust the distinct explicit and implicit definitions of angle in school mathematics. Below I discuss seven types.

A. *Set of two rays.* An angle is defined as a non-ordered set of two rays with the same origin. To measure this type of angle is to evaluate the inclination that the two rays have relative to each other. The outcome, if measured in degrees, is a real number in the interval $]0, 180[$. This is Euclid's (and Hilbert's) definition, and it is traditionally used in the United States and in some other countries. An angle is associated with sides of "infinite length." This is what Freudenthal (1973) calls the elementary geometry angle, and it fits Saxon's first definition (1985).

This definition is very well adapted to an Euclidean or Hilbertian framework, but it conflicts with other areas of mathematics and sciences. It does not encompass angles greater than the straight angle. In fact, neither Hilbert nor Euclid considered the zero angle or the straight angle. Although this definition works well with convex plane figures, it causes problems when analyzing concave polygons, and also in the theorem of center and periphery angles in the circle (Freudenthal, 1973).

B. *Portion of a plane.* An angle is defined as one of the parts of a plane limited by two rays with the same origin. To measure this type of angle involves a comparison between two "infinite areas," and its outcome is a real number in the interval $]0, 360[$. This is what Close (1982) calls the static angle. This approach was partially present in Euclid's work (Freudenthal, 1973). It was revived by the modern math movement and is currently used in Europe, Canada, and Israel (Hershkowitz & Vinner, 1984). Although this definition is more flexible in allowing angles greater than a straight angle and can

consequently be used to discuss internal angles of concave figures, it still cannot accommodate angles produced by more than one turn, nor can it accommodate directed angles.

C. *Modular external rotation*. This angle is associated with a rotation in the plane. Mathematically these angles may be defined as the set of isometries on a plane that preserve orientation. To measure this angle is to measure the amount of rotation factored by whole turns, and its value is contained in the interval $[0, 2\pi[$. Sometimes this is a directed angle because it is necessary to know the direction of turn. Close (1982) identifies this approach as a dynamic angle.

D. *Continuous external rotation*. In some cases that make use of rotations, the number of whole turns is not factored out. This is the case in such real-life situations as describing the position of a screw. The amplitude of these angles is contained in the interval $]-\infty, +\infty[$. Angles in the previous definition are topologically equivalent to a circle, whereas in this one they are equivalent to a line.

E. *Intrinsic rotation*. In the particular case of the Logo environment, an angle can be associated with a specific turtle turn. The student usually types in a number that he or she expects will produce a specific turtle turn. This special type of angle may be defined as the rotation needed to change the direction of a straight path. It takes values in the interval $]-\infty, +\infty[$, and as it makes a difference whether one is talking of a right or a left turn, this type of angle is also oriented. This is the kind of rotation that sailors identify as backing and veering. This type of angle relates to differences with orientation, of heading, whereas the first type of angle, in Definition A, relates to differences between the inclination of two rays.

Definition E is distinct from Definitions C and D from both a mathematical and a psychological point of view. With angles other than 90° , the line segments that are drawn on the screen may not match the angle of turning. For example, the sequence FD 30 LT 120 FD 30 does not draw a representation of a 120° angle, but rather a 60° angle. If BK

30 LT 120 FD 30 is used, the angle is indeed 120° (Figure 19).

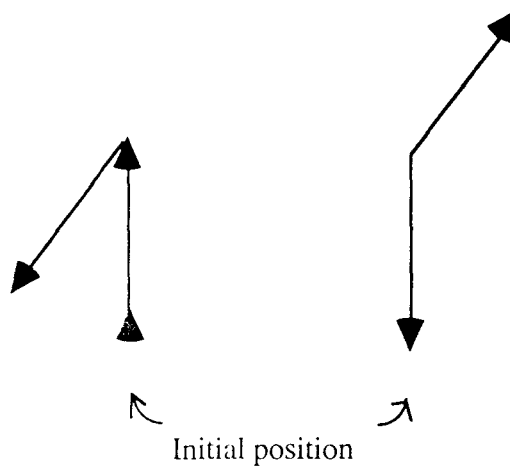


Figure 19. Comparing commands in Logo:

FD 30 LT 120 FD 30BK 30 LT 120 FD 30

In other words, if one is walking along a path composed of distinct line segments, the angle associated with one's turns may not be the same as the angle formed by adjacent line segments on the path.

Definition E allows the construction of a geometry with a different set of priorities than Euclidean geometry. Euclidean geometry focuses on the study of static elements, whereas Turtle Geometry is concerned with dynamic aspects of geometry. It is also an intrinsic geometry, and, for example, one of its natural consequences is to consider the total turn theorem, which states that the total turn on a closed path that does not cross itself is 360° . The theorem about the sum of the internal angles of a polygon, although more complex, is much more natural in terms of Euclidean geometry; that is, if figures are observed from an external viewpoint. We may say, in this case, that computers are enabling us to look at mathematics itself (angles in this specific case) from a different perspective.

F. *Slope*. In analytic geometry the inclination of a line relative to a system of

coordinates is evaluated by the slope. This system needs a fixed system of coordinates, and so instead of evaluating inclinations of pairs of lines, it only needs to evaluate the inclination of each line against the system of coordinates. Crosswhite, Hawkinson, and Sachs (1988), for example, associate the slope with a ratio between the “rise,” which means the change in y , and the “run,” which is the change in x (p. 43). The “inclination” of a line is defined as the angle between the line and the x -axis. The measure of inclination is expressed by a ratio of lengths (a tangent) and does not require any of the previous notions of angle. In fact, in affine geometry there is no need for the concept of angle as developed in Euclid’s *Elements* (Heath, 1956), because differences in direction are evaluated by a dot product function obtained by a product of matrices. Inclination may vary from $]-\infty, +\infty[$, which corresponds to angles $[0, \pi/2[$ or $] \pi/2, \pi[$. Although this definition can accommodate perpendicular lines, it does not handle vertical lines elegantly.

Definition F loses the association between the additivity of angles and the additivity of angle measure. Nevertheless, historically, it is one of the earliest notions of angle. This may be so because it is naturally associated with our perception of inclination; that is, we seem to think about the inclination of each line against the background of a horizontal and vertical orientation.

G Trigonometric angle. Here angles are identified with the length of the arc of a circle of unit radius. This is an oriented angle that allows a distinction between $\pi/2$ and $3\pi/2$. In high school, this type of angle is usually recalled as an internal angle of a triangle and is later extended over real (which later become complex) arguments of trigonometric functions. In Crosswhite et al. (1988), for instance, angles are not even drawn when discussing circular functions. At most, there is the suggestion of a rotation of one point around a circle of radius 1 (see pp. 186-187). The same suggestion of a rotation is made by Saxon (1985), which discusses further, for example, the notion of phase angles (p. 191) which are used in electronics and elementary physics. This type of angle

may be identified with the goniometric angles proposed by Freudenthal (1973).

Definitions F and G have a crucial distinction. In the slope, the measure of an angle is a ratio of lengths, hence it has no dimensions. In the trigonometric angle, the measure is a length and consequently should have a unit. This is especially important in dimensional analysis, where there seems to be no agreement over the characterization of the measure of angles (Krantz, Luce, Suppes, & Tversky, 1971, p. 455).

