Softening Duffing oscillator reliability assessment subject to evolutionary stochastic excitation

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Abstract

An approximate analytical technique for assessing the reliability of a softening Duffing oscillator subject to evolutionary stochastic excitation is developed. Specifically, relying on a stochastic averaging treatment of the problem the oscillator time-varying survival probability is determined in a computationally efficient manner. In comparison with previous techniques that neglect the potential unbounded response behavior of the oscillator when the restoring force acquires negative values, the herein developed technique readily takes this aspect into account by introducing a special form for the oscillator non-stationary response amplitude probability density function (PDF). A significant advantage of the technique relates to the fact that it can readily handle cases of stochastic excitations that exhibit strong variability in both the intensity and the frequency content. Numerical examples include a softening Duffing oscillator under evolutionary earthquake excitation, as well as a softening Duffing oscillator with nonlinear

damping modeling the nonlinear ship roll motion in beam seas. Comparisons with pertinent Monte Carlo simulation data demonstrate the efficiency of the technique.

1 Introduction

Assessing the reliability of structural systems has been a persistent challenge in the field of engineering dynamics with diverse applications. Clearly, the level of difficulty rises when evolutionary stochastic excitation models are considered that exhibit strong variability in both the intensity and the frequency content (e.g. Spanos and Kougioumtzoglou 2012, Kougioumtzoglou 2013). In this regard, it is often desirable for risk assessment applications to estimate the probability (also known as survival probability) that the system response stays within a prescribed domain over a given time interval. Several research efforts, ranging from purely numerical techniques to approximate analytical methodologies, have focused on addressing the aforementioned challenge which is also known in the literature as the first-passage problem.

Indicatively, advanced Monte Carlo simulation (MCS) methodologies such as importance sampling, subset simulation and line sampling have been developed for reliability assessment applications; see Bucher 2011, Au et al. 2007, Au and Beck 2001, Schueller 2004 for some indicative references. Note, however, that there are cases of complex systems where MCS can be a computationally demanding, or even a prohibitive task; thus, there is a need for developing efficient approximate analytical and/or numerical techniques for addressing the first-passage problem such as Poisson distribution based approximations (e.g. Vanmarcke 1975, Barbato and Conte 2001), probability density evolution schemes (e.g. Li and Chen 2009), and stochastic averaging/linearization approaches (e.g. Spanos and Kougioumtzoglou 2014a, 2014b]). Further, a promising framework for stochastic response determination and reliability assessment of structural systems relates to the Wiener path integral (WPI) concept (e.g. Wiener 1921, Feynman 1948, Chaichian and Demichiev 2001). Recently, a WPI based methodology was developed for determining the non-stationary response probability density function (PDF) of nonlinear/hysteretic multi-degree-of-freedom-system (MDOF) systems (e.g. Kougioumtzoglou and Spanos 2012, 2014a) and of systems comprising fractional derivative elements (Di Matteo et al. 2014) Furthermore, a WPI technique was developed in Zhang and Kougioumtzoglou 2014 for determining the survival probability and first-passage PDF of nonlinear oscillators in a computationally efficient manner. Note in passing that the aforementioned WPI technique should not be confused with alternative numerical schemes (commonly referred to as numerical path integral schemes) which rely, in essence, on a discrete version of the Chapman-Kolmogorov (C-K) equation for propagating in time the system response PDF and for determining first-passage statistics (e.g. Wehner and Wolfer 1983, Naess and Johnsen 1993, Iourtchenko et al. 2008, Di Paola and Santoro 2008, Pirrotta and Santoro 2011, Kougioumtzoglou and Spanos 2013).

The softening Duffing oscillator is a nonlinear oscillator possessing a linear-plus-cubic restoring force so that the spring has a softening characteristic. This oscillator has received considerable attention in the literature primarily due to its importance in describing the roll motion of a ship model in beam seas (e.g. Spyrou and Thomson 2000, Belenky and Sevastianov 2007). Note, however, that the softening Duffing oscillator has found applications in diverse other fields of engineering dynamics such as structural system vibration isolation (e.g. Fu et al. 2014), energy harvesting (e.g. Vandewater and Moss 2013) and dynamics of timber structures (e.g. Reynolds et al. 2014).

Further, although several research efforts have focused on studying the oscillator response under deterministic excitation (e.g. Szemplinska–Stupnicka 1988, Nayfeh and Sanchez

1989, Brennan et al. 2008), limited results exist regarding the response analysis of the oscillator when it is subjected to stochastic excitation (e.g. Roberts 1986, Roberts and Vasta 2000, Cottone et al. 2010). Specifically, most of the results are based on rather heuristic approaches which inherently assume stationarity and that the probability the response leaves the stable region is extremely small; thus, neglecting important aspects of the analysis such as the possible unbounded response behavior when the restoring force acquires negative values. Recently, a numerical path integral approach was developed in Kougioumtzoglou and Spanos 2014b for determining the survival probability of a softening Duffing oscillator subject to stochastic excitation. The unbounded character of the response was rigorously taken into account by introducing a special form for the conditional response PDF, while the solution was propagated by utilizing a discrete version of the C-K equation. Note, however, that, in general, numerical path integral schemes based on discrete versions of the C-K equation can be computationally demanding; this is due to the fact that the solution needs to be advanced in short time steps, while convolution integrals need to be numerically evaluated at every time step as well.

In this paper, an efficient approximate analytical technique for determining the survival probability of a softening Duffing oscillator subject to evolutionary stochastic excitation is developed. Specifically, relying on a stochastic averaging treatment of the problem and introducing a special form for the oscillator response PDF, the technique developed in Spanos and Kougioumtzoglou 2014b is adapted and generalized herein to account for the special case of the softening Duffing oscillator. A significant advantage of the technique is that it can readily handle cases of evolutionary stochastic excitation with arbitrary evolutionary power spectrum (EPS) forms, even of the non-separable kind. Numerical examples include a softening Duffing oscillator under evolutionary earthquake excitation, as well as a softening Duffing oscillator with nonlinear damping modeling the nonlinear ship roll motion in beam seas. Comparisons with pertinent Monte Carlo simulations demonstrate the reliability of the technique.

2 Mathematical formulation

2.1 Softening Duffing oscillator response analysis

Consider the softening Duffing oscillator whose motion is governed by the equation

$$\ddot{x} + 2\zeta_0\omega_0\dot{x} + \omega_0^2 x + \varepsilon\omega_0^2 x^3 = w(t), \quad \varepsilon < 0, \tag{1}$$

where a dot over a variable denotes differentiation with respect to time t; ε denotes a negative constant representing the magnitude of the nonlinearity degree; ζ_0 is the ratio of critical damping; ω_0 is the natural frequency corresponding to the linear oscillator (i.e. $\varepsilon = 0$) and $w(t)$ represents a Gaussian, zero-mean non-stationary stochastic process possessing an evolutionary broad-band power spectrum $S_w(\omega, t)$. Examining Eq.(1), it can be readily seen that there exist values of the response displacement $x(t)$ for which the oscillator restoring force $F(x) = \omega_0^2 x + \varepsilon\omega_0^2 x^3 = \omega_0^2 x(1 + \varepsilon x^2)$ reaches zero, and even negative values. Clearly, this may lead to unbounded system response, and a special treatment is necessary to account for this behavior. Next, bearing this qualitative behavior in mind, and focusing on lightly damped systems (i.e. $\zeta_0 \ll 1$), it can be argued (e.g. Spanos and Lutes 1980) that for $F(x) = \omega_0^2 x(1 + \varepsilon x^2) \geq 0$, or equivalently $x^2 \geq -1/\varepsilon$, the oscillator response exhibits a pseudo-harmonic behavior described by the equations

$$x(t) = a\cos[\omega(a)t + \phi(t)], \tag{2}$$

and

$$\dot{x}(t) = -\omega(a)\, a\sin[\omega(a)t + \phi(t)]. \tag{3}$$

In Eqs.(2-3), ϕ and a represent a slowly varying with time phase and a slowly varying with time response amplitude, respectively. Manipulating Eqs.(2-3) yields an expression for the oscillator response amplitude; that is,

$$a(t) = \sqrt{x^2(t) + \frac{\dot{x}^2(t)}{\omega(a)}}. \quad (4)$$

It is primarily the assumption of light damping that allows a combination of deterministic and stochastic averaging to be performed next and to approximate the second-order stochastic differential equation (SDE) (Eq.(1)) by a first-order SDE governing the response amplitude process a. A more detailed presentation/discussion of the assumptions involved and the corresponding assumed pseudo-harmonic behavior of the response process $x(t)$ can be found in references (e.g. Spanos and Lutes 1980, Roberts and Spanos1986, Zhu 1996, Kougioumtzoglou and Spanos 2009). Next, following a stochastic averaging/linearization approach (e.g. Kougioumtzoglou and Spanos 2009, Roberts and Spanos 2003) a linearized version of Eq.(1) becomes

$$\ddot{x} + 2\zeta_0\omega_0\dot{x} + \omega^2(a)x = w(t), \quad (5)$$

where the equivalent natural frequency $\omega(a)$ is given by the expression

$$\omega^2(a) = \frac{\omega_0^2}{\pi a}\int_0^{2\pi} cos\psi(a\cos\psi + \varepsilon(a\cos\psi)^3)\, d\psi = \omega_0^2\left(1 + \frac{3}{4}\varepsilon a^2\right). \quad (6)$$

