Thin Trees in Some Families of Graphs

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Let $G = (V, E)$ be a graph and let T be a spanning tree of G. The thinness parameter of T denoted by $\rho(T)$ is the maximum over all cuts of the proportion of the edges of T in the cut. Thin trees play an important role in some recent papers on the Asymmetric Traveling Salesman Problem (ATSP). Goddyn conjectured that every graph of sufficiently large edge-connectivity has a spanning tree T such that $\rho(T) \leq \epsilon$.

In this thesis, we study the problem of finding thin spanning trees in two families of graphs, namely, (1) distance-regular graphs (DRGs), and (2) planar graphs.

For some families of DRGs such as strongly regular graphs, Johnson graphs, Crown graphs, and Hamming graphs, we give a polynomial-time construction of spanning trees T of maximum degree ≤ 3 such that $\rho(T)$ is determined by the parameters of the graph.

For planar graphs, we improve the analysis of Merker and Postle ("Bounded Diameter Arboricity", arXiv:1608.05352v1) and show that every 6-edge-connected planar graph has two edge-disjoint spanning trees T, T' such that $\rho(T), \rho(T') \leq \frac{14}{15}$. For 8-edge-connected planar graphs G, we present a simplified version of the techniques of Merker and Postle and show that G has two edge-disjoint spanning trees T, T' such that $\rho(T), \rho(T') \leq \frac{12}{13}$.

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Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Thin Trees

In this thesis, we study a special kind of spanning tree known as a thin tree. We often use the term *thin tree* to mean a thin spanning tree. First, we review some standard notation from graph theory. Let $G = (V, E)$ be a graph; G may have multiedges, but G has no loops. We use n to denote the number of vertices, thus $n = |V|$. For $F \subseteq E$ and $\emptyset \neq S \subsetneq V$, we use $\delta_F(S)$ to denote the set of all edges in F with one endpoint in S, and the other endpoint in the complement of S. If $|V| = n > 1$ and $G' = (V, E \setminus F)$ is connected for every set $F \subseteq E$ of fewer than k edges, then G is called k-edge-connected. Notice that a graph G is k-edge-connected if and only if $|\delta_G(S)| \geq k$ for all $\emptyset \neq S \subsetneq V$. For $S \subseteq V$, we use $E[S]$ to denote the set of all edges with both endpoints in S. For $F \subseteq E$, let $d_F(v)$ be the number of edges of F incident to the vertex v. We denote the maximum degree of F by $\Delta(F) := \max_{v \in V} d_F(v)$. A subset of edges $M \subseteq E$ is a *matching* if no two edges of M are incident to the same vertex. For $i, j \in V$ and $i \neq j$ let w_{ij} be the number of edges in E with endpoints i and j. Note that if there is no edge with endpoints i and j, then $w_{ij} = 0$. The adjacency matrix $A = (a_{ij})_{n \times n}$ of G is defined by

$$a_{ij} = \begin{cases} w_{ij}, & \text{if } i \neq j, \\ 0, & \text{otherwise.} \end{cases}$$

Let d_i denote the degree of vertex i in G, i.e., $d_i = \sum_{j \in V \setminus \{i\}} w_{ij}$. The Laplacian matrix $L_G = (l_{ij})_{n \times n}$ of G is defined by:

$$l_{ij} = \begin{cases} -w_{ij}, & \text{if } i \neq j, \\ d_i, & \text{if } i = j. \end{cases}$$

Now, we define a thinness parameter of a subset of edges $F \subseteq E$. The thinness parameter of F is the maximum over all cuts of the proportion of the edges of F in the cut.

Definition 1.1.1. *Let $G = (V, E)$ be a connected graph, and let F be a subset of E. We define the thinness parameter of F to be*

$$\rho(F) := \max_{\forall S \subsetneq V, S \neq \emptyset} \frac{|\delta_F(S)|}{|\delta_G(S)|}.$$

Notice that $0 \leq \rho(F) \leq 1$. We say a spanning tree T is ϵ-thin if $\rho(T) \leq \epsilon$. By a thin spanning tree, we mean a spanning tree T such that $0 < \rho(T) < 1$. In Example 1.2.4, we determine the thinness parameter of a particular spanning tree of the Octahedral graph.

Given a connected graph G, we are interested in constructing a thin tree in G with thinness parameter as small as possible. Thin trees play an important role in some recent papers on the Asymmetric Traveling Salesman Problem (ATSP), see [AG15], [GS11], and [AGM+10]. Goddyn [God] conjectured that every graph of sufficiently large edge-connectivity has an ϵ-thin spanning tree. Here is the precise statement of the conjecture.

Conjecture 1.1.2 (Goddyn [God])**.** *There exists a function f such that, for any $0 < \epsilon < 1$, every $f(\epsilon)$-edge-connected graph has an ϵ-thin spanning tree.*

Now, we discuss some recent developments on this conjecture. Asadpour et al. [AGM+10] showed that there is a polynomial-time algorithm that finds a $\frac{1}{k} \cdot O(\frac{\log n}{\log \log n})$-thin spanning tree of a given k-edge-connected graph. Later, Oveis Gharan and Saberi [GS11] designed a polynomial-time algorithm that given a connected graph embedded on an orientable surface with genus γ and dual-girth g^* finds a $\frac{2 \cdot \alpha}{g^*}$-thin spanning tree where $\alpha := 4 + \lfloor 2 \log{(\gamma + \frac{3}{2})} \rfloor$. Thus, in order to show the existence of a thin tree T (with $\rho(T) < 1$) in a planar graph, [GS11] requires the edge-connectivity to be ≥ 11. Very recently, Merker and Postle [MP] showed that every 6-edge-connected planar graph has two edge-disjoint $\frac{18}{19}$-thin spanning trees (see Theorem 3.1.1).

Notational remark: We use numbers in parentheses just after a theorem, lemma, etc., to indicate the part that is being referenced, e.g., Theorem 1.2.5(2) refers to part (2) of Theorem 1.2.5.

1.2 Spectrally Thin Trees

We define a strengthening of thin trees called spectrally thin trees. After we state the definition we discuss why this notion is stronger than the notion of (combinatorially) thin trees. Let $G = (V, E)$ be a graph with n vertices. For any edge $e = \{u, v\} \in E$ (assume u precedes v in our ordering of V), the signed incidence vector $b_e \in \mathbb{R}^n$ is the vector that has precisely two nonzero entries, namely, the entry corresponding to b_u is -1, and the entry corresponding to b_v is $+1$. Observe that $L_G = \sum_{e \in E} b_e b_e^T$.

Definition 1.2.1. *Let G be a graph on n vertices. We say a spanning tree T of G is α-spectrally thin if $L_T \preceq \alpha \cdot L_G$, i.e., for all $x \in \mathbb{R}^n$, $x^T L_T x \leq \alpha \cdot x^T L_G x$.*

As before, we are interested in α-spectrally thin spanning trees where $0 < \alpha < 1$. Let $G = (V, E)$ be a connected graph, and let T be a spanning tree of G. If T is α-spectrally thin, then $\rho(T) \leq \alpha$. This can be seen as follows: For each set $S \subseteq V$, let χ_S be the characteristic vector of the set S, i.e., the vector in $\mathbb{R}^n$ whose i-th entry is 1 if $i \in S$ and is 0 otherwise. It can be seen that $|\delta_T(S)| = \chi_S^T L_T \chi_S$ and $|\delta_G(S)| = \chi_S^T L_G \chi_S$ (since for any edge e, $(\chi_S^T b_e)(b_e^T \chi_S)$ is 1 if $e \in \delta(S)$ and is 0 otherwise). Hence, if T is α-spectrally thin, then we have $|\delta_T(S)| = \chi_S^T L_T \chi_S \leq \alpha \chi_S^T L_G \chi_S = \alpha |\delta_G(S)|$. But the converse may not hold (see Example 1.2.4 and note that the spanning tree T is α-spectrally thin for some $0.77 \leq \alpha \leq 0.78$ but $\rho(T) = 0.66$).

The results that will be discussed in this section are based on the notion of effective resistance that is defined below.

Definition 1.2.2. *Let $G = (V, E)$ be a graph, and let $e = \{u, v\}$ be an edge of G. We define the effective resistance of e to be*

$$\mathrm{Reff}(e) := b_e^T L_G^\dagger b_e, \tag{1.1}$$

where $L_G^\dagger$ is the pseudoinverse of L_G, and b_e is the signed incidence vector of e.

See Example 1.2.4 for an illustration of the effective resistance of an edge in a graph. There is a relation between edge-connectivity and effective resistance. In particular, we have the following observation:

Remark 1.2.3 (Anari and Oveis Gharan [AG15])**.** *Let $e = \{u, v\}$ be an edge of $G = (V, E)$ such that $\mathrm{Reff}(e) \leq \frac{1}{k}$, where k is a positive integer. Then, G has at least k edge-disjoint paths between u and v. However, the converse may not hold.*

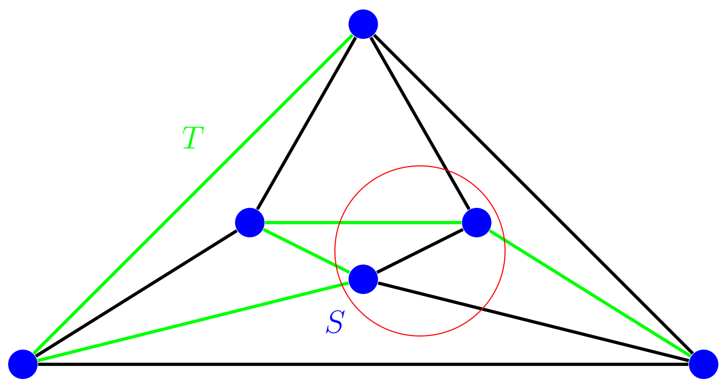

Figure 1.1: Octahedral graph G. The spanning tree T is indicated by the green-colored lines. The set S is indicated by the red-colored circle.

Let us give an example to illustrate the notions that we have introduced so far.

Example 1.2.4. *Let G be the Octahedral graph shown in Figure 1.1. Notice that G is edge-transitive (meaning, for any pair of edges $e, f \in E$, there is an automorphism of G that maps e to f). Hence, by symmetry, the effective resistances of all edges are the same. The effective resistance of each edge in this graph is 0.41. Let T be the spanning tree of G indicated by the green-colored lines, and let S be the subset of vertices indicated by the red-colored circle in Figure 1.1. Then, we have $\rho(T) \geq \frac{|\delta_T(S)|}{|\delta_G(S)|} = \frac{4}{6} = 0.66$. By checking all $2^5 - 1$ cuts, it can be seen that $\rho(T) = \frac{2}{3}$. We can compute the largest eigenvalue λ_1 of $L_G^{-\frac{1}{2}} L_T L_G^{-\frac{1}{2}}$. It turns out that $\lambda_1 \leq 0.78$. Furthermore, let $x = (0, 1, 0, 1, -1, -1)^T$. Then, $\frac{x^T L_T x}{x^T L_G x} \geq 0.77$. Therefore, T is α-spectrally thin for some $0.77 \leq \alpha \leq 0.78$.*

Harvey and Olver [HO14] were able to show the following result by making a stronger assumption than k-edge-connectivity:

Theorem 1.2.5 (Theorem 4.11 & 4.12 in [HO14])**.** *Let G be a graph with n vertices such that $\mathrm{Reff}(e) \leq \frac{1}{k}$ for every edge e, where k is a positive integer. Then, (1) there is a polynomial-time algorithm to construct a $\frac{1}{k} \cdot O(\frac{\log n}{\log \log n})$-spectrally thin spanning tree in G, and (2) there exists an $\frac{O(1)}{k}$-spectrally thin spanning tree in G.*

Notice that by Remark 1.2.3, the assumption in the above theorem is stronger than k-edge-connectivity.

Finally, in very recent work, Anari and Oveis Gharan [AG15] gave an existential result on thin trees in k-edge-connected graphs.

Theorem 1.2.6 (Corollary 1.8 in [AG15]). *Any k-edge-connected graph has a $\frac{poly(\log \log n)}{k}$-thin spanning tree, where n denotes the number of vertices.*

1.3 Contributions of This Thesis

We mainly focus on constructing thin trees for two families of graphs, namely, (1) distance-regular graphs (defined in Section 2.4), and (2) planar graphs.

Chapter 2 addresses distance-regular graphs (DRGs). We know that the effective resistance of each edge in a DRG G of degree d is $\frac{O(1)}{d}$, see Corollary 1 in [KMP13]. So we can apply Theorem 1.2.5(2) to G, and conclude that there exists a spanning tree T in G with $\rho(T) = \frac{O(1)}{d}$. Unfortunately, no efficient algorithm is known for Theorem 1.2.5(2). Moreover, Theorem 1.2.5 is an asymptotic result. In particular, for $d = O(1)$, Theorem 1.2.5(2) could well be useless since the implied value of $\rho(T)$ could be ≥ 1. In Chapter 2, we give a polynomial-time construction of thin trees T with $\rho(T) < 1$ (we determine the value of $\rho(T)$ based on the parameters of the graph) in some families of DRGs such as strongly regular graphs (SRGs), Johnson graphs, Crown graphs, and Hamming graphs (see Section 2.4 for the definitions of these graphs). Our methods have two advantages: (1) there is an efficient algorithm for finding a thin spanning tree T, and (2) there are precise bounds on $\rho(T)$ (see Table 2.1) that allow us to certify any spanning tree with maximum degree ≤ 3 to be a thin tree.

Chapters 3 and 4 deal with planar graphs. In [MP] it is shown that a 6-edge-connected planar graph has two edge-disjoint $\frac{18}{19}$-thin spanning trees. The main technique in the paper is to find a particular edge-coloring of the graph. To this end, they start with a particular vertex-coloring of the graph and then they extend the vertex-coloring to an edge-coloring. In Chapter 3, we present a simplified version of the methods of [MP]. Given an 8-edge-connected planar graph, we directly find an edge-coloring and we use it to prove that there exist two edge-disjoint $\frac{12}{13}$-thin spanning trees. In Chapter 4, we use the same methods as in the paper [MP] but we present a tighter analysis to show that a 6-edge-connected planar graph has two edge-disjoint $\frac{14}{15}$-thin spanning trees.

Chapter 2

Thin Trees in Distance-Regular Graphs

2.1 Introduction

In this chapter, we study thin trees in distance-regular graphs (DRGs). Let $G = (V, E)$ be a graph; multiedges are allowed; let $n = |V|$. Recall that $\rho(F)$ is the thinness parameter of an edge set F. As we discussed in the introductory chapter, if G is a DRG with degree d, then by Theorem 1.2.5(2), there exists an $\frac{O(1)}{d}$-thin spanning tree in G. Unfortunately, no efficient algorithm is known for finding such a spanning tree. Moreover, Theorem 1.2.5 is an asymptotic result. In particular, for $d = O(1)$, Theorem 1.2.5(2) could well be useless since the implied value of $\rho(T)$ could be ≥ 1. In this chapter, we give a polynomial-time construction for thin trees T in some families of DRGs such that $\rho(T)$ is upper bounded by the parameters of the graph.

This chapter is organized as follows: In Section 2, we provide some certificates for giving an upper bound on $\rho(F)$ for an edge set F. Then, in Section 3, given a DRG, we give a candidate for a thin tree in the graph. In fact, we find in polynomial time a spanning tree of maximum degree ≤ 3. Finally, in the last section, we provide upper bounds for the thinness parameter of our candidate in some families of DRGs, see table 2.1 for more information. The DRGs that we are considering in this chapter are strongly regular graphs, Johnson graphs, Crown graphs, and Hamming graphs.

2.2 Certificates for the Thinness Parameter

In this section, we provide two certificates for giving upper bounds on $\rho(F)$ for an edge set F. We start with a certificate given in [Gha].

Lemma 2.2.1. *Let $G = (V, E)$ be a graph, and let F be a subset of E. Suppose $\mathcal{C} = \{C_1, ...C_l\}$ is a set of cycles in G that has the following properties:*

1. *Each cycle in $\mathcal{C}$ has exactly one edge from F.*
2. *Each edge in F is in at least β cycles in $\mathcal{C}$.*
3. *Each edge not in F is in at most α cycles in $\mathcal{C}$.*

Then, we have $\rho(F) \leq \frac{\alpha}{\beta}$.

Proof. We call any cycle in $\mathcal{C}$ a good cycle. Take an arbitrary subset of vertices $\emptyset \neq S \subsetneq V$. By property (2), for each $e \in \delta_F(S)$, there are at least β good cycles that each contains e. So by property (1) there are at least $\beta \cdot |\delta_F(S)|$ good cycles such that each has at least one edge in $\delta_G(S) \setminus F$. By property (3), we conclude that $|\delta_G(S) \setminus F| \geq \frac{\beta}{\alpha}|\delta_F(S)|$. Therefore, we have $\rho(F) \leq \frac{|\delta_F(S)|}{\frac{\beta}{\alpha} \cdot |\delta_F(S)|} = \frac{\alpha}{\beta}$, as desired. □

The next lemma provides the second certificate. This certificate is based on Cheeger's inequality which is a well-known inequality in spectral graph theory.

Lemma 2.2.2. *Let $G = (V, E)$ be a d-regular graph, and let F be a subset of E such that $d_F(v) \leq \alpha$ for all $v \in V$. Let λ_2 be the second largest eigenvalue of the adjacency matrix $A(G)$. Then, $\rho(F) \leq \frac{2 \cdot \alpha}{d - \lambda_2}$.*

Proof. Recall Cheeger's inequality, i.e., for every set $\emptyset \neq S \subsetneq V$, where $|S| \leq \frac{|V|}{2}$ we have

$$\frac{d - \lambda_2}{2} \leq \frac{|\delta_G(S)|}{|S|}. \tag{2.1}$$

Now, suppose $\emptyset \neq S \subsetneq V$. Since $\delta_G(S) = \delta_G(\bar{S})$ and $|S| + |\bar{S}| = |V|$, wlog, we can assume that $|S| \leq \frac{|V|}{2}$. Since $d_F(v) \leq \alpha$ for all $v \in V$, we get

$$|\delta_F(S)| \leq \alpha \cdot |S|. \tag{2.2}$$

From (2.1) and (2.2), we have

$$\frac{|\delta_F(S)|}{|\delta_G(S)|} \leq \frac{2 \cdot \alpha}{d - \lambda_2}. \tag{2.3}$$

Since S was an arbitrary subset of V, from (2.3) we conclude that $\rho(F) \leq \frac{2 \cdot \alpha}{d - \lambda_2}$. □

2.3 A Candidate for a Thin Tree in a DRG

Let $G = (V, E)$ be a d-regular graph, and let T be a spanning tree in G. Suppose $\Delta(T)$ is much smaller than d. Then, intuitively the chance of a vertex $v \in S \subsetneq V$ to contribute a non-tree edge to $\delta(S)$ increases. In fact, this is our main idea for finding thin trees in DRGs. In this section, we give a polynomial-time construction for spanning trees with small maximum degree. Then, we show that by considering bounded-degree spanning trees in some d-regular graphs with edge-connectivity d, each vertex in a subset of vertices $S \subsetneq V$ contributes at least a constant number of non-tree edges to $\delta(S)$. This constant number can be used to give upper bounds on the thinness parameter of the tree, see Theorems 2.3.4 & 2.3.5.

The next theorem is due to Singh and Lau [SL15]; it gives an algorithm for finding a bounded-degree spanning tree.

Theorem 2.3.1 (Theorem 1.2 of [SL15])**.** *Let $G = (V, E)$ be a graph, and let $B_v \in N_+$ be a degree bound for each vertex $v \in V$. If the following LP is feasible, then we can find in polynomial time a spanning tree T in G such that $d_T(v) \leq B_v + 1$ for all $v \in V$.*

$$
\begin{array}{llcll}
\min & \sum_{e \in E} x_e & & & \\
s.t. & \displaystyle\sum_{e \in E} x_e & = & n - 1 & \\
& \displaystyle\sum_{e \in E[S]} x_e & \leq & |S| - 1 & \forall S \subsetneq V, S \neq \emptyset \\
& \displaystyle\sum_{e \in \delta(v)} x_e & \leq & B_v & \forall v \in V \\
& x_e & \geq & 0 & \forall e \in E
\end{array}
\tag{2.4}
$$

Note that the LP in (2.4) is bounded. If the LP is feasible, then by fundamental theorem of linear programming, we know that the LP has an optimal solution. So the above theorem is equivalent to Theorem 1.2 of [SL15].

