On Absence and Existence of the Anomalous Localized Resonance without the Quasi-static Approximation

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Abstract

The paper considers the transmission problems for Helmholtz equation with bodies that have negative material parameters. Such material parameters are used to model metals on optical frequencies and so-called metamaterials. As the absorption of the materials in the model tends to zero the fields may blow up. When the speed of the blow up is suitable, this is called the Anomalous Localized Reconance (ALR). In this paper we study this phenomenon and formulate a new condition, the weak Anomalous Reconance (w-AR), where the speed of the blow up of fields may be slower. Using this concept, we can study the blow up of fields in the presence of negative material parameters without the commonly used quasi-static approximation. We give simple geometric conditions under which w-AR or ALR may, or may not appear. In particular, we show that in a case of a curved layer of negative material with a strictly convex boundary neither ALR nor w-AR appears with non-zero frequencies (i.e. in the dynamic range) in dimensions $d \geq 3$. In the case when the boundary of the negative material contains a flat subset we show that the w-AR always happens with some point sources in dimensions $d \geq 2$.

1 Introduction and statement of main results

Consider a pair of bounded C^∞-domains D and Ω of $\mathbb{R}^d$, $d \geq 2$, such that the closure of D is included in Ω. Given complex wave numbers k_e and k_i, $\operatorname{Im} k_\mathrm{e}, \operatorname{Im} k_\mathrm{i} \geq 0$, we consider the properties of the following transmission problem

$$\begin{aligned} -(\Delta + k_\mathrm{e}^2)v_1 &= 0 \text{ in } D, \quad -(\Delta + k_\mathrm{i}^2)v_2 = 0 \text{ in } \Omega \setminus \overline{D} \\ -(\Delta + k_\mathrm{e}^2)v_3 &= f \text{ in } \mathbb{R}^d \setminus \Omega, \quad f \in \mathcal{E}'(\mathbb{R}^d \setminus \overline{\Omega}), \end{aligned} \tag{1.1}$$

where on the *interior boundary* $\Gamma_1 = \partial D$ we have the boundary conditions

$$v_1|_{\Gamma_1} = v_2|_{\Gamma_1}, \quad \tau_1 \partial_\nu v_1|_{\Gamma_1} = \partial_\nu v_2|_{\Gamma_1} \tag{1.2}$$

and on the *exterior boundary* $\Gamma_2 = \partial\Omega$ we have

$$v_2|_{\Gamma_2} = v_3|_{\Gamma_2}, \quad \tau_2 \partial_\nu v_3|_{\Gamma_2} = \partial_\nu v_2|_{\Gamma_2}. \tag{1.3}$$

Above, ν is the exterior unit normal vector of $\Omega \setminus \overline{D}$. We also assume that the exterior field v_3 satisfies the (outgoing) Sommerfeld radiation condition at infinity,

$$\begin{aligned} k_{\rm e} \neq 0,\ d \geq 2: &\qquad v_3(x) = O\left(|x|^{2-d}\right), \quad (\partial_r - ik_{\rm e})v_3(x) = o\left(|x|^{2-d}\right), \\ &\qquad \text{as } |x| \to \infty \text{ uniformly in } x/|x| \in S^{d-1}, \\ k_{\rm e} = 0,\ d \geq 3: &\qquad v_3(x) = O\left(|x|^{2-d}\right) \text{ as } |x| \to \infty \text{ uniformly in } x/|x| \in S^{d-1}, \\ k_{\rm e} = 0,\ d = 2: &\qquad v_3(x) = o\,(1) \text{ as } |x| \to \infty \text{ uniformly in } x/|x| \in S^1, \end{aligned} \tag{1.4}$$

where $\partial_r = \frac{x}{|x|} \cdot \nabla$. Also, if $d = 2$ and $k_e = 0$, we assume that the compactly supported source $f \in \mathcal{E}'(\mathbb{R}^2 \setminus \overline{\Omega})$ satisfies the vanishing condition

$$\langle f, 1 \rangle = 0.$$

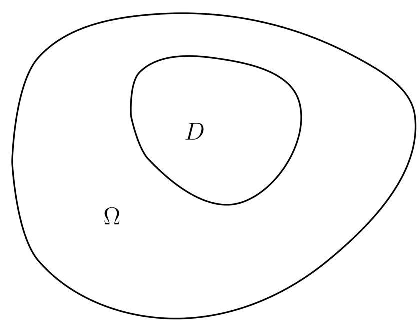

Figure 1: Setting of the paper: Domain $\Omega \subset \mathbb{R}^d$ that contains the closure of domain D. In the set $\Omega \setminus \overline{D}$ the material parameters approach negative value and are positive outside this set.

We will also consider the equations (1.1)-(1.3) in divergence form. To this end, we define a piecewise constant function a_η by

$$a_\eta(x) = a_{\rm e} > 0 \text{ in } D \text{ and } \mathbb{R}^3 \setminus \overline{\Omega}, \tag{1.5}$$

$$a_\eta(x) = a_{\rm i} = a_{\rm e}(-1 + \eta) \text{ in } \Omega \setminus \overline{D}, \quad \eta \in \mathbb{C}, \tag{1.6}$$

and

$$\tau_1 = \tau_2 = \tau = \frac{a_{\rm e}}{a_{\rm i}} = (-1 + \eta)^{-1}. \tag{1.7}$$

Typically, the parameter η above will be small. Also note that in the electromagnetic case we would have $a_\eta = 1/\varepsilon$, and hence it is natural to assume that $\operatorname{Im} \eta \leq 0$. A weak solution of

$$\nabla \cdot a_\eta(x) \nabla u + \omega^2(\chi_{D \cup \mathbb{R}^d \setminus \overline{\Omega}} + b\chi_{\Omega \setminus \overline{D}})\mu_0 u = f \quad \text{in } \mathbb{R}^d,\ f \in \mathcal{E}'(\mathbb{R}^d \setminus \overline{\Omega}), \tag{1.8}$$

where b is a complex constant, is obtained from $v_1 = u|_D$, $v_2 = u|_{\Omega \setminus \overline{\mathbb{D}}}$ and $v_3 = u|_{\mathbb{R}^d \setminus \overline{\Omega}}$ solving (1.1)–(1.3), where the transmission coefficients satisfy (1.7). Note that since outside the interfaces u solves a Helmholtz–equation, it has one sided weak normal derivatives on both interfaces. Also, the wave numbers are determined by

$$k_e^2 = \omega^2 \mu_0 a_{\mathrm{e}}^{-1},\ k_i^2 = k_i(\eta)^2 := \omega^2 \mu_0 a_{\mathrm{i}}^{-1} b, \tag{1.9}$$

and depending on our choice of b and η the sign of $\operatorname{Re} k_i^2$ may vary. We will in particular consider two physically interesting cases. In the first case, $b = 1$, and $\operatorname{Re} k_i^2 \leq 0$. In the second case, $b = -1$ and $\operatorname{Re} k_i^2 \geq 0$. For more on the physical relevance of these cases, please see the Appendix in the attached Supplementary Material. We also denote

$$k_{i,0}^2 = k_i(\eta)^2|_{\eta=0} = -\omega^2 \mu_0 a_{\mathrm{e}}^{-1} b. \tag{1.10}$$

We are especially interested in the behavior of the solutions – and of course in the unique solvability – as $\eta \to 0$ when the ellipticity of (1.1)–(1.4) degenerates. Physically this corresponds to having a layer of (meta)material in $\Omega \setminus \overline{D}$. More precisely, as explained in the Appendix in the Supplementary Material, in $\mathbb{R}^2$ this problem comes up when considering time-harmonic TE-polarized waves in the cylinder $\mathbb{R}^2 \times \mathbb{R}$ with the dielectric constants given by a piecewise constant a_η^{-1}.

It is known that in the case when $\Omega = B(0, R_1)$ and $D = B(0, 1)$ are discs (see [2, 3, 23, 24]) and $\omega = 0$, that when $i\mathbb{R} \ni \eta \to 0$ there is a limit radius $R^* > 0$ s.t. if

$$\operatorname{supp}(f) \subset (\mathbb{R}^2 \setminus \overline{\Omega}) \cap \{x; |x| > R^*\}$$

the solution of (1.1)–(1.4) will have a bounded H^1-norm in $\Omega \setminus \overline{D}$ as $\eta \to 0$, but when

$$\operatorname{supp}(f) \subset (\mathbb{R}^2 \setminus \overline{\Omega}) \cap B(0, R^*)$$

the $H^1(\Omega \setminus \overline{D})$–norm of u_2 blows up at least as $\mathcal{O}(|\eta|^{-1/2})$. This phenomenon is called *anomalous localized resonance* (ALR). To clarify the results of this paper, we make the following formal definitions:

Definition 1.1. *Let v_i^η, $i = 1, 2, 3$, be the unique solutions of (1.1)–(1.4) for $\eta \neq 0$ with a given, fixed source term $f \in \mathcal{E}'(\mathbb{R}^d \setminus \overline{\Omega})$.*

1. *If $\limsup_{\eta \to 0} \|v_2^\eta\|_{H^1(\Omega \setminus \overline{D})} = \infty$, we say that the* Weak Anomalous Resonance *(w-AR) occurs.*
2. *If η is purely imaginary, and $\lim_{i\mathbb{R} \ni \eta \to 0} |\eta|^{1/2} \|\nabla v_2^\eta\|_{L^2(\Omega \setminus \overline{D})} = \infty$, we say that the* Anomalous Localized Resonance *(ALR) occurs.*

In this paper we show that neither ALR nor w-AR happens in $\mathbb{R}^d$, $d \geq 3$, when the boundaries of Ω and D are strictly convex as embedded hypersurfaces of $\mathbb{R}^d$. We also prove that if the exterior boundary has a flat part then w-AR will occur even without the quasi-static approximation. Numerical simulations explained in the Appendix support the hypothesis that w-AR is a weaker phenomenon than ALR. Note, that in [2] the authors define a condition called weak-CALR. In our case this is equivalent to having

$$\limsup_{\eta \to 0} |\eta|^{1/2} \|v_2^\eta\|_{H^1(\Omega \setminus \overline{D})} = \infty$$

and hence is stronger than w-AR.

