Cover Time in Edge-Uniform Stochastically-Evolving Graphs*

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March 1, 2018

Abstract

We define a general model of stochastically-evolving graphs, namely the *Edge-Uniform Stochastically-Evolving Graphs*. In this model, each possible edge of an underlying general static graph evolves independently being either alive or dead at each discrete time step of evolution following a (Markovian) stochastic rule. The stochastic rule is identical for each possible edge and may depend on the past $k \geq 0$ observations of the edge's state. We examine two kinds of random walks for a single agent taking place in such a dynamic graph: (i) The *Random Walk with a Delay* (*RWD*), where at each step the agent chooses (uniformly at random) an incident possible edge, i.e., an incident edge in the underlying static graph, and then it waits till the edge becomes alive to traverse it. (ii) The more natural *Random Walk on what is Available* (*RWA*) where the agent only looks at alive incident edges at each time step and traverses one of them uniformly at random. Our study is on bounding the *cover time*, i.e., the expected time until each node is visited at least once by the agent. For *RWD*, we provide the first upper bounds for the cases $k = 0, 1$ by correlating *RWD* with a simple random walk on a static graph. Moreover, we present a modified electrical network theory capturing the $k = 0$ case and a mixing-time argument toward an upper bound for the case $k = 1$. For *RWA*, we derive the first upper bounds for the cases $k = 0, 1$, too, by reducing *RWA* to an *RWD*-equivalent walk with a modified delay. Further, we also provide a framework, which is shown to compute the exact value of the cover time for a general family of stochastically-evolving graphs in exponential time. Finally, we conduct experiments on the cover time of *RWA* in Edge-Uniform graphs and compare the experimental findings with our theoretical bounds.

1 Introduction

In the modern era of Internet, modifications in a network topology can occur extremely frequently and in a disorderly way. Communication links may fail from time to time, while connections amongst terminals may appear or disappear intermittently. Thus, classical (static) network theory fails to capture such ever-changing processes. In an attempt to fill this void, different research communities have given rise to a variety of theories on *dynamic networks*. In the context of algorithms and distributed computing, such networks are usually referred to as *temporal graphs* [17]. A temporal graph is represented by a (possibly infinite) sequence of subgraphs of the same static graph. That is, the graph is *evolving* over a set of (discrete) time steps under a certain group

*This work was supported by the University of Liverpool, EEE/CS School, NeST Initiative

of deterministic or stochastic rules of evolution. Such a rule can be edge- or graph-specific and may take as input some graph instances observed in previous time steps.

In this paper, we focus on stochastically-evolving temporal graphs. We define a model of evolution, where there exists a single stochastic rule, which is applied *independently* to each edge. Furthermore, our model is general in the sense that the underlying static graph is allowed to be a general connected graph, i.e., with no further constraints on its topology, and the stochastic rule can include any finite number of past observations.

Assume now that a single mobile agent is placed on an arbitrary node of a temporal graph evolving under the aforementioned model. Next, the agent performs a simple random walk; at each time step, after the graph instance is fixed according to the model, the agent chooses uniformly at random a node amongst the neighbors of its current node and visits it. The *cover time* of such a walk is the expected number of time steps until the agent has visited each node at least once. Herein, we prove some first bounds on the cover time for a simple random walk as defined above, mostly via the use of Markovian theory.

Random walks constitute a very important primitive in terms of distributed computing. Examples include their use in information dissemination [1] and random network structure [4]; also, see the short survey in [8]. In this work, we consider a single random walk as a fundamental building block for other more distributed scenarios to follow.

1.1 Related Work

A paper which is very relevant with respect to ours is the one of Clementi, Macci, Monti, Pasquale and Silvestri [10], where they consider the flooding time in *Edge-Markovian* dynamic graphs. In such graphs, each edge independently follows a one-step Markovian rule and their model appears as a special case of ours (matches our case $k = 1$). Further work under this Edge-Markovian paradigm includes [5, 11].

Another work related to our paper is the one of Avin, Koucký and Lotker [3], who define the notion of a *Markovian Evolving Graph*, i.e., a temporal graph evolving over a set of graphs $G_1, G_2, \ldots$, where the process transits from G_i to G_j with probability p_{ij}, and consider random walk cover times. Note that their approach becomes computationally intractable if applied to our case; each of the possible edges evolves independently, thence causing the state space to be of size 2^m, where m is the number of possible edges in our model.

Clementi, Monti, Pasquale and Silvestri [12] study the broadcast problem, when at each time step the graph is selected according to the well-known $G_{n,p}$ model. Also, Yamauchi, Izumi and Kamei [22] study the rendezvous problem for two agents on a ring, when each edge of the ring independently appears at every time step with some fixed probability p. Lastly, there exist a few papers considering random walks on different models of stochastic graphs, e.g., [16, 19, 20], but without considering the cover time.

In the analysis to follow, we employ several seminal results around the theory of random walks and Markov chains. For random walks, we base our analysis on the seminal work in [1] and the electrical network theory presented in [9, 13], while for results regarding the mixing time of a Markov chain we cite textbooks [15, 18].

1.2 Our Results

We define a general model for stochastically-evolving graphs, where each possible edge evolves independently, but all of them evolve following the same stochastic rule. Furthermore, the stochastic rule may take into account the last k states of a given edge. The motivation for such a model lies in several practical examples from networking where the existence of an edge in the recent past means it is likely to exist in the near future, e.g., for telephone or Internet links. In some other cases, existence may mean that an edge has "served its purpose" and is now unlikely to appear in the near future, e.g., due to a high maintenance cost.

Special cases of our model have appeared in previous literature, e.g., in [12, 22] for $k = 0$ and in the line of work starting from [10] for $k = 1$, however they only consider special graph topologies (like ring and clique). On the other hand, the model we define is general in the sense that no assumptions, aside from connectivity, are made on the topology of the underlying graph and any amount of history is allowed into the stochastic rule. Thence, we believe it can be valued as a basis for more general results to follow capturing search or communication tasks in such dynamic graphs.

We hereby provide the first known upper bounds relative to the cover time of a simple random walk taking place in such stochastically evolving graphs for $k = 0$ and $k = 1$. To do so, we make use of a simple, yet fairly useful, modified random walk, namely the *Random Walk with a Delay* (*RWD*), where at each time step the agent is choosing uniformly at random from the incident edges of the static underlying graph and then waits for the chosen edge to become alive in order to traverse it. Moreover, we consider the natural random walk on such graphs, namely the *Random Walk on What's Available* (*RWA*), where at each time step the agent only considers the currently alive incident edges and chooses to traverse one out of them uniformly at random.

For the case $k = 0$, that is, when each edge appears at each round with a fixed probability p regardless of history, we prove that the cover time for *RWD* is upper bounded by C_G/p, where C_G is the cover time of a simple random walk on the (static) underlying graph G. The result can be obtained both by a careful mapping of the *RWD* walk to its corresponding simple random walk on the static graph and by generalizing the standard electrical network theory literature in [9, 13]. Later, we proceed to prove that the cover time for *RWA* is between $C_G/(1-(1-p)^{\Delta})$ and $C_G/(1-(1-p)^{\delta})$ where δ (respectively Δ) is the minimum (respectively maximum) degree of the underlying graph. The main idea here is to reduce *RWA* to an *RWD* walk, where at each step the traversal delay is lower (respectively upper) bounded by $(1-(1-p)^{\delta})$ (respectively $(1-(1-p)^{\Delta})$).

For $k = 1$, the stochastic rule takes into account the previous (one time step ago) state of the edge. If an edge was not present, then it becomes alive with probability p, whereas if it was alive, then it dies with probability q. Let τ_{mix} stand for the mixing time of this process. We prove that the *RWD* cover time is upper bound by $\tau_{mix} + C_G \cdot (p^2 + q)/(p^2 + pq)$ by carefully computing the expected traversal delay at each step after mixing is attained. Moreover, we show another C_G/ξ_{min} upper bound by considering the minimum probability guarantee of existence at each round, i.e., $\xi_{min} = \min\{p, 1-q\}$, and we discuss the trade-off between these two bounds. Similarly, we show a C_G/ξ_{max} lower bound, where $\xi_{max} = \max\{p, 1-q\}$. As far as *RWA* is concerned, we upper (respectively lower) bound its cover time by $C_G/(1-(1-\xi_{min})^{\delta})$ (respectively $C_G/(1-(1-\xi_{max})^{\Delta})$) again by a reduction to an *RWD*-equivalent walk.

