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Abstract When a subdivision scheme is factorised into *lifting* steps, it admits an in–place and invertible implementation, and it can be the predictor of many multiresolution biorthogonal wavelet transforms. In the regular setting where the underlying lattice hierarchy is defined by $\mathbb{Z}^s$ and a dilation matrix M, such a factorisation should deal with every vertex of each subset in $\mathbb{Z}^s/M\mathbb{Z}^s$ in the same way. We define a subdivision scheme which admits such a factorisation as being uniformly elementary factorable. We prove a necessary and sufficient condition on the directions of the Box spline and the arity of the subdivision for the scheme to admit such a factorisation, and recall some known keys to construct it in practice.

Keywords Box spline · Subdivision scheme · Biorthogonal filter bank · Elementary factorisation · Lifting scheme

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1 Introduction

A subdivision scheme adds to a discrete topological object (signal, polygonal line, mesh,. . .) more discrete elements (samples, vertices, edges, faces,. . .) so that, when applied iteratively, it defines a sequence of discrete objects which converges to a continuous object of the same dimension (function, curve, surface,. . .). It is made up with a local topological operator, which describes where to insert new discrete elements, and defines a hierarchy of lattices. The value or position of the new

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samples or vertices are defined by a local linear combination of samples or vertices of the previous object in the sequence.

Thus, a subdivision scheme creates a natural multiresolution structure and can be used as a predictor, that is a synthesis operator without any detail, of a multiresolution biorthogonal wavelet transform if it can be factorised into *lifting steps* [15, 28]. It yields then a Finite Impulse Response filter bank with perfect reconstruction, with analysis and synthesis transforms being local and linear without having to solve any global linear system [27]. Moreover, if the subdivision rules come from a refinement equation of given functions, then translates and dilates of these functions are the *scaling functions* of the multiresolution analysis, which gives us a complete knowledge of their properties such as their smoothness. Note also that such a factorisation is interesting for the subdivision scheme itself as it yields an efficient in–place and invertible implementation.

Dahmen and Micchelli showed that such a factorisation is possible for the dyadic subdivision of uniform or non–uniform B–splines [5]. In the regular setting for s–dimensional objects, the coarsest lattice is $\mathbb{Z}^s$ and the hierarchy is defined by a dilation matrix M (an $s \times s$ matrix with integer entries and eigenvalues larger than 1 in modulus). If m is the determinant of this dilation matrix, then each lattice is split into m sub–lattices of equivalent locations in $\mathbb{Z}^s/M\mathbb{Z}^s$. Besides, if the subdivision scheme is based on the refinement equation of a single function associated to this hierarchy, then a single filter (or *subdivision mask*) applies and yields m subdivision rules (or *stencils*), each defining every vertex associated to one of the m sub–lattices. If we consider the Z–transform of these filters, the stencils are the *polyphase components* of the mask, and a lifting step corresponds to an *elementary matrix* [7]. The factorisation must be consistent with this context and every vertex associated to the same sub–lattice must be dealt with exactly in the same way. A subdivision scheme which admits such a uniform factorisation is defined in this article as being *uniformly elementary factorable*.

If the dilation matrix is a diagonal matrix with the same value a in every entry of its diagonal, the subdivision scheme is said to have an arity equal to a. In this article, we show that the subdivision scheme based on the refinement equation of an s–dimensional Box spline is not always uniformly elementary factorable and we give necessary and sufficient condition on the *directions* of the Box–Spline and the arity the subdivision scheme for such a factorisation to be possible. In Section 2, we define the elements involved in this theorem and explain its consequences for implementing a subdivision scheme as for building a multiresolution filter bank. We propose a proof of the theorem in Section 3, and we collect in Section 4 some known results about designing in practice such a factorisation. In particular, if the theorem concerns only Box spline–based subdivision in the regular setting, some factorisation designs presented in this last section apply also to meshes with extraordinary vertices or to schemes which are not based on Box splines.

2 Definitions and interpretation

2.1 Definitions

Let us first introduce a notation that will be used with different variations throughout this article. Let $\{w_1, \ldots, w_\kappa\}$ be a set of vectors of the same dimension, and ξ

a scalar space. Then,

$$[w_1 \dots w_\kappa]\xi^\kappa := \left\{ \sum_{i=1}^{\kappa} \alpha_i w_i, \alpha_i \in \xi \right\}. \tag{1}$$

When $\{w_1, \dots, w_\kappa\}$ are the columns of a matrix W, then $W\xi^\kappa$ will define the same set.

A *Box spline* $B(\cdot|v_1, \dots, v_k)$ is a function from $\mathbb{R}^s$ to $\mathbb{R}$ defined with k *directions* $v_1, \dots, v_k \in \mathbb{R}^s$ such that $(v_1, \dots, v_s)$ is a basis of $\mathbb{R}^s$. There are many ways to define a Box spline [8]. We recall here the inductive definition by successive convolutions, introducing the first s directions together, and then the $k - s$ others one by one:

$$B_s(t) := \begin{cases} 1/|d| & \text{if } t \in [v_1 \dots v_s][0,1[^s, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

$$B_\kappa(t) := \int_0^1 B_{\kappa-1}(t - uv_k)du, \kappa > s, \tag{3}$$

$$B(t|v_1, \dots, v_k) := B_k(t), \tag{4}$$

where d is the determinant of the $s \times s$ matrix whose columns are the first s directions $v_1, \dots v_s$. In order to define a (converging) subdivision scheme, we constrain the directions to live in $\mathbb{Z}^s$, and such that $[v_1 \dots v_k]\mathbb{Z}^k = \mathbb{Z}^s$.

Let a be the arity of the subdivision scheme, and H be the $s \times s$ diagonal matrix with $1/a$ on the diagonal. The matrix H^{-1} is the dilation matrix M mention-Ned in the introduction. The scheme defines new vertices $\lambda[y]$, $y \in H\mathbb{Z}^s$, embedded into some space, by local linear combination of old vertices $\Lambda[x]$, $x \in \mathbb{Z}^s$. The weights used in these linear combinations are gathered into the *subdivision mask* $b_0[y]$, $y \in H\mathbb{Z}^s$, called also, up to a different normalisation, *discrete box spline* [8,13]. Similarly to the continuous function definition, the mask is defined by consecutive discrete convolutions in the directions $v_1, \dots, v_k$. A graphical way to define it is proposed by Dodgson and co–workers [9]. An important property is that the number of non–zero entries in the mask is finite.