Note that to prove that ALR does **not** happen, it is enough to prove that there exists a stable H^1–limit of all the fields v_i^η as $\eta \to 0$. Also, that our **existence** result, Theorem 5.1, is much weaker: first of all, we cannot infer from our result that the resonance is localized. Also, the condition of w-AR only says that there is a sequence $\eta_i \to 0$ such that the corresponding field $v_2^{\eta_i}$ do not remain bounded in H^1–norm.

In the seminal papers by Milton et al [23, 24] it was observed that ALR happens in the two-dimensional case when Ω and D are co-centric disc, i.e., $\Omega \setminus \overline{D}$ is an annulus, in the quasi-static regime. This case corresponds to a "perfect lens" made of negative material with a small conductivity $|\eta|$ when $|\eta| \to 0$. When this device is located in a homogeneous electric field and a polarizable point-like object is taken close to the object, it produces a point source due to the background field. When the object is sufficiently close to the annulus, the induced fields in the annulus blow up as $|\eta| \to 0$. Surprisingly, the fields in the annulus create a field which far away cancels the field produced by the point like-object. This result can be interpreted by saying that the annulus makes the point-like object invisible. Presently, this phenomena is called "exterior cloaking". It is closely related to other type of invisibility cloaking techniques, the transformation optics based cloaking, see [11, 12, 13, 14, 15, 16, 21, 20, 30] and active cloaking, see [35, 36]. These cloaking examples can be considered as counterexamples for unique solvability of various inverse problems that show the limitations of various imaging modalities. [37, 38].

Results of Milton et al [23, 24] raised plenty of interest and motivated many studies on the topic. The cloaking due to anomalous localized resonance is studied in the quasi-static regime for a general domain in [2]. There, it is shown that in $\mathbb{R}^2$ the resonance happens for a large class of the sources and that the resonance occurs not because of system approaching an eigenstate, but because of the divergence of an infinite sum of terms related to spectral decomposition of the Neumann-Poincaré operator. In [19] the ALR is studied in the quasi-static regime in the two dimensional case when the outer domain Ω is a disc and the core D is an arbitrary domain compactly supported in Ω. ALR in the case of confocal ellipses is studied in [7].

In [3], it was shown that the cloaking due to anomalous localized resonance does not happen in $\mathbb{R}^3$ when Ω and D are concentric balls. In [4], cloaking due to anomalous localized resonance is connected to transformation optics and it is shown that ALR may happen in three dimensional case when the coefficients of the equations are appropriately chosen matrix-valued functions, i.e. correspond to non-homogeneous anisotropic material.

Earlier, ALR has been studied without the quasi-static approximation both in the 2 and 3 dimensional cases in [26, 27]. In these papers the appearance of ALR is connected to the compatibility of the sources. The compatibility means that for these sources there exists solutions for certain non-elliptic boundary value problems, that are analogous to the so-called interior transmission problems. We also want to mention the work [5], where the surface plasmon resonance for nanoparticles is studied using the full Maxwell system.

In this paper we show that w-AR either happens, or does not happen, when certain simple geometric conditions hold: We show that w-AR - and hence also ALR - does not happen in the three and higher dimensional case when the boundaries $\partial\Omega$ and ∂D are strictly convex, and also show in section 5 that w-AR does happen in d-dimensional case, $d \geq 2$ with some sources when the boundary of $\partial\Omega$ contains a flat part. These results show that w-AR is directly related to geometric properties of the boundaries.

The first main result deals with the solvablity of the case $\tau_1 = \tau_2 = -1$ when the ellipticity of the transmission problem degenerates. Below we will use the notation $\overline{H}^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \overline{\Omega}) = \{u;\, u = w|_{\mathbb{R}^3\setminus\overline{\Omega}},\, w \in H^1_{\mathrm{loc}}(\mathbb{R}^d)\}$. Also, we assume in both Theorems below that the following injectivity assumption is valid:

- **Injectivity assumption (A)**: Assume that the equation (1.1)–(1.4) has only the trivial solution $v_1 = v_2 = v_3 = 0$ when $f = 0$.

The first main result shows that under certain geometric conditions on the boundary interfaces the limit problem $\eta = 0$ is solvable.

Theorem 1.2. *Let $d \geq 3$. Assume that the interior boundary Γ_1 and the exterior boundary Γ_2 are smooth and strictly convex. Assume also that $\eta = 0$, so that $\tau = -1$, and that $k_e > 0$, $k^2_{i,0}$ is not a Dirichlet eigenvalue of $-\Delta$ in D and $\mathbb{R}^d \setminus \overline{\Omega}$, and k_e^2 is not a Dirichlet eigenvalue of $-\Delta$ in Ω. Then given $K \subset \mathbb{R}^d \setminus \overline{\Omega}$ compact and $f \in H^s(\mathbb{R}^d \setminus \overline{\Omega})$ with supp$(f) \subset K$, the problem (1.1)–(1.4) has a unique solution $v_1 \in H^1(D)$, $v_2 \in H^1(\Omega \setminus D)$ and $v_3 \in \overline{H}^1_{loc}(\mathbb{R}^d \setminus \overline{\Omega})$ such*

$$\|v_2\|_{H^1(\Omega\setminus\overline{D})} \leq C_K \|f\|_{H^s(\mathbb{R}^d\setminus\overline{\Omega})}.$$

This result is generalization to the layered case of a previous result by one of the authors ([29]). We can also prove the following limiting result when $\eta \to 0$.

Theorem 1.3. *Let $d \geq 3$. Assume that the interfaces Γ_1 and Γ_2 are smooth and strictly convex and let f be as in the previous Theorem. Let $\tau_1 = \tau_2 = \tau(\eta) = (-1+\eta)^{-1}$, $\eta \in \mathbb{C}$. Assume also that the wave numbers k_e and k_i are given by (1.9) and (1.10), with $k^2_{i,0}$ is not a Dirichlet eigenvalue of $-\Delta$ in D, and k_e^2 is not a Dirichlet eigenvalue of $-\Delta$ in Ω. Then the problem (1.1)–(1.4) with $\tau = \tau(\eta)$ is uniquely solvable for $|\eta|$ small enough, and if v_i^η, $i = 1,2,3$ are its solutions and v_i, $i = 1,2,3$ the solutions given by Theorem 1.2, we have as $\eta \to 0$ along the imaginary axis $i\mathbb{R}$,*

$$v_1^\eta \xrightarrow[H^1(D)]{} v_1, \quad v_2^\eta \xrightarrow[H^1(\Omega\setminus\overline{D})]{} v_2, \quad v_3^\eta \xrightarrow[\overline{H}^1_{loc}(\mathbb{R}^d\setminus\overline{\Omega})]{} v_3.$$

Both of these theorems will be proven in section four of this paper. Analogous results have been proven in [28] using variational methods.

Some comments are in order. First of all, under the assumptions of the above theorems, given a fixed source distribution f supported in $\mathbb{R}^d \setminus \overline{\Omega}$, the solution v_2^η tends to a H^1–function v_2 as $i\mathbb{R} \ni \eta \to 0$, and thus there is no blow up.

Secondly, in Proposition 4.1 we give sufficient conditions for the injectivity assumption (A) to hold. Especially, if the wave numbers come from a divergence type equation with piecewise constant coefficients, so that (1.5) – (1.10) are valid, the injectivity will hold. Also, both the above results remain true if $k_e = 0$, assuming that the injectivity assumption (A) holds.

2 Layer potentials

As a first step we are going to reduce (1.1)–(1.4) to an equivalent problem on the boundary interfaces by replacing the source f with equivalent boundary currents. So, fix $f \in H_0^s(\mathbb{R}^d \setminus \overline{\Omega})$ and let v be the unique solution of the problem

$$-(\Delta + k_e^2)v = f \text{ in } \mathbb{R}^d \setminus \overline{\Omega}, \tag{2.1}$$

$$v|_{\Gamma_2} = 0 \tag{2.2}$$

that satisfies the Sommerfeld radiation condition (1.4). Then, if we let $u_3 = v_3 - v$ in (1.1)–(1.4) we see that $u_1 = v_1$, $u_2 = v_2$ and u_3 will satisfy the transmission problem

$$-(\Delta + k_{\mathrm{e}}^2)u_1 = 0 \text{ in } D, \qquad -(\Delta + k_{\mathrm{i}}^2)u_2 = 0 \text{ in } \Omega \setminus \overline{D} \tag{2.3}$$

$$-(\Delta + k_{\mathrm{e}}^2)\widetilde{u}_3 = 0 \qquad \text{in } \mathbb{R}^d \setminus \overline{\Omega} \tag{2.4}$$

$$u_1|_{\Gamma_1} = u_2|_{\Gamma_1} - f_1, \quad \tau_1 \partial_\nu u_1|_{\Gamma_1} = \partial_\nu u_2|_{\Gamma_1} - g_1, \tag{2.5}$$

$$u_2|_{\Gamma_2} = u_3|_{\Gamma_2} - f_2, \quad \tau_2 \partial_\nu u_3|_{\Gamma_2} = \partial_\nu u_2|_{\Gamma_2} - g_2, \tag{2.6}$$

$$u_3 \text{ satisfies Sommerfeld condition (1.4).} \tag{2.7}$$

where $f_1 \in H^s(\Gamma_1)$, $g_1 \in H^{s-1}(\Gamma_1)$, $f_2 \in H^s(\Gamma_2)$ and $g_2 \in H^{s-1}(\Gamma_2)$ for some real value of s. In fact, for boundary jumps originating from an exterior source we have $f_1 = g_1 = 0$ and $f_2 = v = 0$ and $g_2 = \partial_\nu v$. Notice also that since the source f is supported away from Γ_2 the solution v of (2.1) will actually be smooth near Γ_2, and hence the boundary jumps f_i and g_i will be C^∞ functions. This will be crucial for our argument.