Examining Eq.(6) it can be readily seen that the stiffness element of the equivalent linear oscillator becomes zero at the critical response amplitude value $a_{cr} = \sqrt{-4/(3\varepsilon)}$. In this regard, the requirement $x^2 \geq -1/\varepsilon$ for the oscillator of Eq.(1) to have a bounded response is equivalently expressed in the following by the requirement $a < a_{cr}$. Bearing this qualitative aspect in mind, a special form for the non-stationary response amplitude PDF $p(a,t)$ is introduced next; that is,

$$p(a,t) = \frac{a}{c(t)}\exp\left(\frac{-a^2}{2c(t)}\right) rect(a) + S(t)\delta(a - a_\infty), \qquad (7)$$

where $rect(a) = u(a) - u(a - a_{cr})$, $u(.)$ denotes the unit step function, $c(t)$ is a time-dependent coefficient to be determined, $\delta(.)$ denotes the Dirac delta function, and a_∞ represents an arbitrary response amplitude value with the property $a_\infty \gg a \in [0, a_{cr}]$. Further, the time-dependent factor $S(t)$ can be determined by applying the normalization condition $\int_0^\infty p(a,t)da = 1$; this yields

$$S(t) = 1 - \int_0^{a_{cr}} \frac{a}{c(t)} \exp\left(\frac{-a^2}{2c(t)}\right) da = \exp\left(-\frac{{a_{cr}}^2}{2c(t)}\right). \qquad (8)$$

Examining the form of the non-stationary response amplitude PDF of Eq.(7), it can be readily seen that it comprises two conceptually different terms. The first one represents a truncated Rayleigh PDF for amplitude values in the range $[0, a_{cr}]$, whereas the factor $S(t)$ in the second term represents the probability at a specific time instant that the response grows unbounded, namely the system response asymptotically approaches infinity. The rationale behind the choice of the truncated time-dependent Rayleigh PDF of Eq.(7) relates to the fact that the linear oscillator stationary response amplitude PDF is a Rayleigh one (see also [42]). In fact, as it was shown in Spanos and Lutes 1980, the non-stationary response amplitude PDF of a linear oscillator subject to Gaussian white noise excitation is a time-dependent Rayleigh PDF of the form $p(a,t) = \frac{a}{c(t)} \exp\left(\frac{-a^2}{2c(t)}\right)$ with the property $\lim_{t\to\infty} p(a,t) = \frac{a}{\sigma^2} exp\left(-\frac{a^2}{2\sigma^2}\right)$; where σ^2 represents the linear oscillator stationary response variance. In [40] it was further shown that the Rayleigh representation is suitable for nonlinear oscillators also and under evolutionary stochastic excitation as well. It is pointed out that a significant difference between adopting a PDF of the form $p(a,t) = \frac{a}{c(t)} \exp\left(\frac{-a^2}{2c(t)}\right)$ in [40] and introducing a PDF form of Eq.(7) in the herein developed technique, is that in the former case $c(t)$ accounts for the variance of the non-stationary response process x, whereas in the latter case $c(t)$ is simply a time-varying coefficient to be determined. Further, note that for the case where the oscillator is assumed to be initially at rest, i.e. $p(a_0, t_0 = 0) = \delta(a_0)$, the amplitude PDF $p(a,t)$ values will be concentrated around $a = 0$ for the very early part of the oscillation duration, or in other words, $\lim_{t\to 0^+} c(t) = 0$ which yields $\lim_{t\to 0^+} S(t) = 0$; that is, the probability that the system response will grow unbounded goes to zero as $t \to 0^+$.

Next, relying on Eq.(7), it can be argued that an alternative to Eq.(5) equivalent linear system is given in the form

$$\ddot{x} + 2\zeta_0\omega_0\dot{x} + \omega_{eq}^2(t)x = w(t), \quad (9)$$

where the time-dependent stiffness element $\omega_{eq}^2(t)$ is defined as (see also Kougioumtzoglou and Spanos 2009, Spanos and Kougioumtzoglou 2014b)

$$\omega_{eq}^2(t) = E[\omega^2(a)] = \int_0^\infty \omega^2(a)p(a,t)da. \quad (10)$$

Note that taking into account the form of the amplitude PDF of Eq.(7), the time-varying equivalent stiffness element of Eq.(10) also has two parts. Specifically, for $a \in [0, a_{cr}]$, $\omega_{eq}^2(t)$ has a bounded part, i.e. $\omega_{eq,B}^2(t)$, whereas for $a > a_{cr}$ the stiffness element $\omega_{eq}^2(t)$ exhibits negative values; thus, yielding negative restoring force values resulting potentially in an unbounded system response behavior. In this regard, utilizing Eq.(7) the bounded part $\omega_{eq,B}^2(t)$ is determined as

$$\omega_{eq,B}^2(t) = \int_0^{a_{cr}} \omega^2(a)p(a,t)da. \quad (11)$$

Analytical determination of the integral in Eq.(11) yields

$$\omega_{eq,B}^2(t) = \omega_0^2\left(1 + \frac{3}{2}\varepsilon c(t)(1 - S(t))\right). \quad (12)$$

Examining Eq.(12) it can be readily seen that the stiffness element $\omega_{eq,B}^2(t)$ is bounded between the values 0 and ω_0^2. Specifically, assuming that the oscillator is initially at rest yields $\lim_{t\to 0^+} p(a,t) = \delta(a_0)$, or in other words, $\lim_{t\to 0^+} c(t) = 0$, which yields $\lim_{c(t)\to 0^+} \omega_{eq,B}^2(t) = \omega_0^2$. This means that for the very early part of the oscillation duration the oscillator features an approximately linear restoring force. Further, as time increases and the transient phase progresses, the truncated Rayleigh PDF of Eq.(7) broadens as the oscillator exhibits higher amplitude values $a(t)$. Equivalently, the time-varying coefficient $c(t)$ increases with time, whereas the equivalent stiffness part $\omega_{eq,B}^2(t)$ decreases with time. Taking into account Eqs.(7) and (12) it can be readily shown that in the extreme case $\lim_{c(t)\to\infty} \omega_{eq,B}^2(t) = 0$. Thus, the equivalent stiffness part $\omega_{eq,B}^2(t)$ is a non-negative and bounded quantity varying with time between the values 0 and ω_0^2. This is in agreement with the fact that $\omega_{eq,B}^2(t)$ corresponds to amplitude values $a \in [0, a_{cr}]$ where the oscillator response is assumed to behave in a bounded manner.

Further, focusing on the case where $a \in [0, a_{cr}]$ and based on a stochastic averaging approach Eq.(9) can be cast in a first-order SDE governing the evolution in time of the amplitude $a(t)$; see Spanos and Lutes 1980, Roberts and Spanos 1986, Zhu 1996, Kougioumtzoglou and Spanos 2009 for a more detailed presentation. Related to this SDE is the Fokker-Planck (F-P) partial differential equation

$$\frac{\partial p(a,t|a_1,t_1)}{\partial t} = -\frac{\partial}{\partial a}[K_1(a,t)p] + \frac{1}{2}\frac{\partial^2}{\partial a^2}[K_2(a,t)p], \qquad (13)$$

where

$$K_1(a,t) = -\zeta_0\omega_0 a + \frac{\pi S\big(\omega_{eq,B}(t),t\big)}{2a\omega_{eq,B}^2(t)}, \qquad (14)$$

and

$$K_2(a,t) = \frac{\pi S\big(\omega_{eq,B}(t),t\big)}{\omega_{eq,B}^2(t)}. \qquad (15)$$

