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**Higher-order modulation theory for resonant flow over topography**

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The flow of a fluid over isolated topography in the long wavelength, weakly nonlinear limit is considered. The upstream flow velocity is assumed to be close to a linear long wave velocity of the unforced flow, so that the flow is near resonant. Higher order nonlinear, dispersive and nonlinear-dispersive terms beyond the Korteweg-de Vries approximation are included, so that the flow is governed by a forced extended Korteweg-de Vries equation. Modulation theory solutions for the undular bores generated upstream and downstream of the forcing are found and used to study the influence of the higher-order terms on the resonant flow, which increases for steeper waves. These modulation theory solutions are compared with numerical solutions of the forced extended Korteweg-de Vries equation for the case of surface water waves. Good comparison is obtained between theoretical and numerical solutions, for properties such as the upstream and downstream solitary wave amplitudes and the widths of the bores.

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## I. INTRODUCTION

The waves generated by the flow of a fluid over topography or by a forcing, such as a ship, on the surface of a fluid or submarine within a stratified fluid, is a classical topic in fluid mechanics and wave theory<sup>1,2</sup>. The majority of this classical theory is based on small amplitude, linear waves, for which there exists a number of detailed accounts<sup>3-6</sup>. However, when the speed of the imposed flow or the speed of the forcing is near the speed of a linear wave mode, energy accumulates at the forcing, so that the flow becomes nonlinear with unsteady nonlinear wavetrains propagating upstream and downstream of the forcing. This flow regime is termed resonant, or transcritical, in the terminology of hydraulic theory<sup>6</sup>. Experimental work by Baines<sup>5,7,8</sup> on the flow of a stratified fluid over topography found large amplitude upstream waves when the flow is near resonance. Baines<sup>5</sup> also noted that the upstream wavetrain took the form of an undular bore. These experimental results for a stratified fluid were confirmed by ship tank experiments<sup>9</sup> and in wave tank experiments on the resonant forcing of surface waves by an obstacle<sup>10</sup>. These experimental studies generated interest in theoretical and numerical analyses of resonant flow. In the weakly nonlinear, long wave regime it has been shown by a number of authors that in the resonant, or transcritical, regime the flow is governed by a forced Korteweg-de Vries (KdV) equation, with the forcing due to the topography or the imposed forcing, such as a pressure distribution<sup>10-14</sup>.

As noted, in the resonant regime, undular bores propagate upstream and downstream of the topography or forcing. In general, an undular bore is a modulated periodic wavetrain with solitons at one edge and linear dispersive waves at the other<sup>15-17</sup>. While such modulated wavetrains are generally termed undular bores in fluids applications, the term dispersive shock waves tends to be used in other nonlinear wave applications<sup>17</sup>. This terminology is a reference to an undular bore being a structure linking two different flow levels, in analogy with a shock wave linking two different flow states in compressible flow. An undular bore differs from a shock in that dispersion resolves the initial jump discontinuity between the two levels, while for compressible flow viscosity plays this role. Dispersion then results in an undular bore spreading as it evolves, while a compressible shock does not spread. Cnoidal waves are the nonlinear travelling wave solutions of the KdV equation and are expressed in terms of Jacobian elliptic functions<sup>2</sup>. In the limit in which the modulus  $m$  of the elliptic function approaches unity, the cnoidal wave becomes the KdV soliton solution and in the

limit in which the modulus goes to zero the cnoidal wave becomes a small amplitude, linear dispersive wave<sup>2</sup>. One edge of the KdV undular bore then consists of solitons with  $m = 1$  and the other edge consists of linear waves with  $m = 0$ <sup>15,16</sup>. However, in general, the bore resulting from resonant flow has a variation from this general structure as the trailing edge of the bore can be fixed at the forcing<sup>13,18</sup>. For instance, near exact resonance, the upstream propagating bore is not a full undular bore, but a partial bore with a minimum modulus  $m_0 > 0$  at the forcing, where the bore is generated, and  $m = 1$  at its leading edge<sup>18</sup>. A full range bore is not generated as part of this bore would propagate downstream. As the minimum modulus  $m_0$ , which is related to the wavenumber of the modulated wave, is close to unity in this exact resonance case, the upstream undular bore can be approximated by a train of solitons, which has been a useful approximation<sup>10,13,19</sup>. However, away from exact resonance, particularly as the flow becomes subcritical in hydraulic terminology, the upstream bore becomes detached from the forcing and propagates upstream. It is then a full undular bore with linear waves at its trailing edge, so that the train of solitons approximation ceases to be valid<sup>18</sup>. The soliton approximation is less useful for the downstream propagating bore as it is a full undular bore for most of the resonant regime<sup>13,18</sup>. This issue of the different flow regimes in which the upstream bore is partial or full will be taken up in detail in this work.

The solution for resonant flow in the weakly nonlinear, long wave regime has then been fully developed in terms of the undular bore solution of the KdV equation and its generalisations. Whitham developed modulation theory to describe slowly varying modulated wavetrains<sup>2,20,21</sup>. In particular, he derived modulation equations for the single phase cnoidal wave solution of the KdV equation<sup>2,20</sup>. These modulation equations form a hyperbolic system for the amplitude, wavenumber and mean height of a modulated cnoidal wave. A particular solution of the modulation equations is a simple wave solution, which corresponds to an undular bore<sup>15,16</sup>. This weakly nonlinear, long wave theory based on the KdV equation has been successful in describing resonant flow. However, there is the question of the influence of higher order corrections to the KdV approximation on the solution for resonant flow, particularly in terms of relating these theoretical solutions to experimental results<sup>6,10</sup>. Lamb and Yan<sup>22</sup> compared numerical solutions of the equations for internal waves in the Boussinesq approximation with solutions of the KdV equation and the extended Korteweg-de Vries (eKdV) equation with the next higher order nonlinear, dispersive and nonlinear-dispersive

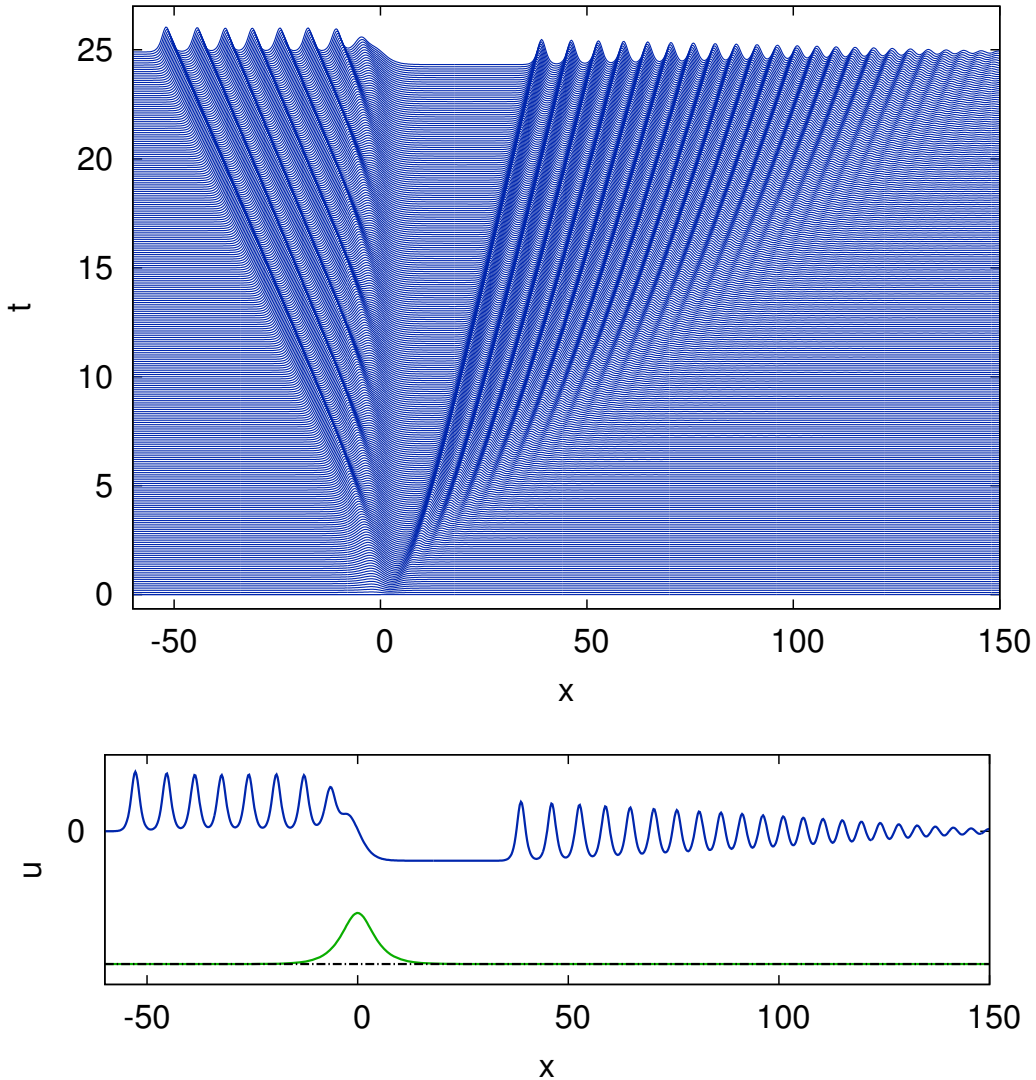


FIG. 1. Solution of the forced eKdV equation (1). Shown is a perspective view in the  $x - t$  plane (top) and the surface profile  $u$  and bathymetry  $G$  versus  $x$  at  $t = 25$  (bottom). The numerical solution of (1) with the initial condition  $u = 0$  is shown. The parameters are  $\alpha = 0.15$  and  $\Delta = 0$ .

terms included. The initial condition was a depression which developed into an undular bore, so that this work has connections with resonant flow over topography. It was found that the inclusion of these higher order terms resulted in better agreement with numerical solutions, except when the waves are of high amplitude, as would be expected. Various studies of resonant flow in the weakly nonlinear, long wave limit have included higher order corrections to the KdV equation under a number of different approximations. Resonant flow governed by the KdV equation with a third order nonlinearity correction, the Gardner

equation, has been studied based on extended modulation equations<sup>23</sup>. Resonant flow based on this eKdV equation with third order nonlinearity, the Gardner equation, was also studied numerically and using hydraulic theory<sup>24</sup>. Finally, a complete description of resonant flow as governed by the Gardner equation has been given<sup>26</sup> as the Gardner equation is integrable and its full Whitham modulation equations can be derived, from which its undular bore solution can be found<sup>27</sup>. A study of fully nonlinear resonant flow was based on the Su-Gardner system<sup>29,30</sup>. This system results from assuming a long wave approximation of the water wave equations, but with no small amplitude expansion in the wave amplitude, so that nonlinearity is included exactly<sup>31</sup>. This work confirmed the qualitative predictions of KdV theory, even for finite amplitude waves.