The next lemma shows that Theorem 2.3.1 can be applied to any d-regular graph with edge-connectivity d to find a spanning tree with maximum degree ≤ 3.

Lemma 2.3.2. *Let $G = (V, E)$ be a d-regular graph on n vertices with edge-connectivity d. Set $B_v := 2$ for all $v \in V$. Let $\bar{x} \in \mathbb{R}^{|E|}$ be a vector such that $\bar{x}_e := \frac{2}{d} \cdot \frac{n-1}{n}$ for all $e \in E$. Then, $\bar{x}$ is feasible for the LP in (2.4).*

Proof. From the value of $\bar{x}_e$ for each e, we have $\sum_{e \in E} \bar{x}_e = (\frac{nd}{2})(\frac{2}{d})(\frac{n-1}{n}) = n - 1$. So the first constraint of the LP holds.

Since G is d-regular, we have $\sum_{e \in \delta(v)} \bar{x}_e = d(\frac{2}{d})(\frac{n-1}{n}) \leq 2$. Hence, the last constraint of the LP holds too.

It remains to show that the middle constraint of the LP holds for $\bar{x}$.

Let $\emptyset \neq S \subsetneq V$. Since G is d-edge-connected, we have $|\delta(S)| \geq d$; hence, $|E[S]| \leq \frac{d(|S|-1)}{2}$. So we can write

$$\sum_{e \in E[S]} \bar{x}_e \leq (\frac{d(|S|-1)}{2})(\frac{2}{d})(\frac{n-1}{n}) \leq |S| - 1,$$

as desired. So we proved that $\bar{x}$ is a feasible solution for the LP. $\square$

By combining Lemma 2.3.2 with Theorem 2.3.1, we conclude that there is a polynomial-time algorithm that given a d-regular graph with edge-connectivity d finds a spanning tree with maximum degree ≤ 3. This tree is our candidate for thin spanning trees in DRGs. So we restate this fact in the following corollary:

Corollary 2.3.3. *Let $G = (V, E)$ be a d-regular graph with edge-connectivity d. Then, in polynomial time, we can find a spanning tree T such that $\Delta(T) \leq 3$.*

Now, we are able to show that under some conditions on d-regular graphs with edge-connectivity d, the spanning tree given by Corollary 2.3.3 is thin in the graph. We should mention that these conditions may not seem natural on general d-regular graphs. However, our goal is to find thin spanning trees in DRGs, and these conditions are related to the parameters of a DRG. The next theorem uses the first certificate (Lemma 2.2.1) to give an upper bound on the thinness factor of our bounded-degree spanning tree.

Theorem 2.3.4. *Let $G = (V, E)$ be a d-regular graph with edge-connectivity d. Suppose each edge is incident to $\hat{a} \geq 11$ triangles. Then, in polynomial time, we can find a spanning tree T in G such that $\rho(T) \leq \frac{6}{\hat{a}-4}$.*

Proof. By Corollary 2.3.3, in polynomial time, we can find a spanning tree T in G such that $\Delta(T) \leq 3$. For convenience, we call the edges of T the green edges, and the edges of $E \setminus E(T)$ the black edges. We construct a set of cycles $\mathcal{C}$ for Lemma 2.2.1 as follows: Let e be an edge of T, then by our assumption, e is in $\hat{a}$ triangles. Notice that at most 4 of these triangles have two green edges. See Figure 2.1. Therefore, each of the other $\hat{a} - 4$ triangles

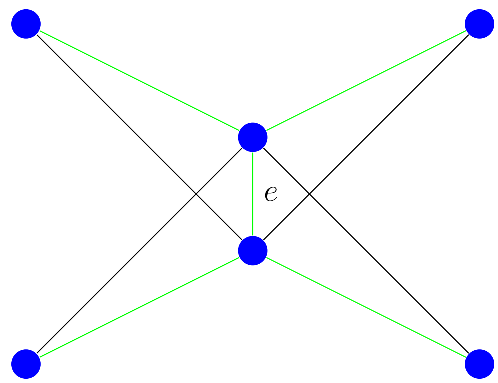

Figure 2.1: There are at most 4 triangles with two green edges incident to a green edge (tree edge) such as e. See the proof of Theorem 2.3.4.

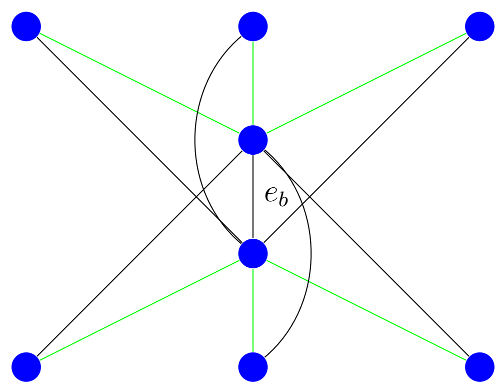

Figure 2.2: A black (non-tree) edge e_b is in at most 6 cycles of $\mathcal{C}$. See the proof of Theorem 2.3.4.

has exactly one green edge and that edge is e. So for each edge of T, we find $\hat{a} - 4$ such triangles and put them in $\mathcal{C}$. Thus, the parameter β in Lemma 2.2.1 is $\hat{a} - 4$.

Now, consider a black edge e_b. Note that e_b is in at most 6 cycles of $\mathcal{C}$, because each endpoint of e_b is incident to at most three green edges, due to the degree bound on T. See Figure 2.2. Therefore, the parameter α in Lemma 2.2.1 is 6.

By Lemma 2.2.1, we conclude that $\rho(T) \leq \frac{6}{\hat{a}-4}$. Notice when $\hat{a} \geq 11$, we have $\frac{6}{\hat{a}-4} < 1$ which is acceptable for the thinness parameter.

$\square$

In the next theorem, we use the second certificate (Lemma 2.2.2) to give an upper bound on the thinness parameter of our bounded-degree spanning tree.

Theorem 2.3.5. *Let G be a d-regular graph with edge-connectivity d, and let λ_2 be the second largest eigenvalue of $A(G)$. Then, in polynomial time, we can find a spanning tree T in G such that $\rho(T) \leq \frac{6}{d-\lambda_2}$.*

Proof. By Corollary 2.3.3, in polynomial time, we can find a spanning tree T in G such that $\Delta(T) \leq 3$. The statement follows by applying Lemma 2.2.2 with $\alpha = 3$. $\square$

2.4 Thin Trees in Some Families of DRGs

In this section, we apply Theorems 2.3.4 & 2.3.5 to find thin spanning trees for some families of DRGs.

A connected graph G with diameter D is called distance-regular if there are constants c_i, a_i, and b_i, the so called intersection numbers, such that for all $i = 0, 1, ..., D$, and all vertices x, y at distance i, among the neighbors of y, there are c_i at distance $i-1$ from x, a_i at distance i from x, and b_i at distance $i+1$ from x (the distance between two vertices u, v is the minimum number of edges in a path between u and v). Notice that a_1 is the number of triangles incident to an edge. Let d be the degree of each vertex of G. Then, the following equation holds: $d = a_i + b_i + c_i$.

Brouwer and Haemers [BH05] proved the following result for DRGs:

Theorem 2.4.1 (Theorem 4.1 in [BH05])**.** *A distance-regular graph with degree d is d-edge-connected.*

We start by constructing thin spanning trees in strongly regular graphs (SRGs); SRGs are known to be distance-regular. Let G be a regular graph that is neither complete nor empty. Then G is said to be strongly regular with parameters

$$(n, d, a, c)$$

if it is d-regular, every pair of adjacent vertices has a common neighbours, and every pair of non-adjacent vertices has c common neighbours. Notice that the parameter a indicates the number of triangles incident to an edge. For more background on SRGs, see Chapter 10 of [GR13].

Note that by Theorem 2.4.1, an SRG with degree d is d-edge-connected. The next result is an immediate consequence of Theorem 2.3.4.

Corollary 2.4.2. *Let G be an SRG with parameters (n, d, a, c), where $a \geq 11$. Then, in polynomial time, we can find a spanning tree T with $\Delta(T) \leq 3$ such that $\rho(T) \leq \frac{6}{a-4}$.*

Next, we provide a thin spanning tree in an SRG with parameters (n, d, a, c), where $a \leq 10$ and $d \geq 18$.

Corollary 2.4.3. *Let G be an SRG with parameters (n, d, a, c), where $d \geq 18$ and $a \leq 10$. Then, in polynomial time, we can find a spanning tree T with $\Delta(T) \leq 3$ such that $\rho(T) \leq \frac{6}{d-\sqrt{d+25}-5}$.*

Proof. We start by giving an upper bound on the second largest eigenvalue λ_2 of $A(G)$. We know that $\lambda_2 = \frac{(a-c)+\sqrt{(a-c)^2+4(d-c)}}{2}$ (see Section 10.2 of [GR13]). It is easy to see that λ_2 as a function of c is decreasing. Thus, we have

$$\lambda_2 \leq \frac{a + \sqrt{a^2 + 4d}}{2}. \tag{2.5}$$

By the assumption of the lemma, i.e., $a \leq 10$, we get

$$\lambda_2 \leq \frac{10 + \sqrt{100 + 4d}}{2}. \tag{2.6}$$

From (2.6), we have

$$\frac{2d - 10 - \sqrt{100 + 4d}}{2} \leq d - \lambda_2. \tag{2.7}$$

Now, by applying Theorem 2.3.5 to G where $d - \lambda_2$ in the theorem satisfies (2.7), in polynomial time, we can find a spanning tree T with $\Delta(T) \leq 3$ such that $\rho(T) \leq \frac{6}{d-\sqrt{d+25}-5}$. Notice that when $d \geq 18$, we have $\frac{6}{d-\sqrt{d+25}-5} < 1$ which is acceptable for the thinness parameter. $\square$

Now, we focus on Johnson graphs $J(n, k)$:

Let X be a set of size n. The vertex set of $J(n, k)$ is the collection of all subsets of X with k elements. Two such subsets u, v are adjacent whenever $|u \cap v| = k - 1$. For more background on Johnson graphs see Section 9.1 of [BCN89].

It is known that the number of vertices of $J(n, k)$ is $\binom{n}{k}$, it is d-regular, where $d = k(n - k)$, and each edge is incident to $n - 2$ triangles. Thus, by applying Theorem 2.3.4 to the Johnson graph $G = J(n, k)$, in polynomial time, we can find a spanning tree T in G with $\Delta(T) \leq 3$ such that $\rho(T) \leq \frac{6}{n-6}$. In order to have $\rho(T) < 1$, we must have $n \geq 13$.

families of DRGs	value of $\rho(T)$
SRG (n, d, a, c), $a \geq 11$	$\frac{6}{a-4}$
SRG (n, d, a, c), $d \geq 18$, $a \leq 10$	$\frac{6}{d-\sqrt{d+25}-5}$
$J(n, k)$, $n \geq 13$	$\frac{6}{n-6}$
$Crown(n)$, $n \geq 9$	$\frac{6}{n-2}$
$H(n, k)$, $k \geq 7$	$\frac{6}{k}$

Table 2.1: The thinness parameter $\rho(T)$ of a spanning tree T with maximum degree ≤ 3 is indicated for some families of DRGs.

Next, we focus on Crown graphs $Crown(n)$:

The vertex set of $Crown(n)$ is $\{x_1, ..., x_n, y_1, ..., y_n\}$. Two vertices x_i and y_j are connected whenever $i \neq j$ for $i, j \in \{1, ..., n\}$.

It is known that $Crown(n)$ is $(n-1)$-regular. It is not hard to see that the second largest eigenvalue of its adjacency matrix is $\lambda_2 = 1$ (by using the fact that $Crown(n)$ is the complement of the Cartesian product of K_2 and K_n). Therefore, by applying Theorem 2.3.5 to the Crown graph $G = Crown(n)$, in polynomial time, we can find a spanning tree T in G with $\Delta(T) \leq 3$ such that $\rho(T) \leq \frac{6}{n-2}$. In order to have $\rho(T) < 1$, we must have $n \geq 9$.

Finally, we focus on Hamming graphs $H(n, k)$:

Let X be a set of size k. The vertex set of $H(n, k)$ is X^n (X^n is the Cartesian product of n copies of X, i.e, the set of all n-tuples where each coordinate belongs to X). Two n-tuples u, v are adjacent whenever u and v differ in precisely one coordinate. For more background on Hamming graph see Section 9.2 of [BCN89].

It is known that $H(n, k)$ is $n(k-1)$-regular, and the second largest eigenvalue of its adjacency matrix is $\lambda_2 = n(k-1) - k$. Therefore, by applying Theorem 2.3.5 to the Hamming graph $G = H(n, k)$, in polynomial time, we can find a spanning tree T in G with $\Delta(T) \leq 3$ such that $\rho(T) \leq \frac{6}{k}$. In order to have $\rho(T) < 1$, we must have $k \geq 7$.

We summarize all the above results in Table 2.1.

Chapter 3

Thin Trees in 8-Edge-Connected Planar Graphs

3.1 Introduction

In this chapter, all graphs are considered to be simple (without multiedges and without loops) unless stated otherwise. In particular, our decomposition result (Theorem 3.1.4) works for simple planar graphs, and our thin tree result (Theorem 3.1.2) works for planar graphs with multiedges (i.e., (multi)graphs). The goal of this chapter is to investigate thin spanning trees for planar graphs. Merker and Postle [MP] showed the following result for thin trees in this family of graphs:

Theorem 3.1.1 (Corollary 3.6 of [MP]). *Every 6-edge-connected planar (multi)graph contains two edge-disjoint $\frac{18}{19}$-thin spanning trees.*

The above theorem is a byproduct of the main result in [MP]. So let us explain it here. A decomposition of a graph $G = (V, E)$ consists of edge-disjoint subgraphs on the vertex set of G, $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$ whose union is G, i.e., $E = E_1 \cup E_2$. Kim et al. [KKW+13] showed that a connected planar graph of girth ≥ 6 has a decomposition into a spanning tree and a star forest (a star forest is a forest in which every component is a bipartite graph $K_{1,t}$ for some $t \in \mathbb{Z}, t \geq 0$), see Theorem 4.2.2. Merker and Postle [MP] showed that for a graph G, if there is a decomposition of G into a spanning tree and a star forest, then G can be decomposed into two forests with diameter at most 18. From this result they are able to prove Theorem 3.1.1.

In this chapter, we present a simplified version of the techniques of [MP] to show that if a graph G has a decomposition into a spanning tree and a matching, then G can be decomposed into two forests with diameter at most 12. Montassier et al. [MdMRZ12] showed that a connected planar graph of girth ≥ 8 has a decomposition into a spanning tree and a matching, see Theorem 3.2.2. Based on this, for any 8-edge-connected planar graph, we are able to give better thin trees than the thin trees given by Theorem 3.1.1. In particular, we have the following result:

Theorem 3.1.2. *Let $G = (V, E)$ be an 8-edge-connected planar (multi)graph. Then, there are two edge-disjoint spanning trees T, T' in G such that $\rho(T), \rho(T') \leq \frac{12}{13}$.*

The connection between thin trees in a planar graph and a decomposition of the graph into bounded diameter forests comes from the notion of *dual* of planar graphs. Consider a planar graph $G = (V, E)$ with a fixed planar embedding. We define a planar graph $G^* = (V^*, E^*)$ as follows. Corresponding to each face f of G there is a vertex f^* of G^*, and corresponding to each edge e of G there is an edge e^* of G^*. Two vertices f^* and g^* are joined by the edge e^* in G^* if and only if the corresponding faces f and g are incident to the edge e in G. The graph G^* is called the *dual* of G.

In the following lemma, we state the connection between thin trees in a planar graph and a decomposition of the graph into two bounded diameter forests. Recall that $\rho(F)$ is the thinness parameter of a set $F \subseteq E$ in $G = (V, E)$.

Lemma 3.1.3 (Lemma 3.3 of [MP])**.** *If $G = (V, E)$ is a planar graph that can be decomposed into two forests F_1, F_2 with diameter at most d, then G^* has two edge-disjoint spanning trees T, T' such that $\rho(T), \rho(T') \leq \frac{d}{d+1}$.*

Notice that in the above lemma, G is a simple graph and its dual could have multiedges, i.e., G^* is a (multi)graph.

Proof. This proof is essentially the same as the proof of [MP, Lemma 3.3] and it is included for completeness.

Let us color the edges of F_1 and F_2 by colors 1 and 2, respectively. Then this coloring gives an edge-coloring of G such that there is no monochromatic cycle and every monochromatic path has length at most d. In G^*, let F_1^* be the subgraph corresponding to F_1. Similarly, we define F_2^*. By the usual bijection $E(G) \to E(G^*)$, this gives a 2-edge-coloring of G^*. Notice that the edges of F_1^* and F_2^* are colored 1 and 2, respectively. We want to prove that $(V(G^*), F_1^*)$ and $(V(G^*), F_2^*)$ are spanning connected subgraphs of G^*. Furthermore, we have $\rho(F_1^*), \rho(F_2^*) \leq \frac{d}{d+1}$.

Let $\emptyset \neq S \subsetneq V(G^*)$. The edges of $\delta_{G^*}(S)$ correspond to an edge-disjoint union of cycles in G. Consider one such cycle C in the union. Since there is no monochromatic cycle in G, both colors appear in C. Thus, both colors appear in $\delta_{G^*}(S)$. Hence, both $(V(G^*), F_1^*)$ and $(V(G^*), F_2^*)$ are spanning connected subgraphs of G^*, i.e., for each $i = 1, 2$ and $\forall S \subsetneq V, S \neq \emptyset$ we have $\delta_{F_i^*}(S) \neq \emptyset$. Furthermore, since each of F_1 and F_2 has diameter at most d, every path in G of length at least $d+1$ contains at least one edge of color 1. This implies that at least $\frac{1}{d+1}|E(C)|$ edges of C are colored 1. Thus, at most $\frac{d}{d+1}|\delta_{G^*}(S)|$ edges of $\delta_{G^*}(S)$ are colored 2. Therefore, we have $\rho(F_2^*) \leq \frac{d}{d+1}$. Similarly, we have $\rho(F_1^*) \leq \frac{d}{d+1}$.

Now pick any spanning tree T_2 in $(V(G^*), F_2^*)$. Then, we have $\rho(T_2) \leq \frac{d}{d+1}$. Similarly, for any spanning tree T_1 in $(V(G^*), F_1^*)$, we have $\rho(T_1) \leq \frac{d}{d+1}$. Since F_1^* and F_2^* are disjoint, so are T_1 and T_2. $\square$

As we discussed earlier, our main contribution in this chapter is the following result:

Theorem 3.1.4. *Let $G = (V, E)$ be a connected planar graph with girth at least 8. Then, G can be decomposed into two forests with diameter at most 12.*

Before we go into the details of the proof of the above theorem, let us combine this theorem and Lemma 3.1.3 to prove Theorem 3.1.2.

Proof of Theorem 3.1.2: This proof is essentially the same as the proof of [MP, Corollary 3.6] and it is included for completeness.

Since G is an 8-edge-connected planar (multi)graph, its dual $G^* = (V^*, E^*)$ is a simple planar graph with girth at least 8. By Theorem 3.1.4, we know that each connected component $G_i^* = (V_i^*, E_i^*)$ of G^* can be decomposed into two forests (V_i^*, F_i^1) and (V_i^*, F_i^2) such that each has diameter at most 12. Let $H_1 := (\bigcup_i V_i^*, \bigcup_i F_i^1)$ and $H_2 := (\bigcup_i V_i^*, \bigcup_i F_i^2)$. Notice that for each $i = 1, 2$, H_i is a forest with diameter at most 12. Thus, G^* can be decomposed into two forests with diameter at most 12. By applying Lemma 3.1.3 to G^* we conclude that G has two edge-disjoint spanning trees T, T' such that $\rho(T), \rho(T') \leq \frac{12}{13}$.

In the next section, we will show that the decomposition of a graph into two bounded diameter forests corresponds to a particular edge-coloring (good edge-coloring) of the graph. Then, we construct an edge-coloring of the graph. Finally, in the last section, we will prove that our edge-coloring is indeed a good edge-coloring and this implies Theorem 3.1.4.