Each of these definitions has its specific domain of application, and it may be conjectured that mathematicians who need to work with these concepts are able to jump pragmatically from one to the other. Children may develop some of these different types of angle, but it is not clear how they see the interactions among them.

Appendix B

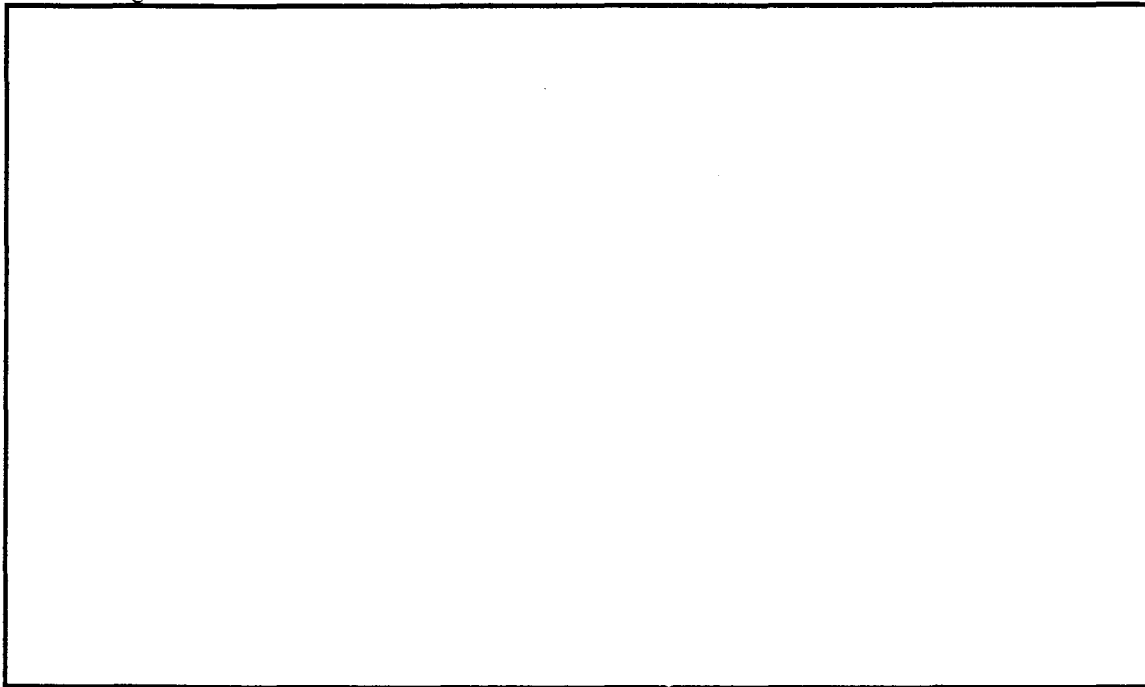
Tests

The fourth-grade test was composed of Questions 1 through 9. The fifth-grade test was composed of Questions 1 through 6, 10 and 11.

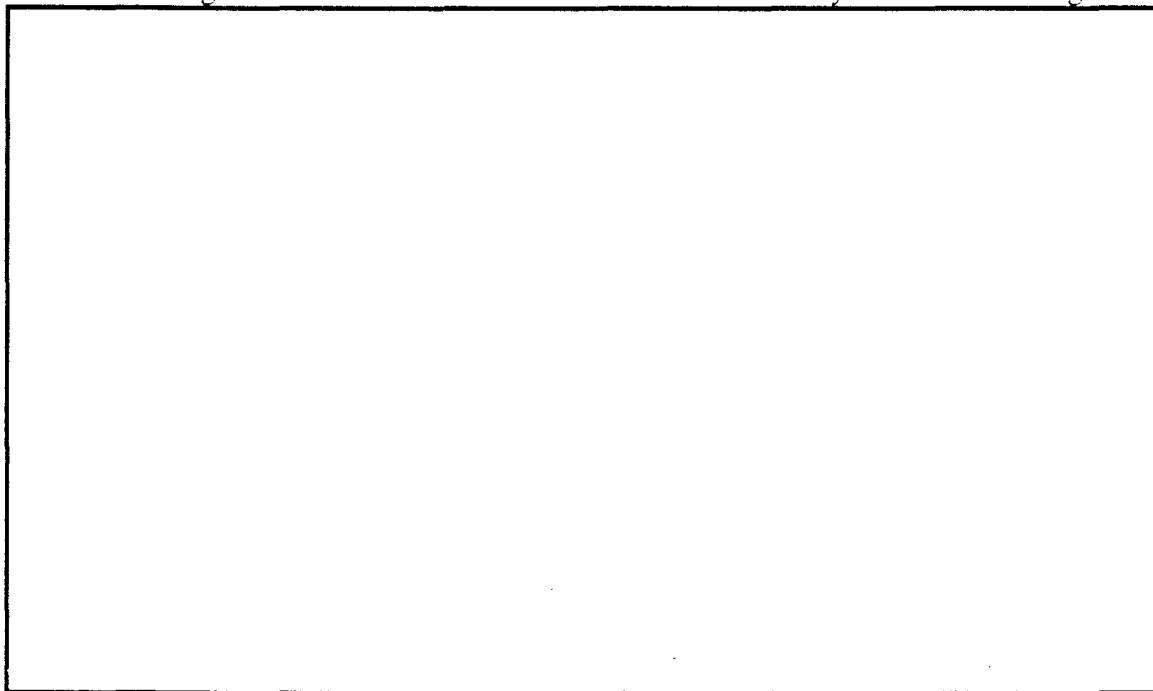
Name: _____

Part 1.

1. Draw an angle in the box below.



2. Draw another angle in the box below that is different in some way from the first angle.



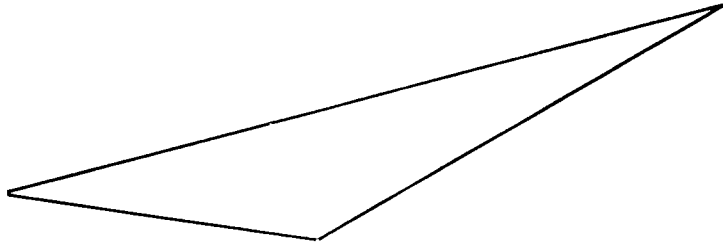
3. How are the angles different?

4. How are the angles alike?

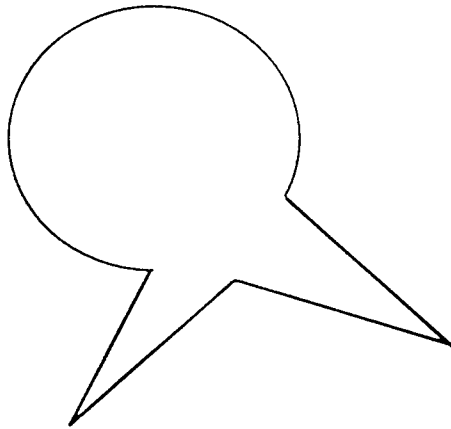
5. How many different angles could you draw?