Consequently, we demonstrate an exact, exponential-time approach to determine the precise cover time value for a general setting of stochastically-evolving graphs, including also the edge-independent model considered in this paper.

Finally, we conduct a series of experiments on calculating the cover time of RWA on several underlying graphs. We compare our experimental results with the achieved theoretical bounds.

1.3 Outline

In Section 2, we provide preliminary definitions and results regarding important concepts and tools that we use in later sections. Then, in Section 3, we define our model of stochastically-evolving graphs in a more rigorous fashion. Afterwards, in Sections 4 and 5, we provide the analysis of our cover time bounds when for determining the current state of an edge we take into account its last 0 and 1 states, respectively. In Section 6, we demonstrate an exact approach for determining the cover time for general stochastically-evolving graphs. Then, in Section 7, we present some experimental results on RWA cover time and compare them to the corresponding theoretical bounds. Finally, in Section 8, we cite some concluding remarks.

2 Preliminaries

Let us hereby define a few standard notions related to a simple random walk performed by a single agent on a simple connected graph $G = (V, E)$. By $d(v)$, we denote the degree, i.e., the number of neighbors, of a node $v \in V$. A simple random walk is a Markov chain where, for $v, u \in V$, we set $p_{vu} = 1/d(v)$, if $(v, u) \in E$, and $p_{vu} = 0$, otherwise. That is, an agent performing the walk chooses the next node to visit uniformly at random amongst the set of neighbors of its current node. Given two nodes v, u, the expected time for a random walk starting from v to arrive at u is called the *hitting time* from v to u and is denoted by H_{vu}. The *cover time* of a random walk is the expected time until the agent has visited each node of the graph at least once. Let P stand for the stochastic matrix describing the transition probabilities for a random walk (or, in general, a discrete-time Markov chain) where p_{ij} denotes the probability of transition from node i to node j, $p_{ij} \geq 0$ for all i, j and $\sum_j p_{ij} = 1$ for all i. Then, the matrix P^t consists of the transition probabilities to move from one node to another after t time steps and we denote the corresponding entries as $p_{ij}^{(t)}$. Asymptotically, $\lim_{t\to\infty} P^t$ is referred to as the *limiting distribution* of P. A *stationary distribution* for P is a row vector π such that $\pi P = \pi$ and $\sum_i \pi_i = 1$. That is, π is not altered after an application of P. If every state can be reached from another in a finite number of steps, i.e., P is *irreducible*, and the transition probabilities do not exhibit periodic behavior with respect to time, i.e., $gcd\{t : p_{ij}^{(t)} > 0\} = 1$, then the stationary distribution is *unique* and it matches the limiting distribution; this result is often referred to as the *Fundamental Theorem of Markov chains.* The *mixing time* is the expected number of time steps until a Markov chain approaches its stationary distribution. Below, let $p_i^{(t)}$ stand for the i-th row of P^t and $tvd(t) = \max_i ||p_i^{(t)} - \pi|| = \frac{1}{2} \max_i \sum_j |p_{ij}^{(t)} - \pi_j|$ stand for the *total variation distance* of the two distributions. We say that a Markov chain is ϵ*-near* to its stationary distribution at time t if $tvd(t) \leq \epsilon$. Then, we denote the mixing time by $\tau(\epsilon)$: the minimum value of t until a Markov chain is ϵ-near to its stationary distribution. A *coupling* (X_t, Y_t) is a joint stochastic process defined in a way such that X_t and Y_t are copies of the same Markov chain P when viewed marginally, and once $X_t = Y_t$ for some t, then $X_{t'} = Y_{t'}$ for any $t' \geq t$. Also, let T_{xy} stand for the minimum expected time until the two copies *meet*, i.e., until $X_t = Y_t$ for the first time, when starting from the initial states $X_0 = x$ and $Y_0 = y$. We can now state the following *Coupling Lemma* correlating the coupling meeting time to the mixing time:

Lemma 1 (Lemma 4.4 [15])**.** *Given any coupling* (X_t, Y_t)*, it holds* $tvd(t) \leq \max_{x,y} \Pr[T_{xy} \geq t]$*. Consequently, if* $\max_{x,y} \Pr[T_{xy} \geq t] \leq \epsilon$*, then* $\tau(\epsilon) \leq t$*.*

Furthermore, asymptotically, we need not care about the exact value of the total variation distance, since, for any $\epsilon > 0$, we can force the chain to be ϵ-near to its stationary distribution after a multiplicative time of $\log \epsilon^{-1}$ steps due to the submultiplicativity of the total variation distance. Formally, it holds $tvd(kt) \leq (2 \cdot tvd(t))^k$.

Lemma 2 (Lemma 2.20 [2])**.** *Suppose* $\tau(\epsilon_0) \leq t$ *for some Markov chain* P *and a constant* $0 < \epsilon_0 < 1$*. Then, for any* $0 < \epsilon < \epsilon_0$*, it holds* $\tau(\epsilon) \leq t \log \epsilon^{-1}$*.*

In order to derive lower bounds for *RWA*, we use the following graph family, commonly known as *lollipop graphs*, capturing the maximum cover time for a simple random walk, e.g. see [7, 14].

Definition 1. *A lollipop graph L_n^k consists of a clique on k nodes and a path on $n-k$ nodes connected with a cut-edge, i.e., an edge whose deletion makes the graph disconnected.*

3 The Edge-Uniform Evolution Model

Let us define a general model of a dynamically evolving graph. Let $G=(V,E)$ stand for a simple, *connected* graph, from now on referred to as the *underlying graph* of our model. The number of nodes is given by $n=|V|$, while the number of edges is denoted by $m=|E|$. For a node $v\in V$, let $N(v)=\{u:(v,u)\in E\}$ stand for the *open neighborhood* of v and $d(v)=|N(v)|$ for the *(static) degree* of v. Note that we make no assumptions regarding the topology of G, besides connectedness. We refer to the edges of G as the *possible edges* of our model. We consider evolution over a sequence of discrete time steps (namely $0,1,2,\ldots$) and denote by $\mathcal{G}=(G_0,G_1,G_2,\ldots)$ the infinite sequence of graphs $G_t=(V_t,E_t)$, where $V_t=V$ and $E_t\subseteq E$. That is, G_t is the graph appearing at time step t and each edge $e\in E$ is either *alive* (if $e\in E_t$) or *dead* (if $e\notin E_t$) at time step t.

Let R stand for a *stochastic rule* dictating the probability that a given possible edge is alive at any time step. We apply R at each time step and at each edge *independently* to determine the set of currently alive edges, i.e., the rule is *uniform* with regard to the edges. In other words, let e_t stand for a random variable where $e_t=1$, if e is alive at time step t, or $e_t=0$, otherwise. Then, R determines the value of $\Pr(e_t=1|H_t)$ where H_t is also determined by R and denotes the history length, i.e., the values of $e_{t-1},e_{t-2},\ldots$, considered when deciding for the existence of an edge at time step t. For instance, $H_t=\emptyset$ means no history is taken into account, while $H_t=\{e_{t-1}\}$ means the previous state of e is taken into account when deciding for its current state.

Overall, the aforementioned *Edge-Uniform Evolution* model (shortly *EUE*) is defined by the parameters G and R. In the following sections, we consider some special cases for R and provide some first bounds for the cover time of G under this model. Each time step of evolution consists of two stages: in the first stage, the graph G_t is fixed for time step t following R, while in the second stage, the agent moves to a node in $N_t[v]=\{v\}\cup\{u\in V:(v,u)\in E_t\}$. Notice that, since G is connected, then the cover time under *EUE* is finite, since R models edge-specific delays.

4 Cover Time with Zero-Step History

We hereby analyze the cover time of G under *EUE* in the special case when no history is taken into consideration for computing the probability that a given edge is alive at the current time step. Intuitively, each edge appears with a fixed probability p at every time step independently of the others. More formally, for all $e\in E$ and time steps t, $\Pr(e_t=1)=p\in[0,1]$.