3.2 A Good Edge-Coloring of the Graph

As we noted in the previous section, there is a correspondence between Theorem 3.1.4 and a particular edge-coloring of planar graphs. We define this edge-coloring below.

Definition 3.2.1 (good edge-coloring). *Let $G = (V, E)$ be a graph, and let $c : E(G) \to \{1, 2\}$ be an edge-coloring of G. We say c is a* ***good edge-coloring*** *if c satisfies the following two conditions:*

1. *There is no edge-monochromatic cycle.*

2. *Any edge-monochromatic path has length at most* 12.

To prove Theorem 3.1.4 we need the following result: Theorem 1 of [WZ11] or Corollary 2 of [MdMRZ12] states that every planar graph of girth at least 8 can be decomposed into a forest F and a matching M. If G is connected, then we can convert F to a tree T by adding as many matching edges as possible. So we have the following result:

Theorem 3.2.2 (Theorem 1 of [WZ11]). *Every connected planar graph with girth at least* 8 *can be decomposed into a spanning tree and a matching.*

We often use *tree* to mean a spanning tree. Let $G = (V, E)$ be a graph that can be decomposed into a tree T and a matching M. Then we denote this decomposition by $G = (V, T \uplus M)$. Starting with $G = (V, T \uplus M)$, we obtain the directed graph $\overrightarrow{G}$ as follows:

Pick a vertex r, and make T a rooted tree with root r. We fix a planar embedding of T with the root at the top and the leaves at the bottom; for each non-leaf vertex v of T, we place the children of v on a horizontal line below v. Then, orient each edge of the tree away from the root. This directed tree is denoted by $\overrightarrow{T}$.
For orienting the matching edges, we first define a labeling for the vertices of G based on T. Set the label of the root equal to 1. We traverse T, and each time we visit a vertex for the first time, we assign to it the smallest natural number that has not been used so far. Our traverse on T is as follows: Start from the root. We always go to the rightmost child that has not yet been visited. Once we could not go further, we backtrack until we find a vertex v with at least one child that has not yet been visited, then we take u to be the rightmost unvisited child of v, and we continue our traverse from u. See Figure 3.1 for an example of this labeling of the vertices of a graph.
Now, we orient the matching edges as follows: For each matching edge orient the edge from the lower label to the higher label. We denote this directed matching with $\overrightarrow{M}$. So we have $\overrightarrow{G} = (V, \overrightarrow{T} \uplus \overrightarrow{M})$.

Since G is simple, any (di)cycle in G or $\overrightarrow{G}$ has length at least 3. Also notice that any (vertex)edge-coloring of $\overrightarrow{G}$ can be considered as an (vertex)edge-coloring of G, and vice versa.

We say $\overrightarrow{uv}$ is an M-edge if $\overrightarrow{uv} \in E(\overrightarrow{M})$, and $\overrightarrow{uv}$ is a T-edge if $\overrightarrow{uv} \in E(\overrightarrow{T})$. We say a T-edge is in depth i if it is oriented from a vertex in depth i to a vertex in depth $i+1$. We fix the depth of the root r to be 1. Define the parent of a T-edge $\overrightarrow{uv}$ of depth ≥ 2 to be the T-edge $\overrightarrow{wu}$.

For the rest of this chapter when we talk about a graph G we mean $G = (V, T \cup M)$ where T is a spanning tree and M is a matching, and $\overrightarrow{G}$ is the digraph described above.

Notice that any edge in $\overrightarrow{G}$ is oriented from a vertex with lower label to a vertex with higher label. Thus, there could not be a dicycle in $\overrightarrow{G}$. We restate this fact in the following lemma:

Lemma 3.2.3. *There is no dicycle in* $\overrightarrow{G}$.

This lemma will help us to show that our edge-coloring satisfies the first condition of a good edge-coloring, see the proof of Corollary 3.3.4.

From now on we work on $\overrightarrow{G}$ and try to give an edge-coloring of it such that every monochromatic dipath in $\overrightarrow{G}$ has length at most 6 (we prove this fact in a series of lemmas, see Lemmas 3.3.10-3.3.12). Once we show this property of our edge-coloring, then we can easily show that it satisfies the second condition of a good edge-coloring of G, see Corollary 3.3.13.

Example 3.2.4. *Let G be the undirected graph underlying the digraph in Figure 3.1. The decomposition of G is shown by two colors, i.e, green edges indicate the tree T, and black edges indicate the matching M. The green edges are oriented away from the root. We define a labeling for the vertices of G as discussed earlier in this section. These labels are denoted by a number next to each vertex. Then, we orient the matching edges with respect to this labeling. Notice that all the edges are oriented from a vertex with lower label to a vertex with higher label (i.e., either downward or right-to-left in the planar embedding).*

We color the edges of $\overrightarrow{G}$ with colors 1 and 2 such that any M-edge and any T-edge that are oriented toward the same vertex have different colors. This way of coloring the edges imposes the so-called indegree property (Fact 3.3.1) on the edge-coloring. We use the indegree property to prove that our edge-coloring is a good edge-coloring.

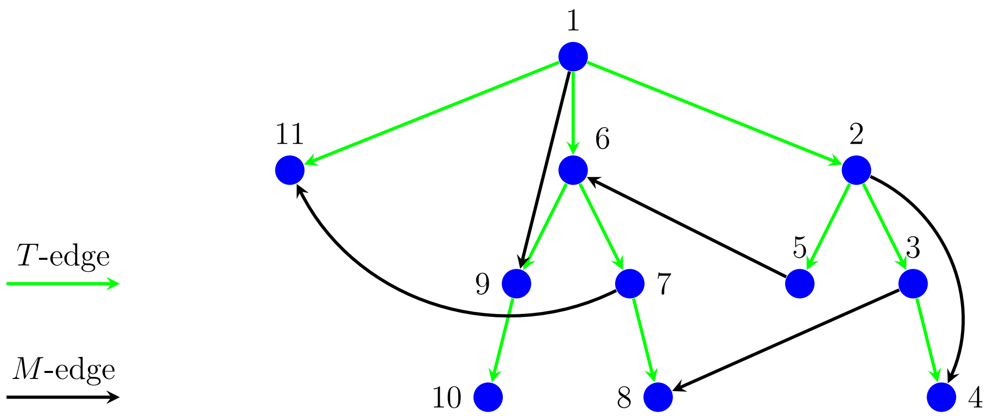

Figure 3.1: Directed graph $\overrightarrow{G}$ and the labeling of the vertices as described before. Notice that each edge is oriented from a lower label to a higher label.

Initial Coloring:

We start by coloring the edges of the tree. We assign color 1 to all edges in depth 1. Suppose we have colored all the edges in depth i. Then, for each edge in depth $i+1$, we color the edge by a different color than its parent. We denote this coloring of the T-edges by c_0. It remains to determine the color of the matching edges. Let $\overrightarrow{uv}$ be an M-edge, then there exists a T-edge $\overrightarrow{wv}$. Let $c_0(\overrightarrow{uv}) := 3 - c_0(\overrightarrow{wv})$.

Notice that the color of the M-edges is determined uniquely based on the color of the edges of T. We say $c_0 : E(\overrightarrow{G}) \to \{1, 2\}$ is the ***initial coloring*** of $\overrightarrow{G}$.

Example 3.2.5. *Figure 3.2 illustrates the initial coloring that we discussed above; the color of each edge is denoted by* 1 *or* 2.

As the above example illustrates, in this coloring, we might have a long monochromatic dipath. In Figure 3.3, this dipath is shown in blue. So to avoid such long monochromatic dipaths, we need to update the coloring c_0. The following definition will be used in the updating process.

Definition 3.2.6. *Let $\overrightarrow{uv}$ be a T-edge such that $c_0(\overrightarrow{uv}) = i$ (where $i = \{1, 2\}$), and there are two M-edges such that one is oriented toward u, and the other one is oriented away from v. Furthermore, both matching edges have color i. Then, we say that $\overrightarrow{uv}$ falls into* ***pattern $\boldsymbol{\pi_i}$***.

When we apply the above definition, the colors of the matching edges need not be the same as their initial colors. Pattern π_1 is shown in Figure 3.4. It is clear that the blue dipath in Figure 3.3 is formed by "linking up" two copies of pattern π_1.

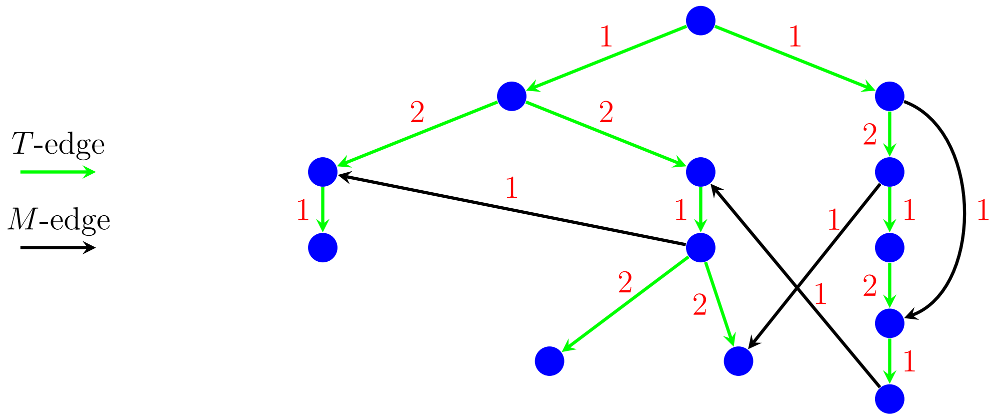

Figure 3.2: Initial coloring of $\overrightarrow{G}$.

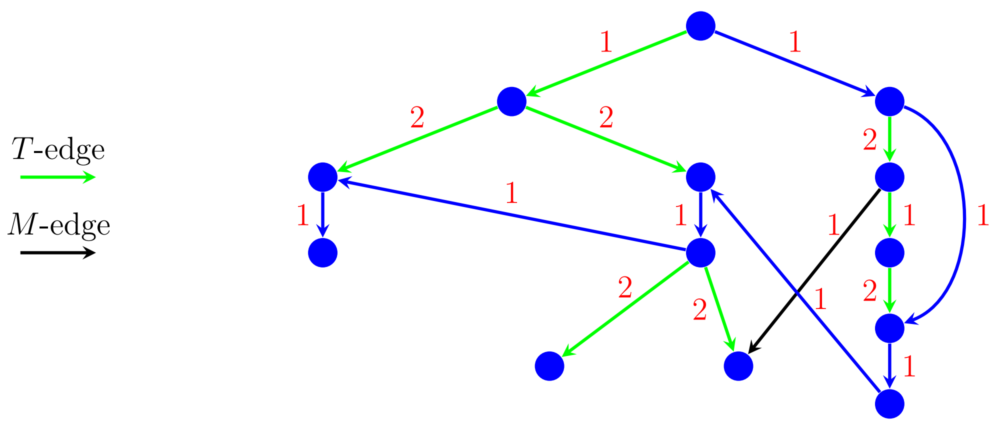

Figure 3.3: The blue edges form a monochromatic dipath.

Figure 3.4: The edge $\overrightarrow{uv}$ falls into pattern π_1.

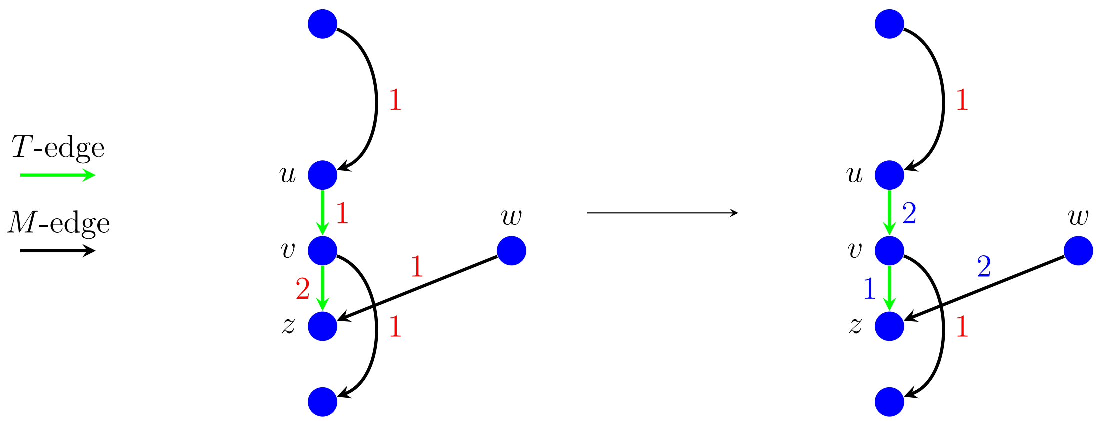

Figure 3.5: In the left subfigure, we see that $\overrightarrow{uv}$ falls into pattern π_1 before the recoloring process enters to the depth of $\overrightarrow{uv}$. The right subfigure shows the final color of $\overrightarrow{uv}$, one of its children $\overrightarrow{vz}$, and the M-edge $\overrightarrow{wz}$ that is oriented toward the head of $\overrightarrow{vz}$.

To avoid such patterns, we update the coloring c_0 to get a new coloring c' as follows:

Recoloring Process:

We start from depth 1 of T and go downward. When we are in depth j, we apply the following step to all T-edges in this depth. If $\overrightarrow{uv}$ is a T-edge in depth j falling into pattern π_i for $i = 1$ or 2, then we flip the color of $\overrightarrow{uv}$ and all of its children. Furthermore, we flip the colors of the matching edges (if any) that are oriented toward the heads of the children of $\overrightarrow{uv}$ (in order to maintain the indegree property, see Fact 3.3.1). Otherwise the color of the edge does not change.
Then, we go to depth $j+1$ and repeat the same procedure. See Figure 3.5 for an example.

Let c' be the edge-coloring of $\overrightarrow{G}$ after the recoloring process terminates; we refer to c' as the ***final coloring*** of $\overrightarrow{G}$. Obviously, the final color of each edge could be different than its initial color or be the same as before.

Definition 3.2.7. *We say an edge $\overrightarrow{uv}$ is a* ***flipped edge*** *if its color has been flipped, i.e., $c_0(\overrightarrow{uv}) = 3 - c'(\overrightarrow{uv})$. If an edge is not a flipped edge then we call it a* ***non-flipped edge****.*

We also use the terms flipped T-edge, flipped M-edge, etc. For example, in the graph of Figure 3.5, edge $\overrightarrow{uv}$ is a flipped T-edge and $\overrightarrow{wz}$ is a flipped M-edge.

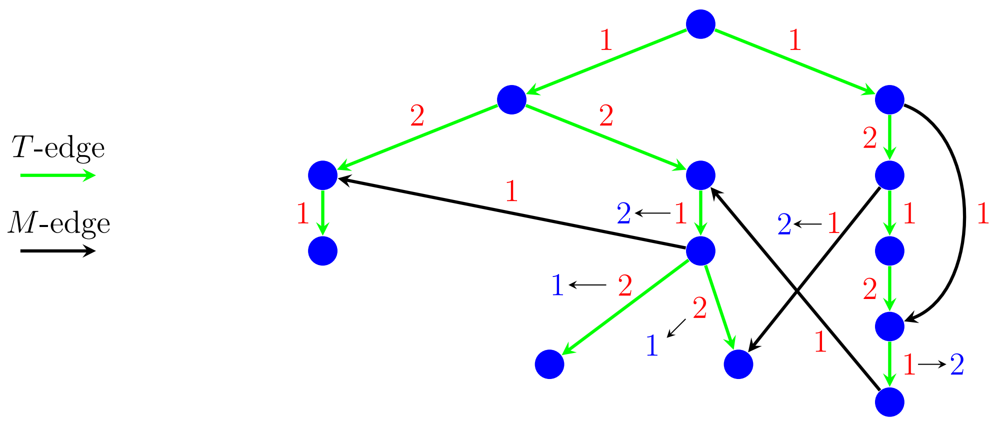

Figure 3.6: The final coloring of $\overrightarrow{G}$.

Example 3.2.8. *Let* $\overrightarrow{G}$ *be the graph in Example 3.2.5, and let* c_0 *be the initial coloring of the graph;* c_0 *is denoted by the red numbers. Figure 3.6 illustrates the final coloring of* $\overrightarrow{G}$ *after applying the recoloring process. The final colors of some of the edges are different from their initial colors; we show such changes by an arrow from the red numbers (initial color) to the blue numbers (the final color).*

3.3 Proof of Theorem 3.1.4

In this section, we prove c' that is introduced in the previous section is a good edge-coloring for G.

We state an important property of the edge-colorings c_0 and c' that was briefly mentioned in the previous section. Let $indeg(v)$ be the indegree of a vertex v, i.e., the number of edges that are oriented toward v, and let $indeg_i(v)$ be the number of edges of color i that are oriented toward v. Notice that $indeg(v) \leq 2$ for all $v \in V$. Furthermore, we have $indeg(v) = 2$ if and only if there is an M-edge $\overrightarrow{uv}$ and a T-edge $\overrightarrow{wv}$ that are oriented toward v. But in both the initial coloring and the recoloring process we made sure that the color of $\overrightarrow{uv}$ is always different than the color of $\overrightarrow{wv}$. So we summarize this fact as follows:

Fact 3.3.1 (indegree property). *The following holds for both coloring* c_0, c'*: For any vertex* $v \in V$ *and* $i \in \{1, 2\}$*, we have* $indeg_i(v) \leq 1$*.*

The next lemma is an immediate consequence of the above fact. This lemma says that in order to show c' is a good edge-coloring for G, it is enough to show that there is no monochromatic (with respect to c') dicycle in $\overrightarrow{G}$, and every monochromatic (with respect to c') dipath has length at most 6. The following lemma is due to Merker and Postle [MP].

Lemma 3.3.2. *Let c' be the edge-coloring of both G and $\overrightarrow{G}$. Then, we have:*

1. *Every monochromatic cycle in G is a dicycle in $\overrightarrow{G}$.*

2. *Every monochromatic path in G is the union of at most two monochromatic dipaths in $\overrightarrow{G}$.*

Proof. Let C be a monochromatic cycle in G. WLOG, we can assume that C has color 1 in c'. Consider the subgraph of $\overrightarrow{G}$ corresponding to C, and denote it by $\overrightarrow{C}$. Then, the sum of the indegrees (with respect to the subgraph $\overrightarrow{C}$) of the vertices of $\overrightarrow{C}$ is $|V(\overrightarrow{C})|$. Either each of the indegrees is 1, thus $\overrightarrow{C}$ is a dicycle, or else there exist vertices of indegrees zero and two. The former case contradicts Lemma 3.2.3, and the latter case contradicts Fact 3.3.1. Thus, there is no monochromatic cycle in G. So part (1) of the lemma holds.
Now suppose we have a monochromatic path P in G. In $\overrightarrow{G}$, let $\overrightarrow{P}$ be the subgraph corresponding to P; observe that $\overrightarrow{P}$ is the union of k (inclusion) maximal dipaths $\overrightarrow{P_1}, ..., \overrightarrow{P_k}$ such that $\overrightarrow{P_i}$ and $\overrightarrow{P_{i+1}}$ have exactly one vertex in common. Notice that either both $\overrightarrow{P_i}$ and $\overrightarrow{P_{i+1}}$ are oriented toward their common vertex or they both are oriented away from their common vertex. Suppose $k \geq 3$. Then, either the end-vertices of $\overrightarrow{P_1}$ and $\overrightarrow{P_2}$, or the end-vertices of $\overrightarrow{P_2}$ and $\overrightarrow{P_3}$ are the same. WLOG, assume that the former case holds. Let v be the end-vertex of $\overrightarrow{P_1}$ and $\overrightarrow{P_2}$. Then, $indeg_1(v) \geq 2$, a contradiction with Fact 3.3.1. Therefore, $k \leq 2$, and this proves part (2) of the lemma. $\square$

Remark 3.3.3. *Notice that the above lemma holds for any graph G with any orientation of its edges and any edge-coloring that satisfies Fact 3.3.1.*

Now, we prove that c' is a good edge-coloring. We start by showing that there is no monochromatic cycle in G (i.e., the first condition of a good edge-coloring holds for c'). By Lemma 3.2.3, there is no dicycle in $\overrightarrow{G}$. On the other hand, by Lemma 3.3.2(1), every monochromatic cycle in G is directed in $\overrightarrow{G}$. So we have the following result:

Corollary 3.3.4. *Let c' be the edge-coloring of G. Then, there is no monochromatic cycle in G.*

For the rest of this section, we prove that the second property of a good edge-coloring holds for c' on G. Notice that by Lemma 3.3.2(2), it is enough to show that every monochromatic (with respect to c') dipath in $\overrightarrow{G}$ has length at most 6. We start by highlighting some important properties of c_0 and c' which will be used in the proofs of our lemmas.