In the present work the resonant flow of a fluid over topography will be considered in the weakly nonlinear, long wave limit. The next higher order nonlinear, dispersive and nonlinear-dispersive corrections to the KdV approximation will be included, so that the flow is governed by a forced eKdV equation. As for resonant flow governed by the forced KdV equation, the forcing generates undular bores which propagate upstream and downstream of it. This is balanced by a flat depression downstream of the forcing to which the downstream bore is attached, so that the total flow consists of a bore upstream of the forcing and a flat depression downstream, followed by the downstream bore. Solutions for the upstream and downstream flows are derived from the Whitham modulation equations for the eKdV equation. These modulation equations are found via an approximate transformation which transforms the eKdV equation to the KdV equation<sup>33,34</sup>. This transformation is approximate in that it does not transform the eKdV to the KdV equation exactly, but the error is of higher order than the eKdV expansion. This transformation also means that the modulation equations for the eKdV equation can be found from those for the KdV equation. As discussed above, the upstream bore is either a full or partial undular bore<sup>18</sup>. Unless the flow is sufficiently supercritical, part of the trailing edge of a full bore would flow downstream of the forcing, which is not observed<sup>13</sup>. This is resolved by making the upstream bore a partial undular bore, that is a full bore which is terminated at the forcing<sup>18</sup>. Similarly, if the flow is not sufficiently subcritical, part of a full downstream bore would propagate upstream. As for the upstream bore, this is resolved by making it a partial bore in this case, so that it is attached to the forcing. In the case of a partial downstream bore, there is no downstream depression. In §II the higher order modulation theory for the forced eKdV equation is developed. In

§III the results of the higher order modulation theory are compared with numerical solutions and excellent agreement is found. The eKdV equation is also used to quantify the effect of the higher order nonlinear, dispersive and nonlinear-dispersive terms on resonant flow. The effect of only certain of these higher order terms has been studied in the past<sup>23,24,26</sup>. In §IV conclusions are given.

## II. HIGHER-ORDER MODULATION THEORY

### A. The forced extended Korteweg-de Vries equation

Let us consider the waves generated by the flow of a stratified flow over an isolated topographic feature<sup>13,23</sup> or by a pressure distribution moving with constant velocity on the surface of a fluid<sup>10-12</sup>. We consider the special case of the flow of an uniform fluid of finite depth. We use a non-dimensional spatial variable  $X$ , scaled by the fluid depth  $h$  and a non-dimensional time  $T$  scaled by  $\sqrt{gh^{-1}}$ , where  $g$  is the acceleration due to gravity. We take the upstream velocity to be  $U$  in the  $X$  direction, with the  $z$  direction upwards, opposite to the direction of gravity. For waves generated by a surface forcing, we take the forcing to move at velocity  $U$  in the negative  $X$  direction, which is equivalent to flow of velocity  $U$  in the  $X$  direction in the frame of reference moving with the forcing. The weakly nonlinear, long wave limit is considered so that the height of the waves is much less than the fluid depth and the wavelength of the waves is much greater than the length scale of the topography. Let  $\alpha$  and  $\epsilon$  be small parameters, where the amplitude of the topography is  $O(\alpha^2)$  and  $\epsilon^{-1}$  is a measure of the horizontal length scale of the topography<sup>13</sup>. The flow will be taken in the resonant regime so that the imposed flow speed  $U$  is close to the linear long wave speed,  $c = 1$  in non-dimensional variables. If the forcing amplitude and horizontal length scales are such that  $\alpha = \epsilon^2$ , then the flow is governed by a forced KdV equation<sup>13</sup>, as considered in previous work<sup>10-13,18</sup>. In this scaling the parameter  $\Delta$ ,  $U = c + \alpha\Delta$ , measures how close the flow is to exact resonance, with  $\Delta = 0$  corresponding to exact resonance. Let us take the topography or forcing to have the functional form  $\alpha^2 G(x)$ ,  $x = \epsilon X$ , due to the assumption about its length and amplitude scales.  $G$  is assumed to have its maximum at  $x = 0$  and to have amplitude  $g_0$ , so that  $G(0) = g_0$ . In the present work, we shall be interested in the influence of the next order nonlinear, dispersive and mixed nonlinear-dispersive terms

in the KdV approximation to the resonant flow<sup>23,24</sup>. At this order, the non-dimensional, normalised equation governing the resonant flow of a fluid over topography is the forced eKdV equation<sup>23</sup>

$$\begin{aligned} & -u_t - \Delta u_x + 6uu_x + u_{xxx} - \alpha c_1 u^2 u_x + \alpha c_2 u_x u_{xx} + \alpha c_3 u u_{xxx} + \alpha c_4 u_{xxxx} \\ & = -(1 + \alpha c_8 \Delta) G_x - \alpha c_6 u G_x - \alpha c_5 G u_x - \alpha c_7 G_{xxx}. \end{aligned} \quad (1)$$

Here,  $\alpha \ll 1$  is the square root of a typical non-dimensional topography height. The flow is assumed to start from the rest state, so that  $u(x, 0) = 0$ . The coefficients of the higher-order terms,  $c_i$ ,  $i = 1, \dots, 8$ , are calculated from the background stratification and have been explicitly calculated for the case of surface water waves<sup>23</sup>, for which

$$c_1 = 1, \quad c_2 = \frac{23}{6}, \quad c_3 = \frac{5}{3}, \quad c_4 = \frac{19}{60}, \quad c_5 = -\frac{4}{3}, \quad c_6 = -\frac{7}{6}, \quad c_7 = \frac{5}{12}, \quad c_8 = \frac{1}{4}. \quad (2)$$

This work focuses on resonant flow for surface water waves. Hence, the coefficients (2) appropriate for surface water waves are used for all the solutions and figures presented in this paper. Modulation theory for resonant flow over topography for internal waves with general higher-order coefficients is also available<sup>25</sup>.

Figure 1 shows a typical solution of the forced eKdV equation (1). Shown are a perspective view of the solution in the  $x - t$  plane (top) and the surface profile  $u$  and bathymetry  $G$  versus  $x$  at  $t = 25$  (bottom). The numerical solution of (1) with the initial condition  $u = 0$  is shown. The other parameters are  $\Delta = 0$  and  $\alpha = 0.15$ , so that the flow is at exact resonance. The solution consists of three parts, a steady hydraulic flow over the topography, a partial undular bore which propagates upstream and a full undular bore downstream of the obstacle. Mass is transported upstream, so a flat depression occurs downstream of the obstacle to conserve mass overall. The downstream bore returns the mean level to zero downstream of the depression. Solutions for the flow in these three regions will be derived in this section.

## B. Steady hydraulic flow over the forcing

The solution of the forced eKdV equation (1) displayed in Figure 1 shows that over the forcing the flow is steady and non-dispersive, as found by Grimshaw and Smyth<sup>13</sup> and Smyth<sup>18</sup>. The flow over the forcing is then the solution of the non-dispersive form of the



forced eKdV equation (1), which is

$$-u_t - \Delta u_x + 6uu_x - \alpha u^2 u_x + (1 + \alpha \frac{1}{4} \Delta) G_x - \alpha \frac{7}{6} u G_x - \alpha \frac{4}{3} G u_x = 0. \quad (3)$$

This hyperbolic equation has two steady solutions and the appropriate solution for the steady flow in this context is

$$u_s = \begin{cases} \frac{1}{6} [\Delta + \alpha \frac{4}{3} (g_0 - \frac{\Delta^2}{12}) + (1 + \alpha \frac{1}{8} \Delta) N(x)] - \alpha \frac{5}{108} [\Delta^2 + \Delta N(x) + 4(g_0 - G)] \\ + \alpha \frac{5}{72} [\Delta^2 + \frac{\Delta}{2} N(x)], & x < 0, \\ \frac{1}{6} [\Delta + \alpha \frac{4}{3} (g_0 - \frac{\Delta^2}{12}) - (1 + \alpha \frac{1}{8} \Delta) N(x)] - \alpha \frac{5}{108} [\Delta^2 - \Delta N(x) + 4(g_0 - G)] \\ + \alpha \frac{5}{72} [\Delta^2 - \frac{\Delta}{2} N(x)], & x > 0. \end{cases}, \quad (4)$$

where  $N(x) = \sqrt{12(g_0 - G)}$ . This solution is comprised of the upper branch for negative  $x$  and the lower branch for positive  $x$  and is continuous at  $x = 0$  at the peak of the forcing. It approaches a positive constant as  $x \rightarrow -\infty$  and a negative constant as  $x \rightarrow \infty$ ; this limiting behaviour is required so that the steady flow over the forcing matches with the bores propagating upstream and downstream<sup>13,18</sup>.