The F-P Eq.(13) governs the evolution in time of the transition PDF $p(a,t|a_1,t_1)$ for $a \in [0, a_{cr}]$ and $a_1 \in [0, a_{cr}]$. Next, a solution of the associated F-P equation $p(a,t|a_1 = 0, t_1 = 0) = p(a,t)$ is attempted in the form of the truncated Rayleigh PDF of Eq.(7). Specifically, substituting the truncated Rayleigh PDF into the associated F-P equation, assuming that the oscillator is initially at rest (i.e. $p(a,t=0) = \delta(a)$), and manipulating yields the first-order nonlinear differential equation

$$\dot{c}(t) = -2\zeta_0\omega_0 c(t) + \frac{\pi S\big(\omega_{eq,B}(t),t\big)}{\omega_{eq,B}^2(t)}, \qquad (16)$$

to be solved numerically for the time-varying coefficient $c(t)$. Obviously, once the time-varying coefficient $c(t)$ is determined, the time-dependent coefficient $S(t)$ can be evaluated via Eq.(8). Further, equations similar to Eq.(16) can be derived for the case of the response amplitude transition PDF in a straightforward manner. Specifically, following a similar analysis as in Spanos and Solomos 1983, the transition amplitude PDF $p(a,t|a_1,t_1)$ is sought in the form

$$p(a,t|a_1,t_1) = \begin{cases} p_{tr}(a,t|a_1,t_1) + R(t,t_1)\delta(a - a_\infty), & 0 < a_1 < a_{cr} \\ \delta(a - a_\infty), & a_1 > a_{cr} \end{cases}, \qquad (17)$$

where

$$p_{tr}(a,t|a_1,t_1) = \frac{a}{c(t,t_1)} \exp\left(-\frac{a^2 + h^2(t,t_1)}{2c(t,t_1)}\right) I_0\left(\frac{ah(t,t_1)}{c(t,t_1)}\right) rect(a), \qquad (18)$$

and $c(t,t_1)$ and $h(t,t_1)$ are time-varying coefficients to be determined. Further, applying the normalization condition $\int_0^\infty p(a,t|a_1,t_1)\, da = 1$ yields the time-varying coefficient

$$R(t,t_1) = 1 - \int_0^{a_{cr}} p_{tr}(a,t|a_1,t_1) da, \qquad (19)$$

where $I_0(.)$ denotes the modified Bessel function of the first kind and of zero order. In a similar manner as before, under the condition that $a \in [0, a_{cr}]$ and $a_1 \in [0, a_{cr}]$ substituting the bounded part of Eq.(17) into Eq. (13) and manipulating yields the first-order differential equations (see Spanos and Solomos 1983 for a more detailed derivation)

$$\frac{dc(t,t_1)}{dt} + 2\zeta_0\omega_0 c(t,t_1) - \frac{\pi S\left(\omega_{eq,B}(t),t\right)}{\omega_{eq,B}^2(t)} = 0, \quad (20)$$

and

$$\frac{dh(t,t_1)}{dt} + \zeta_0\omega_0 h(t,t_1) = 0. \quad (21)$$

Eqs.(20-21) are subject to the initial condition $p(a_2, t_1|a_1, t_1) = \delta(a_2 - a_1)$ which states that no change of state can occur if the transition time is zero.

2.2 Softening Duffing oscillator reliability assessment

In this section the approach developed in Spanos and Kougioumtzoglou 2014b is adapted and generalized herein to account for the special case of the softening Duffing oscillator and to determine the oscillator time-dependent survival probability. This is defined as the probability $P_B(t)$ that the amplitude a stays below the threshold a_{cr} over a given time interval $[t_0, T]$; that is, $Prob[a(t) \leq a_{cr}\ ,\ over\ [t_0, T]|a(t_0)\ < a_{cr}]$. In the following, adopting the dicretization scheme applied in [11] the time domain is divided into intervals of the form

$$[t_{i-1}, t_i\ ], \qquad i = 1,2\ldots, M, t_0 = 0, t_M = T\ and\ t_i = t_{i-1} + d_T\ T_{eq}(t_{i-1}), \quad (22)$$

where T_{eq} denotes the equivalent natural period of the oscillator given by

$$T_{eq}(t) = \frac{2\pi}{\omega_{eq,B}(t)}\ , \quad (23)$$

and d_T is a constant to be selected with the property $d_T \in (0,1]$. In the ensuing analysis, the survival probability is determined assuming that it is approximately constant over the time interval $[t_{i-1}, t_i]$. Clearly, for $d_T = 1$ the time interval $[t_{i-1}, t_i]$ corresponds to the equivalent time-dependent natural period of the oscillator. The choice is justified by the fact that the response amplitude a is assumed to be approximately constant over the interval $[t_{i-1}, t_i]$, owing to its slowly varying character with respect to time (see section 2.1). Thus, the survival probability $P_B(T)$ is assumed to be constant over $[t_{i-1}, t_i]$ as well. Of course, if higher accuracy is required a smaller value for d_T can be chosen. This is especially important for the case of the herein considered softening Duffing oscillator. Specifically, taking into account Eq.(12) it can be readily seen that for large enough values of the excitation intensity and/or of the nonlinearity magnitude, the equivalent time-varying natural frequency $\omega_{eq,B}(t)$ decreases significantly, or equivalently considering Eq.(23), the natural period $T_{eq}(t)$ increases considerably. Thus, the time interval $[t_{i-1}, t_i]$ of Eq.(22) increases substantially yielding potentially unrealistically large time intervals where the survival probability $P_B(T)$ is assumed to be constant. This phenomenon can be readily mitigated by selecting a small enough value for the coefficient d_T.

Further, taking into account the discretization of Eq.(22), the survival probability $P_B(T)$ is given by the equation

$$P_B(T = t_M) = \prod_{i=1}^{M} 1 - F_i, \qquad (24)$$

where F_i is defined as the probability that a will cross the barrier a_{cr} in the time interval $[t_{i-1}, t_i]$, given that no crossings have occurred prior to time t_{i-1}. Next, invoking the Markovian property for the process a and utilizing the standard definition of conditional probability yields

$$F_i = \frac{Prob[a(t_i) \geq a_{cr} \cap a(t_{i-1}) \leq a_{cr}]}{Prob[a(t_{i-1}) \leq a_{cr}]} = \frac{Q_{i-1,i}}{H_{i-1}} \qquad (25)$$

where

$$H_{i-1} = \int_0^{a_{cr}} p(a_{i-1}, t_{i-1}) da_{i-1}, \quad (26)$$

and, by utilizing the relationship $p(a_1, t_1; a_2, t_2) = p(a_1, t_1)p(a_2, t_2|a_1, t_1)$,

$$Q_{i-1,i} = \int_0^{a_{cr}} \left(\int_{a_{cr}}^{+\infty} p(a_i, t_i|a_{i-1}, t_{i-1}) da_i \right) p(a_{i-1}, t_{i-1}) da_{i-1}. \quad (27)$$

Next, taking into account Eqs.(7) and (17), Eqs.(26-27) become

$$H_{i-1} = 1 - \exp\left(\frac{a_{cr}^2}{2c(t_{i-1})} \right), \qquad (28)$$

and

$$Q_{i-1,i} = \int_0^{a_{cr}} \left(\int_{a_{cr}}^{+\infty} \left(p_{tr}(a_i, t_i|a_{i-1}, t_{i-1}) + R(t_i, t_{i-1})\delta(a_i - a_\infty) \right) da_i \right) p(a_{i-1}, t_{i-1}) da_{i-1}, \quad (29).$$

respectively. Taking into account the properties of the Dirac delta function, Eq.(29) becomes

$$Q_{i-1,i} = \int_0^{a_{cr}} R(t_i, t_{i-1})\, p(a_{i-1}, t_{i-1}) da_{i-1}, \quad (30)$$

and utilizing Eq.(19) yields

$$Q_{i-1,i} = \int_0^{a_{cr}} p(a_{i-1}, t_{i-1})\, da_{i-1} - \int_0^{a_{cr}} \left(\int_0^{a_{cr}} p_{tr}(a_i, t_i|a_{i-1}, t_{i-1}) da_i \right) p(a_{i-1}, t_{i-1})\, da_{i-1}. \quad (31)$$

Next, considering Eqs.(26), Eq.(31) takes the form

$$Q_{i-1,i} = H_{i-1} - \int_0^{a_{cr}} \int_0^{a_{cr}} p_{tr}(a_i, t_i|a_{i-1}, t_{i-1}) p(a_{i-1}, t_{i-1}) da_i\, da_{i-1}. \quad (32)$$