The reduction to the boundary will be done using layer potentials. Given $k \in \mathbb{C}$, $\operatorname{Im} k \geq 0$, let

$$G_k(x) = \begin{cases} \frac{i}{4}\left(\frac{k}{2\pi|x|}\right)^{\frac{d-2}{2}} H^{(1)}_{\frac{d-2}{2}}(k|x|), & k \neq 0, \quad d \geq 2 \\ \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{\frac{d}{2}}}|x|^{2-d}, & k = 0, \quad d \geq 3 \\ \frac{1}{2\pi}\ln|x|, & d = 2,\ k = 0. \end{cases}$$

be the fundamental solution of $-(\Delta + k^2)$ in $\mathbb{R}^d$ satisfying the Sommerfeld condition (1.4).

For $\Gamma = \Gamma_1$ or $\Gamma = \Gamma_2$, we define using these kernels the (volume) single layer operators by

$$S_k^\Gamma(\phi)(x) = \int_\Gamma G_k(x-y)\phi(y)\mathrm{d}S(y), \quad x \notin \Gamma, \quad \phi \in C^\infty(\Gamma).$$

Sometimes when we wish to emphasize that we are restricting these operators to a domain Ω, $\Gamma \cap \Omega = \emptyset$, we use the notations $S_k^{\Gamma,\Omega}(\phi) = S_k^\Gamma(\phi)|_\Omega$. These operators define continuous mappings $S_k^\Gamma : C^\infty(\Gamma) \to D'(\mathbb{R}^d \setminus \Gamma)$.

Similarly, we define the (volume) double layer operators by

$$D_k^\Gamma(\phi)(x) = \int_\Gamma \frac{\partial G_k(x-y)}{\partial \nu(y)} \phi(y) \mathrm{d}S(y), \quad x \notin \Gamma, \quad \phi \in C^\infty(\Gamma)$$

where ν will always denote the exterior unit normal to $\Omega \setminus D$. Like for the single layer potentials, we will occasionally denote the restrictions of these to $\Omega \subset \mathbb{R}^d \setminus \Gamma$ by $D_k^{\Gamma,\Omega}$. Also, $D_k^\Gamma : C^\infty(\Gamma) \to D'(\mathbb{R}^d \setminus \Gamma)$ continuously.

Mapping properties of these operators between appropriate Sobolev spaces are also well known (see [6], [22] or [31], page 156, Theorem 4): For all $s \in \mathbb{R}$ we have $S_k^{\Gamma,\Omega} : H^s(\Gamma) \to H^{s+\frac{3}{2}}(\Omega)$ and $D_k^{\Gamma,\Omega} : H^s(\Gamma) \to H^{s+\frac{1}{2}}(\Omega)$ if $\Omega \subset \mathbb{R}^d \setminus \Gamma$ is a bounded domain.

We need traces of these operators on both Γ_1 and Γ_2. Hence, let $\gamma_{0,j}^+$ be the trace operator on Γ_j from the complement of $\Omega \setminus \overline{D}$, that is, $\gamma_{0,j}^+(u) = u|_{\Gamma_j}$ for $u \in H^1((\mathbb{R}^d \setminus \overline{\Omega}) \cup D)$. Respectively let $\gamma_{0,j}^-$ be the trace–opearator on Γ_j from $\Omega \setminus \overline{D}$, that is, $\gamma_{0,j}^-(u) = u|_{\Gamma_j}$ for $u \in H^1(\Omega \setminus \overline{D})$. Then, for $\phi \in H^s(\Gamma_j)$, $s > -1$, we have

$$\gamma_{0,j}^+ S_k^{\Gamma_j} \phi = V_k^{\Gamma_j} \phi = \gamma_{0,j}^- S_k^{\Gamma_j} \phi, \tag{2.8}$$

where

$$V_k^{\Gamma_j} \phi(x) = \int_{\Gamma_j} G_k(x-y) \phi(y) \mathrm{d}S(y) \tag{2.9}$$

is the trace-single-layer operator on Γ_j.

Also, if $\psi \in H^s(\Gamma_j)$, $s > 0$, for the traces of the double layer we have the jump relations

$$\gamma_{0,j}^- D_k^{\Gamma_j} \psi + \frac{\psi}{2} = \gamma_{0,j}^+ D_k^{\Gamma_j} \psi - \frac{\psi}{2} = K_k^{\Gamma_j} \psi, \tag{2.10}$$

where $K_k^{\Gamma_j}$ is the trace-double-layer operator on Γ_j given by

$$K_k^\Gamma \phi(x) = p.v. \int_{\Gamma_j} \frac{\partial G_k(x-y)}{\partial \nu(y)} \phi(y) \mathrm{d}S(y). \tag{2.11}$$

The maps $K_k^{\Gamma_j} : H^s(\Gamma_j) \to H^s(\Gamma_j)$ and $S_k^{\Gamma_j} : H^s(\Gamma_j) \to H^{s+1}(\Gamma_j)$ are continous pseudodifferential operators for any $s \in \mathbb{R}$.

Next, let $\gamma_{1,j}^+$ be the trace of the normal derivative on Γ_j from the complement of $\Omega \setminus \overline{D}$, that is, $\gamma_{1,j}^+(u) = \partial_\nu u|_{\Gamma_j}$ for $u \in H^1((\mathbb{R}^d \setminus \overline{\Omega}) \cup D)$. Respectively let $\gamma_{1,j}^-$ be the trace of the normal derivative on Γ_j from $\Omega \setminus \overline{D}$, that is, $\gamma_{1,j}^-(u) = \partial_\nu u|_{\Gamma_j}$ for $u \in H^1(\Omega \setminus \overline{D})$. For the normal derivatives of the single layer potentials we have the jump relations

$$\gamma_{1,j}^- V_k^{\Gamma_j} \phi - \frac{\phi}{2} = \gamma_{1,j}^+ V_k^{\Gamma_j} \phi + \frac{\phi}{2} = K_k^{*,\Gamma_j} \phi, \tag{2.12}$$

for any $\phi \in H^{-1/2}(\Gamma_j)$, where the operator K_k^{*,Γ_j} is the adjoint trace-double-layer operator on Γ_j given by

$$K_k^{*,\Gamma_j} \phi(x) = p.v. \int_{\Gamma_j} \frac{\partial G_k(x-y)}{\partial \nu(x)} \phi(y) \mathrm{d}S(y), \tag{2.13}$$

For H^1–solutions of an inhomogeneous Helmholz–equation with an L^2–source one can define normal traces weakly using Green's theorems. With this interpretation, for any $\psi \in H^{1/2}(\Gamma_j)$, we also have the traces

$$\gamma_{1,j}^{-} D_k^{\Gamma_j}\psi = \gamma_{1,j}^{+} D_k^{\Gamma_j}\psi = N_k^{\Gamma_j}\psi, \tag{2.14}$$

where the hypersingular integral operator $N_k^{\Gamma_j}$ has (formally) the kernel

$$\frac{\partial^2 G_k(x-y)}{\partial\nu(x)\partial\nu(y)},\ x,\, y \in \Gamma_j,\, x \neq y.$$

The maps $K_k^{*,\Gamma_j} : H^s(\Gamma_j) \to H^s(\Gamma_j)$ and $N_k^{\Gamma_j} : H^s(\Gamma_j) \to H^{s-1}(\Gamma_j)$ are continous pseudodifferential operators for any $s \in \mathbb{R}$.

3 Reduction to the boundary

We will follow the ideas of [18] adapted to our situation, where we have two interfaces instead of just one. Let us consider (2.3)–(2.7). Write an ansaz for u_1 and u_3:

$$u_1 = S_{k_{\rm e}}^{\Gamma_1,D}(\phi), \quad \phi \in H^{-\frac{1}{2}}(\Gamma_1), \quad u_3 = S_{k_{\rm e}}^{\Gamma_2,\mathbb{R}^d\setminus\overline{\Omega}}(\psi), \quad \psi \in H^{-\frac{1}{2}}(\Gamma_2), \tag{3.1}$$

and to

$$u_2 \in \mathcal{L} := \{v \in H^1(\Omega\setminus\overline{D});\ -(\Delta + k_{\rm i}^2)v = 0\}$$

we apply the representation theorem (see for example [8]) to get

$$\begin{aligned} u_2 \quad = \quad & S_{k_{\rm i}}^{\Gamma_1,\Omega\setminus\overline{D}}\left(\left.\frac{\partial u_2}{\partial\nu}\right|_{\Gamma_1}\right) - D_{k_{\rm i}}^{\Gamma_1,\Omega\setminus\overline{D}}\left(u_2|_{\Gamma_1}\right) \\ & + S_{k_{\rm i}}^{\Gamma_2,\Omega\setminus\overline{D}}\left(\left.\frac{\partial u_2}{\partial\nu}\right|_{\Gamma_2}\right) - D_{k_{\rm i}}^{\Gamma_1,\Omega\setminus\overline{D}}\left(u_2|_{\Gamma_2}\right),\ \text{in } \Omega\setminus\overline{D}. \end{aligned} \tag{3.2}$$