The mask gives the weights of a given old vertex into the definition of new vertices:

$$\lambda[y] = \sum_{t \in \mathbb{Z}^s} b_0[y - t]\Lambda[t], \ y \in H\mathbb{Z}^s. \tag{5}$$

As $t \in \mathbb{Z}^s$ and $y \in H\mathbb{Z}^s$, all the entries of the mask are not involved in the definition of each new vertex $\lambda[y]$, but several new vertices use the same subset of its entries. Indeed, $H\mathbb{Z}^s$ is partitioned into $\mathbb{Z}^s$ and its $a^s - 1$ cosets in $H\mathbb{Z}^s/\mathbb{Z}^s$. Let $y_i \in H\mathbb{Z}^s, i = 0, \dots, a^s - 1$ be one element of each subset. The set of new vertices is partitioned accordingly:

$$\{\lambda[y], \quad y \in H\mathbb{Z}^s\} = \bigcup_{i=0}^{a^s-1} \{\lambda_i[x] := \lambda[x + y_i], \ x \in \mathbb{Z}^s\}, \tag{6}$$

and (5) can be written specifically for the vertices of each subset as

$$\lambda_i[x] = \sum_{t \in \mathbb{Z}^s} b_{0,i}[t - x]\Lambda[t], \ x \in \mathbb{Z}^s, \tag{7}$$

where $\{b_{0,i},\ i=0,\dots,a^s-1\}$ are the *stencils* of the subdivision scheme defined as:

$$b_{0,i}[x] := b_0[-x+y_i],\ x\in\mathbb{Z}^s. \tag{8}$$

Note that (5) defines the "signal" of new vertices as the convolution of the "signal" of old vertices with the mask, whereas (7) defines each new vertex as the product to the left of the "column vector" of old vertices with a "row vector" whose non–zero entries are given by one stencil.

With the usual notation, if $z=(z_1,\dots,z_s)\in\mathbb{C}^s$ and $x=(x_1,\dots,x_s)\in\mathbb{Z}^s$, then $z^{-x} := \prod_{i=1}^s z_i^{-x_i}$, let us consider the Laurent polynomials defined as the Z–transforms of these stencils

$$b_{0,i}(z) := \sum_{x\in\mathbb{Z}^s} b_{0,i}[x]z^{-x}, \tag{9}$$

gathered into the vector

$$B_0(z) := \begin{bmatrix} b_{0,0}(z) \\ \vdots \\ b_{0,a^s-1}(z) \end{bmatrix}. \tag{10}$$

We consider two families of invertible matrices with entries in the ring of Laurent polynomials:

- type 1 (*elementary matrices*):

$$E^1(z)\in\{I+c(z)F_{ij}\}, \tag{11}$$

- type 2:

$$E^2(z)\in\{I+(d-1)F_{ii}\}, \tag{12}$$

where F_{ij} is a matrix with just one non–zero element in it

$$(F_{ij})_{k,l} := \begin{cases} 1 \text{ if } (k,l)=(i,j), \\ 0 \text{ otherwise,} \end{cases} \tag{13}$$

$c(z)$ is a Laurent polynomial, $d\in\mathbb{R}\setminus\{0\}$, and I is the identity matrix.

We are now ready for the main definition of this section.

Definition 1 A Box spline subdivision scheme is *uniformly elementary factorable* if the vector of Z–transforms of its stencils can be written as

$$B_0(z) = z^x \prod_{p=1}^{P<\infty} E_p^{t_p}(z) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ x\in\mathbb{Z}^s,\ t_p\in\{1,2\}. \tag{14}$$

2.2 Interpretation

Let us explain the relationship between this property and, firstly, the definition of a multiresolution biorthogonal wavelet filter bank with the subdivision scheme as a predictor and secondly, an in–place and invertible implementation of the subdivision scheme.

2.2.1 Biorthogonal filter bank

From (14) $z^{-x}B_0(z)$ can be completed into an invertible square matrix

$$\prod_{p=1}^{P<\infty} E_p^{t_p}(z),\ t_p \in \{1,2\}\,, \tag{15}$$

which, as explained by Daubechies and Sweldens [7] or by Park [21], can be the polyphase matrix of the synthesis transform of a multiresolution FIR filterbank with perfect reconstruction, whose dual polyphase analysis matrix is its inverse, that is the product, in reverse order, of the inverse of the matrices, which are matrices of the same kind:

$$\begin{cases} (I + c(z)F_{ij})^{-1} = I - c(z)F_{ij}; \\ (I + (d-1)F_{ii})^{-1} = I + (\frac{1}{d} - 1)F_{ii}. \end{cases} \tag{16}$$

Note that factoring a subdivision scheme as (14) corresponds to what Kobbelt and Schröder describe as building it as a wiring diagram consisting of lifting steps [15]. In particular, in terms of the *toolbox* that they proposed, each matrix $E_p^1(z)$ corresponds to a *predict* element and each matrix $E_p^2(z)$ corresponds to a *scale* element. Strictly speaking, only the elementary matrices $E_p^1(z)$ correspond to a *lifting* step as introduced by Sweldens [27], the scaling elements as the phase shift z^{-x} being there to normalise the filters.

As already stated by these previous works, this polyphase synthesis matrix should be multiplied to the right by further elementary matrices if we want to construct a more stable multiresolution framework, for example with vanishing moment wavelet functions. Besides, the scaling functions of this multiresolution transform made up with the translates and dilates of the Box spline at the origin of its construction, are not linearly independent in general, and then do not make up a Riesz basis. For them to be linearly independent, the directions of the Box spline should satisfy $|\det \bar{V}| = 1$, for every $\bar{V} \in \mathcal{B}(V)$, where $V := [v_1 \cdots v_k]$ is the $s \times k$ matrix with the Box spline directions $v_1, \ldots, v_k \in \mathbb{Z}^s$ as columns, and $\mathcal{B}(V)$ is the set of all $s \times s$ invertible matrices with s among the k directions as columns [8]. Biorthogonal Box spline wavelets construction have to satisfy this condition which excludes a lot of Box splines with exotic directions [22]. But the condition given by our Theorem 1 for a Box spline subdivision to be uniformly elementary factorable is less restrictive. It would then be interesting to investigate what kind of bi-framelets could be built from these more exotic Box splines [6,4,23,11]. In particular, as the polyphase matrix is square, the wavelet bi–frames will suffer from the restrictions of the biorthogonal setting such as dual functions with poor properties [10], but the framework could still be useful in practice. However, such a study is outside the scope of this article.