6. Put an A at the point of each angle in the figures below:

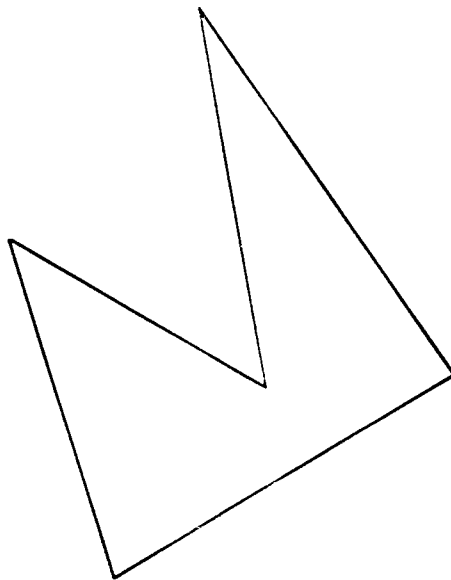
a



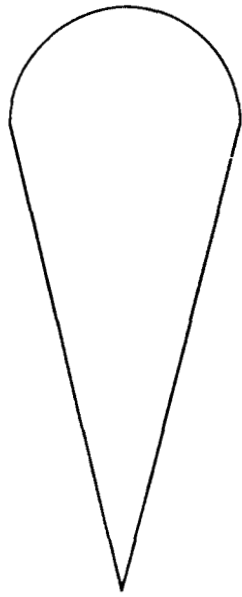
b



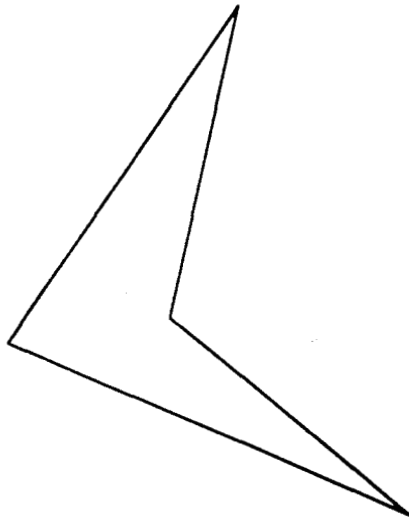
c



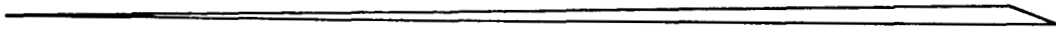
d



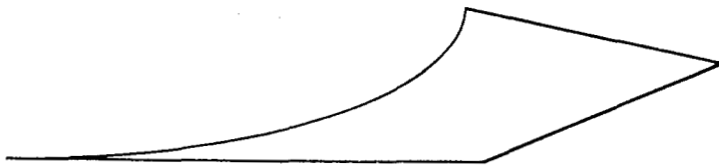
e



f



g



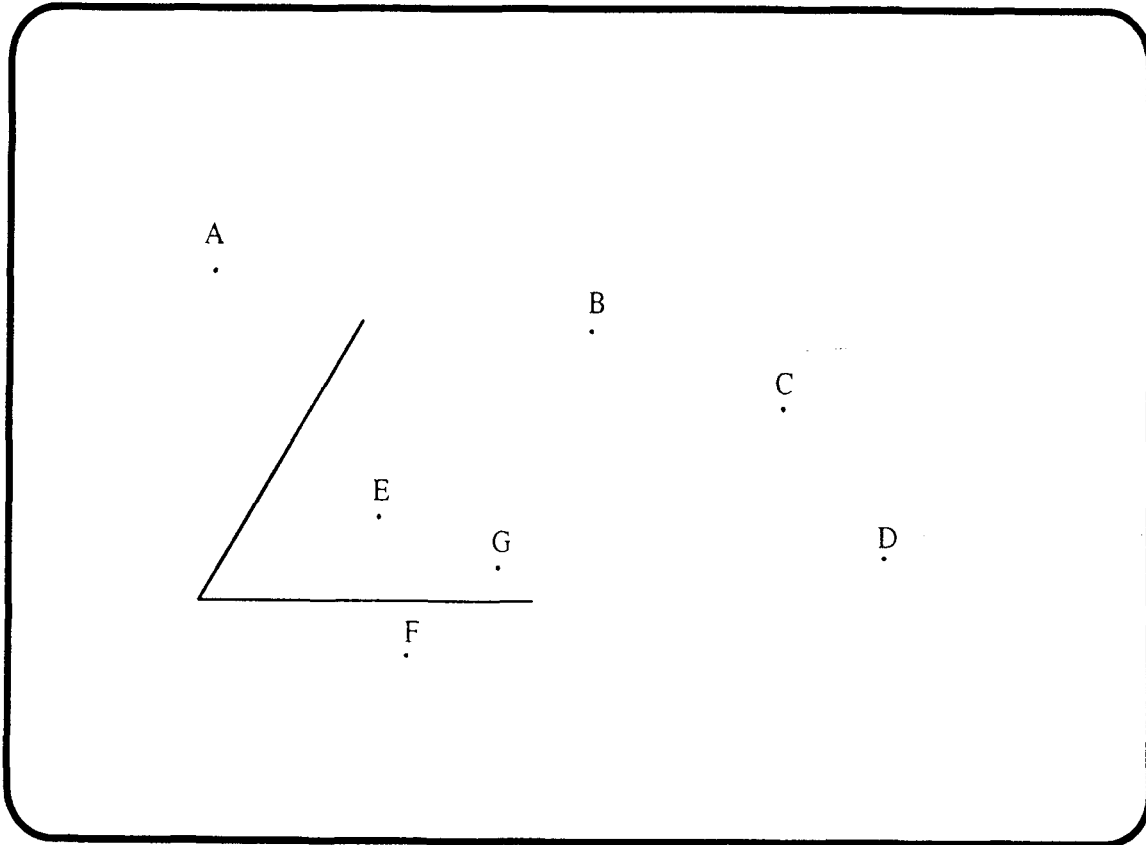
Name: _____

Part 2.(Grade 4)