Since in the recoloring process we do the changes in depth i and we never come back to this depth again, color of a T-edge flips at most once. Furthermore, the color of an M-edge $\overrightarrow{uv}$ flips only if there is a T-edge $\overrightarrow{wv}$ whose color has been flipped. Therefore, color of an M-edge flips at most once too.
Another simple observation is as follows: Let $\overrightarrow{wv}, \overrightarrow{vz}$ be T-edges, and let $\overrightarrow{uv}$ be an M-edge. Since the color of $\overrightarrow{uv}$ only can be flipped when the recoloring process is in the depth of $\overrightarrow{wv}$; together with the fact that $\overrightarrow{wv}$ is the parent of $\overrightarrow{vz}$ (in other words, $\overrightarrow{wv}$ is in a lower depth than $\overrightarrow{vz}$), we conclude that the color of $\overrightarrow{uv}$ is finalized before the recoloring process enters the depth of $\overrightarrow{vz}$. We summarize these observations below.

Property 3.3.5. *The color of an edge flips at most once. Furthermore, let $\overrightarrow{uv}$ be an M-edge, and let $\overrightarrow{vz}$ be a T-edge. The color of $\overrightarrow{uv}$ is finalized before the recoloring process enters the depth of $\overrightarrow{vz}$.*

Suppose $\overrightarrow{uv}$ is an M-edge. Then, there is a T-edge $\overrightarrow{wv}$ such that $c_0(\overrightarrow{wv}) = 3 - c_0(\overrightarrow{uv})$. Thus, all the children of $\overrightarrow{wv}$ have the same initial color as $\overrightarrow{uv}$. So, we have the following observation:

Property 3.3.6. *Let $\overrightarrow{uv}$ be an M-edge, and let $\overrightarrow{wv}$ be a T-edge. Then, for any child $\overrightarrow{vz}$ of $\overrightarrow{wv}$ we have $c_0(\overrightarrow{vz}) = c_0(\overrightarrow{uv})$.*

Notice that the recoloring process can flip the color of a T-edge in two ways. These two ways are as follows:

Property 3.3.7. *Let $\overrightarrow{uv}$ be a T-edge. Then, we have $c'(\overrightarrow{uv}) = 3 - c_0(\overrightarrow{uv})$ if and only if one of the following cases happens:*

- **Case 1.** *when the recoloring process is in the depth of $\overrightarrow{uv}$, there are matching edges of color $c_0(\overrightarrow{uv})$ such that one is oriented toward u, and the other one is oriented away from v.*

- **Case 2.** *the parent of $\overrightarrow{uv}$ falls into Case 1.*

Now, we show that any monochromatic dipath in $\overrightarrow{G}$ has length at most 6 based on two preliminary lemmas (Lemma 3.3.8 and Lemma 3.3.9). We apply these two lemmas extensively in our overall proofs (see Lemmas 3.3.10-3.3.12).

Lemma 3.3.8. *Let $\overrightarrow{P}$ be a monochromatic dipath that contains a flipped T-edge $\overrightarrow{uv}$, and a non-flipped edge $\overrightarrow{vz}$. Then, $\overrightarrow{uv}$ is the start-edge of $\overrightarrow{P}$.*

Proof. WLOG, suppose $\overrightarrow{P}$ has color 1 in c'. Since $\overrightarrow{uv}$ is a flipped T-edge, $\overrightarrow{uv}$ must fall into one of the cases of Property 3.3.7. We prove that $\overrightarrow{uv}$ could only fall into case (2). We show this fact by considering two cases, i.e., when $\overrightarrow{vz}$ is an M-edge, and when $\overrightarrow{vz}$ is a T-edge.

Case 1 (when $\overrightarrow{vz}$ is an M-edge).

Since $\overrightarrow{uv}$ is a flipped edge, and $\overrightarrow{P}$ is in color 1, we have $c_0(\overrightarrow{uv}) = 2$. On the other hand, $\overrightarrow{vz}$ is a non-flipped M-edge (assumption of the lemma). So, we have $c_0(\overrightarrow{vz}) = c'(\overrightarrow{vz}) = 1$. Hence, the color of $\overrightarrow{vz}$ is always 1. Thus, $\overrightarrow{uv}$ could not fall into Property 3.3.7(1).

Case 2 (when $\overrightarrow{vz}$ is a T-edge).

Suppose $\overrightarrow{uv}$ falls into Property 3.3.7(1). Then, all of its children, including $\overrightarrow{vz}$ are flipped edges, a contradiction with the assumption of the lemma.

In both cases, we showed that $\overrightarrow{uv}$ could not fall into Property 3.3.7(1). So it must be the case that $\overrightarrow{uv}$ falls into Property 3.3.7(2); hence, the parent of $\overrightarrow{uv}$ has color 2 in c', and there is an M-edge that is oriented away from u. Thus, there cannot be an edge of $\overrightarrow{P}$ whose head is u which implies that $\overrightarrow{uv}$ is the start-edge of $\overrightarrow{P}$. □

Lemma 3.3.9. *Let $\overrightarrow{P}$ be a monochromatic dipath that contains a flipped M-edge $\overrightarrow{uv}$. Then, $\overrightarrow{uv}$ is the end-edge of $\overrightarrow{P}$.*

Proof. WLOG, assume that $\overrightarrow{P}$ has color 1 in c'. By contradiction, suppose $\overrightarrow{uv}$ is not the end-edge of $\overrightarrow{P}$. So there is a T-edge (since $\overrightarrow{uv}$ is an M-edge) $\overrightarrow{vz}$ such that $c'(\overrightarrow{vz}) = 1$ (because the color of $\overrightarrow{P}$ is 1 in c'). Since $\overrightarrow{uv}$ is a flipped edge and $\overrightarrow{P}$ has color 1 in c', the initial color of $\overrightarrow{uv}$ is 2, i.e., $c_0(\overrightarrow{uv}) = 2$; together with Property 3.3.6, we conclude that $c_0(\overrightarrow{vz}) = 2$. Therefore, $\overrightarrow{vz}$ is a flipped T-edge. However, by Property 3.3.5, when the recoloring process is in the depth of $\overrightarrow{vz}$, the color of the M-edge $\overrightarrow{uv}$ is 1 and it is finalized. So $\overrightarrow{vz}$ does not fall into Property 3.3.7(1). Furthermore, since $\overrightarrow{uv}$ is an M-edge oriented toward v, $\overrightarrow{vz}$ could not fall into Property 3.3.7(2). Thus, $\overrightarrow{vz}$ is not a flipped T-edge, a contradiction. See Figure 3.7 for an example.

□

A monochromatic dipath $\overrightarrow{P}$ falls into one of the following types: (1) $\overrightarrow{P}$ contains no M-edge, (2) $\overrightarrow{P}$ does not have a non-flipped M-edge, and (3) $\overrightarrow{P}$ contains at least one non-flipped M-edge. We give upper bounds for the length of monochromatic dipaths of each type. We start by monochromatic dipaths with no M-edge.

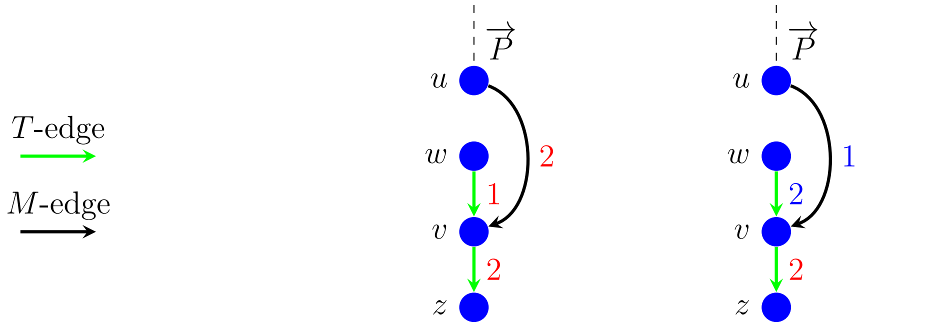

Figure 3.7: The left subfigure shows the initial color of $\overrightarrow{uv}$, $\overrightarrow{wv}$, and $\overrightarrow{vz}$. The right subfigure shows that the color of $\overrightarrow{wv}$ flips, consequently, the color of $\overrightarrow{uv}$ flips too. Therefore, $\overrightarrow{vz}$ falls into neither of the cases in Property 3.3.7, showing that $\overrightarrow{vz}$ is not a flipped edge.

Lemma 3.3.10. *Let $\overrightarrow{P}$ be a monochromatic dipath that contains no M-edge. Then, the length of $\overrightarrow{P}$ is at most* 3.

Proof. WLOG, suppose $\overrightarrow{P}$ has color 1 in c'. Let dipath v_1, v_2, v_3, v_4 be a sub-dipath of $\overrightarrow{P}$. There are two cases based on the initial color of $\overrightarrow{v_1v_2}$, i.e., either $c_0(\overrightarrow{v_1v_2}) = 1$, or else $c_0(\overrightarrow{v_1v_2}) = 2$. We show that the first case could not happen, and in the second case, we prove that the dipath v_1, v_2, v_3, v_4 is indeed equal to the whole dipath $\overrightarrow{P}$ which implies that the length of $\overrightarrow{P}$ is at most 3.

Case 1 (when $c_0(\overrightarrow{v_1v_2}) = 1$).

Since $c_0(\overrightarrow{v_1v_2}) = 1$, we have $c_0(\overrightarrow{v_2v_3}) = 2$ and $c_0(\overrightarrow{v_3v_4}) = 1$. So $\overrightarrow{v_1v_2}, \overrightarrow{v_3v_4}$ are non-flipped T-edges (because of the color of $\overrightarrow{P}$), and $\overrightarrow{v_2v_3}$ is a flipped T-edge. If $\overrightarrow{v_2v_3}$ falls into Property 3.3.7(1), then $\overrightarrow{v_3v_4}$ is a flipped T-edge which is impossible. If $\overrightarrow{v_2v_3}$ falls into Property 3.3.7(2), then $\overrightarrow{v_1v_2}$ is a flipped T-edge again it is impossible. Therefore, $\overrightarrow{v_2v_3}$ could not be a flipped edge; hence, the final color of $\overrightarrow{v_2v_3}$ is 2. Thus, $\overrightarrow{v_2v_3}$ is not in $\overrightarrow{P}$, a contradiction.

Case 2 (when $c_0(\overrightarrow{v_1v_2}) = 2$).

Since $c_0(\overrightarrow{v_1v_2}) = 2$, we have $c_0(\overrightarrow{v_2v_3}) = 1$ and $c_0(\overrightarrow{v_3v_4}) = 2$. Therefore, $\overrightarrow{v_1v_2}, \overrightarrow{v_3v_4}$ are flipped T-edges, and $\overrightarrow{v_2v_3}$ is a non-flipped T-edge. By Lemma 3.3.8, we conclude that $\overrightarrow{v_1v_2}$ is the start-edge of $\overrightarrow{P}$. Furthermore, $\overrightarrow{v_3v_4}$ could only fall into Property 3.3.7(1) (otherwise, $\overrightarrow{v_3v_4}$ falls into Property 3.3.7(2) which implies $\overrightarrow{v_2v_3}$ is a flipped T-edge, a contradiction). Therefore, all the children of $\overrightarrow{v_3v_4}$ are flipped T-edges; together with the fact that $c_0(\overrightarrow{v_3v_4}) = 2$, we conclude that the final color of the children of $\overrightarrow{v_3v_4}$ is 2 which implies that $\overrightarrow{v_3v_4}$ is the end-edge of $\overrightarrow{P}$.

□

Now we give an upper bound on the lengths of the dipaths in the second family of monochromatic dipaths.

Lemma 3.3.11. *Let $\overrightarrow{P}$ be a monochromatic dipath that does not have a non-flipped M-edge. Then, the length of $\overrightarrow{P}$ is at most* 4*.*

Proof. By the assumption of the lemma, we know that $\overrightarrow{P}$ does not have a non-flipped M-edge. If $\overrightarrow{P}$ does not have a flipped M-edge, then $\overrightarrow{P}$ consists of only T-edges. Thus, by Lemma 3.3.10, the length of $\overrightarrow{P}$ is at most 3.
Now suppose $\overrightarrow{P}$ contains a flipped M-edge $\overrightarrow{uv}$. Then by Lemma 3.3.9, we know that $\overrightarrow{uv}$ is the end-edge of $\overrightarrow{P}$. Furthermore, all the edges in $\overrightarrow{P}$ before $\overrightarrow{uv}$ must be T-edges. Therefore, by Lemma 3.3.10 we conclude that the length of $\overrightarrow{P}$ is at most $3 + 1 = 4$. □

The next lemma gives an upper bound on the lengths of the dipaths in the last family of monochromatic dipaths.

Lemma 3.3.12. *Let $\overrightarrow{P}$ be a monochromatic dipath that contains at least one non-flipped M-edge $\overrightarrow{uv}$. Then, the length of $\overrightarrow{P}$ is at most* 6*.*

Proof. WLOG, we assume that $\overrightarrow{P}$ has color 1 in c'. Let $\overrightarrow{P_u}$ be the sub-dipath of $\overrightarrow{P}$ that has the same initial-vertex as $\overrightarrow{P}$ and its end-vertex is u. Let $\overrightarrow{P_v}$ be the sub-dipath of $\overrightarrow{P}$ whose initial-vertex is v and its end-vertex is the same end-vertex as $\overrightarrow{P}$. We show that the length of $\overrightarrow{P_u}$ is at most 2 and the length of $\overrightarrow{P_v}$ is at most 3. Therefore, the length of $\overrightarrow{P}$ is at most $2 + 1 + 3 = 6$, as desired.

Length of $\overrightarrow{P_u}$ is at most 2:

Let $v_1, ..., v_k = u$ be the vertices of $\overrightarrow{P_u}$. Notice that $\overrightarrow{v_{k-1}v_k}$ could only be a T-edge, because there is an M-edge $\overrightarrow{uv}$ incident to $v_k = u$. Depends on the initial color of $\overrightarrow{v_{k-1}v_k}$, there are two cases to check. When $c_0(\overrightarrow{v_{k-1}v_k}) = 2$, we prove that the length of $\overrightarrow{P_u}$ is at most 1, and when $c_0(\overrightarrow{v_{k-1}v_k}) = 1$, we show that the length of $\overrightarrow{P_u}$ is at most 2.

Case 1 (when $c_0(\overrightarrow{v_{k-1}v_k}) = 2$).

In this case, $\overrightarrow{v_{k-1}v_k}$ must be a flipped T-edge (because the final color of the edges of $\overrightarrow{P}$ is 1). Notice that $\overrightarrow{uv}$ is a non-flipped edge (assumption of the lemma). So by Lemma 3.3.8, we conclude that $\overrightarrow{v_{k-1}v_k}$ is the start-edge of $\overrightarrow{P}$ which means that the length of $\overrightarrow{P_u}$ is at most 1. We depict this case in Figure 3.8(a).

Case 2 (when $c_0(\overrightarrow{v_{k-1}v_k}) = 1$).

In this case, $\overrightarrow{v_{k-1}v_k}$ is a non-flipped edge. Suppose $\overrightarrow{v_{k-2}v_{k-1}}$ is an M-edge. We know that its final color is 1. By Property 3.3.5, we conclude that when the recoloring process enters the depth of $\overrightarrow{v_{k-1}v_k}$, the color of $\overrightarrow{v_{k-2}v_{k-1}}$ is 1; together with the fact that the color of $\overrightarrow{uv}$ always is 1, we conclude that $\overrightarrow{v_{k-1}v_k}$ falls into Property 3.3.7(1) which implies that $\overrightarrow{v_{k-1}v_k}$ is a flipped edge, a contradiction. Thus, $\overrightarrow{v_{k-2}v_{k-1}}$ cannot be an M-edge, so we conclude that $\overrightarrow{v_{k-2}v_{k-1}}$ is a T-edge. Therefore, we have $c_0(\overrightarrow{v_{k-2}v_{k-1}}) = 2$ (since $c_0(\overrightarrow{v_{k-1}v_k}) = 1$) and this implies that $\overrightarrow{v_{k-2}v_{k-1}}$ is a flipped T-edge. By Lemma 3.3.8, $\overrightarrow{v_{k-2}v_{k-1}}$ is the start-edge of $\overrightarrow{P_u}$, which means that the length of $\overrightarrow{P_u}$ is at most 2. See Figure 3.8(b).

In both cases we proved that the length of $\overrightarrow{P_u}$ is at most 2.

Length of $\overrightarrow{P_v}$ is at most 3:

Let $v = v_1, ..., v_k$ be the vertices of $\overrightarrow{P_v}$. Since $\overrightarrow{uv}$ is an M-edge, $\overrightarrow{v_1v_2}$ must be a T-edge. Furthermore, by Property 3.3.6, we have $c_0(\overrightarrow{v_1v_2}) = 1$ which implies $\overrightarrow{v_1v_2}$ is a non-flipped T-edge. We consider two cases: (1) when $\overrightarrow{v_2v_3}$ is an M-edge, and (2) when $\overrightarrow{v_2v_3}$ is a T-edge. In the former case, we prove that the length of $\overrightarrow{P_v}$ is at most 2, and in the latter case, we prove that the length of $\overrightarrow{P_v}$ is at most 3.

Case 1 ($\overrightarrow{v_2v_3}$ is an M-edge).

Note that $c'(\overrightarrow{v_2v_3}) = 1$ (because $\overrightarrow{v_2v_3}$ is in $\overrightarrow{P_v}$). If $c_0(\overrightarrow{v_2v_3}) = 1$, it means that the color of $\overrightarrow{v_2v_3}$ is always 1. Thus, $\overrightarrow{v_1v_2}$ falls into Property 3.3.7(1), a contradiction with the fact that $\overrightarrow{v_1v_2}$ is a non-flipped edge. Let us assume that $c_0(\overrightarrow{v_2v_3}) = 2$. Thus, $\overrightarrow{v_2v_3}$ is a flipped M-edge. By Lemma 3.3.9, we conclude that $\overrightarrow{v_2v_3}$ is the end-edge of $\overrightarrow{P_v}$. Thus, the length of $\overrightarrow{P_v}$ is at most 2. This case is shown in Figure 3.9(a).

Case 2 ($\overrightarrow{v_2v_3}$ is a T-edge).

Recall that $c_0(\overrightarrow{v_1v_2}) = 1$ which implies $\overrightarrow{v_1v_2}$ is a non-flipped T-edge and $c_0(\overrightarrow{v_2v_3}) = 2$. So $\overrightarrow{v_2v_3}$ must be a flipped T-edge. However, $\overrightarrow{v_2v_3}$ does not fall into property 3.3.7(2)

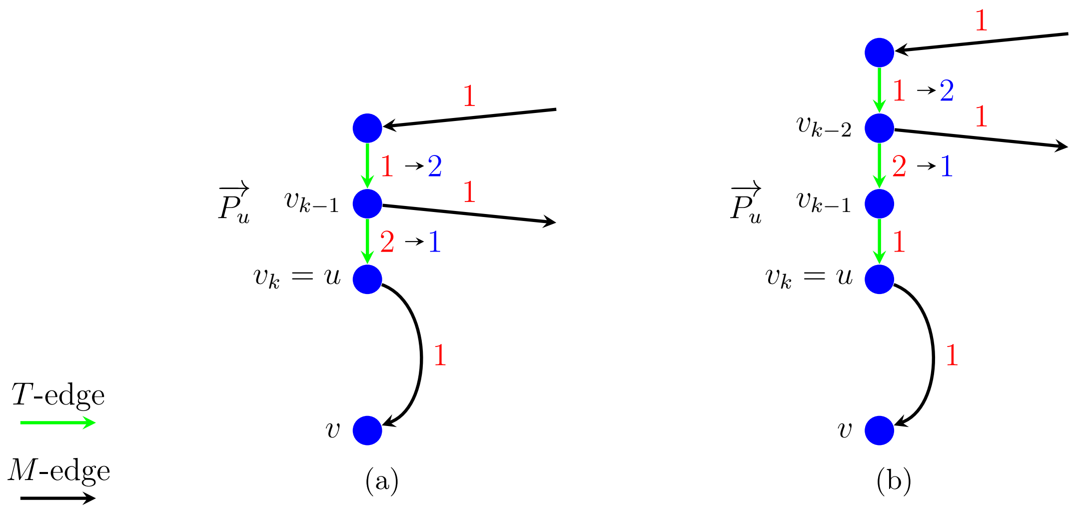

Figure 3.8: (a) When $c_0(\overrightarrow{v_{k-1}v_k}) = 2$, then the parent of $\overrightarrow{v_{k-1}v_k}$ has color 2 in c'. Furthermore, there is an M-edge that is oriented away from v_{k-1}. Thus, $k \leq 2$. (b) When $c_0(\overrightarrow{v_{k-1}v_k}) = 1$, then the parent of $\overrightarrow{v_{k-2}v_{k-1}}$ has color 2 in c'. Furthermore, there is an M-edge that is oriented away from v_{k-2}; hence, $k \leq 3$. Thus, the length of $\overrightarrow{P_u}$ is at most 2. See the proof of Lemma 3.3.12.

