On Geometric Drawings of Graphs

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This thesis is about geometric drawings of graphs and their topological generalizations.

First, we study pseudolinear drawings of graphs in the plane. A *pseudolinear* drawing is one in which every edge can be extended into an infinite simple arc in the plane, homeomorphic to $\mathbb{R}$, and such that every two extending arcs cross exactly once. This is a natural generalization of the well-studied class of *rectilinear drawings*, where edges are straight-line segments. Although, the problem of deciding whether a drawing is homeomorphic to a rectilinear drawing is NP-hard, in this work we characterize the minimal forbidden subdrawings for pseudolinear drawings and we also provide a polynomial-time algorithm for recognizing this family of drawings.

Second, we consider the problem of transforming a topological drawing into a similar rectilinear drawing preserving the set of crossing pairs of edges. We show that, under some circumstances, pseudolinearity is a necessary and sufficient condition for the existence of such transformation. For this, we prove a generalization of Tutte's Spring Theorem for drawings with crossings placed in a particular way.

Lastly, we study drawings of K_n in the sphere whose edges can be extended to an arrangement of pseudocircles. An *arrangement of pseudocircles* is a set of simple closed curves in the sphere such that every two intersect at most twice. We show that (i) there is drawing of K_{10} that cannot be extended into an arrangement of pseudocircles; and (ii) there is a drawing of K_9 that can be extended to an arrangement of pseudocircles, but no extension satisfies that every two pseudocircles intersects exactly twice. We also introduce the notion *pseudospherical drawings of* K_n, a generalization of *spherical drawings* in which each edge is a minor arc of a great circle. We show that these drawings are characterized by a simple local property. We also show that every pseudospherical drawing has an extension into an arrangement of pseudocircles where the "at most twice" condition is replaced by "exactly twice".

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For the amazing hours we spent in the research room, I am very thankful to Gelasio Salazar and Dan McQuillan. I hope our collaboration will continue in the years to come.

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Thanks to János Pach, Therese Biedl, Luke Postle and Jim Geelen. Your insightful comments about the thesis sparked my enthusiasm for looking into future research directions.

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Special thanks to my parents: *no hay palabras para agradecer todo el esfuerzo que han hecho por mí. Gracias por todo el apoyo y amor que me han dado [no words can express my gratitude for all the things you have done for me. Thanks for all the support and love that I have received from you].*

Finally, I would like to thank my wife Diana for always reminding me of the truly important things in life.

Dedication

Para Chichí, que me enseño a disfrutar la vida [For Chichí, who taught me how to enjoy life].

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Chapter 1

Introduction

In 2014, Dan McQuillan and Gelasio Salazar visited the University of Waterloo when I was starting my PhD studies. The purpose of their visit was to begin an investigation with Bruce Richter and me, about one the topics treated in the *Exact Crossing Numbers Workshop*, held at the American Mathematical Institute (AIM) in the same year.

The topic was the recent progress on the Harary-Hill conjecture, stating that the crossing number of K_n is given by the following formula:

$$H(n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

At the AIM meeting we learned about new methods found by Ábrego et al. [1] showing that the Harary-Hill Conjecture holds for some interesting classes of drawings, such as rectilinear, 2-page, cylindrical and monotone drawings.

These methods emphasize the importance of understanding the special features of these classes of drawings. Perhaps, with this in mind, Gelasio Salazar asked us, in our first meeting in Waterloo, the following question: *Is there a simple way, in terms of local properties of the drawing, to characterize rectilinear drawings of K_n?*

As an initial observation, we found that pseudolinear drawings (a class of drawings defined below that generalizes rectilinear drawings) have the property that, if one fixes a 3-cycle T and looks at the vertices drawn in the bounded side of T, then the edges connecting these interior vertices to the three vertices in T are contained in the bounded side of T. Not every topological drawing of K_n has this property; it was reasonable to think that drawings of K_n having this property are pseudolinear.

A *pseudolinear* drawing is one in which every edge can be extended into an infinite simple arc in the plane, homeomorphic to $\mathbb{R}$, and such that every two extending arcs cross exactly once. These extending arcs are called *pseudolines*, and they are the topological analog of the straight lines extending the edges of a rectilinear drawing.

Many of the properties satisfied by rectilinear drawings also hold for pseudolinear drawings. In fact, knowing that a property holds for pseudolinear drawings has its practical advantages: proofs and algorithms for pseudolinear drawings do not rely on geometric parameters such as angles or distances; they rely only on the combinatorial structure of how the extending pseudolines intersect and how they separate the vertices in the drawing.

Our group continued studying drawings of K_n satisfying the local property satisfied by the 3-cycles. We found that *every drawing of K_n satisfying this local property is pseudolinear* [8]. In fact, an equivalent result was independently discovered by Aichholzer et al. [5]. They showed that a drawing of K_n is pseudolinear if and only if it does not include one of the three drawings of K_4 in the plane.

Hernández-Vélez presented a talk in the Crossing-Number Workshop held in Rio de Janeiro (2015) about investigating the difference between the crossing number $\mathrm{cr}(G)$, the rectilinear crossing number $\overline{\mathrm{cr}}(G)$ and the pseudolinear crossing number $\tilde{\mathrm{cr}}(G)$ [18]. He explained how to find graphs where the difference between cr and $\tilde{\mathrm{cr}}$ is large. The key idea was to use some basic non-pseudolinear drawings, that he called *obstructions*. At that time, it was unclear how to construct new obstructions, or how many of them where there.

A few months later, I visited Carsten Thomassen and his group at DTU as part of an internship. I gave a talk about pseudolinear drawings of K_n, where I posed the problem of classifying the minimal forbidden obstructions of pseudolinear drawings of all graphs. Thomassen observed that one of difficulties of this problem is the existence of infinitely many minimal obstructions (In fact, in [29], it was already observed that there are infinitely many obstructions for rectilinear drawings; these are also examples of obstructions for pseudolinear drawings).

This thesis work is about the research following the previous events. In Chapter 2 we show as one of our main results in this thesis, a characterization of the minimal forbidden subdrawings of pseudolinear drawings. In this joint work with Bensmail and Richter, we found a simple description for all the obstructions, leading to a polynomial-time algorithm to recognize pseudolinear drawings.

Understanding pseudolinear drawings can be helpful to know more about geometric drawings of graphs. In [13], Bienstock and Dean showed that a graph G with $\mathrm{cr}(G) \leq 3$, has $\overline{\mathrm{cr}}(G) = \mathrm{cr}(G)$. With this result as our motivation, in Chapter 3 we study the problem of turning a topological drawing into a rectilinear drawing with the same number of crossings. Our main result in Chapter 3 finds conditions where this can be done and, surprisingly, where pseudolinearity plays an essential part of these conditions. For this purpose, we obtained a generalization of Tutte's Spring Theorem that allow some edges to be crossed. The content of Chapter 3 is based on the author's original ideas, but has been somewhat influenced by what will be a more general work in collaboration with Richter and Thomassen.

Chapter 4 is motivated from studying *spherical* drawings of K_n, where vertices are represented as distinct points in the unit sphere and the edges are shortest-arcs connecting these points. Spherical and rectilinear drawings have some properties in common, however, it is still unknown if spherical drawings of K_n satisfy the Harary-Hill conjecture.

A problem in the spotlight of the graph drawing community was to find a class of drawings generalizing spherical drawings of K_n in the same way pseudolinear drawings generalize rectilinear drawings. In Chapter 4, we investigate this by answering two open problems about extending drawings of K_n into arrangements of pseudocircles. Moreover, we propose a definition for pseudospherical drawings that achieves many of the desired properties.

In connection to our previous work in pseudolinear drawings of K_n, we show that a drawing is pseudospherical if and only if for every 3-cycle T there is a disk Δ_T in the sphere bounded by T, where all the edges connecting a vertex inside Δ_T to a vertex in T are drawn inside Δ_T and all the Δ_Ts are structured in a hereditary-way (for more technical details see Definition 4.4.1). Using this local property we show that every edge in a pseudospherical drawing induces a decomposition of the drawing into two pseudolinear pieces (this generalizes the observation that every edge in a spherical drawing of K_n induces a partition into two rectilinear drawings).

Part of the discussion of how to define pseudospherical drawings was focused on whether to include the condition of any two pseudocircles intersecting exactly twice or to simply let them intersect zero or two times. The main result of Chapter 4 ends this discussion by showing that a pseudospherical drawing of K_n admitting an extension where the pseudocircles cross zero or two times, also has one where they cross exactly twice. This last chapter is a joint work with Richter and Sunohara.

Chapter 2

Pseudolinear drawings of graphs

2.1 Introduction

When is a drawing of a graph in the plane homeomorphic to a drawing whose edges are straight line segments? A rectilinear drawing of a graph is one in which edges are drawn using straight line segments, and more generally, a *stretchable* drawing is one that is homeomorphic to a rectilinear drawing. Fáry's Theorem [14, 31, 27], a classic result in graph theory, asserts that drawings of simple graphs with no crossings between edges are stretchable.

A *good drawing* is one in which every two incident edges are not crossed, every pair of edges cross at most once, and no crossing point is in three edges. In [29], Thomassen extended Fáry's Theorem by characterizing stretchable good drawings of graphs in which every edge is crossed at most once. This characterization is in terms of forbidding two configurations, known as the B and W configurations, shown in Figure 2.1.

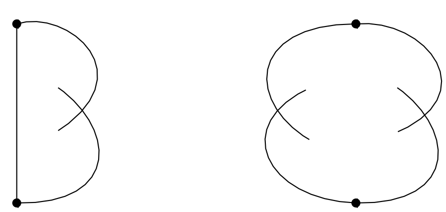


Figure 2.1: B and W configurations.

Thomassen's characterization is a partial answer to the general problem of determining which drawings are stretchable. There is not likely to be a complete characterization, because the closely related problem of stretchability of arrangements of pseudolines is NP-hard.

A *pseudoline* is an unbounded open arc in the plane whose complement is disconnected. In particular, lines are pseudolines, and any pseudoline is the image of a line under a homeomorphism of the plane into itself. An *arrangement of pseudolines* is a set of pseudolines in which every two intersect in exactly one point, and their intersection point is a crossing. We allow multiple pseudolines crossing at one point.

Mnëv showed [21, 22] that it is NP-hard to determine whether an arrangement of pseudolines is stretchable (in fact, he showed that the stretchability problem is $\exists\mathbb{R}$-hard). The problem of stretching arrangements of pseudolines can be reduced to the problem of deciding whether the drawing of a graph is stretchable: given an arrangement of pseudolines, consider a big circle containing all the intersection points between the pseudolines. Cut off the two ends of each pseudoline outside this circle to obtain a drawing of multiple copies of K_2. This drawing is stretchable if and only if the arrangement is stretchable.

Our interest in stretchable drawings comes from studying crossing numbers. The *crossing number* of a graph G is the minimum number $cr(G)$ of crossings in a drawing of G in the plane. If we restrict our drawings to be stretchable, then this minimum is the *rectilinear crossing number* $\overline{cr}(G)$.

One of the main open problems in the study of crossing numbers is the Harary-Hill Conjecture [16], which states that the crossing number of K_n is equal to

$$H(n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor. \tag{2.1}$$

There are drawings of K_n with $H(n)$ crossings. In 2004 it was shown in [2] and in [19] that $\overline{cr}(K_n) \geq H(n)$, and in fact, the tools used to show this lower bound come from the study of arrangements of pseudolines, so naturally the proof generalizes to pseudolinear drawings. A *pseudolinear* drawing of a graph is one in which each edge-arc can be extended to a pseudoline, resulting in an arrangement of pseudolines. Clearly stretchable drawings are pseudolinear.

In the recent study of crossing numbers, pseudolinear drawings have played a predominant role, as they are treated as a combinatorial generalization of rectilinear drawings. For instance, in parallel results from [6] and [9], it was shown that in rectilinear and pseudolinear drawings of K_n with as few crossings as possible, the outer face is bounded by a

3-cycle. This connection led Balogh et al. [9] to conjecture that $\widetilde{cr}(K_n) = \overline{cr}(K_n)$ (where $\widetilde{cr}(K_n)$ denotes the *pseudolinear crossing number* of K_n).

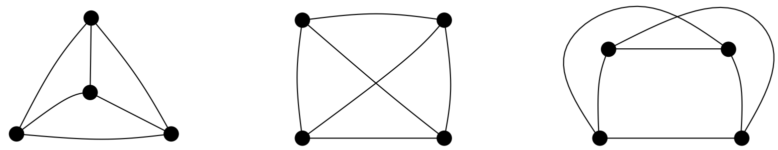


Figure 2.2: Non-isomorphic drawings of K_4 in the plane.

Pseudolinear drawings of K_n can be characterized in terms of a forbidden minimal subdrawing. There are exactly three non-homeomorphic drawings of K_4 (shown in Figure 2.2), and exactly one of them (the one on the right) is not pseudolinear. It was shown in [5] and [8] that a drawing of K_n is pseudolinear if and only if it does not include the non-pseudolinear drawing of K_4.

The aim of this chapter to generalize the previous result by characterizing pseudolinear drawings of all graphs by describing a set of minimal forbidden subdrawings. One of the difficulties to finding such a set of drawings is the fact that there are infinitely many of them, such as the ones in Figure 2.3. Thomassen already made this observation for rectilinear drawings [29].

The description of minimal forbidden drawings requires a refined understanding of drawings and also of what we meant by saying "minimal", leading us to study sets of strings. A *string* σ is the image $f([0,1])$ of a continuous function $f : [0,1] \to \mathbb{R}^2$ that restricted to $(0,1)$ is injective; in other words, strings are arcs that are allowed to self-intersect only at their *ends* $f(0)$ and $f(1)$. If no such self-intersection exists, then σ is *simple*. Most of the time we will consider simple strings, although considering non-simple strings will come in handy for technical reasons.

A set of strings Σ is in *general position* if, for every two strings σ, $\sigma' \in \Sigma$ (i) $\sigma \cap \sigma'$ is a finite set of points in $\mathbb{R}^2$; and (ii) each point in $\sigma \cap \sigma'$ is either a crossing between σ and σ', or an end of either σ or σ'. For instance, the set of edge-arcs of a good drawing of a graph is a set of strings in general position, but not all the sets of strings in general position come in this fashion: a string might include end points of other strings in its interior.

For a set Σ of strings in general position, its *underlying plane graph* $G(\Sigma)$ is the plane graph obtained from Σ by replacing the crossings between strings and the end points of every string in Σ by vertices. Our main result below characterizes when a set of strings in general position can be extended to an arrangement of pseudolines.

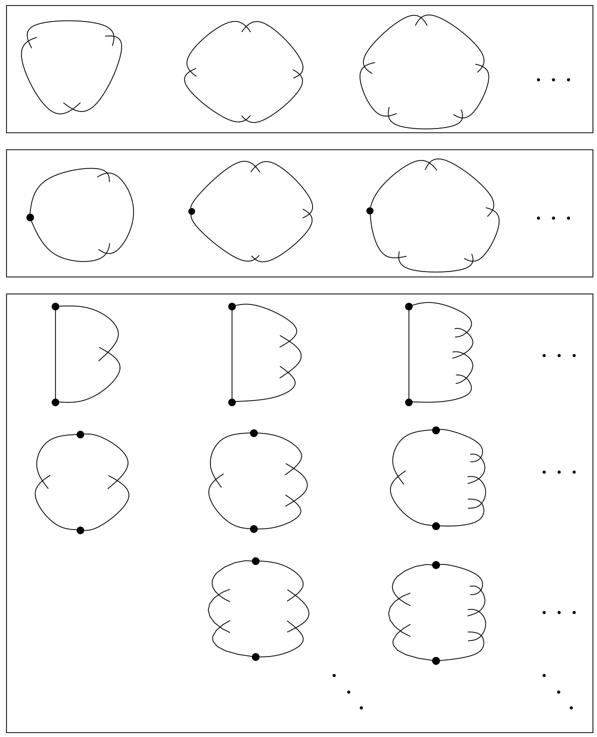


Figure 2.3: Obstructions to pseudolinear drawings.

Theorem 2.1.1. *A set of strings Σ in general position can be extended to an arrangement of pseudolines if and only if, for each cycle C in the underlying plane graph $G(\Sigma)$ of Σ, there are at least three vertices with the property that the edges incident to the vertex that are included in the closed disk bounded by C belong to distinct strings in Σ.*

For instance, let C be the unique cycle in the underlying plane graph in any of the drawings in Figure 2.3. There are at most two vertices of G in C, represented as black dots. The strings incident with such a vertex are distinct and contained in the closed disk bounded by C. The vertices represented as crossings do not satisfy this property: they are incident with four edges in the disk bounded by C, and these four edges consist of two strings that cross at this vertex. Theorem 2.1.1 implies that none of the drawings in Figure 2.3 is pseudolinear. Surprisingly, we will show, as a consequence of Theorem 2.1.1, that every non-pseudolinear drawing contains one of the configurations in Figure 2.3 as a subdrawing.

Theorem 2.1.2. *Let D be a non-pseudolinear good drawing of a graph H. Then there is a* *subset S of edge-arcs in $\{D[e] : e \in E(H)\}$, such that each $\sigma \in S$ has a substring $\sigma' \subseteq \sigma$ for which $\bigcup_{\sigma \in S} \sigma'$ is one of the drawings in Figure 2.3.*

Cycles that have fewer than three vertices as in Theorem 1 are the *obstructions* of $G(\Sigma)$ (this definition will be made more precise at the beginning of Section 2.2). Showing that when $G(\Sigma)$ has obstructions, then Σ cannot be extended to an arrangement of pseudolines, is the first part of Section 2.2. The rest of Section 2.2 is devoted to show that if $G(\Sigma)$ has no obstructions, then Σ can be extended to an arrangement of pseudolines. The proof of two technical lemmas used in the proof of Theorem 2.1.1 are deferred to Section 2.3. In Section 2.4, we describe a simple algorithm that finds an obstruction in polynomial time. In Section 2.5, by applying Theorem 2.1.1, we prove that a drawing of a complete graph K_n is pseudolinear if and only if it does not contain the B configuration in Figure 2.1. This result is equivalent to the characterizations of pseudolinear drawings of K_n given in [5] and [8], but its proof is simpler. At the end, in Section 2.6, we show how Theorem 2.1.2 easily follows from Theorem 2.1.1, together with some concluding remarks.

The present chapter is a collaborative work with Julien Bensmail and Bruce Richter.

2.2 Proof of Theorem 2.1.1

In this section, we use Lemmas 2.2.4 and 2.2.5 (proved in the next section) to prove Theorem 2.1.1. As we enter into the subject, we need some notation that is useful in identifying an obstruction. Let C be a cycle of a plane graph G and let v be a vertex of C. The *rotation at v inside C* is the counterclockwise ordered list $e_0, e_1..., e_k$ of edges incident with v that are included in the closed disk bounded by C, with e_0 and e_k both in C. Likewise, the *rotation at v outside C* is defined as the counterclockwise ordered list e_k, $e_{k+1}, \dots, e_0$ of edges incident with v included in the closure of the exterior of C.

In the case $G = G(\Sigma)$ for some set Σ of strings in general position, a vertex v in a cycle C of $G(\Sigma)$ is *reflecting* in C if at least two edges in the rotation at v inside C belong to the same string (Figure 2.4a). The alternative is that v is a *rainbow*, in which case all the edges of its rotation inside C are in different strings (Figure 2.4b). In these terms, an *obstruction* is a cycle with at most two rainbows.

2.2.1 Sets of strings with obstructions are not extendible

The following observation will be used in this subsection and also in Theorem 2.5.1. If C is a cycle in $G(\Sigma)$, where Σ is a set of strings in general position, then $\delta(C)$ is the set of

(a) A reflecting vertex (b) A rainbow

Figure 2.4: A representation of reflecting and rainbow vertices, where each string in $G(\Sigma)$ has assigned a unique colour.

vertices in C for which their two incident edges in C belong to two distinct strings in Σ. Note that if $|\delta(C)| < 3$ for some cycle C, then either $|\delta(C)| = 2$ and two strings intersect more than once, or $|\delta(C)| \leq 1$ and some string is self-crossed. Both these possibilities are forbidden in good drawings.

Observation 2.2.1. *Let Σ be a set of simple strings in general position in which every two strings intersect at most once. Let:*

(a) C be an obstruction of $G(\Sigma)$ for which $|\delta(C)|$ is as small as possible;

(b) $x \in \delta(C)$;

(c) e be an edge in C incident to x;

(d) $\sigma \in \Sigma$ be the string containing e; and

(e) σ' be the component of $\sigma \setminus e$ containing x.

Then $\sigma' \cap C = \{x\}$.

Proof. By way of contradiction, suppose that $\sigma' \cap C$ includes a point distinct from x. This in particular implies that $\sigma' \neq \{x\}$, and, because $x \in \delta(C)$, the points of $\sigma' \setminus \{x\}$ near x are not in C. Let P be the path in $G(\Sigma)$ obtained by traversing σ', starting at x, and stopping the first time we encounter a point $y \in C \cap (\sigma' \setminus \{x\})$. Note that $y \in V(C)$ and that P is drawn in either the interior or the exterior of C.

First, suppose that P is drawn in the interior of C. Let C_1 and C_2 be the cycles obtained from the union of P and one of the two xy-subpaths in C. We may assume C_1 includes e. Each of $C_1 - P$ and $C_2 - P$ has a vertex in $\delta(C)$; otherwise one of C_1 or C_2 would be included in at most two strings, implying that a string is self-crossing or two strings intersect twice. Therefore $|\delta(C_1)|$ and $|\delta(C_2)|$ are strictly smaller than $|\delta(C)|$. Then, by assumption, C_1 and C_2 are not obstructions.

None of the vertices in $P - y$ is a rainbow for C_1 ($P \subseteq \sigma'$ and x is reflecting in C_1, so the interior rotations of the vertices in $P - y$ include two edges in σ). Since all the vertices in $C_1 - V(P)$ that are rainbow in C_1 are also rainbow in C, C_1 has at most two rainbows in $V(C_1) \setminus V(P)$. These last two observations and the fact that C_1 is not an obstruction, together imply that C_1 has three rainbows: two of them are in $V(C_1) \setminus V(P)$ and the other is y.

Now we look at the rainbows in C_2. Because C has two rainbows in $C_1 - V(P)$ and any rainbow in $V(C_2) \setminus V(P)$ for C_2 is rainbow for C, C_2 has no rainbow in $V(C_2) \setminus V(P)$. All the interior vertices of P are reflecting in C_2, so C_2 has at most two rainbows. This contradicts that C_2 is not an obstruction.

Secondly, suppose that P is drawn in the exterior of C. Let C_{out} be the cycle bounding the outer face of $C \cup P$. The cycle C_{out} is the union of P and one of the two xy-paths in C, and, in both cases, as $x \in \delta(C) \setminus \delta(C_{out})$ and $P - y \subset \sigma$, $|\delta(C_{out})| < |\delta(C)|$. Every vertex in $P - y$ is reflecting in C_{out} (this statement follows from the fact that the rotation of a vertex inside a cycle also includes the edges of the cycle incident with the vertex). Moreover, every vertex in $V(C_{out}) \setminus (V(P - y))$ that is a rainbow in C_{out} is also a rainbow in C. These two facts imply that C_{out} has at most as many rainbows as C; hence C_{out} is an obstruction. This contradicts the fact that C minimizes $|\delta|$. □

Next we show that, if a set of strings contains an obstruction, then it is not pseudolinear.

Observation 2.2.2. *If Σ is a set of strings in general position and $G(\Sigma)$ has an obstruction, then Σ cannot be extended to an arrangement of pseudolines.*

Proof. By way of contradiction, suppose that there is a set of strings Σ that can be extended to an arrangement of pseudolines and $G(\Sigma)$ has an obstruction C. Consider an extension of Σ to an arrangement of pseudolines, and then cut off the two infinite ends of each pseudoline to obtain a set of strings Σ' extending Σ, and in which every two strings in Σ' cross. In $G(\Sigma')$, there is a cycle C' that represents the same simple closed curve as C. Because C' is obtained from subdividing some edges of C, C' has fewer than three rainbows. Therefore, we may assume that $\Sigma = \Sigma'$ and $C = C'$. Now, the ends of every string in Σ are degree-one vertices in the outer face of $G(\Sigma)$.

As every string in Σ is simple, and no two strings intersect more than once, $|\delta(C)| \geq 3$. We will assume that C is chosen to minimize $|\delta(C)|$.

Since C is an obstruction, there is at least one vertex $x \in \delta(C)$ reflecting inside C. Let $e \in E(C)$ be an edge incident to x, and suppose that σ is the string including e. Traversing σ along e through x, we encounter another edge $e' \subseteq \sigma$ incident to x. Because $x \in \delta(C)$, e' is not in C. Suppose that e' is drawn in the outer face of C. As x is reflecting inside C, there exists a string $\bar{\sigma}$ that includes two edges in the rotation at x inside C. However, σ and $\bar{\sigma}$ tangentially intersect at x, contradicting that the strings in Σ are in general position. Therefore e' is drawn inside C.

Let y be the end of σ contained in the component of $\sigma \setminus e$ containing x. Since $|\delta(C)|$ is minimum, Observation 2.2.1 implies that the component of $\sigma \setminus e$ having x and y as ends have all its points, with the exception of x, in the inner face of C. However, y is drawn in the inner face of C, contradicting that the ends of all the strings in Σ are incident with the outer face of $G(\Sigma)$. □

2.2.2 Extending sets of strings with no obstructions

In this subsection we prove that a set of strings with no obstructions can be extended to an arrangement of pseudolines. We restate Theorem 2.1.1 using our new terminology.

Theorem 2.2.3. *A set of strings Σ in general position can be extended to an arrangement of pseudolines if and only if $G(\Sigma)$ has no obstructions.*

Proof. We showed in Observation 2.2.2 that if $G(\Sigma)$ has an obstruction, then Σ cannot be extended to an arrangement of pseudolines. For the converse, suppose that $G(\Sigma)$ has no obstructions.

We start by reducing the proof to the case in which the point set $\bigcup \Sigma$ is connected. If $\bigcup \Sigma$ is not connected, then we add a simple string to Σ, connecting two points in distinct components of $G(\Sigma)$, and so that it is included inside a face of $G(\Sigma)$. This operation: reduces the number of components; does not create obstructions; and ensures that any pseudolinear extension of the new set of strings shows the existence of one for Σ. We continue adding strings in this way until we obtain a connected set of strings and we redefine Σ to be this set. Thus, we may assume $\bigcup \Sigma$ is connected.

Our proof is algorithmic, and consists of repeatedly applying one of the three steps described below.

- **Disentangling Step.** If a string $\sigma \in \Sigma$ has an end a with degree at least 2 in $G(\Sigma)$, then we slightly extend the a-end of σ into one of the faces incident with a.

- **Face-Escaping Step.** If a string $\sigma \in \Sigma$ has an end a with degree 1 in $G(\Sigma)$, and is incident with an inner face, then we extend the a-end of σ until we intersect some point in the boundary of this face.
- **Exterior-Meeting Step.** Assuming that all the strings in Σ have their two ends in the outer face and these ends have degree 1 in $G(\Sigma)$, we extend the ends of two disjoint strings so that they meet in the outer face.

We can always perform at least one of these steps, unless the strings are pairwise intersecting and all of them have their ends in the outer face (in this case we extend their ends to infinity to obtain the desired arrangement of pseudolines). Each step increases the number of pairwise intersecting strings. Henceforth, our aim is to show that, as long as there is a pair of non-intersecting strings, then one of these three steps may be performed without adding an obstruction. The proof is now divided into three parts that can be read independently.

Disentangling Step. *Suppose that $\sigma \in \Sigma$ has an end a with degree at least 2 in $G(\Sigma)$. Then we can extend the a-end of σ into one of the faces incident to a without creating an obstruction.*

Proof. An edge f of $G(\Sigma)$ incident with a is a *twin* if there exists another edge $f' \neq f$ incident with a such that both f and f' are part of the same string in Σ. Observe that the edge $e_0 \subseteq \sigma$ incident with a is not a twin.

The fact no pair of strings tangentially intersect at a tells us that if (f_1, f_1') and (f_2, f_2') are pairs of corresponding twins, then f_1, f_2, f_1', f_2' occur in this cyclic order for either the clockwise or counterclockwise rotation at a. Thus, we may assume that the twins at a are labeled as $f_1, \dots, f_t, f_1', \dots, f_t'$, and that this is their counterclockwise order occurrence when we follow the rotation at a starting at e_0. In such a case, (f_i, f_i') is a pair of corresponding twins for $i = 1, \dots, t$.

In order to avoid tangential intersections when twins are present, every valid extension of σ at a must cross into the angle between f_t and f_1' not containing e_0.

Let $(e_1, \dots, e_k)$ be the list of non-twin edges between f_t and f_1' in the counterclockwise rotation at a; this list might be empty. In the case there are no twins, we set f_t and f_1' both equal to e_0, so $(e_1, \dots, e_k)$ is all the edges incident with a other than e_0.

We consider all the feasible extensions for σ: for each $i \in \{1, ..., k-1\}$, we let Σ_i be the set of strings obtained from extending σ by adding a small bit of arc α_i starting at a, and continuing into the face between e_i and e_{i+1}. Let Σ_0 be the set of strings obtained by

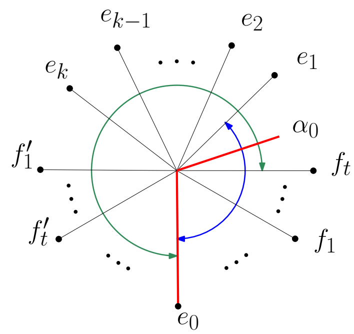


Figure 2.5: Substrings included in the disk bounded by C_0.

adding an arc α_0 in the face between f_t and e_1, and let Σ_k be obtained by adding an arc α_k in the face between e_k and f'_1.