The steady solution (4) terminates in a positive jump upstream of the forcing and a negative jump downstream. As the eKdV equation (1) is a nonlinear, dispersive wave equation, these jumps are smoothed by evolving into undular bores, also termed dispersive shock waves<sup>6,15-17</sup>. It is this dispersive resolution of the discontinuities resulting from the resonant response of the flow over the forcing which generates the upstream and downstream propagating undular bores<sup>13,18</sup>. To match with the upstream and downstream flows, we take the limiting forms of the steady flow (4) as  $x \rightarrow \pm\infty$ , giving

$$u_s = \frac{1}{6} \left[ \Delta + \alpha \frac{4}{3} \left( g_0 - \frac{\Delta^2}{12} \right) \pm \left( 1 + \alpha \frac{1}{8} \Delta \right) \sqrt{12g_0} \right] - \alpha \frac{5}{108} \left[ \Delta^2 \pm \Delta \sqrt{12g_0} + 4g_0 \right] + \alpha \frac{5}{72} \left[ \Delta^2 \pm \frac{\Delta}{2} \sqrt{12g_0} \right], \quad x \rightarrow \mp\infty. \quad (5)$$

These limiting values  $u_s \rightarrow u_u$  as  $x \rightarrow -\infty$  and  $u_s \rightarrow u_d$  as  $x \rightarrow \infty$  will be matched to the upstream and downstream bores in the next section. The resonant flow, characterised by strong upstream and downstream responses in the form of undular bores, only exists for a finite range of  $\Delta$  around the exact resonance at  $\Delta = 0$ <sup>13,18</sup>. As  $\Delta$  increases from zero, the flow eventually becomes supercritical with a localised trapped hump over the forcing and transient waves propagating upstream and downstream<sup>6,13,18</sup>. On the other hand, as  $\Delta$  decreases from zero, the flow eventually becomes subcritical with a localised trapped dip over

the forcing, a steady lee wavetrain downstream and a transient propagating upstream<sup>6,13,18</sup>. The range of the resonant regime can be quantified from the requirement of matching the steady flow (4) over the forcing to the upstream and downstream propagating bores. This is easiest to determine by looking at the limiting values of  $u_s$  as  $x \rightarrow \pm\infty$ . The upstream and downstream states (5) are physically valid when  $u_u > 0$  and  $u_d < 0$ . These requirements give that resonant flow will occur when  $\Delta$  lies in the range

$$-\sqrt{12g_0} + \alpha\frac{1}{9}g_0 < \Delta < \sqrt{12g_0} + \alpha\frac{1}{9}g_0. \quad (6)$$

Note that at  $O(\alpha)$  the width of the resonant band,  $2\sqrt{12g_0}$ , is unchanged from that given by the forced KdV equation, but is translated by  $\alpha\frac{1}{9}g_0$  towards the supercritical regime. Only the coefficients associated with the higher-order non-dispersive terms in (3) contribute to this correction of the resonant range.

### C. Upstream and downstream wavetrains

It was found by Marchant<sup>32-34</sup> that solutions of the eKdV equation

$$-u_t - \Delta u_x + 6uu_x + u_{xxx} - \alpha c_1 u^2 u_x + \alpha c_2 u_x u_{xx} + \alpha c_3 u u_{xxx} + \alpha c_4 u_{xxxx} = 0, \quad (7)$$

can be transformed by

$$\begin{aligned} u &= \eta + \alpha c_9 \eta^2 + \alpha c_{10} \eta_{xx} + \alpha c_{11} \eta_x \int_{Ut}^x (\eta(p, t) - \beta) dp, \\ \tau &= t + \alpha \frac{c_4}{3} x, \quad \xi = x + \alpha c_{11} \beta (x - Ut) + \alpha c_{11} Dt \\ c_9 &= \frac{1}{6}(c_1 + c_3 + 4c_4), \quad c_{10} = \frac{1}{12}(c_1 + c_2 - 6c_4), \quad c_{11} = \frac{1}{3}(8c_4 - c_3), \quad D = Uu - 3u^2 - u_{\xi\xi}, \end{aligned} \quad (8)$$

to solutions of the standard KdV equation

$$\frac{\partial \eta}{\partial \tau} + 6\eta \frac{\partial \eta}{\partial \xi} + \frac{\partial^3 \eta}{\partial \xi^3} = 0, \quad (9)$$

when terms of  $O(\alpha^2)$  are neglected. In this transformation  $\beta$  and  $U$  are the mean level and phase speed of the cnoidal wave solution<sup>2</sup> of the KdV equation (9). This transformation will be used to derive the undular bore solution of the unforced eKdV equation from that of the KdV equation<sup>15,16</sup>.

Note that there is an exact transformation between the modulation equations of the KdV equation and of the Gardner equation, but which is non-invertible<sup>26,27</sup>. The Gardner

equation has a number of different solution types, such as trigonometric bores and solibores, which have no KdV counterparts. In addition, Sprenger and Hofer<sup>28</sup> considered a fifth-order KdV equation and found new types of undular bore solutions due to a resonance between radiation and the bore. The eKdV equation (1) contains both the third order nonlinear term  $u^2u_x$  of the Gardner equation and the fifth order dispersive term  $u_{xxxxx}$  of the fifth order KdV equation. However, the near-identity transformation (8) only generates a classical KdV bore type solution for the eKdV equation as the amplitude parameter  $\alpha$  is assumed small. Hence, the novel bore types which occur for the Gardner and fifth-order KdV equations cannot be found using the methods of the present work.

Modulation theory<sup>2,20,21</sup>, or the method of averaged Lagrangians, is based on finding differential equations for the parameters, such as the mean height, wavenumber and amplitude, of a slowly varying wavetrain. Modulation theory for the KdV equation<sup>2,20</sup> is based on its periodic cnoidal wave solution<sup>2</sup>

$$\eta = \beta + \frac{2a}{m} \left[ 1 - m - \frac{E(m)}{K(m)} + m \operatorname{cn}^2\left(\frac{K(m)}{\pi}\theta, m\right) \right]. \quad (10)$$

Here the phase is  $\theta = k\xi - \omega\tau$ . This travelling wave has wavenumber  $k$ , frequency  $\omega$ , phase speed  $U = \frac{\omega}{k}$ , mean height  $\beta$  and amplitude  $a$ .  $K(m)$  and  $E(m)$  are complete elliptic integrals of the first and second kinds, respectively, while  $m$  is the modulus squared of the elliptic integrals. In the limit  $m \rightarrow 1$  the cnoidal wave solution (10) becomes the KdV soliton and in the limit  $m \rightarrow 0$  it becomes the linear travelling wave solution of the KdV equation.

A particular solution of the hyperbolic KdV modulation equations is a simple wave solution, which corresponds physically to the undular bore solution of the KdV equation<sup>15,16</sup>. This undular bore solution is a modulated cnoidal wave which links the level  $A$  behind the bore to the level  $B$  in front of the bore, with  $A > B$ . The undular bore is then the modulated cnoidal wave

$$\eta = A - (A - B)m + 2(A - B)m \operatorname{cn}^2\left(\frac{K}{\pi}\theta, m\right), \quad (11)$$

with the modulated wave parameters given by

$$\begin{aligned} a &= 2(A - B)m, & \beta &= 2B - A + (A - B) \left( 2\frac{E(m)}{K(m)} + m \right), \\ k &= K^{-1}\pi\sqrt{A - B}, & U &= 2A + 4B + 2(A - B)m, \end{aligned} \quad (12)$$

$$\begin{aligned} p &= A + (A - B)m, & q &= A - (A - B)m, \\ \text{on } \frac{\xi}{\tau} &= \lambda = U - 4(A - B) \frac{m(1 - m)K(m)}{E(m) - (1 - m)K(m)}, & 12B - 6A &\leq \frac{\xi}{\tau} \leq 4A + 2B. \end{aligned} \quad (13)$$

In this undular bore solution the modulus squared  $m$  varies from  $m = 1$  at the leading edge to  $m = 0$  at the trailing edge of the bore. At the leading edge solitons of amplitude  $2(A - B)$  with velocity  $4A + 2B$  propagate on a mean level  $\beta = B$ . At the trailing edge, there are sinusoidal waves of small amplitude on a mean level  $\beta = A$ . Undular bores only occur if there is a step down in mean height, i.e.  $A > B$ . The quantities  $p$  and  $q$  are the peak and trough heights of the wave and are the envelopes of the wavetrain.