(because its parent $\overrightarrow{v_1v_2}$ is a non-flipped edge). So it is the case that, $\overrightarrow{v_2v_3}$ falls into Property 3.3.7(1). Therefore, the color of the children of $\overrightarrow{v_2v_3}$ is 2 in c', and there is an M-edge $\overrightarrow{v_3w}$ such that when the recoloring process is in the depth of $\overrightarrow{v_2v_3}$, the color of $\overrightarrow{v_3w}$ is 2. Therefore, the only possibility for $\overrightarrow{v_3v_4}$ of $\overrightarrow{P_v}$ is $\overrightarrow{v_3w}$. Therefore, if $\overrightarrow{v_3w}$ is not in $\overrightarrow{P_v}$, then we are done. So suppose that $\overrightarrow{v_3w}$ is in $\overrightarrow{P_v}$. Thus, $c'(\overrightarrow{v_3w}) = 1$ which implies that $\overrightarrow{v_3w}$ is a flipped M-edge (remember that the color of $\overrightarrow{v_3w}$ was 2 when the recoloring process was in the depth of $\overrightarrow{v_2v_3}$). By Lemma 3.3.9, we conclude that $\overrightarrow{v_3w} = \overrightarrow{v_3v_4}$ is the end-edge of $\overrightarrow{P_v}$. So the length of $\overrightarrow{P_v}$ is at most 3. This case is shown in Figure 3.9(b).

We proved that in both cases, the length of $\overrightarrow{P_v}$ is at most 3.

□

We put together Lemma 3.3.10, Lemma 3.3.11, and Lemma 3.3.12 to give a bound on the length of a monochromatic path in G.

Corollary 3.3.13. *Let c' be the edge-coloring of G. Then, any monochromatic path in G has length at most* 12.

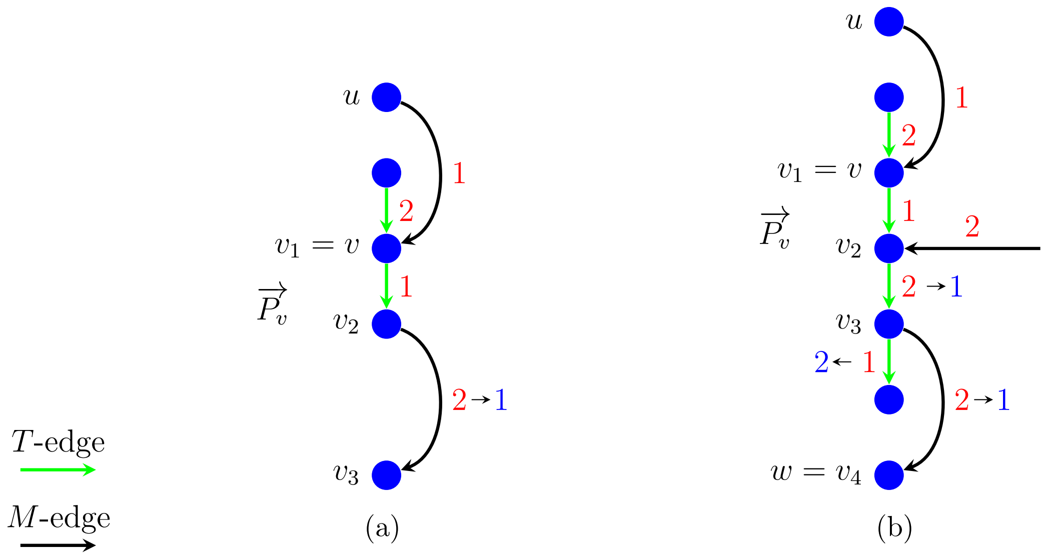

Figure 3.9: (a) When $\overrightarrow{v_2v_3}$ is an M-edge, then $c_0(\overrightarrow{v_2v_3}) = 2$. So by Lemma 3.3.9, we conclude that $\overrightarrow{v_2v_3}$ is the end-edge of $\overrightarrow{P_v}$; hence, $k \leq 3$. (b) When $\overrightarrow{v_2v_3}$ is a T-edge, then all the children of $\overrightarrow{v_2v_3}$ have color 2 in c'. So $\overrightarrow{v_3v_4}$ of $\overrightarrow{P_v}$ could be only an M-edge; hence, $k \leq 4$. Thus, The length of $\overrightarrow{P_v}$ is at most 3. See the proof of Lemma 3.3.12.

Proof. Let P be a path in G, and let $\overrightarrow{P}$ be the subgraph of $\overrightarrow{G}$ corresponding to P. By Lemma 3.3.2, either $\overrightarrow{P}$ is a dipath, or $\overrightarrow{P}$ is the union of two maximal dipaths $\overrightarrow{P_1}, \overrightarrow{P_2}$. By Lemma 3.3.10, Lemma 3.3.11, and Lemma 3.3.12, we conclude that the length of each maximal dipaths of $\overrightarrow{P}$ is at most 6. Thus, the length of P is at most 12. □

By Corollary 3.3.4, together with the above corollary, we conclude that c' is a good edge-coloring of G.

Corollary 3.3.14. *Suppose $G = (V, T \cup M)$ where T and M are a tree and a matching, respectively. Then, G has a good edge-coloring.*

Now Theorem 3.1.4 follows easily.

Proof of Theorem 3.1.4: By Theorem 3.2.2, we know that there exist a tree T and a matching M such that $G = (V, T \cup M)$. So we can apply Corollary 3.3.14 to G, i.e., G has a good edge-coloring. Let F_1 and F_2 be the subgraphs induced by colors 1 and 2, respectively. By the first condition of the good edge-coloring, we know that F_1 and F_2 are acyclic. Furthermore, by the second condition of the good edge-coloring, we conclude that the diameter of F_i for $i = 1, 2$ is at most 12. Therefore, F_1 and F_2 are forests with diameter at most 12, as desired.

Chapter 4

Thin Trees in 6-Edge-Connected Planar Graphs

4.1 Introduction

We continue our study of thin trees in planar graphs. Recall from Chapter 3 that a decomposition of a graph $G = (V, E)$ consists of edge-disjoint subgraphs $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$ whose union is G, i.e., $E = E_1 \cup E_2$. All graphs in this chapter are considered to be simple (without multiedges and without loops) unless stated otherwise. In particular, our decomposition result (Theorem 4.1.1) works for simple planar graphs, and our thin tree result (Theorem 4.1.2) works for planar graphs with multiedges (i.e., (multi)graphs). In Chapter 3, we proved that any 8-edge-connected planar (multi)graph has two edge-disjoint $\frac{12}{13}$-thin spanning trees. Now it is natural to consider thin trees in planar graphs with edge-connectivity < 8. In this chapter, we prove that a 6-edge-connected planar (multi)graph has two edge-disjoint $\frac{14}{15}$-thin spanning trees. For planar graphs with edge-connectivity ≤ 4 it was proved in [MP] that for any $0 < \epsilon < 1$ there is a 4-edge-connected planar graph with no ϵ-thin spanning tree. This leaves us with the family of 5-edge-connected planar graphs. Unfortunately, we are not aware of any result about thin trees in this family of graphs.

As in the previous chapter, the main technique in this chapter is to color the edges of the graph subject to some criteria. We cannot use the procedure of Chapter 3 to obtain our desired edge-coloring on a connected planar graph with girth 6 because there exist such graphs that do *not* have a decomposition into a spanning tree and a matching, see [MdMRZ12] for examples of these graphs. However, Kim et al. [KKW+13] showed that any connected planar graph with girth 6 can be decomposed into a spanning tree and a star forest (a star forest is a forest in which every component is a bipartite graph $K_{1,t}$ for some $t \in \mathbb{Z}, t \geq 0$), see Theorem 4.2.2. Merker and Postle [MP] showed that if a graph G has a decomposition into a spanning tree and a star forest, then there is a decomposition of G into two forests with diameter at most 18. A consequence of this result is that any 6-edge-connected planar (multi)graph has two edge-disjoint $\frac{18}{19}$-thin spanning trees (see Theorem 3.1.1).

In this chapter by using the same techniques as in [MP] but with a more careful analysis, we are able to show that if a graph G has a decomposition into a spanning tree and a star forest, then G can be decomposed into two forests with diameter at most 14; this result implies that there are two edge-disjoint $\frac{14}{15}$-thin spanning trees in any 6-edge-connected planar (multi)graph (see Theorem 4.1.2). The main theorem that we prove in this chapter is as follows:

Theorem 4.1.1. *Let $G = (V, E)$ be a connected planar graph with girth at least 6. Then, G can be decomposed into two forests with diameter at most 14.*

Before we go into the details of the proof of Theorem 4.1.1, let us combine Lemma 3.1.3 and Theorem 4.1.1 to give our result about thin trees.

Theorem 4.1.2. *Suppose G is a 6-edge-connected planar (multi)graph. Then, G has two edge-disjoint spanning trees T, T' such that $\rho(T), \rho(T') \leq \frac{14}{15}$.*

Proof. This proof is essentially the same as the proof of [MP, Corollary 3.6] and it is included for completeness.

Since G is a 6-edge-connected planar (multi)graph, its dual $G^* = (V^*, E^*)$ is a simple planar graph with girth at least 6. By Theorem 4.1.1, we know that each connected component $G_i^* = (V_i^*, E_i^*)$ of G^* can be decomposed into two forests (V_i^*, F_i^1) and (V_i^*, F_i^2) such that each has diameter at most 14. Let $H_1 := (\bigcup_i V_i^*, \bigcup_i F_i^1)$ and $H_2 := (\bigcup_i V_i^*, \bigcup_i F_i^2)$. Notice that for each $i = 1, 2$, H_i is a forest with diameter at most 14. Thus, G^* can be decomposed into two forests with diameter at most 14. By applying Lemma 3.1.3 to G^* we conclude that G has two edge-disjoint spanning trees T, T' such that $\rho(T), \rho(T') \leq \frac{14}{15}$.

By Theorem 4.1.1, we know that G^* can be decomposed into two forests with diameter at most 14. So we can apply Lemma 3.1.3 to G^* and derive the desired result. $\square$

The rest of this chapter is organized as follows: In Section 2, we will show that the decomposition of a graph into two bounded diameter forests corresponds to a particular edge-coloring (good edge-coloring) of the graph. Then, we will construct an edge-coloring for the graph. Finally, in Section 3, we will prove that our edge-coloring given in Section 2 is indeed a good edge-coloring and this implies Theorem 4.1.1.

4.2 A Good Edge-Coloring of the Graph

In this chapter, we use the following modified definition of a good edge-coloring (see Chapter 3).

Definition 4.2.1 (good edge-coloring). *Let $G = (V, E)$ be a graph, and let $c : E(G) \to \{1, 2\}$ be an edge-coloring of G. We say that c is a* ***good edge-coloring*** *if it satisfies the following two conditions:*

1. *There is no edge-monochromatic cycle.*
2. *Any edge-monochromatic path has length at most* 14.

Our proof of Theorem 4.1.1 relies on Theorem 4.2.2 below. The maximum average degree of a graph G, denoted by $\mathrm{Mad}(G)$, is $\max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$ where H is a subgraph of G. Kim et al. [KKW+13] showed that a planar graph G of girth at least 6 has $\mathrm{Mad}(G) < 3$. From this fact and Theorem 7.1 of [KKW+13] it follows that a planar graph of girth ≥ 6 has a decomposition into a forest and star forest. (Theorem 7.1 of [KKW+13] states the following: "If a graph G is $(1, 2)$-sparse, then G has a decomposition (F, S) such that both F, S are forests and every component of S has ≤ 2 edges." The same paper shows that a graph G is $(1, 2)$-sparse if and only if $\mathrm{Mad}(G) < 3$; we refer the reader to [KKW+13], Section 1, for the definition of $(1, 2)$-sparse.)

Let G be a connected planar graph with girth ≥ 6. Then, by Kim et al. [KKW+13] result, G has a decomposition into a forest F and a star forest S. Since G is connected, we can convert F to a spanning tree T by adding as many star edges as possible. We state this result in the following theorem:

Theorem 4.2.2. *Every connected planar graph of girth at least* 6 *has a decomposition into a spanning tree and a star forest.*

Now, we can sketch the proof of Theorem 4.1.1.
Let $G = (V, E)$ be a connected planar graph with girth at least 6. By Theorem 4.2.2, G can be decomposed into a spanning tree T and a star forest S. Then, we will show that any graph that can be decomposed into a spanning tree and a star forest has a good edge-coloring, see Corollary 4.3.9. Now, Theorem 4.1.1 follows easily. Let F_1 and F_2 be the subgraphs induced by colors 1 and 2, respectively. It is clear that F_i for $i = 1, 2$ is a forest with diameter at most 14.

We often use *tree* to mean a spanning tree. Let $G = (V, E)$ be a graph that can be decomposed into a tree T and a star forest S. We denote this decomposition by $G =$

$(V, T \cup S)$. In a star graph, we call a vertex with the highest degree the *center*, and the rest of the vertices the *leaves*. If a star has no edge, i.e., it consists of an isolated vertex, then we call this vertex the center. If a star has one edge, then we arbitrarily pick one vertex to be the center and the other one to be the leaf.

Now we obtain a directed graph $\overrightarrow{G}$ by orienting the edges of G as follows:
Pick a vertex r, and make T a rooted tree with root r. Each edge of the tree is oriented away from the root. Call this directed tree $\overrightarrow{T}$. For the star forest, in each component, we orient the edges from the center to the leaves. We denote this directed star forest by $\overrightarrow{S}$. Clearly, $\overrightarrow{G} = (V, \overrightarrow{T} \cup \overrightarrow{S})$.

Let $C(\overrightarrow{S})$ be the set of all centers of $\overrightarrow{S}$, and $L(\overrightarrow{S})$ be the set of all leaves of $\overrightarrow{S}$. We say that an edge $\overrightarrow{uv}$ is a T-edge if $\overrightarrow{uv} \in E(\overrightarrow{T})$, and an S-edge if $\overrightarrow{uv} \in E(\overrightarrow{S})$.

Since G is simple, any (di)cycle in G or $\overrightarrow{G}$ has length at least 3. Also notice that any (vertex)edge-coloring of $\overrightarrow{G}$ can be considered as an (vertex)edge-coloring of G, and vice versa.

For constructing our good edge-coloring, first we give a vertex-coloring $c : V(\overrightarrow{G}) \to \{1, 2\}$ and then extend it to an edge-coloring c' such that $indeg_i(v) \leq 1$ (recall from Chapter 3 that $indeg_i(v)$ is the number of edges of color i that are oriented toward v), see Fact 4.2.6. In particular, for any vertex v such that $\overrightarrow{uv}$ is an S-edge and $\overrightarrow{wv}$ is a T-edge, we have $c'(\overrightarrow{uv}) = c(v) \neq 3 - c(v) = c'(\overrightarrow{wv})$. Before we define this extension of a vertex-coloring to an edge-coloring, we need some definitions. At the end of this section, we provide some motivation for these definitions (see Examples 4.2.11 & 4.2.12).

From now on, by a graph $G = (V, E)$ we mean $G = (V, T \cup S)$, and $\overrightarrow{G}$ is the digraph described above.

Definition 4.2.3. *We say an S-edge $\overrightarrow{uv}$ is rebellious if $c(u) \neq c(v)$. We also call $v \in V(G)$ rebellious if it is the head of a rebellious edge.*

Definition 4.2.4. *We say a vertex-coloring c is tame, if for every T-edge $\overrightarrow{uv}$ where v is rebellious, we have $c(u) \neq c(v)$ and u is not rebellious.*

Let $x \in \{1, 2\}$, then define $\overline{x}$ to be $\overline{x} := 3 - x$. Now, we define the extension of a vertex coloring to the edges of the graph.

Definition 4.2.5. *Let $c : V(\overrightarrow{G}) \to \{1, 2\}$ be a vertex-coloring of $\overrightarrow{G}$. The extension of c, denoted by $Ext(c)$, is the edge-coloring $c' : E(\overrightarrow{G}) \to \{1, 2\}$ where:*

i. *For all S-edges $\overrightarrow{uv}$, we have $c'(\overrightarrow{uv}) = c(v)$.*

ii. *For all T-edges $\overrightarrow{uv}$, we have*

$$c'(\overrightarrow{uv}) = \begin{cases} c(v), & \textit{if } v \in C(\overrightarrow{S}), c(u) = c(v), \textit{and } u \textit{ is not rebellious,} \\ \overline{c(v)}, & \textit{otherwise.} \end{cases}$$

Notice that for any $v \in C(\overrightarrow{S})$ we have $indeg(v) \leq 1$, and for any $v \in L(\overrightarrow{S})$ we have $indeg(v) \leq 2$. Furthermore, for a leaf $v \in L(\overrightarrow{S})$, we have $indeg(v) = 2$ if and only if there is an S-edge $\overrightarrow{uv}$ and a T-edge $\overrightarrow{wv}$. By Definition 4.2.5, we know that $c'(\overrightarrow{uv}) = c(v)$ and $c'(\overrightarrow{wv}) = \overline{c(v)}$. So we can summarize this property in the following fact:

Fact 4.2.6 (indegree property). *Suppose c is a vertex-coloring of $\overrightarrow{G}$. Let $c' := Ext(c)$. Then, for any $v \in V(\overrightarrow{G})$ and $i = \{1, 2\}$, we have $indeg_i(v) \leq 1$.*

The high level idea of our vertex-coloring is as follows: First we color the vertices in $C(\overrightarrow{S})$. To do this, we construct a digraph called the center graph with respect to $\overrightarrow{G}$ (see the definition below). Then, we extend the coloring of vertices in $C(\overrightarrow{S})$ to $L(\overrightarrow{S})$ such that the resulting coloring is tame.

Definition 4.2.7. *The center graph $Center(\overrightarrow{G})$ of a digraph $\overrightarrow{G} = (V, \overrightarrow{T} \cup \overrightarrow{S})$ is a digraph whose vertex set is $C(\overrightarrow{S})$ and for every pair $u, v \in C(\overrightarrow{S})$ with $u \neq v$, there is an edge $\overrightarrow{uv}$ if $\overrightarrow{uv}$ is a T-edge, or if there is a vertex w such that $\overrightarrow{uw}$ is an S-edge and $\overrightarrow{wv}$ is a T-edge.*

The next lemma provides a coloring for the vertices in $C(\overrightarrow{S})$ such that there is no vertex-monochromatic dicycle in Center$(\overrightarrow{G})$.

Lemma 4.2.8. *The vertices of $Center(\overrightarrow{G})$ can be colored with colors 1 and 2 such that the color 1 vertices form an independent set, and the color 2 vertices induce a subgraph with diameter at most 1.*

Proof. This proof is essentially the same as the proof of [MP, Lemma 2.10] and it is included for completeness.