Taking traces of u_1 on Γ_1, of u_3 on Γ_3, and of u_2 on $\Gamma_1 \cup \Gamma_2$, and using the transmission conditions (2.5)–(2.6) to express the boundary values of u_2 in terms of those of u_1 and u_3 we get a system of boundary integral equations for $(\phi,\psi) \in H^{-\frac{1}{2}}(\Gamma_1) \times H^{-\frac{1}{2}}(\Gamma_2)$:

$$(\mathcal{A} + \mathcal{M})\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \widetilde{f} := \begin{pmatrix}\widetilde{f}_1\\ \widetilde{f}_2\end{pmatrix} \tag{3.3}$$

where

$$\mathcal{A} = \begin{pmatrix}\mathcal{A}_1 & 0\\ 0 & \mathcal{A}_2\end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix}0 & M_1\\ M_2 & 0\end{pmatrix} \tag{3.4}$$

and

$$\begin{aligned} \mathcal{A}_1 \ &= \tfrac{1}{2}\left(V_{k_{\rm e}}^{\Gamma_1} + \tau_1 V_{k_{\rm i}}^{\Gamma_1}\right) + \left(K_{k_{\rm i}}^{\Gamma_1} V_{k_{\rm e}}^{\Gamma_1} - \tau_1 V_{k_{\rm i}}^{\Gamma_1} K_{k_{\rm e}}^{*,\Gamma_1}\right), \\ \mathcal{A}_2 \ &= \tfrac{1}{2}\left(V_{k_{\rm e}}^{\Gamma_2} + \tau_2 V_{k_{\rm i}}^{\Gamma_2}\right) + \left(K_{k_{\rm i}}^{\Gamma_2} V_{k_{\rm e}}^{\Gamma_1} - \tau_2 V_{k_{\rm i}}^{\Gamma_2} K_{k_{\rm e}}^{*,\Gamma_2}\right). \end{aligned}$$

Here the off-diagonal operators M_i are infinitely smoothing. This is the integral equation we are going to study. For the details of the derivation as well as for the explicit expressions of M_i see section A in the Supplementary Material.

The next proposition establishes conditions under which (4.8) is equivalent with the original transmission problem (2.3)–(2.7).

Proposition 3.1. *Assume k_i^2 is not a Dirichlet eigenvalue of $-\Delta$ in D or in $\mathbb{R}^3 \setminus \overline{\Omega}$. If $u_1 \in H^1(D)$, $u_2 \in H^1(\Omega \setminus \overline{D})$ and $u_3 \in \overline{H}^1_{loc}(\mathbb{R}^d \setminus \overline{\Omega})$ solve (2.3)–(2.7), then $\phi \in H^{-\frac{1}{2}}(\Gamma_1)$, $\psi \in H^{-\frac{1}{2}}(\Gamma_2)$ satisfying (3.1) solve (4.8).*

Conversely, assume that $(\phi, \psi) \in H^{-\frac{1}{2}}(\Gamma_1) \times H^{-\frac{1}{2}}(\Gamma_2)$ solve (4.8). Define u_1 and u_3 by (3.1) and u_2 by

$$\begin{aligned} u_2 \quad = \quad & S_{k_i}^{\Gamma_1}\left(\tau_1\left[K_{k_e}^{*,\Gamma_1}(\phi) - \frac{\phi}{2}\right] + g_1\right) - D_{k_i}^{\Gamma_1}\left(V_{k_e}^{\Gamma_1}(\phi) + f_1\right) \\ & + S_{k_i}^{\Gamma_2}\left(\tau_2\left[K_{k_e}^{*,\Gamma_2}(\psi) - \frac{\psi}{2}\right] + g_2\right) - D_{k_i}^{\Gamma_2}\left(V_{k_e}^{\Gamma_2}(\psi) + f_2\right). \end{aligned} \tag{3.5}$$

Then the triplet $(u_1, u_2, u_3) \in H^1(D) \times H^1(\Omega \setminus \overline{D}) \times \overline{H}^1_{loc}(\mathbb{R}^d \setminus \overline{\Omega})$ will solve (2.3)–(2.7).

Proof. It only remains to prove the second claim. So define u_2 by (3.5) and u_1 and u_3 by (3.1) where $\phi \in H^{-\frac{1}{2}}(\Gamma_1)$, $\psi \in H^{-\frac{1}{2}}(\Gamma_2)$ solve (4.8). By taking traces on Γ_1 and Γ_2 we immediately recover the transmission conditions

$$u_1|_{\Gamma_1} = u_2|_{\Gamma_1} - f_1, u_3|_{\Gamma_2} = u_2|_{\Gamma_2} - f_2.$$

To prove the transmission conditions for the normal derivatives we define for $x \in D \cup \mathbb{R}^d \setminus \overline{\Omega}$,

$$\begin{aligned} v(x) \quad = \quad & S_{k_\mathrm{i}}^{\Gamma_1}\left(\tau_1\left[K_{k_\mathrm{e}}^{*,\Gamma_1}(\phi) - \frac{\phi}{2}\right] + g_1\right)(x) - D_{k_\mathrm{i}}^{\Gamma_1}\left(V_{k_\mathrm{e}}^{\Gamma_1}(\phi) + f_1\right)(x) \\ & + S_{k_\mathrm{i}}^{\Gamma_2}\left(\tau_2\left[K_{k_\mathrm{e}}^{*,\Gamma_2}(\psi) - \frac{\psi}{2}\right] + g_2\right)(x) - D_{k_\mathrm{i}}^{\Gamma_2}\left(V_{k_\mathrm{e}}^{\Gamma_2}(\psi) + f_2\right)(x). \end{aligned} \tag{3.6}$$

Then using (4.8) we see that v solves

$$\begin{cases} (\Delta + k_\mathrm{i}^2)v = 0 \text{ in } D \\ v|_{\Gamma_1} = 0 \end{cases}$$

and since we assumed that k_i^2 was not a Dirichlet eigenvalue of $-\Delta$ in D, we get that $v = 0$ in D. Similarly the restriction of v to $\mathbb{R}^3 \setminus \overline{\Omega}$ is a solution of

$$\begin{cases} -(\Delta + k_\mathrm{i}^2)v = 0 \text{ in } \mathbb{R}^d \setminus \overline{\Omega} \\ v|_{\partial\Omega} = 0 \end{cases}$$

satisfying Sommerfeld condition (1.4). Hence by the assumptions, $v = 0$ also in $\mathbb{R}^3 \setminus \overline{\Omega}$. Taking traces of the normal derivative of v from D and recombining the terms we get

$$\begin{aligned}\tau_1\left[K^{*,\Gamma_1}_{k_e}(\phi)-\frac{\phi}{2}\right]+g_1 &= K^{*,\Gamma_1}_{k_i}\left(\tau_1\left[K^{*,\Gamma_1}_{k_e}(\phi)-\frac{\phi}{2}\right]+g_1\right) \\ &+\frac{1}{2}\left(\tau_1\left[K^{*,\Gamma_1}_{k_e}(\phi)-\frac{\phi}{2}\right]+g_1\right)-N^{\Gamma_1}_{k_i}\left(V^{\Gamma_1}_{k_i}(\phi)+f_1\right) \\ &+B_1\left(\tau_2\left[K^{*,\Gamma_2}_{k_e}(\psi)-\frac{\psi}{2}\right]+g_2\right)-S_1\left(V^{\Gamma_2}_{k_e}(\psi)+f_2\right).\end{aligned} \tag{3.7}$$

The left-hand side of (3.7) is $\tau_1\partial_\nu u_1+g_1$. The right-hand side is equal to $\partial_\nu u_2|_{\Gamma_1}$. Hence we have shown the second equation in (2.5). Proceeding similarly, but taking traces of v from $\mathbb{R}^3\setminus\overline{\Omega}$, we get the second equation of (2.6). □

Remark 1. *Note that the Dirichlet-spectrum of $-\Delta$ on D is discrete and positive, so that always, if $k_i^2\le 0$ or if $\operatorname{Im} k_i^2\neq 0$, we have the first condition. Similarly, $\operatorname{Im} k_i\ge 0$, $k_i\neq 0$, is enough to guarantee that the second assumption of Proposition 3.1 is valid. In the case of most interest to us we have $k_e^2=\omega^2 a_e\mu_0>0$ and $k_i^2=\omega^2 b a_i^{-1}\mu_0$ with $a_i=a_e(-1+\eta)$, where $a_e>0$. Hence the assumptions of Proposition 3.1 are always valid if $\operatorname{Im}\eta\neq 0$, or if $b>0$. If $b<0$, then the assumptions remain valid if ω is small enough.*

4 Absence of ALR

As the first step in proving the solvability and stability when $\eta\to 0$ we give a uniqueness result (see for example [9]). The proof is a standard application of Rellich–theorem, and is given in detail in section B of the Supplementary Material:

Proposition 4.1. *Assume $k_e>0$, $\operatorname{Im}(1/\tau_2))\le 0$, $\operatorname{Im}\left(k_i^2/\tau_2\right)\ge 0$ and $\operatorname{Im}(\tau_1/\tau_2)=0$. Then the problem (1.1)–(1.4) has at most one solution. Especially, if the wave numbers are associated to a divergence form equation, i.e. (1.5) – (1.7) are satisfied with a real valued parameter b, the uniqueness holds.*

It is well known (see for example [9] or [33]) that on a smooth compact surface Γ without boundary the single-layer potentials V_k^Γ are classical pseudodifferential operators (ψDO's) of order -1 with principal symbol

$$\sigma_{(-1)}\left(V_k^\Gamma\right)(x,\xi')=c_d|\xi'|^{-1},\quad \xi'\in T^*_*(\Gamma),\quad x\in\Gamma,$$

and that $4N_k^\Gamma$ is the parametrix of V_k^Γ, so that

$$\sigma_{(1)}\left(N_k^\Gamma\right)(x,\xi')=c_d^{-1}|\xi'|/4,\quad \xi'\in T^*_*(\Gamma)\setminus\{0\},\quad x\in\Gamma.$$

Also – and this is important to us – even though formally of order 0, the double-layer and its adjoints are in fact of order -1, and hence compact as operators $H^s(\Gamma)\to H^s(\Gamma)$. For the principal symbol of $K_k^{*,\Gamma}$ we have (see [29] or [33], Proposition C.1, p.453)

$$\sigma_{(-1)}(K_k^{*,\Gamma})(x',\xi')=a_d d_\Gamma(x')|\xi'|^{-3}\left(l_x(\xi',\xi')-\sum_j\lambda_j(x')|\xi'|^2\right),$$

where a_d is a nonzero constant, $d_\Gamma(x')$ is the density of the surface measure on Γ, l_x is the second fundamental form of Γ (embedded in $\mathbb{R}^d$) and $\lambda_j(x')$ are the principal curvatures of Γ, i.e. eigenvalues of l_x. Hence, if Γ is strictly convex, $K_k^{*,\Gamma}$ is an elliptic operator of order -1.

