Instruments for impossible problems: Around the work of Ljubomir Klerić (1844-1910)

Dominique Tournès

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I am interested in mathematical problems that are non-constructible with a certain set of instruments, but that may be constructed by creating new devices. After introducing this issue, I will present a case study centred on a specific instrument conceived at the end of the 19th century by the Serbian engineer, Ljubomir Klerić.

In the first three postulates of Euclid, no physical instrument is mentioned, but everyone understands that they constitute an idealization of the use of a ruler and a compass. In his famous commentary on the first book of the *Elements*, Proclus insists that a straight line, a circle and, more generally, any line, is the trace of a moving point. He describes without any ambiguity the mechanical movement that generates a circle, but for the straight line he offers a more vague "uniform and undeviating flowing", which is not as well-defined.

This flaw has been noticed by Alfred Kempe and many engineers such as Watt and Tchebycheff, when they wanted to mechanically replicate perfect linear motion in steam engines. In his stimulating book entitled *How to draw a straight line*, Kempe says ([2], p. 2): "If we are to draw a straight line with a ruler, the ruler must itself have a straight edge; and how are we going to make the edge straight? We come back to our starting-point." The first satisfactory answer was found by the French engineer Peaucellier and, independently, by the Lithuanian Lipkin: one can construct a straight line with a linkage transforming by inversion a circular motion into a perfect straight-line motion.

Due to its first three postulates, Euclid's geometry concerns the problems that can be solved by constructing a finite number of straight lines and circles. However, in the Antiquity and the Middle Ages, in particular in the Greek and Arabic worlds, mathematicians encountered problems that they could not solve with a ruler and a compass: the most famous of these are the duplication of the cube, the trisection of an angle, and the quadrature of the circle. To solve these problems, they had to introduce new devices and new curves, starting with the conic sections.

In his *Géométrie* of 1637, Descartes remarked that ruler and compass are machines, and so there was no reason to refuse the use of other machines in geometry, as long as they generated curves by a simple, continuous motion. By this definition, he accepted the use of what we call today algebraic curves, which can be mechanically traced, at least locally, by linkages. In a certain sense, we can view a linkage as analogous to a combination of a finite number of compasses.

At the end of the 17th century, algebraic curves were no longer a source of great interest within the realm of infinitesimal calculus, hence Leibniz expended much effort exploring new kinds of continuous movements that could generate transcendental curves. Eventually deciding that tractional motion was the best candidate to revitalize geometry, Leibniz imagined a universal integraph for quadratures and more general differential equations. His idea was to take a tense string and, by a suitable mechanical device, as it moved, impose the concomitant slope given by the differential equation. Thus the motion of the string traced a curve whose tangents are given, in other words, an integral curve solution of the inverse tangent problem. This idea gave birth to a complete theory developed later by Euler and Vincenzo Riccati, and is also at the origin of the conception and the making of actual integraphs, a few in the 18th century, but the majority at the end of the 19th century [6].

Throughout the 19th century, new abstract methods of reasoning were implemented to study the classical problems that remained unsolved. For the first time, the impossibility of defining a solution to a given problem was rigorously established, and the set of problems that could be solved with a given procedure, clearly characterized. Among the results, we should mention here that Wantzel proved in 1837 the impossibility of the duplication of the cube, and the trisection of an angle with a ruler and a compass; also Lindemann established in 1882 the transcendence of π, the consequence of which was to be able to confirm the impossibility of squaring the circle with a ruler and a compass.

In fact, all the problems that were proven as impossible to solve at that time were subsequently solved by the introduction of new instruments. I want to illustrate this by considering the case of Ljubomir Klerić. The starting point for this study was a curious paper published in 1897 in the *Dinglers polytechnisches Journal*, which announced an ambitious program: the construction of the numbers π and e, and all regular polygons [4].

Julius Klery was born in Subotica, Austria-Hungary, on June 29, 1844. His family was of German origin. When he arrived in Belgrade, he decided to adopt a Serbian form for his name: Ljubomir Klerić (or Kleritj). After graduation from high school, he studied engineering at the Belgrade College. In 1865, having received a state scholarship, he was sent to the mining academies in Freiberg and Berlin, and to the Zürich polytechnical school. From 1870 to 1875, he worked for mining companies in Westphalia, Saxony, Upper Silesia and Bohemia. In 1875, he became a professor of the Belgrade College. In 1887, he was elected as a full member of the Serbian Royal Academy. During 1894-1895, he was Minister of education and ecclesiastical affairs, and in the period 1896-1897, he was Minister of the national economy. He died in Belgrade on the 21st of January, 1910 [5].