4.1 Random Walk with a Delay

A first approach toward covering G with a single agent is the following: The agent is randomly walking G as if all edges were present and, when an edge is not present, it just waits for it to appear in a following time step. More formally, suppose the agent arrives on a node $v\in V$ with (static) degree $d(v)$ at the second stage of time step t. Then, after the graph is fixed for time step $t+1$, the agent selects a neighbor of v, say $u\in N(v)$, uniformly at random, i.e., with probability $\frac{1}{d(v)}$. If $(v,u)\in E_{t+1}$, then the agent moves to u and repeats the above procedure. Otherwise, it remains on v until the first time step $t'>t+1$ such that $(v,u)\in E_{t'}$ and then moves to u. This way, p acts as a *delay* probability, since the agent follows the same random walk it would on a static graph, but with an expected delay of $\frac{1}{p}$ time steps at each node. Notice that, in order for such a strategy to be feasible, each node must maintain knowledge about its neighbors in the underlying graph; not just the currently alive ones. From now on, we refer to this strategy for the agent as the *Random Walk with a Delay* (shortly *RWD*).

Now, let us upper bound the cover time of *RWD* by exploiting its strong correlation to a simple random walk on the underlying graph G via Wald's Equation (Theorem 3). Below, let C_G stand for the cover time of a simple random walk on the static graph G.

Theorem 3 ([21]). *Let $X_1, X_2, \dots, X_N$ be a sequence of real-valued, independent and identically distributed random variables where N is a nonnegative integer random variable independent of the sequence (in other words, a stopping time for the sequence). If each X_i and N have finite expectations, then it holds*

$$E[X_1 + X_2 + \ldots + X_N] = E[N] \cdot E[X_1]$$

Theorem 4. *For any connected underlying graph G evolving under the zero-step history* EUE*, the cover time for* RWD *is expectedly C_G/p.*

Proof. Consider a simple random walk, shortly *SRW*, and an *RWD* (under the *EUE* model) taking place on a given connected graph G. Given that *RWD* decides on the next node to visit uniformly at random based on the underlying graph, that is, in exactly the same way *SRW* does, we use a coupling argument to enforce *RWD* and *SRW* to follow the exact same trajectory, i.e., sequence of visited nodes.

Then, let the trajectory end when each node in G has been visited at least once and denote by T the total number of node transitions made by the agent. Such a trajectory under *SRW* will cover all nodes in expectedly $E[T] = C_G$ time steps. On the other hand, in the *RWD* case, for each transition we have to take into account the delay experienced until the chosen edge becomes available. Let $D_i \geq 1$ be a random variable, where $1 \leq i \leq T$ stands for the actual delay corresponding to node transition i in the trajectory. Then, the expected number of time steps till the trajectory is realized is given by $E[D_1 + \ldots + D_T]$. Since the random variables D_i are independent and identically distributed by the edge-uniformity of our model, T is a stopping time for them and all of them have finite expectations, then by Theorem 3 we get $E[D_1 + \ldots + D_T] = E[T] \cdot E[D_1] = C_G \cdot 1/p$. ◻

For an explicit general bound on *RWD*, it suffices to use $C_G \leq 2m(n-1)$ proved in [1].

A Modified Electrical Network. Another way to analyze the above procedure is to make use of a modified version of the standard literature approach of electrical networks and random walks [9, 13]. This point of view gives us expressions for the hitting time between any two nodes of the underlying graph. That is, we hereby (in Lemmata 5, 6 and Theorem 7) provide a generalization of the results given in [9, 13] thus correlating the hitting and commute times of *RWD* to an electrical network analog and reaching a conclusion for the cover time similar to the one of Theorem 4.

In particular, given the underlying graph G, we design an electrical network, $N(G)$, with the same edges as G, but where each edge has a resistance of $r = \frac{1}{p}$ ohms. Let $H_{u,v}$ stand for the hitting time from node u to node v in G, i.e. the expected number of time steps until the agent reaches v after starting from u and following *RWD*. Furthermore, let $\phi_{u,v}$ declare the electrical potential difference between nodes u and v in $N(G)$ when, for each $w \in V$, we inject $d(w)$ amperes of current into w and withdraw $2m$ amperes of current from a single node v. We now upper-bound the cover time of G under *RWD* by correlating $H_{u,v}$ to $\phi_{u,v}$.

Lemma 5. *For all $u, v \in V$, $H_{u,v} = \phi_{u,v}$ holds.*

Proof. Let us denote by C_{uw} the current flowing between two neighboring nodes u and w. Then, $d(u) = \sum_{(u,w)\in E} C_{uw}$ since at each node the total inward current must match the total outward current (Kirchhoff's first law). Moving forward, $C_{uw} = \phi_{uw}/r = \phi_{uw}/(1/p) = p \cdot \phi_{uw}$ by Ohm's law. Finally, $\phi_{uw} = \phi_{uv} - \phi_{wv}$ since the sum of electrical potential differences forming a path is equal to the total electrical potential difference of the path (Kirchhoff's second law). Overall, we can rewrite $d(u) = \sum_{(u,w)\in E} p(\phi_{u,v} - \phi_{w,v}) = d(u) \cdot p \cdot \phi_{u,v} - p \sum_{(u,w)\in E} \phi_{w,v}$. Rearranging gives

$$\phi_{u,v} = \frac{1}{p} + \frac{1}{d(u)} \sum_{(u,w)\in E} \phi_{w,v}.$$

Regarding the hitting time from u to v, we rewrite it based on the first step:

$$H_{u,v} = \frac{1}{p} + \frac{1}{d(u)} \sum_{(u,w)\in E} H_{w,v}$$

since the first addend represents the expected number of steps for the selected edge to appear due to *RWD*, and the second addend stands for the expected time for the rest of the walk.

Wrapping it up, since both formulas above hold for each $u \in V \setminus \{v\}$, therefore inducing two identical linear systems of n equations and n variables, it follows that there exists a unique solution to both of them and $H_{u,v} = \phi_{u,v}$. □

In the lemma below, let $R_{u,v}$ stand for the *effective resistance* between u and v, i.e., the electrical potential difference induced when flowing a current of one ampere from u to v.

Lemma 6. *For all $u, v \in V$, $H_{u,v} + H_{v,u} = 2mR_{u,v}$ holds.*

Proof. Similarly to the definition of $\phi_{u,v}$ above, one can define $\phi_{v,u}$ as the electrical potential difference between v and u when $d(w)$ amperes of current are injected into each node w and $2m$ of them are withdrawn from node u. Next, note that changing all currents' signs leads to a new network where for the electrical potential difference, namely ϕ', it holds $\phi'_{u,v} = \phi_{v,u}$. We can now apply the Superposition Theorem (see Section 13.3 in [6]) and linearly superpose the two networks implied from $\phi_{u,v}$ and $\phi'_{u,v}$ creating a new one where $2m$ amperes are injected into u, $2m$ amperes are withdrawn from v and no current is injected or withdrawn at any other node. Let $\phi''_{u,v}$ stand for the electrical potential difference between u and v in this last network. By the superposition argument, we get $\phi''_{u,v} = \phi_{u,v} + \phi'_{u,v} = \phi_{u,v} + \phi_{v,u}$, while from Ohm's law we get $\phi''_{u,v} = 2m \cdot R_{u,v}$. The proof concludes by combining these two observations and applying Lemma 5. □

Theorem 7. *For any connected underlying graph G evolving under the zero-step history* EUE*, the cover time for* RWD *is at most $2m(n-1)/p$.*

Proof. Consider a spanning tree T of G. An agent, starting from any node, can visit all nodes by performing an Eulerian tour on the edges of T (crossing each edge twice). This is a feasible way to cover G and thus the expected time for an agent to finish the above task provides an upper bound on the cover time. The expected time to cover each edge twice is given by $\sum_{(u,v)\in E_T}(H_{u,v} + H_{v,u})$ where E_T is the edge-set of T with $|E_T| = n-1$. By Lemma 6, this is equal to $2m \sum_{(u,v)\in E_T} R_{u,v} = 2m \sum_{(u,v)\in E_T} \frac{1}{p} = 2m(n-1)/p$. □

4.2 Random Walk on what's Available

Random Walk with a Delay does provide a nice connection to electrical network theory. However, depending on p, there could be long periods of time where the agent is simply standing still on the same node. Since the walk is random anyway, waiting for an edge to appear may not sound very wise. Hence, we now analyze the strategy of a *Random Walk on what's Available* (shortly *RWA*). That is, suppose the agent has just arrived at a node v after the second stage at time step t and then E_{t+1} is fixed after the first stage at time step $t+1$. Now, the agent picks uniformly at random only amongst the alive edges at time step $t+1$, i.e., with probability $\frac{1}{d_{t+1}(v)}$, where $d_{t+1}(v)$ stands for the degree of node v in G_{t+1}. The agent then follows the selected edge to complete the second stage of time step $t+1$ and repeats the strategy. In a nutshell, the agent keeps moving randomly on available edges and only remains on the same node if no edge is alive at the current time step. Below, let $\delta = \min_{v\in V} d(v)$ and $\Delta = \max_{v\in V} d(v)$.