The transformation (8) can now be used to transform the KdV undular bore solution (11) into the undular bore solution of the unforced eKdV equation for surface water waves<sup>23</sup>. The amplitude  $a$ , wavenumber  $k$  and mean height  $\beta$  of the eKdV undular bore are then

$$\begin{aligned}
a &= (A - B)m + \alpha \left[ \frac{59}{90}m + \frac{22}{45}(m^2 - 2m) \right] (A - B)^2, \\
\beta &= 2B - A + (A - B) \left( 2 \frac{E(m)}{K(m)} + m \right) + \alpha \frac{59}{270} (A - B)^2 \left[ 3m^2 - 5m + 2 + 2(2m - 1) \frac{E(m)}{K(m)} \right] \\
&\quad + \alpha \frac{52}{135} (A - B)^2 \left[ 3 \left( 1 - \frac{E(m)}{K(m)} \right)^2 - 2 \left( 1 - \frac{E(m)}{K(m)} \right) (m + 1) + m \right], \\
k &= \pi K^{-1} \sqrt{A - B} \left[ 1 + \alpha \frac{13}{45} (2A(A - B)m - (A - B)^2 m^2 - A^2 + 4AB) \right. \\
&\quad \left. - \alpha \frac{19}{90} (A + 2B + (A - B)m) - \alpha \frac{59}{180} (A + B) \right].
\end{aligned} \tag{14}$$

This is also the undular bore solution of the forced eKdV equation (1) if the sign of the characteristic  $\lambda$  (13) is reversed to account for  $-u_t$  and a shift  $\Delta$  is added to account for  $-\Delta u_x$  in this equation. The undular bore solution of the forced eKdV equation (1) is then (14) on

$$\begin{aligned}
\frac{x}{t} &= \Delta - 2A - 4B - 2(A - B)m + 2(A - B)S + \alpha \frac{59}{45} [A^2 + 2B^2 + (A^2 - B^2)m - (A^2 - B^2)S] \\
&\quad - \alpha \frac{19}{180} [U - 2(A - B)S]^2 + \alpha \frac{13}{45} [2A(A - B)m - (A - B)^2 m^2 - A^2 + 4AB] \\
&\quad - \alpha \frac{26}{45} (A - B)S \left[ 2B - A + (A - B)m + 2(A - B) \frac{E(m)}{K(m)} \right],
\end{aligned} \tag{15}$$

$$\text{where } S(m) = \frac{2m(1 - m)K(m)}{E(m) - (1 - m)K(m)}.$$

Here the higher order phase speed is

$$\begin{aligned}
U &= 2A + 4B + 2(A - B)m - \alpha \frac{13}{45} [2A(A - B)m - (A - B)^2 m^2 - A^2 + 4AB] \\
&\quad + \alpha \frac{19}{90} [A + 2B + (A - B)m]^2 - \alpha \frac{59}{45} [3B^2 + (A^2 - B^2)(1 + m)].
\end{aligned} \tag{16}$$

The envelopes  $p$  and  $q$  of this cnoidal wave are then given by

$$p = A + (A - B)m + \alpha \frac{59}{90} [(A - B)^2 m^2 + 2A(A - B)m - (A^2 - B^2)m] - \alpha \frac{44}{45} (A - B)^2 m, \quad (17)$$

$$q = A - (A - B)m + \alpha \frac{59}{90} [(A - B)^2 m^2 - 2A(A - B)m + (A^2 - B^2)m] + \alpha \frac{44}{45} (A - B)^2 m(1 - m). \quad (18)$$

At the leading edge of the extended undular bore the solitary waves have amplitude  $2(A - B) - \alpha \frac{1}{3}(A - B)^2$  and velocity  $4A + 2B - \alpha \frac{14}{15}A^2 + \alpha \frac{128}{45}AB - \alpha \frac{53}{45}B^2$  and propagate on a mean level  $\beta = B$ . At the trailing edge, there are sinusoidal waves of small amplitude on a mean level  $\beta = A$ .

The extended undular bore solution can now be used to determine the bores propagating both upstream and downstream of the forcing. Let us first consider the upstream propagating bore, as seen in Figure 1. It is clear from matching with the steady solution over the forcing (5) that the upstream propagating bore has  $B = 0$  to match with the initial undisturbed flow. However, it can be seen from the characteristic velocity (15) that the linear trailing edge of the bore propagates downstream if  $\Delta > -6A + O(\alpha)$  as  $m \rightarrow 0$ , which is unphysical and contradicts what is seen in the numerical solution of Figure 1. This is resolved by stopping the upstream undular bore solution at  $x = 0$ , so that the upstream undular bore is a partial bore<sup>18</sup>. The partial bore then has modulus in the range  $m_0 \leq m \leq 1$ , with  $m_0$  the characteristic velocity (15) which sets  $\frac{x}{t} = 0$ . Waves of modulus  $m_0$  are then generated at the forcing<sup>18</sup>, which is what is seen in Figure 1. This gives, on using the characteristic velocity (15), that the minimum modulus  $m_0$  is the solution of

$$\begin{aligned} \Delta &= 2A(1 + m_0 - S_0) + \alpha \frac{19}{45} A^2 (1 + m_0 - S_0)^2 - \alpha \frac{59}{45} A^2 (1 + m_0 - S_0) \\ &+ \alpha \frac{13}{45} A^2 (1 - 2m_0 + m_0^2) - \frac{26}{45} S_0 A^2 \left( 1 - m_0 - 2 \frac{E(m_0)}{K(m_0)} \right), \end{aligned} \quad (19)$$

where  $S_0 = \frac{2m_0(1 - m_0)K(m_0)}{E(m_0) - (1 - m_0)K(m_0)}$ .

The jump height  $A$  can then be found by setting the mean level  $\beta$ , given by (14), of the bore at the forcing  $x = 0$ , so that  $m = m_0$ , equal to the upstream limit  $u_u$  (5) of the steady

solution over the forcing. This results in  $A$  being the solution of

$$\begin{aligned} & \frac{1}{6} \left[ \Delta + \alpha \frac{4}{3} \left( g_0 - \frac{\Delta^2}{12} \right) + \left( 1 + \alpha \frac{1}{8} \Delta \right) \sqrt{12g_0} \right] - \alpha \frac{5}{108} \left[ \Delta^2 + \Delta \sqrt{12g_0} + 4g_0 \right] \\ & + \alpha \frac{5}{72} \left( \Delta^2 + \frac{\Delta}{2} \sqrt{12g_0} \right) = m_0 A - A + 2A \frac{E(m_0)}{K(m_0)} + \alpha \frac{59}{270} A^2 \left[ 3m_0^2 - 5m_0 + 2 + 2(2m_0 - 1) \frac{E(m_0)}{K(m_0)} \right] \\ & + \alpha \frac{52}{135} A^2 \left[ 3 \left( 1 - \frac{E(m_0)}{K(m_0)} \right)^2 - 2 \left( 1 - \frac{E(m_0)}{K(m_0)} \right) (m_0 + 1) + m_0 \right]. \end{aligned} \quad (20)$$

Equations (19) and (20) for the minimum modulus  $m_0$  and the upstream jump height  $A$  can be solved in the limit of small  $\alpha$  to give

$$A = A_0 + \alpha A_1, \quad A_0 = \frac{\Delta + \sqrt{12g_0}}{6T}, \quad T = m_0 - 1 + 2 \frac{E(m_0)}{K(m_0)}, \quad (21)$$

$$\begin{aligned} A_1 = & \frac{1}{216T} \left[ \frac{9}{2} \Delta \sqrt{12g_0} - 10 \left( \Delta^2 + \Delta \sqrt{12g_0} + 4g_0 \right) \right. \\ & \left. + 15 \left( \Delta^2 + \frac{\Delta}{2} \sqrt{12g_0} \right) + 48 \left( g_0 - \frac{\Delta^2}{12} \right) \right] \\ & - \frac{1}{108T^3} \left( \Delta + \sqrt{12g_0} \right)^2 \left[ \frac{59}{90} \left( 3m_0^2 - 5m_0 + 2 + 2(2m_0 - 1) \frac{E(m_0)}{K(m_0)} \right) \right. \\ & \left. + \frac{13}{15} \left( 3 \left( 1 - \frac{E(m_0)}{K(m_0)} \right)^2 - 2 \left( 1 - \frac{E(m_0)}{K(m_0)} \right) (m_0 + 1) + m_0 \right) \right]. \end{aligned} \quad (22)$$

On solving equation (19) for the minimum modulus  $m_0$  it is found that when

$$\Delta = -\frac{1}{2} \sqrt{12g_0} + \alpha \frac{4}{9} g_0, \quad (23)$$

$m_0 = 0$ , so that the partial upstream undular bore becomes a full bore with  $A = u_u$ . For  $\Delta$  below the value (23), a full undular bore propagates upstream. The upstream propagating partial and full undular bores have now been determined. On noting the resonant range (6), we have that for

$$-\sqrt{12g_0} + \alpha \frac{1}{9} g_0 < \Delta \leq -\frac{1}{2} \sqrt{12g_0} + \alpha \frac{4}{9} g_0, \quad (24)$$

a full undular bore propagates upstream, while for

$$-\frac{1}{2} \sqrt{12g_0} + \alpha \frac{4}{9} g_0 < \Delta < \sqrt{12g_0} + \alpha \frac{1}{9} g_0, \quad (25)$$

a partial undular bore propagates upstream. As the upper resonant bound in (25) is approached,  $m_0 \rightarrow 1$  and the partial bore becomes a train of solitons. Even at exact resonance  $\Delta = 0$ ,  $m_0 = 0.64 + O(\alpha)^{18}$ , so that the upstream undular bore can be well approximated by a train of solitary waves<sup>13,18</sup>. For this reason, the upstream wavetrain is often termed a

train of solitons<sup>10,19</sup>, even though this is just an approximation which in fact breaks down as the lower limit of the resonant range is approached.