Relying further on the assumption that $\omega_{eq,B}(t)$ follows a slowly varying with time behavior, the following approximation over a small time interval $[t_{i-1}, t_i]$ is introduced; i.e., $\omega_{eq,B}(t) = \omega_{eq,B}(t_{i-1})$ for $t \in [t_{i-1}, t_i]$. Next, based on the slowly varying with time behavior of the EPS, $S_w(\omega, t)$ is also treated as a constant over the interval $[t_{i-1}, t_i]$. Further, based on the above assumptions, introducing the variable $\tau_i = t_i - t_{i-1}$, and applying a first-order Taylor expansion around point $\tau_i = 0$, Eqs.(20-21) become (see Spanos and Kougioumtzoglou 2014b for a detailed derivation)

$$c(t_{i-1}, t_i) = \frac{\pi S_w\big(\omega_{eq,B}(t_{i-1}), t_{i-1}\big)}{\omega_{eq,B}^2(t_{i-1})}\tau_i, \qquad (33)$$

and

$$h(t_{i-1}, t_i) = a_{i-1}\sqrt{1 - 2\zeta_0\omega_0\tau_i}, \qquad (34)$$

respectively. Furthermore, considering Eqs.(20) and (33) and applying a first-order Taylor expansion for the time-varying coefficient $c_i(t)$ around point $t = t_{i-1}$ yields

$$c(t_i) = c(t_{i-1}, t_i) + c(t_{i-1})(1 - 2\zeta_0\omega_0\tau_i). \qquad (35)$$

Next, setting

$$r_i^2 = \frac{c(t_{i-1})}{c(t_i)}(1 - 2\zeta_0\omega_0\tau_i), \qquad (36)$$

Eq.(35) yields

$$c(t_{i-1}, t_i) = c(t_i)(1 - r_i^2). \qquad (37)$$

Further, taking into account Eq.(32) and expanding the Bessel function $I_0(x)$ in the form (e.g. Spanos and Kougioumtzoglou 2014b)

$$I_0(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{k!\,\Gamma(k+1)}, \qquad (38)$$

analytical treatment of the involved double integral of Eq.(32) is possible yielding

$$Q_{i-1,i} = H_{i-1} - (A_0 + \sum_{n=1}^{N} A_n), \qquad (39)$$

where

$$A_0 = \left(1 - \exp\left(-\frac{a_{cr}^2}{2c(t_i)(1-r_i^2)}\right)\right)\left(1 - \exp\left(-\frac{a_{cr}^2}{2c(t_{i-1})(1-r_i^2)}\right)\right)(1-r_i^2), \quad (40)$$

$$A_n = \frac{{r_i}^{2n}(1-{r_i}^2)}{\prod_{m=1}^{n}(m)^2} L_n = \frac{{r_i}^{2n}(1-{r_i}^2)}{(n!)^2} L_n\,, \qquad (41)$$

and

$$L_n = \left(\Gamma[1+n,0] - \Gamma\left[1+n, \frac{{a_{cr}}^2}{2c(t_{i-1})(1-{r_i}^2)}\right]\right)\left(\Gamma[1+n,0] - \Gamma\left[1+n, \frac{{a_{cr}}^2}{2c(t_i)(1-{r_i}^2)}\right]\right). \quad (42)$$

In Eq.(42) $\Gamma[\gamma, z]$ represents the incomplete Gamma function defined as $\Gamma[\gamma, z] = \int_z^{+\infty} t^{\gamma-1} e^{-t} dt$. A more detailed presentation of the derivations in this section can be found in Spanos and Kougioumtzoglou 2014b.

Concisely, the developed technique comprises the following steps:

i. Determination of the time-varying coefficient $c(t)$ via numerical solution of Eq.(16).

ii. Determination of the bounded equivalent time-varying natural frequency $\omega_{eq,B}(t)$ via Eq.(12).

iii. Determination of the effective natural period $T_{eq}(t)$ (Eq.(23)) and discretization of the time domain via Eq.(22).

iv. Determination of the parameters H_{i-1} and $Q_{i-1,i}$ via Eqs.(28) and (39).

v. Determination of the survival probability $P_B(T)$ via Eq.(24).

3 Numerical examples

3.1 Softening Duffing oscillator under earthquake excitation

As noted in the introductory section, although the softening Duffing oscillator has been widely utilized to model the nonlinear ship rolling motion in beam seas (e.g. Spyrou and Thomson 2000, Belenky and Sevastianov 2007), it has also been used in conjunction with structural dynamics/earthquake engineering applications such as structural system vibration isolation (e.g. Fu et al. 2014), energy harvesting (e.g. Vandewater and Moss 2013) and dynamics of timber structures (e.g. Reynolds et al. 2014). In this regard, the non-separable earthquake excitation EPS of the form

$$S_w(\omega,t) = S\left(\frac{\omega}{5\pi}\right)^2 exp(-0.2t)t^2 exp\left(-\left(\frac{\omega}{10\pi}\right)^2 t\right), \quad (43)$$

is considered in this example. This spectrum, plotted in Fig.(1) for $S = 1$, comprises some of the main characteristics of seismic shaking, such as decreasing of the dominant frequency with time (e.g. Sabetta and Pugliese 1996). Further, survival probabilities determined via the herein developed approximate technique are compared with pertinent Monte Carlo simulation data (10,000 realizations). To this aim, realizations compatible with the EPS of Eq.(43) are generated based on a spectral representation approach (e.g. Liang et al. 2007), while a standard fourth-order Runge-Kutta scheme is employed for solving the nonlinear equation of motion (Eq.(1)). The initial distribution chosen for the response amplitude PDF is the Dirac delta function, i.e., $p(a_0, t_0 = 0) = \delta(a_0)$, assuming the system is initially at rest. In the ensuing analysis the value $N = 60$ is chosen in Eq.(39) for the terms to be included in the expansion.

In Fig.(2), the bounded equivalent natural frequencies (Eq.(17)) of the oscillators with parameter values $(S = 1, \omega_0^2 = \pi^2, \zeta_0 = 0.01, \varepsilon = -1)$, $(S = 1, \omega_0^2 = \pi^2, \zeta_0 = 0.01, \varepsilon = -2)$, and $(S = 1, \omega_0^2 = \pi^2, \zeta_0 = 0.01, \varepsilon = -3)$ are plotted. In Fig.(3), the equivalent natural periods for the above oscillators are plotted, whereas in Fig.(4) the survival probabilities determined by Eqs.(24) are plotted for various barrier levels $a_{cr} = \sqrt{-4/(3\varepsilon)}$; comparisons with MCS (10,000 realizations) demonstrate a quite satisfactory agreement.