Theorem 8. *For any connected underlying graph G with min-degree δ and max-degree Δ evolving under the zero-step history* EUE*, the cover time for* RWA *is at least $C_G/(1-(1-p)^\Delta)$ and at most $C_G/(1-(1-p)^\delta)$.*

Proof. Suppose the agent follows *RWA* and has reached node $u \in V$ after time step t. Then, G_{t+1} becomes fixed and the agent selects uniformly at random a neighboring edge to move to. Let M_{uv} (where $v \in \{w \in V : (u,w) \in E\}$) stand for a random variable taking value 1 if the agent moves to node v and 0 otherwise. For $k = 1, 2, \ldots, d(u) = d$, let A_k stand for the event that $d_{t+1}(u) = k$. Therefore, $\Pr(A_k) = \binom{d}{k} p^k (1-p)^{d-k}$ is exactly the probability k out of the d edges exist since each edge exists independently with probability p. Now, let us consider the probability $\Pr(M_{uv} = 1|A_k)$: the probability v will be reached given that k neighbors are present. This is exactly the product of the probability that v is indeed in the chosen k-tuple (say p_1) and the probability that then v is chosen uniformly at random (say p_2) from the k-tuple. $p_1 = \binom{d-1}{k-1}/\binom{d}{k} = \frac{k}{d}$ since the model is edge-uniform and we can fix v and choose any of the $\binom{d-1}{k-1}$ k-tuples with v in them out of the $\binom{d}{k}$ total ones. On the other hand, $p_2 = \frac{1}{k}$ by uniformity. Overall, we get $\Pr(M_{uv} = 1|A_k) = p_1 \cdot p_2 = \frac{1}{d}$. We can now apply the total probability law to calculate

$$\Pr(M_{uv} = 1) = \textstyle\sum_{k=1}^{d} \Pr(M_{uv} = 1|A_k)\Pr(A_k) = \frac{1}{d}\sum_{k=1}^{d}\binom{d}{k} p^k (1-p)^{d-k} = \frac{1}{d}(1-(1-p)^d)$$

To conclude, let us reduce *RWA* to *RWD*. Indeed, in *RWD* the equivalent transition probability is $\Pr(M_{uv} = 1) = \frac{1}{d}p$, accounting both for the uniform choice and the delay p. Therefore, the *RWA* probability can be viewed as $\frac{1}{d}p'$ where $p' = (1-(1-p)^d)$. To achieve edge-uniformity we set $p' = (1-(1-p)^\delta)$ which lower bounds the delay of each edge and finally we can apply the same *RWD* analysis by substituting p by p'. Similarly, we can set the upper-bound delay $p'' = (1-(1-p)^\Delta)$ to lower-bound the cover time. Applying Theorem 4 completes the proof. □

The value of δ used to lower-bound the transition probability may be a harsh estimate for general graphs. However, it becomes quite more accurate in the special case of a d-regular underlying graph where $\delta = \Delta = d$. To conclude this section, we provide a worst-case lower bound on the cover time based on similar techniques as above.

Lemma 9. *There exists an underlying graph G evolving under the zero-step history* EUE *such that the* RWA *cover time is at least $\Omega(mn/(1-(1-p)^\Delta))$.*

Proof. We consider the $L_n^{2n/3}$ lollipop graph which is known to attain a cover time of $\Omega(mn)$ for a simple random walk [7, 14]. Applying the lower bound from Theorem 8 completes the proof. □

5 Cover Time with One-Step History

We now turn our attention to the case where the current state of an edge affects its next state. That is, we take into account a history of length one when computing the probability of existence for each edge independently. A Markovian model for this case was introduced in [10]; see Table 1. The left side of the table accounts for the current state of an edge, while the top for the next one. The respective table box provides us with the probability of transition from one state to the other. Intuitively, another way to refer to this model is as the *Birth-Death* model: a dead edge becomes alive with probability p, while an alive edge dies with probability q.

Table 1: Birth-Death chain for a single edge [10]

	dead	*alive*
dead	$1-p$	p
alive	q	$1-q$

Let us now consider an underlying graph G evolving under the *EUE* model where each possible edge independently follows the aforementioned stochastic rule of evolution. In order to bound the *RWD* cover time, we apply a two-step analysis. First, we bound the mixing time of the Markov chain defined by Table 1 for a single edge and then for the whole graph by considering all m independent edge processes evolving together. Lastly, we estimate the cover time for a single agent after each edge has reached the stationary state of Birth-Death.

On the other hand, for *RWA*, we make use of the "being alive" probabilities $\xi_{min} = \min\{p, 1-q\}$ and $\xi_{max} = \max\{p, 1-q\}$ in order to bound the cover time by following a similar argument to the one in Theorem 8, starting again from an *RWD* analysis.

5.1 RWD for General (p, q)-Graphs via Mixing

As a first step, let us prove the following upper-bound inequality, which helps us break our analysis to follow into two separate phases.

Lemma 10. *Let $\tau(\epsilon)$ stand for the mixing time for the whole-graph chain up to some total variation distance $\epsilon > 0$, $C_{\tau(\epsilon)}$ for the expected time to cover all nodes after time step $\tau(\epsilon)$ and C for the cover time of G under* RWD*. Then, $C \leq \tau(\epsilon) + C_{\tau(\epsilon)}$ holds.*

Proof. The upper bound is easy to see since *RWD* covers a subset $V_0 \subseteq V$ until mixing occurs and then, after the mixing time $\tau(\epsilon)$, we require *RWD* to cover the whole node-set V; including the already visited V_0 nodes. That is, we discard any progress made by the walk during the first $\tau(\epsilon)$ time steps and require a full cover to occur afterwards. □

The above upper bound discards some walk progress, however, intuitively, this may be negligible in some cases: if the mixing is rapid, then the cover time $C_{\tau(\epsilon)}$ dominates the sum, whereas, if the mixing is slow, this may mean that edges appear rarely and thence little progress can be made anyway.

Phase I: Mixing Time. Let P stand for the Birth-Death Markov chain given in Table 1. It is easy to see that P is irreducible and aperiodic and therefore its limiting distribution matches its stationary distribution and is unique. We hereby provide a coupling argument to upper-bound the mixing time of the Birth-Death chain for a single edge. Let X_t, Y_t stand for two copies of P, where

$X_t = 1$ if the edge is alive at time step t and $X_t = 0$ otherwise. We need only consider the initial case $X_0 \neq Y_0$. For any $t \geq 1$, we compute the meeting probability $\Pr(X_t = Y_t | X_{t-1} \neq Y_{t-1}) = \Pr(X_t = Y_t = 1 | X_{t-1} \neq Y_{t-1}) + \Pr(X_t = Y_t = 0 | X_{t-1} \neq Y_{t-1}) = p(1-q) + q(1-p)$.

Definition 2. *Let $p_0 = p(1-q)+q(1-p)$ denote the meeting probability under the above Birth-Death coupling for a single time step.*

We now bound the mixing time of Birth-Death for a single edge.