The downstream propagating bore seen in Figure 1 can be determined in a similar manner. In this case, to match with the undisturbed flow downstream of the forcing we have  $A = 0$  with  $B < 0$ . In the case of a full downstream undular bore the trailing solitary wave edge of the bore with  $m \rightarrow 1$  matches with the downstream level  $u_d$  of the steady flow given by (5), as seen from Figure 1. The mean level (14) then gives

$$B = \frac{1}{6} \left[ \Delta + \alpha \frac{4}{3} \left( g_0 - \frac{\Delta^2}{12} \right) - \left( 1 + \alpha \frac{1}{8} \Delta \right) \sqrt{12g_0} \right] - \alpha \frac{5}{108} \left( \Delta^2 - \Delta \sqrt{12g_0} + 4g_0 \right) + \alpha \frac{5}{72} \left( \Delta^2 - \frac{\Delta}{2} \sqrt{12g_0} \right). \quad (26)$$

Finally, the general undular bore solution (14) and (15) gives that the amplitude, mean height and wavenumber of the downstream propagating undular bore in the range (6) are given by

$$\begin{aligned} a &= |B|m + \alpha \left[ \frac{59}{90}m + \frac{22}{45} (m^2 - 2m) \right] B^2, \\ \beta &= 2B - B \left( 2 \frac{E(m)}{K(m)} + m \right) + \alpha \frac{59}{270} B^2 \left[ 3m^2 - 5m + 2 + 2(2m - 1) \frac{E(m)}{K(m)} \right] \\ &\quad + \alpha \frac{52}{135} B^2 \left[ 3 \left( 1 - \frac{E(m)}{K(m)} \right)^2 - 2 \left( 1 - \frac{E(m)}{K(m)} \right) (m + 1) + m \right], \\ k &= \frac{\pi \sqrt{|B|}}{K(m)} \left[ 1 - \alpha \frac{13}{45} B^2 m^2 + \alpha \frac{19}{90} Bm - \frac{3}{4} B \right], \end{aligned} \quad (27)$$

on the characteristics

$$\begin{aligned} \frac{x}{t} &= \Delta - 2B(2 - m + S) - \alpha \frac{19}{45} B^2 (2 - m + S)^2 + \alpha \frac{59}{45} B^2 (2 - m + S) \\ &\quad - \alpha \frac{13}{45} B^2 m^2 + \alpha \frac{26}{45} S B^2 \left( 2 - m - 2 \frac{E(m)}{K(m)} \right). \end{aligned} \quad (28)$$

This full downstream undular bore has  $0 \leq m \leq 1$ , and so has extent

$$\Delta - 2B + \alpha \frac{3}{5} B^2 \leq \frac{x}{t} \leq \Delta - 12B - \alpha \frac{22}{3} B^2. \quad (29)$$

This solution gives the full downstream undular bore seen in Figure 1 for  $\Delta = 0$ . However, the solution is only valid if the velocity of the trailing solitary wave edge is positive. It can be found from the characteristic velocity (28) that when

$$\Delta = -\frac{1}{2} \sqrt{12g_0} - \alpha \frac{16}{45} g_0, \quad (30)$$

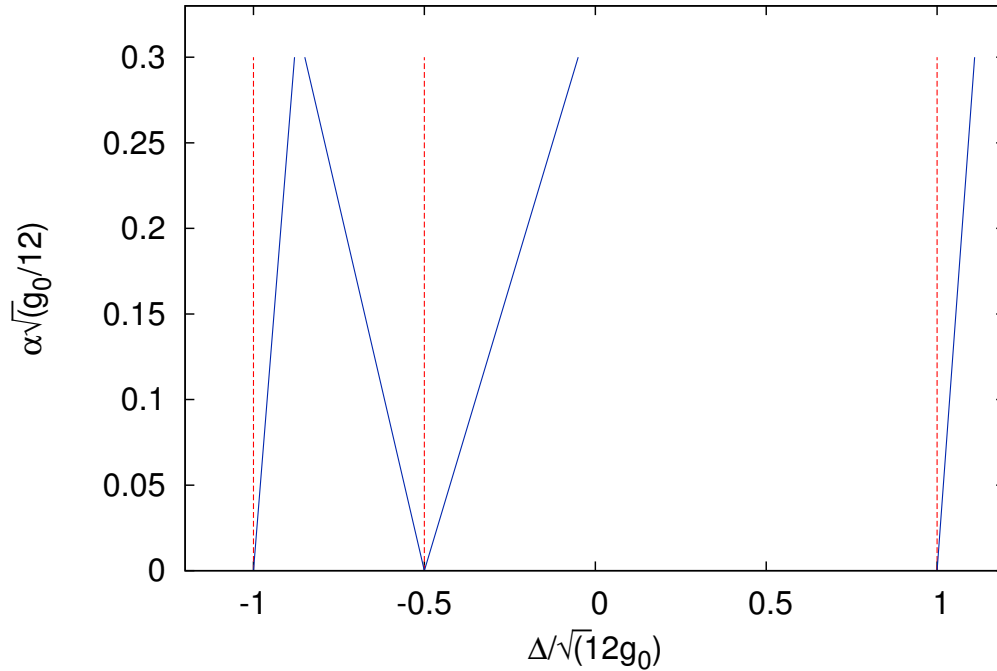


FIG. 2. The parameter ranges in the  $\Delta(12g_0)^{-\frac{1}{2}}$  versus  $\alpha(\frac{12}{g_0})^{\frac{1}{2}}$  plane for full and partial undular bores for surface water waves. Compared are eKdV (blue solid line) and KdV (red dashed lines) modulation theory.

the solitary wave, trailing edge of the bore with  $m = 1$  is stationary and attached to the forcing. For subcritical  $\Delta$  less than this value, the solitary wave edge of the downstream bore would propagate upstream. For example, at the subcritical limit of the resonant range for the KdV case  $\Delta = -\sqrt{12g_0}$  and the trailing edge of the full bore has the negative velocity  $-\frac{1}{3}\sqrt{12g}$ . This is resolved by making the downstream bore a partial bore in this highly subcritical range. The downstream bore is then a full undular bore for  $\Delta$  in the range

$$-\frac{1}{2}\sqrt{12g_0} - \alpha\frac{16}{45}g_0 < \Delta < \sqrt{12g_0} + \alpha\frac{1}{9}g_0 \quad (31)$$

and is a partial bore for

$$-\sqrt{12g_0} + \alpha\frac{1}{9}g_0 < \Delta \leq -\frac{1}{2}\sqrt{12g_0} - \alpha\frac{16}{45}g_0. \quad (32)$$

In the subcritical regime, the solution must match to the linear lee wave solution, which is a stationary wavetrain attached to the forcing, preceded by a transient front<sup>18</sup>. The downstream solution in the range of  $\Delta$  given by (32) is then a stationary cnoidal wavetrain



of modulus  $m_{0d}$  preceded by a partial bore with modulus squared in the range  $m_{0d} > m > 0$  to match with this stationary wavetrain at its trailing edge. The details of this downstream lee wave limit are given in<sup>18</sup>. In particular, the phase velocity of the waves of the partial bore does not approach 0 as the upper limit (32) of the partial bore range is approached, so that a partial bore as for the upstream case is not possible. The stationary cnoidal wavetrain then as mean level  $\beta_l$  and phase velocity  $U_l$  given by

$$\begin{aligned} \beta_l &= 2B - B \left( 2 \frac{E(m_{0d})}{K(m_{0d})} + m_{0d} \right) + \alpha \frac{59}{270} B^2 \left[ 3m_{0d}^2 - 5m_{0d} + 2 + 2(2m_{0d} - 1) \frac{E(m_{0d})}{K(m_{0d})} \right] \\ &+ \alpha \frac{52}{135} B^2 \left[ 3 \left( 1 - \frac{E(m_{0d})}{K(m_{0d})} \right)^2 - 2 \left( 1 - \frac{E(m_{0d})}{K(m_{0d})} \right) (m_{0d} + 1) + m_{0d} \right], \quad (33) \\ U_l &= 2B(2 - m_{0d}) + \alpha \frac{13}{45} B^2 m_{0d}^2 + \alpha \frac{19}{45} B^2 (2 - m_{0d})^2 - \alpha \frac{59}{45} B^2 (2 - m_{0d}), \end{aligned}$$

where the mean level of the stationary wavetrain is the same as the leading edge of the partial undular bore and is equal to the downstream limit of the steady hydraulic flow, i.e.  $\beta_l = u_s$ . Also, the wavetrain must be stationary, so that  $U = \Delta$ . Equations (33) form a pair of equations determining the parameters  $m_{0d}$  and  $B$  for the partial downstream undular bore. The modulation theory solutions for the full and partial undular bores upstream and downstream of the forcing are now complete.

Figure 2 shows the parameter ranges in the  $\Delta(12g_0)^{-\frac{1}{2}}$  versus  $\alpha(\frac{12}{g_0})^{\frac{1}{2}}$  plane for full and partial undular bores for surface water waves. Compared are eKdV and KdV modulation theory. The figure illustrates the ranges (24) and (25) for the upstream bore and (31) and (32) for the downstream bore. Moving from left to right, the three sets of curves show the subcritical limit of the bore, the transition between the partial and full bore and the supercritical limit of the bore. The upstream and downstream bores have the same subcritical and supercritical resonant limits; as  $\alpha$  increases these limits move toward the supercritical range. A full upstream bore is predicted by KdV theory for strongly subcritical flows and a partial bore for weakly subcritical and all supercritical flows. The transition point in KdV theory between full and partial bores occurs at  $\Delta = -\frac{1}{2}\sqrt{12g_0}$ . For the downstream bore the KdV transition point is the same, but the regimes are reversed. A partial downstream bore is predicted by KdV theory for strongly subcritical flows and a full bore for weakly subcritical and all supercritical flows. In the eKdV theory the transition point between the full and partial bores is no longer the same for the upstream and downstream bores. For the upstream bore the transition point moves towards the supercritical range as  $\alpha$  increases,

while for the downstream bore it moves towards the subcritical range. Figures 1 and 5-7, that display bore profiles, all have parameters that correspond to the Figure 2 regime of partial upstream and full downstream undular bores. It appears from the figure that the eKdV subcritical limit and eKdV downstream bore boundaries cross at  $\alpha \approx 0.30$ . However, it should be noted that the extended theory is only valid for small  $\alpha$  and the exact curves will deviate from the small  $\alpha$  predictions as  $\alpha$  increases.