2.2.2 In–place invertible implementation of a subdivision scheme

First of all, we stress that the factorisation (14) is different from the probably more common refine–and–smooth factorisation of a subdivision scheme. Indeed, the main target of refine–and–smooth implementation is to factorise subdivision into sub–steps where combinations involve only neighbouring vertices. If some refine–and–smooth factorisations allow in–place computations [2], it is not the case for all the refine–and–smooth factorisations, and in particular not for the first of them by Lane and Riesenfeld [17].

On the contrary, *uniformly elementary factorisation* of a subdivision scheme yields an in–place and invertible implementation which allows us to go efficiently through the different levels from coarse to fine as from fine to coarse, without any more memory than what is necessary to store the finest mesh. The fact that each step could involve only neighbouring vertices is considered only as a bonus.

Let us explain now how to define such an implementation from factorisation (14). If $S_{a^s*,*}$ is the bi–infinite matrix which transforms old vertices into new vertices (the index notation tells that there are a^s times more rows than columns), then each row is one of the "row vectors" involved in the definition (7) of a given new vertex $\lambda_i[x] = \lambda[x+y_i]$, $x \in \mathbb{Z}^s$, $y_i \in H\mathbb{Z}^s$, whereas each column, corresponding to an old vertex $\Lambda[t]$, $t \in \mathbb{Z}^s$, contains the weights of the mask.

From (7), the Z–transforms of the rows of the subdivision matrix are equal to

$$\sum_{t\in\mathbb{Z}^s} b_{0,i}[t-x]z^{-t} = z^{-x}b_{0,i}(z). \tag{17}$$

As a consequence, the factorisation (14) of $B_0(z)$ corresponds to the following factorisation of $S_{a^s*,*}$

$$S_{a^s*,*} = \prod_{p=1}^{P<\infty} E_p^{t_p} I_{a^s*,*},\ t_p \in \{1,2\}\,, \tag{18}$$

where

- $I_{a^s*,*}$ is a matrix with the same dimensions as $S_{a^s*,*}$ and whose entries $(I_{a^s*,*})_{k,l}$ are zero everywhere except $(I_{a^s*,*})_{k,k} = 1$;
- $E_p^{t_p}$ are square matrices with as many rows as $S_{a^s*,*}$ and such that $E_p^{t_p}A$ updates the rows of A,
  - either by adding to them a finite linear combination of rows corresponding to points belonging to the other subsets ($t_p = 1$),
  - or by multiplying all the rows corresponding to the same subset of $H\mathbb{Z}^s$ by a constant $d \in \mathbb{R}\setminus\{0\}$ ($t_p = 2$).

Thus, each $E_p^{t_p}$ corresponds to an in–place and invertible update of new vertices corresponding to the same subset of $H\mathbb{Z}^s$.

Let us illustrate such a factorisation with the Chaikin subdivision scheme [3]. This dyadic scheme ($a = 2$) is based on the univariate ($s = 1$) Box spline $B(\cdot|1,1,1)$ with $k = 3$ directions all equal to the unity vector, its mask is $b_0[x] = [1/4,\ 3/4,\ 3/4,\ 1/4]$ and the vector of Z–transforms of its stencils is (with $y_0 = 0$ and $y_1 = \frac{1}{2}$)

$$B_0(z) = \begin{bmatrix} 1/4 + 3/4z \\ 3/4 + 1/4z \end{bmatrix}. \tag{19}$$

As we can write, for example,

$$B_0(z) = z\begin{bmatrix} 1 & 1/3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2/3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3/4z^{-1}+1/4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{20}$$

the Chaikin scheme is uniformly elementary factorable. The in–place and invertible implementation of this scheme, corresponding to the factorisation (20) is, after the initialisation $\forall x \in \mathbb{Z}$

$$\lambda[x] := \Lambda[x]; \tag{21}$$
$$\lambda[x+\frac{1}{2}] := 0; \tag{22}$$

the three following steps (note that due to the definition of Z–transform, a negative power of z corresponds to a right neighbour):

$$\lambda[x+\frac{1}{2}] \mathrel{+}= 3/4\ \lambda[x+1] + 1/4\ \lambda[x]; \tag{23}$$
$$\lambda[x] \mathrel{*}= 2/3; \tag{24}$$
$$\lambda[x] \mathrel{+}= 1/3\ \lambda[x+\frac{1}{2}]. \tag{25}$$

In practice, the last two steps (24) and (25) can be implemented in a single step. More generally, the factorisation (20) is not unique and may lead to different in–place implementations. The study of a factorisation leading to an optimal implementation in the number of steps or in the distance between two vertices involved in each of them, is outside the scope of this article.

3 Necessary and sufficient condition for a Box spline subdivision to be uniformly elementary factorable

3.1 Theorem

Not all the Box spline subdivision schemes are uniformly elementary factorable. In this section, we prove the following characterisation of these schemes.

Theorem 1 *Let $V := [v_1 \cdots v_k]$ be the $s\times k$ matrix with the directions $v_1,\ldots,v_k \in \mathbb{Z}^s$ as columns. Let $\mathcal{B}(V)$ the set of all $s\times s$ invertible matrices with s among the k directions as columns. The three following propositions are equivalent:*

(i) the subdivision scheme associated with the Box spline $B(\cdot|v_1,\ldots,v_k)$ and the arity a is uniformly elementary factorable;
(ii) a is relatively prime to $|\det \bar{V}|$, for every $\bar{V} \in \mathcal{B}(V)$;
(iii) the columns $S^x_{a^s*}$, $x \in \mathbb{Z}^s$ of the bi-infinite subdivision matrix $S_{a^s*,*}$ are linearly independent, i.e. if there are $\{\alpha_x \in \mathbb{R}\}_{x\in\mathbb{Z}^s}$ such that

$$\sum_{x\in\mathbb{Z}^s} \alpha_x\ S^x_{a^s*} = 0_{a^s*} \tag{26}$$

where 0_{a^s*} is the column of zeros with the same dimension as the columns of the matrix, then $\forall x \in \mathbb{Z}^s$, $\alpha_x = 0$.

The item *(iii)* has been introduced in order to prove the equivalence between the first two items *(i)* and *(ii)*.

Moreover, we stress the fact that, if linear independence defined in this item has the usual meaning, however, the sum is a series. But due to the finiteness of the subdivision mask, the non–zero entries of each row of $S_{a^s*,*}$ are in finite number, making the computation of the infinite sum of columns (26), an infinite set (one per "row") of finite sums, without any problem of sum convergence.