Lemma 11. *The mixing time of Birth-Death for a single edge is $\mathcal{O}(p_0^{-1})$.*

Proof. Let T_{xy} denote the meeting time of X_t and Y_t, i.e., the first occurrence of a time step t such that $X_t = Y_t$. We now compute the probability the two chains meet at a specific time step $t \geq 1$:

$$\begin{aligned}
\Pr[T_{xy} = t] &= \Pr(X_t = Y_t | X_{t-1} \neq Y_{t-1}, X_{t-2} \neq Y_{t-2}, \ldots, X_0 \neq Y_0) &&= \\
&= \Pr(X_t = Y_t | X_{t-1} \neq Y_{t-1}) \cdot \Pr(X_{t-1} \neq Y_{t-1} | X_{t-2} \neq Y_{t-2}) \cdot \ldots \cdot \Pr(X_1 \neq Y_1 | X_0 \neq Y_0) \cdot \Pr(X_0 \neq Y_0) &&= \\
&= p_0 \cdot (1-p_0)^{t-1}
\end{aligned}$$

where we make use of the total probability law and the one-step Markovian evolution. Finally, we accumulate and then bound the probability the meeting time is greater to some time-value t:

$$\Pr[T_{xy} \leq t] = \sum_{i=1}^{t} \Pr[T_{xy} = i] = \sum_{i=1}^{t} p_0(1-p_0)^{i-1} = p_0 \frac{1-(1-p_0)^t}{p_0} = 1 - (1-p_0)^t$$

Then, $\Pr[T_{xy} > t] = (1-p_0)^t \leq e^{-p_0 t}$, by applying the inequality $1 - x \leq e^{-x}$ for all $x \in \mathbb{R}$. By setting $t = c \cdot p_0^{-1}$ for some constant $c \geq 1$, we get $\Pr[T_{xy} > c \cdot p_0^{-1}] \leq e^{-c}$ and apply Lemma 1 to bound $\tau(e^{-c}) \leq c \cdot p_0^{-1}$. □

The above result analyzes the mixing time for a single edge of the underlying graph G. In order to be mathematically accurate, let us extend this to the Markovian process accounting for the whole graph G. Let G_t, H_t stand for two copies of the Markov chain consisting of m independent Birth-Death chains; one per edge. Initially, we define a graph $G^* = (V^*, E^*)$ such that $V^* = V$ and $E^* \subseteq E$; any graph with these properties is fine. We set $G_0 = G^*$ and $H_0 = \overline{G^*}$ which is a worst-case starting point since each pair of respective G, H edges has exactly one alive and one dead edge. To complete the description of our coupling, we enforce that when a pair of respective edges meets, i.e., when the coupling for a single edge as described in the proof of Lemma 11 becomes successful, then both edges stop applying the Birth-Death rule and remain at their current state. Similarly to before, let $T_{G,H}$ stand for the meeting time of the two above defined copies, that is, the time until all pairs of respective edges have met. Furthermore, let $T^e_{x,y}$ stand for the meeting time associated with edge $e \in E$.

Lemma 12. *The mixing time for any underlying graph G, where each edge independently applies the Birth-Death rule, is at most $\mathcal{O}(p_0^{-1} \log m)$.*

Proof. To start with, we calculate the probability the meeting time is bounded by some value t:

$$\begin{aligned}
\Pr[T_{G,H} \leq t] &= \Pr[\max_{e \in E} T_{x,y} \leq t] &&= \Pr[(T^{e_1}_{x,y} \leq t) \wedge (T^{e_2}_{x,y} \leq t) \wedge \ldots \wedge (T^{e_m}_{x,y} \leq t)] &&= \\
&= \Pr[T_{x,y} \leq t]^m &&= (1-(1-p_0)^t)^m &&\geq \\
&\geq 1 - m(1-p_0)^t &&\geq 1 - m e^{-p_0 t}
\end{aligned}$$

where we successively applied the fact that the edges are independent, Bernoulli's inequality stating $(1+x)^r \geq 1 + rx$ for every r and any $x \geq -1$, and the already seen inequality $1 - x \leq e^{-x}$.

Moving forward, $\Pr[T_{G,H} > t] \leq m e^{-p_0 t}$ and after setting $t = \alpha p_0^{-1} \log m$, for some $\alpha \geq 2$ we derive that $\Pr[T_{G,H} > \alpha p_0^{-1} \log m] \leq m^{1-\alpha}$. Applying Lemma 1 gives $\tau(m^{1-\alpha}) \leq \alpha p_0^{-1} \log m$. □

Phase II: Cover Time After Mixing. We can now proceed to apply Lemma 10 by computing the expected time for *RWD* to cover G after mixing is attained. As before, we use the notation $C_{\tau(\epsilon)}$ to denote the cover time after the whole-graph process has mixed to some distance $\epsilon > 0$ from its stationary state in time $\tau(\epsilon)$. The following remark is key in our motivation toward the use of stationarity.

Fact 1. *Let D be a random variable capturing the number of time steps until a possible edge becomes alive under* RWD *once the agent selects it for traversal. For any time step $t \geq \tau(\epsilon)$, the expected delay for any single edge traversal e under* RWD *is the same and equals $E[D|e_t = 1]\Pr(e_t = 1) + E[D|e_t = 0]\Pr(e_t = 0)$.*

That is, due to the uniformity of our model, all edges behave similarly. Furthermore, after convergence to stationarity has been achieved, when an agent picks a possible edge for traversal under *RWD*, the probability $\Pr(e_t = 1)$ that the edge is alive for any time step $t \geq \tau(\epsilon)$ is actually given by the stationary distribution in a simpler formula and can be regarded independently of the edge's previous state(s).

Lemma 13. *For any constant $0 < \epsilon < 1$ and $\epsilon' = \epsilon \cdot \frac{\min\{p,q\}}{p+q}$, it holds that $C_{\tau(\epsilon')} \leq C_G \cdot (1+2\epsilon)\frac{p^2+q}{p^2+pq}$.*

Proof. We compute the stationary distribution π for the Birth-Death chain P by solving the system $\pi P = \pi$. Thus, we get $\pi = [\frac{q}{p+q}, \frac{p}{p+q}]$.

From now on, we only consider time steps $t \geq \tau(\epsilon')$, i.e., after the chain has mixed, for some $\epsilon' = \epsilon \cdot \frac{\min\{p,q\}}{p+q} \in (0, 1)$. We have $tvd(t) = \frac{1}{2}\max_i \sum_j |p_{ij}^{(t)} - \pi_j| \leq \epsilon'$ implying that for any edge e, we get $\Pr(e_t = 1) \leq (1 + 2\epsilon)\frac{p}{p+q}$. Similarly, $\Pr(e_t = 0) \leq (1 + 2\epsilon)\frac{q}{p+q}$. Let us now estimate the expected delay until the *RWD*-chosen possible edge at some time step t becomes alive. If the selected possible edge exists, then the agent moves along it with no delay, i.e., we count 1 step. Otherwise, if the selected possible edge is currently dead, then the agent waits till the edge becomes alive. This will expectedly take $1/p$ time steps due to the Birth-Death chain rule. Overall, the expected delay is at most $1 \cdot (1 + 2\epsilon)\frac{p}{p+q} + \frac{1}{p} \cdot (1 + 2\epsilon)\frac{q}{p+q} = (1 + 2\epsilon)\frac{p^2+q}{p^2+pq}$, where we condition on the above cases.

Since for any time $t \geq \tau(\epsilon)$ and any edge e, we have the same expected delay to traverse an edge, we can extract a bound for the cover time by considering an electrical network with each resistance equal to $(1 + 2\epsilon)\frac{p^2+q}{p^2+pq}$. Applying Theorem 4 completes the proof. ☐

The following theorem is directly proven by plugging into the inequality of Lemma 10 the bounds computed in Lemmata 12 and 13.

Theorem 14. *For any connected underlying graph G and the Birth-Death rule, the cover time of* RWD *is $\mathcal{O}(p_0^{-1} \log m + C_G \cdot (p^2 + q)/(p^2 + pq))$.*

5.2 RWD and RWA for General (p, q)-Graphs via Min-Max

In the previous subsection, we employed a mixing-time argument in order to reduce the final part of the proof to the zero-step history case. Let us hereby derive another upper bound for the cover time of *RWD* (and then extend it for *RWA*) via a min-max approach. The idea here is to make use of the "being alive" probabilities to prove lower and upper bounds for the cover time parameterized by $\xi_{min} = \min\{p, 1 - q\}$ and $\xi_{max} = \max\{p, 1 - q\}$.

Let us consider an *RWD* walk on a general connected graph G evolving under *EUE* with a zero-step history rule dictating $\Pr(e_t = 1) = \xi_{min}$ for any edge e and time step t. We refer to this walk as the *Upper Walk with a Delay*, shortly *UWD*. Respectively, we consider an *RWD* walk when the stochastic rule of evolution is given by $\Pr(e_t = 1) = \xi_{max}$. We refer to this specific walk as the *Lower Walk with a Delay*, shortly *LWD*. Below, we make use of *UWD* and *LWD* in order to bound the cover time of *RWD* and *RWA* in general (p, q)-graphs.