Proposition 4.2. *Assume $\tau_1, \tau_2 \neq -1$. Then the integral operator $\mathcal{A} + \mathcal{M}$ defined by (4.8)–(4.9) is an elliptic ψDO of order -1, and hence a Fredholm operator*

$$\mathcal{A} + \mathcal{M} : \begin{array}{c} H^s(\Gamma_1) \\ \oplus \\ H^s(\Gamma_2) \end{array} \to \begin{array}{c} H^{s+1}(\Gamma_1) \\ \oplus \\ H^{s+1}(\Gamma_2) \end{array}$$

for all $s \in \mathbb{R}$. Also, $\operatorname{ind} \mathcal{A} = 0$.

Proof. The principal symbol of $\mathcal{A} + \mathcal{M}$ is

$$\begin{pmatrix} \frac{1}{2}(1+\tau_1)|\xi'|^{-1} & 0 \\ 0 & \frac{1}{2}(1+\tau_1)|\xi'|^{-1} \end{pmatrix}$$

proving the ellipticity. Also, if $k = 0$, then V_0^Γ is self-adjoint on Γ, and hence $\operatorname{Ind}\left(V_0^\Gamma\right) = 0$. Since $V_k^\Gamma - V_0^\Gamma$ is of order < -1, and $K_k^{\Gamma_j}$ and K_k^{*,Γ_j} are of order -1,

$$\operatorname{Ind}(\mathcal{A} + \mathcal{M}) = \operatorname{Ind} \begin{pmatrix} V_0^{\Gamma_1} & 0 \\ 0 & V_0^{\Gamma_2} \end{pmatrix} = 0. \quad \square$$

From now on we only consider the case $d \geq 3$. We have the following lemma, whose proof can be found in section C in the attached Supplementary Material:

Lemma 4.3. *For the difference of single layer potentials we have*

$$V_{k_e}^{\Gamma_j} - V_{k_i}^{\Gamma_j} \in \Psi_{cl}^{-3+\epsilon}(\Gamma_j) \quad \text{for all} \quad \epsilon > 0.$$

Proposition 4.4. *Assume $\tau_1 = \tau_2 = -1$ and that Γ_1 and Γ_2 are strictly convex smooth hypersurfaces of $\mathbb{R}^d$ with $d \geq 3$. Then $(\mathcal{A} + \mathcal{M})$ is an elliptic ψDO of order -2 with index* 0.

Proof. As $\tau_1 = \tau_2 = -1$,

$$\mathcal{A} = \begin{pmatrix} K_0^{\Gamma_1} V_0^{\Gamma_1} + V_0^{\Gamma_1} K_0^{*,\Gamma_1} & 0 \\ 0 & K_0^{\Gamma_2} V_0^{\Gamma_2} + V_0^{\Gamma_2} K_0^{*,\Gamma_2} \end{pmatrix} \mod \Psi_{\mathrm{cl}}^{-3+\epsilon}$$

Since $K_0^{\Gamma_j} V_0^{\Gamma_j} + V_0^{\Gamma_j} K_0^{*,\Gamma_j}$ is self-adjoint, $\operatorname{Ind}(\mathcal{A} + \mathcal{M}) = 0$. Also, by Calderon's identities (see [9])

$$K_0^{\Gamma_j} V_0^{\Gamma_j} = V_0^{\Gamma_j} K_0^{*,\Gamma_j}$$

and hence the principal symbol of $\mathcal{A}$ is

$$C d_{\Gamma_1}(x') |\xi'|^{-4} \begin{pmatrix} a_1(x', \xi') & 0 \\ 0 & a_2(x', \xi') \end{pmatrix}$$

where

$$a_k(x',\xi') = l_{x'}^{\Gamma_k}(x',\xi') - \sum_{j=1}^{d-1} \lambda_j^{\Gamma_k}(x')|\xi'|^2,$$

$l_x^{\Gamma_k}$ is the second scalar fundamental form of Γ_k, and $\lambda_j^{\Gamma_k}$ are the principal curvatures of Γ_k, i.e., eigenvalues of l^{Γ_k}. If Γ_k is strictly convex, then for all $x \in \Gamma_k$, $\lambda_j^{\Gamma_k}(x)$ are either positive or negative, and since λ_j^k are eigenvalues of $l_x^{\Gamma_k}(\xi',\xi')$, $l_{x'}^{\Gamma_k}(\xi',\xi') - \sum \lambda_j^{\nu}(x')|\xi'|^2$ is correspondingly either negative or positive definite, so $\mathcal{A}+\mathcal{M}$ is elliptic of order -2. $\square$

Next we consider the unique solvability of the boundary integral equation. We start by proving the uniquenes, and for this we make no additional assumptions on τ_1 and τ_2.

Lemma 4.5. *Assume that the conditions on the wavenumbers k_e and k_i of Propositions 4.1 and 3.1 hold, k_e^2 is not a Dirichlet eigenvalue of Ω, and*

$$(\mathcal{A}+\mathcal{M})\begin{pmatrix}\phi\\ \psi\end{pmatrix} = 0, \qquad \begin{pmatrix}\phi\\ \psi\end{pmatrix} \in H^s(\Gamma_1)\times H^s(\Gamma_1).$$

Then $\phi_1 = \phi_2 = 0$.

Proof. Assume

$$(\mathcal{A}+\mathcal{M})\begin{pmatrix}\phi\\ \psi\end{pmatrix} = 0.$$

Then by Proposition 3.1

$$\begin{cases} u_1 = S_{k_e}^{\Gamma_1,D}(\phi) \\ u_2 = S_{k_i}^{\Gamma_1}\left(\tau_1\left[K_{k_e}^{*,\Gamma_1}(\phi) - \frac{\phi}{2}\right]\right) - D_{k_i}^{\Gamma_1}V_{k_e}^{\Gamma_1}(\phi) \\ \qquad + S_{k_i}^{\Gamma_2}\left(\tau_2\left[K_{k_e}^{*,\Gamma_2}(\psi) - \frac{\psi}{2}\right]\right) - D_{k_i}^{\Gamma_2}V_{k_e}^{\Gamma_2}(\psi), \qquad \text{in } \Omega\setminus\overline{D} \\ u_3 = S_{k_e}^{\Gamma_2,\mathbb{R}^3\setminus\overline{\Omega}}(\psi) \end{cases}$$

will solve (1.1)–(1.4) with $f = 0$, so by Proposition 4.1 we have $u_1 = 0$, $u_2 = 0$ and $u_3 = 0$. Hence $u_1|_{\Gamma_1} = V_{k_e}^{\Gamma_1}(\phi) = 0 = u_3|_{\Gamma_2} = V_{k_e}^{\Gamma_2}(\psi)$. Also

$$\partial_\nu u_1|_{\Gamma_1} = K_{k_e}^{*,\Gamma_1}(\phi) - \frac{\phi}{2} = 0 = \partial_\nu u_3|_{\Gamma_2} = K_{k_e}^{*,\Gamma_2}(\psi) - \frac{\psi}{2}. \tag{4.1}$$

Define now

$$\widetilde{u}_1 = S_{k_e}^{\mathbb{R}^d\setminus\overline{D}}(\phi) \text{ in } \mathbb{R}^3\setminus D, \quad \widetilde{u}_2 = S_{k_e}^{\Omega}(\psi) \text{ in } \Omega.$$

Then $\widetilde{u}_1|_{\Gamma_1} = V_{k_e}^{\Gamma_1}(\phi) = 0\widetilde{u}_2|_{\Gamma_2} = V_{k_e}^{\Gamma_2}(\psi)$. Since the exterior Dirichlet problem is always uniquely solvable if $k_e > 0$ (see [8]), $\widetilde{u}_1 \equiv 0$ in $\mathbb{R}^d\setminus\overline{D}$, so by taking traces of $\partial_\nu\widetilde{u}_1$ on Γ_1 we get

$$0 = K_{k_e}^{*,\Gamma_1}(\phi) + \frac{\phi}{2}. \tag{4.2}$$

Thus (4.1) and (4.2) imply that $\phi = 0$. Similarly, since k_e^2 is not an interior Dirichlet eigenvalue of Ω, we get $\psi = 0$. $\square$