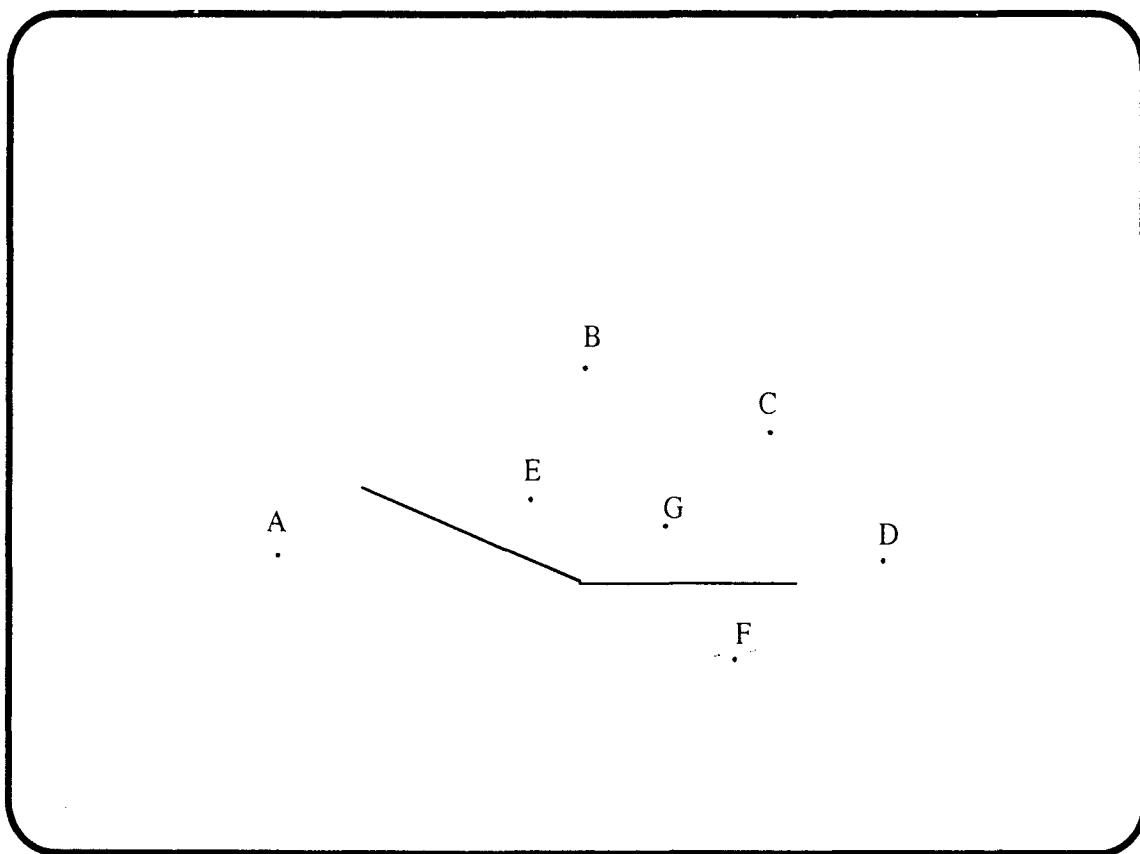
7 – Circle the points that are inside the angle below.



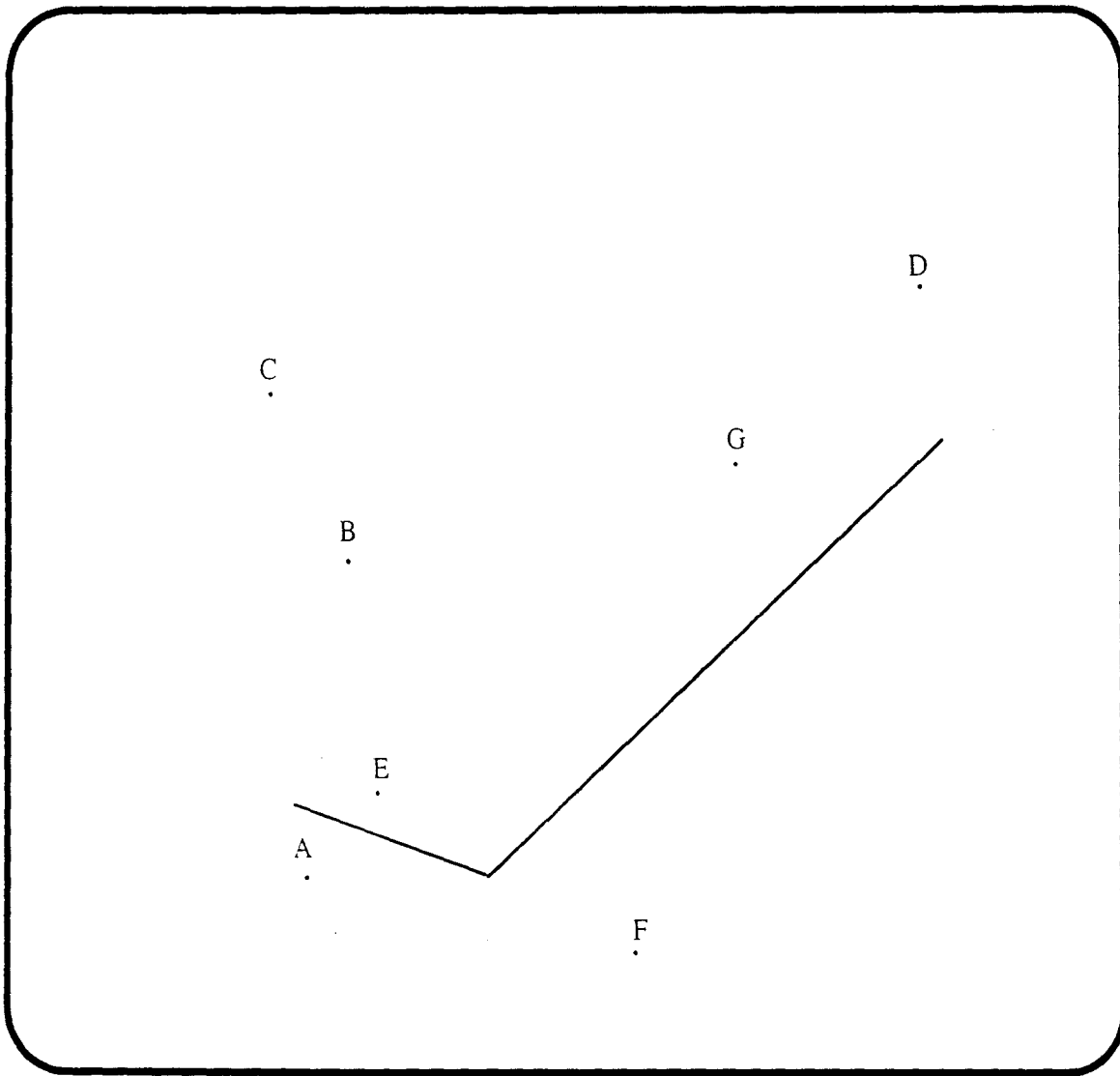
Example: Circle the points that are inside the angle below



8 – Circle the points that are inside the angle below.



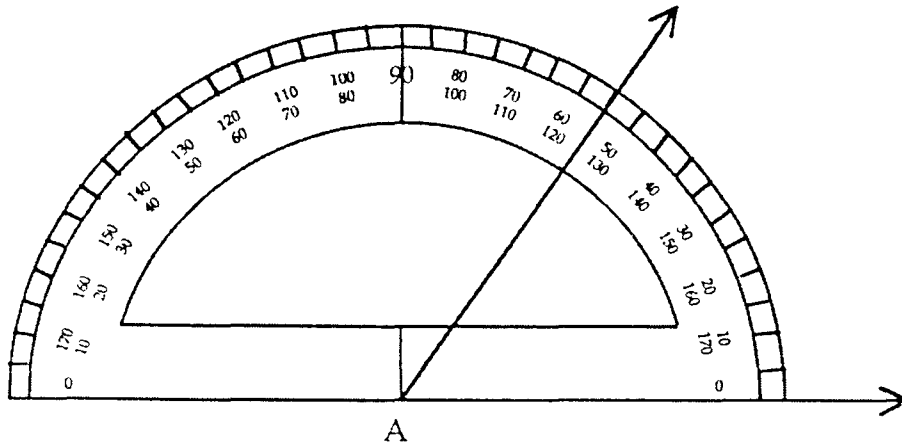
9 – Circle the points that are inside the angle below.



