Recognizing Shrinkable Complexes Is NP-Complete

Dominique Attali, Olivier Devillers, Marc Glisse, Sylvain Lazard

RECOGNIZING SHRINKABLE COMPLEXES IS NP-COMPLETE*

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Abstract. We say that a simplicial complex is *shrinkable* if there exists a sequence of admissible edge contractions that reduces the complex to a single vertex. We prove that it is NP-complete to decide whether a (two-dimensional) simplicial complex is shrinkable. Along the way, we describe examples of contractible complexes that are not shrinkable.

1 Introduction

Edge contraction is a useful operation for simplifying simplicial complexes. An edge contraction consists in merging two vertices, the result being a simplicial complex with one vertex less. By repeatedly applying edge contractions, one can thus reduce the size of a complex and significantly accelerate many computations. For instance, edge contractions are used in computer graphics to simplify triangulated surfaces for fast rendering [16, 18]. For such an application, it may be unimportant to modify topological details and ultimately reduce a surface to a single point since this corresponds to what a sufficiently far away observer is expected to see [23]. However, for other applications, it may be desirable that every edge contraction preserves the topology. This is particularly true in the field of machine learning when simplicial complexes are used to approximate shapes that live in high-dimensional spaces [2, 7, 9, 12]. Such shapes cannot be visualized easily and their comprehension relies on our ability to extract reliable topological information from the simplicial complexes approximating them [8, 13, 22].

In this paper, we are interested in edge contractions that preserve the topology, more precisely, the homotopy type of simplicial complexes. It is known that edge contractions that satisfy the so-called *link condition* preserve the homotopy type of simplicial complexes [15] and, moreover, for triangulated surfaces and piecewise-linear manifolds, the link condition *characterizes* the edges whose contractions produce a complex that is homeomorphic to the original one (a constraint that is stronger than preserving the homotopy type) [14, 21]. An edge ab satisfies the *link condition* if the link of ab is equal to the intersection of the links of a and b, where the link of a face f is a simplicial complex defined as follows (see Fig. 1): consider the smallest simplicial complex that contains all the faces containing f, i.e. the

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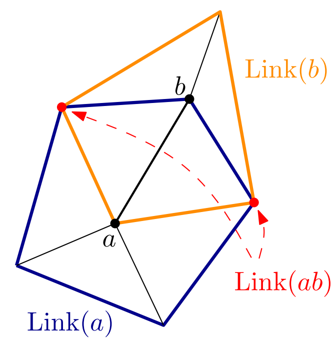


Figure 1: The link condition

star of f; the link of f is the set of faces disjoint from f in that simplicial complex [14].[1]

We only consider contractions of edges that satisfy the link condition, which implies that the homotopy type is preserved. We refer to such edge contractions as *admissible*; an admissible edge contraction is also called a *shrink* and the corresponding edge is said to be *shrinkable*. After some sequence of shrinks, the resulting complex (possibly a point) does not admit any more shrinkable edges and the complex is called (shrink) *irreducible*.

We are interested in long sequences of shrinks because they produce irreducible complexes of small size and it is natural to ask, in particular, whether a simplicial complex can be reduced to a point using admissible edge contractions. If this is the case, the simplicial complex is called *shrinkable*.

Barnette and Edelson [4] proved that a topological disk is always shrinkable (all possible sequences of shrinks work). They use this property to prove that a compact 2-manifold (orientable or not) of fixed genus admits finitely many triangulations that are (shrink) irreducible [4, 5]. For instance, the number of irreducible triangulations of the torus is 21 [19] and it is at most 396 784 for the double torus [24]. In this paper, we address the problem of recognizing whether an arbitrary simplicial complex is shrinkable.

Tancer [25] recently addressed a similar problem where he considered *admissible simplex collapses* instead of admissible edge contractions. An admissible simplex collapse (called elementary collapse in [25]) is the operation of removing a simplex and one of its facets (e.g., a triangle and one of its edges, or an edge and one of its vertices) if this facet belongs to no other simplex.[2] Such collapses preserve the homotopy type. Similarly to edge contractions, collapses are often used to simplify simplicial complexes, and a sim-

[1] In other words, in an *abstract* simplicial complex, the link of σ is the set of faces λ disjoint from σ such that $\sigma \cup \lambda$ is a face of the complex.

[2] Strictly speaking, Tancer's elementary collapse is the removal of a non-empty non-maximal face σ and the removal of all the faces containing σ if σ is contained in a unique maximal face of the simplicial complex, where maximality is considered for the inclusion in an abstract simplicial complex. Such an elementary collapse may thus correspond to a sequence of our admissible simplex collapses; for instance, an elementary collapse of a vertex, its two incident edges and its (unique) incident triangle corresponds to an admissible simplex collapse of one of these edges and the triangle, followed by the collapse of the vertex and the other edge.

plicial complex is said *collapsible* if it can be reduced to a single vertex by a sequence of admissible collapses. Tancer proved that it is NP-complete to decide whether a given (three-dimensional) simplicial complex is collapsible [25]. The proof is by reduction from 3-SAT and gadgets are obtained by altering Bing's house [6], a space that is contractible but whose triangulations are not collapsible.