Combining Lemma 4.5 and Propositions 4.2 and 4.4 we now get

Proposition 4.6. *Assume again that the conditions on the wavenumbers k_e and k_i of Propositions 4.1 and 3.1 hold, and that k_e^2 is not a Dirichlet eigenvalue of Ω. Let*

$$\begin{pmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{pmatrix} \in \begin{matrix} H^s(\Gamma_1) \\ \oplus \\ H^s(\Gamma_2) \end{matrix},$$

where $s > -1$. Then if either

a) $\tau_1, \tau_2 \neq -1$,

or

b) $\tau_1 = \tau_2 = -1$, $d \geq 3$ and ∂D and $\partial\Omega$ are strictly convex,

the boundary integral equation (4.8) has a unique solution

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \begin{matrix} H^{s-1}(\Gamma_1) \\ \oplus \\ H^{s-1}(\Gamma_2) \end{matrix} \text{ in case a) or } \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \begin{matrix} H^{s-2}(\Gamma_1) \\ \oplus \\ H^{s-2}(\Gamma_2) \end{matrix}, \text{ in case b) respectively.} \qquad \square$$

Remark 2. *Notice that if $\widetilde{f}_1$ and $\widetilde{f}_2$ are given by formulas (A5) and (A8) in the attached supplementary material respectively, with f_i and g_i determined by the source $f \in H^s(\mathbb{R}^d \setminus \overline{\Omega})$ with a compact support contained in $\mathbb{R}^d \setminus \overline{\Omega}$ as described at the beginning of the section 2, then $\widetilde{f}_1$ and $\widetilde{f}_2$ will be smooth functions and hence the above proposition holds with any $s > -1$. Hence especially the field u_2 will belong to $H^1(\Omega \setminus \overline{D})$, and we have proven Theorem 1.2.*

To prove Theorem 1.3 we need the following result:

Proposition 4.7. *Consider the transmission problem (2.3) – (2.7) with the transmission coefficients given by*

$$\tau_1 = \tau_2 = (-1 + \eta)^{-1},$$

where $\eta = i\delta$ is purely imaginary, and the wave numbers by

$$k_e^2 = \omega^2 \mu_0 a_e^{-1},\ k_i^2 = k_i(\eta)^2 := \omega^2 \mu_0 a_i^{-1} b,$$

with b and μ_0 real, and $a_i/a_e = -1 + \eta$. Assume also that the conditions on the wavenumbers k_e and $k_i(\eta)$ of Propositions 4.1 and 3.1 hold, and that k_e is not a Dirichlet eigenvalue of Ω, and also that the interfaces ∂D and $\partial\Omega$ are strictly convex. Let

$$\widetilde{f(\eta)} := \begin{pmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{pmatrix} \in \begin{matrix} H^{1/2}(\Gamma_1) \\ \times \\ H^{1/2}(\Gamma_2) \end{matrix}$$

be the equivalent boundary source term as given by formulas (A.5) and (A.8) in the supplementary material. Note that these depend on η through the interior wave number k_i. Let

$$\begin{pmatrix} \phi_1(\eta) \\ \phi_2(\eta) \end{pmatrix} \in \begin{matrix} H^{-1/2}(\Gamma_1) \\ \oplus \\ H^{-1/2}(\Gamma_2) \end{matrix}$$

be the unique solution of (4.8) with $\tau_1 = \tau_2 = \tau(\eta)$. Also, let

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \begin{matrix} H^{-3/2}(\Gamma_1) \\ \oplus \\ H^{-3/2}(\Gamma_2) \end{matrix}$$

be the unique solution of (4.8) with $\tau_1 = \tau_2 = -1$.
(a) As $i\mathbb{R} \ni \eta \to 0$ we have

$$\begin{pmatrix} \phi_1(\eta) \\ \phi_2(\eta) \end{pmatrix} \to \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

in $H^{-3/2-\rho}(\Gamma_1) \oplus H^{-3/2-\rho}(\Gamma_2)$ for any positive value of ρ.
(b) If in addition $\widetilde{f(\eta)} \in H^s(\Gamma_1) \times H^s(\Gamma_2)$ with $s > 3/2$, then

$$\begin{pmatrix} \phi_1(\eta) \\ \phi_2(\eta) \end{pmatrix} \to \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

in $H^{-1/2}(\Gamma_1) \oplus H^{-1/2}(\Gamma_2)$ as $\eta \to 0$ along the imaginary axis.

Before the proof we give the following lemma:

Lemma 4.8. *Consider the transmission problem (2.3) – (2.7). Assume that the transmission coefficients are given by*

$$\tau_1 = \tau_2 = (-1+\eta)^{-1},$$

and the wave numbers by

$$k_e^2 = \omega^2 \mu_0 a_e^{-1},\ k_i^2 = k_i(\eta)^2 := \omega^2 \mu_0 a_i^{-1} b,$$

with b and μ_0 real, and $a_i/a_e = -1+\eta$. Under these assumptions we have an priori bound, for $R > 0$ large enough and for all $|\eta| \le \eta_0$ with η_0 small enough,

$$\|u_1\|_{H^1(D)} + \|u_2\|_{H^1(\Omega\setminus\overline{D})} + \|u_3\|_{H^1(B_R(0)\setminus\overline{\Omega})} \le \frac{C(f_1, f_2, g_1, g_2)}{|Im\,\eta|},$$

with some constant $C(f_1, f_2, g_1, g_2)$ depending continuously on $\|g_j\|_{H^{-1/2}(\Gamma_j)}$ and $f_j\|_{H^{1/2}(\Gamma_j)}$, $j = 1, 2$.

Proof. Denote $\tau = (-1+\eta)^{-1}$. Integrating repeatedly by parts and using the transmission conditions we get, since u_3 is outgoing,

$$\begin{aligned} \int_{\Omega\setminus D} |\nabla u_2|^2 - k_i^2 |u|^2 \, dx &= \tau \int_D |\nabla u_1|^2 - k_e^2 |u_1|^2 \, dx \\ &+ \tau \lim_{R\to\infty} \int_{B_R(0)\setminus\overline{\Omega}} |\nabla u_3|^2 - k_e^2 |u_3|^2 \, dx \\ &+ \sum_{j=1,2} \int_{\Gamma_j} f_j \partial_\nu \overline{u}_2 + g_j \overline{u}_2 \, dS. \end{aligned}$$

Note now that $k_i^2 = \tau k_e^2 b$, so dividing by τ and taking imaginary parts we get

$$\mathrm{Im}\,(-1+\eta) \int_{\Omega\setminus\overline{D}} |\nabla u_2|^2 \, dx = \mathrm{Im}\left((-1+\eta) \sum_{j=1,2} \int_{\Gamma_j} f_j \partial_\nu \overline{u}_2 + g_j \overline{u}_2 \, dS \right).$$

Hence we have an L^2–bound for the gradient of u_2,

$$\int_{\Omega\setminus\overline{D}} |\nabla u_2|^2\, dx \leq \frac{C'}{|\mathrm{Im}\,\eta|} \sum_j (\|u_2\|_{H^{1/2}(\Gamma_j)} + \|\partial_\nu u\|_{H^{-1/2}(\Gamma_j)}). \tag{4.3}$$

with $C' = C(\sum_j \|f_j\|_{H^{-1/2}(\Gamma_j)} + \|g_j\|_{H^{-1/2}(\Gamma_j)})$ To get a similar bound for the L^2–norms of u_i's we argue as follows. First of all, for all $\phi \in H_0^1(\Omega \setminus \overline{D})$ we have

$$\int_{\Omega\setminus\overline{D}} \nabla u_2 \nabla\phi - k_i^2 u_2 \phi\, dx = 0,$$

so that

$$\left|\int_{\Omega\setminus\overline{D}} k_i^2 u_2 \phi\, dx\right| \leq C\|\nabla u_2\|_{L^2(\Omega\setminus\overline{D})} \|\phi\|_{H^1(\Omega\setminus\overline{D})},$$

If $\omega > 0$, we may divide by k_i^2, and then take supremun over $\phi \in H_0^1(\Omega \setminus \overline{D})$ to conclude

$$\|u_2\|_{H^{-1}(\Omega\setminus\overline{D})} \leq C\|\nabla u_2\|_{L^2(\Omega\setminus\overline{D})}. \tag{4.4}$$

Hence

$$\Delta u_2 = -k_1^2 u_2 \in H^{-1}(\Omega \setminus \overline{D}).$$

Recall now the weak definition of the normal derivative of u_2: Given $h \in H^1(\Omega \setminus \overline{D})$ the normal derivatives of u_2 are defined by duality

$$\int_{\Gamma_1\cup\Gamma_2} \partial_\nu u_2 h\, dS = \int_{\Omega\setminus\overline{D}} (\Delta u_2)h + \langle \nabla u_2, \nabla h\rangle,$$

and we have an estimate

$$\|\partial_\nu u_2\|_{H^{-1/2}(\Gamma_2)} \leq C(\|\Delta u_2\|_{H^{-1}(\Omega\setminus\overline{D})} + \|\nabla u_2\|_{L^2(\Omega\setminus\overline{D})}). \tag{4.5}$$

If $\omega > 0$, we can use (4.4) to deduce

$$\|\partial_\nu u_2\|_{H^{-1/2}(\Gamma_2)} \leq C\|\nabla u_2\|_{L^2(\Omega\setminus\overline{D})}, \tag{4.6}$$

possibly with another constant. If $\omega = 0$, this already follows from the weak definition of $\partial_\nu u_2$ since then u_2 is harmonic. Hence the H^1 stability of the interior and exterior Neumann–problems gives, for $R > 0$ large enough,

$$\|u_1\|_{H^1(D)} + \|u_3\|_{H^1(B_R(0)\setminus\overline{\Omega})} \leq C(\|\nabla u_2\|_{L^2(\Omega\setminus\overline{D})} + \sum_j \|f_j\|_{H^{1/2}} + \|g_j\|_{H^{-1/2}}), \tag{4.7}$$