Seeking a contradiction, suppose that, for each $i \in \{0, ..., k\}$, $G(\Sigma_i)$ contains an obstruction C_i. The cycle C_i does not include the bit of arc α_i as an edge, so C_i is a cycle in $G(\Sigma)$. This cycle is not an obstruction in $G(\Sigma)$, although it becomes one when we add α_i. The reason explaining this conversion is simple: in $G(\Sigma)$, C_i has exactly three vertices not reflecting, and one of them is a. After α_i is added, a is now reflecting in C_i (witnessed by σ).

Understanding how cycles with exactly three rainbows may behave in an obstructionless set of strings is a crucial piece of the proof. In general, if v is a vertex in the underlying plane graph of a set of strings in general position, then a *near-obstruction at* v is a cycle with exactly three rainbows, and one of them is v. Each of the cycles C_0, C_1,...,C_k above is a near-obstruction at a in $G(\Sigma)$.

Both e_0 and α_0 are on the disk bounded by C_0, and since α_0 is not part of C_0, either $e_0, f_1, f_2, \ldots, f_t, e_1$ are on the same side of C_0 (blue bidirectional arrow in Figure 2.5) or all of $f_t, e_1, \ldots, e_k, f'_1, f'_2, \ldots, f'_1, e_0$ are in the same side of C_0 (green bidirectional arrow in Figure 2.5). Because α_0 is the only edge between f_t and e_1, we see that e_1 belongs to the first sublist (the blue one) and f_t belongs to the second list (the green one). In the second case both f_t and f'_t are in the disk bounded by C_0, showing that a is not a rainbow for C_0 in Σ. Therefore, all of $e_0, f_1, f_2, \ldots, f_t, e_1$ are in the disk bounded by C_0.

Regardless of the presence or absence of twins, we know that (e_0, e_1) occurs as a sublist of the rotation of a inside C_0. A symmetric argument shows that (e_k, e_0) occurs as a sublist of the rotation of a inside C_k.

Since (e_0, e_1) is a substring of the rotation at a inside C_0 and $(e_0, e_1, \ldots, e_k, f'_1)$ is not inside C_k, there is a largest $i \in \{0, 1, \ldots, k-1\}$ such that $(e_0, \ldots, e_{i+1})$ is inside C_i. The choice of i implies $(e_{i+1}, \dots, e_k, e_0)$ is inside C_{i+1}.

The next lemma states that the existencce of such a pair of cycles C_i and C_{i+1} is impossible, completing the proof. $\square$

Lemma 2.2.4. *Let Σ be a set of strings in general position. Suppose that C_1 and C_2 are cycles in $G(\Sigma)$ that are near-obstructions at v, so that the rotation at v inside C_1 includes (as a sublist) the rotation at v outside C_2, and that the rotation at v inside C_2 includes the rotation at v outside C_1. Then $G(\Sigma)$ has an obstruction.*

We defer the proof of Lemma 2.2.4 to Section 2.3 as it is technical and it deviates our attention from the proof of Theorem 2.1.1.

Face-Escaping Step. *Suppose that there is a string σ that has an end a with degree 1 in $G(\Sigma)$, and a is incident to an inner face F. Then there is an extension σ' of σ from its a-end to a point in the boundary of F such that the set $(\Sigma \setminus \{\sigma\}) \cup \{\sigma'\}$ has no obstruction.*

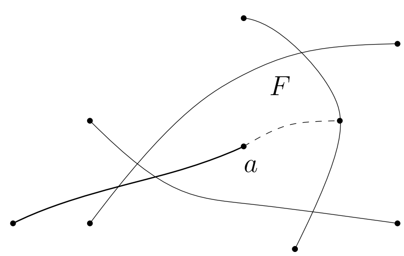


Figure 2.6: Face-Escaping Step.

Proof. Let W be the closed boundary walk $(x_0, e_1, \dots, e_n, x_n)$ of F such that $x_0 = x_n = a$ and F is to the left as we traverse W.

Sometimes it is only possible to extend σ by using an arc connecting a to a vertex in the middle of an edge. Figure 2.6 shows an example of this situation. Joining a to any x_i produces a second crossing between two strings; the simple closed curves enclosed by the union of these two strings is an obstruction. So for each edge e_i, let m_i be a point in e_i, between x_{i-1} and x_i.

Let P denote the list of points $(m_1, x_1, m_2, x_2, \dots, x_{n-1}, m_n)$. For each point p in P, let α_p denote an arc in F connecting a to p. (A given element of the point set of $G(\Sigma)$ might occur more than once in P. In particular, it is always the case that $x_1 = x_{n-1}$; see Figure 2.7.)

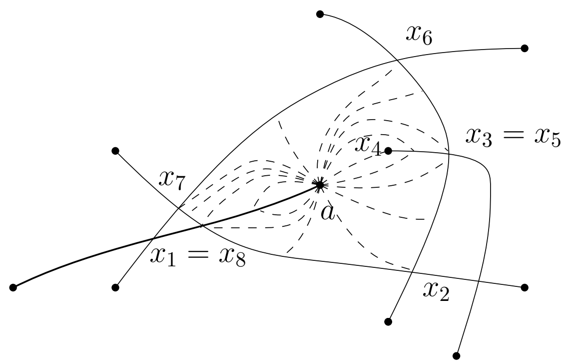


Figure 2.7: All possible extensions in the Face-Escaping Step.

For each point p in P, we consider the set of strings Σ_p obtained from Σ by extending σ using α_p. Let f_p be the edge $e_1 \cup \alpha_p$ in $G(\Sigma_p)$; it has ends x_1 and p. Also, let $\sigma^p = \sigma \cup \alpha_p$. The existence of obstructions in $G(\Sigma_p)$ is independent of how we draw α_p inside F. We will take advantage of this fact later on in the proof.

Seeking a contradiction, suppose that each $G(\Sigma_p)$ has an obstruction. Our next claim gives two sufficient conditions on p that imply that all the obstructions in $G(\Sigma_p)$ contain f_p.

Claim 1. *Let $p \in P$ be either one of $m_1, \dots, m_n$ or not in σ. Then every obstruction in $G(\Sigma_p)$ includes f_p.*

Proof. Let $p \in P$ be such that there is an obstruction C in $G(\Sigma_p)$ not including f_p.

First, we show that p is not a vertex in the middle of an edge in W. By contradiction, suppose that $p = m_i$ for some $i \in \{1, ..., n\}$. Since m_i is the only vertex whose rotation in $G(\Sigma)$ differs from its rotation in $G(\Sigma_{m_i})$, $m_i \in V(C)$. Consider the cycle C' of $G(\Sigma)$ obtained by replacing the subpath x_{i-1}, m_i, x_i of C by the edge $x_{i-1}x_i$. The inside rotation of each vertex in C' is the same as their rotation inside C. This shows that C' is an obstruction in $G(\Sigma)$, a contradiction.

Now suppose that p is not in the middle of an edge in W. Then C is a cycle in $G(\Sigma)$ and is not an obstruction in $G(\Sigma)$. The only vertex in $G(\Sigma_p)$ that has a rotation that is different from its rotation in $G(\Sigma)$ is p. Therefore p is a point in C that is reflecting inside C (witnessed by two edges included in σ^p), and is not reflecting in C with respect to $G(\Sigma)$. Exactly one of the two witnessing edges is in $G(\Sigma)$. So $p \in \sigma$. $\square$

More can be said about the obstructions in $G(\Sigma_p)$ for each point in P, but for this we need some terminology. If we orient an edge e in a plane graph, then the *sides* of e are either the points near e that are to the right of e, or the points near e to the left of e. Our next lemma shows that if $p \in P$, then all the obstructions in $G(\Sigma_p)$ include the same side of f_p in its interior face. We defer its proof to Section 2.3 to keep the flow of the current proof. For the convenience of the reader, we provide all the hypotheses in the statement.

Lemma 2.2.5. *Let Σ be a set of strings in general position. Let C_1 and C_2 be obstructions in $G(\Sigma)$ with $e \in E(C_1) \cap E(C_2)$. If C_1 and C_2 include distinct sides of e in their interior faces, then $G(\Sigma)$ has an obstruction not including e.*

The condition on the two cycles C_1 and C_2 containing distinct sides of e implies that e is incident with only interior faces of $C_1 \cup C_2$. The perspective of the cycles being on distinct sides of e is useful in the application, but what we really use in the proof of Lemma 2.2.5 is that e is not incident with the outer face of $C_1 \cup C_2$.

For each point $p \in P$, we will consider an obstruction C_p containing f_p; the choice of C_p will be more specific when $p \in \sigma$ (see below). It is important to know what side of f_p our obstruction C_p includes in its interior, so from now on we will assume f_p is oriented from x_1 to p. If $p \notin \sigma$, then we let C_p be any obstruction in $G(\Sigma_p)$ (Claim 1 guarantees that $f_p \in E(C_p)$). In case $p \in \sigma$, we choose C_p to be the unique cycle included in the drawing of σ^p. Note that for $p = x_1$, the interior of C_p includes the right of f_p, while for $p = x_{n-1}$, the interior of C_p includes the left of f_p (here we use the fact that the face F is bounded).

Our last observation implies that there are two consecutive vertices x_{i-1}, x_i in $W - a$ such that the interior of $C_{x_{i-1}}$ includes the right of $f_{x_{i-1}}$ and the interior of C_{x_i} includes the left of f_{x_i}.

Without loss of generality, suppose that the interior of C_{m_i} includes the left of f_{m_i} (otherwise we reflect our drawing in a mirror). To make the notation simpler, we let $x = x_{i-1}$ and $m = m_i$. We may assume that f_m is drawn near the left of f_x.

The next claim is the last ingredient to obtain a final contradiction.

Claim 2. *Exactly one of the following holds:*

(a) $x \in \sigma$ and $G(\Sigma_m)$ has an obstruction containing f_m whose interior includes a side that is distinct from the side included by C_m; or

(b) $x \notin \sigma$ and $G(\Sigma_x)$ has an obstruction containing f_x whose interior includes a side of f_x that is distinct from the side included by C_x.

Proof. First, suppose that $x \in \sigma$. For (2.a) we have two cases depending on whether $x_{i-1}x_i$ is an edge in C_x.

Case a.1 *$x_{i-1}x_i$ is not in C_x.*

In this case we consider the cycle C'_m obtained by replacing in C_x the edge f_x by the path $P = (x_1,\ f_m,\ m,\ mx,\ x)$. Since $x \in \sigma$, by the choice of C_x, all the edges in C_x are in σ^x. Therefore all the edges in C'_m, with the possible exception of mx, are in σ^m. Thus C'_m is an obstruction in $G(\Sigma_m)$.

It remains to show that the interior of C'_m includes the right side of f_m. Note that $C_x \cup P$ consists of three internally disjoint x_1x-paths, and because some points in P are near the left side of f_x, P is in the outer face of C_x. The face of $f_x \cup P$ that is to the right of f_m is included in the inner face F, so it is bounded. This implies the interior face of C'_m includes the right of f_m. Since the interior of C_m includes the left of f_m, C'_m and C_m are obstructions including distinct sides of f_m.

Case a.2. *$x_{i-1}x_i$ is in C_x.*

In this case, $(x_1,\ f_x,\ x,\ xx_i,\ x_i)$ is a subpath of C_x. We let C'_m be the cycle obtained by replacing this path by $P = (x_1, f_m, m, mx_i, x_i)$. Since $x \in \sigma$, the way we choose C_x implies that all the edges in C_x are in σ^x. So all the edges in C'_m are in σ^m, and C'_m is an obstruction. An argument similar to the one given in the previous case shows that the interior of C'_m includes the right side of f_m. Thus the interior of C_m and C'_m include distinct sides of f_m.

Turning to (2.b), let us suppose that $p \notin \sigma$. We split the proof into two cases depending on whether x is in C_m.

Case b.1. *x is in C_m.*

First, we redraw f_x and f_m inside F so that $f_x \cap f_m = \{x_1\}$. Let T be the triangle bounded by f_x, f_m and xm. The interior face of T is to the left of f_x and to the right of f_m. Consider the mx-path P of C_m that does not include the edge f_m. Since the interior face of T is a subset of F, P is drawn in the closure of the exterior of T (possibly $P = (m, mx, x)$).

Let C be the simple closed curve bounded by $P \cup f_x \cup f_m$. We claim that the interior of C is on the left of f_x. In the alternative, suppose that the interior of C is on the right of f_x. Then $C' = P + xm$ is a cycle of $G(\Sigma_m)$ including f_x and f_m in its interior. The xx_1-path P' of C_m that does not include m, is an arc connecting x_1 to x inside C'. Thus, $V(C') \subseteq V(C_m)$ and the closed disk bounded by C' includes C_m. These two observations together imply that C' has at most as many rainbows as C_m, and hence, C' is an obstruction of $G(\Sigma_m)$ not including f_m. Claim 1 asserts that all the obstructions in $G(\Sigma_m)$ include f_m, a contradiction. Thus the interior of C is on the left of f_m.

From our last observation, it follows that P' is an arc connecting x_1 and x in the exterior of C. Because the interior of $C_m = P' \cup f_m \cup P$ is on the left of f_m, the interior of the cycle $C'_x = P' + f_x$ is on the left of f_x.

Now we show that C'_x is an obstruction. Note that $V(C'_x) \subseteq V(C_m)$ and that the closed disk bounded by C'_x includes C_m. Then, every rainbow in C'_x is a rainbow in C_m, and hence C'_x is an obstruction. The cycles C_x and C'_x are obstructions including distinct sides of f_x in their interiors, as claimed.

Case b.2. *x is not in C_m.*

In this case we let C'_x be the cycle obtained by replacing the path (x_1, f_m, m, mx_i, x_i) in C_m by the path $P = (x_1, f_x, x, xx_i, x_i)$ in $G(\Sigma_x)$. Let α be the subarc of P joining x_1 to m. As the points of α near x_1 are drawn on the left of f_m, and α is internally disjoint to C_m, α connects x_1 and m in the exterior of C_m. Since the interior face of $\alpha \cup f_m$ is on the left of f_x, the interior face of C'_x is on the left of f_x.

To show that C'_x is an obstruction, note that the disk bounded by C'_x includes C_m and that $V(C'_x) \setminus \{x\} \subseteq V(C_m)$. Thus all the rainbows of C'_x in $V(C'_x) \setminus \{x\}$ are also rainbows in C_m. The rotation of x inside C'_x is the list (xx_i, f_x), and, because $x \notin \sigma$, x is a rainbow in C'_x, and is not a vertex of C_m. To compensate, we note that m is a rainbow in C_m that is not in $V(C_x)$: if m is not rainbow, both f_m and xx_i are included in σ, implying that $x \in \sigma$. This shows that C'_x has at most as many rainbows as C_m. Thus C'_x is an obstruction. Again, the interiors of C_x and C'_x include distinct sides of f_x. □

By Claim 2, for some $p \in \{x, m\}$, $G(\Sigma_p)$ has obstructions including both sides of f_p (and when $p = x$, we can guarantee that $p \notin \sigma$). Lemma 2.2.5 implies that $G(\Sigma_p)$ has an obstruction not including f_p. Since either $p \notin \sigma$ or $p = m$, this last statement contradicts Claim 1. □

Exterior-Meeting Step. *Suppose that all the strings in Σ have their ends on the outer face of $G(\Sigma)$ and that all the ends have degree 1 in $G(\Sigma)$. Then either all the strings are pairwise intersecting, and then Σ can be extended to an arrangement of pseudolines, or we can extend two disjoint strings so that these strings intersect without creating an obstruction.*

Proof. We start by considering a simple closed curve $\mathcal{O}$ containing all the ends of the strings in Σ, and that is otherwise disjoint from $\bigcup \Sigma$. We construct this curve by connecting each pair of vertices with degree 1 that are consecutive in the boundary walk of the outer face. To connect these pairs we use an arc whose interior is included in the outer face, near the portion of the boundary walk between the two vertices.

Suppose σ_1, σ_2 are two disjoint strings in Σ. For $i = 1, 2$, let a_i, b_i be the ends of σ_i. Since σ_1 and σ_2 do not intersect inside $\mathcal{O}$, their ends do not alternate as we traverse $\mathcal{O}$ in counterclockwise order. We may assume, by relabeling if necessary, that the ends occur in the order a_1, b_1, b_2, a_2.

We extend the a_i-ends of σ_1 and σ_2 so that they meet in a point p in the outer face. We do this extension so that the two added segments are in the outer face, and, more importantly, so that the interior face of the simple closed curve bounded by the added segments and the a_2a_1-arc in $\mathcal{O}$ not containing $\{b_1, b_2\}$, does not include the inner face of $\mathcal{O}$. In Figure 2.8 we show the right and wrong way to extend, respectively.

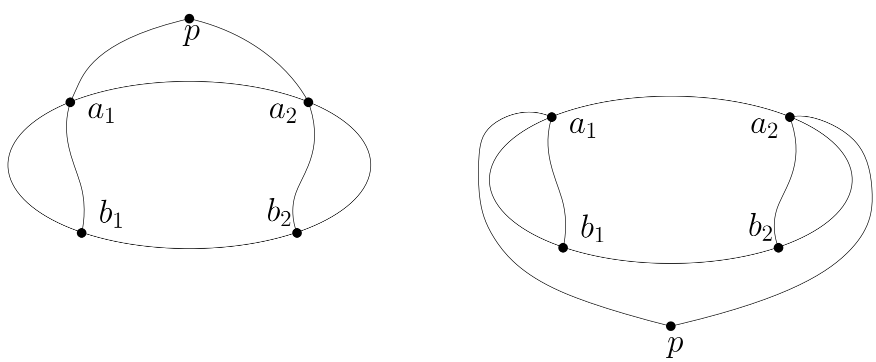


Figure 2.8: The right and wrong way to extend in the Exterior-Meeting Step.

We denote the new set of strings obtained as above by Σ'. To show that Σ' has no obstruction, we consider a cycle C in $G(\Sigma')$. If C does not contain p, then C is a cycle in $G(\Sigma)$, and so is not an obstruction in $G(\Sigma')$. Now suppose that p is in C.

The idea is to find three rainbows in C. To get the first one, we consider the path P_1 obtained by traversing C, starting at p, continuing along the path induced by σ_1, and stopping just before we reach a first vertex not in σ_1. Let c_1 be the last vertex in P_1, and let d_1 be the neighbour of c_1 in C that is not in P_1.

Claim 1. *The cycle C has a rainbow included in the disk Δ_1 bounded by σ_1 and the a_1b_1-arc of $\mathcal{O}$ not containing a_2.*

Proof. The vertex d_1 is in one of the two bounded faces of $\mathcal{O} \cup \sigma_1$. Suppose that d_1 is in the face F that is bounded by σ_1 and the a_1b_1-arc of $\mathcal{O}$ containing a_2 and b_2. The rotation at c_1 inside C does not include two edges in the same string σ, as otherwise σ and σ_1 tangentially intersect at c_1. Therefore, when $d_1 \in F$, c_1 is a rainbow of C in Δ_1.

Now suppose that d_1 is in Δ_1. Let P'_1 be the path of C starting at c_1 and the edge c_1d_1, and ending at the first vertex we encounter that is in σ_1. The cycle C' enclosed by P'_1 and σ_1 is not an obstruction, so it has at least three rainbows. The vertices in $C' - V(P'_1)$ are reflecting inside C' because their rotations inside C' contain two edges in σ. Hence at least one internal vertex of P'_1 is a rainbow in C'. This vertex is also a rainbow in C, and is included in Δ_1. □

Considering σ_2 instead of σ_1, Claim 1 yields a second rainbow in C inside an analogous disk Δ_2. The third rainbow is p, showing that C is not an obstruction. □

Since the Disentangling Step, Face-Escaping Step and Exterior-Meeting Step can be performed without creating new obstructions, either: one of these steps can be performed to increase the number of pairwise intersecting strings in Σ; or the strings in Σ are pairwise intersecting and all of them have their ends in the outer face, which implies that Σ can be extended to an arrangement of pseudolines. □

2.3 Proof of Lemmas 2.2.4 and 2.2.5

We deferred the proofs of Lemmas 2.2.4 and 2.2.5, both essential in the proof of Theorem 2.1.1, to this section.

Our next observation follows immediately from the definition of rainbow, and it will be repeatedly used in the next proofs.

Useful Fact. *Let Σ be set of strings in general position. Let v be a vertex that is in both the cycles C and C' of $G(\Sigma)$ such that the rotation at v inside C includes the rotation at v inside C'. If v is a rainbow in C, then v is a rainbow in C'.*

Recall that a *near-obstruction* at v is a cycle C (in the underlying graph of a set of strings) that has precisely three rainbows, one of which is v. In Figure 2.9, we depict (up to symmetries) how two near-obstructions may intersect at v. In each of the nine diagrams, v is represented as a black dot, while the interiors of the near-obstructions are represented as dotted and dashed lines. In our next lemma, we will consider two near-obstructions at

v that intersect only as in the last three diagrams, where every small open disk centered at v is included in the union of the disks bounded by the two near-obstructions. In the statement, an equivalent description is given in terms of the local rotation at v.

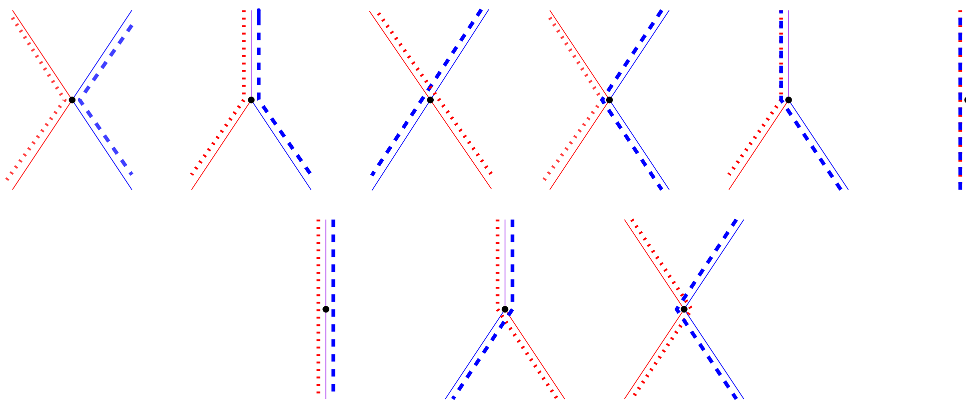


Figure 2.9: Two near obstuctions at v.

Lemma 2.2.4. *Let Σ be a set of strings in general position. Suppose that C_1 and C_2 are cycles in $G(\Sigma)$ that are near-obstructions at v, so that the rotation at v inside C_1 includes (as a sublist) the rotation at v outside C_2, and that the rotation at v inside C_2 includes the rotation at v outside C_1. Then $G(\Sigma)$ has an obstruction.*

Proof. In order to obtain a contradiction, suppose that $G(\Sigma)$ has no obstructions and that it contains such cycles C_1, C_2. The conditions on the rotation at v imply that every edge incident with v is in the interior of either C_1 or C_2. Thus, v is not incident with the outer face of $C_1 \cup C_2$.

Our next goal is to show that $C_1 \cap C_2$ has at least two vertices. If e is an edge of C_1 incident with v, then either e is an edge of C_2 or e is inside C_2. In the former case $|V(C_1) \cap V(C_2)| \geq 2$, thus we may assume that both edges of C_1 incident with v are inside C_2. If v is the only vertex in $V(C_1) \cap V(C_2)$, then $C_1 - v$ is in the interior of C_2, and hence the edges of C_2 incident with v are not in the rotation at v inside C_1, a contradiction. Thus $|V(C_1) \cap V(C_2)| \geq 2$. It follows that $C_1 \cup C_2$ is 2-connected; in particular, its outer face is bounded by a cycle C_{out}.

The Useful Fact applied to $C = C_{out}$ and to each $C' \in \{C_1, C_2\}$, shows that every vertex that is a rainbow in C_{out} is also a rainbow in each of the cycles in $\{C_1, C_2\}$ containing it. By assumption, C_{out} is not an obstruction, so it has at least three rainbows. The preceding two sentences imply that we may choose the labelling such that two of them, say p and q, are also rainbows in C_1. Neither p nor q is v and C_1 is a near-obstruction. Thus, p and q are the only rainbows of C_{out} that are in C_1.

Since $v \notin V(C_{out})$, C_1 has a subpath P_v containing v in which only the ends of P_v are in C_{out}. Since v is not in the outer face of C_{out}, P_v is included in the inner face of C_{out}. We let u and w be the ends of P_v, and let Q^1_{out}, Q^2_{out} be the uw-paths of C_{out}. The cycle C_1 is inside one of the two disks bounded by P_v and one of Q^1_{out} and Q^2_{out}. By symmetry, we may assume that C_1 is included in the disk bounded by $Q^1_{out} \cup P_v$. In this case Q^2_{out} is a subpath of C_2.

Our desired contradiction will be obtained by finding three rainbows in C_2 distinct from v. The first is relatively easy to find: if $C_1 - (P_v)$ is the uw path in C_1 distinct from P_v, we consider the cycle $(C_1 - (P_v)) \cup Q^2_{out}$. The disk bounded by $(C_1 - (P_v)) \cup Q^2_{out}$ contains the one bounded by C_1. Then the Useful Fact applied to $C = (C_1 - (P_v)) \cup Q^2_{out}$ and $C' = C_1$, implies that each vertex in $C_1 - (P_v)$ that is rainbow in $(C_1 - (P_v)) \cup Q^2_{out}$ is also rainbow in C_1. Since C_1 has at most two rainbows in $C_1 - (P_v)$, namely p and q, $(C_1 - (P_v)) \cup Q^2_{out}$ must have a third rainbow r_1 in the interior of Q^2_{out}. The interiors of the disks bounded by C_2 and $(C_1 - (P_v)) \cup Q^2_{out}$ are on the same side of Q^2_{out}; thus r_1 is a rainbow for C_2.

To find another rainbow in C_2, consider the edge e_u of C_2 incident to u and not in Q^2_{out}. We claim that either u is a rainbow in C_2 or that e_u is not included in the closed disk bounded by $P_v \cup Q^2_{out}$. Looking for a contradiction, suppose that u is reflecting in C_2 and that e_u is included in the disk. Then we can find two edges in the rotation at u, included in the disk bounded by $P_v \cup Q^2_{out}$, that belong to the same string σ. The vertex u is not reflecting in C_1, as else, we would find another pair of edges in the rotation at u inside $Q^1_{out} \cup P_v$, and included in a different string σ'; in this case, σ and σ' tangentially intersect at u, a contradiction. Therefore u is a rainbow in C_1, so u is one of p and q. This implies that u is a rainbow in C_{out}, and hence, a rainbow in C_2, a contradiction.

If u is a rainbow in C_2, then this is the desired second one. Otherwise, the preceding paragraph shows that e_u is not in the closed disk bounded by $P_v \cup Q^2_{out}$. In this latter case, e_u is in a path P_u that starts at u, ends at u' in P_v and is otherwise disjoint from P_v.

Note that $u' \neq w$, as otherwise $C_2 = P_u \cup Q^2_{out}$ and we have the contradiction that v is not in C_2. Let C_u be the cycle consisting of P_u and the uu'-subpath uP_vu' of P_v.

Claim 1. *If P_u does not have a rainbow of C_u in its interior, then:*

(a) C_u and C_2 are near-obstructions at v satisfying the conditions in Lemma 2.2.4; and

(b) the closed disk bounded by the outer cycle of $C_u \cup C_2$ contains fewer vertices than the disk bounded by C_{out}.

Proof. Suppose that all the rainbows of C_u are located in uP_vu'. Since C_u is not an obstruction, at least one of them is an interior vertex of uP_vu'. Each vertex in the interior of uP_vu' that is a rainbow in C_u, is also a rainbow in C_1. As v is the only vertex in the interior of P_v that is a rainbow in C_1, v is the only rainbow of C_u that is in the interior of uP_vu'. Since C_u is not an obstruction, u, u' and v are the only rainbows of C_u, and C_u is a near-obstruction at v. The rotation at v inside C_u is the same as inside C_1, so C_u and C_2 satisfy the conditions in Lemma 2.2.4.

Let C'_{out} be the outer cycle of $C_u \cup C_2$. Since $C_u \cup C_2 \subseteq C_1 \cup C_2$, the exterior of C_{out} is included in the exterior of C'_{out}. This shows that the disk bounded by C_{out} includes the disk bounded by C'_{out}.

If both p and q are in C_2, then p, q and r_1 are rainbows in C_2, and also distinct from v, contradicting that C_2 is a near-obstruction for v. Thus, we may assume $p \notin C_2$. Then p is not in $P_u \subseteq C_2$ and, since p is not an interior vertex of P_v, $p \notin V(C_u)$. Since p is in C_{out}, and p is not in $C_u \cup C_2$, p is in the outer face of $C_u \cup C_2$. Then p is in the disk bounded by C_{out} but not by C'_{out}, as required. □

The proof of the existence of the additional two rainbows in C_2 is by induction on the number of vertices in the closed disk bounded by C_{out}. If u is not a rainbow in C_2 and P_u does not have a rainbow of C_2 in its interior, then Claim 1 implies C_u and C_2 make a smaller instance and we are done. Thus, we may assume one of them yields the next additional rainbow.

In the same way, either the induction applies or the last rainbow comes by considering the edge of $C_2 - Q^2_{out}$ incident with w. It follows that v, r_1, and these two other vertices are four different rainbows in C_2, contradicting the fact that C_2 is a near-obstruction. □

Although the statements and proofs of Lemmas 2.2.4 and 2.2.5 are similar, some subtle differences make it hard to find a statement encapsulating both results. For instance, Lemma 2.2.5 assumes that $G(\Sigma)$ has obstructions, while finding an obstruction is the conclusion of Lemma 2.2.4. We sketch the proof of Lemma 2.2.5, emphasizing such differences. It would be interesting to find a common theory behind these two lemmas.

Lemma 2.2.5. *Let Σ be a set of strings in general position. Let C_1 and C_2 be obstructions in $G(\Sigma)$ with $e \in E(C_1) \cap E(C_2)$. If C_1 and C_2 include distinct sides of e in their interior faces, then $G(\Sigma)$ has an obstruction not including e.*

Sketch of the proof. We start assuming that such cycles exist and that every obstruction includes e.