Both questions of collapsibility and shrinkability are related to the question of contractibility: given a simplicial complex, is it contractible? This question is known to be undecidable for simplicial complexes of dimension four. A proof given in Tancer's paper [25, Appendix] relies on a result of Novikov [26, page 169], which says that there is no algorithm to decide whether a given five-dimensional triangulated manifold is the five-sphere. We thus cannot expect shrinks and collapses, even combined, to detect all contractible complexes, but they still provide useful heuristics towards this goal (e.g. [3]) and can even be sufficient in specific situations [15]. It is always possible to reduce a contractible simplicial complex to a point if we allow another homotopy preserving operation: the anti-collapse (the reverse operation of collapse) [10] but, of course, the undecidability of contractibility implies that the length of such a sequence cannot be bounded by a computable function.

Contributions. A shrinkable simplicial complex is clearly contractible and the converse is not true because of the above undecidability result. We first present a simple shrink-irreducible contractible simplicial complex with 7 vertices. This simple complex is interesting in its own right and it inspires the proof of our main result, that it is NP-complete to decide whether a given (two-dimensional) simplicial complex is shrinkable. Our proof uses a reduction from 3-SAT similarly as in Tancer's NP-completeness proof of collapsibility [25] but, noticeably, our gadgets are much smaller than those used for collapsibility.

Our NP-completeness result on shrinkability together with Tancer's analog on collapsibility naturally raises the question of whether it is also NP-complete to decide if a given simplicial complex can be reduced to a single vertex by a sequence *combining* admissible edge contractions and admissible simplex collapses. In this direction, we present a contractible simplicial complex with 12 vertices that is irreducible both for shrinks and collapses.

2 Preliminaries

In this paper, simplicial complexes are *abstract* and their elements are (abstract) simplices, that is, finite non-empty collections of vertices. We can associate to every abstract simplicial complex a *geometric realization* that maps every abstract simplex to a geometric simplex of the same dimension. The union of the geometric simplices forms the *underlying space* of the complex.

As mentioned in the introduction, given a simplicial complex, we are interested in operations that preserve the homotopy type of the underlying space. A popular way of simplifying simplicial complexes is the edge contraction, where the two vertices of an edge are identified, simplices containing the edge decrease in dimension and may become identical to already existing simplices (in such a case we keep only one occurrence). An edge contraction does not always preserve the homotopy type, but, as mentioned above, the edge is called admissible or shrinkable if the link condition is verified. Shrinkability is a sufficient condition of preservation of the homotopy type [15]. Below, we give a useful characterization of shrinkable edges in terms of blockers. Let $\mathcal{K}$ be a simplicial complex and recall that a *face* of a simplex is a non-empty subset of the simplex. The face is *proper* if it is distinct from the simplex.

Definition 1. *A* blocker *of $\mathcal{K}$ is a simplex that does not belong to $\mathcal{K}$ but whose proper faces all belong to $\mathcal{K}$.*

A blocker is also sometimes called a *missing face* [20], a *minimal non-face* [15], or a *simplicial hole* [17].

Lemma 2 ([15])**.** *An edge ab of $\mathcal{K}$ is shrinkable if and only if ab is not contained in any blocker of $\mathcal{K}$.*

Proof. One direction comes from the fact that if σ is a blocker containing ab, then $\sigma \setminus \{a, b\} \in \text{Link}\,(a) \cap \text{Link}\,(b)$ but $\sigma \setminus \{a, b\} \notin \text{Link}\,(ab)$. In the other direction, if $(\text{Link}\,(a) \cap \text{Link}\,(b)) \neq \text{Link}\,(ab)$ then $(\text{Link}\,(a) \cap \text{Link}\,(b)) \setminus \text{Link}\,(ab) \neq \emptyset$ and if τ is an inclusion-minimal face in this set, then $\tau \cup \{a, b\}$ is a blocker. □

As we contract shrinkable edges, blockers may appear or disappear and therefore edges may become non-shrinkable or shrinkable. For instance, consider the simplicial complex $\mathcal{L} = \{a, b, c, d, ab, bc, cd, da\}$ whose edges form a 4-circuit and the cone $\mathcal{K}$ on $\mathcal{L}$ with apex w, that is, $\mathcal{L}$ augmented by w and the set of simplices of the form $\{w\} \cup \sigma$ where $\sigma \in \mathcal{L}$. The complex $\mathcal{K}$ does not contain any blocker and therefore all edges are shrinkable. Note however that the contraction of edge ab creates a blocker acd which disappears as we contract wa. Hence, as we simplify the complex, an edge that used to be shrinkable (or not) may change its status several times during the course of the simplification. Interestingly, the only blockers we need to consider in this paper are triangles since higher-dimensional blockers cannot exist in contractible complexes of dimension two (the only complexes we consider in our constructions).

3 A simple non-shrinkable contractible simplicial complex

To construct a contractible simplicial complex that is shrink-irreducible, we start with the triangulation of the torus with 7 vertices described in Fig. 2 (it can be geometrically realized as the boundary of Császár's polyhedron [11]). Notice that the vertices and edges of this triangulation form a complete graph. Thus, every triple of vertices forms a cycle in this graph, which may or may not bound a face.