Figure 3 shows (a) the modulus squared,  $m_0$ , and (b) the leading and trailing edges of the downstream undular bore,  $\frac{x}{t}$ , versus the detuning parameter  $\Delta$  for surface water waves. Compared are modulation theory for the forced eKdV and KdV equations. The other parameter is  $\alpha = 0.15$ . Figure 3(a) shows that the upstream undular bore lies in the range  $m_0 \leq m \leq 1$ . If  $m_0 = 0$ , a full bore occurs upstream, while as  $m_0 \rightarrow 1$  the bore becomes a train of solitary waves. At exact linear resonance  $\Delta = 0$  the modulus squared,  $m_0 = 0.64$ , is the same for both the eKdV and the KdV theories. In the supercritical case, for positive  $\Delta$ , the modulus squared  $m_0$  of the eKdV theory is slightly greater than that for KdV theory. For example at  $\Delta = 3$ ,  $m_0 = 0.87$  and  $0.84$  for eKdV and KdV theories, respectively. For the subcritical case, for negative  $\Delta$ , modulus squared  $m_0$ , of the eKdV theory is slightly lower than the KdV theory. For example, at  $\Delta = -0.5$ ,  $m_0 = 0.53$  and  $0.54$  for the eKdV and KdV theories, respectively. Also, the resonant regimes, for a partial upstream bore, are slightly different,  $-1.73 < \Delta < 3.46$  and  $-1.67 < \Delta < 3.48$  for the KdV and eKdV cases, respectively.

In Figure 3(b) the comparison is given for the ranges of the full downstream bore, which are  $-1.73 < \Delta < 3.46$  and  $-1.78 < \Delta < 3.48$  for the KdV and eKdV equations, respectively. The trailing edge has zero velocity and is at the forcing when  $\Delta = -1.73$  and  $\Delta = -1.78$  for the KdV and eKdV cases, respectively. Modulation theory shows the eKdV bore is up to 12% narrower than the KdV bore and that the velocity of the trailing edge of the eKdV bore is significantly lower. This result is in agreement with the internal wave bore<sup>22</sup>. Partial downstream bores occur in the ranges  $-3.46 < \Delta < -1.73$  and  $-3.44 < \Delta < -1.78$  for the KdV and eKdV equations, respectively. In these ranges the trailing edge is a steady cnoidal wave of modulus squared  $m_{0d}$ . Solving equations (33) gives  $m_{0d} = 1$  at the transition between a full and partial bore, with  $1 > m_{0d} > 0.99$  over the whole range of  $\Delta$  for which the bore is partial. Hence, the stationary wavetrain is composed of near solitary waves.

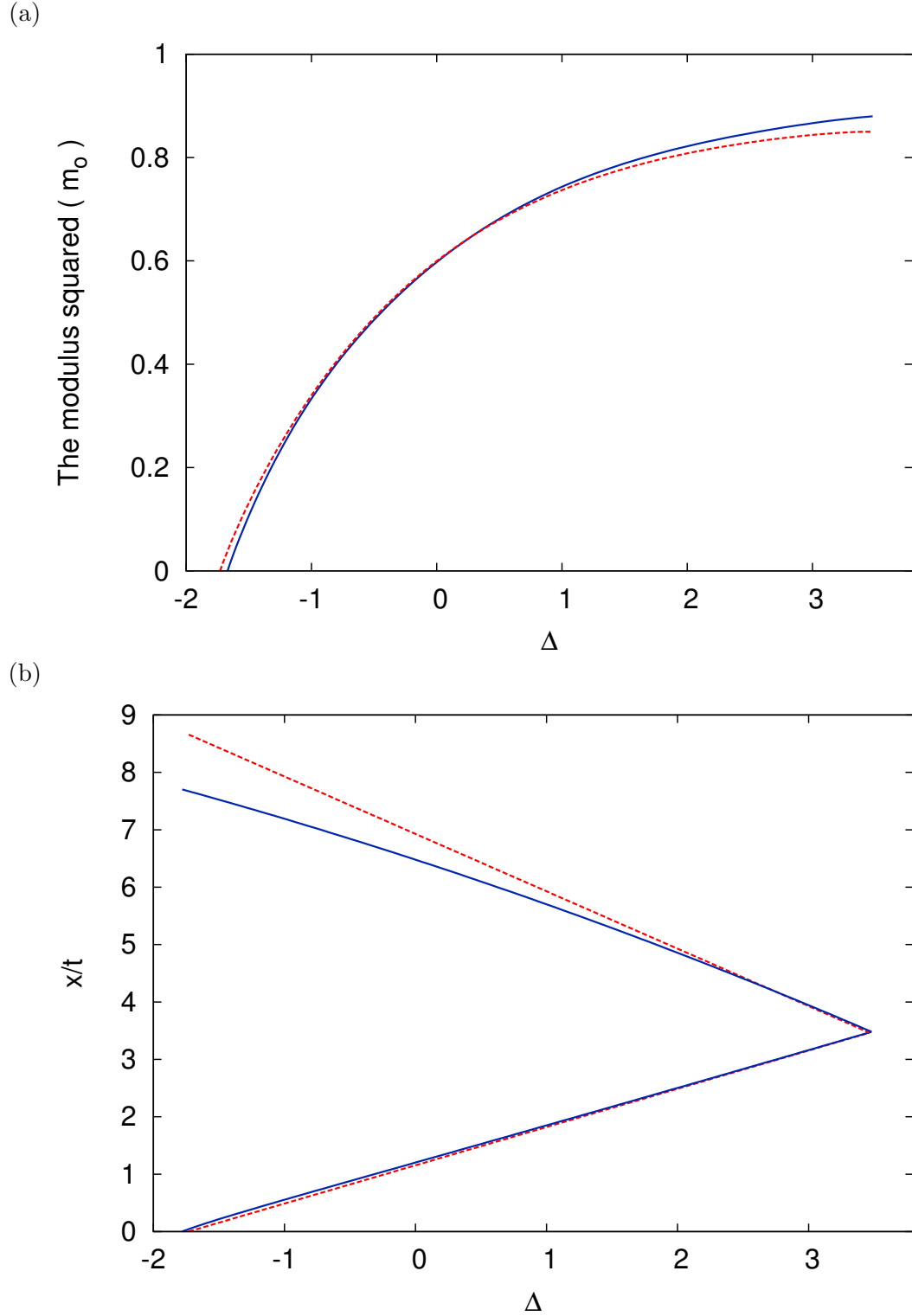


FIG. 3. (a) The modulus squared  $m_0$  and (b) the leading and trailing edges of the downstream undular bore,  $\frac{x}{t}$ , versus the detuning parameter  $\Delta$  for surface water waves. Compared are eKdV (blue solid line) and KdV (red dashed lines) modulation theory. The other parameter is  $\alpha = 0.15$ .

### III. COMPARISON WITH NUMERICAL RESULTS

In this section the higher order modulation theory solution will be compared with numerical solutions of the forced eKdV equation (1). The forced eKdV equation was solved numerically by the classical explicit leapfrog method of Zabusky and Kruskal<sup>35</sup>, which has truncation error  $O(\Delta t^2, \Delta x^2)$  and is stable for small  $\Delta t = O(\Delta x^3)$ . The forcing  $G(x)$  was chosen as

$$G(x) = g_0 \operatorname{sech}^2(Wx), \quad (34)$$

see Grimshaw and Smyth<sup>13</sup>. The parameter values  $g_0 = 1.0$  and  $W = 0.3$  were used for the numerical solutions of the present work. These values give solutions which are representative of the general behaviour<sup>13,18</sup>. Also the higher-order coefficients (2) for surface water waves are used for all the examples.

Figure 4 shows the upstream (a) and downstream (b) solitary wave amplitudes versus the detuning parameter  $\Delta$ , for surface water waves. Compared are the eKdV and KdV modulation theories and the corresponding numerical results. The other parameter is  $\alpha = 0.15$ . The upstream solitary wave amplitudes as predicted by eKdV modulation theory are greater than the KdV predictions. The predictions are similar in the subcritical regime, but the difference between the theoretical predications increases in the supercritical regime, with a difference of 10% at  $\Delta = 3.46$ . The variations between the numerical and theoretical results are small for the strongly subcritical and supercritical cases, but are slightly larger in the middle of the resonant band, with errors up to 10%. Overall, the amplitude results as given by modulation theory are in agreement with the numerical values. The downstream solitary wave amplitudes as predicted by the eKdV theory are similar to the KdV theory for supercritical flows and higher than KdV theory for subcritical flows, by up to 3%. There are small variations between the theoretical and numerical results for both theories, with a maximum error of 3%. The results show that for lower amplitude waves, the KdV and eKdV predictions are very similar. However, as wave amplitudes increase, for upstream solitary waves when the flow is supercritical, and for downstream solitary waves when the flow is subcritical, then the higher-order terms included in the eKdV model play a more significant role, with the eKdV predictions higher than the KdV ones.