Name: _____

Part 2.(Grade 5)

10. How many degrees in angle A?



11. You **can't** draw a triangle that has two of these angles!

What kind of angle is that? _____

Draw one.

Appendix C

Tasks

Task V1: Describe an angle

Purpose: To find attributes that students use when describing and comparing angles

Area: Verbal

Materials: None

Description:

The researcher asks:

1. "Suppose you were talking to a friend over the phone who had never learned about angles, and you wanted to explain him (or her) what an angle is. What would you say?"
2. "Can you give an example of something that is not an angle, but that a younger kid would think is?"
3. "Now I am going to ask you about the relationships between angles, corners, turns, and triangles. How are corners different from angles? How are corners the same as angles? How are turns different from angles? How are turns the same as angles? How are angles different from triangles? How are angles the same as triangles?"
4. "What was most difficult to learn about angles?"

Task V2: Verbalizing turns

Purpose: To find attributes that students use when describing and comparing turns

Area: Verbal

Materials: Spinner

Description:

1. Enact the performance of a full turn with the spinner.
2. Ask the student to perform a turn with the spinner.
3. Ask:

"How would you describe this turn to a friend over the phone?"
4. Say:

“Can you make a turn that is very different from the one you just did? How is it different?”

5. Perform a quarter turn with the spinner. Ask:

“How would you describe this turn to a friend over the phone?”

Task B3: Drawing turns

Purpose: Identify characteristics of students’ drawings of turns

Area: Drawing

Materials: Spinner, protractor, paper, D-Stix, pencil, ruler.

Description:

Ask the student to draw the quarter turn used in step 5 of Task V2.

Task B4: Drawing angles

Purpose: Identify characteristics of students’ drawings of angles

Area: Drawing

Materials: Paper, pencil, ruler, protractor, D-Stix

Description:

Use students’ answers to Questions 1 through 5 from the test to phrase the following questions:

1. Ask:

“Is there some other way to make angles different, other than just turning them?”

(Or “making sides longer” or “shorter.” depending on the attributes in students’ answers to the test questions.)

2. Referring to the student’s answer to Question 4, ask:

“Can you tell me the ways in which all angles are alike?”

3. If the student has not mentioned it yet, ask:

“Does it make a difference whether your sides are larger or smaller? Does it make a difference whether you draw it as you did, or like this?” (Rotate his or her

drawing 180° .)

Note: Ask about any marks the student used to indicate interior angle, vertex, or continuation of rays.

Task I4: Identification of angles

Purpose: To find attributes that students use when identifying angles in figures

Area: Visual

Materials: Protractor, ruler, pencil, a set of configurations (Addend 1), and a set of pairs of angles (Addend 2)

Description:

For all students:

1. Use students' answers to question 6 on the test and ask:

“Why is there an angle in here? Why there is not an angle in here?”

Focus specially on the angles with curved sides, and on concave figures.

Only for students who included curved sides:

2. Give the student the configurations in Addend 1 in random order.

Ask the student to select the figures that show only one angle.

3. Ask: “Can you always tell if an angle is greater than another?”

4. Present students successively with the following pairs of angles (Addend 2):

15, 19; 13, 14; 1, 12; 3, 13; 29, 29.

For each pair ask student which angle is greater, or if they are equal.

In pair (29, 29), probe the students' answer by asking why.

For all students:

5. Hold configuration 5 (Addend 1) and ask:

“Can you pick an angle that is larger than this angle?”

“Can you pick an angle that is smaller than this angle?”

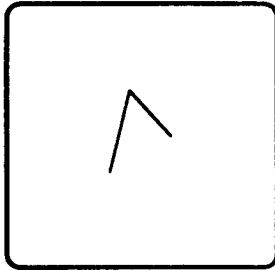
6. Ask:

“Is there an angle that is smaller than all the angles?”

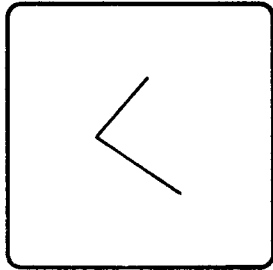
“Is there an angle that is larger than all the angles?”

Addend 1 to Task I4 – Configurations for Step 2

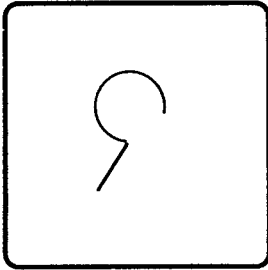
(The lengths in the figures are 25% of the actual size)