We now modify the complex as follows. The idea is to add two triangles so that every (arbitrary) cycle on the modified torus is contractible and to remove a triangle so as to open the cavity; see Fig. 3-(Left). Namely, we add triangles 012 and 035 and remove triangle 145; see Fig. 3-(Middle). The resulting complex is contractible because it is collapsible; indeed all edges and vertices inside the "square" and on the boundary of the (expanding) hole can

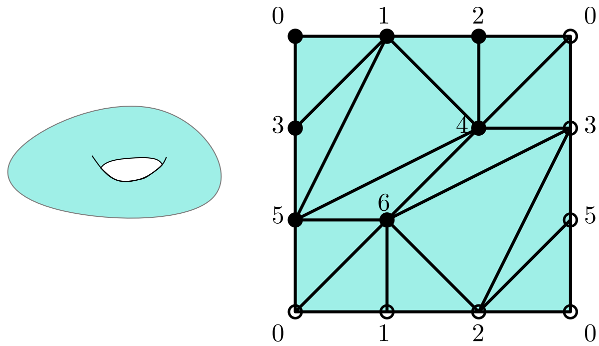


Figure 2: A torus embedded in $\mathbb{R}^3$ and its (unfolded) triangulation with 7 vertices.

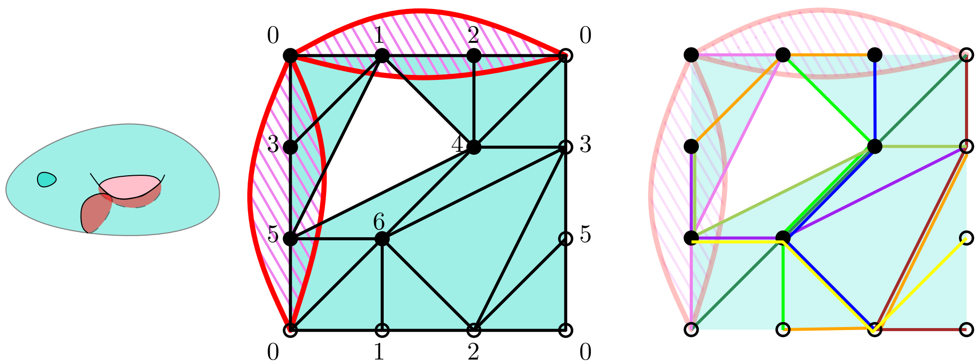


Figure 3: A contractible non-shrinkable simplicial complex. Left: embedding in $\mathbb{R}^3$. Middle: triangulation (unfolded). Right: we highlight 9 blockers (015, 023, 046, 123, 146, 246, 256, 345, 356) that suffice to cover the 21 edges.

be collapsed until the hole fills the entire square, then only triangles 012 and 035 remain, which can be trivially collapsed onto a single vertex.

To see that the resulting complex is shrink irreducible, note that every edge is incident to at most 3 triangles; indeed, every edge is incident to 2 triangles in the initial triangulation of the torus, and we only added two triangles, which do not share edges. On the other hand, every edge belongs to exactly 5 cycles of length 3 since the graph is complete on 7 vertices. Hence, every edge belongs to at least 2 blockers, which implies that no edge is shrinkable, by Lemma 2. Fig. 3-(Right) shows some of these blockers.

4 NP-completeness of shrinkability

Theorem 3. *Given an abstract simplicial complex of dimension two whose underlying space is contractible, it is NP-complete to decide whether the complex can be reduced to a point by a sequence of admissible edge contractions.*

Given a sequence of shrinks, it is easy to check in polynomial time that this sequence reduces the simplicial complex, so the problem is obviously in NP.

The proof of NP-hardness is given in this section by reduction from 3-SAT. We show that any Boolean formula in 3-conjunctive normal form (3CNF) can be transformed, in polynomial time, to a contractible two-dimensional simplicial complex, such that a satisfying assignment exists if and only if the complex is shrinkable.

Note that, in our reduction, the simplicial complexes are contractible because they are collapsible by construction. This implies that Theorem 3 can be stated in a slightly stronger form where the given simplicial complex is not only contractible but also collapsible.

4.1 Gadget design

In the following, the gadgets are defined as *abstract* 2D simplicial complexes but, for clarity, we describe a geometric realization[3] of these gadgets in $\mathbb{R}^3$. Then the gadgets are assembled by identifying some triangles of one gadget with triangles of another; as explain in Section 4.2.1 this operation preserves the blockers and thus the unshrinkability of edges. A shrinkable edge remains shrinkable if it does not belong to the identified triangles or if it was shrinkable in both gadgets. We shall see that gluing two disjoint simplicial complexes in this way creates a new connected simplicial complex and trivially preserves contractibility.

We describe below the variable gadget and clause gadget. All these gadgets are two-dimensional and 3D embeddable, but in the final assembly of all gadgets, 3D embeddability is no longer possible.

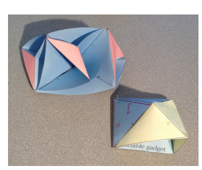
4.1.1 Variable gadget

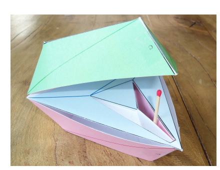
Properties. The variable gadget associated to a variable x has three special edges: $X, \bar{X}$ and L (lock). At the beginning, only these three edges are shrinkable and whatever edge is shrunk first, there is a sequence of shrinks that reduces the gadget to a single point. Furthermore, X and $\bar{X}$ cannot both be shrunk strictly before L (edges can be shrunk simultaneously if they have been identified by previous shrinks). There is also a sequence of collapses that reduces the gadget to the two triangles in which $X, \bar{X}$ and L belong.