By assumption, $C_1 \cap C_2$ has at least two vertices and, therefore, $C_1 \cup C_2$ is 2-connected. Thus, its outer face is bounded by a cycle C_{out}.

The Useful Fact shows that every rainbow in C_{out} is a rainbow in each of the cycles C_1 and C_2 containing it.

Since C_1 and C_2 include different sides of e, it follows that e is not in C_{out}. Therefore C_{out} is not an obstruction. Thus, C_{out} has at least three rainbows, and by our previous observation, we may choose the labelling such that two of them, say p and q, are also rainbows in C_1. Because C_1 is an obstruction, p and q are the only rainbows in C_1.

Then C_1 has a subpath P_e of containing e and in which only the ends u and w of P_e are in C_{out}. Let Q^1_{out} and Q^2_{out} be the uw-paths of C_{out}. We may assume that C_1 is drawn in the disk bounded by $Q^1_{out} \cup P_e$.

Let $C_1 - (P_e)$ be uw-path in C_1 that is not P_e. Note that p and q are the only vertices in $C_1 - (P_e)$ that are rainbows in $(C_1 - (P_e)) \cup Q^2_{out}$. Since $(C_1 - (P_e)) \cup Q^2_{out}$ is not an obstruction, the interior Q^2_{out} has a vertex r_1 that is a rainbow of $(C_1 - (P_e)) \cup Q^2_{out}$. This vertex r_1 is also a rainbow of C_2.

Let e_u be the edge incident to u in C_2 that is not in Q^2_{out}. As we did in Lemma 2.2.4, we can show that either u is a rainbow in C_2 or that e_u is not included in the disk bounded by $P_e \cup Q^2_{out}$. We assume the latter situation, as in the former we found our desired second rainbow in C_2.

Let P_u be the subpath of C_2 starting at u, continuing on e_u, and ending on the first vertex $u' \in V(P_e) \cap V(C_2)$ distinct from u. Note that $u' \neq w$, as otherwise $C_2 = P_u \cup Q^2_{out}$ and we have the contradiction that e is not in C_2. Let C_u be the cycle consisting of P_u and the uu'-subpath uP_eu' of P_e.

We claim that either P_u has an interior vertex that is a rainbow in C_2 or that there is a pair of cycles C'_1 and C'_2 satisfying the conditions in Lemma 2.2.5, but with fewer vertices in the closed disk bounded by the outer cycle of $C'_1 \cup C'_2$ than in the disk bounded by C_{out}.

Suppose that none of the interior vertices in P_u is a rainbow in C_2. Because the interior of P_e has no vertices that are rainbows in C_1 (as p and q are the only rainbows of C_1), the interior of uP_eu' has no vertices that are rainbows in C_u. Therefore C_u is an obstruction, and $C'_1 = C_u$ and $C'_2 = C_2$ is a pair of obstructions including both sides of e. As $C_u \cup C_2 \subseteq C_1 \cup C_2$, the closed disk bounded by the C_{out} contains the closed disk bounded by the outer cycle of $C_u \cup C_2$. Not both of p and q are in the outer cycle of $C_u \cup C_2$, as both p and q would be part of C_2, concluding that C_2 has three rainbows p, q and r_1, and contradicting that C_2 is an obstruction.

From the previous paragraph, either C_u and C_2 is a smaller instance, and we are done by induction on the number of vertices in the closed disk bounded by C_{out}, or we found our second rainbow of C_2 in the interior of P_u.

In the same way, either the induction applies or the last rainbow comes by considering an edge of $C_2 - Q^2_{out}$ incident with w. It follows that r_1, and these other two vertices are three different rainbows in C_2, contradicting that C_2 is an obstruction. $\square$

2.4 Finding obstructions in polynomial time

In this section we describe a polynomial-time algorithm that determines whether a set of strings has an obstruction. We will assume that our input is the underlying plane graph $G(\Sigma)$ of a set Σ of simple strings in general position, and that every string in Σ is identified as a path in $G(\Sigma)$ (see notation below).

The key idea behind the algorithm is simple: either find an obstruction in the outer boundary of $G(\Sigma)$ or find a vertex in the outer boundary whose removal reduces our problem into a smaller instance.

We start by describing the vertex removal operation. Suppose that x is a vertex of $G(\Sigma)$ incident to the outer face of $G(\Sigma)$. For each $\sigma \in \Sigma$, we consider the path P_σ of $G(\Sigma)$ representing σ. Let $P_\sigma - x$ be the plane graph obtained from P_σ by removing x and the edges of P_σ incident to x (if $x \notin P_\sigma$, then $P_\sigma - x = P_\sigma$). Each component of $P_\sigma - x$ is either a vertex that represents an end of σ, or a string. Let $S_{\sigma,x}$ be the set of string components of $P_\sigma - x$ and let $\Sigma - x = \bigcup_{\sigma\in\Sigma} S_{\sigma,x}$. Note that $G(\Sigma - x)$ can be obtained from $G(\Sigma)$ by removing x and the edges incident to x, and then suppressing the degree-2 vertices whose incident edges belong to the same string in Σ, as well as removing remaining degree-0 vertices (Figure 2.10 illustrates this process).

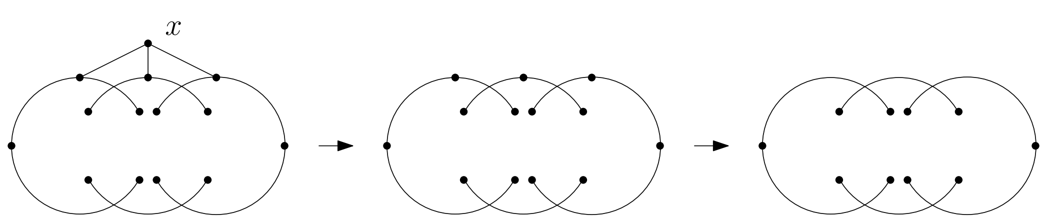


Figure 2.10: From Σ to $\Sigma - x$.

The next lemma is the key property used in the algorithm.

Lemma 2.4.1. *Let Σ be a set of simple strings in general position and let x be a vertex incident with the outer face. Then there is a $1-1$ correspondence between the obstructions in $G(\Sigma)$ not containing x and the obstructions of $G(\Sigma - x)$. Moreover, corresponding obstructions are the same simple closed curve.*

Proof. In general, there is a natural correspondence between cycles in $G(\Sigma)$ not containing x and cycles in $G(\Sigma - x)$: if C is a cycle in $G(\Sigma)$ not containing x, then every edge of C is not incident with x, and hence every edge is part of a string in $\Sigma - x$. Thus, there is a cycle C' in $G(\Sigma - x)$ that represents the same simple closed curve as C. Conversely, each cycle C' in $G(\Sigma - x)$ is a simple closed curve in $\bigcup(\Sigma - x) \subseteq \bigcup \Sigma$, and hence, there is a cycle C in $G(\Sigma)$ representing the same simple closed curve as C'.

To complete the proof it is enough to show that any two cycles C, C' that correspond as above have the same rainbows. Since $G(\Sigma - x)$ is obtained from suppressing and removing vertices in a subgraph of $G(\Sigma)$, $V(C') \subseteq V(C)$. Thus, $V(C) \setminus V(C')$ consists of suppressed and removed vertices in the process of converting $G(\Sigma)$ into $G(\Sigma - x)$. Since $x \notin V(C)$, if $v \in V(C)$ is suppressed, then the two edges of C incident to v belong to the same string in Σ. Therefore, none of the vertices in $V(C) \setminus V(C')$ is a rainbow in C.

Every rainbow in C is also a rainbow in C' because every two edges of $G(\Sigma - x)$ that are included in distinct strings of Σ are also included in distinct strings in $\Sigma - x$.

Conversely, suppose that $v \in V(C) \cap V(C')$ is reflecting in C. Let $\sigma \in \Sigma$ be a string including two edges of $G(\Sigma)$ in the rotation at v inside C. Since x is drawn in the exterior of C, these two edges are part of the same string in $\Sigma - x$, and hence v is reflecting inside C'.

Therefore every rainbow of C' is a rainbow of C, and thus, C and C' have the same rainbows. □

A vertex in $G(\Sigma)$ is an *outer-rainbow* if it is in the outer boundary and all the edges in its rotation belong to different strings. Note that every outer-rainbow is a rainbow for all the cycles in $G(\Sigma)$ that contain it.

An *outer cycle* is a cycle of $G(\Sigma)$ that has all its edges incident to the outer face of $G(\Sigma)$. For any graph $G(\Sigma)$, a *block* of $G(\Sigma)$ is a maximal connected subgraph of $G(\Sigma)$ with no cut-vertex. If $G(\Sigma)$ is connected with at least two vertices, then each block is either an edge or is 2-connected. In the latter case, the outer face of the block is bounded by a cycle of the block.

We find obstructions by solving an auxiliary problem: finding obstructions including one or two fixed outer-rainbows. The next subroutine (Algorithm 1) describes how to find an obstruction containing two fixed outer-rainbows. Below we discuss its correctness.

Algorithm 1: Finding obstructions through two fixed outer-rainbows.

Data: $G(\Sigma)$ and two outer-rainbows x and y.

Result: Either an obstruction containing x and y or that no such obstruction exists.

```
1  repeat
2      if there is no cycle containing x and y then
3          return G(Σ) has no obstruction containing x and y;
4      end
5      Find the outer cycle C containing x and y;
6      while C is not the outer boundary of G(Σ) do
7          Pick w ∈ V(G(Σ)) \ V(C) incident with the outer face;
8          Σ ⟵ Σ − w
9      end
10     if C has a rainbow z ∉ {x, y} in G(Σ) then
11         Σ ⟵ Σ − z;
12     else
13         return C;
14     end
15 until V(G(Σ)) = {x, y};
16 return G(Σ) has no obstruction containing x and y.
```

To see that Algorithm 1 is correct, observe that when Step 2 does not apply, then Step 5 can be performed: if there is a cycle containing x and y, then, as x and y are incident to the outer face of $G(\Sigma)$, the outer boundary of the block containing x and y is an outer cycle C containing x and y. Every obstruction $\mathcal{C}$ through x and y is drawn in the closed disk bounded by C. Lemma 2.4.1 guarantees that if we remove a vertex in the outer boundary that is not in C (Step 7) and we update Σ (Step 8), then $\mathcal{C}$ (or more precisely, the cycle in the new $G(\Sigma)$ that is the same simple closed curve as $\mathcal{C}$) is an obstruction through x and y.

In Step 10, if x and y are the only rainbows of C, then C is an obstruction returned in Step 13. Else, C has a rainbow $z \notin \{x, y\}$. Any obstruction $\mathcal{C}$ through x and y does not contain z, and hence removing z and updating Σ (Step 11) does not change the fact that $\mathcal{C}$ is an obstruction in the new $G(\Sigma)$. This algorithm terminates as the number of vertices in $G(\Sigma)$ is always decreasing.

We now turn to Algorithm 2, used as subroutine in the main algorithm. Its correctness again easily follows from Lemma 2.4.1.

Algorithm 2: Finding obstructions through a fixed outer-rainbow.

Data: $G(\Sigma)$ and an outer-rainbow vertex x.

Result: Either an obstruction containing x or that no such obstruction exists.

repeat
    if *there is no cycle containing* x **then**
        return $G(\Sigma)$ has no obstruction containing x;
    end
    Find an outer cycle C containing x;
    if C *has an outer-rainbow* $y \neq x$ **then**
        Run Algorithm 1 on $(G(\Sigma), x, y)$;
        if $G(\Sigma)$ *has an obstruction* D *including* x *and* y, **then**
            return D;
        end
        $\Sigma \longleftarrow \Sigma - y$;
    else
        return C;
    end
until $V(G(\Sigma)) = \{x\}$;
return $G(\Sigma)$ *has no obstruction containing* x.

Finally we present the algorithm to find obstructions, whose correctness also relies on Lemma 2.4.1.

Algorithm 3: Finding obstructions.

Data: $G(\Sigma)$.

Result: Either finds an obstruction or that no such obstruction exists.

repeat
    if $G(\Sigma)$ *has no cycles* **then**
        return $G(\Sigma)$ has no obstructions;
    end
    Find an outer cycle C;
    if C *has no rainbows* **then**
        return C;
    end
    Pick a rainbow x in C (x is outer-rainbow in $G(\Sigma)$);
    Run Algorithm 2 on $(G(\Sigma), x)$;
    if $G(\Sigma)$ *has an obstruction* D *including* x **then**
        return D;
    end
    $\Sigma \longleftarrow \Sigma - x$;
until $G(\Sigma) = \emptyset$;
return $G(\Sigma)$ *has no obstructions.*

2.5 Pseudolinear drawings of K_n

In this section we present a simple proof of a characterization of pseudolinear drawings of complete graphs (Theorem 2.5.1), equivalent to the ones given in [5] and [8]. One of the equivalences, Corollary 2.5.2, is shown at the end of this of this section. The other equivalence (Lemma 4.4.8) is deferred to Subsection 4.4.2 of Chapter 4, where it plays an important role in the study of pseudospherical drawings of K_n.

Theorem 2.5.1. *A good drawing of a complete graph is pseudolinear if and only if it does not include the B configuration (see Figure 2.1).*

Proof. The unique cycle in a B configuration is an obstruction, so, by Theorem 2.1.1, no pseudolinear drawing of K_n can include it. Conversely, suppose that D is a good drawing of K_n that is not pseudolinear. Let $\Sigma = \{D[e] \; : \; e \in E(K_n)\}$ be the set of edge-arcs, and let $G(\Sigma)$ be its underlying plane graph. In order to avoid confusion between vertices and edges of K_n and $G(\Sigma)$, vertices in $G(\Sigma)$ are called *points*, and edges of $G(\Sigma)$ are *segments*. Because D is good, each point is either in $V(K_n)$ or a crossing.

For every cycle C in $G(\Sigma)$, we let $\delta(C)$ be the set of points in C for which their two incident segments in D belong to distinct edges in Σ. Theorem 2.1.1 implies that $G(\Sigma)$ has an obstruction C. We choose our obstruction C so that $|\delta(C)|$ is as small as possible.

Since D is good, $|\delta(C)| \geq 3$ and, because C is an obstruction, at most two vertices in $\delta(C)$ are rainbows in C. Consider a point $x \in \delta(C)$ that is reflecting inside C. Note that x is a crossing. Let σ_1 and σ_2 be the two edge-arcs in Σ crossed at x. We traverse σ_1, starting at x, continuing on the segment of σ_1 included in the interior of C, until an end $a_1 \in V(K_n)$ of σ_1 is reached. Likewise we define a_2 for σ_2. Henceforth, we refer to a_1 and a_2 as the *internal vertices* corresponding to the crossing x. The following claim explains why we call them "internal".

Claim 1. *Let $x \in \delta(C)$ be a point reflecting inside C. Then the two internal vertices corresponding to x are in the interior of C.*

Proof. Let a_1 be an internal vertex corresponding to x, and suppose σ_1 is the edge-arc including both x and a_1. Let σ_1' be the substring of σ, having x and a_1 as endpoints. Applying Observation 2.2.1 to our obstruction C, with $\sigma = \sigma_1$ and $\sigma' = \sigma_1'$, we obtain that $\sigma_1' \cap C = \{x\}$. Since points of σ_1' near x are in the interior face of C, $\sigma_1' \setminus \{x\}$ is included in the interior of C. In particular, a_1 is in such a face. □

Now we look at the points in $\delta(C)$ that are not reflecting inside C. If x is one of them, then x is a vertex or a crossing. Suppose that x is a crossing. Let σ_1, σ_2 be the edge-arcs crossing at x. Because x is not reflecting inside C, one of the two segments at x included in σ_1 is in the outer face of C. We traverse σ_1, starting in x, and continuing in the outer face until we reach an end b_1 of σ_1. Likewise we define b_2 for σ_2. These vertices b_1, b_2 are the *external* vertices corresponding to the crossing x.

Claim 2. *Let x be a crossing in $\delta(C)$ that is not reflecting inside C, and let σ be an edge-arc including x and an external vertex b of x. If σ' is the substring of σ connecting x to b, then $\sigma' \setminus \{x\}$ is included in the outer face of C.*

Proof. Applying Observation 2.2.1 to C, σ_1, and σ', we see that $\sigma' \cap C = \{x\}$. Since the points of σ_1' near x are in the outer face of C, $\sigma_1' \setminus \{x\}$ is included in the outer face of C. □

It is convenient, in the case when x is a vertex of K_n, to let x be its own external vertex.

Henceforth we refer to the vertices of K_n that are internal to some crossing in C as the *internal vertices* of C, and likewise, the *external vertices* of C are the vertices of K_n that are external to some crossing or to a vertex in C.

Claim 3. *Every segment in C is included in an edge-arc whose ends are either internal or external vertices of C.*

Proof. Any segment s of C is contained in a subpath P of C whose ends are in $\delta(C)$ but is otherwise disjoint from $\delta(C)$. This path P is part of an edge-arc $\sigma \in \Sigma$. Let $a \in V(K_n)$ be one of the ends of σ, and suppose that x is the first end of P that we encounter when we traverse σ from a to the other end of σ. If σ is reflecting at x, then a is internal. If σ is not reflecting at x, then a is external. Likewise, the other end of σ is internal or external. □

Suppose that K_n has a vertex y that is neither external nor internal to C. Then, by our previous claim, the underlying plane graph of $D[K_n - y]$ contains a cycle whose drawing is $D[C]$ and is an obstruction. Thus, $D[K_n - y]$ is not pseudolinear, and applying induction on n, we obtain that $D[K_n - y]$ has a B configuration. Henceforth we assume that all the vertices of K_n are either internal or external to C.

Claim 4. *Either the outer face of D is bounded by a cycle of K_n or D has a B configuration.*

Proof. Suppose that the outer face of D is not bounded by a cycle of K_n. Then the outer face is incident to a crossing $\times$ between two edge-arcs σ_1 and σ_2. Let K be the crossing K_4 induced by the ends of σ_1 and σ_2. The drawing $D[K]$ has exactly five faces, four of them incident to $\times$. Exactly one of the faces incident to $\times$ includes the outer face of D. Such a face of $D[K]$ is bounded by portions of σ_1, σ_2, and an edge e of K_n connecting an end of σ_1 to an end of σ_2. The drawing induced by σ_1, σ_2 and $D[e]$ is a B configuration. □

Claims 1 and 4 imply that the outer cycle of D consists of only external vertices of C. Every external vertex either is associated with a crossing that is not reflecting inside C, or is itself a vertex of K_n in C. Because C has at most two points not reflecting inside C, and each of them has at most two external vertices, there are at most four points in the outer cycle of D. Thus the outercycle is a 3- or 4-cycle of K_n.

As C has at least three external vertices (in the outer cycle), $\delta(C)$ has precisely two points p and q not reflecting inside C. The outer cycle of D has an edge uv, where u is external to p and v is external to q (possibly $u = p$ or $q = v$).

Consider the pq-path P in $G(\Sigma)$, starting at p, continuing on the edge-arc connecting p to u, then following the edge uv until we reach v, and ending by following the edge-arc connecting v to q. We finish our proof by considering two cases, depending on whether uv is a segment of C.

Case. *uv is not a segment of C.*

In this case, there exists a point $w \in D[uv] \setminus D[C]$. As $D[uv]$ is part of the outer cycle, it contains neither crossings nor vertices in its interior, so the arcs in $D[uv]$ connecting w to the ends u and v are internally disjoint from C. From Claim 2, it follows that the pu- and the qv-subpaths of P are internally disjoint from C. Thus P is an arc connecting p and q in the outer face of C.

Consider the cycle C' obtained from the union of P and the pq-path of C that lies in the outer face of $D[C \cup P]$.

We will show that C' is an obstruction by showing that u and v are the only rainbows of C'. If $p \neq u$, then the edge-arc σ connecting p and u shows that every point in $(P - u) \cap \sigma$ is reflecting inside C'. Analogously, if $q \neq v$, the points distinct from v in the edge-arc connecting q and v, are reflecting inside C'. Thus the internal points in P, with the exception of u and v, are not reflecting. The same holds for the points in $C' - P$, as these points are not reflecting inside C (recall that p and q are the only rainbows of C). Thus u and v are the only rainbows of C'.

Note that all the segments of C' are included in edges whose ends are u, v or interior points of C. So if y is a vertex in the outercycle of D distinct from u and v, $D[K_n - y]$ also includes $D[C']$ as an obstruction, implying that $D[K_n - y]$ is not pseudolinear. Again, by induction on n, we obtain that $K_n - y$ has a B configuration.

Case. *uv is a segment of C.*

In this case, as u, v are vertices of K_n in C, they are rainbows of C. Since p and q are the only rainbows, $p = u$, $q = v$, and $D[uv]$ is a segment of C. Then, all the segments of C are included in edge-arcs whose ends are u, v or interior points of C. Again, remove a vertex in the outer cycle of D distinct from u and v to obtain a non-pseudolinear drawing of K_{n-1} in which, by induction, we find a B configuration. □

We conclude this section by showing a slightly different way to characterize pseudolinear drawings in terms of a four points property. This local property plays a central role in Chapter 4.

Corollary 2.5.2. *A good drawing D of K_n in the plane is pseudolinear if and only if, for every 3-cycle T and for every vertex v drawn in the bounded face of $D[T]$, the three edges connecting v to the vertices of T are contained in the disk bounded by $D[T]$.*

Proof. Suppose first that D is not a pseudolinear drawing of K_n. From Theorem 2.5.1 we know that there exists a path $P = (x, y, z, w)$ that is drawn in D as a B configuration (Figure 2.1). Since $D[\{x, y, z, w\}]$ is a crossing K_4 in which xy crosses zw, the edge xz is not crossed in $D[\{x, y, z, w\}]$, and hence it connects x to z through the bounded face of $D[P]$. In this case (x, y, z, x) is a 3-cycle, containing w in its interior face, and such that $D[wz]$ is not contained in the disk bounded by this 3-cycle.

Conversely, suppose that there exist a 3-cycle $T = (x, y, z, x)$ and a vertex w for which at least one of the edges joining w to a vertex of T, say $D[wz]$, is not contained in the disk bounded by $D[T]$. The edge wz crosses at least one edge of T, and because D is good, it only crosses xy. Exactly one of x and y is not incident with the outer face F of $D[T + wz]$. Remove from $D[T + wz]$ the edge connecting z to the one of x and y not incident with F to obtain a B configuration. This shows that D is not pseudolinear. □

2.6 Concluding remarks

In our initial attempts to formulate Theorem 2.1.1, we intended to characterize non-pseudolinear good drawings of graphs by means of having at least one of the configurations in Figure 2.3 as a subdrawing. We obtain this as an easy consequence of Theorem 2.1.1. We sketch its proof.

Theorem 2.1.2. *Let D be a non-pseudolinear good drawing of a graph H. Then there is a subset S of edge-arcs in $\{D[e] : e \in E(H)\}$, such that each $\sigma \in S$ has a substring $\sigma' \subseteq \sigma$ for which $\bigcup_{\sigma \in S} \sigma'$ is one of the drawings in Figure 2.3.*

Proof. Take C an obstruction of the underlying plane graph associated to D. We choose C so that $|\delta(C)|$ is as small as possible. Decompose C into a cyclic sequence of paths $P_0, \dots, P_m$, where P_i connects two points in $\delta(C)$ and it is otherwise disjoint from $\delta(C)$. By using Observation 2.2.1, one can show that $P_0, \dots, P_m$ belong to distinct edge-arcs $\sigma_0, \dots, \sigma_m$, respectively. For each P_i, we consider the string σ_i', obtained by slightly extending the ends of P_i that are reflecting in C; we extend them along σ_i.

Let $x \in \delta(C)$ be an end shared by P_{i-1} and P_i. If x is reflecting in C, then x is a crossing between σ_{i-1} and σ_i. Moreover, the arcs added to P_{i-1} and P_i at x to obtain σ_{i-1}'

and σ_i' are in the interior of C. If x is a rainbow in C, then P_i and P_{i-1} are not extended at x, and x acts as one of the black dots in Figure 2.3. The rest of the points in $\delta(C)$ are crossings in $\bigcup_{i=0}^{m} \sigma_i'$ facing the interior of C. Since C has at most two rainbows, $\bigcup_{i=0}^{m} \sigma_i'$ is one in Figure 2.3. $\square$

There are pseudolinear drawings that are not stretchable. For instance, consider the Non-Pappus configuration in Figure 2.11. Nevertheless, as an immediate consequence of Thomassen's main result in [29], pseudolinear and stretchable drawings are equivalent, under the assumption that every edge is crossed at most once.

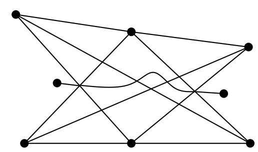


Figure 2.11: Non-Pappus configuration.

Corollary 2.6.1. *A drawing of a graph in which every edge is crossed at most once is stretchable if and only it is pseudolinear.*

Proof. Let D be a drawing of a graph in which every edge is crossed at most once. If D is stretchable then clearly it is pseudolinear. To show the converse, suppose that D is pseudolinear. Then D does not contain any obstruction, and in particular, neither of the B and W configurations in Figure 2.1 occur in D. In [29], it was shown that not containing the B and W configurations is equivalent to being rectilinear. $\square$

One can construct more general examples of pseudolinear drawings that are not stretchable by considering non-strechable arrangements of pseudolines. However, such examples seem to inevitably have edges crossing several times. This leads to two natural questions.

Question 1. *Is it true that if D is a pseudolinear drawing in which every edge is crossed at most twice, then D is stretchable?*

Question 2. *Is it true that if D is a pseudolinear drawing in which all the crossings involve a fixed edge, then D is stretchable?*

Chapter 3

Straight Line Drawings of Graphs with Crossings

3.1 Introduction

In 1993, Bienstock and Dean showed that every graph G with $\mathrm{cr}(G) \leq 3$, has $\mathrm{cr}(G) = \overline{cr}(G)$ [13]. A key idea used in their proof was that, under certain circumstances, one can modify an optimal drawing of a graph G (a drawing with fewest crossing pairs of edges as possible) to obtain a rectilinear drawing of G with the same number of crossings. This raises a natural question.

Question 3. *Given a drawing D of a graph G, can we find a rectilinear drawing $\overline{D}$ of G with the same crossing pairs of edges as D?*

For example, consider an n-cycle C drawn as a simple closed curve, with some of its chords drawn in the bounded side of C. Although, there are non-stretchable drawings of this type produced by considering a non-stretchable arrangement of pseudolines, as in Figure 3.1a, drawing C as a convex polygon in the plane induces a rectilinear drawing with the same pairs of crossing edges (Figure 3.1b).

This problem becomes more interesting when we let some chords of the n-cycle C be drawn in the outer face of C. We add the constraint that chords in the outer face are not crossing. In this case, it is less clear whether there exists a rectilinear drawing of the same graph with the chords drawn on the same sides of C. For instance, the 5-cycle with chords of Figure 3.2a is not pseudolinear. To obtain a rectilinear version of such a drawing, one would need to choose a different face to be the unbounded face, as in Figure 3.2b.

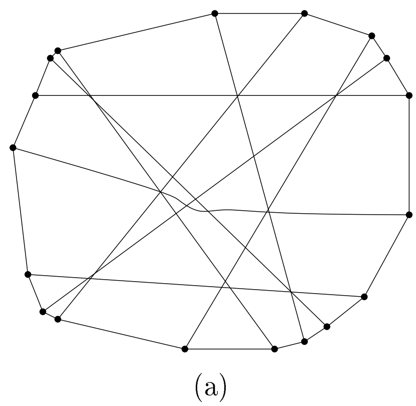


Figure 3.1: A non-stretchable and a rectilinear drawing with the same pair of crossing edges.

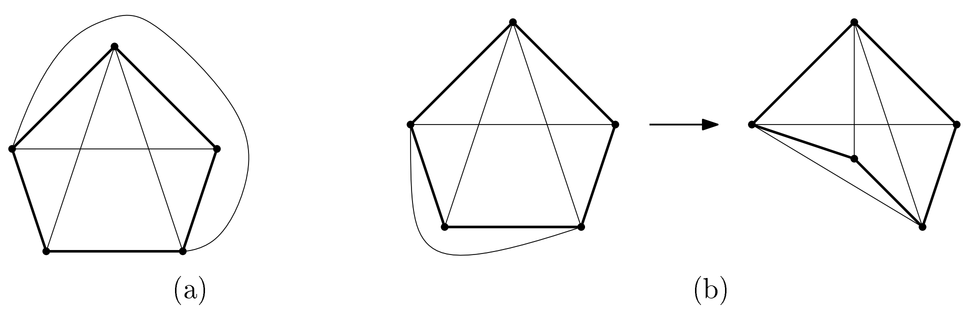


Figure 3.2

Our interest in this class of drawings (cycles with chords drawn in distinct sides of the cycle) came from realizing that many of the cases considered in the proof of Bienstock and Dean's Theorem can be argued by applying Tutte's Spring Theorem (Theorem 3.3.1) and by solving a more restricted version of Question 3 (explained below) for small instances of the drawings of cycles and chords. The main result in this chapter answers this restricted version of Question 3 for an extension of this class of drawings. To state this result, we need some terminology used throughout this chapter.

Let D be a drawing of a graph G. The subgraph induced by the edges of G not crossed in D is denoted as $\mathrm{pl}_D(G)$, or more simply as $\mathrm{pl}(G)$, if the drawing D is clear from the context. For a cycle C of $\mathrm{pl}(G)$, let:

- $\mathrm{Int}_D(C)$ be the subgraph induced by the vertices and edges of G drawn in the closed disk bounded by $D[C]$;

- $\mathrm{int}_D(C)$ be the subgraph induced by the vertices and edges of G drawn in the open disk bounded by $D[C]$;

- $\mathrm{Ext}_D(C)$ be the subgraph induced by the vertices and edges of G drawn in the closure of the outer face of $D[C]$; and

- $\mathrm{ext}_D(C)$ be the subgraph induced by the vertices and edges of G drawn in the outer face of $D[C]$.