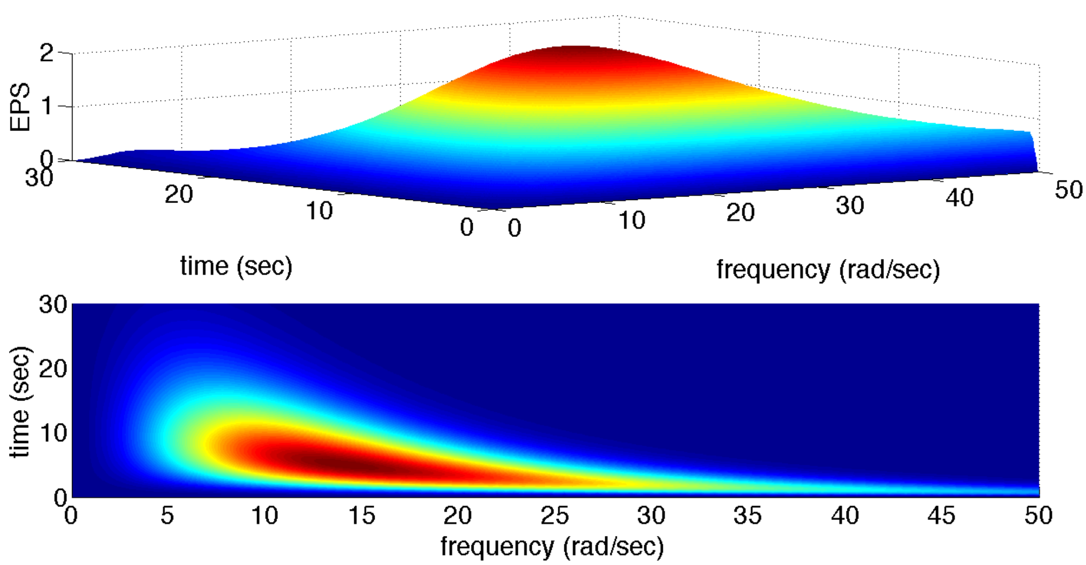

Fig.(1). Non-separable earthquake excitation evolutionary power spectrum

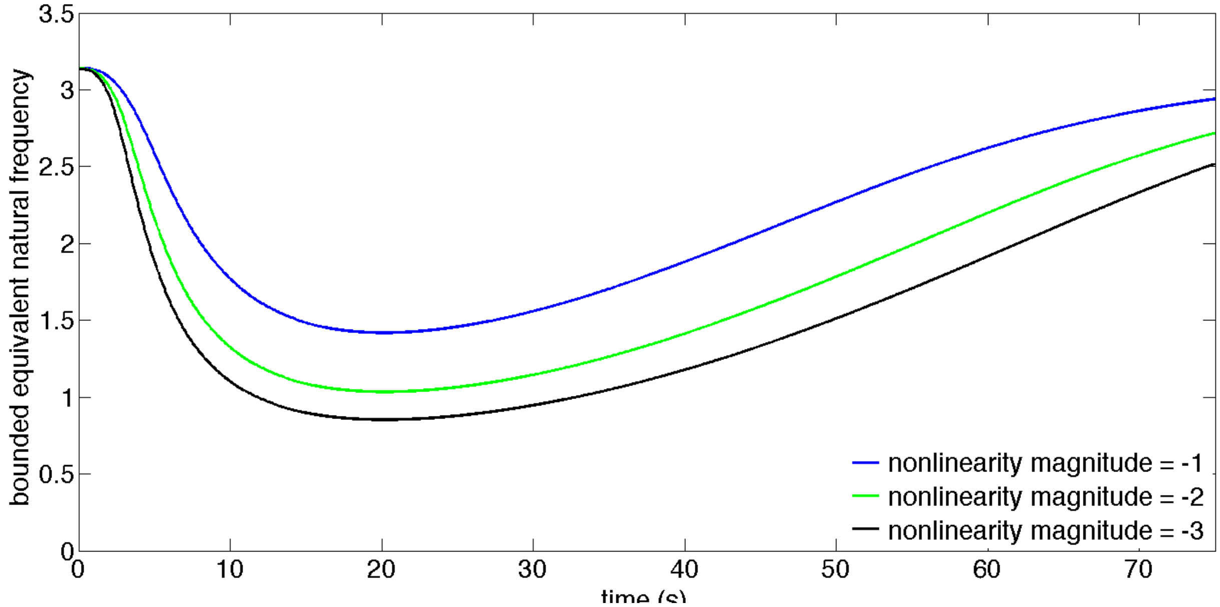

Fig.(2). Bounded equivalent time-varying natural frequency $\omega_{eq,B}(t)$ for a softening Duffing oscillator $(S = 1, \omega_0^2 = \pi^2, \zeta_0 = 0.01)$ under earthquake excitation

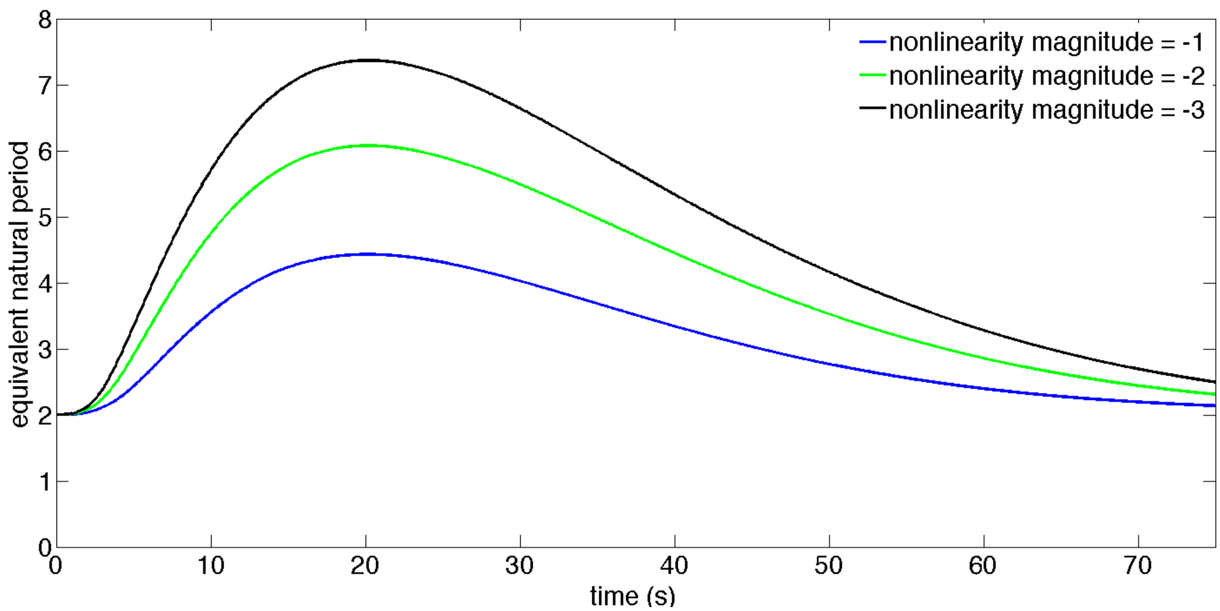

Fig.(3) Equivalent natural period $T_{eq}(t)$ for a softening Duffing oscillator $(S = 1, \omega_0^2 = \pi^2, \zeta_0 = 0.01)$ under earthquake excitation

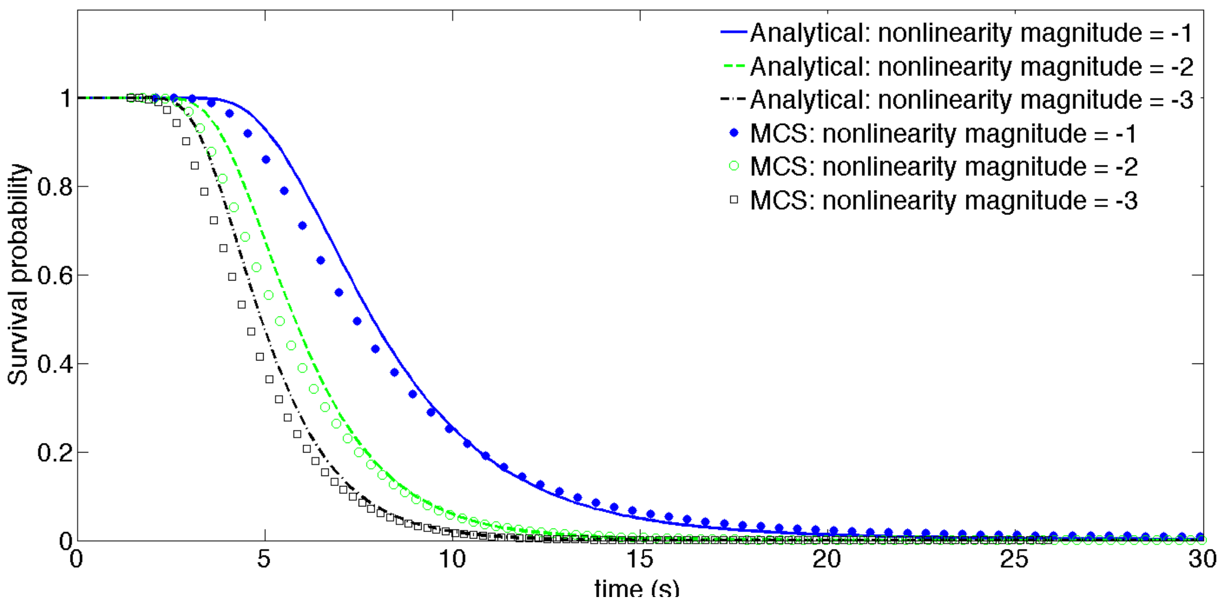

Fig.(4). Survival probability for a softening Duffing oscillator ($S = 1, \omega_0^2 = \pi^2, \zeta_0 = 0.01, d_T = 0.125$) under earthquake excitation; comparisons with MCS (10,000 realizations)

3.2 Softening Duffing oscillator under sea wave excitation

Considering the rolling motion of a ship in unidirectional beam waves enables one to approximate reasonably the motion as uncoupled with respect to other motions such as sway, pitch and heave; see Spyrou and Thomson 2000, Belenky and Sevastianov 2007, Arnold et al. 2004, Ibrahim et al. 2007 for a detailed presentation of the topic. Further, to take into account the viscous and vortex components of roll damping, a nonlinear expression for the damping force of the form $M_D = 2\zeta_0\omega_0(\dot{\phi} + \varepsilon_1\dot{\phi}^3), \varepsilon_1 > 0$, where ϕ is the ship rolling angle, is commonly adopted in the literature; indicatively, see also Spanos and Chen 1980, Dalzell 1978, Taylan 2000, Mamontov and Naess 2009 for some alternative polynomial and other approximations. As far as the nonlinear restoring moment is concerned, several approximations exist in the literature with the expression $M_D = \omega_0^2(\phi + \varepsilon_2\phi^3), \varepsilon_2 < 0$, being among the most commonly adopted choices (e.g. Taylan 1999); see also Senjanovic et al. 2000, Surendran et al. 2007. The aforementioned expression, although phenomenological, manages to capture to an adequate degree the qualitative behavior and basic physics of nonlinear ship rolling motion under beam waves (e.g. Spyrou and Thomson 2000, Belenky and Sevastianov 2007).