Usage. Given a truth assignment, true (resp. false), for variable x, the edge X (resp. $\bar{X}$) of the associated gadget is contracted before the other edge $\bar{X}$ (resp. X). Gluing the lock edge to some key edges (see the clause gadgets), we ensure that once an assignment is chosen for the variable, the other edge, $\bar{X}$ (resp. X), cannot be contracted unless all the keys needed to open the lock have been released (i.e., all the blockers passing through L have been removed).

Realization. Refer to Fig. 4. We first consider a square-based skeleton pyramid of apex

[3] You can build your own 3D model with the additional material available on the journal web site.





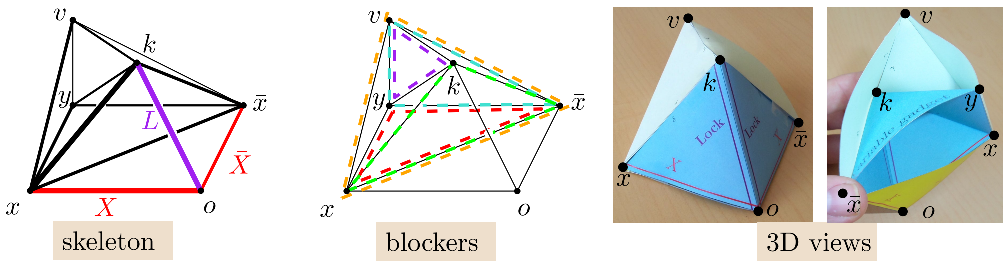


Figure 4: The variable gadget

k and base vertices x, o, $\bar{x}$, and y, in which we add the edge $x\bar{x}$ to "triangulate" the base. We define the special edges $X = ox$, $\bar{X} = o\bar{x}$, and $L = ok$. We also add an extra vertex v (above y at the height of k) and connect it to all vertices but o.

For the faces, we first add all those incident to X or $\bar{X}$ (i.e., oxk, $o\bar{x}k$, $ox\bar{x}$) so that X and $\bar{X}$ are shrinkable. The idea is to make X or $\bar{X}$ unshrinkable after the contraction of $\bar{X}$ or X, using the fact that the cycle of length four $xo\bar{x}y$ becomes a blocker after the contraction; we thus do not add face $x\bar{x}y$.

To prevent the contraction of the 10 edges different from X, $\bar{X}$ and L, we also do not add faces $x\bar{x}v$, vyk and $vy\bar{x}$.

Finally, we want L to remain shrinkable at all time (i.e., before and after X or $\bar{X}$ has been shrunk). To see that this is the case, note that all triangles incident to L are already part of the gadget. After X or $\bar{X}$ has been shrunk, L is identified with kx or $k\bar{x}$ and, to ensure that L is still shrinkable, we add all faces incident to kx and $k\bar{x}$ (except $kx\bar{x}$ since this face appears when X or $\bar{X}$ shrinks; e.g., when X is shrunk, face $o\bar{x}k$ identifies with $x\bar{x}k$), which are kxv, kxy, $k\bar{x}v$, and $k\bar{x}y$.

It remains to consider cycles vyx, for which we add a face, and $kx\bar{x}$, for which we do not. This is done so that the gadget can be reduced by a sequence of collapses to the two triangles formed by $X, \bar{X}$ and L. We give below such a sequence of admissible simplex collapses, that is, a list of (σ, Σ) where σ belongs to a unique simplex Σ and σ is a facet of Σ: $(v\bar{x}, v\bar{x}k)$, (vk, vkx), (vy, vyx), (xy, xyk), $(\bar{x}y, \bar{x}yk)$, (v, vk), (y, yk), $(x\bar{x}, x\bar{x}o)$.

This concludes the description of the realization of the gadget and it remains to prove that the required properties of the gadget are satisfied. We just proved that the gadget can be reduced by a sequence of collapses to the two triangles formed by $X, \bar{X}$ and L. Initially, only edges X, $\bar{X}$ and L are shrinkable, thus we only have to prove that X and $\bar{X}$ cannot both be shrunk strictly before L and that, whatever edge is shrunk first, there is a sequence of shrinks that reduces the gadget to a single point.

If X is shrunk first, then L is identified with xk and $\bar{X}$ is identified with $x\bar{x}$. The only shrinkable edges are now L and $k\bar{x}$ because the blockers $x\bar{x}v$, vyk, $vy\bar{x}$ and $x\bar{x}y$ remain. Shrinking $k\bar{x}$ identifies L and $\bar{X}$. Hence if X is shrunk first, L must be shrunk before $\bar{X}$, or at the same time. Similarly, if $\bar{X}$ is shrunk first, L must be shrunk before X, or at the same time. Finally, after any of the sequences of shrinks $(X, L, \bar{X})$, $(\bar{X}, L, X)$ and $(L, X, \bar{X})$, the gadget reduces to triangle yvo which can be shrunk to a point.