The *chords* of C are the edges in $E(G) \setminus E(C)$ between two vertices in $V(C)$. A chord of C is *external* if it belongs to $\mathrm{Ext}_D(C)$, and *internal* if it belongs to $\mathrm{Int}_D(C)$. A cycle of G is *facial* if it bounds a face of D.

A drawing D of a graph G is *bundled* if it is good, and every crossing in D is the crossing between two internal chords of a facial cycle of $\mathrm{pl}_D(G)$. Every bundled drawing can be obtained from a planar embedding by selecting a set of facial cycles, and then adding, for each selected cycle, internal chords such that each of them cross at least once (they must cross in order not to be part of $\mathrm{pl}(G)$).

In a bundled drawing D of G, a *special cycle* S is a facial cycle of $\mathrm{pl}(G)$ bounding an inner-face and containing crossings in its interior. Every cycle bounding a face of $D[\mathrm{pl}(G)]$ is either facial or special. In Section 3.3, Figure 3.4a depicts a bundled drawing with the special cycles bounding coloured regions.

With the necessary terminology in hand, now we phrase the restricted version of Question 3: *Given a bundled drawing D of a graph G, is there a rectilinear drawing $\overline{D}$ of G satisfying the following conditions?*

(i) $pl_D(G) = pl_{\overline{D}}(G)$;

(ii) $D[pl_D(G)]$ *and* $\overline{D}[pl_{\overline{D}}(G)]$ *are homeomorphic; and*

(iii) for every special cycle S *of* D, $Int_D(S)$ *and* $Int_{\overline{D}}(S)$ *are the same graph.*

These three conditions strengthen the requirement of D and $\overline{D}$ having the same pairs of crossing edges. For the drawing in Figure 3.2a, the answer to this question is negative, while for the one in Figure 3.2b, the answer is positive.

One would expect that, as D and $\overline{D}$ satisfying (i)-(iii) are not very different from each other, a necessary condition to guarantee the existence of $\overline{D}$ is that D is pseudolinear. Our main result in this chapter shows that pseudolinearity is a necessary and sufficient condition when the special cycles are well-behaved.

Theorem 3.1.1. *Let D be a bundled drawing of a graph G. Suppose that for every two distinct special cycles S_1 and S_2, the intersection between S_1 and S_2 is either empty; a vertex; or an edge. Then D is pseudolinear if and only if there exists a rectilinear drawing $\overline{D}$ of G such that:*

(i) $pl_D(G) = pl_{\overline{D}}(G)$;

(ii) $D[pl_D(G)]$ *and* $\overline{D}[pl_{\overline{D}}(G)]$ *are homeomorphic; and*

(iii) for every special cycle S of D, $Int_D(S)$ and $Int_{\overline{D}}(S)$ are the same graph.

The results used in the proof Theorem 3.1.1 are presented in different sections of this chapter. In Section 3.2 we show that non-pseudolinear bundled drawings have a specific type of obstructions. In Section 3.3 we study a family of bundled drawings satisfying additional connectivity conditions; we refer to them as *crossing-webs*. We also introduce the notion of *elastic* crossing-web, that can be informally described as a drawing satisfying a generalized version of Tutte's Spring Theorem. We conclude Section 3.3 by showing a family of elastic crossing-webs. This family is part of the basis of the induction in the proof of Theorem 3.4.4, the main result of Section 3.4.

In Section 3.4 we show that a *full crossing-web* is elastic if and only if is pseudolinear. In Section 3.5 we conclude the proof of Theorem 3.1.1 by showing that every pseudolinear bundled drawing can be extended into a pseudolinear full crossing-web.

This section is based on author's original ideas, but it has been somewhat influenced by what will a more general work in collaboration with Richter and Thomassen.

3.2 Obstructions in bundled drawings

In this section we characterize non-pseudolinear bundled drawings of graphs. This characterization has the benefit of depending only on how special cycles and their chords are drawn. This will be used in the proofs of Observation 3.3.3 and Theorem 3.3.5.

Given a good drawing D of a graph G, let $H(D)$ be its underlying plane graph obtained from D, by replacing crossings with degree 4 vertices. In Chapter 2, we showed that D is pseudolinear if and only if $H(D)$ has no obstructions (obstructions are defined in Section 2.2). A *generalized* B is an obstruction C that contains an entire edge $D[xy]$ of G as a subpath of C. The obstructions in the figures in the first row of the third box in Figure 2.3 on page 7 are generalized Bs.

In Lemma 3.2.2 we will show that a bundled drawing, satisfying the conditions in Theorem 3.1.1, is not pseudolinear if and only if contains a generalized B. In fact, with the purpose of using this characterization in the next sections, we show that a more specific condition holds; this condition is phrased in Definition 3.2.1 and depicted in Figure 3.3.

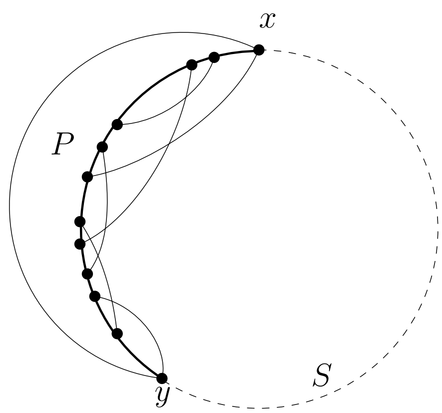


Figure 3.3: An external chord xy inducing a generalized B in S.

Definition 3.2.1. *Let D be a bundled drawing of a graph G and let S be a special cycle. Suppose that xy is an external chord of S and let P be the xy-path of S such that P is incident with the two inner faces of $D[S + xy]$. Then xy* induces a generalized B, *if, in the subdrawing D' of D obtained from $D[P + xy]$ by adding the internal chords of S with both ends in P, x and y are the only vertices of P in the outer boundary of D'.*

Observe that, if xy induces a generalized B, then the outer cycle of D' is a generalized B including the edge $D[xy]$.

Lemma 3.2.2. *Let D be a bundled drawing of a graph G, such that for every two distinct special cycles S_1 and S_2, their intersection is either empty, a vertex or an edge. Then D is not pseudolinear if and only there exists a special cycle S and an outer chord of S inducing a generalized B.*

Proof. From the remark preceding Lemma 3.2.2, it is clear that if an outer chord of some special cycle induces a generalized B, then D is not pseudolinear.

Conversely, suppose that D is not pseudolinear. Let $H(D)$ be the underlying plane graph of D (to avoid confusion, we call $V(H(D))$ and $E(H(D))$ the *points* and *segments* of $H(D)$). Let C be an obstruction of $H(D)$ minimizing $|\delta(C)|$ (recall from Subsection 2.2.1 that $\delta(C)$ is the set of points in $V(C)$ for which their two incident segments in C are included in distinct edge-arcs of $E(G)$).

Every point in C is either a vertex of G or a crossing in D. As every element in $V(C) \cap V(G)$ is a rainbow for C, $|V(C) \cap V(G)| \leq 2$.

Claim 1. *$|V(C) \cap V(G)| = 2$.*

Proof. By way of contradiction, suppose that $|V(C) \cap V(G)| \leq 1$. Then, for every pair of crossings that are points in $V(C)$, there is a path in C avoiding $V(G)$. Thus, every edge of $E(G)$ involved in a crossing in $V(C)$ is an inner chord of the same special cycle S. This in particular implies that the inner face F of C is part of the inner face of $D[S]$.

Let $\times$ be a point in $\delta(C)$ reflecting inside C ($\times$ is a crossing in D because elements in $V(G)$ are rainbows). The minimality of $|\delta(C)|$ and Observation 2.2.1 imply that at least two ends of the edges crossed at $\times$ are drawn in F. However, these two vertices cannot be drawn in the inner face of the special cycle $D[S]$, a contradiction. □

Thus, we may assume that there are two vertices x, $y \in V(C) \cap V(G)$. Since C is an obstruction and x and y are rainbows for C, all the points in $V(C) \setminus \{x, y\}$ are reflecting inside C. Let P_1 and P_2 be the xy-paths of C. Since G is simple, at least one of these paths, say P_1, contains a point in $\delta(C)$ distinct from x and y. The edges of $E(G)$ involved in a crossing in $V(P_1)$ are inner chords of the same special cycle S_1. Since x and y are ends of two of these edges, $\{x, y\} \subseteq V(S_1)$.

Claim 2. *P_2 is the edge $D[xy]$.*

Proof. In the alternative, there exist points of $V(P_2) \setminus \{x, y\}$ that are in $\delta(C)$. The edges of $E(G)$ involved in crossings of $V(P_2)$ are inner chords of the same special cycle S_2, and consequently $\{x, y\} \subseteq V(S_2)$.

Suppose that $S_1 = S_2$. Then, if we consider an element $\times \in \delta(C)$, distinct from x and y, Observation 2.2.1 implies that at least two of the four ends of the pair of edges crossing at $\times$ are drawn in the inner face F bounded by C. However, because $S_1 = S_2$, F is part of the inner face of $D[S_1]$, and consequently, there are vertices of G drawn in the interior of $D[S_1]$, a contradiction. Thus, we may assume that $S_1 \neq S_2$.

Since S_1 and S_2 have x and y in common, the assumption of how the special cycles intersect implies that $xy \in E(S_1) \cap E(S_2)$. In this case, xy is not crossed in D, and hence it is drawn as an xy-arc in one of the two faces of C. Regardless of how xy is drawn in D, either $P_1 \cup D[xy]$ or $P_2 \cup D[xy]$ is an obstruction C'. Moreover, since $D[xy]$ neither contains crossings nor vertices of G in its interior, $|\delta(C')| < |\delta(C)|$, contradicting the minimality of $|\delta(C)|$. This last contradiction implies that P_2 is the edge $D[xy]$. □

Claim 3. *The edge xy is an outer chord of S_1.*

Proof. Suppose not. Since x and y are vertices of S_1, xy is either an inner chord of S_1 or $xy \in E(S_1)$. Either of these possibilities, and the fact that $P_2 = D[xy]$, imply that the inner face F bounded by C is included in the inner face of $D[S_1]$. Now, the existence of crossings in $\delta(C)$ as interior points of P_1 and Observation 2.2.1 imply the existence of vertices of G drawn in F. Then there are vertices of G drawn in the inner face of $D[S_1]$, a contradiction. □

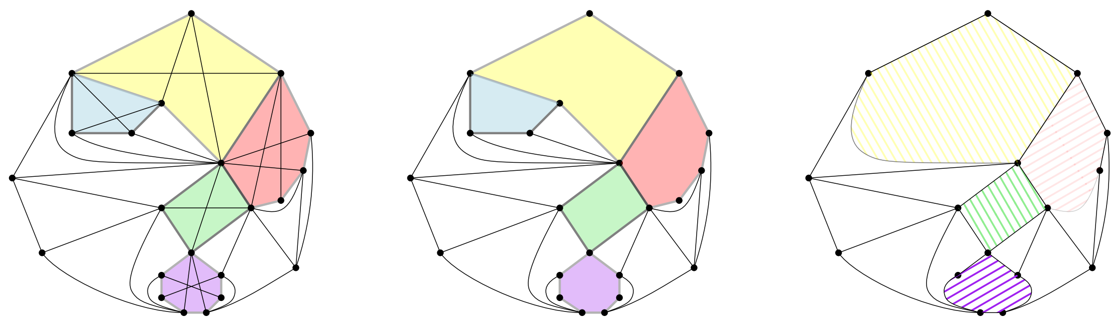
Finally, to see that xy induces a generalized B, consider the xy-path P of S_1 that is incident with the two inner faces of $D[S_1 + xy]$. Since P_1 is an xy-arc drawn in the inner face of $D[S_1]$, $D[P]$ is in the inner face of $D[xy] \cup P_1$.

From Observation 2.2.1 it follows that every edge $e \in E(G)$ that has a subsegment in P_1 has its ends in $V(P)$. Thus, every such e is an inner chord of S_1 with both ends in P. The simple closed curve C shows that x and y are the only vertices of $V(P)$ incident with the outer face of the subdrawing induced by $P + xy$ and the inner chords of S_1 with both ends in P. Thus xy induces a generalized B, as desired. □

3.3 Crossing-webs

In this section we introduce crossing-webs, which are bundled drawings satisfying additional connectivity conditions. In Lemma 3.5.1 of Section 3.5 we will show that every bundled drawing can be included as part of a crossing-web (with the further condition of preserving pseudolinearity). We also introduce the notion of *elastic* crossing-webs, and characterize in Theorem 3.3.5, when a *basic* crossing-web is elastic.

A *crossing-web* is a triple $\mathcal{W} = (G, D, \mathcal{O})$, where D is a drawing of a graph G such that $\mathrm{pl}(G)$ is spanning and 2-connected; and the outer face of D is bounded by the cycle $\mathcal{O}$ of $\mathrm{pl}(G)$. We remark that in a crossing-web, D is a bundled drawing of G. We will refer to the special cycles of D also as the *special cycles* of $\mathcal{W}$ (Figure 3.4a exhibits a crossing-web with its special cycles bounding distinct coloured regions).



(a) $D[G]$ (b) $\mathrm{pl}_D(G)$ (c) $\mathrm{hole}(\mathcal{W})$

Figure 3.4: A crossing-web $\mathcal{W} = (G, D, \mathcal{O})$

Let ϕ be a rectilinear embedding of a cycle C. A *corner* is a vertex of C with angle in $\phi[C]$ distinct from 180°. A *facet* is a path P of C joining two consecutive corners. We also refer to $\phi[P]$ as a *facet* of $\phi[C]$. In case all the angles in $\phi[C]$ are at most 180°, then ϕ is a *weakly convex* embedding of C, and if all the angles are less than 180°, then ϕ is a *strictly convex* embedding of C.

There are extensions to Fary's Theorem that improve the conditions imposed on the planar embeddings. Perhaps the most celebrated one is the following result by Tutte [30]:

Theorem 3.3.1 (Tutte's Spring Theorem)**.** *Let D be an embedding of a 3-connected planar graph with outer face $\mathcal{O}$. Then, for every strictly convex embedding ϕ of $\mathcal{O}$, there exists a rectilinear embedding $\overline{D}$ of G such that:*

- $\phi[\mathcal{O}] = \overline{D}[\mathcal{O}]$*;*
- *D is homeomorphic to $\overline{D}$; and*
- *the boundary of every inner face is strictly convex in $\overline{D}$.*

This theorem serves as our inspiration for the following definition:

Definition 3.3.2. *A* crossing-web $\mathcal{W} = (G, D, \mathcal{O})$ *is* elastic *if, for every strictly convex embedding ϕ of $\mathcal{O}$, there exists a rectilinear embedding $\overline{D}$ of G such that:*

(i) $\phi[\mathcal{O}] = \overline{D}[\mathcal{O}]$*;*

(ii) $pl_D(G) = pl_{\overline{D}}(G)$ *and* $\overline{D}[pl(G)]$ *is homeomorphic to* $D[pl(G)]$*;*

(iii) for every special cycle S in $\mathcal{W}$, $Int_D(S)$ and $Int_{\overline{D}}(S)$ are the same graph; and

(iv) every face of D bounded by a cycle of G is strictly convex in $\overline{D}$.

The natural example of an elastic crossing web $(G, D, \mathcal{O})$ arises from letting D be an embedding of a 3-connected planar graph G. The next observation shows that pseudolinearity is a necessary condition for a crossing-web to be elastic.

Observation 3.3.3. *If $(G, D, \mathcal{O})$ is an elastic crossing-web, then D is pseudolinear.*

Proof. Suppose that $\mathcal{W}$ is elastic. Then, for an arbitrary strictly convex embedding ϕ of $\mathcal{O}$, there exists a rectilinear drawing $\overline{D}$ of G satisfying conditions (i)-(iv) in Definition 3.3.2. Conditions (ii) and (iii) imply that D and $\overline{D}$ have the same special cycles, and each of them has the same external and internal chords. Moreover, these conditions also imply that an outer chord of a special cycle S induces a generalized B in D if and only if it induces a generalized B in $\overline{D}$. Because $\overline{D}$ is pseudolinear, Lemma 3.2.2 implies that D is pseudolinear. $\square$

If S is a special cycle of a crossing-web $(G, D, \mathcal{O})$, then $\mathrm{ec}(S)$ is the graph obtained from S by adding the external chords of S in $D[G]$. The outer cycle of $D[\mathrm{ec}(S)]$ is denoted as $\mathcal{O}_S$.

Definition 3.3.4. *A crossing-web $\mathcal{W} = (G, D, \mathcal{O})$ is* basic *if*

- *$\mathcal{W}$ has exactly one special cycle S;*
- *$\mathcal{O} = \mathcal{O}_S$; and*
- *for every facial cycle C of $D[ec(S)]$ distinct from S and $\mathcal{O}$, $Int_D(C)$ is a 3-connected planar graph embedded in the disk bounded by $D[C]$.*

Next we show that pseudolinearity is also a sufficient condition for a basic crossing-web to be elastic.

Theorem 3.3.5. *A basic crossing-web $(G, D, \mathcal{O})$ is elastic if and only if D is pseudolinear.*

For proving Theorem 3.3.5 we need two results. The first is an extension of Theorem 3.3.1.

Theorem 3.3.6. *[28] Let D be an embedding of 2-connected graph G in the plane with outer cycle $\mathcal{O}$. Suppose that ϕ is a weakly convex embedding of $\mathcal{O}$. If, for every 2-cut $\{x, y\}$, $\{x, y\} \subseteq V(\mathcal{O})$ and x and y are not in the same facet, then there exists a rectilinear embedding $\overline{D}$ of G such that:*

- *$\phi[\mathcal{O}] = \overline{D}[\mathcal{O}]$;*
- *$\overline{D}$ is homeomorphic to D; and*
- *every inner face of $\overline{D}$ is bounded by a strictly convex cycle.*

In particular, if ϕ is a strictly convex embedding of $\mathcal{O}$, such $\overline{D}$ exists if and only if for every 2-cut $\{x, y\}$, $\{x, y\} \subseteq V(\mathcal{O})$ and $xy \notin E(\mathcal{O})$.

The second result that we need is an observation about slightly modifying a rectilinear drawing of a graph. We sketch its proof.

Observation 3.3.7. *Let G be a graph and let $\rho : V(G) \to \mathbb{R}^2$ be a map inducing a rectilinear good drawing of G. Then, there exists $\epsilon > 0$ such that, if $\rho' : V(G) \to \mathbb{R}^2$ is a map with $||\rho(v) - \rho'(v)|| < \epsilon$ for all $v \in V(G)$, then ρ' induces a rectilinear drawing of G such that:*

(i) $\rho[G]$ and $\rho'[G]$ are homeomorphic;

(ii) every cycle bounding a face of $\rho[G]$ that is strictly convex in ρ is strictly convex in ρ'.

Proof's sketch. Suppose that, for some $\epsilon > 0$ to be chosen later, $\rho' : V(G) \to \mathbb{R}^2$ is such that $||\rho(v) - \rho'(v)|| < \epsilon$ for all $v \in V(G)$.

For $v \in V(G)$, let $(v_1, v_2, \ldots, v_r)$ be the cyclic counterclock-wise order in which v is joined to its neighbours $v_1, \ldots, v_r$ in $\rho[G]$. This sequence $(v_1, v_2, \ldots, v_r)$ is the *rotation* at v in $\rho[G]$.

- For each $v \in V(G)$, there is a sufficently small $\epsilon > 0$ for which the rotation at v in $\rho[G]$ is the same as in $\rho'[G]$.
- For each pair $(e, f) \in E(G) \times E(G)$ with $e \neq f$, there is a sufficiently small $\epsilon > 0$ such that $\rho[e]$ and $\rho[f]$ are either crossing, incident or disjoint if and only if the same holds for $\rho'[e]$ and $\rho'[f]$. Moreover, if e and f are crossing at $\times$, and $H_{e,f}$, $H'_{e,f}$ are the underlying plane graphs of $\rho[e \cup f]$ and $\rho'[e \cup f]$, respectively, then there is a sufficiently small ϵ such that the rotations at $\times$ in $H_{e,f}$ and $H'_{e,f}$ are the same.

- For every edge $xy \in E(G)$, and every pair $\times_1$ and $\times_2$ of crossings occurring in this linear order in $D[xy]$ (when xy is oriented from x to y), there is a sufficiently small ϵ, such that the corresponding crossings $\times_1'$ and $\times_2'$ occur in the same order in $\rho'[xy]$.
- For every cycle C such that $\rho[C]$ is strictly convex, there is a sufficiently small ϵ guaranteeing that $\rho'[C]$ is also strictly convex.

There are finitely many elements considered in each of the previous items. Thus, for a sufficiently small choice of $\epsilon > 0$, ρ' satisfy all of the previous items. This, in particular implies the conclusion in Observation 3.3.7. $\square$

In the proof of Theorem 3.3.5, Observation 3.3.7 is used every time we slightly move some points in a given rectilinear drawing, so we will not make an explicit reference to this observation.

Proof of Theorem 3.3.5. Let $\mathcal{W} = (G, D, \mathcal{O})$ be a basic crossing-web with a unique special cycle S. Observation 3.3.3 shows that if $\mathcal{W}$ is elastic, then D is pseudolinear.

Conversely suppose that D is pseudolinear. Let ϕ be a strictly convex embedding of $\mathcal{O}$.

An *intermediate cycle* is a cycle C of $\mathrm{ec}(S)$ such that $D[S]$ is drawn in the disk bounded by $D[C]$. A *partial drawing* with respect to C, is a function ρ that assigns to each vertex in $V(\mathrm{Ext}_D(C)) \cup V(S)$ a point in the plane with the following properties:

(i) For every $v \in V(\mathcal{O})$, $\rho(v) = \phi(v)$;

(ii) the map ρ restricted to $V(\mathrm{Ext}_D(C))$ induces a rectilinear embedding homeomorphic to $D[\mathrm{Ext}_D(C)]$, in which the boundary cycle of every inner face, with the exception of C, is strictly convex;

(iii) no two edges of $\rho[C]$ are collinear;

(iv) if $xy \in E(C) \setminus E(S)$ and P is the xy-path of S incident with the two inner faces of $D[S + xy]$, then $\rho[P]$ is embedded as a line segment connecting $\rho(x)$ and $\rho(y)$; and

(v) if uv is an internal chord of S in D such that $\rho(u)$ and $\rho(v)$ are not drawn in the same facet of $\rho[S]$, then the open line segment joining $\rho(u)$ to $\rho(v)$ is included in the inner face of $\rho[S]$.

For example, there is a partial embedding $\rho_0 : V(S) \to \mathbb{R}^2$ with respect to the intermediate cycle $\mathcal{O}$, obtained from letting $\phi_0(v) = \phi(v)$ for every $v \in V(\mathcal{O})$; and, for every edge $xy \in E(\mathcal{O}) \setminus E(S)$, the path P as in Item (iv) is embedded in ρ_0 as a line segment connecting $\phi(x)$ to $\phi(y)$. Note that ρ_0 is not necessarily a drawing: for every $xy \in E(\mathcal{O}) \setminus E(S)$, the internal vertices of P are drawn over the edge $\phi[xy]$.

It is useful to think a partial drawing as a rectilinear embedding ρ of $\mathrm{Ext}_D(C)$, in which $\rho[C]$ induces a rectilinear drawing of $\rho[S]$ (where the edges of C determine the facets of S), and $\rho[S]$ satisfies the visibility constraints for the inner chords of S given in (v).

Let C be an intermediate cycle for which there exists a partial drawing ρ with respect to C. We choose C such that $|V(\mathrm{Ext}_D(C))|$ is maximum. If $C = S$, then ρ is our desired rectilinear embedding

By way of contradiction suppose that $C \neq S$. Then C has an edge $xy \in E(C) \setminus E(S)$. For this edge xy, let P be the xy-path as in Item (iv).

Claim 1. *There exists an interior vertex z of P such that every internal chord uv of S with $u, v \in V(P)$, either both u and v are in the xz-path of P, or they are in the zy-path of P.*

Proof. Suppose not. Consider the subdrawing D' of D obtained from $D[P + xy]$ by adding all the internal chords of S that have both ends in P. By assumption, x and y are the only vertices in the outer face of D'. Thus xy induces a generalized B, contradicting that D is pseudolinear (Lemma 3.2.2). □

Seeking a contradiction, our goal is to find an intermediate cycle C' and a partial drawing ρ' with respect to C', for which $\mathrm{Ext}_D(C)$ is a proper subgraph of $\mathrm{Ext}_D(C')$.

To find C', consider the subgraph H_P of $\mathrm{ec}(S)$, obtained from P by adding all the external chords of S with both ends in P. Let E_{xy} be the edges in H_P whose ends are not both in $P[x, z]$ and not both in $P[z, y]$. Note that z and xy are incident with the same inner face F_{xy} of $D[H_P - (E_{xy} \setminus \{xy\})]$. Let P' be the xy-path in the boundary of F_{xy} distinct from the edge xy, and let C' be the cycle obtained from replacing xy in C by P'.

We start describing our first approximation to ρ': For each vertex v in $\mathrm{Ext}_D(C)$, let $\rho_1(v) = \rho(v)$. Note that vertices in P are located in ρ as points in the line segment joining $\rho(x)$ to $\rho(y)$. Let $\rho_1(z)$ be a point near $\rho(z)$, drawn in the inner face bounded by $\rho[C]$. Locate in ρ_1 the vertices of P in such a way that the xz-path of P is embedded as a line segment joining $\rho_1(x)$ to $\rho_1(z)$, and the zy-path of P is embedded as a line segment joining $\rho_1(z)$ to $\rho_1(y)$. It is important to keep the images of vertices in $P - \{x, y\}$ under ρ_1 near their images under ρ.

As consequence of Claim 1, we can pick $\rho_1(z)$ close enough $\rho(z)$, so that ρ_1 satisfies the visibility requirements for the inner chords in Item (v). The map ρ_1 is not necessarily a partial embedding: for instance, ρ_1 may not satisfy Item (iii) with respect to C', as some edges of P' might be drawn as part of xz-facet or as part of zy-facet of $\rho_1[C']$. Moreover, we have not defined where the vertices of $int_D(P' + xy)$ are located. To fix this last issue, we need the following observation.

Claim 2. *The map ρ_1 induces a rectilinear embedding of $P' + E_{xy}$ in which the inner faces are bounded by weakly convex polygons.*

Proof. Let $C_{xy} = P' + xy$. By definition, $\rho_1[C_{xy}]$ is embedded as a (weakly convex) triangle.

To show that ρ_1 induces an embedding of $P' + E_{xy}$, first observe that every edge E_{xy} is drawn in ρ_1 as a line segment connecting a point in the xz-facet of $\rho_1[C_{xy}]$ to a point in the zy-facet of $\rho_1[C_{xy}]$. Thus, two edges x_1y_1, $x_2y_2 \in E_{xy}$ cross in ρ_1, if and only if, the vertices x_1, x_2, y_1, y_2 occur in this cyclic order as we follow some orientation of C_{xy}. However, this same condition would imply that x_1y_1, x_2y_2 cross in $D[P' + E_{xy}]$, and $D[P' + E_{xy}]$ is an embedding. Thus ρ_1 induces a rectilinear embedding of $P' + E_{xy}$.

The way edges in E_{xy} are drawn in ρ_1 induces a linear order on E_{xy}: an edge x_1y_1 precedes x_2y_2 if and only if the the triangle with corners $\rho_1(x_1)$, $\rho_1(z)$ and $\rho_1(y_1)$ contains $\rho_1[x_2y_2]$ as inner chord. In this case, each face of $\rho_1[P' + E_{xy}]$ is either bounded by a weakly convex 4-gon (with two consecutive edges of E_{xy} its boundary), or bounded by a weakly convex triangle (with the last edge of E_{xy} and z in its boundary). □

Thus, ρ_1 induces a rectilinear drawing of the graph obtained from the union of $\mathrm{Ext}_D(C)$ and $P' + E_{xy}$. To find a rectilinear drawing of $\mathrm{Ext}_D(C')$, it remains to determine where the vertices of G drawn in the inner faces of $D[P' + E_{xy}]$ are located under ρ_1.

For every inner face F of $D[P' + E_{xy}]$, let C_F be its boundary cycle. Since $\mathcal{W}$ is basic, $D[\mathrm{Int}_D(C_F)]$ is a 3-connected planar graph embedded in the interior for $D[C_F]$. Apply Theorem 3.3.6 to extend $\rho_1[C_F]$ to a rectilinear embedding of $\mathrm{Int}_D(C_F)$ in which every inner face is strictly convex. Such an embedding is how ρ_1 is defined in its restriction to $\mathrm{Int}_D[C_F]$.

At this stage, ρ_1 satisfies the conditions for being a partial embedding with the exception of (iii), as either

- the edges in the xz-facet P'_{xz} of $\rho_1[C']$ are in the same line; or
- the edges in the zy-facet of P'_{zy} of $\rho_1[C']$ are in the same line.

To resolve this issue, we slightly push the interior vertices of each P'_{xz} and P'_{zy} towards the outer face of $\rho_1(C')$. To retain visibility of the inner chords imposed by (v), we push them so that the cycles $P'_{xz} + xz$ and $P'_{zy} + zy$ form strictly convex polygons. Let ρ' be the new function obtained from ρ_1 after pushing these vertices; extend ρ' to all vertices in S in a way that, if $uv \in E(P') \setminus E(S)$ and Q is the uv-path of S incident with the inner faces of $D[S + uv]$, then $\rho'[Q]$ is drawn as line segment connecting $\rho'(u)$ to $\rho'(v)$.

Now ρ' is a partial drawing of G with respect to C', and $|V(\mathrm{Ext}_D(C'))| > |V(\mathrm{Ext}_D(C))|$. This contradicts our choice of C. Thus, there exists a partial embedding in which S is the intermediate cycle, showing that $\mathcal{W}$ is elastic. □

3.4 Full crossing-webs

In this section we characterize when a full crossing-web is elastic. This is the main ingredient used in the proof of Theorem 3.1.1.