In this regard, consider next the uncoupled ship roll motion given by the equation

$$\ddot{\phi} + 2\zeta_0\omega_0\dot{\phi} + \varepsilon_1 2\zeta_0\omega_0\dot{\phi}^3 + \omega_0^2\phi + \varepsilon_2\omega_0^2\phi^3 = w(t), \qquad \varepsilon_1 > 0, \qquad \varepsilon_2 < 0, \quad (44)$$

where $w(t)$ represents a Gaussian, zero-mean non-stationary stochastic process possessing an evolutionary broad-band power spectrum $S_w(\omega, t)$ of the form

$$S_w(\omega, t) = |g(t)|^2|F_{roll}(\omega)|^2 S_E(\omega). \quad (45)$$

In Eq.(45) $S_E(\omega)$ denotes the stationary wave energy spectrum, whereas the function $F_{roll}(\omega)$ relates the wave energy spectrum to the roll moment excitation spectrum (e.g. Jiang et al. 2000). Although, in general, wave energy spectra, such as the Jonswap (e.g. Hasselmann et al. 1976), are narrow-band with a distinct peak, it has been shown that the resulting roll moment excitation spectrum is significantly more broad-band than the corresponding wave energy spectrum (e.g. Senjanovic et al. 2000). This broad-band characteristic of the stationary roll moment excitation power spectrum $|F_{roll}(\omega)|^2 S_E(\omega)$ is in agreement with the assumptions and justifies to a certain extent the applicability of the approach developed in section 2. In the following, the Pierson-Moskowitz (P-M) spectrum Pierson and Moskowitz 1964, i.e. a special case of the Jonswap spectrum of the form

$$S_E(\omega) = \frac{A}{\omega^5}\exp\left(-\frac{B}{\omega^4}\right), \quad (46)$$

is used for the wave energy spectrum $S_E(\omega)$, where $A = 1 \times 10^{-2} g^2, B = 120(\frac{g}{u})^4, u = 15 m/s, g = 9.8\, m/s^2$. As far as the function $F_{roll}(\omega)$ is concerned, this is chosen to be of the rather general form (e.g. Senjanovic et al. 2000) $|F_{roll}(\omega)|^2 = C\omega^4$ where the constant C is associated with beam sea and oscillator characteristics. In the following, the value $C = 3$ is used. Thus, due to the effect of multiplying Eq.(46) with the term "ω^4" the resulting stationary roll moment excitation spectrum $|F_{roll}(\omega)|^2 S_E(\omega)$ becomes relatively broad-band as shown in Fig.(5).

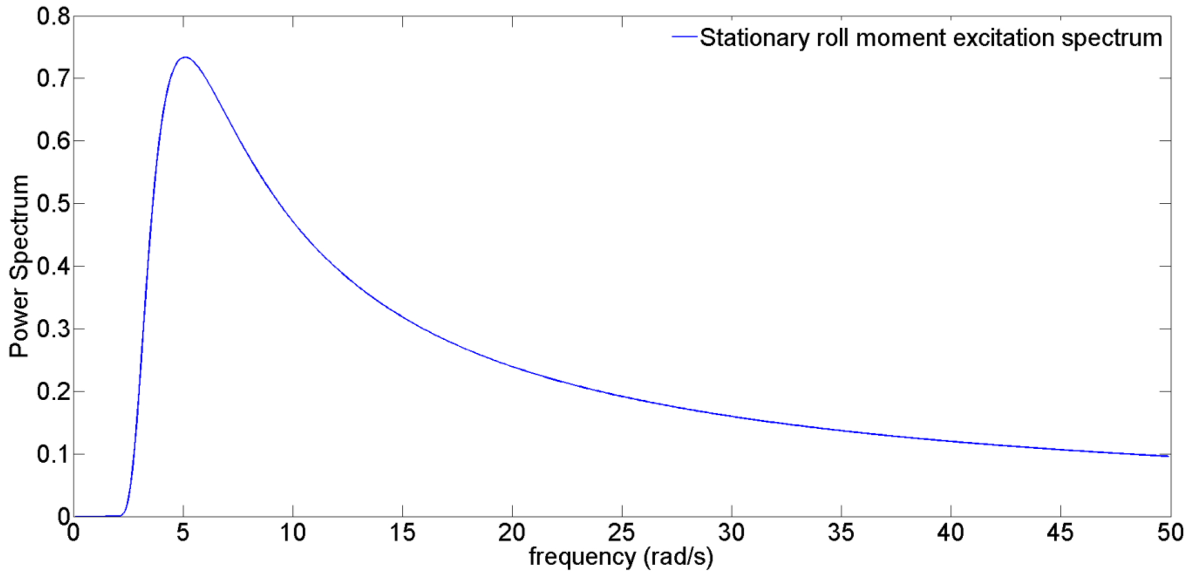

Fig.(5). Stationary roll moment excitation spectrum $|F_{roll}(\omega)|^2 S_E(\omega)$

Further, to demonstrate the versatility of the technique for addressing cases of non-stationary excitations, a time-modulating function $g(t)$ of the form

$$g(t) = \left(0.2 + 0.8 * \left[\frac{t}{a}\exp\left(1 - \frac{t}{a}\right)\right]^b\right)^{0.5} \qquad (47)$$

is utilized next, where $a = 20, b = 5$. As it is shown in Fig.(6) the function $g(t)$ varies slowly with time suggesting a low level of non-stationarity. In Fig.(7) the excitation EPS of Eq.(45) is plotted.