4.1.2 Clause gadget

Properties. The clause gadget has four special edges: three literals U, V, and W and a key K. We enforce that the key is not contracted before one of the three literals. Namely, at the beginning U, V, and W are shrinkable and K is not. K cannot be contracted before one of U, V, or W and there is a sequence of shrinks that contracts U, V, W, and K in any order where K is not first. There is also a sequence of collapses that reduces the gadget to the three triangles formed by K and each of U, V and W.

Realization. The gadget has eleven vertices: o, k, u, v, $w, \ldots$; thirty edges; twenty faces; and thirteen blockers. All vertices, but o, are coplanar and their connections are shown in Fig. 5. All vertices are linked to o by an edge. An edge in the plane is colored red if it makes a triangle with o and dashed black otherwise (then it makes a blocker with o). A first remark is that there are no tetrahedral blockers since each of the four triangles in the plane contains a black edge, which means that one face of the tetrahedron formed by the triangle and o does not belong to the gadget. Possible shrinks are visible on the figure as follows: an edge from o is shrinkable if its other vertex is not incident to a black edge; an edge in the plane is shrinkable if it is red and not part of a blocker in the plane.

We define $K = ok$, $U = ou$, $V = ov$, and $W = ow$ and refer to Fig. 5. At the beginning the only shrinkable edges are U, V, W, and kw. Shrinking kw makes K coincide with W and the only shrinkable edges become U and V. Hence, K cannot be contracted before one of U, V, or W.

We now show that if any of U, V, or W is shrunk first, there is a sequence of shrinks that contracts all the other edges in $\{U, V, W, K\}$ in any order. Shrinking U makes K shrinkable immediately. Shrinking V allows a sequence of two other shrinks (on neither U nor W) that makes K shrinkable. Finally, a sequence of 5 shrinks, starting with W (and not involving U and V) makes K shrinkable. After these first sequences of shrinks, all the other edges in $\{U, V, W, K\}$ are shrinkable in any order and the remaining complex is easily shrinkable by using sequences of shrinks similar to those depicted in Fig. 5.

Finally, it is straightforward to design a sequence of elementary collapses that reduces the gadget to the three triangles formed by K and each of U, V and W. Indeed, collapsing the four dashed black edges in a (trivial) correct order removes the four pink triangles, then the complex is a cone of apex o and collapsing all red edges except ku, kv, kw and all vertices except u, v, w, k, o yields to the desired complex.

4.2 Wrap up

4.2.1 3-SAT and shrinkability.

Given a 3CNF Boolean formula that cannot be decomposed into independent subproblems and where each variable appears at most once in each clause,[4] we build a clause gadget per

[4]3-SAT remains NP-complete with this restriction. Indeed, a clause with x and $\neg x$ can be removed and, in a clause where x appears twice, one of its occurrences can be replaced by a new variable u forced to false by adding two new variables v and w and the clauses $\neg u \vee v \vee w$, $\neg u \vee \neg v \vee w$, $\neg u \vee v \vee \neg w$, and $\neg u \vee \neg v \vee \neg w$.