Marchant and Smyth<sup>23</sup> drew similar graphs for the case when only the higher-order nonlinear term  $c_1 u^2 u_x$  was included in their Gardner-type eKdV equation, see their Figure

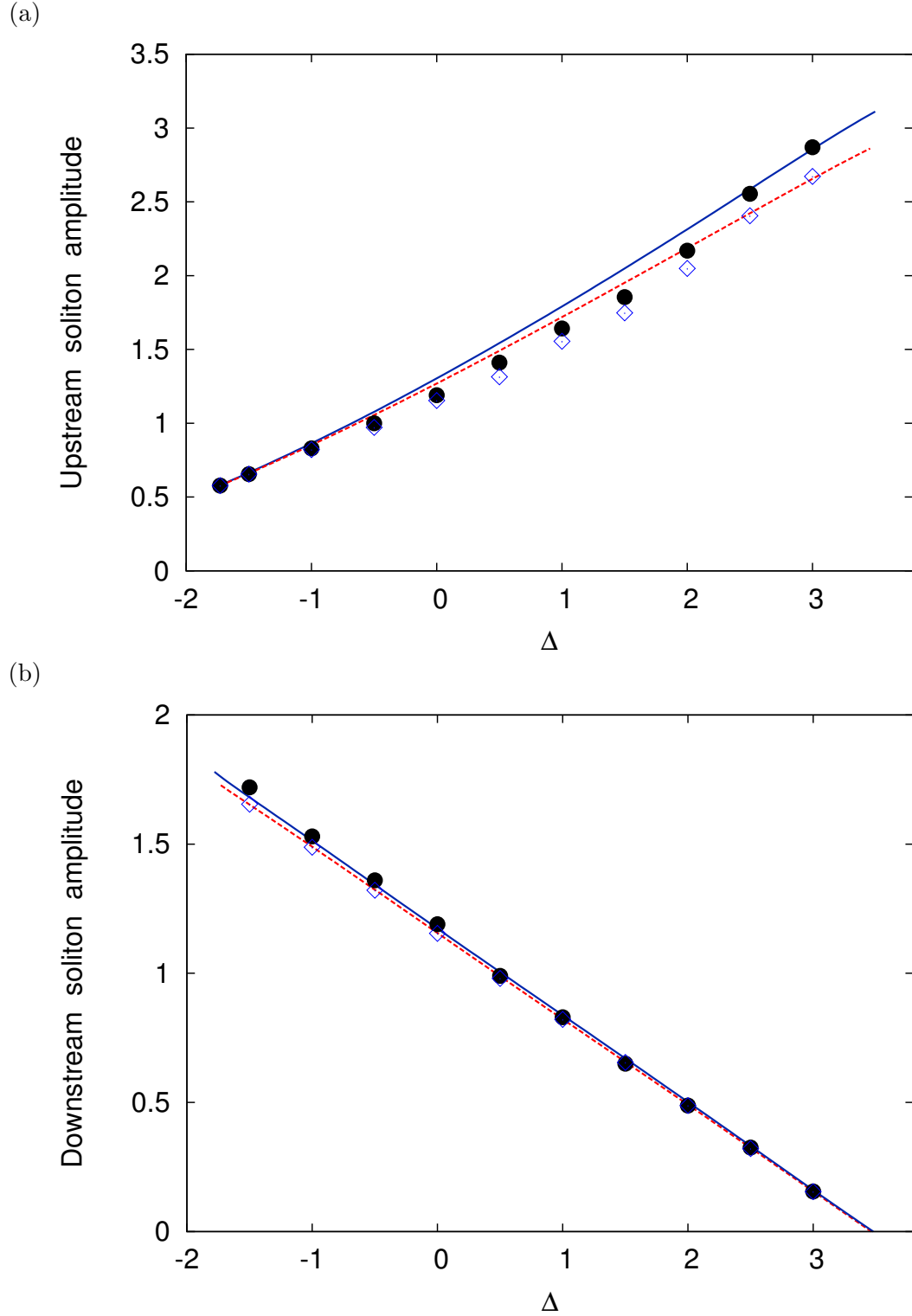


FIG. 4. (a) The upstream and (b) downstream solitary wave amplitudes versus the detuning parameter  $\Delta$  for surface water waves. Compared are eKdV (blue solid line) and KdV (red dashed lines) modulation theory and eKdV ( $\bullet$ ) and KdV ( $\diamond$ ) numerical solutions. The other parameter is  $\alpha = 0.15$ .

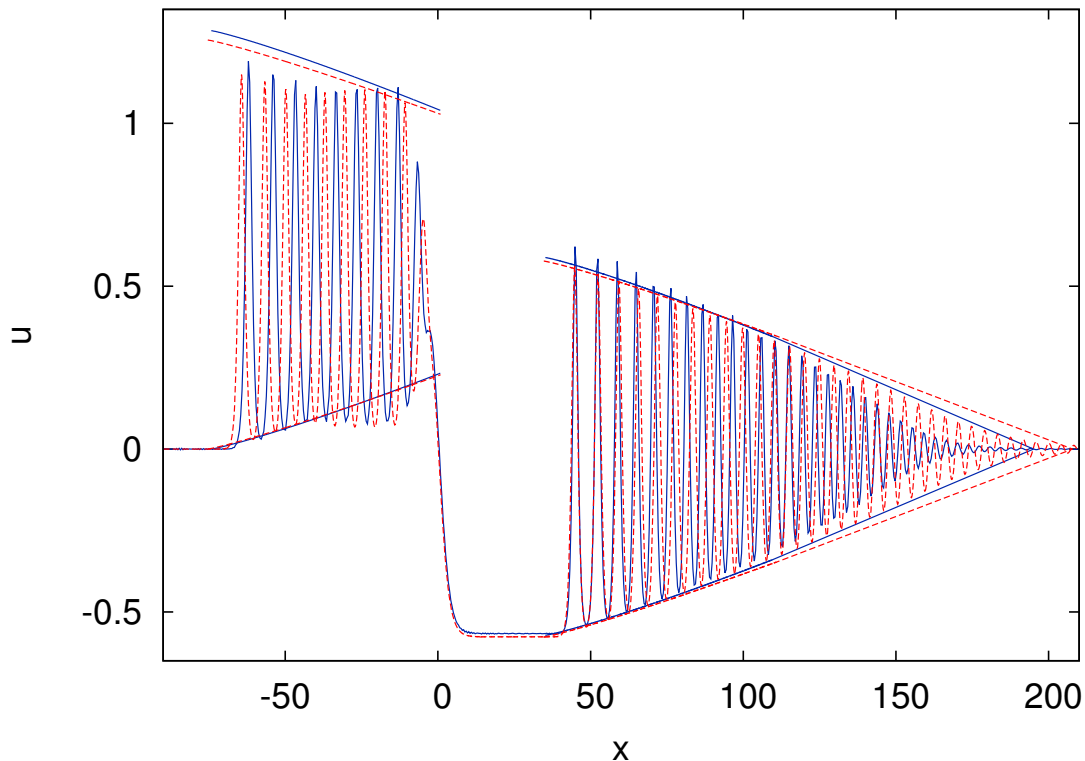


FIG. 5. The solution  $u$  versus  $x$  at  $t = 30$ , for surface water waves. Compared are numerical solutions of the forced eKdV equation (blue solid line) and the forced KdV equation (red dashed line). The other parameters are  $\alpha = 0.15$  and  $\Delta = 0$ . Also shown are the wave envelopes for the upstream and downstream undular bores for eKdV (blue solid line) and KdV (red dashes) modulation theories.

3. They found that the upstream eKdV solitary wave amplitudes were higher than KdV theory for supercritical cases, which is qualitatively similar to the results found here for the eKdV theory which includes the full set of higher-order terms. For the downstream solitary wave amplitude, they found their eKdV results to be higher (lower) for supercritical (subcritical) cases which is different to the results obtained here. Hence the full set of higher-order terms included in the eKdV theory results in qualitatively different results to those for the Gardner equation, which only includes the higher-order nonlinear term.

Figure 5 shows the solution  $u$  versus  $x$  at  $t = 30$ , for surface water waves. Compared are numerical solutions of the forced eKdV and KdV equations and the wave envelopes for the upstream and downstream KdV and eKdV modulation theory wavetrains. The other parameters are  $\Delta = 0$  and  $\alpha = 0.15$ . The upstream solitary wave amplitude from eKdV

modulation theory is  $A = 1.29$ , compared with the numerical value of  $A_n = 1.19$ . For the KdV theory,  $A = 1.25$  and  $A_n = 1.15$ . The eKdV results are about 3% higher than the KdV results for both the theoretical and the numerical solutions, while the variation between the theoretical and the numerical results is about 8%. It should be noted for the comparison shown in Figure 5 that the approach of the leading edge of the upstream bore to the steady state is slow. If the upstream bore is propagated until its leading edge amplitude settles to its steady state, then the variations between the theoretical and numerical solutions are reduced to 5%. The downstream solitary wave amplitude as given by the eKdV modulation theory is  $B = 1.16$ , while the steady numerical value is  $B_n = 1.19$ . Furthermore, the downstream solitary wave amplitude given by KdV modulation theory is  $B = 1.15$ , whereas the numerical value is  $B_n = 1.15$ . The errors for the downstream solitary wave amplitude as given by the eKdV and KdV modulation theories are 3% and 0%, respectively.

The eKdV theory predicts that the downstream bore is located in the region  $36 < x < 195$  and has width  $w = 159$ , while KdV theory gives  $35 < x < 208$  and  $w = 173$ . The eKdV numerical solution lies in the region  $44 < x < 184$  with width  $w = 140$ , while the KdV numerical solution lies in  $44 < x < 224$  with width  $w = 180$ . It is noted that the width of the downstream eKdV bore is significantly smaller than that of the KdV bore by more than 10%, again in agreement with the results of Lamb and Yan<sup>22</sup> who compared results for KdV and eKdV internal wave undular bores. The higher-order coefficients are different to those for surface water waves, but their eKdV results indicate a much narrower bore, in agreement with the results found here.