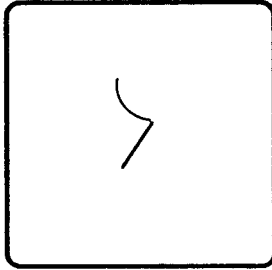
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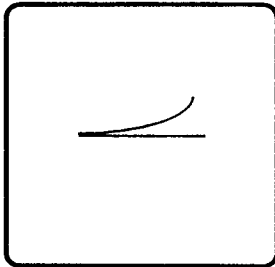
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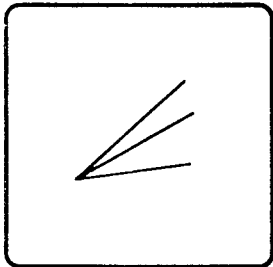
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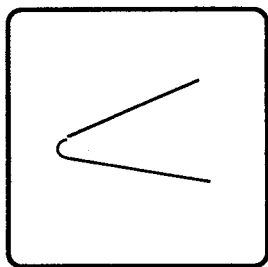
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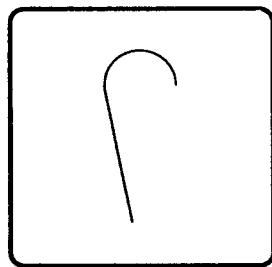
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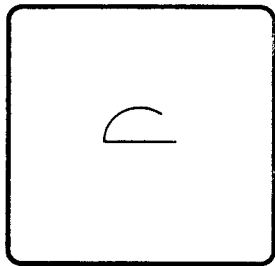
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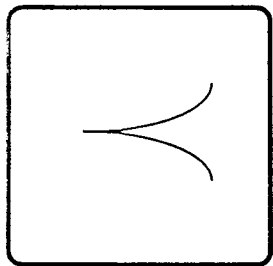
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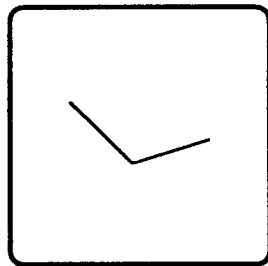
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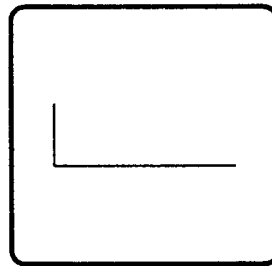
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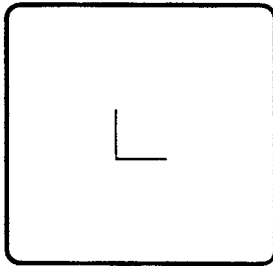
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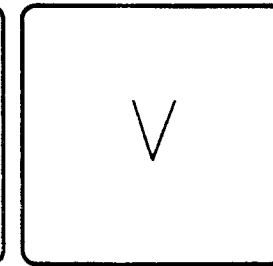
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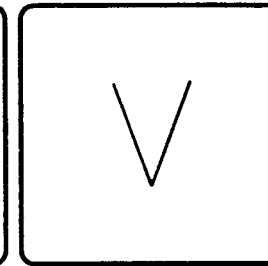
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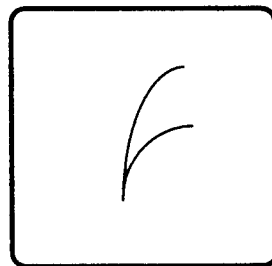
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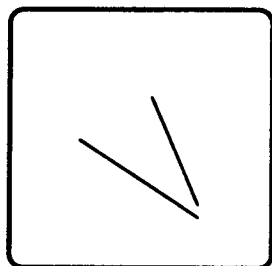
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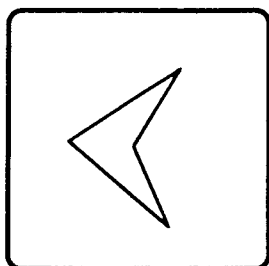
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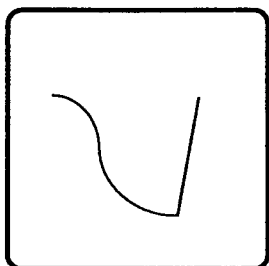
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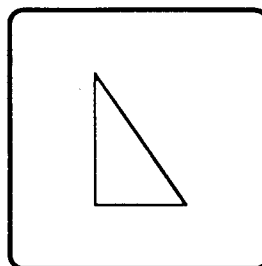
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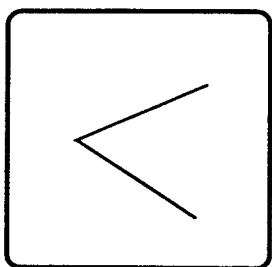
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27



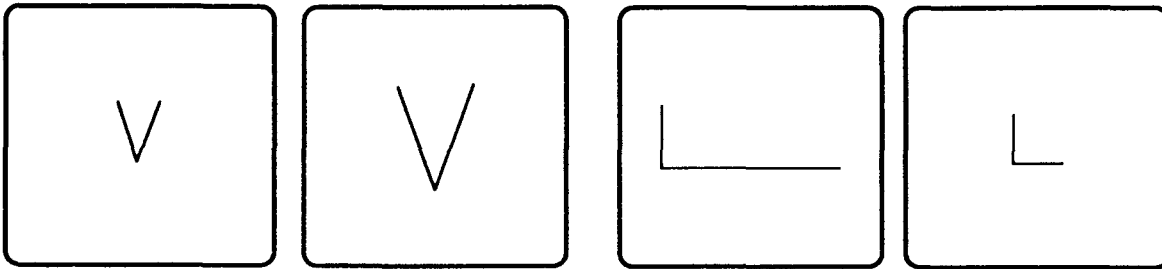
28



29

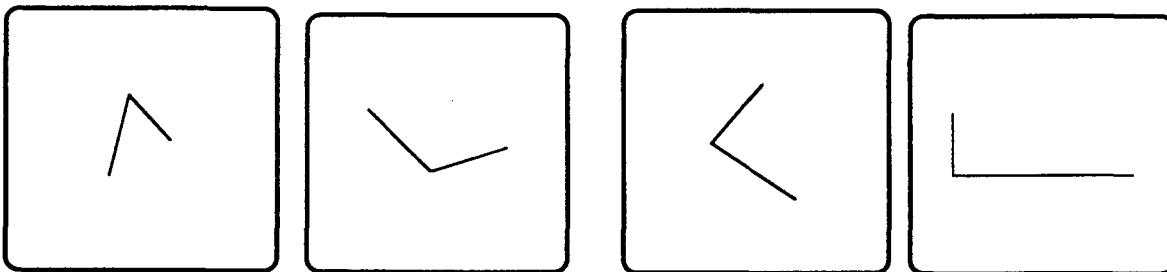
Addend 2 to Task I4 – Pairs of angles for Step 4

(The lengths in the figures are 25% of the actual size)



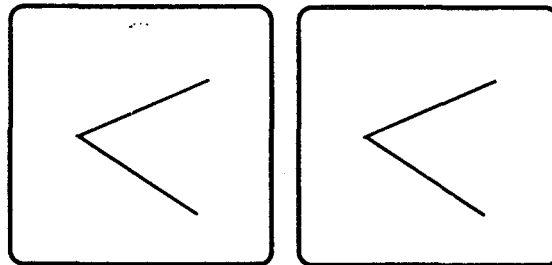
Angles 15 and 19

Angles 13 and 14



Angles 1 and 12

Angles 3 and 13



Two equal angles, Angle 29 29

Task I2: Angles in solids

Purpose: To find attributes that students use when identifying angles in solids

Area: Visual

Materials: A cube, a pyramid, a cylinder, a cone, and a sphere

Description:

Show each material successively to the student and ask him or her to show an angle.

If the student seems to be referring to a solid angle in the pyramid or in the cone, ask:

“How many angles are there in this shape?”

Task I5: Components of Angles

Purpose: Detect students’ concepts of the components of angles

Area: Visual

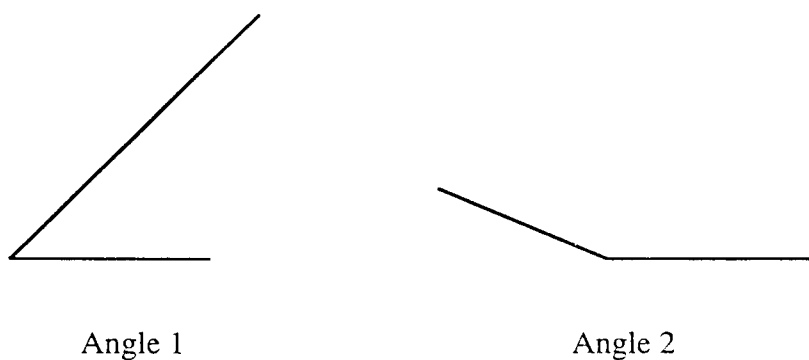
Materials: Set of angles: Angles 1 and 2 shown on the same sheet of paper; Angle 3 drawn so as the sides extend to the edges of the paper

Description:

For all students

1. Show student Angles 1 and 2 (shown below reduced 50%). Ask:

“How does the vertex of the larger angle differ from the vertex of the smaller angle?”