Let H be the undirected graph obtained by removing the direction of the edges of Center$(\overrightarrow{G})$. We will prove each connected component of H has at most one cycle. For now, suppose this holds. If a component does not have an odd cycle, then it is bipartite so we can choose a proper 2-coloring of its vertices. If a component H' has an odd cycle C, then this is the only cycle of H'. Pick an edge uv in $E(C)$. By removing uv from H', we get a bipartite graph H''; hence, we can choose a proper 2-coloring of the vertices of H''

such that the color of u is 2. Thus, the resulting coloring of $V(H) = V(\overrightarrow{G})$ satisfies the statement of the lemma.

It remains to show that each component of H has at most one cycle.
Suppose C and C' are two cycles in a connected component of H. Let $\overrightarrow{C}$ and $\overrightarrow{C'}$ be the subgraphs in Center$(\overrightarrow{G})$ corresponding to C and C', respectively. Since C and C' are in the same component, there is a path P such that $\exists u \in V(P) \cap V(C)$ and $\exists v \in V(P) \cap V(C')$. Let $\overrightarrow{P}$ be the subgraph in Center$(\overrightarrow{G})$ corresponding to P. Notice that the indegree of each vertex in the center graph is at most 1 (this follows from Definition 4.2.7); hence, $\overrightarrow{C}$ and $\overrightarrow{C'}$ are dicycles which implies that there is an edge in $\overrightarrow{C}$ oriented toward u, and there is an edge in $\overrightarrow{C'}$ oriented toward v. Therefore, $\overrightarrow{P}$ must be oriented away from u and v (otherwise, the indegree of u and v is ≥ 2, a contradiction). Notice that the sum of the indegrees (with respect to $\overrightarrow{P}$) of vertices in $V(\overrightarrow{P})$ is $|V(\overrightarrow{P})| - 1$. Since the indegrees (with respect to $\overrightarrow{P}$) of u, v are zero, there is a vertex in $V(\overrightarrow{P})$ with indegree 2, a contradiction with the fact that the indegree of any vertex in Center$(\overrightarrow{G})$ is at most 1. $\square$

Now, we extend this vertex-coloring of Center$(\overrightarrow{G})$ given by the previous lemma to all vertices of $\overrightarrow{G}$ such that the resulting vertex-coloring is tame.

Lemma 4.2.9. *Let c_0 be the vertex-coloring of Center$(\overrightarrow{G})$ described in Lemma 4.2.8. We can extend this coloring to the leaves such that the resulting coloring is a tame vertex-coloring of $\overrightarrow{G}$. Furthermore, the length of a vertex-monochromatic dipath consisting of only leaves is at most* 1.

Proof. This proof is essentially the same as the proof of [MP, Lemma 2.10] and it is included for completeness.

If the root of T is in $L(\overrightarrow{S})$, color it arbitrarily with color 1 or 2. Suppose we have colored all the leaves at depth i. Let v be a leaf at depth $i+1$, and let $\overrightarrow{wv}$ be an S-edge. Let u be the parent of v, i.e., $\overrightarrow{uv}$ is a T-edge. We color v as follows:

$$c_0(v) = \begin{cases} c_0(u), & \text{if } u \text{ is rebellious and } c_0(u) = c_0(w), \\ \overline{c_0(u)}, & \text{otherwise.} \end{cases} \tag{4.1}$$

<u>Claim: the vertex-coloring c_0 is tame.</u>

<u>Proof of the claim:</u> Suppose $\overrightarrow{uv}$ is a T-edge, v is a leaf, and w & $\overrightarrow{wv}$ are as above. Suppose v is rebellious, i.e, $c_0(v) = \overline{c_0(w)}$. Then, $c_0(u) = \overline{c_0(v)}$ and u is not rebellious. The reasons are as follows:

First suppose that $c_0(u) = c_0(v) = \overline{c_0(w)}$. Hence, the first case of (4.1) does not hold which implies that $c_0(v) = \overline{c_0(u)}$, a contradiction.

Notice that in the previous paragraph, we proved that $c_0(u) = \overline{c_0(v)} = c_0(w)$. Now if u is rebellious, then the first case of (4.1) holds which implies that $c_0(u) = c_0(v)$, a contradiction. It follows that c_0 is a tame vertex-coloring.

Proof of the "furthermore" part: Let $\overrightarrow{P}$ be a vertex-monochromatic dipath of length at least 2 whose vertices are leaves. Let dipath u, v, z be a sub-dipath of $\overrightarrow{P}$. Notice that $\overrightarrow{uv}, \overrightarrow{vz}$ are T-edges (since $u, v, z \in L(\overrightarrow{S})$). Since $c_0(u) = c_0(v)$, the first case of (4.1) implies that u is rebellious. On the other hand, since $c_0(v) = c_0(z)$, the first case of (4.1) implies that v is rebellious. Thus, we have a T-edge $\overrightarrow{uv}$ such that both u and v are rebellious, a contradiction with the above claim (i.e., c_0 is a tame vertex-coloring). ☐

The above proof, gives us an important fact that will be used later in Lemmas 4.3.5 & 4.3.7.

Claim 4.2.10. *Let c_0 be the vertex-coloring given by Lemma 4.2.9, and let $\overrightarrow{uv}$ be a T-edge where $u, v \in L(\overrightarrow{S})$. If $c_0(u) = c_0(v)$, then u is rebellious and v is not.*

Proof. Let w be a center such that $\overrightarrow{wv}$ is an S-edge. Since $c_0(v) = c_0(u)$, the first case of (4.1) must have applied. Thus, we know that u is rebellious. Furthermore, $c_0(v) = c_0(u) = c_0(w)$; hence, v is not rebellious. ☐

Let c_0 be the vertex-coloring of Center($\overrightarrow{G}$) given by Lemma 4.2.9. In the next section, we prove that $\hat{c} := Ext(c_0)$ is a good edge-coloring of G.

In the next two examples, we provide some intuition behind the tame property of a vertex-coloring.

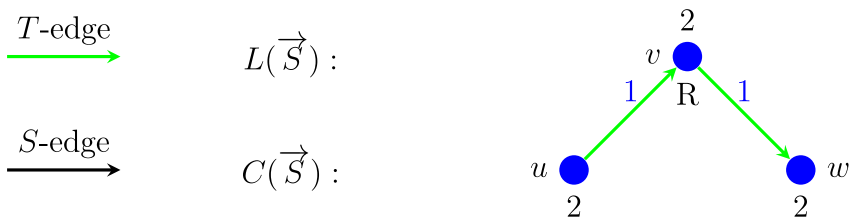

Figure 4.1: Bottom vertices are in $C(\overrightarrow{S})$, and top vertices are in $L(\overrightarrow{S})$. The number (1 or 2) next to a vertex v or an edge $\overrightarrow{uv}$ denotes $c(v)$ or $c'(\overrightarrow{uv})$. Label R next to a vertex denotes Rebellious. T-edges are shown with green color and S-edges are shown with black color.

Example 4.2.11. *Suppose $G = (V, E)$ is a graph that has a decomposition into a tree and a star forest. Let $\overrightarrow{G}$ be the directed graph obtained from G as described earlier in this section. Let dipath u, v, w be a subgraph of $\overrightarrow{G}$, where u, w are centers, v is a leaf, and $\overrightarrow{uv}, \overrightarrow{vw}$ are T-edges. See Figure 4.1 for an illustration. Let c be a vertex-coloring of $\overrightarrow{G}$ such that $c(v) = c(w) = 2$. Furthermore, assume v is rebellious. Let $c' := Ext(c)$ be an edge-coloring of $\overrightarrow{G}$. Then, by Definition 4.2.5, we have $c'(\overrightarrow{uv}) = \overline{c(v)} = 1 = \overline{c(w)} = c'(\overrightarrow{vw})$. Suppose c does not satisfy the first condition of tame coloring. In particular, suppose we have $c(u) = 2$ (notice that $\overrightarrow{uv}$ is a T-edge where $c(u) = c(v) = 2$ and v is rebellious; hence, c does not satisfy the first condition of tame coloring).*

Since $c(u) = c(w) = 2$ and u, w are both centers, by "linking up" many copies of the dipath u, v, w, we can create an arbitrarily long edge-monochromatic dipath. On the other hand, if c is tame, then $c(u) = 1 \neq 2 = c(w)$. Thus, we cannot get a long edge-monochromatic dipath by "linking up" copies of the dipath u, v, w.

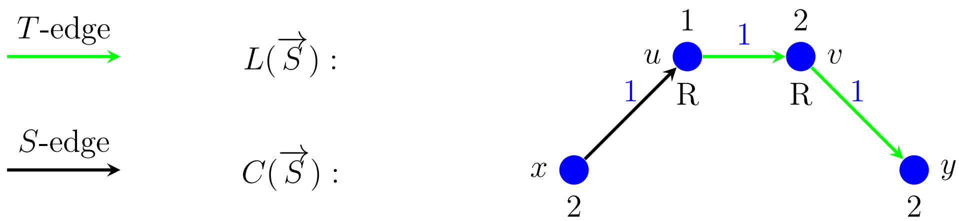

Figure 4.2: Bottom vertices are in $C(\overrightarrow{S})$, and top vertices are in $L(\overrightarrow{S})$. The number (1 or 2) next to a vertex v or an edge $\overrightarrow{uv}$ denotes $c(v)$ or $c'(\overrightarrow{uv})$. Label R next to a vertex stands for Rebellious. T-edges are shown with green color and S-edges are shown with black color.

Example 4.2.12. *Suppose $G = (V, E)$ is a graph that has a decomposition into a tree and a star forest. Let $\overrightarrow{G}$ be the directed graph obtained from G as described earlier in this section. Let dipath x, u, v, y be a subgraph of $\overrightarrow{G}$, where x, y are centers, u, v are leaves, $\overrightarrow{xu}$ is an S-edge, and $\overrightarrow{uv}, \overrightarrow{vy}$ are T-edges. See Figure 4.2 for an illustration. Let c be a vertex-coloring of $\overrightarrow{G}$ such that $c(v) = c(y) = 2$ and $c(u) = 1$. Furthermore, assume v is rebellious. Let $c' := Ext(c)$ be an edge-coloring of $\overrightarrow{G}$. Then, by Definition 4.2.5, we have $c'(\overrightarrow{xu}) = c(u) = 1$, $c'(\overrightarrow{uv}) = \overline{c(v)} = 1$, and $c'(\overrightarrow{vy}) = \overline{c(y)} = 1$.*

Suppose c does not satisfy the second condition of tame coloring. In particular, suppose u is rebellious, i.e., $c(x) = 2 \neq 1 = c(u)$. Thus, there is a T-edge $\overrightarrow{uv}$ whose endpoints are rebellious. Since $c(x) = c(y) = 2$ and x, y are both centers, by "linking up" many copies of the dipath x, u, v, y, we can create an arbitrarily long edge-monochromatic dipath. On the other hand, if c is tame, then $c(x) = 1 \neq 2 = c(y)$. Thus, we cannot get a long edge-monochromatic dipath by "linking up" copies of the dipath x, u, v, y.

4.3 Proof of Theorem 4.1.1

We fix $c_0 : V(\overrightarrow{G}) \to \{1, 2\}$ as given by Lemma 4.2.9 for the vertex-coloring of $\overrightarrow{G} = (V, \overrightarrow{T} \cup \overrightarrow{S})$, and $\hat{c}$ is the extension of c_0 to the edges of $\overrightarrow{G}$, i.e., $\hat{c} = Ext(c_0)$. The goal of this section is to show two properties of $\hat{c}$ on $\overrightarrow{G}$, i.e., (1) there is no monochromatic dicycle (Lemma 4.3.3) and (2) every monochromatic dipath has length at most 7 (Lemma 4.3.5 and Lemma 4.3.7). At the end, these properties will show that $\hat{c}$ is a good edge-coloring for G, see Corollary 4.3.9.

We start by characterizing edge-monochromatic dipaths of color 1 where the initial-

Assumptions	Conclusion
$\overrightarrow{v_{k-3}v_{k-2}}$ is T-edge, $\hat{c}(\overrightarrow{v_{k-3}v_{k-2}}) = 1$, $v_{k-2} \in L(\overrightarrow{S})$	$c_0(v_{k-2}) = 2$
$\overrightarrow{v_{k-2}v_{k-1}}$ is T-edge, $\hat{c}(\overrightarrow{v_{k-2}v_{k-1}}) = 1$, $v_{k-1} \in L(\overrightarrow{S})$	$c_0(v_{k-1}) = 2$
$\overrightarrow{v_{k-1}v_k}$ is T-edge, $\hat{c}(\overrightarrow{v_{k-1}v_k}) = 1$, $v_{k-1} \in L(\overrightarrow{S})$, $v_k \in C(\overrightarrow{S})$, $c_0(v_{k-1}) = 2$	$c_0(v_k) = 2$, v_{k-1} is rebellious

Table 4.1: This table is used in the proof of $k \leq 3$ in Lemma 4.3.1.

vertex and the end-vertex of the dipath are in $C(\overrightarrow{S})$, and the interior vertices are in $L(\overrightarrow{S})$.

Lemma 4.3.1 (Lemma 2.7 of [MP]). *Let $\overrightarrow{P} = v_0, v_1, ..., v_k$ be a dipath in $\overrightarrow{G}$ whose edges are colored 1. Suppose $v_0, v_k \in C(\overrightarrow{S})$ and $v_i \in L(\overrightarrow{S})$ for $1 \leq i \leq k-1$. Then we have $k \leq 3$. Furthermore,*

i. *If $c_0(v_0) = c_0(v_k)$, then $\overrightarrow{v_0v_k} \in E(Center(\overrightarrow{G}))$.*

ii. *If $c_0(v_0) \neq c_0(v_k)$, then $c_0(v_0) = 1$ and $c_0(v_k) = 2$.*

Proof. This proof is essentially the same as the proof of [MP, Lemma 2.7] and it is included for completeness.

For the proof of this lemma we use tables where each row can be verified independently, i.e., by using only the assumptions in the row and Definition 4.2.5.

We start by showing that $k \leq 3$. Suppose $k > 3$, then we have $v_{k-3}, v_{k-2}, v_{k-1} \in L(\overrightarrow{S})$ and $\overrightarrow{v_{k-3}v_{k-2}}, \overrightarrow{v_{k-2}v_{k-1}} \in E(\overrightarrow{T})$. Also notice that the edges of $\overrightarrow{P}$ are colored 1. From Table 4.1, we have $c_0(v_{k-2}) = c_0(v_{k-1}) = 2$ (row 1 and row 2). Now since $c_0(v_{k-1}) = 2$ and v_k is a center (assumption of the lemma), we could apply row 3 of the table. So we conclude that v_{k-1} is rebellious. Therefore, we have a T-edge $\overrightarrow{v_{k-2}v_{k-1}}$ whose endpoints are colored 2, and v_{k-1} is rebellious, a contradiction because c_0 is tame. Thus, we have $k \leq 3$.

Now we prove the "furthermore" part of the lemma. We are not going to use any fact from the previous paragraph or Table 4.1 for the rest of the proof.

We consider $k = 1, 2,$ and 3 separately. In each case, we show that part (i) and part (ii) of the lemma hold. We start with $k = 1$.

Assumptions	Conclusion
$\overrightarrow{v_0v_1}$ is T-edge, $\hat{c}(\overrightarrow{v_0v_1}) = 1$, $v_0, v_1 \in C(\overrightarrow{S})$, $c_0(v_1) = 1$	$c_0(v_0) = 1$
$\overrightarrow{v_0v_1}$ is T-edge, $\hat{c}(\overrightarrow{v_0v_1}) = 1$, $v_0, v_1 \in C(\overrightarrow{S})$, $c_0(v_1) = 2$	$c_0(v_0) = 1$

Table 4.2: This table is used in the case that $k = 1$ in Lemma 4.3.1.

Assumptions	Conclusion
$\overrightarrow{v_1v_2}$ is T-edge, $\hat{c}(\overrightarrow{v_1v_2}) = 1$, $v_1 \in L(\overrightarrow{S})$, $v_2 \in C(\overrightarrow{S})$, $c_0(v_2) = 1$	$c_0(v_1) = 1$, v_1 is not rebellious
$\hat{c}(\overrightarrow{v_0v_1}) = 1$, $v_1 \in L(\overrightarrow{S})$, $c_0(v_1) = 1$	$\overrightarrow{v_0v_1}$ is S-edge

Table 4.3: This table is used in the case that $k = 2$ in Lemma 4.3.1.

***k* = 1:** Since $v_0, v_1 \in C(\overrightarrow{S})$, we know that $\overrightarrow{v_0v_1}$ is a T-edge. So $\overrightarrow{v_0v_1} \in E(\text{Center}(\overrightarrow{G}))$; hence, part (i) holds. Let us determine the color of v_0 based on the color of v_1. From Table 4.2, regardless of the color of v_1, we always have $c_0(v_0) = 1$. Hence, part (ii) holds too.

***k* = 2:** Here we have $\overrightarrow{P} = v_0, v_1, v_2$ where v_1 is a leaf and v_0, v_2 are centers. So $\overrightarrow{v_1v_2}$ must be a T-edge.

First, we show that part (i) holds when $c_0(v_0) = c_0(v_2) = 1$. From the first row of Table 4.3, we have $c_0(v_1) = 1$. So we can apply the second row of the table. Thus, $\overrightarrow{v_0v_1}$ is an S-edge; together with the fact that $\overrightarrow{v_1v_2}$ is a T-edge, we have $\overrightarrow{v_0v_2} \in E(\text{Center}(\overrightarrow{G}))$. So part (i) holds when the centers are colored 1.

Now suppose centers are colored 2, i.e., $c_0(v_0) = c_0(v_2) = 2$. Then, from Table 4.4, we see that if $c_0(v_1) = 2$, then $\overrightarrow{v_0v_1}$ is a T-edge (the first row of Table 4.4) whose endpoints are colored 2, and v_1 is rebellious (the second row of Table 4.4), contradicting the tame property of c_0. So we proved that if $c_0(v_0) = c_0(v_2) = 2$, then $c_0(v_1) = 1$; hence, $\overrightarrow{v_0v_1}$ is an S-edge (otherwise $\hat{c}(\overrightarrow{v_0v_1}) = 2$ because $v_1 \in L(\overrightarrow{S})$, a contradiction with the coloring of $\overrightarrow{P}$). Recall that $\overrightarrow{v_1v_2}$ is a T-edge; hence, $\overrightarrow{v_0v_2} \in E(\text{Center}(\overrightarrow{G}))$. Thus, part (i) holds when the centers are colored 2.

Assumptions	Conclusion
$\hat{c}(\overrightarrow{v_0v_1}) = 1$, $v_1 \in L(\overrightarrow{S})$, $c_0(v_1) = 2$	$\overrightarrow{v_0v_1}$ is T-edge
$\overrightarrow{v_1v_2}$ is T-edge, $\hat{c}(\overrightarrow{v_1v_2}) = 1$, $v_1 \in L(\overrightarrow{S})$, $v_2 \in C(\overrightarrow{S})$, $c_0(v_2) = 2$, $c_0(v_1) = 2$	v_1 is rebellious

Table 4.4: This table is used in the case that $k = 2$ in Lemma 4.3.1.

Assumptions	Conclusion
$\overrightarrow{v_1v_2}$ is T-edge, $\hat{c}(\overrightarrow{v_1v_2}) = 1$, $v_1 \in L(\overrightarrow{S})$, $v_2 \in C(\overrightarrow{S})$, $c_0(v_2) = 1$	$c_0(v_1) = 1$, v_1 is not rebellious
$\hat{c}(\overrightarrow{v_0v_1}) = 1$, $v_1 \in L(\overrightarrow{S})$, $c_0(v_1) = 1$	$\overrightarrow{v_0v_1}$ is S-edge
$\overrightarrow{v_0v_1}$ is S-edge, $2 = c_0(v_0) \neq c_0(v_1) = 1$	v_1 is rebellious

Table 4.5: This table is used in the case that $k = 2$ in Lemma 4.3.1.