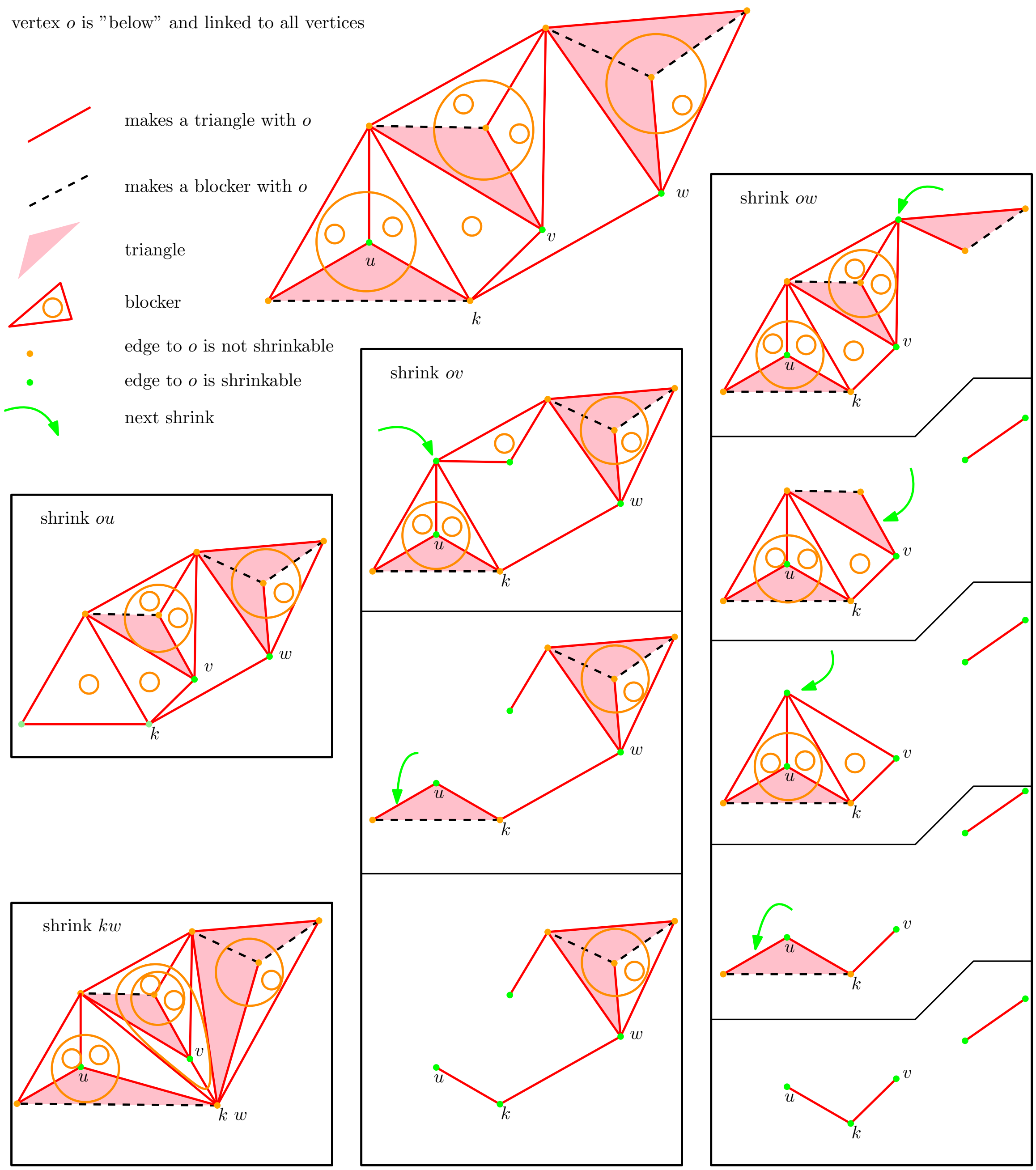


Figure 5: Clause gadget and sequences of shrinks that release the key.

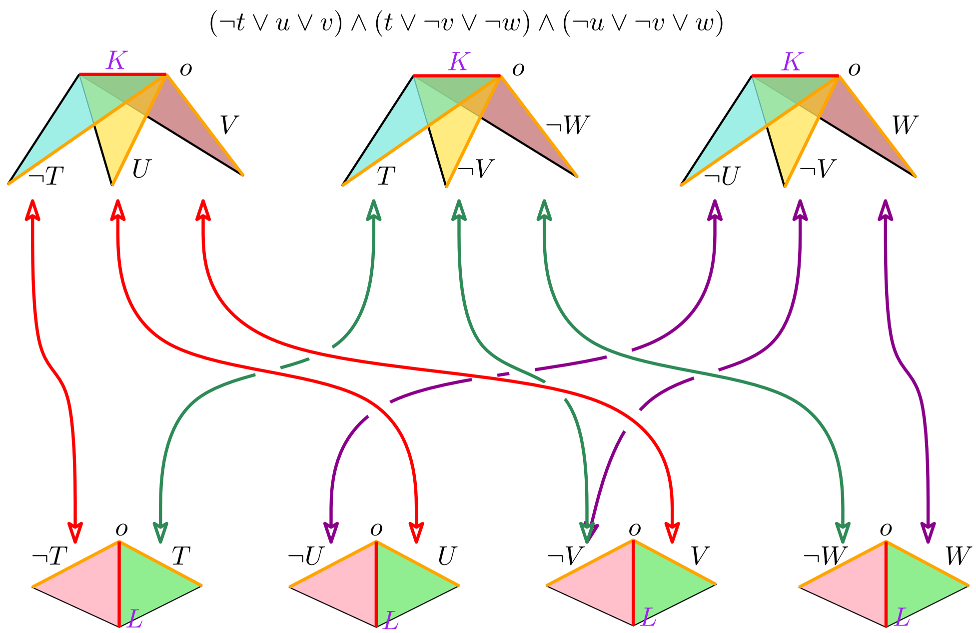


Figure 6: Schematic figure of triangles identification for SAT problem: $(\neg t \vee u \vee v) \wedge (t \vee \neg v \vee \neg w) \wedge (\neg u \vee \neg v \vee w)$. Notice that all red edges are identified in a single edge $K = L$.

clause and a variable gadget per variable. We refer in the following to the schematic Fig. 6. The literal edge of each clause gadget is glued to the relevant edge of the variable gadget, that is, the edge corresponding to a literal x (resp. $\neg x$) is glued to the edge X (resp. $\bar{X}$) of the variable gadget associated to x. The lock edge of each variable gadget is glued to the key edge of each clause it appears in.

In this construction, (i) edges K and L of all gadgets are identified and become a single edge in the final complex and (ii) their endpoints o in all gadgets are also identified and become a single vertex. Indeed, the construction glues the K and L of the clause and variable gadgets corresponding to *one* clause in the Boolean formula and, for any other clause that involves some already considered variable x, the key and lock edges associated to that clause are glued to the lock edge associated to x. Claim (i) follows since the Boolean formula cannot be decomposed into independent subproblems. For Claim (ii), observe that, for any clause gadget, edges K and one of U, V and W are glued, respectively, to L and one of X and $\bar{X}$ in some variable gadget. Claim (ii) follows since, in the clause gadget, o is incident to K, U, V and W and, in the variable gadget, o is incident to L, X and $\bar{X}$.

Notice also that any pair of edges (key, literal) belongs to a triangle in the clause gadget and that each pair of edges (lock, X) and (lock, $\bar{X}$) also belongs to a triangle in the variable gadget. Thus, the third edges of these triangles are also glued.