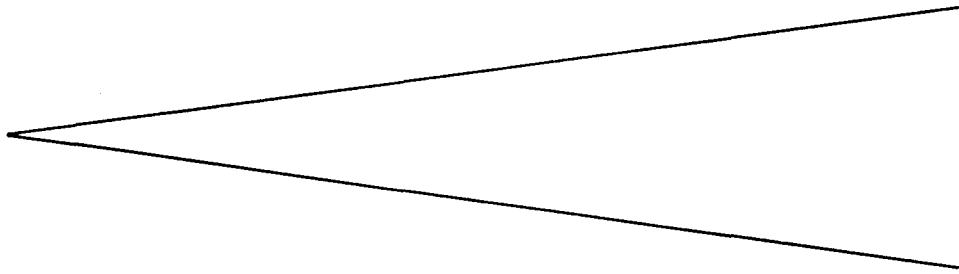
2. Ask:

“How would one of those vertices look when seen through a microscope?”

3. Show Angle 3. Ask:

“There was not enough room on the paper to draw completely the sides of this

angle. How long do you think they could go on? Would that change the angle?"



Angle 3

For fifth-grade students

4. Ask Questions 7, 8, and 9 from the fourth-grade version of the test.

Task C4: Comparing turns

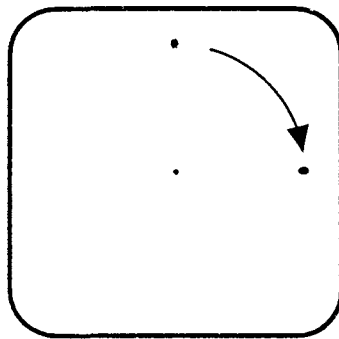
Purpose: Detect attributes that students use to compare turns

Area: Visual

Materials: Spinner, set of drawings

Description:

1. Show the student turn 1. Explain what is the starting position, the ending position, and the purpose of the circle arrow.



Turn 1 (lengths reduced 25%)

2. Present the student successively with the following pairs of turns (Addend 1):

11, 13; 13, 14; 7, 8; 3, 6.

For each pair of turns ask student which turn turns more. In drawing 8 explain that it represents more than one turn. In the case of the pair (3, 6) explain that the points are in similar positions, the only difference being the direction of turn.

4. Pick two exemplars of turn 20 (Addend 1) and say:

“These drawings are the same, but this one turned very fast and this one very slowly. Which one turned more?”

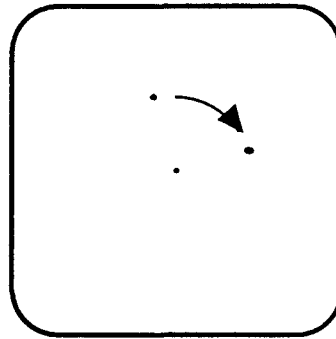
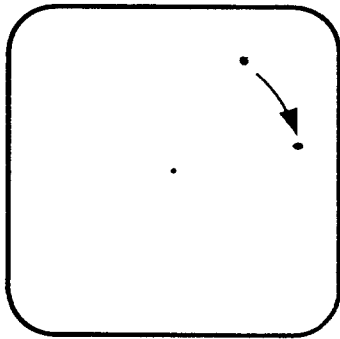
5. Ask:

“Can you show me a turn smaller than all the turns?”

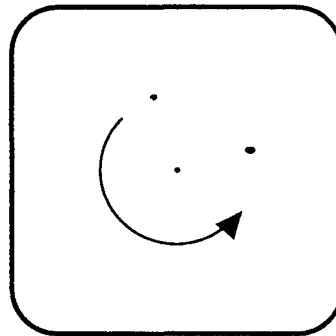
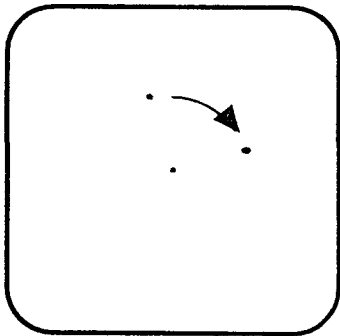
“Can you show me a turn larger than all the turns?”

Addend 1 for Task C4 – Pairs of drawings of turns

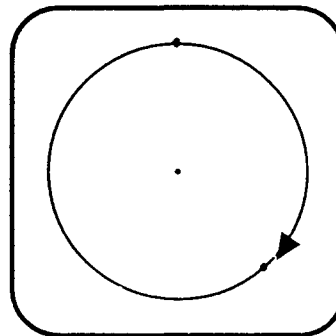
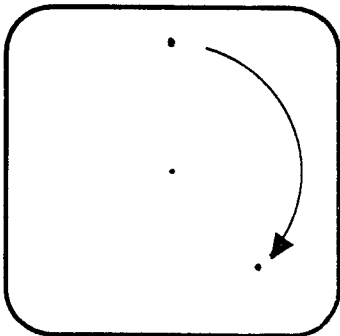
(The lengths in the figures are 25% of the actual size)



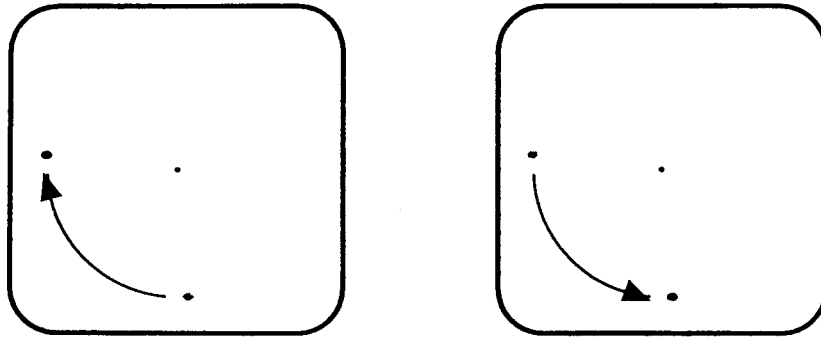
Turns 11 and 13



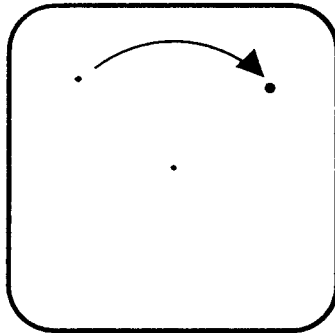
Turns 13 and 14



Turns 7 and 8



Turn 3 and 6



Turn 20 (two exemplars)

Task D2: Dividing angles

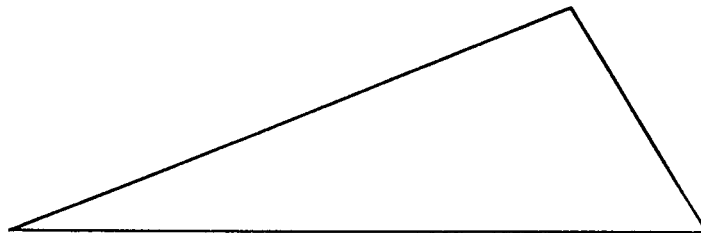
Purpose: Detect students' understanding of the partition of angles

Area: Visual, applied

Materials: Several cutouts of triangles, spinner

Description:

1. The researcher picks the triangle below and says:

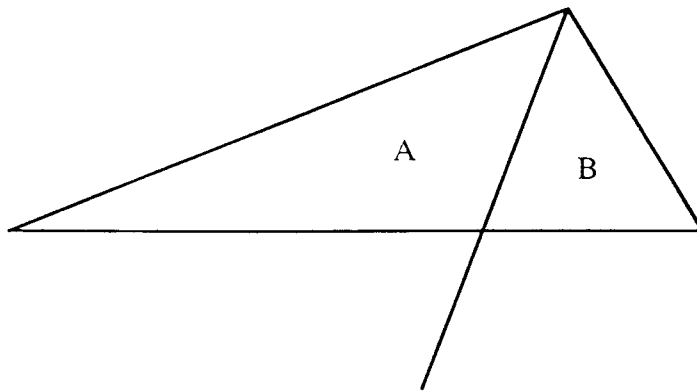


“This triangle has three angles, and let us choose this angle. [Points to the angle opposite the base.]