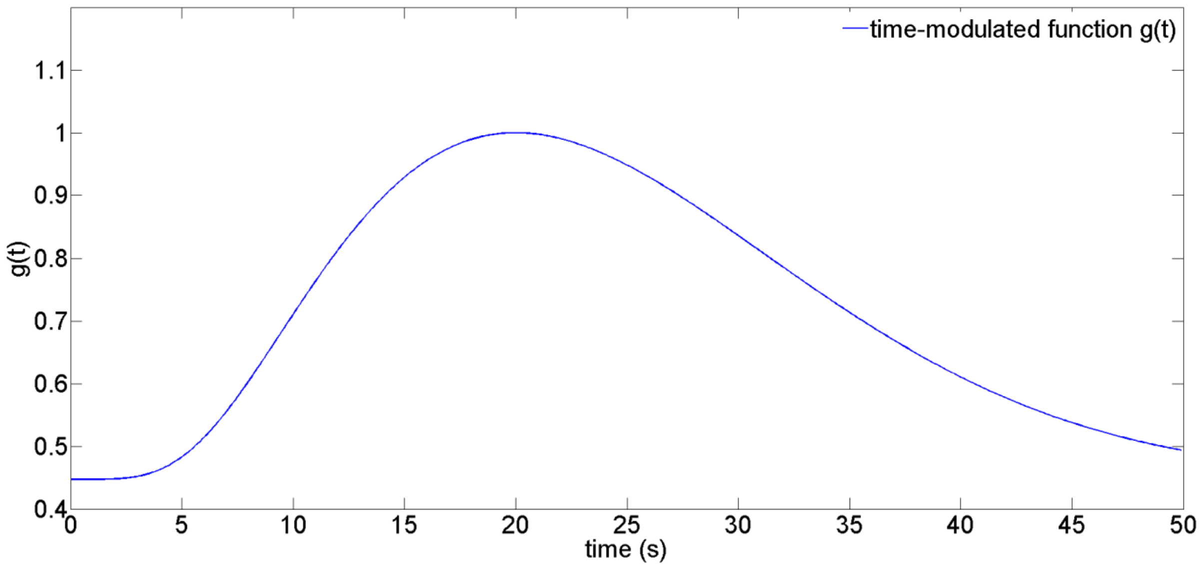

Fig.(6). Time-modulating function $g(t)$

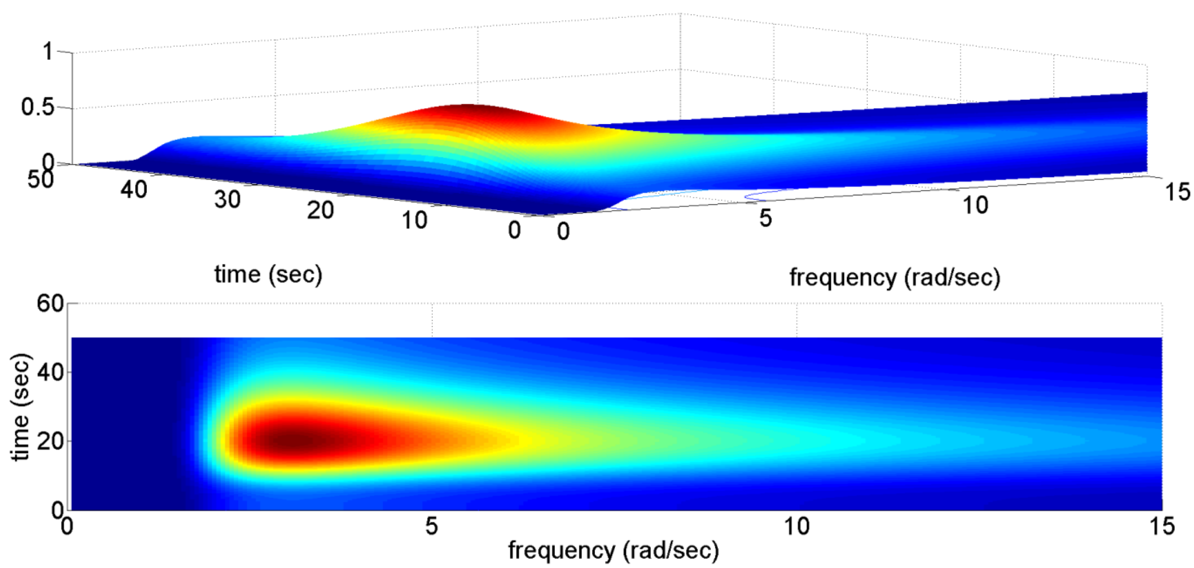

Fig. (7). Time-modulated roll moment excitation spectrum

It can be readily seen that the only qualitative difference between Eq.(44) and the softening Duffing oscillator of Eq.(1) is the nonlinear damping term; thus, following Kougioumtzoglou and Spanos 2009 (see also Spanos and Kougioumtzoglou 2014b) an equivalent linear oscillator is given in the form

$$\ddot{x} + \beta_{eq}(t)\dot{x} + \omega_{eq}^2(t)x = w(t), \quad (48)$$

where the time-dependent stiffness element $\omega_{eq}^2(t)$ is given by Eq.(10), and the time-dependent damping element $\beta_{eq}(t)$ is given by

$$\beta_{eq}(t) = E[\beta(a)] = \int_0^\infty \beta(a)p(a,t)da. \quad (49)$$

Following a stochastic averaging/linearization treatment (e.g. Spanos and Kougioumtzoglou 2014b, Kougioumtzoglou and Spanos 2009) $\beta(a)$ in Eq.(49) is given by

$$\beta(a) = 2\zeta_0\omega_0 - \frac{1}{\pi a\omega(a)}\int_0^{2\pi} sin\psi(\varepsilon_1 2\zeta_0\omega_0(-\omega(a)asin\psi)^3 + \omega_0^2 a\cos\psi + \omega_0^2\varepsilon(a\cos\psi)^3)\,d\psi = 2\zeta_0\omega_0\left(1+\varepsilon_1\frac{3}{4}\omega^2(a)a^2\right). \quad (50)$$

It can be readily seen that the time-dependent damping element $\beta_{eq}(t)$ depends on $\beta(a)$ which in turn depends on the stiffness element $\omega^2(a)$; thus, following the development in section 2, a bounded part $\beta_{eq,B}(t)$ is defined as

$$\beta_{eq,B}(t) = \int_0^{a_{cr}} \beta(a)p(a,t)da. \quad (51)$$

Substituting Eq. (50) into (51), and taking into account Eq.(7) yields

$$\beta_{eq,B}(t) = 2\zeta_0\omega_0\left\{1 - S(t) + \frac{3}{4}\varepsilon_1\omega_0^2[2c(t) - S(t)(2c(t) + a_{cr}^2)] + \frac{9}{16}\omega_0^2\varepsilon_1\varepsilon_2[8c(t)^2 - S(t)(a_{cr}^4 + 4c(t)a_{cr}^2 + 8c(t)^2)]\right\}. \quad (52)$$

Further, Eqs. (14), (16), (20), (21), and (36) are updated accordingly (see also Spanos and Kougioumtzoglou 2014b, Kougioumtzoglou and Spanos 2009) taking the form

$$K_1(a,t) = -\frac{1}{2}\beta_{eq,B}(t)a + \frac{\pi S\big(\omega_{eq,B}(t),t\big)}{2a\omega_{eq,B}^2(t)}, \quad (53)$$

$$\dot{c}(t) = -\beta_{eq,B}(t)c(t) + \frac{\pi S\left(\omega_{eq,B}(t), t\right)}{\omega_{eq,B}^2(t)}, \qquad (54)$$

$$\frac{dc(t,t_1)}{dt} + \beta_{eq,B}(t)c(t,t_1) - \frac{\pi S\left(\omega_{eq,B}(t), t\right)}{\omega_{eq,B}^2(t)} = 0, \quad (55)$$

$$\frac{dh(t,t_1)}{dt} + \frac{1}{2}\beta_{eq,B}(t)h(t,t_1) = 0, \quad (56)$$

and

$$r_i^2 = \frac{c(t_{i-1})}{c(t_i)}\left(1 - \beta_{eq,B}(t_{i-1})\tau_i\right), \qquad (57)$$

respectively. As in section 3.1 survival probabilities are determined via the herein developed approximate technique and are further compared with spectral representation based (e.g. Liang et al. 2007) pertinent Monte Carlo simulation data (10,000 realizations). The oscillator is assumed to be initially at rest, whereas the value $N = 60$ is chosen in Eq.(39) for the terms to be included in the expansion. In Fig.(8), the bounded equivalent natural frequencies $\omega_{eq,B}(t)$ of the oscillators of Eq.(44) with parameter values $(\zeta_0 = 0.01, \omega_0^2 = \pi^2, \varepsilon_1 = 0.1, \varepsilon_2 = -1)$, $(\zeta_0 = 0.01, \omega_0^2 = \pi^2, \varepsilon_1 = 0.1, \varepsilon_2 = -2)$ and $(\zeta_0 = 0.01, \omega_0^2 = \pi^2, \varepsilon_1 = 0.1, \varepsilon_2 = -4)$ are plotted. In Fig.(10), the equivalent natural periods for the above oscillators are plotted, whereas in Fig.(11) the survival probabilities determined by Eq.(24) are plotted for various barrier levels $a_{cr} = \sqrt{-4/(3\varepsilon_2)}$; comparisons with MCS (10000 realizations) demonstrate a quite satisfactory agreement.

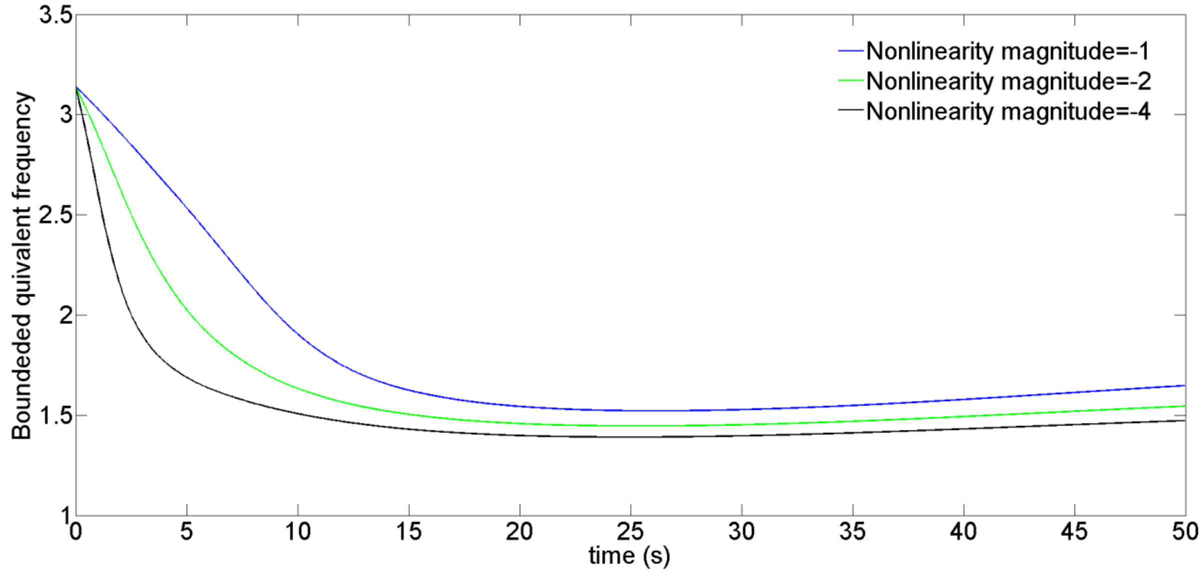

Fig.(8). Bounded equivalent time-varying natural frequency $\omega_{eq,B}(t)$ for a softening Duffing oscillator with nonlinear damping ($\zeta_0 = 0.01, \omega_0^2 = \pi^2, \varepsilon_1 = 0.1$) under sea wave excitation