Using now the H^1 – stability of interior Dirichlet–problem and the Trace–theorem for Sobolev spaces, we conclude, in view of the above and (4.3),

$$\begin{aligned} \|u_2\|_{H^1(\Omega\setminus\overline{D})} \quad &\leq \quad C(\sum_j \|f_j\|_{H^{1/2}} + \|g_j\|_{H^{-1/2}})^{1/2} \times \\ &\times \quad \left(\frac{\|u_2\|^{1/2}_{H^1(\Omega\setminus\overline{D})}}{|\mathrm{Im}\,\eta|^{1/2}} + (\sum_j \|f_j\|_{H^{1/2}} + \|g_j\|_{H^{-1/2}})^{1/2} \right). \end{aligned}$$

This estimate proves the claim for u_2 when η_0 is chosen small enough, and the estimates for u_1 and u_3 then follow from the H^1–stability of the interior and exterior Dirichlet–problems. $\Box$

Proof of Proposition 4.7: Recall that (4.8) is given by

$$(\mathcal{A}_\eta + \mathcal{M}_\eta)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \widetilde{f}_\eta, \tag{4.8}$$

where

$$\mathcal{A}_\eta = \begin{pmatrix}\mathcal{A}_{\eta,1} & 0\\ 0 & \mathcal{A}_{\eta,2}\end{pmatrix}, \quad \mathcal{M}_\eta = \begin{pmatrix}0 & M_{\eta,1}\\ M_{\eta,2} & 0\end{pmatrix} \tag{4.9}$$

with

$$\begin{aligned}\mathcal{A}_{\eta,1} &= \tfrac{1}{2}\left(V^{\Gamma_1}_{k_e} + \tau_1 V^{\Gamma_1}_{k_i}\right) + \left(K^{\Gamma_1}_{k_i} V^{\Gamma_1}_{k_e} - \tau_1 V^{\Gamma_1}_{k_i} K^{*,\Gamma_1}_{k_e}\right),\\ \mathcal{A}_{\eta,2} &= \tfrac{1}{2}\left(V^{\Gamma_2}_{k_e} + \tau_2 V^{\Gamma_2}_{k_i}\right) + \left(K^{\Gamma_2}_{k_i} V^{\Gamma_1}_{k_e} - \tau_2 V^{\Gamma_2}_{k_i} K^{*,\Gamma_2}_{k_e}\right).\end{aligned}$$

Here the off-diagonal operators M_i were infinitely smoothing, and we have explicitly indicated the η–dependence of $k_i = k_i(\eta)$. Denote $\tau = -1+\rho$, so that $\eta = i\delta \to 0$ precisely when $\rho \to 0$, and for some positive constant c we have $c^{-1}|\rho| \le |\eta| \le c|\rho|$. We write $A_{\eta,j}$ in the form

$$\mathcal{A}_{\eta,j} = \frac{1}{2}\left(V^{\Gamma_1}_{k_e} - V^{\Gamma_1}_{k_i}\right) + \left(K^{\Gamma_1}_{k_i} V^{\Gamma_1}_{k_e} - \tau_1 V^{\Gamma_1}_{k_i} K^{*,\Gamma_1}_{k_e}\right) \tag{4.10}$$

$$= -\frac{\rho}{2} V_{k_i} + \frac{1}{2}\left(V_{k_e} - V_{k_i}\right) + \mathcal{K}_\eta. \tag{4.11}$$

Note now that analyticity of the layer potentials with respect to the wave number implies

$$\|V_{k_e} - V_{k_i}\|_{\mathcal{L}(H^{-3/2},H^{1/2})} \le C\rho,$$

and

$$\|\mathcal{K}_\eta - \mathcal{K}_0\|_{\mathcal{L}(H^{-3/2},H^{1/2})},\ \|\mathcal{M}_\eta - \mathcal{M}_0\|_{\mathcal{L}(H^{-3/2},H^{1/2})} \le C\rho.$$

Also, note that the remainder $\frac{1}{2}\left(V_{k_e} - V_{k_i}\right) + \mathcal{K}_\eta$ is infact of order -2. We now have the equations

$$\begin{aligned}\left(-\frac{\rho}{2}\begin{pmatrix}V_{k_i} & 0\\ 0 & V_{k_i}\end{pmatrix} + B_\eta\right)\begin{pmatrix}\phi_1(\eta)\\ \phi_2(\eta)\end{pmatrix} &= \widetilde{f}_\eta,\\ \mathcal{B}_0\begin{pmatrix}\phi_1(0)\\ \phi_2(0)\end{pmatrix} &= \widetilde{f}_0,\end{aligned}$$

where

$$\mathcal{B}_0 := \begin{pmatrix}\frac{1}{2}(V_{k_e} - V_{k_i(0)}) + \mathcal{K}_0 & \mathcal{M}_{0,1}\\ M_{0,2} & \frac{1}{2}(V_{k_e} - V_{k_i(0)}) + \mathcal{K}_0\end{pmatrix}$$

is invertible, and

$$B_\eta := \begin{pmatrix}\frac{1}{2}(V_{k_e} - V_{k_i}) + \mathcal{K}_\eta & \mathcal{M}_{\eta,1}\\ M_{\eta,2} & \frac{1}{2}(V_{k_e} - V_{k_i(\eta)}) + \mathcal{K}_\eta\end{pmatrix}.$$

Substracting the two equations above from each other, we see that the difference

$$\Psi(\eta) = \Phi(\eta) - \Phi(0) := \begin{pmatrix}\phi_1(\eta)\\ \phi_2(\eta)\end{pmatrix} - \begin{pmatrix}\phi_1(0)\\ \phi_2(0)\end{pmatrix}$$

satisfies the equation

$$\mathcal{B}_0\Psi(\eta) = \widetilde{f}(\eta) - \widetilde{f}_0 + \frac{\rho}{2}\begin{pmatrix} V_{k_i} & 0 \\ 0 & V_{k_i} \end{pmatrix}\Phi(\eta) - (\mathcal{B}_\eta - B_0)\Phi(\eta). \tag{4.12}$$

By the previous lemma and the definition of our unknown boundary densities, $\|\Phi(\eta)\|_{H^{-1/2}} \leq C|\mathrm{Im}\,\eta|^{-1}$, so that we have for some other constant C,

$$\|\mathcal{B}_0\Psi(\eta)\|_{H^{1/2}} \leq C, \text{uniformly for } |\eta| \text{ small enough.}$$

The invertibility of $\mathcal{B}_0$ then implies the uniform bound

$$\|\Psi(\eta)\|_{H^{-3/2}} \leq C',\ |\eta| \leq \eta_0,$$

so by compactness every sequence $\Psi(\eta_j)$, $\eta_j \to 0$, has a subsequence $\Psi(\eta_{j_k})$ converging in $H^{-3/2-\rho}(\Gamma_1) \times H^{-3/2-\rho}(\Gamma_2)$ to a limit Ψ_0 solving the equation $\mathcal{B}_0(\Psi_0) = 0$, and hence invertibility of $\mathcal{B}_0$ implies $\Psi_0 = 0$. Thus every sequence $\eta_j \to 0$ has a subsequence such that $\Psi(\eta_{j_k}) \to 0$, and so $\|\Phi(\eta) - \Phi(0)\|_{H^{-3/2-\rho}} \to 0$ as $\eta \to 0$ along the imaginary axis.

Assume now, that $\widetilde{f}(\eta) \in H^{s+1/2}(\Gamma_1) \times H^{s+1/2}(\Gamma_2)$ with $s > 0$. For η small enough V_{k_i} are classical, strongly elliptic invertible ψDO's, so we can define the pseudo differential (complex) powers as

$$\Lambda^s = \begin{pmatrix} V_{k_i} & 0 \\ 0 & V_{k_i} \end{pmatrix}^{-s}.$$

These are isomorphisms of Sobolev order $\mathrm{Re}\, s$ commuting with each other. Consider the equation

$$\left(\mathcal{A}_\eta + \mathcal{M}_\eta\right)\left(\Phi(\eta)\right) = \widetilde{f}_\eta.$$

The order reducing operators Λ^s commute with the principal part of $\mathcal{A}$, and since these both are diagonal operators modulo smoothing, we see that $\widetilde{\Phi}(\eta) := \Lambda^s\Phi(\eta)$ solves

$$(\mathcal{A}_\eta + \mathcal{M}_\eta)\widetilde{\Phi}(\eta) = \Lambda^s\widetilde{f}_\eta - [B_\eta - \mathcal{M}_\eta, \Lambda^s]\Phi(\eta).$$

where the commutator $[B_\eta - \mathcal{M}_\eta, \Lambda^k]$ is of order $s-3$, and hence

$$\Lambda^s\widetilde{f} - [B_\eta - \mathcal{M}_\eta, \Lambda^k]\Phi(\eta) \to \Lambda^s\widetilde{f}_0 - [B_0 - \mathcal{M}_0, \Lambda^s]\Phi(0)$$

in $H^{1/2}$ if $-3/2 - \rho + 3 - s > 1/2$ for some positive ρ, i.e . if $s < 1$. The first part of the proof then gives that $\Lambda^s(\Phi(\eta) - \Phi(0)) \to 0$ in $H^{-3/2-\rho}$, i.e that $\Phi(\eta) - \Phi(0) \to 0$ in $H^{s-3/2-\rho}$. Iteration of this argument then proves that if $\widetilde{f}(\eta) \to \ \widetilde{f}(0)$ in the $H^{3/2+\alpha}$–norm for some positive α, then actually $\Phi(\eta) - \Phi(0) \to 0$ in $H^{-1/2}$–norm. This proves the final claim.