Let us illustrate this theorem with two subdivision schemes. The first one is the dyadic scheme ($a = 2$) based on the univariate ($s = 1$) Box spline $B(\cdot|1, 1, 2)$ with $k = 3$ directions. Its mask is $b_0[x] = [1/4,\ 1/2,\ 1/2,\ 1/2,\ 1/4]$ and the vector of Z–transforms of its stencils is (with $y_0 = 0$ and $y_1 = \frac{1}{2}$)

$$B_0(z) = \begin{bmatrix} 1/4 + 1/2z + 1/4z^2 \\ 1/2 + 1/2z \end{bmatrix}. \tag{27}$$

The matrix V is simply $V = [\ 1\ 1\ 2\ ]$ and the set $\mathcal{B}(V) = \{[1], [2]\}$. For $\bar{V} := [2]$, $\det \bar{V} = 2 = a$, the item *(ii)* is false and the subdivision scheme is not uniformly elementary factorable. Moreover, the bi–infinite subdivision matrix is (where only the non–zero entries are written)

$$S_{a^s*,*} = \begin{bmatrix} \ddots & \vdots & \vdots & & & & \\ & 1/2 & 1/2 & & & & \\ & 1/4 & 1/2 & 1/4 & & & \\ & & 1/2 & 1/2 & & & \\ & & 1/4 & 1/2 & 1/4 & & \\ & & & 1/2 & 1/2 & & \\ & & & 1/4 & 1/2 & 1/4 & \\ & & & & 1/2 & 1/2 & \\ & & & & \vdots & \vdots & \ddots \end{bmatrix} \tag{28}$$

and we can notice that a column $S^x_{a^s*}$ is equal to the alternate sum of the others:

$$S^x_{a^s*} = \sum_{t \in \mathbb{Z} \setminus \{0\}} (-1)^{1+|t|}\ S^{x+t}_{a^s*}, \tag{29}$$

confirming that item *(iii)* is false. Note however that an in–place implementation is still possible, but not invertible. Indeed, $B_0(z)$ can be written as

$$B_0(z) = z \begin{bmatrix} 1 & 1/2 + 1/2z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/2z^{-1} + 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{30}$$

yielding the following implementation

$$\lambda[x] \ := \ \Lambda[x]; \tag{31}$$
$$\lambda[x + \frac{1}{2}] \ := \ 0; \tag{32}$$
$$\lambda[x + \frac{1}{2}] \mathrel{+}= 1/2\ \lambda[x + 1] + 1/2\ \lambda[x]; \tag{33}$$
$$\lambda[x] \mathrel{*}= 0; \tag{34}$$
$$\lambda[x] \mathrel{+}= 1/2\ \lambda[x + \frac{1}{2}] + 1/2\ \lambda[x - \frac{1}{2}]. \tag{35}$$

whose step (34) cannot be inverted.

Let us consider now the triadic scheme ($a = 3$) based on the same univariate ($s = 1$) Box spline $B(\cdot|1, 1, 2)$ with $k = 3$ directions. Its mask is $b_0[x] = [1/9,\ 2/9,\ 4/9,\ 4/9,\ 5/9,\ 4/9,\ 4/9,\ 2/9,\ 1/9]$ and the vector of Z–transforms of its stencils is (with $y_0 = 0$, $y_1 = \frac{1}{3}$ and $y_2 = \frac{2}{3}$)

$$B_0(z) = \begin{bmatrix} 1/9 + 4/9z + 4/9z^2 \\ 2/9 + 5/9z + 2/9z^2 \\ 4/9 + 4/9z + 1/9z^2 \end{bmatrix}. \tag{36}$$

The matrix $V = [\ 1\ 1\ 2\ ]$ and the set $\mathcal{B}(V) = \{[1], [2]\}$ are the same than for the previous scheme, but as the arity is now $a = 3$, the item *(ii)* is true and the subdivision scheme is uniformly elementary factorable. Indeed, we can write

$$B_0(z) = z \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2/3 + 4/3z \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1/3 + 8/3z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{37}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2/3z^{-1} - 1/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{38}$$

yielding

$$\lambda[x] \ := \ \Lambda[x]; \tag{39}$$

$$\lambda[x + \frac{1}{3}] \ := \ 0; \tag{40}$$

$$\lambda[x + \frac{2}{3}] \ := \ 0; \tag{41}$$

$$\lambda[x] \ *= \ -1/2; \tag{42}$$

$$\lambda[x + \frac{2}{3}] \ += \ -2/3\ \lambda[x + 1] - 1/3\ \lambda[x]; \tag{43}$$

$$\lambda[x] \ += \ 1/3\ \lambda[x + \frac{2}{3}] + 8/3\ \lambda[x - \frac{1}{3}]; \tag{44}$$

$$\lambda[x + \frac{1}{3}] \ += \ 2/3\ \lambda[x + \frac{2}{3}] + 4/3\ \lambda[x - \frac{1}{3}]; \tag{45}$$

$$\lambda[x + \frac{2}{3}] \ += \ 1/2\ \lambda[x + \frac{1}{3}]. \tag{46}$$

3.2 Further definitions and known results

To prove Theorem 1, we use known results on discrete Box splines proved or recalled by Jia [13] and others from commutative algebra. Let us recall these results, and in particular by translating those on discrete Box splines to subdivision matrix properties.

3.2.1 Integer translates of discrete Box splines

We have already pointed out in Section 2.1 that, up to a different normalisation, discrete Box splines defined by Jia [13] are identical to the subdivision masks. Therefore, there is a direct link between columns of the subdivision matrix, and integer translates of discrete Box splines, described thereafter. Let us start with a definition which will be useful for this translation.

Definition 2 For a given set of rows $\mathcal{R}$ of a possibly bi–infinite matrix, the *associated local columns* are made up with all columns having at least one non–zero entry in $\mathcal{R}$, and reduced to their entries belonging to $\mathcal{R}$.

We recall that the rows of the subdivision matrix $S_{a^s*,*}$ are associated with points in $H\mathbb{Z}^s$. The *local linear independence* of the integer translates of a discrete Box spline defined in Section 7 in [13], can then be translated into the following definition.

Definition 3 Let $S_{a^s*,*}$ be the subdivision matrix associated with the Box spline $B(\cdot|v_1,\ldots,v_k)$ and the arity a. Its columns are *locally linearly independent* if, for any subset Ω of $\mathbb{R}^s$, the local columns associated with the rows which correspond to the points $\big(\Omega - [\frac{1}{a}v_1 \ldots \frac{1}{a}v_k][0;1]\big) \cap H\mathbb{Z}^s$ are linearly independent.