Lemma 15. *For any connected underlying graph G and the Birth-Death rule, the cover time of* RWD *is at least C_G/ξ_{max} and at most C_G/ξ_{min}.*

Proof. Regarding *UWD*, one can design a corresponding electrical network where each edge has a resistance of $1/\xi_{min}$ capturing the expected delay till any possible edge becomes alive. Applying Theorem 4, gives an upper bound for the *UWD* cover time.

Let C' stand for the *UWD* cover time and C stand for the cover time of *RWD* under the Birth-Death rule. It now suffices to show $C \leq C'$ to conclude.

In Birth-Death, the expected delay before each edge traversal is either $1/p$, in case the possible edge is dead, or $1/(1-q)$, in case the possible edge is alive. In both cases, the expected delay is upper-bounded by the $1/\xi_{min}$ delay of *UWD* and therefore $C \leq C'$ follows since any trajectory under *RWD* will take at most as much time as the same trajectory under *UWD*.

In a similar manner, the cover time of *LWD* lower bounds the cover time of *RWD* and, by applying Theorem 4, we derive a lower bound of C_G/ξ_{max}. $\square$

Notice that the upper bound in Lemma 15 improves over the one in Theorem 14 for a wide range of cases, especially if q is really small. For example, when $q = \Theta(m^{-k})$ for some $k \geq 2$ and $p = \Theta(1)$, then Lemma 15 gives $O(mn)$ whereas Theorem 14 gives $O(m^k)$ since the mixing time dominates the whole sum. On the other hand, for relatively big values of p and q, e.g., in $\Omega(1/m)$, then mixing is rapid and the upper bound in Theorem 14 proves better.

Let us now turn our attention to the *RWA* case with the subsequent results.

Theorem 16. *For any connected underlying graph G evolving under the Birth-Death rule, the cover time for* RWA *is at least $C_G/(1-(1-\xi_{max})^{\Delta})$ and at most $C_G/(1-(1-\xi_{min})^{\delta})$.*

Proof. Suppose the agent follows *RWA* with some stochastic rule R of the form $\Pr(e_t = 1|H_t)$ which incorporates some history H_t when making a decision about an edge at time step t. Let us now proceed in fashion similar to the proof of Theorem 8. Assume the agent follows *RWA* and has reached node $u \in V$ after time step t. Then G_{t+1} becomes fixed and the agent selects uniformly at random an alive neighboring node to move to. Let M_{uv}, where v is a neighbor to u, stand for a random variable taking value 1 if the agent moves to v at time step $t+1$ and 0 otherwise. For $k = 0, 1, 2, \ldots, d(u) = d$, let $A_k(H_t)$ stand for the event that $d_{t+1} = k$ given some history H_t about all incident possible edges of u. We compute $\Pr(M_{uv} = 1) = \sum_{k=1}^{d} \Pr(M_{uv} = 1|A_k(H_t)) \Pr(A_k(H_t))$. Similarly to the proof of Theorem 8, $\Pr(M_{uv} = 1|A_k(H_t)) = p_1 \cdot p_2 = 1/d$ where p_1 is the probability v is indeed in the chosen k-tuple (which is k/d) and p_2 is the probability it is chosen uniformly at random from the k-tuple (which is $1/k$). Thus, we get $\Pr(M_{uv} = 1) = \frac{1}{d} \sum_{k=1}^{d} \Pr(A_k(H_t)) = \frac{1}{d}(1 - \Pr(A_0(H_t)))$ where A_0 is the event no edge becomes alive at this time step.

Moving forward, by definition, *LWD* and *UWD* both depict zero-step history *RWD* walks. Let us denote by *LWA* and *UWA* their corresponding *RWA* walks. Furthermore, let P_L (respectively P_U) be equal to the probability $\Pr(M_{uv} = 1)$ under the *LWA* (respectively *UWA*) walk. Then, we can substitute p by ξ_{max} and ξ_{min} respectively in order to apply Theorem 8 and get $P_L = \frac{1}{d}(1-(1-\xi_{max})^d)$ and $P_U = \frac{1}{d}(1-(1-\xi_{min})^d)$. In the Birth-Death model, we know $(1-\xi_{max})^d \leq \Pr(A_0(H_1)) \leq (1-\xi_{min})^d$ since each possible edge becomes alive with probability at least ξ_{min} and at most ξ_{max}. Thus, it follows $P_U \leq \Pr(M_{uv} = 1) \leq P_L$.

To wrap up, *LWA* and *UWA* are viewed as *RWD* walks with delay probabilities $(1-(1-\xi_{max})^d)$ and $(1-(1-\xi_{min})^d)$, which lower and upper bound the $(1-\Pr(A_0(H_t))$ delay probability associated with *RWA*. Inverting the inequality to account for the delays, we have $C_L \leq C \leq C_U$ for the cover times. Finally, Theorem 8 gives $C_L \geq C_G/(1-(1-\xi_{max})^{\Delta})$ and $C_U \leq C_G/(1-(1-\xi_{min})^{\delta})$. □

Lemma 17. *There exists an underlying graph G evolving under the Birth-Death rule such that the* RWA *cover time is at least $\Omega(mn/(1-(1-\xi_{max})^{\Delta}))$.*

Proof. We consider an *RWA* walk on $L_n^{2n/3}$ to get the $\Omega(mn)$ term in the cover time [7, 14]. Applying the lower bound from Theorem 16 completes the proof. □

Given the above results, we can derive a general observation correlating the cover time of *RWA* with the cover time of a simple random walk on the (static) underlying graph G.

Corollary 1. *For any connected underlying graph G evolving under the Birth-Death rule with $\xi_{min} \geq c \cdot \frac{1}{\delta}$, for some constant $c > 0$, the cover time of* RWA *is in $\mathcal{O}(C_G)$.*

Proof. By Theorem 16, the *RWA* cover time is at most $C_G/(1-(1-\xi_{min})^{\delta})$. Assume $\xi_{min} \geq c/\delta$ for some constant $c > 0$. By applying $1-x \leq e^{-x}$ and the restriction on ξ_{min}, we get $(1-\xi_{min})^{\delta} \leq e^{-\xi_{min}\cdot\delta} \leq e^{-c}$. Thus, for the cover time, we get $C_G/(1-(1-\xi_{min})^{\delta}) \leq C_G/(1-e^{-c}) \in \mathcal{O}(C_G)$. □

Intuitively, for any (nearly) complete underlying graph with n nodes where $\delta = \Theta(n)$, the condition $\xi_{min} \in \Omega(1/n)$ indicates that, for any time step t, the graph instance G_t has a "huge" connected component, since G_t can be viewed as "lower-bounded" by a $G(n, \xi_{min})$ Erdős-Rényi random graph. Given the existence of a "huge" connected component at each step, an *RWA* walk evolves in a similar fashion to the static case, without being significantly affected by the temporal restrictions.

6 An Exact Approach

So far, we have established upper and lower bounds for the cover time of edge-uniform stochastically-evolving graphs. Our bounds are based on combining extended results from simple random walk theory and careful delay estimations. In this section, we describe an approach to determine the *exact* cover time for temporal graphs evolving under *any* stochastic model. Then, we apply this approach to the already seen zero-step history and one-step history cases of *RWA*.

The key component of our approach is a Markov chain capturing both phases of evolution: the graph dynamics and the walk trajectory. In that case, calculating the cover time reduces to calculating the hitting time to a particular subset of Markov states. Although computationally intractable for large graphs, such an approach provides the exact cover time value and is hence practical for smaller graphs.