Now, we show part (ii) of the lemma holds. So we are assuming $c_0(v_0) \neq c_0(v_2)$. Suppose part (ii) does not hold, i.e., $c_0(v_0) = 2$ and $c_0(v_2) = 1$. Notice that since v_1 is a leaf, $\overrightarrow{v_1v_2}$ is a T-edge. Then, from the first row of Table 4.5, we have $c_0(v_1) = 1$ and v_1 is not rebellious. Since $c_0(v_1) = 1$, we can apply the second row of the table and conclude that $\overrightarrow{v_0v_1}$ is an S-edge. Thus, we can apply the third row of the table which implies that v_1 is rebellious, a contradiction. Therefore, if $c_0(v_0) \neq c_0(v_2)$, then $c_0(v_0) = 1$ and $c_0(v_2) = 2$, so part (ii) holds.

Now we consider the last case, i.e., the length of $\overrightarrow{P}$ is 3.

$\boldsymbol{k = 3}$**:** Here we have $\overrightarrow{P} = v_0, v_1, v_2, v_3$. In this case we show that always we have $1 = c_0(v_0) \neq c_0(v_3) = 2$ which implies part (ii). Since v_1, v_2 are leaves, we conclude that $\overrightarrow{v_1v_2}, \overrightarrow{v_2v_3}$ are T-edges. Furthermore, since the edges of $\overrightarrow{P}$ are colored 1, we could apply the first row of Table 4.6. So we have $c_0(v_2) = 2$. Then, from the second row, we conclude that $c_0(v_3) = 2$ and v_2 is rebellious. Since $\overrightarrow{v_1v_2}$ is a T-edge and $c_0(v_2) = 2$, v_1 is not rebellious and $c_0(v_1) = 1$ (tame property of c_0). By applying the third row of the table we conclude that $\overrightarrow{v_0v_1}$ is an S-edge; together with the fact that v_1 is not rebellious, we have $c_0(v_0) = c_0(v_1) = 1$. Hence, if $k = 3$, then we have

Assumptions	Conclusion
$\overrightarrow{v_1v_2}$ is T-edge, $\hat{c}(\overrightarrow{v_1v_2}) = 1$, $v_1, v_2 \in L(\overrightarrow{S})$,	$c_0(v_2) = 2$
$\overrightarrow{v_2v_3}$ is T-edge, $\hat{c}(\overrightarrow{v_2v_3}) = 1$, $v_2 \in L(\overrightarrow{S})$, $v_3 \in C(\overrightarrow{S})$, $c_0(v_2) = 2$	$c_0(v_3) = 2$, v_2 is rebellious
$\hat{c}(\overrightarrow{v_0v_1}) = 1$, $v_0 \in C(\overrightarrow{S})$, $v_1 \in L(\overrightarrow{S})$, $c_0(v_1) = 1$,	$\overrightarrow{v_0v_1}$ is S-edge

Table 4.6: This table is used in the case that $k = 3$ in Lemma 4.3.1.

$1 = c_0(v_0) \neq c_0(v_1) = 2$, so (ii) holds, and (i) vacuously holds.

□

Similarly to the previous lemma, we have the following characterization of edge-monochromatic dipaths of color 2 where the initial-vertex and the end-vertex are in $C(\overrightarrow{S})$, and the interior vertices are in $L(\overrightarrow{S})$. The proof of the following lemma is omitted since it is similar to the proof of the previous lemma.

Lemma 4.3.2. *Let $\overrightarrow{P} = v_0, v_1, ..., v_k$ be a dipath in $\overrightarrow{G}$ whose edges are colored 2. Suppose $v_0, v_k \in C(\overrightarrow{S})$ and $v_i \in L(\overrightarrow{S})$ for $1 \leq i \leq k-1$. Then we have $k \leq 3$. Furthermore,*

i. *If $c_0(v_0) = c_0(v_k)$, then $\overrightarrow{v_0v_k} \in E(Center(\overrightarrow{G}))$.*

ii. *If $c_0(v_0) \neq c_0(v_k)$, then $c_0(v_0) = 2$ and $c_0(v_k) = 1$.*

The next lemma proves that there is no edge-monochromatic dicycle in $\overrightarrow{G}$. To show this, we use the fact that there is no vertex-monochromatic dicycle in Center($\overrightarrow{G}$) (see Lemma 4.2.8).

Lemma 4.3.3. *There is no edge-monochromatic dicycle in $\overrightarrow{G}$.*

Proof. This proof is essentially the same as the proof of [MP, Lemma 2.8] and it is included for completeness.

Suppose not. Let $\overrightarrow{Q}$ be an edge-monochromatic dicycle in $\overrightarrow{G}$. WLOG, assume the edges of $\overrightarrow{Q}$ are colored 1. We set $Q_C = V(\overrightarrow{Q}) \cap C(\overrightarrow{S})$ and $Q_L = V(\overrightarrow{Q}) \cap L(\overrightarrow{S})$. Both Q_C and Q_L are non-empty because $\overrightarrow{Q}$ must contain an S-edge as $\overrightarrow{T}$ is a tree. Let $v_0 \in Q_C$ and label the remaining vertices of Q_C by $v_1, ..., v_k$ as they appear in $\overrightarrow{Q}$ starting from v_0.

First suppose that not all the vertices in Q_C are colored the same. Then, there exists $i \in \{0, ..., n\}$ such that $c_0(v_i) = 2$ and $c_0(v_{i+1}) = 1$ (indices are considered modulo $k + 1$), a contradiction with Lemma 4.3.1(ii). Therefore, all the vertices in Q_C are colored the same. By Lemma 4.3.1(i), we have $\overrightarrow{v_i v_{i+1}} \in E(\text{Center}(\overrightarrow{G}))$ for all $i \in \{0, ..., k\}$. Therefore, $v_0, ..., v_k, v_0$ correspond to a vertex-monochromatic dicycle in Center($\overrightarrow{G}$), a contradiction with Lemma 4.2.8 (i.e., there is no vertex-monochromatic dicycle in Center($\overrightarrow{G}$)). □

We need the following fact which is the immediate consequence of the furthermore part of Lemma 4.2.9.

Corollary 4.3.4. *Let $\overrightarrow{P}$ be an edge-monochromatic dipath whose vertices are in $L(\overrightarrow{S})$. Then, the length of $\overrightarrow{P}$ is at most 2.*

Proof. Suppose $\overrightarrow{P} = v_0, v_1, ..., v_k$ is a dipath in $\overrightarrow{G}$. WLOG, assume the edges of $\overrightarrow{P}$ are colored 1. Since $v_i \in L(\overrightarrow{S})$ for $0 \leq i \leq k$, we have $c_0(v_i) = 2$ for $i \geq 1$ (possibly, $c_0(v_0) = 1$). Hence, by the furthermore part of Lemma 4.2.9, the length of the dipath $v_1, v_2, ..., v_k$ is at most 1. Thus, the length of $\overrightarrow{P}$ is at most $1 + 1 = 2$. □

It remains to prove that every monochromatic dipath in $\overrightarrow{G}$ has length at most 7. The next lemma proves this fact for monochromatic dipaths whose edges are colored 1, and Lemma 4.3.7 deals with color 2.

Lemma 4.3.5. *Let $\overrightarrow{P}$ be a dipath whose edges are colored* 1*. Then, the length of $\overrightarrow{P}$ is at most* 7*.*

Proof. For the proof of this lemma we use tables where each row can be verified independently, i.e., by using only the assumptions in the row and Definition 4.2.5.

The sketch of the proof is as follows: First we show that the length of any edge-monochromatic dipath whose initial-vertex is a center of color 2 is at most 4 (Claim 4.3.6). Then, we show the number of centers of color 1 in $\overrightarrow{P}$ is at most one. Finally, with case analysis (depending on whether $\overrightarrow{P}$ contains a center of color 1 or not), we finish the proof.

We start by giving an upper bound on the length of an edge-monochromatic dipath whose edges are colored 1 and the initial-vertex is a center of color 2.

Claim 4.3.6. *Let $\overrightarrow{P_u} = u, u_1, ..., u_k$ be a dipath whose edges are colored* 1*, and let u be a center of color* 2*. Then, the length of $\overrightarrow{P_u}$ is at most* 4*.*

Proof of Claim 4.3.6: Notice that if u is the only center in $\overrightarrow{P_u}$, then we have $u_i \in L(\overrightarrow{S})$ for $i \geq 1$. Thus, by Corollary 4.3.4, we conclude that the length of dipath $u_1, ..., u_k$ is at most 2 which implies the length of $\overrightarrow{P_u}$ is at most $1 + 2 = 3$.

Now suppose $\overrightarrow{P_u}$ contains a center different than u. Assume $w \neq u$ is the center with the smallest index in $\{u_1, ..., u_k\}$. So there is no center in $\overrightarrow{P_u}$ between u and w. Furthermore, the edges of $\overrightarrow{P_u}$ are colored 1 and $c_0(u) = 2$. Thus, Lemma 4.3.1(ii) could not be applied. So by Lemma 4.3.1(i), we have $c_0(w) = 2$ and w is adjacent to u in Center($\overrightarrow{G}$); hence, either $w = u_1$, or $w = u_2$. We consider only the worst case, i.e., $w = u_2$. In the other case (i.e., $w = u_1$), using similar arguments, we can prove the same upper bound on the length of $\overrightarrow{P_u}$. Suppose there is a center u_i for $i \geq 3$ in $\overrightarrow{P_u}$. Let z be the center with the smallest index in $\{u_3, ..., u_k\}$. So there is no center in $\overrightarrow{P_u}$ between w and z. Again, by Lemma 4.3.1(i), we conclude that $\overrightarrow{wz} \in E(\text{Center}(\overrightarrow{G}))$, and $c_0(z) = 2$. Now the dipath u, w, z has length 2 in Center($\overrightarrow{G}$) such that its vertices are colored 2, a contradiction with Lemma 4.2.8. So we can assume that $u_i \in L(\overrightarrow{S})$ for $i \geq 3$.

Suppose $c_0(u_3) = 2$. Then by the first row of Table 4.7, we conclude $\overrightarrow{u_2u_3}$ is a T-edge. Since c_0 is tame, u_3 is not rebellious (note that $c_0(u_2) = c_0(u_3) = 2$). We prove that u_3 is the end-vertex of $\overrightarrow{P_u}$. Suppose not. So there is a leaf u_4 in $\overrightarrow{P_u}$ (recall that u_i's are leaves for $i \geq 3$) and $c_0(u_4) = 2$ (in order to have $\hat{c}(\overrightarrow{u_3u_4}) = 1$). Therefore, we have a T-edge $\overrightarrow{u_3u_4}$ such that $c_0(u_3) = c_0(u_4) = 2$ and u_3 is not rebellious, a contradiction with Claim 4.2.10. Thus u_3 is the end-vertex of $\overrightarrow{P_u}$ which implies that the length of $\overrightarrow{P_u}$ is at most 3.

Suppose $c_0(u_3) = 1$. Then from the second row of Table 4.7, we know $\overrightarrow{u_2u_3}$ is an S-edge. Therefore, u_3 is rebellious (note that $2 = c_0(u_2) \neq c_0(u_3) = 1$). If u_3 is the end-vertex of $\overrightarrow{P_u}$, then we are done. So suppose there is a leaf u_4 in $\overrightarrow{P}$. Note that $\overrightarrow{u_3u_4}$ is a T-edge, and u_3 is rebellious. So u_4 is not rebellious (tame property). We prove that u_4 is the end-vertex of $\overrightarrow{P_u}$. Suppose not. Then, there is a leaf u_5 in $\overrightarrow{P}$ such that $\overrightarrow{u_4u_5}$ is a T-edge. Hence, there is a T-edge $\overrightarrow{u_4u_5}$ such that u_4 is not rebellious and $c_0(u_4) = 2 = c_0(u_5)$ (in order to have $\hat{c}(\overrightarrow{u_3u_4}) = 1 = \hat{c}(\overrightarrow{u_4u_5})$), a contradiction with Claim 4.2.10. So u_4 is the end-vertex of $\overrightarrow{P_u}$ which implies that the length of $\overrightarrow{P_u}$ is at most 4. Figure 4.3 depicts $\overrightarrow{P_u}$ when it achieves the maximum possible length.

Now we prove the lemma. Notice there could be at most one center of color 1 in $\overrightarrow{P}$. Suppose there are at least two such centers. Let u, w be two centers of color 1 in $\overrightarrow{P}$ such

Assumptions	Conclusion
$\hat{c}(\overrightarrow{u_2u_3}) = 1,\ u_2 \in C(\overrightarrow{S}),\ u_3 \in L(\overrightarrow{S}),$ $c_0(u_3) = 2$	$\overrightarrow{u_2u_3}$ is T-edge
$\hat{c}(\overrightarrow{u_2u_3}) = 1,\ u_2 \in C(\overrightarrow{S}),\ u_3 \in L(\overrightarrow{S}),$ $c_0(u_3) = 1$	$\overrightarrow{u_2u_3}$ is S-edge

Table 4.7: This table is used in the proof of Claim 4.3.6.

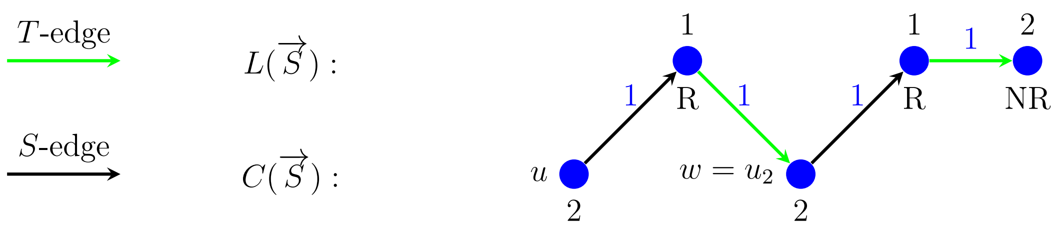

Figure 4.3: The length of $\overrightarrow{P_u}$ reaches its maximum when $w = u_2$ and $c_0(u_3) = 1$.

that there is no center of color 1 in $\overrightarrow{P}$ between u and w.
If there is no center of color 2 in $\overrightarrow{P}$ between u and w, then Lemma 4.3.1(i) implies that u and w are adjacent in Center($\overrightarrow{G}$). However, Lemma 4.2.8 implies vertices of color 1 form an independent set in Center($\overrightarrow{G}$), a contradiction.
If there is a center of color 2 between u and w, then we have a dipath whose edges are colored 1 from a center of color 2 to a center of color 1 such that its interior vertices are leaves, a contradiction with Lemma 4.3.1(ii). Therefore, $\overrightarrow{P}$ has at most one center of color 1. So we only need to consider two cases: (1) $\overrightarrow{P}$ does not contain a center of color 1, and (2) $\overrightarrow{P}$ has a center of color 1. Let $\overrightarrow{P} = v_0, v_1, ..., v_k$.

Case (1) (when $\overrightarrow{P}$ does not contain a center of color 1).

If $\overrightarrow{P}$ does not contain a center of color 2, together with our assumption that $\overrightarrow{P}$ does not contain a center of color 1, we conclude that $\overrightarrow{P}$ consists of only leaves. By Corollary 4.3.4, the length of $\overrightarrow{P}$ is at most 2 and we are done.

Suppose u is the first center of color 2 that appears in $\{u_0, ..., u_k\}$. Let $\overrightarrow{P_u}$ be the sub-dipath of $\overrightarrow{P}$ whose initial-vertex and end-vertex are u and v_k, respectively. Then, by Claim 4.3.6, we know the length of $\overrightarrow{P_u}$ is at most 4. Since we are assuming u is

Assumptions	Conclusion
$v_{i-1}, v_i \in L(\overrightarrow{S})$, $c_0(v_i) = 1$	$\hat{c}(\overrightarrow{v_{i-1}v_i}) = 2$
$\hat{c}(\overrightarrow{v_iu}) = 1$, $u \in C(\overrightarrow{S})$, $v_i \in L(\overrightarrow{S})$, $c_0(u) = c_0(v_i) = 2$	v_i is rebellious
$v_{i-2}, v_{i-1} \in L(\overrightarrow{S})$, $c_0(v_{i-1}) = 1$	$\hat{c}(\overrightarrow{v_{i-2}v_{i-1}}) = 2$

Table 4.8: This table is used in the proof of Lemma 4.3.5 for Case (1).

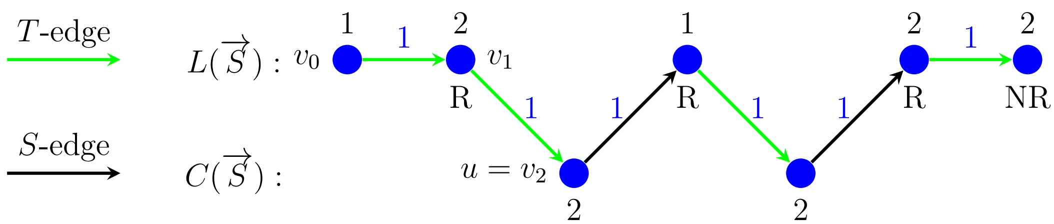

Figure 4.4: There is no center of color 1 in $\overrightarrow{P}$. Suppose $\overrightarrow{P}$ contains a center u of color 2, then the number of edges between u and the end-vertex of $\overrightarrow{P}$ is at most 4 (Claim 4.3.6). Let v_i be a leaf of color 2 before u in $\overrightarrow{P}$. Then $c_0(v_{i-1}) = 1$ and this implies that v_{i-1} is the initial-vertex of the dipath.

the first center of color 2, and $\overrightarrow{P}$ does not contain a center of color 1, all the vertices before u are leaves. Suppose there is a leaf v_i such that $\overrightarrow{v_iu}$ is in $\overrightarrow{P}$.

If $c_0(v_i) = 1$, then by the first row of Table 4.8, there could not be a leaf in $\overrightarrow{P}$ before v_i (if there exists such leaf v_{i-1}, then the color of edge $\overrightarrow{v_{i-1}v_i}$ is 2, a contradiction with the coloring of $\overrightarrow{P}$). Hence, $v_i = v_0$ and the length of $\overrightarrow{P}$ is at most $1 + 4 = 5$.

If $c_0(v_i) = 2$, then by the second row of Table 4.8, we know that v_i is rebellious. Since c_0 is tame, $c_0(v_{i-1}) = 1$ (if v_{i-1} exists). From the last row of the table, we conclude that there could not be a leaf before v_{i-1} in $\overrightarrow{P}$ (if v_{i-2} exists, then $c_0(\overrightarrow{v_{i-2}v_{i-1}}) = 2$, a contradiction with the coloring of $\overrightarrow{P}$). So the length of $\overrightarrow{P}$ is at most $2 + 4 = 6$. Figure 4.4 shows $\overrightarrow{P}$ when it achieves its maximum possible length 6.

Case (2) (when $\overrightarrow{P}$ contains a center of color 1).

Let u be the center of color 1, then $u = v_0$. Otherwise, there is $\overrightarrow{v_iu}$ such that $\hat{c}(\overrightarrow{v_iu}) = 1$. Thus, v_i is either a center of color 1, or v_i is a leaf of color 1 that is not rebellious.

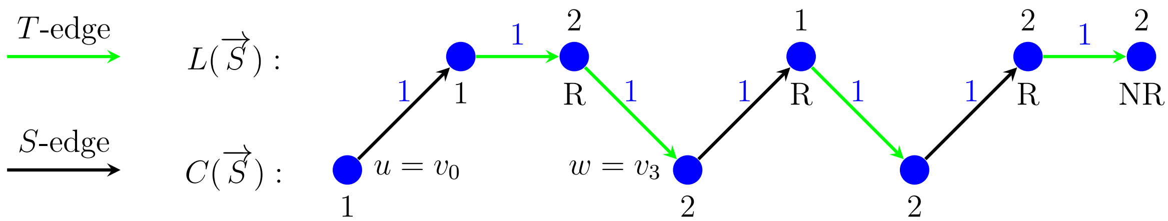

Figure 4.5: There is a center u of color 1 in $\overrightarrow{P}$. Then u is the initial-vertex of the dipath. If there is a center w of color 2 in the dipath, then the number of edges between u and w is at most 3. Furthermore, the longest monochromatic dipath that we could have with initial-vertex w has length 4 (Claim 4.3.6).

Both cases imply there are two centers of color 1 adjacent to each other which is a contradiction with Lemma 4.2.8 (notice that in the latter case, since $v_i \in L(\overrightarrow{S})$, there is a center z such that $\overrightarrow{zv_i}$ is an S-edge. Therefore, $\overrightarrow{zu} \in E(\text{Center}(\overrightarrow{G}))$. Furthermore, Since v_i is not rebellious, we have $c_0(z) = c_0(v_i) = 1$; hence, two centers of color 1 are adjacent in the center graph). So we proved that $u = v_0$ is the initial-vertex of $\overrightarrow{P}$. Recall the fact from the beginning of the proof that $\overrightarrow{P}$ has at most one center of color 1; hence, there is no center of color 1 in $\{v_1, ..., v_k\}$.