With this definition, we can translate Theorem 7.1 and Corollary 4.3 from [13] into the following theorem, with V and $\mathcal{B}(V)$ defined as in Theorem 1.

Theorem 2 *The three following propositions are equivalent:*

(i) *the columns of $S_{a^s*,*}$ are linearly independent;*
(ii) *the columns of $S_{a^s*,*}$ are locally linearly independent;*
(iii) *a is relatively prime to $|\det \bar{V}|$, for every $\bar{V} \in \mathcal{B}(V)$.*

3.2.2 Commutative algebra

The definitions and theorems recalled in this section can be found, for example, in Park's thesis [21] or in Lam's book [16].

Let $GL_p(\mathcal{P}_s)$ be the *general linear group* of invertible $p \times p$ matrices over $\mathcal{P}_s$, the ring of s-variate polynomials. Park proved that the Quillen–Suslin theorem is equivalent to the *unimodular completion* defined as follows [21].

Theorem 3 *Let A_0 be a $p \times q$ unimodular matrix ($p \geq q$) with polynomial entries. A_0 can be completed into a square $p \times p$ unimodular matrix $A \in GL_p(\mathcal{P}_s)$ by adding $p-q$ columns to A_0.*

Besides, the Quillen–Suslin theorem has been extended to $\mathcal{L}_s$, the commutative ring of Laurent polynomials, by Swan [26] or Park [21].

As said in the previous definition, when $p = q$, a $p \times q$ matrix is *unimodular* if it is invertible. When $p > q$, we are only interested in the case $q = 1$, which has the following characterisation.

Definition 4 An $s \times 1$ vector $B_0(z)$ with elements in $\mathcal{L}_s$, is *unimodular* if there is a $1 \times s$ vector $C(z)$ such that $C(z)\ B_0(z) = 1$.

Let $SL_p(\mathcal{L}_s)$ be the *special linear group* of invertible $p \times p$ matrices over $\mathcal{L}_s$ of determinant 1. Let $E_p(\mathcal{L}_s)$ be the subgroup of $SL_p(\mathcal{L}_s)$ generated by the elementary matrices defined in Section 2.1 as invertible matrices of type 1. These two groups are linked by the following theorems.

Theorem 4 (Suslin's stability [25]) *$SL_p(\mathcal{L}_s) = E_p(\mathcal{L}_s)$, $\forall p \geq 3$, $\forall s \geq 1$.*

Theorem 5 (proved e.g. by Lam [16]) *Since $\mathcal{L}_1$ is a Euclidean ring, $SL_p(\mathcal{L}_1) = E_p(\mathcal{L}_1)$, $\forall p$.*

3.3 Proof

We are now ready to prove Theorem 1. The equivalence between items *(ii)* and *(iii)* is given by the equivalence between items *(i)* and *(iii)* in Theorem 2. Let us prove the equivalence between items *(i)* and *(iii)* of Theorem 1.

(i) $\Rightarrow$ *(iii)*
As explained in Section 2.2, *(i)* implies in particular the existence of the factorisation (18) of the subdivision matrix $S_{a^s*,*}$ which itself would directly imply the linear independence of the columns if the matrix was finite. But as it is bi-infinite, we have to take care that each step of the proof is well-defined.

Let us suppose that there is a column vector α_* with as many entries as the number of columns in $S_{a^s*,*}$ such that

$$S_{a^s*,*}\ \alpha_* = 0_{a^s*} \tag{47}$$

where 0_{a^s*} is the column of zeros with the same dimension as the columns of the matrix.

We recall that each $E_p^{t_p}$ in (18) represents an elementary row operation. Thus, it is invertible by a matrix of the same kind. Indeed, if the row operation is the addition of each row corresponding to a point of a given subset of $H\mathbb{Z}^s$ by a finite linear combination of rows corresponding to points of other subsets, it is inverted by the subtraction of the same combinations from the same rows. If the row operation is the multiplication of all rows corresponding to the same subset of $H\mathbb{Z}^s$ by a constant $d \in \mathbb{R} \setminus \{0\}$, it is inverted by the multiplication of the same rows by $\frac{1}{d}$.

Besides, from this definition of these inverses, as each row of $S_{a^s*,*}$ contains a finite number of non–zero entries, so does each row of $E := (\prod_{p=P<\infty}^{1} (E_p^{t_p})^{-1} S_{a^s*,*})$ and the product $E\ \alpha_*$ is well-defined. In particular, in the combination that it defines, all entries of each column of E are multiplied by the same value. As a consequence,

$$E\ \alpha_* = \left(\prod_{p=P<\infty}^{1} (E_p^{t_p})^{-1} \right)\ (S_{a^s*,*}\ \alpha_*), \tag{48}$$

which yields, by assumption (47),

$$E\ \alpha_* = \left(\prod_{p=P<\infty}^{1} (E_p^{t_p})^{-1} \right)\ 0_{a^s*} \tag{49}$$

$$= 0_{a^s*}, \tag{50}$$

and from (18),

$$I_{a^s*,*}\ \alpha_* = 0_{a^s*}, \tag{51}$$

where $I_{a^s*,*}$ is a matrix with the same dimensions as $S_{a^s*,*}$ and whose entries $(I_{a^s*,*})_{k,l}$ are zero everywhere except $(I_{a^s*,*})_{k,k} = 1$. Thus, finally,

$$\alpha_* = 0_*, \tag{52}$$

which means that the columns of the subdivision matrix are linearly independent.

(iii) ⇒ (i)
From Theorem 2, if the columns of the subdivision matrix $S_{a^s*,*}$ are linearly independent, then they are locally linearly independent. From Definitions 2 and 3, it yields the existence of a set $\mathcal{R}$ of rows of $S_{a^s*,*}$ such that, the bi-finite matrix made up with their associated local columns has its columns linearly independent. As a consequence, there is a linear combination of the rows in $\mathcal{R}$ which is equal to a row with zero–entries except one equal to 1.

Several rows of $\mathcal{R}$ may be associated with points of the same subset of $H\mathbb{Z}^s$. So, if we consider the Z–transforms of these rows, this linear combination defines a $1 \times s$ vector $C(z)$ with entries in the commutative ring of Laurent polynomial $\mathcal{L}_s$, such that $C(z)\ B_0(z) = 1$.

From Definition 4, the vector of Z–transforms of the stencils $B_0(z)$ is then unimodular. From Theorem 3 and its extension to $\mathcal{L}_s$, it can be completed into a matrix $B(z) \in GL_{a^s}(\mathcal{L}_s)$ whose first column is $B_0(z)$.