Suppose we are given an underlying graph $G = (V, E)$ and a set of stochastic rules R capturing the evolution dynamics of G. That is, R can be seen as a collection of probabilities of transition from one graph instance to another. We denote by k the (longest) history length taken into account by the stochastic rules. Like before, let $n = |V|$ stand for the number of nodes and $m = |E|$ for the number of possible edges of G. We define a Markov chain M with states of the form (H, v, V_c), where

- $H = (H_1, H_2, \ldots, H_k)$, is a k-tuple of *temporal graph instances*, that is, for each $i = 1, 2, \ldots, k$, H_i is the graph instance present $i-1$ time steps before the current one (which is H_1)
- $v \in V(G)$ is the current position of the agent

- $V_c \subseteq V(G)$ is the set of already covered nodes, i.e., the set of nodes which have been visited at least once by the agent

As described earlier for our edge-uniform model, we assume evolution happens in two phases. First, the new graph instance is determined according to the rule-set R. Second, the new agent position is determined based on a random walk on what's available. In this respect, consider a state $S = (H, v, V_c)$ and another state $S' = (H', v', V'_c)$ of the described Markov chain M. Let $\Pr[S \to S']$ denote the transition probability from S to S'. We seek to express this probability as a product of the probabilities for the two phases of evolution. The latter is possible, since, in our model, the random walk strategy is independent of the graph evolution.

For the graph dynamics, let $\Pr[H \xrightarrow{R} H']$ stand for the probability to move from a history-tuple H to another history-tuple H' under the rules of evolution in R. Note that, for $i = 1, 2, \ldots, k-1$, it must hold $H'_{i+1} = H_i$ in order to properly maintain history, otherwise the probability becomes zero. On the other hand, for valid transitions, the probability reduces to $\Pr[H'_1 | (H_1, H_2, \ldots, H_k)]$, which is exactly the probability that H'_1 becomes the new instance given the history $H = (H_1, H_2, \ldots, H_k)$ of past instances (and any such probability is either given directly or implied by R).

For the second phase, i.e., the random walk on what's available, we denote by $\Pr[v \xrightarrow{H_j} v']$ the probability of moving from v to v' on some graph instance H_j. Since, the random walk strategy is only based on the current instance, we can derive a general expression for this probability, which is independent of the graph dynamics R. Below, let $N_{H_j}(v)$ stand for the set of neighbors of v in graph instance H_j. If $\{v, v'\} \notin E(G)$, that is, if there is no possible edge between v and v', then for any temporal graph instance H_j, it holds $\Pr[v \xrightarrow{H_j} v'] = 0$. The probability is also zero for all graph instances H_j where the possible edge is not alive, i.e., $\{v, v'\} \notin E(H_j)$. In contrast, if $\{v, v'\} \in E(H_j)$, then $\Pr[v \xrightarrow{H_j} v'] = |N_{H_j}(v)|^{-1}$, since the agent chooses a destination uniformly at random out of the currently alive ones. Finally, if $v = v'$, then the agent remains still, with probability 1, only if there exist no alive incident edges. We summarize the above facts in the following equation:

$$\Pr[v \xrightarrow{H_j} v'] = \begin{cases} 1 & \text{, if } N_{H_j}(v) = \emptyset \text{ and } v' = v \\ |N_{H_j}(v)|^{-1} & \text{, if } v' \in N_{H_j}(v) \\ 0 & \text{, otherwise} \end{cases} \tag{1}$$

Overall, we combine the two phases in M and introduce the following transition probabilities.

- If $|V_c| < n$:

$$\Pr[(H, v, V_c) \to (H', v', V'_c)] = \begin{cases} \Pr[H \xrightarrow{R} H'] \cdot \Pr[v \xrightarrow{H'_1} v'] & \text{, if } v' \in V'_c \text{ and } V'_c = V_c \\ \Pr[H \xrightarrow{R} H'] \cdot \Pr[v \xrightarrow{H'_1} v'] & \text{, if } v' \neq v, v' \notin V'_c \text{ and } V'_c = V_c \cup \{v'\} \\ 0 & \text{, otherwise} \end{cases}$$

- If $|V_c| = n$:

$$\Pr[(H, v, V_c) \to (H', v', V'_c)] = \begin{cases} 1 & \text{, if } H = H', v = v', V_c = V'_c \\ 0 & \text{, otherwise} \end{cases}$$

For $|V_c| < n$, notice that only two cases may have a non-zero probability with respect to the growth of V_c. If the newly visited node v' is already covered, then V'_c must be identical to V_c since no new nodes are covered during this transition. Further, if a new node v' is not yet covered, then V_c' is updated to include it as well as all the covered nodes in V_c.

For $|V_c|= n$, the idea is that once such a state has been reached, and so all nodes are covered, then there is no need for further exploration. Therefore, such a state can be made *absorbing*. In this respect, let us denote the set of these states as $\Gamma = \{(H, v, V_c) \in M \ : \ |V_c|= n\}$.

Definition 3. *Let* ECT(G, R) *be the problem of determining the exact value of the cover time for an* RWA *on a graph G stochastically evolving under rule-set R.*

Theorem 18. *Assume all probabilities of the form* $\Pr[H \xrightarrow{R} H']$ *used in M are exact reals and known a priori. Then, for any underlying graph G and stochastic rule-set R, it holds that* ECT$(G, R) \in$ EXPTIME.

Proof. For each temporal graph instance, H_i, in the worst case, there exist 2^m possibilities, since each of the m possible edges is either alive or dead at a graph instance. For the whole history H, the number of possibilities becomes $(2^m)^k = 2^{k \cdot m}$ by taking the product of k such terms. There are n possibilities for the walker's position v. Finally, for each $v \in V(G)$, we only allow states such that $v \in V_c$. Therefore, since we fix v, there are up to $n-1$ nodes to be included or not in V_c leading to a total of $\mathcal{O}(2^{n-1})$ possibilities for V_c. Taking everything into account, M has a total of $\mathcal{O}(2^{k \cdot m+n-1}n)$ states.

Let $H_{s,\Gamma}$ stand for the hitting time of Γ when starting from a state $s \in M$. Assuming exact real arithmetic, we can compute all such hitting times by solving the following system (Theorem 1.3.5 [18]):

$$\begin{cases} H_{s,\Gamma} = 0 & , \forall s \in \Gamma \\ H_{s,\Gamma} = 1 + \sum_{s' \notin \Gamma} \Pr[s \to s'] \cdot H_{s',\Gamma} & , \forall s \notin \Gamma \end{cases}$$

Let C stand for the cover time of an *RWA* on G evolving under R. By definition, the cover time is the expected time till all nodes are covered, regardless of the position of the walker at that time. Consider the set $S = \{(H, v, \{v\}) \in M : v \in V(G)\}$ of start positions for the agent as depicted in M. Then, it follows $C = \max_{s \in S} H_{s,\Gamma}$, where we take the worst-case hitting time to a state in Γ over any starting position of the agent. In terms of time complexity, computing C requires computing all values $H_{s,\Gamma}$, for every $s \in S$. To do so, one must solve the above linear system of size $\mathcal{O}(2^{k \cdot m+n-1}n)$, which can be done in time exponential to input parameters n, m and k. □

It's noteworthy to remark that this approach is general in the sense that there are no assumptions on the graph evolution rule-set R besides it being stochastic, i.e., describing the probability of transition from each graph instance to another given some history of length k. In this regard, Theorem 18 captures both the case of Markovian Evolving Graphs [3] and the case of Edge-Uniform Graphs considered in this paper. We now proceed and show how the aforementioned general approach applies to the zero-step and one-step history cases of Edge-Uniform Graphs. To do so, we calculate the corresponding graph-dynamics probabilities. The random walk probabilities are given in Equation 1.

RWA on Edge-Uniform Graphs (Zero-Step History). Based on the general model, we rewrite the transition probabilities for the special case when *RWA* takes place on an edge-uniform graph without taking into account any memory, i.e., the same case as in Section 4. Notice that, since past instances are not considered in this case, the history-tuple reduces to a single graph instance H. We rewrite the transition probabilities, for the case $|V_c|< n$, as follows:

$$\Pr[(H,v,V_c)\to(H,v',V_c')] = \begin{cases} \Pr[H'|H]\cdot\Pr[v \xrightarrow{H'} v'] & \text{, if } v'\in V_c' \text{ and } V_c'=V_c \\ \Pr[H'|H]\cdot\Pr[v \xrightarrow{H'} v'] & \text{, if } v'\neq v, v'\notin V_c' \text{ and } V_c'=V_c\cup\{v'\} \\ 0 & \text{, otherwise} \end{cases}$$

Let α stand for the number of edges alive in H'. Since there is no dependence on history and each edge appears independently with probability p, we get $\Pr[H'|H]=\Pr[H']=p^{\alpha}\cdot(1-p)^{m-\alpha}$.