“Can you divide this angle into two equal angles? Show me how.”

2. If the student successfully divides the angle, ask him or her:

“Some students told me that these angles would not be equal, because this area is different from this area. [Points to Areas A and B in the figure below.] What do you think?”



Task P3: Mystery angle

Purpose: Detect students' ability to identify an angle given its properties

Area: Logical

Materials: Set of cards with properties, set of angles, set of cards with the names: right angles, acute angle, obtuse angle

Description:

The researcher should warn the student that some of the questions he is about to ask have more than one correct answer or no answer at all.

1. The researcher tells the student that they are going to play a game. The researcher has an angle in mind, and he is going to show the student one card that gives some information about this angle. The student then will choose a card with the possible angle.

The researcher shows the student a card containing one condition and asks the student to choose an angle that satisfies it.

Conditions used:

Two of these angles are less than a right angle.

This angle can be found in a square.

Two of these angles equal a full turn.

Triangles cannot have two of these angles.

This angle cannot be found in a square.

This angle measures 130 degrees.

For each of these questions, the researcher should ask after the student has answered:

“Can you find some other angle?”

2. The researcher tells the student that they are going to play a different game now. The researcher has a whole set of angles in mind, and he is going to show the student one card that gives some information about this set of angles. The student then is supposed to guess what kind of angle it could be.

The researcher shows the student a card containing one condition and asks the student to choose a card with the type of angle that satisfies the condition.

Conditions used:

Rectangles have four of these angles.

All triangles have at least one of these.

Any two of these angles add up to more than a half turn.

Two of these angles may add up to be an obtuse angle.

These are not right angles.

If you cut any of these angles in half you get two acute angles.

These angles have two sides.

These angles are less than a right angle.

For each of these questions, the researcher should ask the following, after the student has

answered:

“Can you find some other type of angle?”

Note: If the student does not recall clearly the meaning of the terms *obtuse* and *acute*, the researcher will briefly give some examples.

Task P4: Solve a problem

Purpose: Detect students’ ability to solve problems requiring the use of geometrical properties

Area: Applied

Materials: Rectangle made of D-Stix, a D-Stix bar

Description:

1. The researcher shows the rectangle to the student and says:

“If you add up all the internal angles of this rectangle, how much are you going to get?”

2. If the student gives the correct answer, the researcher should tilt the rectangle so that there is now a parallelogram with no right angles and ask:

“Did the angles change?”

“How much do the angles add up to now?”

If the student does not give the correct answer, the researcher will remind him or her that the angles add up to 360° and proceed to ask the questions above.

3. If the student seems unsure of how to answer the question in step 2, the researcher should say:

“I am going to give you a hint. Notice that we can divide this figure in two, like this:”

The researcher now places the D-Stix bar on the diagonal and asks:

“I am going to place this bar here. How much do the angles add up to?”

Appendix D

Criteria for the van Hiele Levels for the Concept of Angle

Level 1 : In general, the student identifies, characterizes, and operates on angles according to their appearance.

1. Draws angles.
2. Identifies, names, or labels angles in a simple drawing or more complex figure by relying on visual clues rather than properties of the angle. The student may use standard or non-standard language (such as referring to angles as corners).
3. Includes irrelevant properties or relationships when describing angles, such as length of ray.
4. Exclude relevant properties or relationships when characterizing angles, such as straightness.
5. Sorts angles on the basis of their appearance as a whole, specifically not having the 90 degree referent, making inconsistent sortings, or sorting by an irrelevant attribute.
6. Analyzes or compares angles (in tasks including, but not limited to, turning angles, congruent angles, complementary angles, or supplementary angles) on a looks-like basis.

Level 2: The student establishes properties of angles and uses properties to solve problems.

1. Analyzes and compares angles in terms of their properties.
2. Identifies relationships among angles within figures.
3. Recalls and uses appropriate vocabulary for relationships, such as when parallel lines are cut by a transversal the corresponding angles are congruent.
4. Describes angles with a litany of properties or insufficient properties rather than necessary and sufficient properties.

5. Is able to decentrate (orient turning relative to a spinner's position rather than to his or her own body's position) in a task to determine angle measure, as indicated by accurate estimates of angle turn and by deciding which way to turn.

6. Accurately estimates angle measure by using known properties (such as right angles measure 90 degrees) or by insightful approaches.

7. Formulates and uses generalizations about properties of angles in problem-solving situations and may use related language (*all, every, none*).

Level 3: The student formulates and uses definitions, gives informal arguments that order previously discovered properties, and follows and gives deductive arguments.

1. Identifies necessary and sufficient properties in the context of a justification.

2. Formulates and uses definitions, (a) explicitly referring to them, (b) accepting equivalent forms of definitions, and (c) accepting new definitions of previously learned concepts.

3. Is able to conceive of an infinite number of angles.

4. Explicitly describes relationships between properties, including sub-class relations.

5. Presents an informal argument or informal proof, justifying the conclusion using logical relationships of properties: orders properties, interrelates several properties, or discovers new properties by deduction.

6. Presents an informal argument or informal proof deductively (implicitly using such logical forms as the chain rule and modus ponens, or explicitly using "if/then," for example).

Appendix E

Topic of Each Observed Lesson

Grade	Teacher	Topic	Materials
4	H	Developing the concept of angle I, II	L2.06, L2.07
4	A	Developing the concept of angle I, II	L2.06, L2.07
5	Z	Measuring angles with wedges	T p. 284, L2.11
		Measuring angles	T pp. 284-285
		Classification of angles	T p. 288, posters
		Measuring angles	T pp. 286-287
		Classification of triangles	T p. 290
5	D	Developing the concept of angle I	L2.06
		Measuring angles with wedges	L2.11
		Constructing a protractor	worksheets, L2.12
		Using a conventional protractor	L2.13
		Comparing angle sums in triangles and quadrilaterals	L2.10
		Special triangles and quadrilaterals	L2.29
		Special triangles and quadrilaterals	L2.29
		Side/angle relationships in triangles	L2.30
		Side/angle relationships in triangles	L2.30

Note. *L* refers to Geometry and Measurement lessons coding (McKillip & Wilson, 1990);
T refers to the textbook (Thoburn, Forbes, & Bechtel, 1982b)