Definition 3.4.1. *A crossing-web $\mathcal{W} = (G, D, \mathcal{O})$ is* full *if*

- *every cycle of G bounding an inner face of D is a 3-cycle; and*
- *for every two distinct special cycles S_1 and S_2 in $\mathcal{W}$, the intersection between S_1 and S_2 is either empty, a vertex, or an edge.*

Let $\mathcal{W} = (G, D, \mathcal{O})$ be a crossing-web. A special cycle S is *maximal* if no other special cycle S' contains S in $\mathrm{Int}_D(\mathcal{O}_{S'})$ (recall that $\mathcal{O}_S$ is the outer cycle of the subgraph of G induced by S and its external chords). The special cycle bounding the yellow region in Figure 3.4a is maximal, while the one bounding the blue region is not. We denote $\mathrm{hole}(\mathcal{W})$ as the plane graph obtained from $\mathrm{pl}(G)$ by removing, for each maximal special cycle S in $\mathcal{W}$, the vertices in $\mathrm{int}(\mathcal{O}_S)$ (see Figure 3.4c).

Lemma 3.4.2. *Let $\mathcal{W}$ be a full crossing-web. Then*

(i) $\mathrm{hole}(\mathcal{W})$ is 2-connected; and

(ii) if $\{x, y\}$ is a 2-cut of G, then $\{x, y\} \subseteq V(\mathcal{O})$ and $xy \notin E(\mathcal{O})$.

Proof. We start by showing the following general result.

Claim 1. *In a 2-connected plane graph, if we remove all vertices drawn the interior of a cycle, then the resulting graph is 2-connected.*

Proof. Let C be a cycle of a 2-connected plane graph H. As G is connected, for every vertex in $\mathrm{Ext}(C)$, there is a path in $\mathrm{Ext}(C)$ connecting such a vertex to some vertex in C. Thus Ext(C) is connected. Every face of Ext(C) is bounded by a cycle because it is either a face of H, or is bounded by C. Therefore $\mathrm{Ext}(C)$ is 2-connected. $\square$

Item (i) follows from repeatedly applying Claim 1.

Let $\{x, y\}$ be a 2-cut of hole($\mathcal{W}$). Since $D[\text{hole}(\mathcal{W})]$ is a planar embedding, there exists a simple closed curve γ in the plane intersecting $D[\text{hole}(\mathcal{W})]$ at $\{D[x], D[y]\}$, and such that there is a vertex of hole($\mathcal{W}$) drawn on each side of γ.

Since hole($\mathcal{W}$) is 2-connected, there exists a cycle C_γ containing vertices drawn on distinct sides of γ. Such C_γ necessarily contains x and y, but $xy \notin E(C_\gamma)$. The existence of C_γ shows that (a) the two xy-arcs of γ are drawn in distinct faces F_1 and F_2 of $D[\text{hole}(\mathcal{W})]$ incident with x and y; and (b) if $xy \in E(G)$, then $D[xy]$ is drawn in exactly one side of $D[C_\gamma]$.

Claim 2. *At least one of F_1 and F_2 is the outer face of $D[\mathit{hole}(\mathcal{W})]$.*

Proof. Suppose that both F_1 and F_2 are inner faces. Each of F_1 and F_2 is either a face of D bounded by a 3-cycle of G or, for some maximal special cycle S, is a face bounded by $\mathcal{O}_S$.

Suppose that $xy \in E(G)$. From (b) and the fact that F_1 and F_2 are symmetric, we may assume that $D[xy]$ is drawn in the side of $D[C_\gamma]$ containing F_2. In this case, F_1 is not bounded by a 3-cycle of G, and hence there is a special cycle S_1, for which $\mathcal{O}_{S_1}$ is the boundary of F_1. However, xy is an external chord of S_1 drawn in the exterior of $\mathcal{O}_{S_1}$. This contradicts that $\mathcal{O}_{S_1}$ is the outer cycle of $\mathrm{ec}(S_1)$.

Therefore $xy \notin E(G)$. This implies that, for $i = 1, 2$, there exists a special cycle S_i such that F_i is bounded by $\mathcal{O}_{S_i}$. However, $\{x, y\} \subseteq V(S_1) \cap V(S_2)$ and $xy \notin E(G)$, a contradiction. $\square$

We may assume that F_1 is the outer face of $D[\text{hole}(\mathcal{W})]$. This guarantees that $\{x, y\} \subseteq V(\mathcal{O})$, so we only need to show that $xy \notin E(\mathcal{O})$.

Suppose that $xy \in E(\mathcal{O})$. In this case, xy is incident with the outer face F_1, and from (a) and (b) it follows that xy is not in the boundary of F_2. Then F_2 is not a face of D

bounded by a 3-cycle of G. In this case, there is a maximal special cycle S_2 with $\mathcal{O}_{S_2}$ bounding F_2. However, xy is an external chord of S_2 drawn in the outer face of $\mathcal{O}_{S_2}$, our final contradiction. Thus, $xy \notin E(\mathcal{O})$. $\square$

Let $\mathcal{W} = (G, D, \mathcal{O})$ be a crossing-web and let C be a cycle in $\mathrm{pl}_D(G)$. Then $(\mathrm{Int}_D(C), D[\mathrm{Int}_D(C)], C)$ is also a crossing-web that we denote as $\mathcal{W}|_C$.

Lemma 3.4.3. *Let $\mathcal{W} = (G, D, \mathcal{O})$ be a full crossing-web that has a special cycle S_0 for which $\mathcal{O} = \mathcal{O}_{S_0}$. Let $\mathcal{W}'$ be a crossing-web obtained from $\mathcal{W}$ by replacing, for each facial cycle C of ec(S_0) distinct from S_0 and $\mathcal{O}$, $\mathcal{W}|_C$ by hole$(\mathcal{W}_C)$. Then $\mathcal{W}'$ is basic.*

Proof. Let C be a facial cycle of $\mathrm{ec}(S_0)$ distinct from S_0 and $\mathcal{O}$. It suffices to show that $\mathrm{hole}(\mathcal{W}|_C)$ is 3-connected.

Let $\mathcal{W}|_C = (G', D', C)$. Since $\mathcal{W}|_C$ is full, Lemma 3.4.2(i) implies that $\mathrm{hole}(\mathcal{W}|_C)$ is 2-connected.

Suppose that $\{x, y\}$ is a 2-cut of G'. From Item (ii) of Lemma 3.4.2 we know that $\{x, y\} \subseteq V(C)$ and $xy \notin E(C)$. Since C is a cycle of $\mathrm{ec}(S_0)$, $\{x, y\} \subseteq V(S_0)$. Because x and y are in incident with the facial cycle C of $\mathrm{ec}(S_0)$ and because $xy \notin E(C)$, we see that $xy \notin E(\mathrm{ec}(S_0))$.

Since $\{x, y\}$ is a 2-cut in G', there exists a face F in D' distinct from the outer face of D' that is incident with x and y. As xy is not in the boundary of F (because $xy \notin E(\mathrm{ec}(S_0))$), F is not bounded by a 3-cycle of G. Therefore, there exists a special cycle S' such that $\mathcal{O}_{S'}$ is the boundary of F. However, this implies that S_0 and S' have x and y in common, but $xy \notin E(S_0) \cap E(S')$, a contradiction. $\square$

Now we are ready to characterize elastic full crossing-webs.

Theorem 3.4.4. *A full crossing-web $\mathcal{W} = (G, D, \mathcal{O})$ is elastic if and only if D is pseudolinear.*

Proof. From Observation 3.3.3 it follows that $(G, D, \mathcal{O})$ is elastic when D is pseudolinear. Conversely we assume that D is pseudolinear.

We will show that $\mathcal{W}$ is elastic by induction on $|V(G)|$, where the statement clearly holds for $|V(G)| \leq 3$.

Let ϕ be a strictly convex embedding of $\mathcal{O}$. From Lemma 3.4.2, we know that $\mathrm{hole}(\mathcal{W})$ is 2-connected and, for every 2-cut $\{x, y\}$, $\{x, y\} \subseteq V(\mathcal{O})$ and $xy \notin E(\mathcal{O})$. Thus, the "in particular" part of Theorem 3.3.6 shows that there exists an rectilinear embedding θ of $\mathrm{hole}(\mathcal{W})$ such that

- $\theta[\mathcal{O}] = \phi[\mathcal{O}]$;
- $\theta[\text{hole}(\mathcal{W})]$ is homemorphic to $D[\text{hole}(\mathcal{W})]$
- every facial cycle C of θ is strictly convex.

Let $\mathcal{S}_{max}$ be the set of maximal special cycles in $\mathcal{W}$.

Case. *hole*$(\mathcal{W}) \neq \mathcal{O}$.

In this case, for each special cycle $S \in \mathcal{S}_{max}$, $\mathcal{O}_S \neq \mathcal{O}$. Thus, for each $S \in \mathcal{S}_{max}$, $\mathcal{W}|_{\mathcal{O}_S}$ is a full crossing-web with fewer vertices than $\mathcal{W}$. The induction hypothesis implies that $\mathcal{W}|_{\mathcal{O}_S}$ is elastic; thus, for each $S \in \mathcal{S}_{max}$, there is a rectilinear drawing $\overline{D}_S$ of $\text{Int}_D(\mathcal{O}_S)$ satisfying Conditions (i)-(iv) in Definition 3.3.2 with respect to $\mathcal{W}|_{\mathcal{O}_S}$ and $\theta[\mathcal{O}_S]$. Let $\overline{D}$ be the extension of θ, obtained from letting, for each $S \in \mathcal{S}_{max}$ and for each vertex v in $\text{Int}_D(\mathcal{O}_S)$, $\overline{D}[v] = \overline{D}_S[v]$. Then $\overline{D}$ satisfies Conditions (i)-(iv) in Def. 3.3.2 with respect to $\mathcal{W}$ and ϕ.

Case. *hole*$(\mathcal{W}) = \mathcal{O}$.

In this case, there exists a special cycle S_0 for which $\mathcal{O}_{S_0} = \mathcal{O}$. Every other special cycle S is drawn in the closure of a face of $\text{ec}(S_0)$. Consider the crossing-web $\mathcal{W}' = (G', D', \mathcal{O})$ obtained from $\mathcal{W}$ by replacing, for each facial cycle C of $\text{ec}(S_0)$ distinct from S_0 and $\mathcal{O}$, $\mathcal{W}|_C$ by $\text{hole}(\mathcal{W}|_C)$. Lemma 3.4.3 shows that $\mathcal{W}'$ is basic and Theorem 3.3.5 implies that $\mathcal{W}'$ is elastic. Let $\overline{D'}$ be a rectilinear drawing satisfying Conditions (i)-(iv) in Definition 3.3.2 with respect to $\mathcal{W}'$ and ϕ.

Now, for each special cycle S of $\mathcal{W}$ for which $\mathcal{O}_S$ bounds a face of $\overline{D'}$, $\mathcal{W}|_{\mathcal{O}_S}$ is a full crossing-web, and, since it has fewer vertices than $\mathcal{W}$, is elastic. Since $\overline{D'}[\mathcal{O}_S]$ is a convex drawing of $\mathcal{O}_S$, we can extend this to a rectilinear drawing of $\text{Int}_D(\mathcal{O}_S)$ satisfying conditions (i)-(iv) in Definition 3.3.2. If we do this for every S in which $\mathcal{O}_S$ bounds a face of $\overline{D'}$, then we obtain a rectilinear drawing $\overline{D}$ of G satisfying conditions (i)-(iv) in Def. 3.3.2. $\square$

3.5 Proof of Theorem 3.1.1

In this section we conclude the proof of Theorem 3.1.1 by showing that every pseudolinear bundled drawing with well-behaved special cycles is part of a pseudolinear full crossing-web.

Lemma 3.5.1. *Let D be a bundled drawing of a graph G. Suppose that, for every two distinct special cycles S_1 and S_2, the intersection between S_1 and S_2 is either empty, a vertex or an edge. Then there exists a full crossing-web $(G', D', \mathcal{O})$ such that $G \subseteq G'$, $D \subseteq D'$, and for which D is pseudolinear if and only D' is pseudolinear.*

Proof. Let D' be a drawing obtained by adding a cycle $\mathcal{O}$ in the outer face of D, such that D is in the inner face of $D'[\mathcal{O}]$. As long as $(G', D', \mathcal{O})$ is not a full crossing-web, apply one of the following steps:

1. If $\mathrm{pl}_{D'}(G')$ is not connected, then there exist two vertices u and v of G' in different components of $\mathrm{pl}_{D'}(G')$, and incident with the same face F in $D'[\mathrm{pl}_{D'}(G')]$. Add an arc connecting $D[u]$ to $D[v]$ through F, and extend G' and D' to include this new edge.

2. Suppose that $\mathrm{pl}_{D'}(G')$ is connected and that there is a face F of $\mathrm{pl}_{D'}(G')$ whose boundary is neither a special cycle nor a 3-cycle. Suppose that $(w_0, w_1, \ldots, w_{m-1}, w_0)$ is the boundary walk of F. Then add a cycle $(v_0, v_1, \ldots, v_{m-1}, v_0)$ drawn as a simple closed curve in F, and for $i = 0, 1, \ldots, m-1$, connect v_i to w_i and to w_{i+1} (where the indices are read mod m). Extend G' and D' to include the new vertices and edges.

Step 1 reduces the number of components of $\mathrm{pl}_{D'}(G')$, while Step 2 reduces the number of inner faces of $\mathrm{pl}_{D'}(G')$ whose boundary is neither a special cycle nor a 3-cycle. When Steps 1 and 2 do not apply, $\mathrm{pl}_{D'}(G)$ is connected and the boundary of every inner face of $\mathrm{pl}_{D'}(G')$ is either a special cycle or a 3-cycle. Moreover, as every face of $\mathrm{pl}_{D'}(G')$ is bounded by a cycle, $\mathrm{pl}_{D'}(G')$ is 2-connected. Thus $(G', D', \mathcal{O})$ is a crossing-web with $G \subseteq G'$ and $D \subseteq D'$.

Steps 1 and 2 do not change the set of special cycles nor the way external and internal chords are drawn for a fixed special cycle. Thus, D has an external chord xy of some special cycle S inducing a generalized B if and only xy induces a generalized B in D'. From Lemma 3.2.2, this is equivalent to the statement: D is pseudolinear if and only if D' is pseudolinear. □

Proof of Theorem 3.1.1. Let D be a nice drawing of a graph satisfying the conditions in Theorem 3.1.1.

Suppose that there is a rectilinear drawing $\overline{D}$ satisfying Conditions (i)-(iii) of Theorem 3.1.1. Then D and $\overline{D}$ has the same special cycles, and each of them have the same external and internal chords. Moreover, (i)-(iii) imply that an outer chord of a special cycle S

induces a generalized B in D if and only if induces a generalized B in $\overline{D}$. Because $\overline{D}$ is pseudolinear, Lemma 3.2.2 shows that D is pseudolinear.

Suppose now that D is pseudolinear. Let $\mathcal{W} = (G', D', \mathcal{O})$ be a full crossing-web as in Lemma 3.5.1. Then, since D' is pseudolinear, $\mathcal{W}$ is elastic. Thus, for some arbitrary strictly convex embedding ϕ of $\mathcal{O}$, there exists a rectilinear drawing $\overline{D}$ of G' satisfying (i)-(iv) in Def. 3.3.2. The restriction $\overline{D}[G]$ satisfies (i)-(iii) of Theorem 3.1.1. □

3.6 Concluding remarks

The results in this chapter aimed to investigate Question 3 for bundled drawings. The motivation was to develop new tools for proving results such as Bienstock and Dean's Theorem [13].

In Theorem 3.1.1 we found sufficient conditions for a drawing D implying the existence of a rectilinear drawing $\overline{D}$ having the same pairs of crossing edges. For this result, we assumed that D is bundled and that every pair of special cycles is either disjoint or that they intersect in a vertex or an edge. After some investigation, we believe that these conditions can be weakened. For instance:

Question 4. *Does the conclusion of Theorem 3.1.1 holds for every bundled drawing?*

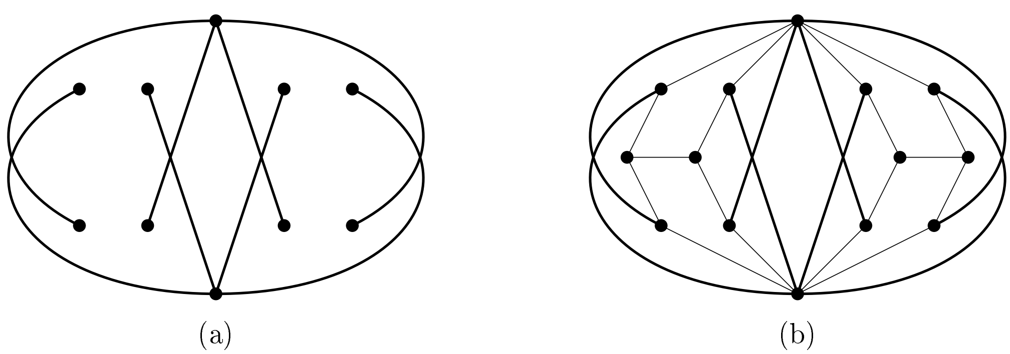


Figure 3.5: Steps to construct a graph with $cr(G) = 4$ and with $\overline{\mathrm{cr}}(G) > 4$.

In [13], Bienstock and Dean showed the existence of graphs with crossing number 4 and arbitrarily large rectilinear crossing number. Hernández-Vélez, Leaños and Salazar [18] observed that these examples are obtained from a specific non-pseudolinear drawing, such as the one in Figure 3.5a, by adding a planar subdrawing, and replacing uncrossed edges by edge-disjoint paths (in Figure 3.5b, the thin edges are replaced by edge disjoint paths). This standard technique has the purpose of making some edges undesirable to cross in an optimal drawing of the graph. The authors of [18] are inclined to believe that this is only way to build these examples, and they conjectured the following.

Conjecture 3.6.1. *There is a function f such that for every 3-connected graph G, $\overline{cr}(G) \leq f(cr(G))$.*

The next question can be taken as a initial approach to this conjecture.

Question 5. *Is it true that every 3-connected graph with $cr(G) = 4$, has $\overline{cr}(G) = cr(G)$? More specifically: can we find a set of minimal subdrawings $\mathcal{F}$ such that every optimal drawing of graph with $cr(G) = 4$ and $\overline{cr}(G) > 4$ has an element of $\mathcal{F}$ as a subdrawing?*

Let $\widetilde{\mathrm{cr}}(G)$ denote the *pseudolinear crossing number* of G. In Schaefer's survey, *The graph crossing number and its variants* [26], he asks the question of whether there is a function f such that $\overline{\mathrm{cr}}(G) \leq f(\widetilde{\mathrm{cr}}(G))$. Hernández-Vélez, Leaños and Salazar showed that a separation between $\overline{\mathrm{cr}}$ and $\widetilde{\mathrm{cr}}$ exists, as there are arbitrarily large 3-connected graphs G with $\overline{\mathrm{cr}}(G) \geq (145/144)\widetilde{\mathrm{cr}}(G)$. This contrasts with Theorem 3.1.1, where the existence of a pseudolinear drawing implies the existence of a similar rectilinear drawing with the same set of crossing pair of edges. There are instances of this problem where there is no clear separation between rectilinear and pseudolinear crossing numbers; for example, Balogh et al. conjectured in [9] that $\overline{\mathrm{cr}}(K_n) = \widetilde{\mathrm{cr}}(K_n)$.

Question 6. *Can we identify features of a pseudolinear drawing D implying the existence of a rectilinear drawing $\overline{D}$ of the same graph with $cr(\overline{D}) \leq cr(D)$ (or with $cr(\overline{D}) \leq f(cr(D))$ for some fixed function f)?*

The concept of elasticity plays a crucial role in the proof of Theorem 3.1.1. Elasticity is the ability of transforming a given drawing into a similar rectilinear drawing by extending any given strictly convex outer boundary and preserving pairs of crossing edges. Theorem 3.3.5 can be considered as an extension Tutte's Spring Theorem allowing some crossings. It is desirable to find extensions of Theorem 3.3.5.

Chapter 4

Pseudospherical Drawings of K_n

4.1 Introduction

Apart from rectilinear drawings, is there any other interesting class of geometric drawings that deserve being studied? To answer this, we recall the Harary-Hill Conjecture, stating that the crossing number of the complete graph K_n is equal to:

$$H(n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor. \tag{4.1}$$

The British artist, Anthony Hill, is acknowledged for discovering a family of drawings of K_n having exactly $H(n)$ crossings. These drawings are beautifully described in the historical survey by Beineke and Wilson, *The Early History of the Brick Factory Problem* [12]:

> Label the vertices $1, 2, \ldots, n$, and arrange the odd numbered ones equally around the inner of two concentric circles and the even ones around the outer circle. Then join all pairs of odd vertices inside the inner circle, join all pairs of even vertices outside the outer circle, and join even vertices to odd ones in the region between the circles.

The drawings described above, cannot be achived by using shortest-arcs in the plane (i.e. straight line segments), but can be achieved by using shortests arcs in the sphere.

In general, if $\mathcal{S}$ is the unit sphere in $\mathbb{R}^3$, then a *spherical drawing* of a graph G is one in which the vertices of G are represented as distinct points in $\mathcal{S}$, and every edge is a shortest-arc connecting its corresponding ends.

A shortest arc connecting two points p and q in $\mathcal{S}$ is obtained as follows. Consider a plane through p, q and the centre of $\mathcal{S}$. The intersection of this plane with $\mathcal{S}$ is a *great circle* through p and q. The shortest pq-arc in the great circle is the shortest arc between p and q in the sphere. If p and q are not *antipodal*, that is, when they are not in the same line through the origin, then the great circle through p and q is unique, and so is the shortest pq-arc.

Although the Harary-Hill Conjecture is known to be true for certain classes of drawings of K_n, it is yet unknown that spherical drawings have at least $H(n)$ crossings. In [23], Moon added more mystery to this problem by showing that the crossing number of a random spherical drawing of K_n is $\frac{1}{4}\binom{n}{2}\binom{n-1}{2}\binom{n-2}{2}\binom{n-3}{2}$.

In the proofs of [2] and [19] showing that rectilinear drawings of K_n have at least $H(n)$ crossings, a crucial point was to relate the number of crossings in a given drawing to the separation properties of the $\binom{n}{2}$ lines extending the edges. Understanding these separation properties, but for the curves extending the edges in spherical drawings, serve as our motivation for studying arrangement of pseudocircles.

An *arrangement of pseudocircles* is a set of simple closed curves in the sphere in which every two curves intersect at most twice, and every intersection is a crossing. If γ is a simple closed curve, then a *side* of γ is one of the two disks in $\mathcal{S}$ bounded by γ. In spherical drawings, the great circles extending the edge-arcs form an arrangement of pseudocircles.

With the aim of finding a combinatorial extension of spherical drawings analogous to how pseudolinear drawings extend rectilinear drawings, there have been two significant questions under active consideration by the graph drawing community in recent years:

(Q1) Do the edges of every good drawing of K_n in the sphere extend to an arrangement of pseudocircles?

(Q2) If the edges of a drawing of K_n extend to an arrangement of pseudocircles, is there an extending arrangement in which any two pseudocircles intersect exactly twice?

In Subsection 4.2, we provide a negative answer to these questions by showing that:

1. the drawing of K_{10} in Figure 4.1 does not have any extension to an arrangement of pseudocircles;

2. the drawing of K_9 in Figure 4.2 has an extension to an arrangement of pseudocircles, but there is no extension in which every two pseudocircles cross twice.

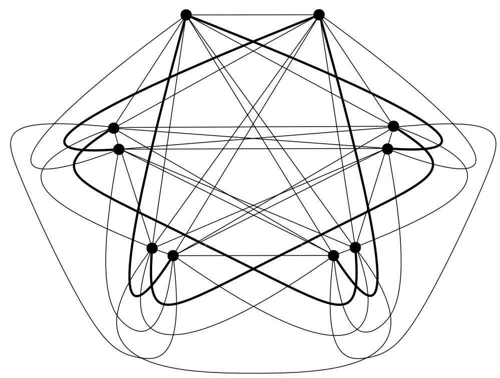


Figure 4.1: A drawing of K_{10} whose edges cannot be extended to an arrangement of pseudocircles.

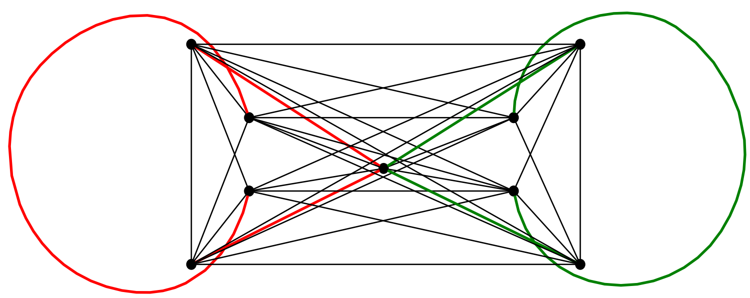


Figure 4.2: A drawing of K_9 whose edges can be extended to an arrangement of pseudocircles, but there is no extension in which the pseudocircles are pairwise intersecting.

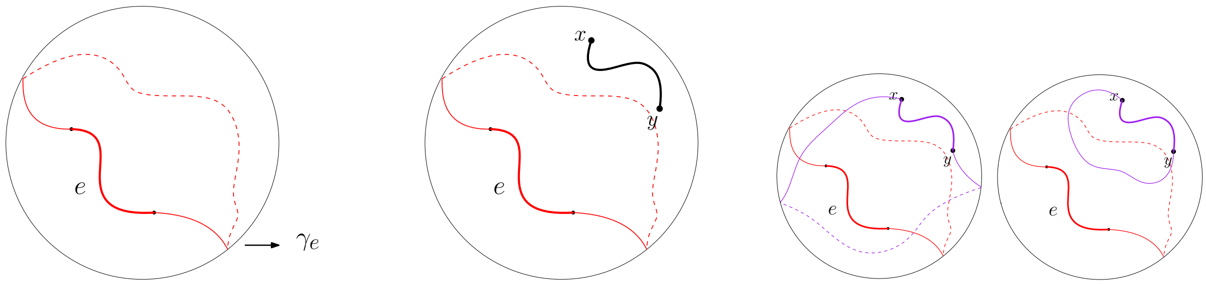
In an attempt to describe a combinatorial generalization of spherical drawings, we introduce the following as the definition of pseudospherical drawings.

Definition 4.1.1. *A drawing D of K_n in the sphere $\mathcal{S}$ is* pseudospherical *if there exists a family $\{\gamma_e : e \in E(K_n)\}$ of simple closed curves in $\mathcal{S}$ such that*

(PS1) $D[e] \subseteq \gamma_e$ *and no vertex other than the ends of e is contained in γ_e;*

(PS2) for every extending curve γ_e, if two vertices x and y are drawn on the same side of γ_e, then $D[xy]$ is completely drawn on that side of γ_e; and

(PS3) if γ_e and γ_f are curves extending the edges e and f, respectively, then $|\gamma_e \cap \gamma_f| \leq 2$, and every intersection point in $\gamma_e \cap \gamma_f$ is a crossing.



(a) PS1 (b) PS2 (c) PS3

Figure 4.3: Pseudospherical drawings.

Throughout this chapter, we will see how this definition captures many of the essential features of spherical drawings.

Drawings for which there exists an extending set of curves only satisfying (PS1) and (PS3) are the ones considered in questions (Q1) and (Q3). Property (PS2) is an analog to the geometric fact that, if two points are drawn in the same hemisphere of $\mathcal{S}$, then the shortest-arc connecting them is also contained in that hemisphere. Adding (PS2) to our list of properties has a big impact on the structure of pseudospherical drawings that brings them closer to spherical.

As our major result for this chapter, we will show that the extending arrangement of pseudospheres can be chosen to resemble a set of great circles; this result answers (Q2) in a positive manner when (PS2) is part of our assumptions.

Theorem 4.1.2. *Let D be a pseudspherical drawing of K_n. Then there exists a set of simple closed curves satisfying (PS1), (PS2), and*

(PS3') if γ_e and γ_f are curves extending the edges e and f, respectively, then $|\gamma_e \cap \gamma_f| = 2$, and every intersection point in $\gamma_e \cap \gamma_f$ is a crossing.

The difference between PS3 and PS3' is that in that in PS3' every two pseudocircles intersect exactly twice.

Questions (Q1) and (Q2) are answered in Section 4.2. In Section 4.3 we prove a technical theorem about arrangements of pseudocircles that will be used in the proof of Theorem

4.1.2. This technical theorem was the inspiration for the gadgets used to answer (Q1) and (Q2).

In Section 4.4 we introduce a class of drawings equivalent to pseudospherical drawings: hereditarily-convex drawings of K_n. The advantage of this definition is that it focuses only on properties of the drawing rather than in the pseudocircular extensions. Moreover, in [7] it was shown the existence of a $O(n^6)$ algorithm to recognize whether a drawing of K_n is hereditarily-convex [7].

Section 4.4 is divided into four subsections. In Subsection 4.4.1 we show that pseudospherical drawings are hereditarily-convex. In Subsections 4.4.2, 4.4.3 and 4.4.4 we show that hereditarily-convex drawings are pseudospherical, and simultaneously give a structural description of hereditarily-convex drawings. This description is crucial in Section 4.5, where we present the proof of Theorem 4.1.2. Finally, in Section 4.6 we give some concluding remarks an open questions related to this chapter.

The present chapter is a collaborative work with Bruce Richter and Matthew Sunohara.

4.2 Solutions to Q1 and Q2

In this section we show that: (1) the drawing in Figure 4.1 cannot be extended to an arrangement of pseudocircles; and (2) the drawing in Figure 4.2 can be extended to an arrangement of pseudocircles, but there is no extension in which the pseudocircles are pairwise intersecting.