The determinant of an invertible matrix in $\mathcal{L}_s$ is a monomial. Let $dz^x :=$ $\det(B(z))$ and $D(z)$ be the diagonal $a^s \times a^s$ matrix with 1 on the diagonal except $(D(z))_{0,0} := \frac{1}{d} z^{-x}$. Then, $B(z)\ D(z) \in SL_{a^s}(\mathcal{L}_s)$. From Theorems 4 and 5, $B(z)\ D(z) \in E_{a^s}(\mathcal{L}_s)$. As a consequence, there are matrices $E_p^1(z)$ such that

$$B(z)\ D(z) = \prod_{p=1}^{P-1<\infty} E_p^1(z). \tag{53}$$

Let us define $E_P^2(z)$ as a diagonal matrix with 1 on the diagonal except in $(0,0)$ where the entry is equal to d. Then we can write

$$B_0(z) = z^x \prod_{p=1}^{P-1<\infty} E_p^1(z)\ E_P^2(z) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{54}$$

which finishes the proof.

4 In practice

Theorem 1 gives a necessary and sufficient condition for a Box spline subdivision scheme to be uniformly elementary factorable. But is there a procedure to construct such a factorisation when it is possible? Following what was proposed by Daubechies and Sweldens [7] for factoring wavelet transforms, a procedure based on the Euclidean algorithm is possible. But as the Euclidean algorithm exists on $\mathcal{L}_s$ only if $s = 1$, heuristics have to be used for scheme with higher dimension parameter space.

However, let us illustrate how the Euclidean algorithm can be used in the univariate case ($s = 1$). The principle is the same as the one used in the particular case of $a = 2$ by Daubechies and Sweldens [7] for factoring wavelet transforms, or by Dahmen and Micchelli [5] for factoring a dyadic B–spline subdivision matrix (more precisely its transpose). Let us define the sequence of vectors $(B_0^j(z))_{j=0,\ldots,J}$ with

$$B_0^j(z) = \begin{bmatrix} b_{0,0}^j(z) \\ \vdots \\ b_{0,a-1}^j(z) \end{bmatrix}, \tag{55}$$

where $B_0^0(z) := B_0(z)$ and $B_0^{j+1}(z)$ is computed from $B_0^j(z)$ as follows. Let $(i_j, k_j) \in \{0, \ldots, a-1\}^2$ be such that $b_{0,i_j}^j(z)$ is an entry of $B_0^j(z)$ with maximal degree and $b_{0,k_j}^j(z) \neq 0$. The Euclidean division between $b_{0,i_j}^j(z)$ and $b_{0,k_j}^j(z)$ writes as

$$b_{0,i_j}^j(z) = q^j(z)\ b_{0,k_j}^j(z) + r^j(z) \tag{56}$$

and we define

$$\begin{cases} b_{0,i}^{j+1}(z) = b_{0,i}^j(z) \text{ if } i \neq i_j, \\ b_{0,i_j}^{j+1}(z) = r^j(z) \quad \text{otherwise.} \end{cases} \tag{57}$$

In particular, with the notation introduced in Section 2.1,

$$B_0^{j+1}(z) = \tilde{E}_j^1(z)\ B_0^j(z) \tag{58}$$

with

$$\tilde{E}_j^1(z) = I - q^j(z)\ F_{i_j,k_j}. \tag{59}$$

The sequence ends when all the entries of the vector are null except one, $b_{0,i_J}^J(z)$, which contains a greatest common divisor of the entries of $B_0(z)$ (which is defined up to a factor z^x). As written in Section 3.3, if the subdivision scheme is uniformly elementary factorable, then the vector of Z–transforms of its stencils $B_0(z)$ can be completed into an invertible matrix $B(z) \in GL_a(\mathcal{L}_1)$ whose determinant is a monomial. Any common divisor of the entries of $B_0(z)$ divides also this determinant, and so, is also a monomial. Thus, $b_{0,i_J}^J(z) = dz^x$. Finally, if $i_J = 0$, we get (14) with $P = J + 1$ and

$$E_p^1(z) = (\tilde{E}_{J+1-p}^1(z))^{-1},\ \forall p = 2, \ldots, P \tag{60}$$

$$E_1^2(z) = I + (d-1)F_{0,0}. \tag{61}$$

Otherwise, we get (14) with $P = J + 3$ and

$$E_p^1(z) = (\tilde{E}_{J+3-p}^1(z))^{-1},\ \forall p = 4, \ldots, P \tag{62}$$

$$E_3^1(z) = I - F_{0,i_J}, \tag{63}$$

$$E_2^1(z) = I + F_{i_J,0}, \tag{64}$$

$$E_1^2(z) = I + (d-1)F_{0,0}. \tag{65}$$

In practice, these first steps may be gathered into the initialisation of the implementation:

$$\lambda[x + \frac{i_J}{a}] := d\ \Lambda[x]; \tag{66}$$

$$\lambda[x + \frac{i}{a}] := 0,\ \forall i \in \{0, \ldots, a-1\} \setminus \{i_J\}. \tag{67}$$

This construction based on the Euclidean algorithm has been used to factorise the triadic scheme associated with $B(\cdot|1,1,2)$ in Section 3.1, with the following successive Euclidean divisions:

$$b^0_{0,2}(z) = (1/2)\ b^0_{0,1}(z) + (1/3 + 1/6z), \tag{68}$$
$$b^1_{0,1}(z) = (2/3 + 4/3z)\ b^1_{0,2}(z) + (0), \tag{69}$$
$$b^2_{0,0}(z) = (1/3 + 8/3z)\ b^2_{0,2}(z) + (-1/2z), \tag{70}$$
$$b^3_{0,2}(z) = (-2/3z^{-1} - 1/3)\ b^3_{0,0}(z) + (0). \tag{71}$$

Euclidean division between other entries of $B^j_0(z)$ could have been chosen, yielding another factorisation. And other factorisations, not based on the Euclidean algorithm, are still possible, as the one chosen for the Chaikin scheme in Section 2.2.

When the dimension s of the parameter space is greater that 1, a similar construction can be tried, but it is no longer certain to succeed as the Euclidean algorithm exists on $\mathcal{L}_s$ only if $s = 1$. As explained by Park about its special 1-input p-output case [19–21], Gröbner bases can be used for such a construction. But in practice, it does not provide the simplest complement filters. However, as seen in Section 3.1, if the subdivision scheme is uniformly elementary factorable, then $B_0(z)$ admits a left inverse $C(z)$, and Fabianska and Quadrat [12] propose possible constructions when $B_0(z)$ or $C(z)$ have certain properties (see Section 3.3.3 of their report). We recall here one of them, which is convenient for most of the common bi–variate ($s = 2$) subdivision schemes.