Figure 6 shows a supercritical solution  $u$  versus  $x$  at  $t = 30$  for surface water waves. Shown are numerical solutions for the forced eKdV and KdV equations. Also shown are the wave envelopes for the upstream and downstream modulated wavetrains as given by the eKdV and KdV modulation theories. The parameters are  $\alpha = 0.15$  and  $\Delta = 1$ . The upstream solitary wave amplitude for the eKdV modulation theory is  $A = 1.79$ , compared with the numerical value of  $A_n = 1.65$ . For the KdV theory,  $A = 1.72$  and the numerical value is  $A_n = 1.56$ . The eKdV amplitudes are then 5% higher than the KdV amplitudes, indicating that the higher order corrections to the KdV equation have a moderate effect on the upstream wavetrain. The difference between the theoretical and the numerical results is about 8%, which reduces to 5% for steady state numerical solutions. The downstream solitary wave amplitude given by eKdV modulation theory is  $B = 0.81$ , compared with the

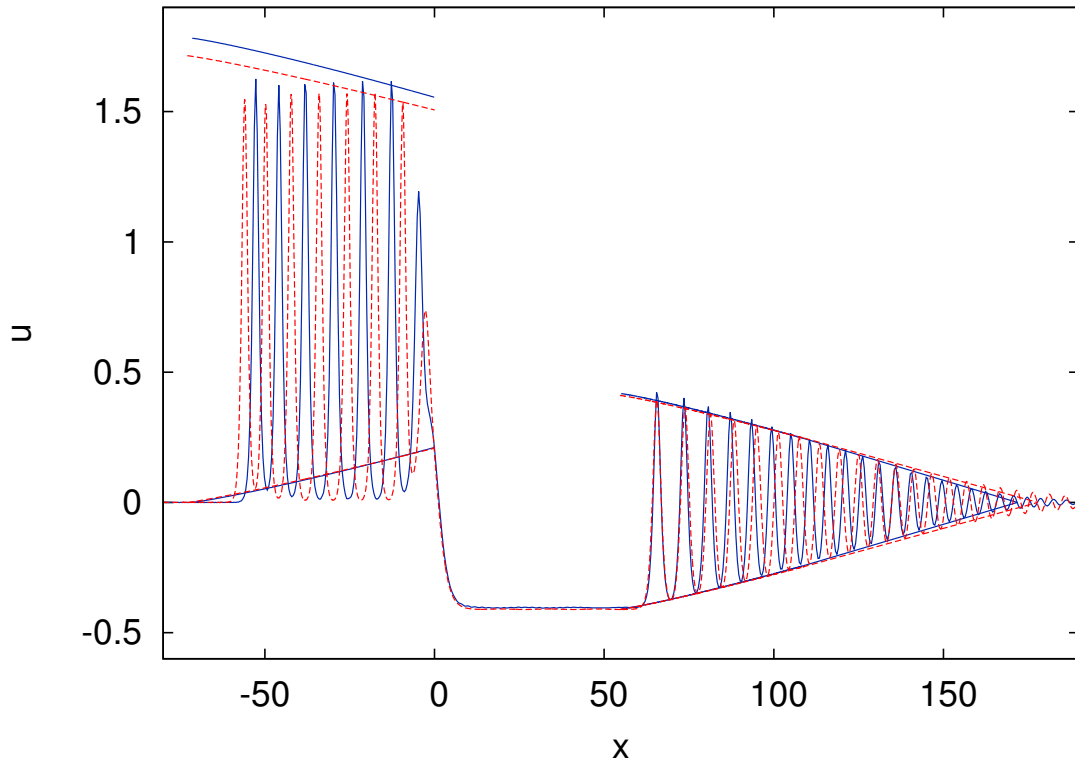


FIG. 6. Solution  $u$  versus  $x$  at  $t = 30$  for surface water waves. Compared are numerical solutions of the forced eKdV equation (blue solid line) and the forced KdV equation (red dashed line). The other parameters are  $\alpha = 0.15$  and  $\Delta = 1$ . Also shown are the wave envelopes for the upstream and downstream undular bores given by eKdV (blue solid line) and KdV modulation theories (red dashes).

numerical value  $B_n = 0.82$ . The value for the KdV equation given by both modulation theory and the numerical solution is  $B = 0.82$ . There is then only a 2% difference between the downstream solitary wave amplitudes as given by the eKdV and KdV equations, so that the higher order corrections to the KdV equation do not have a great effect on the downstream wave amplitude in the supercritical regime. Modulation theory for the eKdV equation gives that the downstream bore lies in  $55 < x < 168$  with width  $w = 113$ , while modulation theory for the KdV equation gives  $55 < x < 178$  and width  $w = 123$ . These are compared with the eKdV numerical results  $65 < x < 181$  with width  $w = 116$  and KdV numerical results  $64 < x < 192$  with width  $w = 128$ . Higher order effects again result in a narrowing of the bore<sup>22,29</sup>.



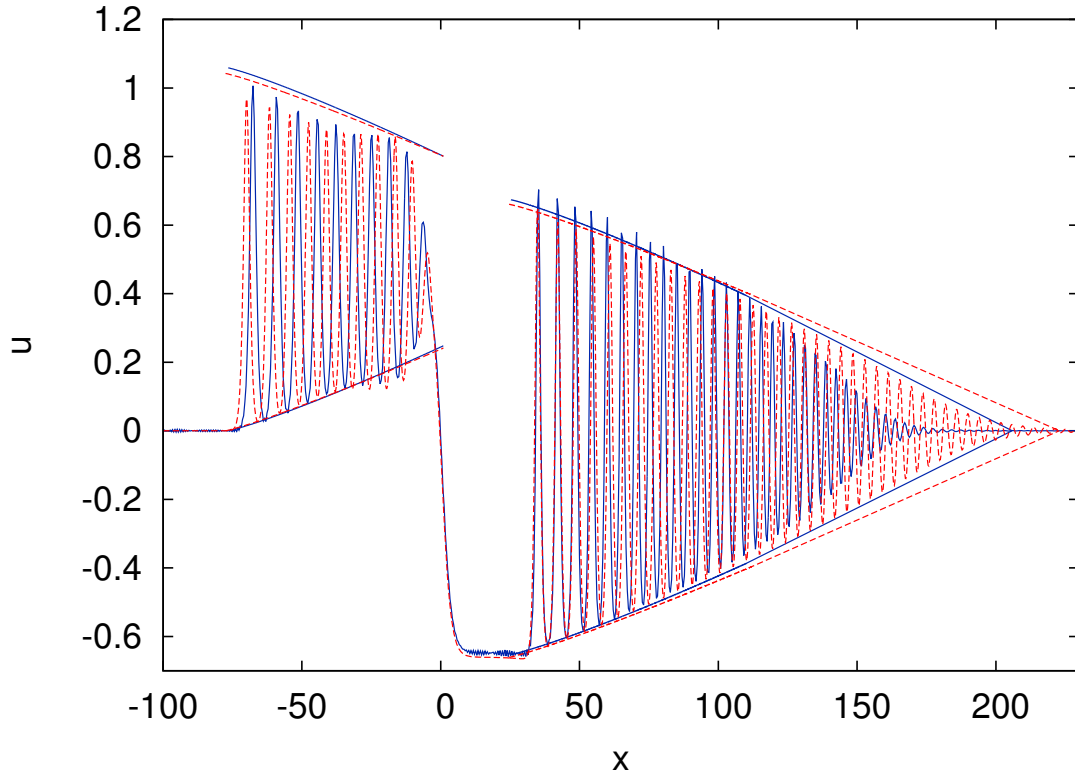


FIG. 7. Solution  $u$  versus  $x$  at  $t = 30$  for surface water waves Compared are numerical solutions of the forced eKdV equation (blue solid line) and the forced KdV equation (red dashed line). The other parameters are  $\alpha = 0.15$  and  $\Delta = -0.5$ . Also shown are the wave envelopes for the upstream and downstream undular bores given by eKdV (blue solid line) and KdV modulation theories (red dashes).

Figure 7 shows a subcritical solution  $u$  versus  $x$  at  $t = 30$ , for surface water waves. Compared are numerical solutions for the forced eKdV and KdV equations. Also shown are the wave envelopes for the upstream and downstream modulated wavetrains from the eKdV and the KdV modulation theories. The other parameters are  $\alpha = 0.15$  and  $\Delta = -0.5$ . The amplitude of the lead solitary wave of the upstream bore as given by eKdV modulation theory is  $A = 1.07$  compared with the numerical value  $A_n = 1.01$ . These amplitudes given by the KdV theory are a modulation amplitude  $A = 1.04$  and a numerical amplitude  $A_n = 0.97$ . The eKdV theory then gives amplitudes which are 3% higher than the KdV results, indicating the effect of higher order corrections to the KdV equation. The difference between the theoretical and the numerical results is about 7%, which reduces to 2% for steady state

numerical solutions. The downstream solitary wave amplitude as given by eKdV modulation theory is  $B = 1.34$ , while the numerical amplitude is  $B_n = 1.36$ . Numerical solutions and modulation theory for the KdV equation give the equivalent amplitude  $B = B_n = 1.32$ . Again, the higher order corrections to the KdV equation give small changes in the upstream and downstream amplitudes, 3% from numerical solutions and 1% from modulation theory. In addition, modulation theory gives that the downstream eKdV bore lies in  $27 < x < 207$  with width  $w = 180$  and the KdV bore lies in  $25 < x < 223$  with width  $w = 198$ , as compared with the numerical values  $34 < x < 181$  with width  $w = 147$  for the eKdV equation and  $35 < x < 208$  with width  $w = 173$  for the KdV equation. As for the previous exactly resonant and supercritical examples, higher order effects again result in a narrower bore<sup>22,29</sup>.

#### IV. CONCLUSIONS

In this paper we have studied resonant flow over topography using the framework of the forced eKdV equation (1) in order to gauge the effect of higher order corrections to the standard KdV approximation for weakly nonlinear long waves. Our results show that the eKdV predictions, which include these higher order terms, vary from the KdV predictions when the wave amplitudes are large. This occurs in the supercritical regime for upstream waves and the subcritical regime for downstream waves; in both cases the eKdV predictions are higher than the KdV ones. The widths of the eKdV downstream undular bores are significantly reduced compared with KdV theory, in agreement with results based on the higher-order internal wave bore<sup>22</sup>. This reduction is predicted in both flow settings even though the internal wave bore has different higher-order coefficients to those for surface water waves. It was further found that the inclusion of higher order corrections to the KdV approximation has greater effects on the upstream bore than on the downstream bore. For the amplitude scale  $\alpha = O(0.1)$  the effects of the higher order corrections are up to 10%. These differences can be significant when comparisons are made with solutions of the full water wave equations and with experimental and observational results.

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