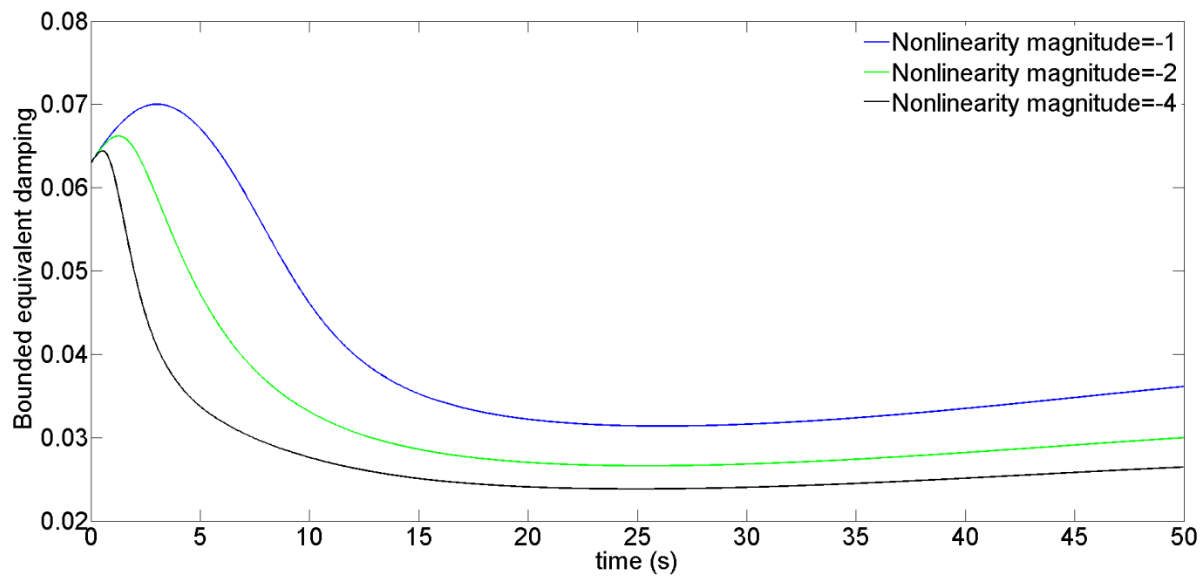

Fig.(9). Bounded equivalent time-varying damping $\beta_{eq,B}(t)$ for a softening Duffing oscillator with nonlinear damping ($\zeta_0 = 0.01, \omega_0^2 = \pi^2, \varepsilon_1 = 0.1$) under sea wave excitation

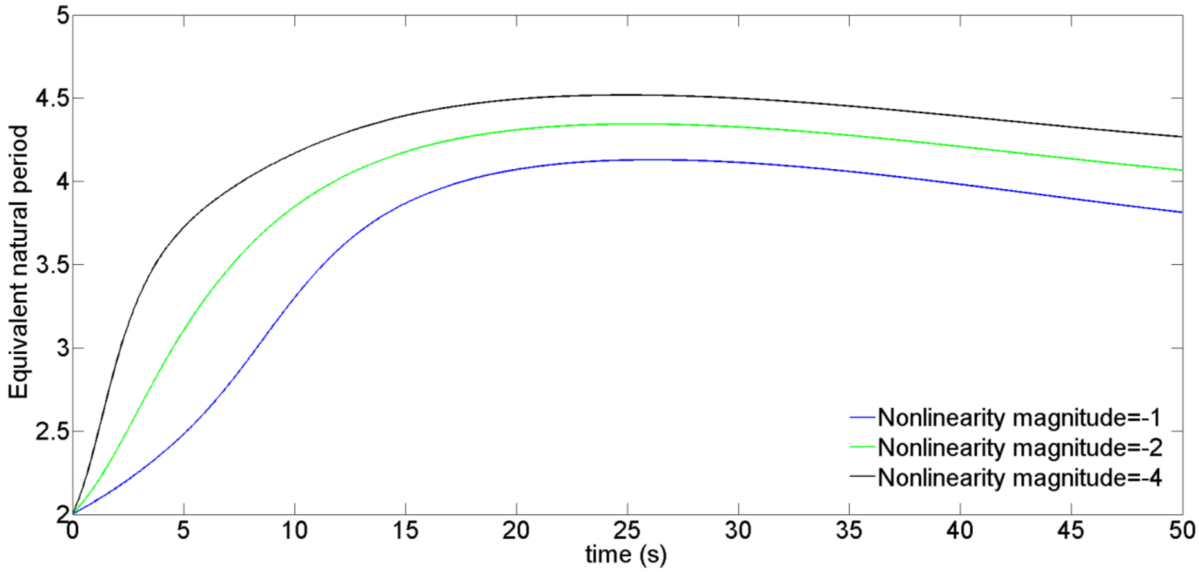

Fig.(10) Equivalent natural period $T_{eq}(t)$ for a softening Duffing oscillator with nonlinear damping ($\zeta_0 = 0.01, \omega_0^2 = \pi^2, \varepsilon_1 = 0.1$) under sea wave excitation

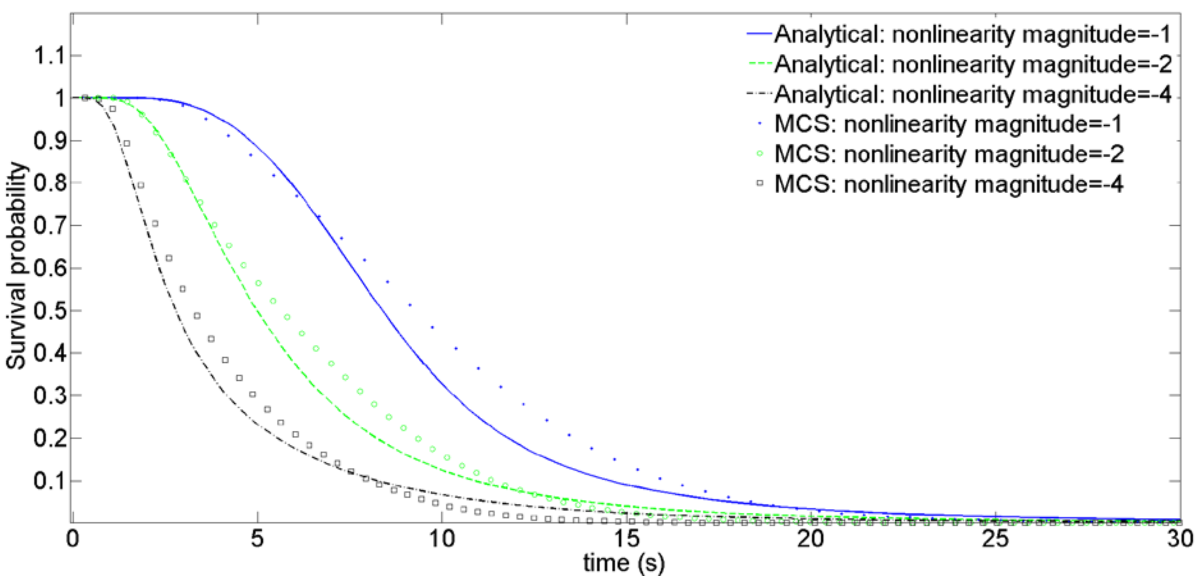

Fig.(11). Survival probability for a softening Duffing oscillator with nonlinear damping ($\zeta_0 = 0.01, \omega_0^2 = \pi^2, \varepsilon_1 = 0.1, d_T = 0.125$) under sea wave excitation; comparisons with MCS (10,000 realizations)

4 Concluding Remarks

An approximate analytical technique has been developed for determining the survival probability of a softening Duffing oscillator subject to evolutionary stochastic excitation. In the context of nonlinear stochastic dynamics, the Duffing oscillator with softening nonlinearity has been so far treated in a manner which disregarded important aspects of the analysis, such as the unbounded behavior the response process experiences when the restoring force acquires negative values. In this paper, introducing a special form for the oscillator non-stationary response amplitude PDF and relying on stochastic averaging a rigorous, as well as a computationally efficient, treatment of the problem has been provided. A significant advantage of the technique relates to the fact that it can readily handle cases of stochastic excitations that exhibit strong variability in both the intensity and the frequency content. Numerical examples have included a softening Duffing oscillator under evolutionary earthquake excitation, as well as a softening Duffing oscillator with nonlinear damping modeling the nonlinear ship roll motion in beam seas. Survival probability estimates have been determined for various levels of nonlinearity magnitude and compared with pertinent Monte Carlo simulations.

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