By construction, the complex is collapsible hence contractible. To see this, first define the *core* subcomplex of a gadget. For a variable gadget, it is defined by the 2 triangles that contain the lock and a literal. For the clause gadget, it is defined by the 3 triangles that contain the key and a literal. Each gadget can be collapsed to its core and corresponding sequences of collapses only involve simplices that are not in the core and that are thus never glued to other gadgets. The result of gluing gadgets is thus a simplicial complex that can be collapsed to a subcomplex obtained by gluing just the cores. This subcomplex consists in a set of triangles that share the edge $K = L$ and is thus obviously collapsible.

Our construction is two-dimensional, thus it can be embedded easily in $\mathbb{R}^5$ using general position for the vertices since, in dimension five, two generic flats of dimension two do not intersect. Actually, embeddability in $\mathbb{R}^4$ can be deduced for any collapsible two-dimensional complex, which is the case of our construction, from a result of Adiprasito and Benedetti [1].

4.2.2 From a sequence of shrinks to a truth assignment.

Consider a sequence of shrinks that reduces the simplicial complex to a point. For every variable gadget, if edge X (resp. $\bar{X}$) is contracted before $\bar{X}$ (resp. X), we assign true (resp. false) to the variable associated to the gadget. All clauses are satisfied by this assignment since K cannot be contracted before all clause gadgets have one of their literal edges contracted and if X (resp. $\bar{X}$) is shrunk first, then $\bar{X}$ (resp. X) cannot be contracted before K by construction of the variable gadget.

4.2.3 From a truth assignment to a sequence of shrinks.

When gluing the gadgets, vertex o and k of all gadgets are identified. The other core vertices u, v, and w of a clause gadget are identified to core vertices x or $\bar{x}$ of variable gadgets. Since we do not allow clauses of type $x \vee \neg x \vee y$, vertices u and v cannot be both identified to vertices x and $\bar{x}$ of one and the same variable gadget. Hence, edge uv cannot be added in a clause gadget during the gluing process, and similarly for vw and uw. Thus no blockers can be built through vertices of different gadgets.

Given a truth assignment, for every variable that is assigned true (resp. false), edge X (resp. $\bar{X}$) is contracted in the associated gadget. For every clause gadget, as soon as one edge U, V or W is shrunk, we execute the sequence of shrinks shown in Fig. 5 which makes K shrinkable. These sequences of shrinks do not create edges of type uv, vw, or uw (unless one of their extremities is identified with o, e.g., if U is shrunk then uv identifies with $ov = V$, but this was already an edge). Hence, no trans-gadget blockers of the type $ouv = oxy$ are created and when $K = L$ is shrinkable in all gadgets, it is shrinkable for the whole complex.

We proceed by first shrinking K, and then the negations of the truth values, i.e. $\bar{X}$ (resp. X). At this point all gadgets have been reduced to subcomplexes that are only connected by the point o and can be shrunk separately as described at Sections 4.1.1 and 4.1.2.

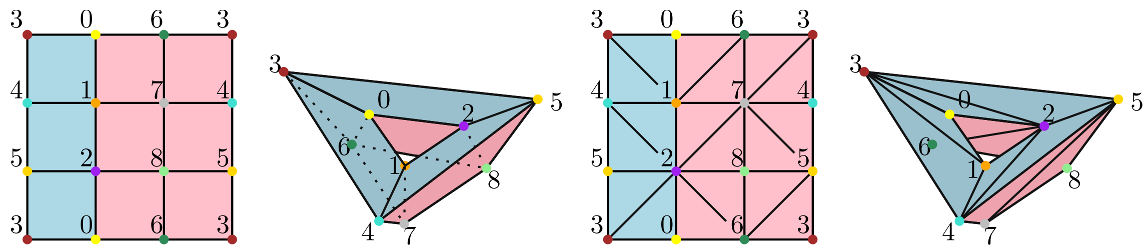


Figure 7: Triangulation of a torus with 9 vertices. From left to right: the torus represented as a square with opposite edges identified and its embedding in $\mathbb{R}^3$ as a polyhedron with 9 trapezoidal faces; a non-shrinkable triangulation; and its embedding.

5 A non-shrinkable Bing's house

In this section, we construct a contractible simplicial complex which is irreducible, both for shrinks and for collapses.[3] The idea is to triangulate carefully Bing's house [6], in such a way that no edge is shrinkable. Bing's house has two rooms, one above the other. The only access to the upper room is through an underground tunnel that passes through the lower room and the only access to the lower room is through a chimney that passes through the upper room.

We start by constructing our two rooms as two triangulated tori as in Fig. 7. These two tori are assembled by gluing their blue faces together, as described in Fig. 8-left (be careful that the right and left sides are no longer identified together but to some edges inside). Then, similarly to the construction of Fig. 3-left, we add face 012 (blue hashed in Fig. 8-right) to fill the hole of both tori, and faces 036 (red hashed) and 036' (green hashed) to transform the inside of each torus into a topological ball. It remains to dig the chimney in the upper room (and symmetrically the tunnel in the lower room) to get Bing's house. The chimney is created by adding the face 236, which splits the room in two parts: the main room and the chimney; then we open the top of the chimney by removing face 026 and its bottom by removing face 023. Notice that edge 36 is now pinched and incident to 4 faces of the complex. We proceed symmetrically for the tunnel and the result is depicted in Fig. 8-right.