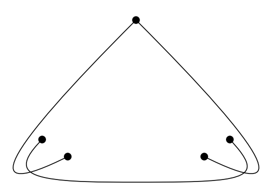


Figure 4.4: The drawing D_1.

The crucial point is to use the drawing in Figure 4.4, denoted as D_1, as a gadget forcing some pseudocircles be drawn in a specific region. We label the face of D_1 incident with all of the five vertices as the *interior face*, while the face incident only with the degree 2 vertex is the *exterior face*. Denote the edge incident with two degree 1 vertices as e_1, while the other two edges are e_0 and e_2.

Consider an arrangement $\{\gamma_0, \gamma_1, \gamma_2\}$ of pseudocircles extending the edges of D_1, where γ_i extends e_i, for $i = 0, 1, 2$.

Claim 1. *Either γ_1 is included in the closure of the interior face, or γ_0 and γ_2 are contained in the closure of the interior face.*

Proof. Suppose that γ_1 is not included in the closure of the interior face. Because γ_1 has points in the exterior face of D_1 and because γ_1 intersects each of γ_0 and γ_2 at the crossings between the pairs (e_1, e_0), (e_1, e_0), $\gamma_1 \setminus e_1$ crosses each of e_0 and e_2 exactly once. This determines, up to homeomorphism, how γ_1 is drawn in the sphere (see Figure 4.5). In particular, this determines all crossings between γ_1 and each of γ_0 and γ_2.

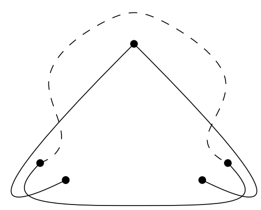


Figure 4.5: The only way γ_1 can be drawn if it has points in the exterior face.

If γ_0 has points in the exterior face, then, as we follow $\gamma_0 \setminus e_0$ from the degree 1 end of e_0, $\gamma_0 \setminus e_0$ crosses into the exterior face. This crossing occurs in the interior of e_2, and together with the degree 2 vertex of D_1, they account for the intersection points between γ_0 and γ_2. However, the ends of e_2 are on distinct sides of a simple closed curve included in $\gamma_0 \cup \gamma_2 \cup e_2$, so the arc $\gamma_2 \setminus e_2$ connects these ends without intersecting this closed curve, a contradiction. $\square$

We overlap two copies of D_1 to obtain the drawing D_2 shown in Figure 4.6.

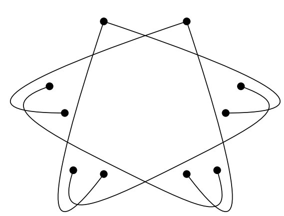


Figure 4.6: The drawing D_2.

Claim 2. *There is no arrangement of pseudocircles extending the edges of D_2.*

Proof. Claim 1 yields four possibilities of how the pseudocircles are drawn in the closure of the interior faces of their corresponding copies of D_1. It is routine to verify that in each of these four cases, there is a pair of pseudocircles that intersect four times (Figure 4.7). $\square$

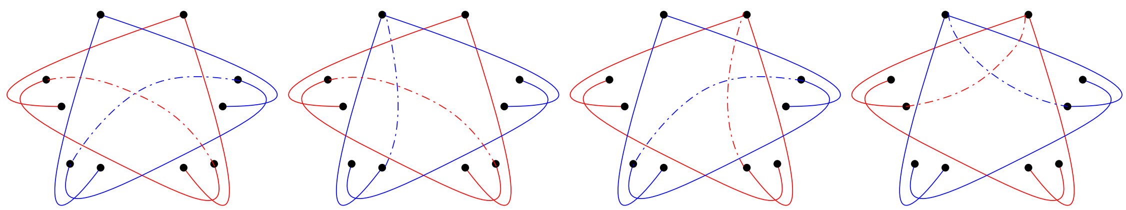


Figure 4.7: The four cases in Claim 2.

The drawing of K_{10} in Figure 4.1, contains D_2, and hence no arrangement of pseudocircles extends its edges.

The drawing of K_9 in Figure 4.2 contains two copies of D_1 where their interior faces are disjoint. This observation and Claim 1 imply that in any extension of this drawing to an arrangement of pseudocircles, there is a pair of disjoint pseudocircles.

Finally, to show that the drawing in Figure 4.2 has an extension to an arrangement of pseudocircles, note that the edges of this drawing, with the exception of two that are drawn as circular arcs, are straight line segments. Extend each straight line segment into a line, and each of the circular arcs into a circle. Map the extending curves from the plane to the sphere by using a stereographic projection; lines are mapped into pairwise intersecting great circles. If a line in the plane crosses one of the two circular extensions, then it crosses each circle exactly twice. Thus, all the extensions intersect at most twice.

4.3 Arcs in Arrangements of Pseudocircles

In this section we prove a technical result (Theorem 4.3.3) that restrics how crossings occur along an arc contained in the point set of an arrangement of pseudocircles. This result is an important piece in the proof of Theorem 4.5.1.

Let Γ be an arrangment of pseudocircles. We use $P(\Gamma)$ to denote the point set $\bigcup_{\gamma\in\Gamma}\gamma$. We are interested in describing properties of the arcs of $P(\Gamma)$, that is, of homeomorphs of $[0,1]$ in $P(\Gamma)$. In Figure 4.8 we depict an arc of an arrangement of pseudocircles. We introduce some standard notation helpful for referring to the features of any such arc.

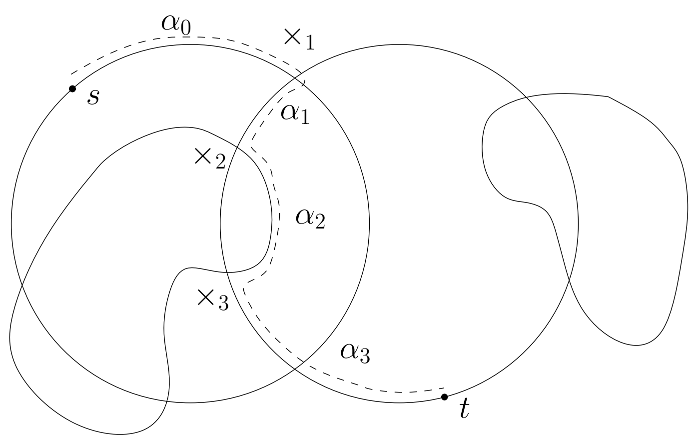


Figure 4.8: An arc in $P(\Gamma)$

Definition 4.3.1. *Let Γ be an arrangement of arcs, let A be an arc of $P(\Gamma)$, and suppose s and t are the ends of A. The* decomposition of A in Γ *is the unique sequence $\alpha_0, \alpha_1, \ldots, \alpha_m$ of subarcs of A such that*

(i) s is an end of α_0, t is an end of α_m;

(ii) for each $i = 0, 1 \ldots, m$, there is $\gamma_i \in \Gamma$ for which $\alpha_i \subseteq \gamma_i$;

(iii) for $i = 1, \ldots, m$, the curves γ_{i-1} and γ_i are distinct, and $\alpha_{i-1} \cap \alpha_i$ is a crossing between γ_{i-1} and γ_i;

(iv) for $i = 1, 2, \ldots m$, the crossing $\alpha_{i-1} \cap \alpha_i$ is denoted as $\times_i$;

(v) for convenience, we let $\times_0 = s$ and $\times_{m+1} = t$; and

(v) the weight *of A is m.*

Consider an arc A with a decomposition as in Definition 4.3.1, and suppose A is oriented from s to t. This orientation defines the sides of A, that is, the points near A that are either on the left side or the right side. For $i = 1, 2, \ldots, m$, since γ_{i-1} and γ_i intersect at $\times_i$, the ends of $\gamma_{i-1} \setminus \alpha_{i-1}$ and $\gamma_i \setminus \alpha_i$ near $\times_i$ are drawn on the same side of A; we denote this as *the side $\times_i$ is facing*. In Figure 4.8, $\times_1$ is facing the left, while $\times_2$ and $\times_3$ are facing the right of A.

Definition 4.3.2. *Let Γ be an arrangement of pseudocircles, and let A be an arc with decomposition $\alpha_0\alpha_1 \cdots \alpha_m$. For $i = 0, 1, \ldots, m$:*

(i) the extensions α_i^- *and* α_i^+ *of* α_i *are the components of* $\gamma_i \setminus A$ *incident to* $\times_{i-1}$ *and* $\times_i$*, respectively (the extensions of* α_1 *and* α_3 *are presented in Figure 4.9);*

(ii) for $\varepsilon \in \{-,+\}$*,* α_i^ε *is* coherent *if the segments of* α_i^ε *near its ends are on the same side of* A *(in Figure 4.9,* α_1^- *is the only non-coherent extension; the other three are coherent);*

(iii) the extensions α_0^- *and* α_m^+ *are not coherent;*

(iv) for $\varepsilon \in \{-,+\}$*,* y_i^ε *is the end of* α_i^ε *distinct from* $\times_i$ *when* $\varepsilon = -$*, and distinct from* $\times_{i+1}$ *when* $\varepsilon = +$*; and*

(v) α_i *is coherent if* $\alpha_i^- \neq \alpha_i^+$*, and at least one of its extensions is coherent (in Figure 4.8,* α_2^- *and* α_2^+ *are coherent, but* α_2 *is not as* $\alpha_2^- = \alpha_2^+$*);*

If for every $i \in \{0\ldots,m\}$*,* α_i *is coherent, then* A *is* coherent.

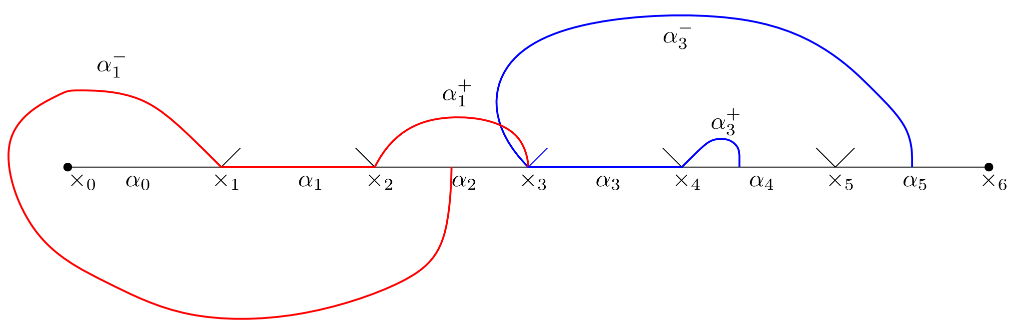


Figure 4.9: An arc in $P(\Gamma)$ and some extensions

The next is a surpizingly technical result about arrangements of pseudocircles. Although, its proof is not particularly difficult, it plays a central role on the proof of Theorem 4.5.1. This result was also the inspiration for the gadgets in Figures 4.4, 4.6 used to answer (Q1) and (Q2).

Theorem 4.3.3. *Let* Γ *be an arrangement of pseudocircles in the sphere. Suppose that* A *is a coherent arc of* $P(\Gamma)$ *with decomposition* $\alpha_0\alpha_1\cdots\alpha_m$*. Then not all of* $\times_1, \times_2, \ldots, \times_m$ *face the same side of* A*.*

Proof. We proceed by induction on the weight m of A. When $m = 0$, none of the extensions of α_0 is coherent (Definition 4.3.2(iii)); thus the base case vacuously holds. Let us assume that $m \geq 1$.

It is convenient to consider the linear order $\prec$ over the point set A, where $p \prec q$ if p occurs first when we traverse A from s to t. We write $p \preccurlyeq q$ if either $p \prec q$ or $p = q$. Recall that, for $j = 0, 1, \ldots, m$, y_j^- is the end of α_j^- distinct from $\times_j$, and y_j^+ is the end of α_j^+ distinct from $\times_{j+1}$. The extension α_j^+ is *forwards* if $\times_{j+1} \prec y_j^+$ and otherwise is *reversed*. Likewise, and extension α_j^- is *forwards* if $y_j^- \prec \times_j$ and otherwise is *reversed*. Note that 'forwards' and 'reversed' does not depend on whether we orient A from s to t or t to s.

In Claim 2 we rule out the existence of reversed coherent extensions; but first we need to understand more about forward extensions.

Claim 1. *Let α_j^+ be a forward coherent extension of α_j with $\times_{j+2} \prec y_j^+$. Suppose that for every $k > j$, α_k^- is not a reversed coherent extension. Then*

(i) for every $\times_i$ with $\times_{j+1} \prec \times_i \prec y_j^+$, $\alpha_i^- \cap \alpha_j^+ = \emptyset$; and

(ii) $\alpha_{j+1}^+ \cap \alpha_j^+ \neq \emptyset$.

Likewise, let α_j^- be a forward coherent extension of α_j with $y_j^- \prec \times_{j-1}$. Suppose that, for every $k < j$, α_k^+ is not a reversed coherent extension. Then

(iii) for every $\times_i$ with $y_i^- \prec \times_i \prec \times_j$, $\alpha_i^+ \cap \alpha_j^- = \emptyset$; and

(iv) $\alpha_{j-1}^- \cap \alpha_j^- \neq \emptyset$.

Proof. We show (i) and (ii), as (iii) and (iv) are the same, but for the traversal of A in reverse direction.

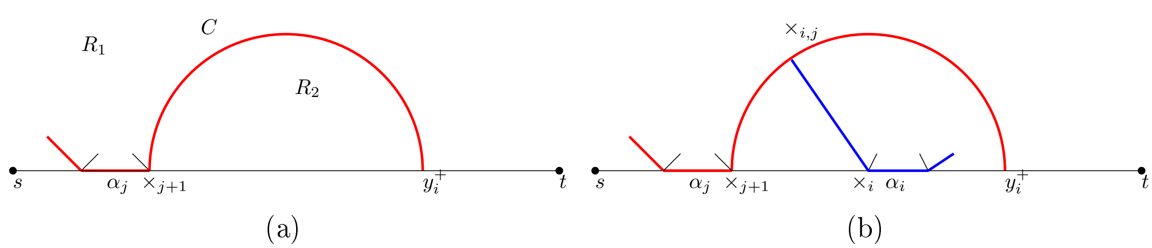


Figure 4.10: Extensions in Claim 1.

To show (i), suppose that for some $\times_i$ with $\times_{j+1} \prec \times_i \prec y_i^+$, $\alpha_i^- \cap \alpha_j^+ \neq \emptyset$. Let C be the simple closed curve obtained from the union of the two $\times_{j+1}y_j^+$-arcs in $A \cup \alpha_j^+$. This curve C defines two closed regions R_1 and R_2; one of these regions, say R_1, contains s and t (see Figure 4.10a).

Let $\times_{j,i}$ be the first point we encounter in α_j^+, when we traverse α_i^- from $\times_i$ to y_i^-. Since $\times_i$ faces the same side of A as the ends of α_j^+, the $\times_i\times_{j,i}$-arc of α_i^- is included in R_2 (Figure 4.10b). Moreover, since α_i^- is not reversed, $\times_{j,i} \neq y_j^+$.

Let α_j^* be the arc obtained by following α_j from $\times_j$ to $\times_{j+1}$, and then continuing on α_j^+ until we reach $\times_{j,i}$. Similarly, we let α_i^* be the arc obtained by following α_i from $\times_{i+1}$ to $\times_i$ and continuing on α_i^- until we reach $\times_{j,i}$.

Let A' be the arc obtained from A by replacing its $\times_j\times_{i+1}$-subarc by $\alpha_j^* \cup \alpha_i^*$. Note that A' decomposes into $\alpha_0\alpha_1\cdots\alpha_{j-1}\alpha_j^*\alpha_i^*\alpha_{i+1}\cdots\alpha_m$, and it has smaller weight than A.

The crossings of A' as in Definition 4.3.1(iii) are given by the sequence $\times_0, \times_1, \ldots, \times_j, \times_{j,i}, \times_{i+1}, \ldots, \times_{m+1}$. Note that $\times_{i,j}$ faces the same side as the rest of the crossings in A', since the points of the extension $(\alpha_j^*)^+$ near $\times_{i,j}$, are precisely the points in $\alpha_j^+ \setminus \alpha_j^*$ near $\times_{i,j}$.

Let α' be any segment of A'. To show α' is coherent, note that α' contains a segment α_k of A. Let α_k^ε be a coherent extension of α_k. The extension $(\alpha')^\varepsilon$ is obtained by following α_k^ε from its end in α_k (or from $\times_{j,i}$ if $\alpha_k^\varepsilon \in \{\alpha_j^+, \alpha_i^-\}$) and stopping at the first point in A' that we encounter. An encounter with A' is guaranteed: either $y_k^\varepsilon \in A \cap A'$, or $y_k^\varepsilon \in A \setminus A'$, and since the ends of $A' \setminus A$ are on the same side of A as the ends of α_k^ε, α_k^ε intersects $A' \setminus A$ when $y_k^\varepsilon \in A \setminus A'$. Moreover, in any of the two cases, the ends of $(\alpha')^\varepsilon$ are on the same side of A', and hence $(\alpha')^\varepsilon$ is coherent.

To complete the proof that α' is coherent we need to show that $(\alpha')^- \neq (\alpha')^+$. First, observe that if both $(\alpha')^-$ and $(\alpha')^+$ are coherent, then $(\alpha')^- \subseteq \alpha_k^-$ and $(\alpha')^+ \subseteq \alpha_k^+$. Since $\alpha_k^- \neq \alpha_k^+$, $(\alpha')^- \neq (\alpha')^+$. Thus A' is a coherent arc of smaller weight than A, and with all its crossings facing the same side, contradicting the induction hypothesis. Then (i) holds.

For (ii), consider the $\times_{j+1}y_j^+$-arc A'' of A. Let $\times_{j+2}, \times_{j+3}\ldots, \times_\ell$ be the sequence of crossings of A in the interior of A''. Then A'' decomposes into $\alpha_{j+1}\alpha_{j+2}\ldots\alpha_{\ell-1}(\alpha_\ell \cap A'')$. From (i) we know that, for $i = j+2, \ldots, \ell$, α_i^- is coherent in A'', and hence all of $\alpha_{j+2}, \ldots, \alpha_\ell$ are coherent in A''. Since A'' has smaller weight than A, A'' is not coherent. Then α_{j+1} is not coherent in A'', and since α_{j+1}^+ intersects $\alpha_j^+ \cup A'$ at least twice, $\alpha_{j+1}^+ \cap \alpha_j^+ \neq \emptyset$. □

Claim 2. *A has no reversed coherent extensions.*

Proof. By way of contradiction, suppose that A has a reversed coherent extension. By possibly reorienting A, we may assume that α_j^+ is a reversed coherent extension for some $j \in \{1, \ldots, m\}$, and that j is the smallest index with that property (note that α_0^+ is coherent but not reversed).

Let C be the simple closed curve bounded in $A \cup \alpha_j^+$. The ends of α_j^+ face the same side of A than $\times_j$, thus, if we follow the extension α_{j-1}^+ from $\times_j$ to y_{j-1}^+, then we either encounter a point in α_j^+ or in $C \cap A$. As α_{j-1}^+ intersects C at least twice and is not reversed, α_{j-1}^+ intersects $\alpha_j \cup \alpha_j^+$ in a point $\times$ distinct from $\times_j$ and y_j^+. Thus $\gamma_{j-1} \cap \gamma_j = \{\times_j, \times\}$, and hence $y_j^+ \notin \alpha_{j-1}$ (note that $y_j^+ \neq \times_j$ because $\alpha_j^- \neq \alpha_j^+$).

The ends of α_j^+ face the same side of A than $\times_{j-1}$, thus α_j^- intersects C twice. Since α_j^- does not intersect the interior of $\alpha_j \cup \alpha_j^+$, $\alpha_j^- \cap C = \{y_j^-, \times_j\}$. Because $\gamma_{j-1} \cap \gamma_j = \{\times_j, \times\}$, $y_j^- \notin \alpha_{j-1}$, and hence $y_j^+ \preccurlyeq y_j^- \prec \times_{j-1}$. Thus α_j^- is a forward coherent extension of α_j with $y_j^- \prec \times_{j-1}$. The minimality of j implies that for every $k < j$, α_k^+ is not a reversed coherent extension. From Claim 1 (iv), we know that α_{j-1}^- and α_j^- have non-empty intersection, and since $\alpha_{j-1}^- \neq \alpha_{j-1}^+$, they intersect in point distinct from $\times_j$ and $\times$. This contradicts that $|\gamma_{j-1} \cap \gamma_j| = 2$. □

Our next goal is to show in Claim 4 that forward coherent extensions are "small jumps". But first we need to show the following.

Claim 3. *Let $k \in \{1, 2 \ldots, m\}$. Suppose α_{k-1}^- is not a coherent extension and that $\gamma_k \setminus \alpha_k^-$ has an intersection with γ_{k-1}. Then α_k^- is not coherent. Likewise, if α_k^+ is not coherent and $\gamma_{k-1} \setminus \alpha_{k-1}^+$ has an intesection with γ_k, then α_{k-1}^+ is not coherent.*

Proof. We prove the first statement; the second is the same, but for the traversal of A in the reverse direction.

Let $\times$ be an intersection between $\gamma_k \setminus \alpha_k^-$ and γ_{k-1}. Since $\times_k \in \alpha_k^-$ is also an intersection between γ_{k-1} and γ_k, $\gamma_{k-1} \cap \gamma_k = \{\times, \times_k\}$.

Consider a simple closed curve C obtained from the union of α_{k-1}^- and the subarc of A connecting the ends of α_{k-1}^-. Let p be a point of α_k^- near $\times_k$, and let A' be the $\times_0 \times_{k-1}$-arc of A. Consider an arc δ, starting on p, following A alongside α_{k-1}, crossing α_{k-1}^-, and continuing beside A' until we reach a point near $\times_0$. Since α_{k-1}^- is not coherent, δ crosses C exactly once, and hence p is on a side of C, different from the one containing the points near A' on the side of A faced by the crossings in its decomposition.

By way of contradiction, suppose that α_k^- is coherent. Claim 2 implies that $y_k^- \prec \times_k$. Since α_k^- does not intersect $\gamma_{k-1} \setminus \{\times_k\}$, α_k^- is disjoint from C. This and the last observation in the previous paragraph together imply that $y_k^- \in \alpha_{k-1} \setminus \{\times_k\}$. Then y_k^-, $\times_k$ and $\times$ are three different points in $\gamma_{k-1} \cap \gamma_k$, a contradiction. □

In the final step we get an accurate description of all the coherent extensions.

Claim 4. *For $j \in \{0, 1, \ldots, m-1\}$, if α_j^+ is a coherent extension, then $y_j^+ \in \alpha_{j+1}$. Likewise, for $j \in \{1, \ldots, m\}$, if α_j^- is a coherent extension, then $y_j^- \in \alpha_{j-1}$.*

Proof. We prove the first statement; the second is the same, but for the traversal of A in the reverse direction.

Suppose by the way of contradiction that the first statement is false. Let j be the smallest index for which α_j^+ is coherent and $y_j^+ \notin \alpha_{j+1}$.

We claim that, for $k = 0, 1, \ldots, j$, α_k^- is not coherent. The proof is by induction on k, where the case $k = 0$ holds as α_0^- is not coherent (see Definition 4.3.2 (iii)). Now, inductively suppose that α_{k-1}^- is not coherent for $k \in \{0, \ldots, j\}$. Since α_{k-1}^+ is coherent, the choice of j implies that $y_{k-1}^+ \in \alpha_k$. From this we know that $\gamma_k \setminus \alpha_k^-$ and γ_{k-1} intersect at y_{k-1}^+. Claim 3 implies that α_k^- is not coherent, as desired. Now we know that α_j^- is not coherent.

Claim 2 and the fact that α_j^+ is coherent imply that $y_j^+ \in \alpha_\ell \setminus \{\times_\ell\}$, for some $\ell \geq j+2$. By applying Claims 1 and 2, we obtain that (i) α_{j+2}^- is disjoint from α_j^+; and (ii) $\alpha_{j+1}^+ \cap \alpha_j^+ \neq \emptyset$.

From (i) and the fact that α_{j+2}^- is not reversed, we know that $y_{j+2}^- \in \alpha_{j+2}$, and hence $\gamma_{j+1} \cap \gamma_{j+2} = \{y_{j+2}^-, \times_{j+2}\}$.

Observation (ii) and Claim 3 imply that α_{j+1}^- is not coherent, and hence α_{j+1}^+ is coherent. From Claim 2 and the fact that $\gamma_{j+1} \cap \gamma_{j+2} = \{y_{j+2}^-, \times_{j+2}\}$ it follows that, for some $r \geq j+3$, $y_{j+1}^+ \in \alpha_r \setminus \{\times_r\}$. Since $\times_{j+2} \prec \times_{j+3} \prec y_{j+1}^+$, Claim 1 implies that $\alpha_{j+2}^+ \cap \alpha_{j+1}^- \neq \emptyset$. However, this contradicts that $\gamma_{j+1} \cap \gamma_{j+2} = \{y_{j+2}^-, \times_{j+2}\}$; therefore Claim 4 holds. □

Definition 4.3.2 (iii) tells us that α_0^- is not coherent, and hence α_0^+ is coherent. Likewise, α_m^- is coherent. Then there exists $j \in \{0, \ldots, m-1\}$ for which α_j^+ and α_{j+1}^- are coherent. Claim 4 implies that $y_j^+ \in \alpha_{j+1}$ and $y_{j+1}^- \in \alpha_j$. But this means that γ_j and γ_{j+1} intersect in at least three points, the final contradiction. □

4.4 Hereditarily-convex drawings

The aim of the next four subsections is to show that pseudospherical drawings of K_n are characterized by a simple local property of the drawing known as *convexity*.

4.4.1 Pseudospherical drawings are hereditarily-convex.

In this subsection we introduce the notion of convexity and show that pseudospherical drawings of K_n are hereditarily-convex.

Definition 4.4.1. *Let D be a drawing of K_n in the sphere.*

- *Let T be a 3-cycle of K_n. A* convex side *of T is a disk Δ bounded by $D[T]$ such that, for every vertex v drawn inside Δ, the three edges connecting v to the vertices of T are drawn in Δ.*
- *A* choice of convex sides *is a set $\{\Delta_T : T \text{ is a 3-cycle of } K_n\}$ such that each Δ_T is a convex side of T.*
- *The drawing D is* convex *if it has a choice of convex sides.*
- *The drawing D is* hereditarily-convex *or simply* h-convex*, if there is a choice of convex sides $\{\Delta_T : T \text{ is a 3-cycle}\}$ such that, if T, T' are 3-cycles for which $D[T'] \subseteq \Delta_T$, then $\Delta_{T'} \subseteq \Delta_T$.*
- *The drawing D is* face-convex *or simply* f-convex*, if there is a face F such that, if for every 3-cycle T we let Δ_T be its side for which $\Delta_T \cap F = \emptyset$, then $\{\Delta_T : T \text{ is a 3-cycle}\}$ is a choice of convex sides.*

It is immediate that face-convex drawings are hereditarily-convex. In connection with Chapter 2, we will see in Subsection 4.4.2 that face-convex drawings are equivalent to pseudolinear drawings of K_n in the plane. Convex drawings of K_n have received recent attention from the community: during the Crossing Numbers Workshop 2017 in Osnabrück, Aichholzer found, using computer assistance, that, for $n \leq 12$, every optimal drawing of K_n is convex. We refer to [7] for a more detailed treatment on convex drawings of K_n. The aim of this section is to show the following:

Theorem 4.4.2. *A drawing of K_n is pseudospherical if and only if it is hereditarily-convex.*

Is it worth noting that in [7] it was shown that h-convex drawings are those from obtained from excluding two drawings of K_5 and one drawing of K_6. This observation and Theorem 4.4.2 imply the existence of an $O(n^6)$ algorithm for testing whether a drawing is pseudospherical.

In this subsection we prove the easy direction of Theorem 4.4.2: pseudospherical drawings are hereditarily-convex (Lemma 4.4.4). In Subsection 4.4.4 we complete the proof by showing that hereditarily-convex drawings are pseudospherical.

We start with a basic observation.

Lemma 4.4.3. *In a pseudospherical drawing of K_n, if e and f are distinct edges and γ_e is the pseudocircle extending e, then γ_e intersects f at most once.*

Proof. Suppose that e, f are edges for which γ_e intersects f at least twice. Since γ_f intersects γ_e in at most two points, γ_e intersects f exactly twice. Then the ends of f are in the same side of γ_e, but f is not contained in that side, contradicting (PS2). □

Lemma 4.4.4. *Let D be a pseudospherical drawing of K_n. Then every 3-cycle T has a side Δ_T whose interior is disjoint from the three pseudocircles extending the edges of T. Moreover, $\{\Delta_T : T$ is a 3-cycle$\}$ is a choice of convex sides witnessing that D is h-convex.*

Proof. Let e be an edge of T. The pseudocircle γ_e extending e intersects the other two edges of T at the ends of e. Lemma 4.4.3 implies that $\gamma_e \setminus D[e]$ is drawn in the interior of a side Δ'_T of $D[T]$.

For any edge f of T distinct from e, γ_f intersects γ_e at the end that e and f have in common. Since this intersection is a crossing, the points of $\gamma_f \setminus D[f]$ near this crossing are in the interior of Δ'_T, and so is $\gamma_f \setminus D[f]$. Then the other side Δ_T of $D[T]$ has its interior disjoint from the three pseudocircles extending the edges of T.

To show that Δ_T is convex, consider a vertex x drawn in the interior of Δ_T. Note that Δ_T is the intersection of three disks bounded by pseudocircles, each of them going through one of the edges of T. This and (PS2) imply that any edge having both ends in Δ_T is drawn in Δ_T. In particular, this holds for the edges connecting x to the three vertices in T, and hence Δ_T is convex.

Finally, in order to show that $\{\Delta_T : T$ is a 3-cycle$\}$ witness that D is h-convex, we consider a pair of 3-cycles T_1, T_2 such that $D[T_2] \subseteq \Delta_{T_1}$.