Klerić fits the definition of an "ingénieur-savant", that is an engineer with a strong training in mathematics, able to create new mathematics by himself for the needs of his practice, as well as being active in the scientific institutions of his country and publishing in scientific journals. Between 1872 and 1907, he actually produced 48 articles and books in several domains of the engineering sciences and in mathematics. In particular, his work included the invention of some new instruments: a new typewriter named a "polypantograph"; a new compass named a "tractoriograph" or "logarithmograph", and several measuring instruments including a precision curvometre and a logarithmometre.

The second of these instruments concerns us. Klerić describes it in these terms ([4], p. 234): "In 1891, I invented a very simple instrument with which, for all kinds of plane curves, one can describe their tractrix at a constant distance, that is with a constant tangent. I called this instrument 'tractoriograph'. This instrument is made at the Mechanical Institute of Oskar Leuner in Dresde, and costs 22 M." In fact, Klerić's device (see Fig. 1) is a variant of the famous Prytz planimeter, a very simple instrument that allows calculation of areas, not exactly, but to a very good approximation sufficient for most practical applications. This planimeter was a great success because it was cheap and easy to use.

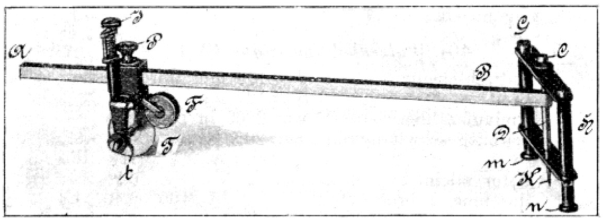

FIGURE 1. Klerić's tractoriograph ([4], p. 234)

Klerić's originality is that he does not use his instrument for calculating areas of surfaces but, in a purely theoretical sense, for the construction of impossible mathematical problems. The major part of these constructions is based on the "circular tractrix", that is the tractrix of a circle traced with the condition that the length or the tractoriograph is equal to the radius of the circle. A spectacular property of this curve is that, if it is joined to a ruler and a compass, it allows the rectification of any arc of the circle. From that, it is easy to construct the number π, to rectify and to square the circle, and to inscribe in it a regular polygon with any number of sides. Incidentally, by using the tractrix of a straight line, Klerić proved that the number e is also constructible with his tractoriograph.

It is amazing to note that at the same time, a different solution of the quadrature of the circle was published independently by Felix Klein in his *Famous Problems of Elementary Geometry* ([4], p. 78): "An actual construction of π can be effected only by the aid of a transcendental curve. If such a construction is desired, we must use besides straight edge and compasses a 'transcendental' apparatus which shall trace the curve by continuous motion. Such an apparatus is the *integraph*, recently invented and described by a Russian engineer, Abdank-Abakanowicz, and constructed by Coradi of Zürich."

It is true that in the solution of classical problems, some instruments like Descartes' linkages or Leibniz's integraphs seem to be thought experiments or imaginary, ideal instruments conceived to solve theoretical problems, but never constructed for actual use. From a different perspective, we have seen that other people, such as Klerić or Klein, have exploited actual physical devices to solve the same problems.

In the 1970s, the Russian engineer Ivan Ivanovitch Artobolevsky published an encyclopedia of mechanisms in five volumes [1]. The aim of this treatise is to provide an inventory and describe all the elementary mechanisms that engineers can use and combine to create complex machines. In this collection, we find algebraic mechanisms: linkages to trace the three conic sections, a linkage to extract cubic roots (which is in fact the old device attributed to Plato), the trisector of Descartes, a linkage to trace the conchoid of Nicomedes, and another to trace the cissoid of Diocles. We can also find transcendent mechanisms: the polar planimeter of Amsler, the integraph of Abdank-Abakanowicz, tractional instruments to trace the logarithmic curve, and the spiral of Archimedes.

In this inheritance accumulated in mechanical engineering practice, we recognize the major part played by the famous devices that have been conceived since Antiquity to solve the classical problems. Most of them were probably ideal machines when they were first used by mathematicians. However, in Artobolevski's catalogue, they are also physical and perfectly efficient machines. All these devices are clearly at the crossroads of mathematics, mechanics and technology. They can be both imaginary and physical, and it seems important to me to study them from both of these perspectives.

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Tractional constructions as foundation of differential equations: Ancient open issues, new results, possible fallouts

Pietro Milici

Machines play various roles in mathematics: they can embody mathematical concepts to be transferred to real-world applications and foster deeper understanding (while conceiving, constructing and using them). But devices can also play a very relevant foundational role, as seen in the geometry of Euclid or Descartes: "simple" machines can be idealized to become the quintessence of fundamental concepts still keeping a strict contact with concrete experience and allowing manipulation (so