If $v_1, ..., v_k \in L(\overrightarrow{S})$, then by Corollary 4.3.4, the length of the dipath $v_1, ..., v_k$ is at most 2 which implies that the length of $\overrightarrow{P}$ is at most $1 + 2 = 3$.

Now, suppose there is a center of color 2 in $\{v_1, ..., v_k\}$. Let w be the center of color 2 with smallest index in $\{v_1, ..., v_k\}$. Then, by Lemma 4.3.1, the number of edges of $\overrightarrow{P}$ between $u = v_0$ and w is at most 3. On the other hand, since w is a center of color 2, by Claim 4.3.6 we conclude that the number of edges of $\overrightarrow{P}$ between w and v_k is at most 4. Therefore, the length of $\overrightarrow{P}$ is at most $3 + 4 = 7$. This case is shown in Figure 4.5.

□

The next lemma provides an upper bound on the length of a dipath whose edges are colored 2 by the edge-coloring $\hat{c}$. Although the main idea of the proof of this lemma is the same as the main idea of the proof of Lemma 4.3.5, in details they are different. This is due to difference in the induced subgraphs by color 1 and color 2 in the center graph.

Lemma 4.3.7. *Let $\overrightarrow{P}$ be a dipath whose edges are colored 2. Then, the length of $\overrightarrow{P}$ is at most 7.*

Assumptions	Conclusion
$\hat{c}(\overrightarrow{uu_1}) = 2$, $u \in C(\overrightarrow{S})$, $u_1 \in L(\overrightarrow{S})$, $c_0(u_1) = 1$	$\overrightarrow{uu_1}$ is T-edge
$\overrightarrow{u_1u_2}$ is T-edge, $\hat{c}(\overrightarrow{u_1u_2}) = 2$, $u_1, u_2 \in L(\overrightarrow{S})$	$c_0(u_2) = 1$

Table 4.9: This table is used in Claim 4.3.8.

Proof. For the proof of this lemma we use tables where each row can be verified independently, i.e., by using only the assumptions in the row and Definition 4.2.5.

The sketch of the proof is as follows: First we show that the length of any dipath whose initial-vertex is a center of color 1 is at most 2 (Claim 4.3.8). Then, we prove that the number of centers of color 1 in $\overrightarrow{P}$ is at most one. Furthermore, the number of centers of color 2 in $\overrightarrow{P}$ is at most two. We finish the proof with case analysis.

Claim 4.3.8. *Let $\overrightarrow{P_u} = u, u_1, ..., u_k$ be a dipath whose edges are colored 2. Suppose u is a center of color 1. Then, the length of $\overrightarrow{P_u}$ is at most 2.*

Proof of Claim 4.3.8: Suppose $\overrightarrow{P_u}$ contains a center different than u. Assume $w \neq u$ is the center with the smallest index in $\{u_1, ..., u_k\}$. So there is no center in $\overrightarrow{P_u}$ between u and w. Since the edges of $\overrightarrow{P_u}$ are colored 2 and $c_0(u) = 1$, Lemma 4.3.2(ii) could not be applied. Thus, by Lemma 4.3.2(i), we have $c_0(w) = 1$ and w is adjacent to u in Center($\overrightarrow{G}$), a contradiction with Lemma 4.2.8 (i.e., the vertices of color 1 form an independent set in Center($\overrightarrow{G}$)). So far we have proved that u is the only center of $\overrightarrow{P_u}$, i.e., $u_i \in L(\overrightarrow{S})$ for $i \geq 1$.

If $c_0(u_1) = 1$, then by the first row of Table 4.9 we conclude that $\overrightarrow{uu_1}$ is a T-edge. Since $c_0(u) = c_0(u_1)$ and the coloring is tame, we conclude that u_1 is not rebellious. If u_1 is not the end-vertex of $\overrightarrow{P_u}$, then there is a leaf u_2 such that $\overrightarrow{u_1u_2}$ is a T-edge; hence, by the second row of Table 4.9 we have $c_0(u_2) = 1$. Thus, we have a T-edge $\overrightarrow{u_1u_2}$ such that $c_0(u_1) = c_0(u_2) = 1$ and u_1 is not rebellious, a contradiction with Claim 4.2.10. Hence, u_1 is the end-vertex of $\overrightarrow{P_u}$ which implies that the length of $\overrightarrow{P_u}$ is at most 1. Figure 4.6(a) shows $\overrightarrow{P_u}$ when $c_0(u_1) = 1$.

Now if $c_0(u_1) = 2$, then by the first row of Table 4.10 we know $\overrightarrow{uu_1}$ is an S-edge. Since $c_0(u) = 1 \neq 2 = c_0(u_1)$, u_1 is rebellious. Thus, if u_1 is not the end-vertex of $\overrightarrow{P_u}$, then

Assumptions	Conclusion
$\hat{c}(\overrightarrow{uu_1}) = 2$, $u \in C(\overrightarrow{S})$, $u_1 \in L(\overrightarrow{S})$, $c_0(u_1) = 2$	$\overrightarrow{uu_1}$ is S-edge
$\overrightarrow{u_1u_2}$ is T-edge, $\hat{c}(\overrightarrow{u_1u_2}) = 2$, $u_1, u_2 \in L(\overrightarrow{S})$	$c_0(u_2) = 1$
$\overrightarrow{u_2u_3}$ is T-edge, $\hat{c}(\overrightarrow{u_2u_3}) = 2$, $u_2, u_3 \in L(\overrightarrow{S})$	$c_0(u_3) = 1$

Table 4.10: This table is used in Claim 4.3.8.

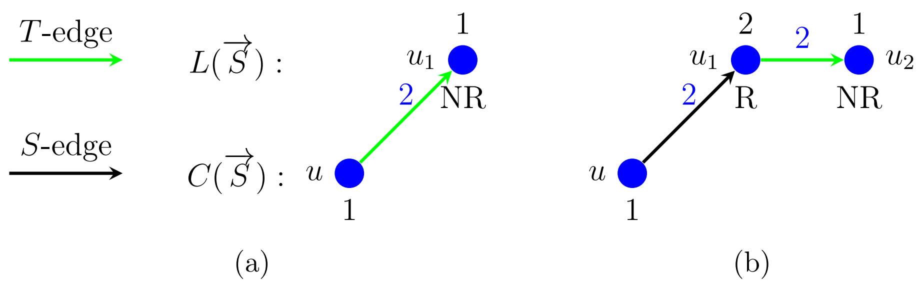

Figure 4.6: (a) $\overrightarrow{P_u}$ when $\overrightarrow{uu_1}$ is a T-edge. (b) $\overrightarrow{P_u}$ when $\overrightarrow{uu_1}$ is an S-edge. See the proof of Claim 4.3.8.

there is a leaf u_2 such that u_2 is not rebellious (tame property). From the second row of Table 4.10, we have $c_0(u_2) = 1$. We will prove that u_2 is the end-vertex of $\overrightarrow{P_u}$ which implies that the length of $\overrightarrow{P_u}$ is at most 2.
So suppose u_2 is not the end-vertex of $\overrightarrow{P_u}$. Then, there is a leaf u_3 in $\overrightarrow{P_u}$ such that $\overrightarrow{u_2u_3}$ is a T-edge. By the third row of Table 4.10, we conclude that $c_0(u_3) = 1$. Now we have a T-edge $\overrightarrow{u_2u_3}$ such that $c_0(u_2) = c_0(u_3) = 1$ and u_2 is not rebellious, a contradiction with Claim 4.2.10. Figure 4.6(b) shows $\overrightarrow{P_u}$ when $c_0(u_1) = 2$.

Now, we prove the lemma. Notice that there could be at most one center of color 1 in $\overrightarrow{P}$. Suppose there are at least two such centers. Let u, w be two centers of color 1 in $\overrightarrow{P}$. WLOG, assume there is no center of color 1 in $\overrightarrow{P}$ between u and w.
If there is no center of color 2 between u and w, then Lemma 4.3.2(i) implies that u and w are adjacent in Center($\overrightarrow{G}$), a contradiction with Lemma 4.2.8 (i.e., the vertices of color 1 form an independent set in Center($\overrightarrow{G}$)). If there is a center of color 2 in $\overrightarrow{P}$ between u and

w, then we have a dipath whose edges are colored 2 from a center of color 1 to a center of color 2 such that its interior vertices are leaves, a contradiction with Lemma 4.3.2(ii). Therefore, there is at most one center of color 1 in $\overrightarrow{P}$.

Now we show that there are at most two centers of color 2 in $\overrightarrow{P}$. Suppose there are at least three centers u, v, and w of color 2 in $\overrightarrow{P}$ such that v is between u and w in $\overrightarrow{P}$. WLOG, assume there is no center of color 2 in $\overrightarrow{P}$ between u and v, and there is no center of color 2 in $\overrightarrow{P}$ between v and w.
If there is a center of color 1 either between u and v or between v and w, then we have a dipath whose edges are colored 2 from a center of color 1 to a center of color 2 such that its interior vertices are leaves, a contradiction with Lemma 4.3.2(ii).
If there is no center of color 1 in $\overrightarrow{P}$ between u and w, then by Lemma 4.3.2(i) we have $\overrightarrow{uv}, \overrightarrow{vw} \in E(\text{Center}(\overrightarrow{G}))$, a contradiction with Lemma 4.2.8 (i.e., the vertices of color 2 induce a subgraph with diameter at most 1 in Center($\overrightarrow{G}$)).

So we proved there is at most one center of color 1 and there are at most two centers of color 2 in $\overrightarrow{P}$. Notice that if $\overrightarrow{P}$ does not have a center, then by Lemma 4.3.4 the length of $\overrightarrow{P}$ is at most 2 and we are done. So we consider three cases such that in each case $\overrightarrow{P}$ has at least one center. Let $\overrightarrow{P} = v_0, v_1, ..., v_k$.

Case (1) ($\overrightarrow{P}$ contains one center of color 1 but no center of color 2).

Let v_i be the center of color 1. So $\{v_0, ..., v_k\} \setminus \{v_i\} \in L(\overrightarrow{S})$. By Corollary 4.3.4, the length of dipath $v_0, v_1, ..., v_{i-1}$, and the length of dipath $v_{i+1}, v_{i+2}, ..., v_k$ are at most 2. Therefore, the length of $\overrightarrow{P}$ is at most $2 + 2 + 2 = 6$.

Case (2) ($\overrightarrow{P}$ contains one center of color 2).

Let v_i for some $0 \leq i \leq k$ be the center of color 2. Suppose there is a center w in $\{v_0, ..., v_{i-1}\}$. Since we are assuming v_i is the only center of color 2 in the dipath, we conclude that $c_0(w) = 1$. Therefore, there is a dipath whose edges are colored 2 from a center of color 1 to a center of color 2 such that its interior vertices are leaves, a contradiction with Lemma 4.3.2(ii). Therefore, $\{v_0, ..., v_{i-1}\} \subseteq L(\overrightarrow{S})$. From the first row of Table 4.11, we conclude that $c_0(v_{i-1}) = 2$. If v_{i-1} is not the initial-vertex of the dipath, then from the second row of Table 4.11, we get that $\hat{c}(\overrightarrow{v_{i-2}v_{i-1}}) = 1$, a contradiction with the coloring of $\overrightarrow{P}$. Therefore, v_{i-1} is the initial-vertex of the dipath; hence $i \leq 1$. We consider only the worst case, i.e., $i = 1$ and v_1 is a center of color 2. In the other case (i.e., $i = 0$ and v_0 is a center of color 2), using similar arguments, we can prove the same upper bound on the length of $\overrightarrow{P}$.

Assumptions	Conclusion
$\hat{c}(\overrightarrow{v_{i-1}v_i}) = 2$, $v_{i-1} \in L(\overrightarrow{S})$, $v_i \in C(\overrightarrow{S})$, $c_0(v_i) = 2$	$c_0(v_{i-1}) = 2$, v_{i-1} is not rebellious
$v_{i-2}, v_{i-1} \in L(\overrightarrow{S})$, $c_0(v_{i-1}) = 2$	$\hat{c}(\overrightarrow{v_{i-2}v_{i-1}}) = 1$

Table 4.11: This table is used in the proof Lemma 4.3.7 for Case (2) & (3).

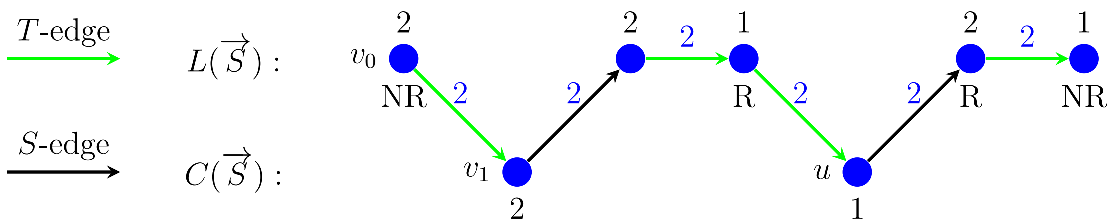

Figure 4.7: When $\overrightarrow{P}$ contains one center of color 2 and one center of color 1, the maximum length of $\overrightarrow{P}$ is 6.

If there is no center of color 1 in $\{v_2, ..., v_k\}$, then $v_2, ..., v_k$ are leaves. By Lemma 4.3.4, we know that the length of dipath $v_2, ..., v_k$ is at most 2. Therefore, the length of $\overrightarrow{P}$ is at most $1 + 1 + 2 = 4$.

Suppose there is a center of color 1 in $\{v_2, ..., v_k\}$. Let u be the center of color 1 with smallest index in $\{v_2, ..., v_k\}$. Then, by Lemma 4.3.2, the number of edges of $\overrightarrow{P}$ between v_1 and u is at most 3. On the other hand, by Claim 4.3.8 the number of edges of $\overrightarrow{P}$ between u and v_k is at most 2. Therefore, the length of $\overrightarrow{P}$ is at most $1 + 3 + 2 = 6$. This case is shown in Figure 4.7.

Case (3) ($\overrightarrow{P}$ contains two centers of color 2).

Let v_i for some $0 \leq i \leq k$ be the center of color 2 with smallest index and $v_j \neq v_i$ is the second center of color 2 in $\overrightarrow{P}$. Notice that if there is a center of color 1 in $\overrightarrow{P}$ between v_i and v_j, then there is a dipath whose edges are colored 2 from a center of color 1 to a center of color 2 such that its interior vertices are leaves, a contradiction with Lemma 4.3.2(ii). Thus, there is no center in $\overrightarrow{P}$ between v_i and v_j. By Lemma 4.3.2, we conclude that v_i and v_j are adjacent in Center($\overrightarrow{G}$). Therefore, either $j = i + 1$ or $j = i + 2$. We consider only the worst case, i.e., $v_j = v_{i+2}$ is the center of color 2. In the other case (i.e., $v_j = v_{i+1}$ is the center of color 2), using similar arguments, we

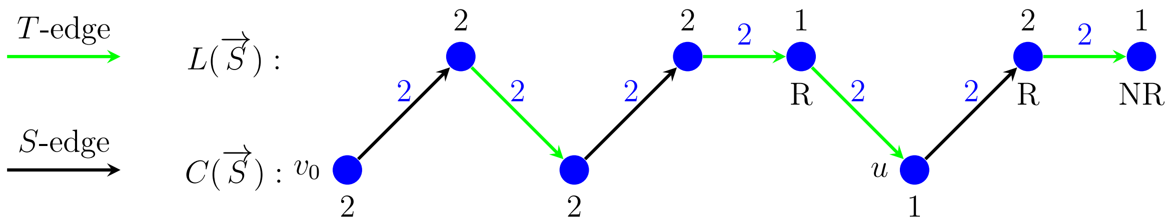

Figure 4.8: When $\overrightarrow{P}$ contains two centers of color 2 and one center of color 1, then its length is at most 7.

can prove the same upper bound on the length of $\overrightarrow{P}$.

Notice that there is no center in $\{v_0, ..., v_{i-1}\}$ because if there is any, it must be a center of color 1 (we know that $\overrightarrow{P}$ contains at most two centers of color 2, i.e., v_i and v_{i+2}). Therefore, there is a dipath whose edges are colored 2 from a center of color 1 to a center of color 2 such that its interior vertices are leaves, a contradiction with Lemma 4.3.2(ii). So we can assume that $\{v_0, ..., v_{i-1}\} \subseteq L(\overrightarrow{S})$.

Since v_{i-1} is a leaf, there is a center w such that $\overrightarrow{wv_{i-1}}$ is an S-edge. On the other hand, by the first row of Table 4.11, we conclude that $c_0(v_{i-1}) = 2$ and v_{i-1} is not rebellious. Therefore, $c_0(w) = c_0(v_{i-1}) = 2$ and $\overrightarrow{wv_i} \in E(\text{Center}(\overrightarrow{G}))$. So dipath w, v_i, v_{i+2} is in the center graph whose vertices are colored 2, a contradiction with Lemma 4.2.8 (i.e., color 2 vertices in Center($\overrightarrow{G}$) induce a subgraph with diameter at most 1). Thus, there is no leaf before v_i, i.e., $i = 0$. So far we have proved that v_0 and v_2 are the only centers of color 2 in $\overrightarrow{P}$. Note that v_1 is a leaf (otherwise, there is a dipath from a center of color 1 to a center of color 2, a contradiction with Lemma 4.3.2(ii)).

If there is no center of color 1 in $\{v_3, ..., v_k\}$, then $v_3, ..., v_k$ are leaves. So by Corollary 4.3.4, dipath $v_3, ..., v_k$ has length at most 2. Thus, the length of $\overrightarrow{P}$ is at most $2 + 1 + 2 = 5$.

If there is a center u of color 1 in $\{v_3, ..., v_k\}$, then by Lemma 4.3.2, the number of edges of $\overrightarrow{P}$ between v_2 and u is at most 3. On the other hand, by Claim 4.3.8, the number of edges of $\overrightarrow{P}$ between u and v_k is at most 2. Thus, the length of $\overrightarrow{P}$ is at most $2 + 3 + 2 = 7$. This case is shown in Figure 4.8.

□

Now it is easy to see that G has a good edge-coloring.

Corollary 4.3.9. *Let $G = (V, T \uplus S)$ where T and S are a tree and a star forest, respectively. Then, G has a good edge-coloring.*

Proof. Let c_0 be the vertex-coloring of both G and $\overrightarrow{G}$ given by Lemma 4.2.9, and let $\hat{c} = Ext(c_0)$ be the edge-coloring for G and $\overrightarrow{G}$. Notice that by Fact 4.2.6, together with Remark 3.3.3, we conclude that Lemma 3.3.2 holds for $\hat{c}$, i.e., (1) every edge-monochromatic cycle in G is an edge-monochromatic dicycle in $\overrightarrow{G}$, and (2) every edge-monochromatic path in G is the union of at most two edge-monochromatic dipaths in $\overrightarrow{G}$.
Since there is no edge-monochromatic dicycle in $\overrightarrow{G}$ (Lemma 4.3.3), by (1) there is no edge-monochromatic cycle in G. Furthermore, let P be an edge-monochromatic path in G. From (2), we know P is the union of at most two edge-monochromatic dipaths $\overrightarrow{P'}, \overrightarrow{P''}$. By Lemmas 4.3.5 & 4.3.7, both $\overrightarrow{P'}$ and $\overrightarrow{P''}$ have length at most 7. Thus, the length of P is at most 14. □

Now Theorem 4.1.1 follows easily.

Proof of Theorem 4.1.1: By Theorem 4.2.2, G can be decomposed into a tree T and a star forest S, i.e, $G = (V, T \uplus S)$. By Corollary 4.3.9, G has a good edge-coloring. Let F_1 and F_2 be the subgraphs induced by colors 1 and 2, respectively. Since there is no edge-monochromatic cycle in the graph, F_1 and F_2 are forests. Furthermore, any edge-monochromatic path has length at most 14, so the diameter of F_1 and F_2 is at most 14.

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