Remarks. (a) The lemma above implies the following upper limit for the solutions of the transmission problems: As $|\mathrm{Im}\,\eta| \to 0$, for $R > 0$ large enough,

$$\|\nabla u_1\|_{H^1(D)},\ \|\nabla u_2\|_{H^1(\Omega\setminus\overline{D})},\ \|\nabla u_3\|_{H^1(B_R(0)\setminus\overline{\Omega})} \leq C|\mathrm{Im}\,\eta|^{-1}.$$

Hence even in the presence of the ALR the blow–up can't be stronger than $\mathcal{O}(|\mathrm{Im}\,\eta|^{-1})$. Note that the ALR requires a blow–up rate stronger than $|\mathrm{Im}\,\eta|^{-1/2}$.

(b) Assume that the source f is supported in $\mathbb{R}^3\setminus\overline{\Omega}$ and that the boundary interfaces are strictly convex as embedded hypersurfaces. Then $\widetilde{f}(\eta) \in H^s$ with any $s \in \mathbb{R}$, and the case (b) above implies $\Phi(\eta) \to \Phi(0)$ in the $H^{-1/2}$–norm. This implies that the original fields u_i, $i = 1, 2, 3$, converge to the limit field in H^1–norm, and hence there is no ALR, or even a weaker resonance like w-AR.

5 Presence of w–AR

In this section we consider dimension $d \geq 2$ and frequencies $\omega \geq 0$. We denote by $a_\eta(x)$ the piecewise constant function in $\mathbb{R}^d$ that is $a_\eta(x) = a_{\rm e}$ for $x \in (\mathbb{R}^d \setminus \Omega) \cup D$ and $a_\eta(x) = a_{\rm e}(-1+\eta)$ for $x \in \Omega \setminus D$, $\eta \in i\mathbb{R}_- \cup \{0\}$. Also, $b = -1$ in equation (1.8).

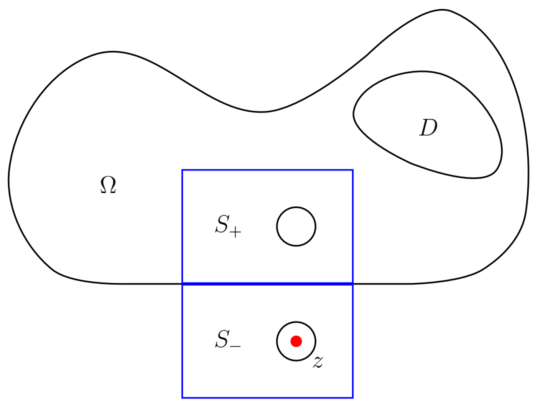

Figure 2: Setting of the Theorem 5.1: Domain $\Omega \subset \mathbb{R}^d$ that contains domain D. The material parameters approach in the set $\Omega \setminus D$ negative value and are positive outside this set. Coordinates are chosen so that x_1–direction is vertical.

Theorem 5.1. *Assume $D \subset \Omega \subset \mathbb{R}^d$, $d \geq 2$ that the interfaces $\Gamma_1 = \partial D$ and $\Gamma_2 = \partial\Omega$ are smooth and Γ_2 contains a flat subset $S_0 = \{y_1\} \times B$, where $y_1 \in \mathbb{R}$ and $B = \{x' \in \mathbb{R}^{d-1};\ |x'| \leq R_0\}$. Also, assume that $S_+ \subset \Omega \setminus \overline{D}$ and $S_- \subset \mathbb{R}^d \setminus \overline{\Omega}$ are compact sets given by $S_+ = [y_1, y_1 + a] \times B$ and $S_- = [y_1 - a, y_1] \times B$, $a > 0$.*

Moreover, let $f = \delta_z$ with $z \in S_-$. Also, let $\tau_\eta = -1 + \eta$, $\eta \in i\mathbb{R}_+$, and assume that $k_e = k_i \in \mathbb{R}_+ \cup \{0\}$, i.e., $b = -1$ in equation (1.8), and (1.5), (1.6), and (1.7) are valid. Let $0 < |\eta| \leq \eta_0$ for some positive fixed η_0. Assume the problem (1.1)–(1.4) with $\tau = \tau_\eta$ is uniquely solvable and that v_i^η, $i = 1, 2, 3$ are its solutions. Let $r_1 > 0$ be such that $B(z, r_1) \subset S_-$. Then as $\eta \to 0$,

$$\lim_{\eta \to 0} \left(\|v_2^\eta\|_{H^1(S_+)} + \|v_3^\eta\|_{H^1(S_- \setminus B(z,r_1))} \right) = \infty.$$

Proof. Denote $S = S_+ \cup S_0 \cup S_-$. Let $z^- = (z_1, z') \in S_-$ and $z^+ = (2y_1 - z_1, z') \in S_+$. By (1.3), we see that there are functions $w^\eta \in H^1(S \setminus \overline{B}(z^-, r_1))$ such that $w^\eta|_{S_+} = v_2^\eta|_{S_+}$ and $w^\eta|_{S_-} = v_3^\eta|_{S_-}$. To show the claim, assume the opposite: We assume that there is a sequence $\eta_j \to 0$ and constants C_0 and C_{r_1} such that

$$\|w^{\eta_j}\|_{H^1(S_+)} \leq C_0,\ \|w^{\eta_j}\|_{H^1(S \setminus \overline{B}(z^-, r_1))} \leq C_{r_1}$$

Hence, using the weak compactness and after replacing η_j with a suitable subsequence, that we continue to denote by η_j, we can then assume that w_j converges in $H^1(S \setminus \overline{B}(z^-, r_1))$ weakly to some function W. For all $\phi \in C_0^\infty(S \setminus \overline{B}(z^-, r_1))$ we now have

$$\int_{\mathbb{R}^d} (a_0(x)\nabla W\cdot\nabla\phi - \omega^2 a_0(x)\mu_0\ W\phi)dx$$
$$= \lim_{j\to\infty}\int_{\mathbb{R}^d} (a_{\eta_j}(x)\nabla w_{\eta_j}\cdot\nabla\phi - \omega^2 a_{\eta_j}(x)\mu_0\ w_{\eta_j}\phi)dx = 0.$$

Hence in the domain $S\setminus\overline{B}(z^-,r_1)$ we have

$$\nabla\cdot(a_0(x)\nabla W) + \omega^2 a_0(x)\mu_0\ W = 0 \tag{5.1}$$

in the weak sense. In particular, this yields that W satisfies an elliptic equation in a neighbourhood of S_0 with trace $W|_{S_0}\in H^{1/2}(S_0)$ and thus W has well defined one-sided normal derivatives on $S_0\subset\Gamma_2$ with values in $H^{-1/2}(S_0)$. Applying integration by parts in domains $S_-\setminus\overline{B}(z^-,r_1))$ and S_+ we obtain for $\phi\in C_0^\infty(S\setminus\overline{B}(z^-,r_1))$

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d} (a_0(x)\nabla W\cdot\nabla\phi - \omega^2 a_0(x)\mu_0\ W\phi)dx \\ &= -\int_{S_0} (\nu\cdot\nabla W|_{S_0-} + \nu\cdot\nabla W|_{S_0+})\phi\, dS, \end{aligned}$$

where $\nu=(0,0,\dots,0,1)$ is normal vector of S_0 pointing from S_- to S_+. Thus we see that $\partial_\nu W|_{S_0-} = -\partial_\nu W|_{S_0+}$. Summarizing, we have

$$W|_{S_0-} = W|_{S_0+},\quad \partial_\nu W|_{S_0-} = -\partial_\nu W|_{S_0+}. \tag{5.2}$$

Using this, equation (5.1), and the fact that $S\setminus(\overline{B}(z^-,r_1)\cup\overline{B}(z^-,r_1))$ is connected implies that for $x=(x_1,x')\in F:=S\setminus(\overline{B}(z^-,r_1)\cup\overline{B}(z^+,r_1))$ we have the symmetry

$$W(x_1,x') = W(2y_1-x_1,x'),\quad x=(x_1,x')\in F. \tag{5.3}$$

Now, using the Gauss theorem we observe

$$\int_{\partial B(z^-,r_1)} \partial_\nu w_{\eta_j}(x)dS(x) = 1-\omega^2\mu_0\int_{B(z^-,r_1)} w_{\eta_j}\,dx. \tag{5.4}$$

On the other hand, symmetry of the limit W implies

$$\lim_{j\to\infty}\int_{\partial B(z^-,r_1)} \partial_\nu w_{\eta_j}(x)dS(x) = \lim_{j\to\infty}\int_{\partial B(z^+,r_1)} \partial_\nu w_{\eta_j}(x)dS(x) = 0.$$

Hence, for any $r_1>0$ small enough,

$$1 = \omega^2\mu_0\lim_{j\to\infty}\int_{B(z^-,r_1)} w_{\eta_j}\,dx = \omega^2\mu_0\int_{B(z^-,r_1)} W\,dx$$

On the other hand, the sequence (w_{η_j}) was bounded in $H^1(S_+)$ with a bound independent of r_1, so again using the symmetry of the limit W we get

$$\Big|\int_{B(z^-,r_1)} W\,dx\Big| \le Cr_2^{d/2},$$

which yields a contradiction as $r_2,\, r_1\to 0$. $\square$

The results of the previous sections, together with the earlier results of Milton et al. ([23, 24]) and Ammari et al. ([2]) show that for strictly convex bodies ALR may appear only for bodies so small that the quasi-static approximation is realistic. This gives limits for size of the objects for which invisibility cloaking methods based on ALR may be used. However, the results of this section show that the weak AR may appear if the body $\Omega \setminus \overline{D}$ has double negative material parameters and its external boundary contains flat parts.

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