Claim 1. *If T_1 and T_2 have at least one vertex in common, then $\Delta_{T_2} \subseteq \Delta_{T_1}$.*

Proof. Let $u \in V(T_1) \cap V(T_2)$ and let e be an edge of T_1 incident with u. If $T_1 = T_2$, then the claim trivially holds, so we may assume that T_2 has a vertex v not in T_1. The choice of Δ_{T_1} and the hypothesis on T_2 imply that γ_e and $D[T_2]$ are drawn in distinct sides of $D[T_1]$. Since $D[u]$ is a crossing between γ_e and γ_{uv} (PS3), there are points of γ_{uv} on both sides of γ_e. Consequently, γ_{uv} is drawn on the side of $D[T_2]$ not contained in Δ_{T_1}; the same holds for any pseudocircle that extends an edge of T_2. Thus $\Delta_{T_2} \subseteq \Delta_{T_1}$. □

From Claim 1, we may assume that T_1 and T_2 have no vertices in common. Let $u \in V(T_2) \setminus V(T_1)$ and let $V(T_1) = \{x, y, z\}$. Since Δ_{T_1} is convex and u is drawn in the interior of Δ_{T_1}, the three edges connecting u to x, y and z are drawn in Δ_{T_1}. If one of these three edges, say ux, intersects the edge in $T_2 - u$, then the side of T_2 containing x is not convex. Because Δ_{T_2} is convex, Δ_{T_2} is the side of T_2 included in Δ_{T_1}. Thus, we may assume that none of ux, uy and uz cross the edge in $T_2 - u$. This implies that the two vertices of $T_2 - u$ are drawn in one of the three faces of $D[T_1 + u]$ included in Δ_{T_1}. By symmetry, we may assume that the side Δ of $T_1' = (x, y, u)$ included in Δ_{T_1} is the one in which T_2 is drawn. Claim 1 applied to T_1 and T_1' implies that $\Delta_{T_1'} = \Delta$, and applied to T_1' and T_2 implies that $\Delta_{T_2} \subseteq \Delta_{T_1'}$. Then $\Delta_{T_2} \subseteq \Delta_{T_1}$, and D is h-convex. $\square$

4.4.2 Face-convex drawings

A drawing D of K_n is face-convex if there is a face F such that, for every 3-cycle T, the side of T disjoint from F is convex. In this subsection we explain why face-convex drawings are equivalent to pseudolinear drawings of K_n in the plane. We will see in the next section that face-convex drawings are the basic pieces in the description of the structure of an h-convex drawing. In Lemma 4.4.8 and Corollary 4.4.9 we investigate some properties of face-convex drawings that we need in the next section.

A drawing D in the sphere $\mathcal{S}$ is *equivalent* to a drawing D' in the plane $\mathbb{R}^2$ if there exists a point p in a face of D and an homeomorphism $f : \mathcal{S} \setminus \{p\} \to \mathbb{R}^2$ such that the image $f(D)$ is homeomorphic to D'. In Section 2.5 of Chapter 2, we characterized pseudolinear drawings of K_n in the plane in terms of forbidding the drawing of a path of length 3 known as the B configuration. From this characterization we obtained the following.

Corollary 2.5.2. *A good drawing D of K_n in the plane is pseudolinear if and only if, for every 3-cycle T and for every vertex v drawn in the bounded face of $D[T]$, the three edges connecting v to the vertices of T are contained in the disk bounded by $D[T]$.*

From this corollary it immediately follows the main result of [8].

Theorem 4.4.5. *[8] A drawing of K_n is face-convex if and only if it is equivalent to a pseudolinear drawing of K_n in the plane.*

In the remainder of this section we prove Lemma 4.4.8, an alternative characterization of pseudolinear drawings of K_n due to Aichholzer et al. [5].

Notation 4.4.6. *Let D be a drawing of K_n and let J be a crossing K_4 of D. If C_J denotes the 4-cycle consisting of the uncrossed edges in $D[J]$, then the* crossing side of J *is the disk bounded by $D[C_J]$ that contains the crossing of $D[J]$.*

Our next observation immediately follows from the definition of convex drawing.

Observation 4.4.7. *Let D is a convex drawing of K_n and let J be a crossing K_4 in D. Then each 3-cycle in J has a unique convex side: the one included in the crossing side of J.*

Lemma 4.4.8. *Let D be a drawing of K_n in the sphere, and let F be a face of D. Then the following are equivalent:*

(i) F witnesses face-convexity; and

(ii) the crossing side of every crossing K_4 is disjoint from F.

Proof. Suppose that (i) holds. By way of contradiction, suppose that J is a crossing K_4 for which its crossing side is not disjoint from F. Then F is included in a face F' of $D[J]$ incident with the crossing of J. The face F' is bounded by two segments of the edges crossing in J and by an uncrossed edge e. Let T be any 3-cycle of J having e as an edge. From Observation 4.4.7 it follows that the unique convex side of T is the one containing F. However, this contradicts that F witnesses face-convexity.

Conversely, suppose that (ii) holds. Let T be a 3-cycle and let Δ_T be the side of T disjoint from F. If Δ_T is not convex, then there exists a vertex u drawn in the interior of Δ_T, and a vertex $v \in V(T)$ for which $D[uv] \not\subseteq \Delta_T$. Thus, uv crosses at least one edge in T, and since D is a good drawing, it crosses only the edge in $T - v$. Then $T + u$ induces a crossing K_4. Moreover, since uv and the edge in $T - v$ are the only crossing edges in $T + u$, the edges connecting u to the vertices in $T - v$ are included in Δ_T. This shows that the face of $D[T + u]$ bounded by a 4-cycle is included in Δ_T, and hence the crossing side of $T + u$ is not disjoint from F, contradicting (ii). Thus Δ_T is convex for every T, and hence F witnesses face-convexity. □

Corollary 4.4.9. *Let n be an integer with $n \geq 3$. If F is a face of a drawing D of K_n witnessing that D is f-convex, then F is bounded by a cycle of K_n.*

Proof. Consider the underlying plane graph G of D obtained by replacing each crossing of D by a degree 4 vertex. From [24, Lemma 5] it follows that G is 2-connected (in fact, the authors showed that, for $n \geq 5$, G is 3-connected). Consequently F is bounded by a cycle of G.

If the boundary cycle of F is not a cycle of K_n, then there is at least one crossing $\times$ of D incident with F. Let J be the crossing K_4 induced by the ends of the pair of edges crossing at $\times$. Since $\times$ is incident with F, one of the four faces of $D[J]$ incident with $\times$ contains F. Then the crossing side of J is not disjoint from F, and this contradicts Lemma 4.4.8. □

4.4.3 The face-convex sides of an hereditarily-convex drawing

In this section we describe the structure of hereditarily-convex drawings needed to complete the proof of Theorem 4.4.2. We will see that, in any hereditarily-convex drawing of K_n, every edge induces a natural decomposition into two face-convex (or pseudolinear) drawings. This decomposition generalizes the fact that, in a spherical drawing, if we consider a great circle γ extending an edge, then each of the two drawings induced by the vertices on one side of γ is equivalent to a rectilinear drawing in the plane.

Notation 4.4.10. *Let D be a drawing of K_n with a choice of convex sides $\{\Delta_T : T$ is a 3-cycle$\}$ witnessing h-convexity. For each $e \in E(K_n)$, let $\overrightarrow{e}$ be an arbitrary orientation of e.*

- *The set Σ_e^1 consists on the vertices v in K_n not incident with e, for which the 3-cycle T containing v and e has its chosen convex side Δ_T to the left of $\overrightarrow{e}$. The set Σ_e^2 consists on the vertices v not incident with e, for which the 3-cycle T containing v and e has Δ_T to the right of $\overrightarrow{e}$.*
- *For $i = 1, 2$, we let D_e^i be the subdrawing of D induced by the vertices in Σ_e^i and e.*

We should make few points about this notation. First, the orientation that we choose for each edge is irrelevant to us; it only helps us distinguish the sides of e. Second, we will assume that every choice of convex sides witnessing h-convexity also comes with an arbitrary orientation of the edges of K_n, so that the 1- and 2-sides of e are predetermined, as well as Σ_e^1 and Σ_e^2.

The next result will be used in the proofs of 4.4.14 and 4.4.16.

Lemma 4.4.11. *Let D be an h-convex drawing of K_n with respect to a given choice of convex sides. Suppose that J is a crossing K_4 in D, and let $v \in V(K_n) \setminus V(J)$ be a vertex drawn in the crossing side of J. Then* *(i) the four edges connecting v to the vertices of J are drawn in the crossing side of J; and*

(ii) these four edges partition the crossing side of J into four regions; each of them is the chosen convex side of their boundary.

Proof. Suppose that x, y, z, w are the vertices of J, and we may choose these labels so that xz crosses yw. Without loss of generality, suppose that v is in the face bounded by $D[xy]$ and by some subsegments of $D[xz]$ and $D[yw]$.

Suppose that $\{\Delta_T : T \text{ is a 3-cycle}\}$ is the given choice of convex sides. Observation 4.4.7 implies that Δ_{xyz} and Δ_{xyw} are included in the crossing side of J, and consequently, so are the edges connecting v to the vertices of J. Then (i) holds.

To show (ii), consider any of the four closed regions of the crossing side of J, bounded by an edge $e \in E(J)$ and by the edges connecting v to the ends of e. If $e \neq zw$ then this region is contained in either Δ_{xyz} or Δ_{xyw}, and hence is convex. Otherwise, this region is bounded by the 3-cycle (v, z, w), and is contained in the crossing side of the K_4 induced by v, x, z and w. Now the result follows from Observation 4.4.7. □

Lemma 4.4.12. *Let D be a an h-convex drawing of K_n with respect to a given choice of convex sides. Let $e, f \in E(K_n)$. If f crosses e, then the ends of f are in distinct ones of Σ_e^1 and Σ_e^2. Consequently, for $i = 1, 2$, e is not crossed in D_e^i.*

Proof. Let J be the crossing K_4 containing the four ends of e and f. If T_1 and T_2 are the 3-cycles of J containing e, then the disks bounded by T_1 and T_2 included in the crossing side of J contain distinct sides of e. These disks are the chosen convex sides by Observation 4.4.7, and hence Lemma 4.4.12 holds. □

Our previous lemma shows that, in any h-convex drawing D of K_n, every edge e is incident to two faces of D_e^i; each of them includes one side of e.

Notation 4.4.13. *Let D be an h-convex drawing of K_n with respect to a given choice of convex sides. For $e \in E(K_n)$ and for $i \in \{1, 2\}$, we let F_e^i be the face of D_e^i incident with e that includes the $(3 - i)$-side of e.*

Lemma 4.4.14. *Let D be an h-convex drawing of K_n with respect to a choice $\{\Delta_T : T \text{ is a 3-cycle}\}$ of convex sides, and let $e \in E(K_n)$. If, for $i \in \{1, 2\}$, $|\Sigma_e^i| \geq 1$, then every 3-cycle T in D_e^i has $\Delta_T \cap F_e^i = \emptyset$. In particular D_e^i is f-convex.*

Proof. Let J be any crossing K_4 in D_e^i and suppose F_J is the face of $D[J]$ bounded by a 4-cycle. By Lemma 4.4.8, it is enough to show that $F_e^i \subseteq F_J$.

Lemma 4.4.12 implies that e does not cross any edge in J. In particular, if $e \in E(J)$, then e is part of the 4-cycle bounding F_J. Moreover, Observation 4.4.7 tells us that, if $e \in E(J)$, then the 3-cycles of J through e have their chosen convex sides disjoint from F_J; in this case $F_e^i \subseteq F_J$, as desired.

Now suppose that $e \notin E(J)$. In this case, e has an end $u \notin V(J)$, drawn in a face of $D[J]$. If $D[u] \in F_J$, then, since e does not cross any edge in J, e is drawn in the closure of F_J. Because F is incident with e in D_e^i, $F_e^i \subseteq F_J$.

By way of contradiction suppose that u is drawn in a face F' of $D[J]$ distinct from F_J.

Lemma 4.4.11 implies that the four edges from u to the vertices in J are drawn inside the crossing side of J. These four edges partition the crossing side of J into four closed regions. Each of these four regions is the convex side of the 3-cycle bounding the region. If e is one of the four edges connecting u to a vertex in J, then two of the 3-cycles bounding these four regions have their convex sides containing distinct sides of e. This contradicts that all the vertices of J are in D_e^i. Thus, we may assume that e is none of the four edges connecting u to a vertex in J.

Let v be the end of e distinct from u. Since e does not cross any edge of J, v is located in one of the four previously described regions; call it Δ. Let T be the 3-cycle of $J + u$ bounding Δ. By Lemma 4.4.11, Δ is convex, and hence e and the other two edges connecting v to the vertices in $T - u$ are drawn in Δ. Now, as D is h-convex, the 3-cycles in $T + v$ containing e have their chosen convex sides including distinct sides of e, a contradiction. $\square$

Recall that, in every f-convex drawing of K_n, a face witnessing f-convexity in D is bounded by a cycle of K_n (Corollary 4.4.9). With this and Lemma 4.4.14 in hand, we introduce the following.

Notation 4.4.15. *Let D be an h-convex drawing with respect to a given choice of convex sides. Let $e \in E(K_n)$. For $i = 1, 2$, C_e^i denotes the cycle bounding F_e^i. We let Δ_e^i be the disk bounded by $D[C_e^i]$ disjoint from F_e^i.*

The remainder of this subsection is devoted to refining our structural description of h-convex drawings, by showing that for every edge e, $\Delta_e^1 \cap \Delta_e^2 = D[e]$ (Lemma 4.4.18). The proof of this lemma is achieved in a sequence of three small steps.

Our first step is to show that each disk Δ_e^i does not contain vertices from $\Sigma_e^{(3-i)}$.

Lemma 4.4.16. *Let D be an h-convex drawing of K_n with respect to a given choice of convex sides. Let $e \in E(K_n)$. For $i = 1, 2$, if $x \in \Sigma_e^{3-i}$, then $D[x] \notin \Delta_e^i$.*

Proof. By way of contradiction, suppose that, for some $i = 1, 2$, there exists $x \in \Sigma_e^{3-i}$ with $D[x] \in \Delta_e^i$.

First, suppose that C_e^i is a 3-cycle. Then x is drawn in the convex side Δ_e^i. Thus, the two edges connecting x to the ends of e are also drawn in Δ_e^i. Since D is h-convex, the convex side of the 3-cycle containing e and x is the one included in Δ_e^i. This implies that $x \in \Sigma_e^i$, a contradiction.

In the alternative, suppose that C_e^i has length at least 4. For any edge f of C_e^i not incident with e, the K_4 induced by the ends of e and f is drawn in Δ_e^i, and hence is crossing. From Lemmas 4.4.14 and 4.4.8 it follows that every crossing K_4 obtained in this way has its crossing side included in Δ_e^i. Moreover, Δ_e^i is the union of the crossing sides of the K_4s obtained in this way, thus at least one of them, say J, contains $D[x]$ in its crossing side.

Since $D[e]$ is part of the boundary of the crossing side of J, Lemma 4.4.11 implies that the 3-cycle containing x and e, as well as its chosen convex side, is included in Δ_e^i. Therefore $x \in \Sigma_e^i$, a contradiction. □

In our second step, we show that edges belonging to distinct ones of D_e^1 and D_e^2 do not cross.

Lemma 4.4.17. *Let D be an h-convex drawing of K_n with a given choice of convex sides. Let $e \in E(K_n)$. Then no edge of D_e^1 crosses any edge of D_e^2.*

Proof. We start with the following observation.

Claim 1. *If an edge f_1 of D_e^1 crosses an edge f_2 of D_e^2, then at least one of f_1 and f_2 has no ends in common with e.*

Proof. Suppose that each of f_1 and f_2 has an end in common with e. The ends of this pair of edges induce a crossing K_4 for which e is part the 4-cycle bounding a face. Observation 4.4.7 shows that the ends of f_1 and f_2 that are not in e are in the same one of Σ_e^1 and Σ_e^2. Thus f_1 and f_2 are in the same of D_e^1 and D_e^2, a contradiction. □

By way of contradiction, suppose that x_1y_1 is an edge of D_e^1 crossing an edge x_2y_2 of D_e^2. Claim 1 implies that at least one of these edges, say x_1y_1, is not incident with e. Let J_1 be the K_4 induced by the ends of x_1y_1 and e.

First, suppose that one of the ends of x_2y_2, say y_2, is an end of e. As x_1y_1 crosses x_2y_2, Lemma 4.4.12 shows that $x_2y_2 \neq e$, and hence $x_2 \in \Sigma_e^2$. Claim 1 implies that x_1y_1 is the only edge of J_1 that crosses x_2y_2. Lemma 4.4.16 implies that $D[x_2] \notin \Delta_e^1$. Thus, when we traverse $D[x_2y_2]$ from x_2 to y_2, the first point of $D[J_1]$ that we encounter is the crossing $\times$ between x_2y_2 and x_1y_1. This implies that x_1y_1 is incident with the face of $D[J_1]$ containing x_2. Hence J_1 is a crossing K_4 where x_1y_1 and e are in the 4-cycle of $D[J_1]$ bounding a face containing $D[x_2]$.

If we continue traversing x_2y_2 across $\times$, then we enter into a face F of $D[J_1]$ included in the crossing side of J_1. When we follow x_2y_2 until we reach y_2, x_2y_2 must cross another edge that has a segment in the boundary of F. However, the existence of a second crossing contradicts Claim 1.

Now we assume that e and x_2y_2 have no ends in common. Let J_2 be the K_4 induced by the ends of e and x_2y_2. From what we just showed, we may assume that the only crossing between an edge in J_1 and an edge in J_2 is between x_1y_1 and x_2y_2. Since $D[x_2] \notin \Delta_e^1$, when we traverse $D[x_2y_2]$ from x_2 to y_2, the first point we encounter is the crossing $\times$ between x_2y_2 and x_1y_1. Again, this implies that $D[J_1]$ is a crossing K_4, having x_1y_1 and e in the 4-cycle bounding a face F_J containing $D[x_1]$.

Since the crossing side of J_1 is included in Δ_e^1, and $D[y_2] \notin \Delta_e^1$, $D[y_2]$ is also drawn in F_J. As we continue traversing x_1y_2 across $\times$, we enter into a face F of $D[J_1]$ included in the crossing side of J_1. Since $D[x_2y_2]$ crosses an even number of times the boundary of F, x_2y_2 crosses some other edge of J_1, a contradiction. $\square$

We reached our third and final step towards understanding the structure of h-convex drawings.

Lemma 4.4.18. *Let D be an h-convex drawing with a given choice of convex sides, and let $e \in E(K_n)$. If Δ_e^1 and Δ_e^2 are the disks defined as in Notation 4.4.15, then $\Delta_e^1 \cap \Delta_e^2 = D[e]$.*

Proof. Let u and v be the ends of e. Since $\Sigma_e^1 \cap \Sigma_e^2 = \emptyset$, u and v are the only vertices that C_e^1 and C_e^2 have in common. Then $D[C_e^1 \cup C_e^2]$ is the drawing of three internally-disjoint uv-paths. Moreover, Lemmas 4.4.12 and 4.4.17 imply that $D[C_e^1 \cup C_e^2]$ is drawn as three internally disjoint uv-arcs in the sphere.

As the open uv-arc $D[C_e^1] \setminus D[e]$ contains at least one vertex of Σ_e^1, Lemma 4.4.16 guarantees that such an arc is not drawn in Δ_e^2. Likewise, $D[C_e^2] \setminus D[e]$ is not drawn in Δ_e^1. So each of Δ_e^1 and Δ_e^2 is the closure of a face in $D[C_e^1 \cup C_e^2]$, and these disks only intersect at $D[e]$. $\square$

4.4.4 Hereditarily-convex drawings are pseudospherical

In this subsection we conclude the proof of Theorem 4.4.2 (pseudospherical and h-convex drawings are equivalent), and simultaneously pave our way towards the proof of Theorem 4.1.2 (pseudospherical drawings have an extension satisfying (PS3')). We assume that the reader is familiar with Notation 4.4.10-4.4.15 given in the previous subsection.

Our aim in this subsection is to extend the edges of an hereditarily-convex drawing D of K_n to an arrangement of pseudocircles. Lemmas 4.4.14 and 4.4.18 describe how every edge e induces a partition into two f-convex drawings that intersect in $D[e]$. This partition suggests that, if γ_e is a pseudocircle extending e, then Δ_e^1 and Δ_e^2 are on distinct sides of γ_e. This condition is the same as assuming that $\gamma_e \setminus D[e]$ is an arc connecting the ends of e in $\mathcal{S} \setminus (\Delta_e^1 \cup \Delta_e^2)$. This last condition is what we define as the basic property that any "feasible" pseudocircle extending e must satisfy. We generalize this notion of "feasibility" to allow multiple pseudocircles extending a given subset of edges in K_n.

Notation 4.4.19. *Let D be an h-convex drawing of K_n with a given choice of convex sides and let $e \in E(K_n)$. We denote as F_e the face of $D_e^1 \cup D_e^2$ bounded by $(C_e^1 - e) \cup (C_e^1 - e)$.*

Notice that $F_e = F_e^1 \cap F_e^2 = \mathcal{S} \setminus (\Delta_e^1 \cup \Delta_e^2)$.

Definition 4.4.20. *Let D be an h-convex drawing of K_n in the sphere, and let $J \subseteq E(K_n)$.* A set of pseudocircles extending J *is a set $\{\gamma_f : f \in J\}$ of simple closed curves satisfying (PS1)-(PS3) and such that, for each $f \in J$:*

(i) $\gamma_f \setminus D[f]$ is an open arc connecting the ends of f in F_f; and

(ii) for each $e \in E(K_n)$, the intersection of γ_f with $D[e]$ is either empty; a crossing with the interior of $D[e]$; a single vertex that is a common end of e and f; or $D[e]$ and $f = e$.

The next remark follows from the fact that, for each edge $f \in J$, Δ_f^1 and Δ_f^2 are contained in distinct sides of γ_f.

Remark 4.4.21. *If γ_f is an element of a set of pseudocircles extending J, and γ_f crosses an edge, then the ends of such an edge are in distinct ones of Σ_f^1 and Σ_f^2. Moreover, if e_1 and e_2 are edges crossed by γ_f, and x_1, x_2 are ends of e_1 and e_2, respectively, then x_1 and x_2 belong to the same one of Σ_f^1 and Σ_f^2 if and only if they are on the same side of γ_f.*

We aim to show that, if $J \subseteq E(K_n)$ and $e \in E(K_n)\setminus J$, then every set Γ_J of pseudocircles extending J can be enlarged to one extending $J \cup \{e\}$. For this, we consider a pseudocircle γ_e extending e, obtained as the union of $D[e]$ and an arc joining the ends of e in F_e, very near to $D[C_e^1 - e]$. In Lemmas 4.4.23 and 4.4.24 we will show that such γ_e can be chosen such that $\Gamma_J \cup \{\gamma_e\}$ extends $J \cup \{e\}$.

Our next observation will be constantly used in the proofs of Lemmas 4.4.23 and 4.4.24.

Lemma 4.4.22. *Let D be an h-convex drawing with a given choice of convex sides. Suppose that e and f are distinct edges of $E(K_n)$. Then there are no four vertices x_1, x_2, y_1, y_2 occuring in this cyclic order around C_e^1, and such that x_i and y_i are drawn in D_f^i, for $i = 1, 2$.*

Proof. If such four vertices exists occur in this cyclic order, then the edges x_1y_1 and x_2y_2 are drawn in Δ_e^1, and hence they cross. However, the edge x_1y_1 is in D_f^1, while x_2y_2 is in D_f^2, contradicting Lemma 4.4.12. ☐

Lemma 4.4.23. *Let D be an h-convex drawing of K_n with a given choice of convex sides and let $J \subseteq E(K_n)$. Suppose that $\{\gamma_f : f \in J\}$ is a set of pseudocircles extending J. Let e, f be distinct edges of K_n with $e \notin J$ and $f \in J$. Then there exists a simple closed curve γ_e with $D[e] \subseteq \gamma_e$ and $\gamma_e \setminus D[e]$ drawn in F_e, sufficiently near $D[C_e^1 - e]$, such that $|\gamma_f \cap \gamma_e| \leq 2$.*

Proof. As γ_e is constructed by considering a curve in F_e near $D[C_e^1 - e]$, we note that any forced crossing between $\gamma_e \setminus D[e]$ and γ_f arises from any short segment σ of γ_f, with one end $z \in D[C_e^1]$, and the rest of $\sigma \setminus z$ drawn in F_e.

Having the previous paragraph under consideration, let $Z = \gamma_f \cap (D[C_e^1] \setminus D[e])$, and for each $z \in Z$, consider a small open arc α of γ_f centered at z. For $z \in Z$, define the contribution $c(z)$ of z to be the number of components of $\alpha \setminus z$ drawn in F_e. In such case, $c(z)$ is either 0, 1 or 2, and it counts the forced crossings near $D[z]$.

Thus, by letting $c(\gamma_f) = \sum_{z \in Z} c(z)$, we can choose γ_e sufficiently near C_e^1 such that

$$|\gamma_e \cap \gamma_f| = c(\gamma_f) + |\gamma_f \cap D[e]|.$$

Since $|\gamma_e \cap \gamma_f|$ is even and $|\gamma_f \cap D[e]| \leq 1$, it suffices to show that $c(\gamma_f) \leq 2$.

Claim 1. *If for some $z \in Z$, $c(z) = 2$, then $c(\gamma_f) \leq 2$.*

Proof. By assumption, if γ_f intersects the interior of an edge, then it crosses that edge. Because γ_f does not cross from F_e into Δ_e^1 at z, z must be a vertex of C_e^1. Moreover, since γ_f satisfies Item (ii) in Definition 4.4.20, z is an end of f.

Because $c(z) = 2$, the points of $D[f]$ near z are drawn in F_e. As every edge that has both ends in D_e^1 is drawn in Δ_e^1, the end of f distinct from z is in Σ_e^2.

Suppose that there exists $z' \in Z \setminus \{z\}$ with $c(z') > 0$. From what we just observed, z' is not an end of f, and hence is a crossing between γ_f and an edge e' of $C_e^1 - e$. Traverse γ_f, starting at z', continuing in the interior of Δ_e^1, until we reach a point z'' in $D[C_e^1]$. Since $c(z) = 2$, $z'' \notin \{z, z'\}$. Thus z'' is also a crossing between γ_f and an edge e'' in $C_e^1 - e'$.

Let x_1 and y_1 be ends of e' and e'', respectively, and we choose them such that they are in the same component of $D[C_e^1] \setminus \{z', z''\}$ containing z. Let x_2 be the end of e' distinct from x_1. As γ_f crosses each of e' and e'' exactly once, Remark 4.4.21 shows that x_1 and y_1 are on the same one of Σ_f^1 and Σ_f^2. By reorienting f, if necessary, we may assume x_1, $y_1 \in \Sigma_f^1$. In this case $x_2 \in \Sigma_f^2$. By letting $y_2 = z$, we obtain vertices x_1, x_2, y_1, y_2 occurring in this cyclic order in C_e^1, and such that x_i, y_i belong to D_e^i, for $i = 1, 2$. This contradicts Lemma 4.4.22. □

Claim 2. *If $f \in E(C_e^1)$, then $c(\gamma_f) \leq 2$.*

Proof. As $D[f] \subseteq \gamma_f$, each end of f has a contribution to c of at most 1. Suppose that there exists $z \in Z$, that is not an end of f, with $c(z) > 0$. Then, from Claim 1, we may assume that $c(z) = 1$. Note that z is a crossing between γ_f and an edge e' of $C_e^1 - e$ not incident with f. As the ends of e' belong to distinct ones of Σ_e^1 and Σ_e^2, and the ends of f belong to both D_f^1 and D_f^2, we can label the ends of e' and f using labels x_1, x_2, y_1, y_2 such that they occur in this cyclic order in C_e^1, and such that x_i, y_i are drawn in D_e^i, for $i = 1, 2$. This contradicts Lemma 4.4.22. □

From the previous two claims, we may assume that $f \notin E(C_e^1)$ and that each element in Z with positive contribution has contribution 1.

Let $z \in Z$ with $c(z) = 1$. Follow γ_f from F_e into Δ_e^1 across z, and continue along γ_z inside Δ_e^1, until we reach a point z' just before we cross into F_e. Since $c(z') = 1$, this shows that the elements in Z with positive contribution are paired in such a way that a pair corresponds to the ends of an arc-component of $\gamma_f \cap \Delta_e^1$.

By way of contradiction, suppose that $c(\gamma_f) > 2$. It follows that there are four vertices z_1, z_1', z_2, z_2' with contribution 1, and such that, for $i = 1, 2$, z_i and z_i' are ends of the same arc-component α_i of $\gamma_f \cap \Delta_e^1$.

Claim 3. *At least one of α_1 and α_2 is disjoint from the ends of f.*

Proof. If both ends of f are in Δ_e^1, then $D[f] \subseteq \Delta_e^1$, and hence they belong to the same arc-component of $\gamma_f \cap \Delta_e^1$. In this case the claim holds.

In the alternative, at most one end of f is part of Δ_e^1. Since α_1 and α_2 are disjoint subsets of Δ_e^1, at least one of them is disjoint from the ends of f. □

By symmetry, we may assume that the ends of f are not in α_1, and hence z_1 and z_1' are crossings between γ_f and distinct edges e_1, $e_1' \in E(C_e^1)$.

Let x_1 and y_1 be ends of e_1, e_1', respectively, and chosen such that they are in the same component of $D[C_e^1] \setminus \{z_1, z_1'\}$ containing $\{z_2, z_2'\}$. Let x_2 be the end of e_1 distinct from x_1. In this case, x_1 and y_1 belong to the same one Σ_f^1 and Σ_f^2. By reorienting f, if necessary, we may assume that x_1, $y_1 \in \Sigma_f^1$. Because γ_f crosses e_1, $x_2 \in \Sigma_f^2$.