If one entry of $C(z) = [c_0(z) \cdots c_{a^s-1}(z)]$ is invertible in $\mathcal{L}_s$, and thus is a monomial, *e.g.* $c_0(z) = \frac{1}{d} z^{-x}$, then

$$b_{0,0}(z) = \left(1 - \sum_{p=1}^{a^s-1} c_p(z)\ b_{0,p}(z)\right) / (c_0(z)), \tag{72}$$

$$z^{-x}\ b_{0,0}(z) = d\ + \sum_{p=1}^{a^s-1} (-c_p(z)/c_0(z))\ z^{-x}\ b_{0,p}(z), \tag{73}$$

and $B_0(z)$ can be factorised as (14) with $P = 2a^s - 1$, and

$$E^1_p(z) = I + z^{-x}\ b_{0,p}(z)\ F_{p,0},\ \text{for } p = 1, \dots, a^s - 1 \tag{74}$$
$$E^2_{a^s}(z) = I + (d-1)\ F_{0,0}, \tag{75}$$
$$E^1_{a^s+p}(z) = I + (-c_p(z)/c_0(z))\ F_{0,p},\ \text{for } p = 1, \dots, a^s - 1. \tag{76}$$

Let us illustrate it with the scheme proposed by Loop [18] which, on regular regions of triangular meshes made up with vertices with six neighbours, is the scheme with arity $a = 2$ based on the Box spline defined with the matrix of directions

$$V = \begin{bmatrix} 1\ 1\ 0\ 0\ 1\ 1 \\ 0\ 0\ 1\ 1\ 1\ 1 \end{bmatrix}. \tag{77}$$

We can observe in particular, that $|\det \bar{V}| = 1$, $\forall \bar{V} \in \mathcal{B}(V)$ and so, this scheme is indeed uniformly elementary factorable. Its mask is

$$b_0[x] = \frac{1}{16} \begin{bmatrix} & & 1 & 2 & 1 \\ & 2 & 6 & 6 & 2 \\ 1 & 6 & 10 & 6 & 1 \\ 2 & 6 & 6 & 2 & \\ 1 & 2 & 1 & & \end{bmatrix} \tag{78}$$

and the vector of Z–transforms of its stencils is (with $y_0 = (0;0)$, $y_1 = (\frac{1}{2};0)$, $y_2 = (0;\frac{1}{2})$, $y_3 = (\frac{1}{2};\frac{1}{2})$ and $z = (z_1, z_2)$)

$$B_0(z) = \frac{1}{16} \begin{bmatrix} 1 + z_1 + z_2 + 10z_1z_2 + z_1^2z_2 + z_1z_2^2 + z_1^2z_2^2 \\ 2 + 6z_2 + 6z_1z_2 + 2z_1z_2^2 \\ 2 + 6z_1 + 6z_1z_2 + 2z_1^2z_2 \\ 6 + 2z_1 + 2z_2 + 6z_1z_2 \end{bmatrix}. \tag{79}$$

The aim is to find a left inverse of $B_0(z)$ with one entry being a monomial, or equivalently as in (72), to write one of the entries of $B_0(z)$ as a combination of the others and a monomial:

$$b_{0,0}(z) = \frac{2}{5}z_1z_2 + \frac{1+z_1}{10}b_{0,1}(z) + \frac{1+z_2}{10}b_{0,2}(z) + \frac{1+z_1z_2}{10}b_{0,3}(z), \tag{80}$$

$$z_1^{-1}z_2^{-1}\ b_{0,0}(z) = \frac{2}{5} + \frac{1+z_1}{10}z_1^{-1}z_2^{-1}\ b_{0,1}(z) + \frac{1+z_2}{10}z_1^{-1}z_2^{-1}\ b_{0,2}(z) \tag{81}$$

$$+\frac{1+z_1z_2}{10}z_1^{-1}z_2^{-1}\ b_{0,3}(z), \tag{82}$$

which allows us to apply the above factorisation.

Note that this construction can be generalised to some schemes which are not based on Box spline like the one proposed by Kobbelt [14] or, in the case of Loop's scheme, to the neighbourhood of an *extraordinary vertex* with $n \neq 6$ neighbours. Let us explain this generalisation with this latter case. Let Λ_0 be such an extraordinary vertex, Λ_i, $i = 1, \ldots, n$, its neighbours, λ_0 the new vertex which takes the place of Λ_0 and λ_i, $i = 1, \ldots, n$, its neighbours:

$$\begin{cases} \lambda_0 = \alpha_n\Lambda_0 + \sum_{i=1}^{n} \frac{1-\alpha_n}{n}\Lambda_i; \\ \lambda_i = 6\Lambda_0 + 2\Lambda_{i-1} + 2\Lambda_{i+1} + 6\Lambda_i\ \forall i = 1, \ldots, n; \end{cases} \tag{83}$$

where the indices are considered modulo n and

$$\alpha_n = \left(\frac{3}{8} + \frac{1}{4}\cos\frac{2\pi}{n}\right)^2 + \frac{3}{8}. \tag{84}$$

A similar in–place implementation as above is still possible since we can write as an equivalent of (73),

$$\lambda_0 = \left(\frac{8\alpha_n - 3}{5}\right)\Lambda_0 + \sum_{i=1}^{n}\frac{8(1-\alpha_n)}{5n}\lambda_i. \tag{85}$$

This is the factorisation used by Bertram [1] and Sauvage [24] for their construction of biorthogonal Loop–subdivision wavelets.

However, this construction does not apply, for example with the $4-8$–scheme proposed by Velho and Zorin [29]. Moreover, when applied 2 steps at a time, this scheme is the one with arity $a = 2$ based on the Box spline defined with the matrix of directions

$$V = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \tag{86}$$

and is not uniformly elementary factorable. Indeed, the following matrix of $\mathcal{B}(V)$

$$\bar{V} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{87}$$

has its determinant $|\det \bar{V}| = 2$.

5 Conclusion

We have introduced the definition of uniformly elementary factorable Box spline subdivision scheme, that admits an implementation made up with local, in–place and invertible computations that apply uniformly on the parameter space. Besides, such a factorisation allows the construction of a multiresolution framework with perfect reconstruction and with the subdivision scheme as prediction operator.