To see that the resulting complex is shrink-irreducible, notice that, in the triangulated torus we start with, none of the edges is shrinkable (e.g., edge 27 is covered by blocker 276). During the modification, the only way an edge may become shrinkable is if there are more triangles incident to that edge that are added than the ones that are removed. The only edges that fulfil that condition are 12, 36 and 36' and one can check that they are still covered by blockers at the end: 123, 136 and 136' respectively.

Similarly, one can check that the complex has no collapsible vertices nor collapsible edges. Indeed, it is easy to see on Fig. 8-right that all edges belong to at least two faces.

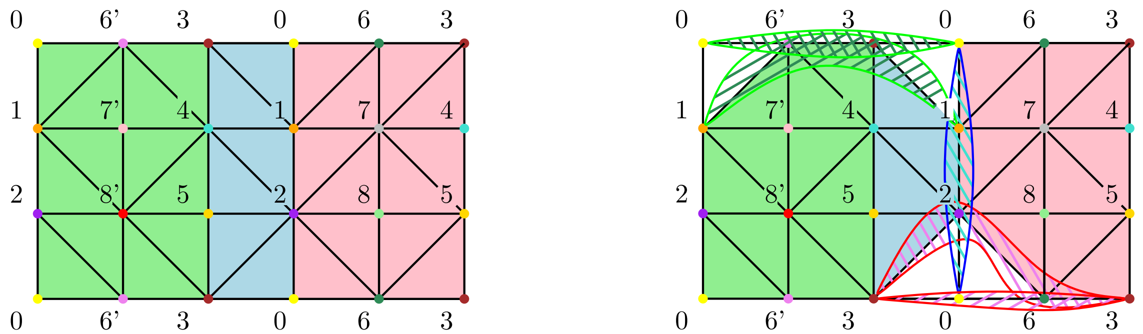


Figure 8: Building the Bing's house. Left: triangulation of the two glued tori. Right: Triangulation of Bing's house.

6 Concluding remarks

Emulating collapses with shrinks and anti-shrinks. Similar to anti-collapse being the reverse of the collapse operation [10], it is natural to introduce anti-shrink, the reverse of a shrink. The anti-shrink of a vertex x introduces a new vertex y, the edge xy and splits the link of x in three parts: $\mathrm{Link}\,(x) \setminus \mathrm{Link}\,(y)$, $\mathrm{Link}\,(y) \setminus \mathrm{Link}\,(x)$, and $\mathrm{Link}\,(x) \cap \mathrm{Link}\,(y)$.

Consider an admissible simplex collapse (σ, Σ), i.e., σ is a facet of Σ and σ belongs to no other simplices. Observe that such a collapse with $x \in \sigma$, $y = \Sigma \setminus \sigma$, and z a new vertex "in the center of Σ", can be obtained using first an anti-shrink of xz that creates simplices with z and all faces of Σ except σ; then a shrink of yz terminates the collapse.

Since we can simulate the collapse and anti-collapse with shrinks and anti-shrinks, it directly follows that any contractible complex can be reduced using shrinks and anti-shrinks, but the length of such a sequence cannot be bounded by a computable function.

Mixing shrinks and collapses. Since collapsibility and shrinkability are both NP-complete, it is natural to conjecture that reducibility by a sequence mixing shrinks and collapses is also NP-complete. One approach to tackle this conjecture would be to construct a "prevent-collapse" gadget. This gadget would need a unique shrinkable edge and no collapsible edges, such that gluing this edge on the boundary edges of our other gadgets would prevent collapses without adding or removing shrinkable edges. The result would then follow. Unfortunately, we have not been able to construct a prevent-collapse gadget with the required properties.

Notice also that our example of a non-shrinkable Bing's house is not minimal in number of vertices; we actually have a nine-vertices example described in Fig. 9.

Dimension. For a one-dimensional complex (a graph), being contractible, shrinkable and collapsible are equivalent and easy to determine in linear time (a contractible graph is a tree). Since the gadgets in our NP-completeness proof are two-dimensional, our result is tight for the dimension of the complex.

Another question is the dimension of the embedding. All our gadgets and examples are embedded in dimension three, but assembling the gadgets requires extra dimensions.

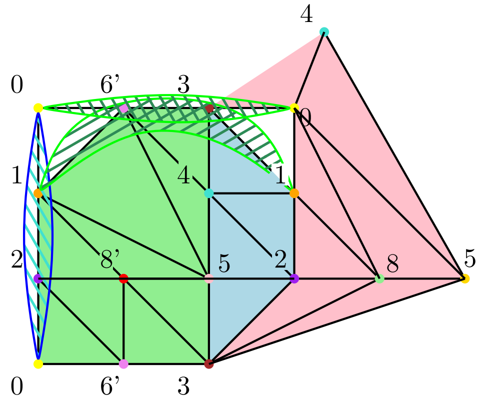


Figure 9: Non-shrinkable Bing's house with 9 vertices.

Hence, an interesting open question is the complexity of shrinkability recognition for simplicial complexes embedded in $\mathbb{R}^3$.

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