RWA on Edge-Uniform Graphs (One-Step History). We hereby rewrite the transition probabilities for a Markov chain capturing an *RWA* taking place on an edge-uniform graph where, at each time step, the current graph instance is taken into account to generate the next one. This case is related to the results in Section 5. Due to the history inclusion, the transition probabilities become more involved than those seen for the zero-history case. Again, we consider the non-absorbing states, where $|V_c|< n$.

$$\Pr[((H_1,H_2),v,V_c)\to((H_1',H_2'),v',V_c')] = \begin{cases} \Pr[(H_1,H_2)\to(H_1',H_2')]\cdot\Pr[v \xrightarrow{H_1'} v'] & \text{, if } v'\in V_c' \text{ and } V_c'=V_c \\ \Pr[(H_1,H_2)\to(H_1',H_2')]\cdot\Pr[v \xrightarrow{H_1'} v'] & \text{, if } v'\notin V_c' \text{ and } V_c'=V_c\cup\{v'\} \\ 0 & \text{, otherwise} \end{cases}$$

If $H_2'\neq H_1$, i.e., if it does not hold that, for each $e\in G$, $e\in H_2'$ if and only if $e\in H_1$, then $\Pr[(H_1,H_2)\to(H_1',H_2')]=0$, otherwise the history is not properly maintained. On the other hand, if $H_2'=H_1$, then $\Pr[(H_1,H_2)\to(H_1',H_2')]=\Pr[(H_1,H_2)\to(H_1',H_1)]=\Pr[H_1'|H_1]$. To derive an expression for the latter, we need to consider all edge (mis)matches between H_1' and H_1, and properly apply the Birth-Death rule (Table 1). Below, we denote by $D(H)=E(G)\setminus E(H)$ the set of possible edges of G, which are dead at instance H. Let $c_{00}=|D(H_1)\cap D(H_1')|$, $c_{01}=|D(H_1)\cap E(H_1')|$, $c_{10}=|E(H_1)\cap D(H_1')|$ and $c_{11}=|E(H_1)\cap E(H_1')|$. Each of the c_{00} edges was dead in H_1 and remained dead in H_1', with probability $1-p$. Similarly, each of the c_{01} edges was dead in H_1 and became alive in H_1', with probability p. Also, each of the c_{10} edges was alive in H_1 and died in H_1', with probability q. Finally, each of the c_{11} edges was alive in H_1 and remained alive in H_1', with probability $1-q$. Overall, due to the edge-independence of the model, we get $\Pr[H_1'|H_1]=(1-p)^{c_{00}}\cdot p^{c_{01}}\cdot q^{c_{10}}\cdot(1-q)^{c_{11}}$.

7 Experimental Results

In this section, we discuss some experimental results to complement our previously-established theoretical bounds. We simulate an *RWA* taking place in graphs evolving under the Birth-Death model (Table 1). We provide an experimental estimation of the value of the cover time for such a walk. To do so, for each specific graph and values p, q considered, we repeat the experiment a large number of times, e.g., 1000 times. In the first experiment, we start from a graph instance with no alive edges. At each step, after the graph evolves according to Birth-Death, the walker picks uniformly at random an incident alive edge to traverse. The process continues till all nodes are visited at least once. Each next experiment commences with the last graph instance of the previous experiment as its first instance.

We construct underlying graphs in the following fashion: given a natural number n, we initially construct a path of n nodes, namely $v_1, v_2, \ldots, v_n$. Afterward, for each two distinct nodes v_i

Table 2: Experimental Results for Complete Graphs ($randomThreshold = 1.0$)

Size	p	q	Experimental Cover Time	Lower Bound	Upper Bound
10	1.0	0.0	**26**	26	26
10	0.3	0.5	27	27	28
10	0.1	0.5	38	26	42
10	0.05	0.5	61	27	73
10	0.01	0.5	278	27	312
10	0.001	0.5	2802	27	3012
10	0.001	0.99	2861	312	3012
100	1.0	0.0	**541**	541	541
100	0.1	0.99	547	526	834
100	0.1	0.999	554	522	5535
100	0.01	0.999	830	826	5524
100	0.005	0.999	1311	1326	5503
200	1.0	0.0	**1184**	1184	1184
200	0.1	0.5	1250	1202	1202
200	0.02	0.9	1277	1180	1201
200	0.005	0.9	1832	1223	1937
200	0.001	0.995	6436	1873	6547

and v_j, we add an edge $\{v_i, v_j\}$ with probability equal to a $randomThreshold$ parameter. For instance, $randomThreshold = 0$ means the graph remains a path. On the other hand, for $randomThreshold = 1$, the graph becomes a clique.

In Tables 2 and 3, we display the average cover time, rounding it to the nearest natural number, computed in some indicative experiments for $randomThreshold$ equal to 1.0 and 0.5, respectively. Consequently, we provide estimates for a lower and an upper bound on the temporal cover time. In this respect, we experimentally compute a value for the cover time of a simple random walk in the underlying graph, i.e., the *static* cover time. Then, we plug in this value in place of C_G to apply the bounds given in Theorem 16 based on $\xi_{min} = \min\{p, 1-q\}$ and $\xi_{max} = \max\{p, 1-q\}$. The case $p = 1.0$ and $q = 0.0$ corresponds to the static cover time of the underlying graph, since every graph instance will be the same as the underlying graph after 1-2 steps. Notice that, since our bounds correspond to expected values and are based on an experimental estimation of the static cover time, it may be the case sometimes that the experimental temporal cover time is slightly outside one of the two bounds.

The results confirm Corollary 1 in the sense that, for large values of $p, 1-q$, the cover times computed are either equal or only slightly differ from the static cover time estimation. On the other hand, only when, e.g., p approaches the value $1/\delta$, do we see the experimental cover time starting to significantly diverge from the static one. Furthermore, the cover times computed appear to be roughly within their corresponding lower and upper bounds. In terms of accuracy, the experimental cover time seems to lie nearer to the bound affected by the value of p, rather than the one affected by $1-q$.

Table 3: Experimental Results for Randomly-Produced Graphs ($randomThreshold = 0.5$)

Size	δ	Δ	p	q	Experimental Cover Time	Lower Bound	Upper Bound
10	4	8	1.0	0.0	31	31	31
10	3	7	0.5	0.5	38	38	43
10	5	8	0.2	0.5	37	30	45
10	4	8	0.1	0.8	64	42	101
10	2	6	0.05	0.9	162	75	359
10	4	6	0.01	0.9	614	62	736
100	36	62	1.0	0.0	532	532	532
100	42	64	0.09	0.9	554	545	558
100	40	61	0.02	0.95	823	575	992
100	40	63	0.01	0.95	1348	584	1695
200	79	116	1.0	0.0	1225	1225	1225
200	84	123	0.1	0.8	1285	1216	1216
200	76	123	0.01	0.9	1875	1215	2275
200	79	120	0.005	0.99	3069	1747	3743
200	78	116	0.001	0.995	12472	2756	16185

8 Conclusions

We defined the general *Edge-Uniform Evolution* model for a stochastically-evolving graph, where a single stochastic rule is applied, but to each edge independently, and provided lower and upper bounds for the cover time of two random walks taking place on such a graph.

Our results can directly be extended for any history length considered by the stochastic rule; even non-Markovian stochastic rules could be approximated using a long-enough window of Markovian history. Of course, if we wish to take into account the last k states of a possible edge, when making a decision about its next state, then we need to consider 2^k possible states, thus making some tasks computationally intractable for large k. On the other hand, the min-max guarantee is easier to deal with for any value of k, since we only care about the minimum and the maximum "being alive" probabilities.

Our model seems to be on the opposite end of the Markovian evolving graph model introduced in [3]. There, the evolution of possible edges directly depends on the family of graphs selected as possible instances. Thus, a new research direction we suggest is to devise another model of *partial* edge-dependency. That is, we would wish the stochastic rule for one edge to depend on a proper subset of the edge-set; neither on no other edge nor on every other edge. Such a model may prove interesting in terms of community-partitioned networks or other block-defined graphs.

Acknowledgements. We would like to acknowledge an anonymous reviewer who identified an important technical error in a previous conference version of this article. Also, we acknowledge another anonymous reviewer, who suggested the use of Theorem 4 as an alternative to electrical network theory and several other useful modifications.

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