We have proved a necessary and sufficient condition on the directions of the Box spline and the arity of the subdivision for the scheme to be uniformly elementary factorable. As for factoring one–dimensional wavelets, the Euclidean algorithm can always be used to define one possible factorisation, in the univariate case. But for higher dimension parameter space, heuristics have to be used. We have shown how one of them can be generalised to the neighbourhood of an extraordinary vertex, or to schemes which are not based on a Box spline.

In future work, it would be interesting to investigate the properties of such a framework when the directions have coordinates greater than one, and in particular what kind of bi–framelets can be designed. As many factorisations are possible for a single scheme, the existence and construction of a factorisation with a minimum number of elementary steps, or with steps defining filters of minimum width, could also be studied.

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References

1. Bertram, M.: Biorthogonal Loop-subdivision wavelets. Computing **72**(1), 29–39 (2004). DOI 10.1007/s00607-003-0044-0
2. Cashman, T.J., Dodgson, N.A., Sabin, M.A.: A symmetric, non-uniform, refine and smooth subdivision algorithm for general degree B-splines. Computer Aided Geometric Design **26**(1), 94 – 104 (2009). DOI http://dx.doi.org/10.1016/j.cagd.2007.12.001
3. Chaikin, G.M.: An algorithm for high-speed curve generation. Computer Graphics and Image Processing **3**, 346–349 (1974)

4. Chui, C.K., He, W., Stöckler, J.: Compactly supported tight and sibling frames with maximum vanishing moments. Applied and Computational Harmonic Analysis **13**(3), 224 – 262 (2002). DOI http://dx.doi.org/10.1016/S1063-5203(02)00510-9
5. Dahmen, W., Micchelli, C.A.: Banded matrices with banded inverses, II: Locally finite decomposition of spline spaces. Constructive Approximation **9**, 263–281 (1993). 10.1007/BF01198006
6. Daubechies, I., Han, B., Ron, A., Shen, Z.: Framelets: MRA-based constructions of wavelet frames. Applied and Computational Harmonic Analysis **14**(1), 1 – 46 (2003). DOI 10.1016/S1063-5203(02)00511-0
7. Daubechies, I., Sweldens, W.: Factoring wavelet transforms into lifting steps. J. Fourier Anal. Appl. **4**(3), 245–267 (1998)
8. De Boor, C., Höllig, K., Riemenschneider, S.: Box Splines. No. vol. 98 in Applied Mathematical Sciences. Springer-Verlag (1993)
9. Dodgson, N.A., Augsdörfer, U.H., Cashman, T.J., Sabin, M.A.: Deriving Box-spline subdivision schemes. In: E.R. Hancock, R.R. Martin, M.A. Sabin (eds.) Mathematics of Surfaces XIII, pp. 106–123. Springer Berlin Heidelberg (2009). DOI 10.1007/978-3-642-03596-8_7
10. Ehler, M.: The construction of nonseparable wavelet bi-frames and associated approximation schemes. Ph.D. thesis, Universität Marburg (2007)
11. Ehler, M.: On multivariate compactly supported bi-frames. Journal of Fourier Analysis and Applications **13**, 511–532 (2007). 10.1007/s00041-006-6021-1
12. Fabianska, A., Quadrat, A.: Applications of the Quillen-Suslin theorem to multidimensional systems theory. Research Report RR-6126, INRIA (2007)
13. Jia, R.Q.: Multivariate discrete splines and linear diophantine equations. Transactions of the American Mathematical Society **340**(1), pp. 179–198 (1993)
14. Kobbelt, L.: $\sqrt{3}$-subdivision. In: Proceedings of SIGGRAPH 2000, ACM, pp. 103–112 (2000)
15. Kobbelt, L., Schröder, P.: A multiresolution framework for variational subdivision. ACM Trans. Graph. **17**(4), 209–237 (1998). DOI 10.1145/293145.293146
16. Lam, T.: Serre's Problem On Projective Modules. Springer Monographs in Mathematics. Springer (2006)
17. Lane, J.M., Riesenfeld, R.F.: A theoretical development for the computer generation and display of piecewise polynomial surfaces. IEEE Transactions on Pattern Analysis and Machine Intelligence **PAMI-2**(1), 35–46 (1980). DOI 10.1109/TPAMI.1980.4766968
18. Loop, C.: Smooth subdivision surfaces based on triangles. Master's thesis, University of Utah (1987)
19. Park, H.: Symbolic computation and signal processing. Journal of Symbolic Computation **37**(2), 209 – 226 (2004). DOI 10.1016/j.jsc.2002.06.003
20. Park, H., Kalker, T., Vetterli, M.: Gröbner bases and multidimensional FIR multirate systems. Multidimensional Syst. Signal Process. **8**(1-2), 11–30 (1997). DOI 10.1023/A:1008299221759
21. Park, H.J.: A computational theory of laurent polynomial rings and multidimensional FIR systems. Ph.D. thesis, University of California, Berkeley (1995). AAI9602697
22. Riemenschneider, S.D., Shen, Z.: Construction of compactly supported biorthogonal wavelets: II. In: M.A. Unser, A. Aldroubi, A.F. Laine (eds.) Proc. SPIE Wavelet Applications in Signal and Image Processing VI, vol. 3813, pp. 264–272 (1999)
23. Salvatori, M., Soardi, P.M.: Affine frames of multivariate box splines and their affine duals. Journal of Fourier Analysis and Applications **8**, 269–290 (2002). 10.1007/s00041-002-0013-6
24. Sauvage, B., Hahmann, S., Bonneau, G.P.: Volume preservation of multiresolution meshes. Computer Graphics Forum **26**(3), 275–283 (2007). DOI 10.1111/j.1467-8659.2007.01049.x
25. Suslin, A.A.: On the structure of the special linear group over polynomail rings. Math. USSR Izvestija **11**(2), 221 – 238 (1977)
26. Swan, R.G.: Projective modules over laurent polynomial rings. Transactions of the American Mathematical Society **237**, 111–120 (1978)
27. Sweldens, W.: The lifting scheme: A construction of second generation wavelets. SIAM J. Math. Anal. **29**(2), 511–546 (1997)
28. Sweldens, W., Schröder, P.: Building your own wavelets at home. In: Wavelets in Computer Graphics, ACM SIGGRAPH Course Notes (1996)
29. Velho, L., Zorin, D.: 4-8 subdivision. Computer Aided Geometric Design **18**(5), 397 – 427 (2001). DOI http://dx.doi.org/10.1016/S0167-8396(01)00039-5