If one of z_2 and z_2' is an end of f, then we label such an end of f as y_2. In the alternative, both z_2, z_2' are crossings. If e_2 is the edge of C_e^1 that crosses γ_f at z_2, then one of the two ends of e_2 is in Σ_f^2; in this case, we label such an end as y_2.

In any case, we obtained four vertices x_1, x_2, y_1, y_2 occurring in this cyclic order in C_e^1, and such that x_i, y_i are drawn in D_e^i, for $i = 1, 2$, contradicting Lemma 4.4.22. This shows that $c(\gamma_2) \leq 2$, as desired. □

The next Lemma guarantees that, in the process of enlarging a set of pseudocircles extending some edges of K_n, the added pseudocircle satisfies (ii) in Definition 4.4.20.

Lemma 4.4.24. *Let D be an h-convex drawing of K_n with a given choice of convex sides, and let e and e' be distinct edges of K_n. Then there exists a simple closed curve γ_e with $D[e] \subseteq \gamma_e$ and $\gamma_e \setminus D[e]$ drawn in F_e, sufficiently near $D[C_e^1 - e]$, such that the intersection between γ_e and $D[e']$ is either empty; a crossing with the interior of $D[e']$; or a single vertex that is a common end of e and e'.*

Proof. If e and e' have an end in common, then $D[e'] \subseteq \Delta_e^1 \cup \Delta_e^2$, and hence the intersection of any curve γ_e, with $\gamma_e \setminus D[e] \subseteq F_e$, is $D[e] \cap D[e']$. In this case, Lemma 4.4.24 holds.

Henceforth we assume that e and e' have no ends in common. For this case, the proof is similar to the proof of Lemma 4.4.23: let $Z = D[e'] \cap D[C_e^1]$. For each $z \in Z$, we consider a short arc α of $D[e']$ centered at z; in case z is an end of $D[e']$, we instead let α be a short segment of $D[e']$ starting at z. In any case, the contribution $c(z)$ of z is the number of components $\alpha \setminus z$ drawn in F_e.

Observe that $c(z)$ is the number of forced crossings near z, between $D[e']$ and a pseudocircle extending $D[e]$ drawn near $D[C_e^1]$. Since every two edges are either crossing, have an edge in common or are disjoint, $c(z) \in \{0, 1\}$ for all $z \in Z$.

Thus, if we let $c(e') = \sum_{z \in Z} c(z)$, and we choose γ_e near enough $D[C_e^1]$, then

$$|\gamma_e \cap D[e']| = c(e').$$

It is enough to show that $c(e') \leq 2$.

If e' has both ends in the same one of Σ_e^1 and Σ_e^2, then $D[e'] \subseteq \Delta_e^1 \cup \Delta_e^1$, and consequently, $c(e') = 0$. Thus, we may assume that e' has ends on each of Σ_e^1 and Σ_e^2.

By way of contradiction, suppose that $c(e') > 2$. Let z, $z', z'' \in Z$ such that $c(z) = c(z') = c(z'') = 1$. At least two of these points in Z, say z and z', are not ends of e'. In this case, each of z, z' is a crossing between $D[e']$ and distinct edges f, $f' \in E(C_e^1)$. Let x_1 and y_1 be ends of f and f', respectively. We choose x_1 and y_1 to be in the same component of $C_e^1 - f - f'$ containing z''. In this case, x_1 and y_1 belong to the same one of $\Sigma_{e'}^1$ and $\Sigma_{e'}^2$. Let x_2 be the end of f distinct from x_1. By reorienting e', if necessary, we may assume that x_1 and y_1 are in $\Sigma_{e'}^1$, while $x_2 \in \Sigma_{e'}^2$.

Either z'' is an end of e', and then we let $y_2 = z''$, or z'' is a crossing between e' and an edge f'' of C_e^1, and then we choose y_2 to be the end of f'' in $\Sigma_{e'}^2$. In any case, the vertices x_1, x_2, y_1, y_2 occur in this cyclic order in C_e^1, and such that x_i, y_i belong to D_e^i, for $i = 1, 2$. This contradiction to Lemma 4.4.22 shows that $c(e') \leq 2$, as desired. □

The next result follows immediately from Lemma 4.4.23 and 4.4.24.

Lemma 4.4.25. *Let D be an h-convex drawing with a given choice of convex sides, and let $J \subseteq E(K_n)$. Suppose that $\{\gamma_f : f \in J\}$ is a set of pseudocircles extending J. Then, for every $e \in E(K_n) \setminus J$, there exists a simple closed curve γ_e, sufficiently near $D[C_e^1]$, such that $\{\gamma_f : f \in J \cup \{e\}\}$ is a set of pseudocircles extending $J \cup \{e\}$.* □

The following Corollary completes the proof Theorem 4.4.2, stating that h-convex drawings are pseudospherical, and it is a straightforward application of Lemma 4.4.25.

Corollary 4.4.26. *Every h-convex drawing of K_n is pseudospherical.* □

4.5 Finding curves that also satisfy (PS3')

In this section we conclude the proof of Theorem 4.1.2. For this, we will use the fact that pseudospherical drawings are h-convex and we will prove the following.

Theorem 4.5.1. *Every h-convex drawing of K_n has a set of pseudocircles extending $E(K_n)$ and satisfying (PS3').*

In Subsection 4.4.2 we saw that every f-convex drawing D of K_n is equivalent, via a stereographic projection ϕ, to a pseudolinear drawing D' in the plane. Then, by considering the inverse image ϕ^{-1} of each pseudoline in an arrangement extending the edges in D', and adding to each of these images the point of the sphere from which we are projecting in ϕ, we get an arrangement of pseudocircles extending $E(K_n)$ and satisfying (PS3'). Thus, in the proof of Theorem 4.5.1, we will assume that the drawing we are considering is not f-convex.

The next lemma, the first ingredient we need to show Theorem 4.5.1, tells us more about a set of extending pseudocircles in an h-convex drawing that is not f-convex.

Lemma 4.5.2. *Let D be an h-convex drawing and let $J \subsetneq E(K_n)$. Suppose that $\{\gamma_f : f \in J\}$ is a set of pseudocircles extending J and that $e \in E(K_n) \setminus J$. Then either D is f-convex, or for every $f \in J$, $\gamma_f \cap F_e \neq \emptyset$.*

Proof. Suppose that for some $f \in J$, $\gamma_f \cap F_e = \emptyset$. This implies that $\gamma_f \subseteq \Delta_e^1 \cup \Delta_e^2$. From Lemma 4.4.18 we know that $\Delta_e^1 \cap \Delta_e^2 = D[e]$, and since γ_f crosses $D[e]$ at most once, $\gamma_f \subseteq \Delta_e^\ell$ for some $\ell \in \{1,2\}$. In particular, (i) f has both ends in Δ_e^ℓ; and (ii) every vertex of C_e^ℓ is in the same one of Δ_f^1 and Δ_f^2.

Claim 1. *f is an edge of C_e^ℓ.*

Proof. If an edge xy crosses f, then x, y, are in different ones of Σ_f^1 and Σ_f^2. Thus, (ii) implies that no edge joining two vertices in C_e^ℓ crosses f. In particular, if we let z be an end of e, and let xy run through the edges of $C_e^\ell - z$, then, as the chosen convex sides of the 3-cycles xyz cover Δ_e^ℓ, it follows that one of these contains $D[f]$. Let xyz be such a cycle whose chosen convex side includes $D[f]$.

By way of contradiction suppose that f has an end u distinct from x, y and z. Since u is drawn in the chosen convex side Δ_{xyz} of xyz, so also are the edges ux, uy, uz. By h-convexity, the chosen convex sides of the triangles uxy, uxz, uyz are included in Δ_{xyz}.

Let v be the end of f distinct from u. If $v \in \{x,y,z\}$, then the two vertices of $\{x,y,z\} \setminus \{v\}$ are on distinct ones of Σ_f^1 and Σ_f^2, contradicting (ii). Thus, v is in the chosen convex side bounded by one of uxy, uxz, uyz, and the two of $\{x,y,z\}$ in such boundary are on distinct ones of Σ_f^1 and Σ_f^2, a contradiction. Thus, both ends of f are in C_e^ℓ.

If $f \notin E(C_e^\ell)$, then f is a chord of C_e^ℓ drawn in Δ_e^ℓ. Then there is an edge xy with ends in C_e^ℓ that crosses f. In such case, x and y are on distinct ones of Σ_f^1 and Σ_f^2, a contradiction. □

Lemma 4.4.14 shows that D_e^ℓ is f-convex. Since f is an edge in C_e^ℓ, and C_e^ℓ is the boundary of the face F_e^ℓ witnessing f-convexity in D_e^ℓ, all the vertices of D_f^ℓ not incident with f are in Σ_f^k, for some $k \in \{1,2\}$. As $\gamma_f \subseteq \Delta_e^\ell$, every vertex of $\Delta_e^{3-\ell}$ is in the same Σ_f^k as the ends of e not incident with f. Then all the vertices of K_n are in D_f^k, and hence $D = D_f^k$. It follows from Lemma 4.4.14 that D is f-convex. □

We are ready to prove Theorem 4.5.1.

Proof of Theorem 4.5.1. Let D be an h-convex drawing of K_n. From the discussion preceeding Lemma 4.5.2, we may assume that D is not f-convex.

Let J be a set of $E(K_n)$ for which there exists a set $\{\gamma_f : f \in J\}$ of pairwise intersecting pseudocircles extending J. From Lemma 4.4.24, we may choose J to have at least one edge.

If $J = E(K_n)$, then J is our desired set of pseudocircles. Otherwise, let $e \in E(K_n) \setminus J$. Our aim is to find a pseudocircle γ_e extending e, and such that $\{\gamma_f : f \in J\} \cup \{\gamma_e\}$ is a set of pairwise intersecting pseudocircles extending $J \cup \{e\}$.

Let M be the set of edges $g \in E(K_n) \setminus (J \cup \{e\})$ such that $D[g] \cap F_e \neq \emptyset$. We repeatedly apply Lemma 4.4.25 to the edges g in M to obtain pseudocircles δ_g. This yields a set $\Gamma = \{\gamma_f : f \in J\} \cup \{\delta_g : g \in M\}$ of pseudocircles extending $J \cup M$.

For our initial approximation to γ_e, we consider the simple closed curve γ_e^0 extending e, obtained from applying Lemma 4.4.25 to Γ and e. We let I_0 be the set of edges in $J \cup M$, for which their corresponding curves in Γ intersect γ_e^0. We remark that $M \subseteq I_0$: every edge $g \in M$ cannot have both ends in the same one of Δ_e^1 and Δ_e^2. Thus, if we follow $D[g]$ from a point in F_e towards the end of $D[g]$ in Δ_e^1, $D[g]$ crosses $D[C_e^1]$. As γ_e^0 is chosen to be near $D[C_e^1]$, δ_g crosses γ_e^0.

A *feasible extension for* e is a simple closed curve γ including $D[e]$, such that $\Gamma \cup \{\gamma\}$ is a set of pseudocircles extending $J \cup M \cup \{e\}$ in D. For every feasible extension γ, the arc $\gamma \setminus D[e]$ partitions F_e into two open regions; we refer to the region bounded by $(\gamma \setminus D[e]) \cup D[C_e^2 - e]$ as Θ_γ (see Figure 4.11a). We use $cr(\Theta_\gamma)$ to denote the number of crossings of Γ inside Θ_γ.

Informally speaking, $cr(\Theta_\gamma)$ measures how close is γ is to C_e^2. The idea of the proof is to start with the feasible extension $\gamma = \gamma_e^0$ (that indeed is far from C_e^2); then we slide γ

towards C_e^2. At each step, either γ intersects more curves in Γ or $cr(\Theta_\gamma)$ decreases. The key is to show that if there is a moment where γ cannot be pushed closer to C_e^2 it is because our current curve intersects all the curves in Γ.

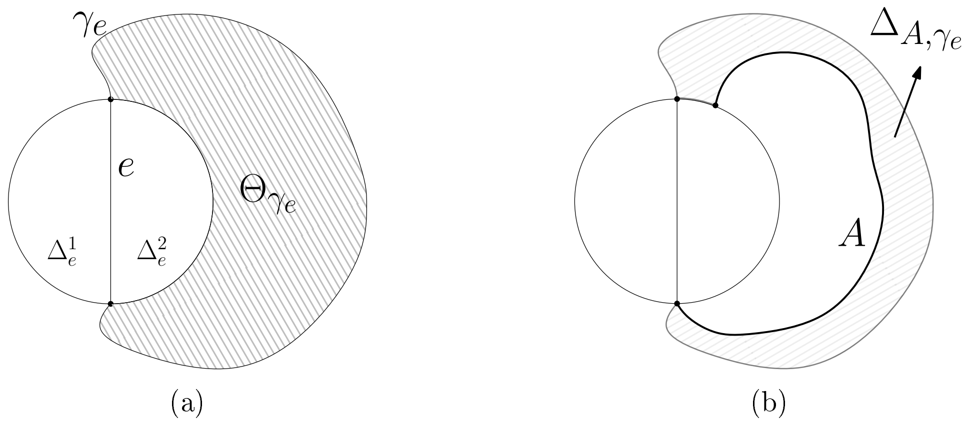


Figure 4.11: Definitions in the proof of Theorem 4.5.1.

We formalize this by considering a critical curve: among all the feasible extensions of e, we pick γ_e such that:

(i) the set $I \subseteq J \cup M$ of edges whose corresponding curves in Γ intersect γ_e is such that $I_0 \subseteq I$;

(ii) subject to (i), I is as large as possible; and

(iii) subject to (i) and (ii), $cr(\Theta_{\gamma_e})$ is as small as possible.

We aim to show that $J \subseteq I$. By way of contradiction suppose that $J \setminus I$ is not empty.

Claim 1. *Let $f \in J \setminus I$. Then $\gamma_f \cap \Theta_{\gamma_e} \neq \emptyset$.*

Proof. Since D is not f-convex, Lemma 4.5.2 implies that $\gamma_f \cap F_e \neq \emptyset$. Thus, if γ_f is disjoint from Θ_{γ_e}, then γ_f has points in the open subregion of F_e bounded by $D[C_e^1 - e] \cup (\gamma_e \setminus D[e])$. Since the ends of f are in $\Delta_e^1 \cup \Delta_e^2$, γ_f intersects the boundary of this open subregion. Then γ_f either intersects $D[C_e^1 - e]$ or γ_e. Any curve in Γ that crosses from F_e into $D[C_e^1]$ intersects δ_e^0, and hence is in I_0. Thus, in either case of γ_f intersecting $D[C_e^1 - e] \cup (\gamma_e \setminus D[e])$, condition (i) implies that γ_f intersects γ_e, contradicting that $f \in J \setminus I$. $\square$

We claim that, for every $f \in J \setminus I$, there is no face F of $\bigcup \Gamma$ for which both $\gamma_e \setminus D[e]$ and γ_f have arcs incident with F. In the alternative, we can apply a Reidemeister Type II move to shift a portion of $\gamma_e \setminus D[e]$ across γ_f, contradicting our choice of γ_e (see (ii)).

Therefore, if we let $\Gamma_I = \Gamma \setminus \{\gamma_f \ : \ f \in J \setminus I\}$, then there exists an arc A in $\bigcup \Gamma_I$ with ends in the closed arc $D[C_e^2 - e]$ and otherwise contained in Θ_{γ_e}, separating γ_e from $(\bigcup(\Gamma \setminus \Gamma_I)) \cap \Theta_{\gamma_e}$.

Let Δ_{A,γ_e} be the closure of the component of $\Theta_{\gamma_e} \setminus A$ incident with $\gamma_e \setminus D[e]$ (see Figure 4.11b). We choose A so that Δ_{A,γ_e} is minimal.

Our goal now is to show that we can slide γ_e towards C_e^2 without having a curve in Γ_i disjoint from the new γ_e. For this, we start by describing how the arcs in Γ_I intersect Δ_{A,γ_e}.

Claim 2. *Let $\gamma \in \Gamma_I$. Then every arc-component of $\gamma \cap \Delta_{A,\gamma_e}$ has an end in the interior of $\gamma_e \setminus D[e]$ and an end not in the interior of $\gamma_e \setminus D[e]$.*

Proof. Let β be an arc of $\gamma \cap \Delta_{A,\gamma_e}$. If β has both ends not in the interior of $\gamma_e \setminus D[e]$, then there is an arc A' in $\bigcup \Gamma_I$, with Δ_{A',γ_e} properly contained in Δ_{A,γ_e}, contradicting the choice of A.

Suppose now that β has both ends in the interior of $\gamma_e \setminus D[e]$. Clearly $\gamma \cap \gamma_e$ is not empty, so, for some $f \in I$, $\gamma = \gamma_f$. Since $|\gamma \cap \gamma_e| = 2$, $\gamma \setminus \beta$ is contained in the side of γ_e that contains Δ_e^1. Therefore $f \notin M$.

Let $f' \in J \setminus I$. Since $\gamma_e \cap \gamma_{f'} = \emptyset$, Claim 1 implies that $\gamma_{f'}$ is entirely drawn in the side of γ_e containing Δ_e^2. Moreover, $\gamma_{f'}$ does not intersect β, as the arc A separates $\gamma_e \setminus D[e]$ from $\gamma_{f'} \cap \Theta_{\gamma_e}$. This implies that $\gamma_f \cap \gamma_{f'} = \emptyset$, contradicting that the arcs in $\{\gamma_f \ : \ f \in J\}$ are pairwise intersecting. □

We return to the context of Subsection 4.3, and consider a decomposition of A with respect to the arrangement Γ.

Claim 3. *At least one crossing in the decomposition of A faces the side of A included in Δ_{A,γ_e}.*

Proof. By way of contradiction suppose that all the crossings in A face the side of A disjoint from Δ_{A,γ_e}. We will arrive to a contradiction by showing that A is coherent.

Let α be an arc in the decomposition of A. Since A is an arc in $\bigcup \Gamma_I$, α is included in some curve $\gamma \in \Gamma$ such that $\gamma \cap \gamma_e \neq \emptyset$.

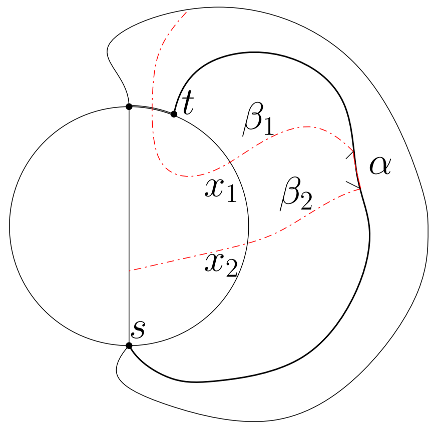


Figure 4.12: Illustrating some points in Claim 3.

Let β_1 and β_2 be the two disjoint arcs of $\gamma \setminus \alpha$, connecting an end of α to a point in γ_e, and having no internal points in γ_e (Figure 4.12).

Let s and t be the ends of A, and let B be the st-arc of $D[C_e^2 - e]$. For $i = 1, 2$, let a_i be the end of β_i in α. In case $a_i \notin \{s, t\}$, when we traverse β_i from a_i to its end in γ_e, we have a first encounter with $A \cup B$ at a point that we denote as x_i (for this part, we are using the fact that the crossings in the decomposition of A are facing towards the side of A disjoint from Δ_{A,γ_e}) . If $a_i \in \{s, t\}$, we simply let $x_i = a_i$.

If, for some $i = 1, 2$, x_i is in the interior of A, then the $a_i x_i$-arc of β_i is a coherent extension of α. Thus, as our goal is to show that α is coherent, we may assume that x_1 and x_2 are in B.

Let $Z = \gamma \cap D[C_e^2]$. For each $z \in Z$, consider a short arc σ of γ centered at z. Let $c(z)$ be the number of components of $\sigma \setminus z$ drawn in F_e. By reversing the roles of Σ_e^1 and Σ_e^2 in Lemma 4.4.24, we know that there is a simple closed curve δ_e extending e, with $\delta_e \setminus D[e]$ drawn in F_e near $D[C_e^2 - e]$, and such that $|\delta_e \cap \gamma| \leq 2$. This implies that $\sum_{z \in Z} c(z) \leq 2$.

Since γ intersects $D[e]$ at most once, at least one of β_1 and β_2, say β_1, has one end in $\gamma_e \setminus D[e]$. Each of x_1 and x_2 is in Z; furthermore, for $i = 1, 2$, there are points of $\gamma \cap F_e$ near x_i, and thus $c(x_i) > 0$. If we traverse β_1 from x_1 towards its end in $\gamma_e \setminus D[e]$, there is first point x_1' from which β_1 crosses from C_e^1 into F_e. Such a point has $c(x_1') > 0$. Either $x_1' \neq x_1$, and there are three points in Z having positive contribution c, or $x_1' = x_1$, and $c(x_1) = 2$. In any case, $\sum_{z \in Z} c(z) > 2$, contradicting the conclusion of the previous paragraph.

Thus, each arc α has a coherent extension, and hence A is coherent. However, Theorem

4.3.3 implies that not all crossings in the decomposition of A face the same side of A. This final contradiction shows that at least one of the crossings faces the side of A included in Δ_{A,γ_e}. □

Recall that Γ_I is the set of curves in Γ that intersect γ_e. From what we just showed, we may assume that there exists distinct curves γ_1, γ_2 in Γ_I, crossing at $\times_{1,2}$, where $\times_{1,2}$ is a crossing in the interior of A facing the side of A included in Δ_{A,γ_e}. For $i = 1, 2$, let β_i be the arc of $\gamma_i \cap \Delta_{A,\gamma_e}$ having $\times_{1,2}$ as an end.

From Claim 2 we know that, for $i = 1, 2$, the end a_i of β_i distinct from $\times_{1,2}$ is in the interior of $\gamma_e \setminus D[e]$.

Since $\beta_1 \subseteq \gamma_1$ and $\beta_2 \subseteq \gamma_2$, $|\beta_1 \cap \beta_2| \leq 2$. If β_1 and β_2 cross twice, we denote their crossing distinct from $\times_{1,2}$ as $\times^*_{1,2}$; otherwise, let $\times^*_{1,2} = \times_{1,2}$.

Consider the simple closed curve θ obtained by traversing β_1 from $\times^*_{1,2}$ to a_1, then following γ_e until we reach a_2, and returning back to $\times^*_{1,2}$ by following β_2. We claim that none of the curves in Γ cross the interior of each of $\theta \cap \beta_1$ and $\theta \cap \beta_2$.

Suppose that some curve $\gamma_3 \in \Gamma$ crosses the interior of $\theta \cap \beta_1$ at $\times_{1,3}$. Let β_3 be the arc-component of $\gamma_3 \cap \Delta_{A,\gamma_e}$ containing $\times_{1,3}$. Claim 2 implies that β_3 has a subarc β'_3 connecting $\times_{1,3}$ to an end of β_3 not in the interior of $\gamma_e \setminus D[e]$. We may assume that β'_3 has no further intersection with β_1.

The set $A \cup \beta'_3 \cup \beta_1$ contains an arc A' such that Δ_{A',γ_e} is properly contained in Δ_{A,γ_e}, contradicting the choice of A. Thus no arc in Γ intersects the interior of $\theta \cap \beta_1$, and likewise no arc in Γ intersects the interior of $\theta \cap \beta_2$. Moreover, no arc in Γ intersects $\theta \cap \gamma_e$ as consequence of Claim 2.

From the previous discussion we conclude that no arc in Γ has a point in the side of θ contained in Δ_{A,γ_e}. Therefore we can perform a Redeimeister III move, shifting the portion of γ_e in θ across $\times^*_{1,2}$. The obtained pseudocircle γ'_e extending $D[e]$ intersects the same curves in Γ as γ_e, and it has $cr(\Theta_{\gamma'_e}) < cr(\Theta_{\gamma_e})$, contradicting the choice of γ_e.

This last contradiction shows that a feasible extension γ_e of e satisfying (i), (ii) and (iii), intersects all the curves in $\{\gamma_f : f \in J\}$. Thus, there is a set of pairwise intersecting pseudocircles extending $J \cup \{e\}$. The result now follows from applying an easy induction. □

4.6 Concluding remarks

In this section we present some observations and questions related to pseudospherical drawings that can be interesting topics for future research.

A classic result of Bárány and Füredi [10], states that given n points in general position in the plane, there are at least $n^2 + o(n^2)$ triangles formed by taking three of these n points, that contain none of the other $n-3$ points in their interior. This is the problem of counting the number of *empty triangles* in a configuration of points in the plane. There are configurations with as few as approximately $1.6n^2 + o(n^2)$ empty triangles [11], and it is an open problem to find the right coefficient for n^2.

In a more general setting, given a good drawing of K_n, an empty triangle is a 3-cycle that contains no vertices in the interior of one of its sides. Harborth found a family of good drawings of K_n with at most $2n-4$ empty triangles [17], and Aichholzer et al. showed that every good drawing of K_n has at least n empty triangles [3]. In contrast, the Bárány and Füredi's result can be extended to pseudolinear drawings of K_n [8], so pseudolinear drawings of K_n also have least $n^2 + o(n^2)$ empty triangles. The next is a lower bound for the number of empty trianges in a pseudospherical drawing of K_n. Its proof uses the fact that in every pseudospherical drawing of K_n, each edge partitions the drawing into two pseudolinear drawings (Lemma 4.4.14).

Theorem 4.6.1. *If D is a pseudospherical drawing of K_n then D has at least $\frac{3}{4}n^2 + o(n^2)$ empty triangles.*

Proof. If D is f-convex, then it has $n^2 + o(n^2)$ empty triangles. Thus, we may assume D is not f-convex. Since D is h-convex, we consider an h-convex choice $\{\Delta_T \ : \ T \text{ is a 3-cycle}\}$ of convex sides.

Claim 1. *Every edge of K_n is in at least two empty triangles.*

Proof. Let $f \in E(K_n)$. For $i = 1, 2$, consider D_f^i the drawing induced by the vertices drawn in Δ_f^i (see Notation 4.4.10-4.4.15). For $i = 1, 2$, let $v_i \in \Sigma_f^i$ and let T_{v_i} be the 3-cycle containing v_i and f. Since $\Delta_{T_{v_i}}$ is convex, every vertex v_i' drawn in the interior of $\Delta_{T_{v_i}}$, has its $\Delta_{T_{v_i'}}$ properly included in $\Delta_{T_{v_i}}$, but $\Delta_{T_{v_i'}}$ has fewer vertices than $\Delta_{T_{v_i}}$ in its interior. Thus, by choosing vertices $v_1 \in \Sigma_f^1$ and $v_2 \in \Sigma_f^2$ minimizing the numbers of vertices in the interior of $\Delta_{T_{v_1}}$ and $\Delta_{T_{v_2}}$, we find our two desired empty triangles T_{v_1} and T_{v_2}. □

Let e be an edge of K_n. Suppose that there are k of the vertices of K_n in Δ_e^1 and $n-k+2$ vertices in Δ_e^2. Then, because the drawings D_e^1 and D_e^2 are f-convex (Lemma 4.4.14), each of these drawings exhibit at least $k^2+o(n^2)$ and $(n-k+2)^2+o(n^2)$ empty triangles.

Consider the pairs (f,T), where f is an edge with ends on distinct ones of Σ_e^1 and Σ_e^2, and T is an empty triangle containing f. Claim 1 implies that each edge with one end in Σ_e^1 and the other in Σ_e^2 is in at least two of these pairs. Thus, there are at least $2(k-2)(n-k-2)$ of such pairs. Moreover, at most two of such pairs contain a fixed triangle T. This counting gives $(k-2)(n-k-2)$ additional empty triangles.

In total, there are at least $k^2+(n-k+2)^2+(k-2)(n-k-2)+o(n^2)$ empty triangles. By optimizing k, it follows that this number is at least $\frac{3}{4}n^2+o(n^2)$. □

So naturally we have the following question:

Question 7. *Can we improve the coefficient $\frac{3}{4}$ in Theorem 4.6.1?*

In [25], Rafla stated the following beautiful conjecture:

Conjecture 4.6.2. *Every good drawing of K_n in the sphere has a Hamilton cycle drawn without self-crossings.*

There is a folklore proof for Rafla's Conjecture for rectilinear drawings (that can be easily easily adapted for pseudolinear drawings). We extend this result by showing that Rafla's conjecture holds for pseudospherical drawings.

Theorem 4.6.3. *Every pseudospherical drawing of K_n has a Hamilton cycle drawn without self-crossings.*

Proof. Suppose that D is a pseudospherical drawing of K_n, and let Γ be an arrangement of pseudocircles extending the edges of K_n. For an edge $e\in E(K_n)$, define its weight $w(e)$ to be the number of pseudocircles in Γ crossing e.

Let H be a Hamilton cycle minimizing $\sum_{e\in E(H)} w(e)$. Suppose by way of contradiction that H is self-crossed. Let xz and yw be edges of H crossing at $\times$, and denote the crossing K_4 containing these edges as J. The graph $H-xz-yw$ consists of two vertex-disjoint paths connecting two vertices of J, that by symmetry, we may assume that one of the paths connects x to y, while the other path connects z to w.

Consider the Hamilton cycle $H'=(H-xz-yw)+xw+yz$. Let α be the arc obtained by following $D[xz]$ from x to $\times$, and continuing along $D[yw]$ from $\times$ to w. Since every pseudocircle in Γ intersects the simple closed curve $D[xw] \cup \alpha$ an even number of times, and none of them can intersect xw more than once, every curve in Γ crossing xw, crosses at least one of xz and yw. Likewise, every curve in Γ crossing yz crosses at least one of xz and yw. Moreover, because xz crosses yw, but xw does not cross yz, $\sum_{e\in E(H')} w(e) \leq \sum_{e\in E(H)} w(e) - 2$, a contradiction. $\square$

It interesting to note that the fact that Γ satisfies (PS3) is not used in the previous proof.

Our last question is about extending the notion of pseudospherical to graphs that are not necessarily K_n. In particular:

Question 8. *Can we characterize drawings of graphs whose edges can be extended to a set of simple closed curves satisfying (PS1), (PS2) and (PS3)? Can we find their minimal obstructions?*

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