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Stability of a self-gravitating rotating stratified plasma

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Abstract : The hydromagnetic stability of a self-gravitating incompressible finitely conducting rotating viscous plasma of variable density has been examined in the presence of finite ion Larmor radius (FLR) effects. The prevailing uniform magnetic field has been assumed to be along the horizontal direction. Proper solutions have been obtained through the variational method for a semi-infinite plasma in which there is exponential density stratification along the vertical direction. The dispersion relation has been derived and solved numerically for different values of the physical parameters involved. It is found that viscosity, rotation and FLR effects have all stabilizing influence on the growth rate of the unstable mode of disturbance.

Keywords : Stratified plasma, hydromagnetic stability, viscosity, rotation, FLR effects.

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1 Introduction

The problem of hydromagnetic stability of a magnetized plasma of variable density is of importance in several astrophysical situations such as in the theories of sunspot magnetic fields, in heating of solar corona and in the stability of stellar atmospheres in magnetic fields. Several researchers have investigated stability problems during the last few years, and a comprehensive account of these problems of hydrodynamic and hydromagnetic stability has been given by Chandrasekhar (1968). The importance of such a study in astronomical context has also been pointed out by Ognesyan (1960) and Shafranov (1960). Sundaram (1968) studied the problem of self-gravitational instability of a viscous finitely conducting fluid of variable density. Bajwa and Srivastava (1972) extended the same problem to include the effects of rotation. Earlier Rosenbluth *et al* (1962), Roberts and Taylor (1962) and Jukes (1964) demonstrated the stabilizing influence of finite ion Larmor radius (FLR) effects on the plasma instabilities. Sankhla and Bansal (1983) have studied the combined influence of the effects of viscosity and FLR on the instability of a stratified layer of a rotating self-gravitating plasma of infinite electrical conductivity for a transverse mode of wave propagation. The effects of finite electrical conductivity are of importance in astrophysical situations.

In this paper we have examined the combined influence of the effects of rotation, FLR, viscosity and finite electrical conductivity on the stability of a self-gravitating plasma of variable density. It has been shown that rotation, viscosity and FLR effects all have a stabilizing influence on the growth rate of unstable mode of disturbance. The differential equation incorporating the effect of magnetic resistivity is uncoupled from the remaining equations in the transverse mode of disturbance considered here and thus does not affect the growth rate of the disturbance.

2. Perturbation equations

The relevant linearized perturbation equations governing the motion of an incompressible, finitely conducting rotating viscous plasma are

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} = -\nabla \delta P + [(\nabla \times \boldsymbol{h}) \times \boldsymbol{H}] + 2\rho(\boldsymbol{v} \times \Omega) + \mu_0 \nabla^2 \boldsymbol{v} + \delta\rho \nabla \phi + \rho \nabla \delta\phi \tag{1}$$

$$\frac{\partial}{\partial t} \delta\rho + (\boldsymbol{v}.\nabla)\rho = 0 \tag{2}$$

$$\nabla^2 \delta\phi = -G\delta\rho \tag{3}$$

$$\frac{\partial \boldsymbol{h}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{H}) + \eta \nabla^2 \boldsymbol{h} \tag{4}$$

$$\nabla.\boldsymbol{v} = 0 \quad \text{and} \quad \nabla.\boldsymbol{h} = 0 \tag{5}$$

where $\boldsymbol{v}(u, v, \omega)$, $\delta\rho$, δP, $\delta\phi$ and $\boldsymbol{h}(h_x, h_y, h_z)$ denote respectively the perturbations in velocity, density ρ, stress tensor $P(P_{ij})$, gravitational potential ϕ, and the magnetic field $\boldsymbol{H}$. Here μ_0 is the coefficient of viscosity and η is the magnetic resistivity.

We assume that the ambient magnetic field is horizontal i.e., $\boldsymbol{H} = (H, 0, 0)$. The components of stress tensor taking into account the FLR effects as given by Roberts and Taylor (1962), are

$$\begin{aligned}
P_{xx} &= p \\
P_{yy} &= p - \rho\nu\left(\frac{\partial \omega}{\partial y} + \frac{\partial v}{\partial z}\right) \\
P_{zz} &= p + \rho\nu\left(\frac{\partial \omega}{\partial y} + \frac{\partial v}{\partial z}\right) \\
P_{xy} &= P_{yx} = -2\rho\nu \frac{\partial u}{\partial z} \\
P_{xz} &= P_{zx} = 2\rho\nu \frac{\partial u}{\partial y} \\
P_{yz} &= P_{zy} = \rho\nu\left(\frac{\partial v}{\partial y} - \frac{\partial \omega}{\partial z}\right)
\end{aligned} \tag{6}$$

where p is the scalar part of the pressure and $\rho\nu=\frac{NT}{4\omega_H}$, ω_H denoting gyration frequency while N and T denote respectively the number density and temperature of the ions.

We assume that the plasma is rotating about the direction of the magnetic field i.e., $\Omega=(\Omega, 0, 0)$. Further we assume that all the perturbed quantities are of the form,

$$f(z) \exp (ik_y+nt), \tag{7}$$

where $f(z)$ is some function of z, k is wave number of the harmonic disturbance along the y-axis and n (may be complex) is the growth rate of the perturbation. We are thus analysing here the transverse mode of disturbance.

On substituting the expression (7) in eqs. (1) to (5) and using eq. (6) we get

$$n\rho u=-2i\nu kuD\rho+\mu_0(D^2-k^2)u, \tag{8}$$

$$n\rho v=2\rho\Omega\omega-ik\delta p-\nu(ikv-D\omega)D\rho+\nu\rho(D^2-k^2)\omega$$
$$+\mu_0(D^2-k^2)v+\rho ik\delta\phi+2\rho\Omega\omega \tag{9}$$

$$n\rho\omega=-2\rho\Omega v-D\delta p-\nu(D\rho)(dv)-\gamma\rho(D^2-k^2)v-ik\gamma(D\rho)\omega$$
$$+\mu_0(D^2-k^2)\omega+\rho D(\delta\phi)-\frac{(D\rho)(D\phi)\omega}{n}, \tag{10}$$

$$(D^2-k^2)\delta\phi=\frac{G\omega(D\rho)}{n} \tag{11}$$

$$[n-\eta(D^2-k^2)]h=0 \tag{12}$$

$$ikv+D\omega=0\ ;\ ikh_y+Dh_z=0 \tag{13}$$

where $D\equiv d/dz$.

Eliminating u, v, δp from the above equations, we finally get an equation in ω and $\delta\phi$ as

$$n[D(\rho D\omega)-\rho k^2\omega]+2i\nu k[D\{(D\rho)(D\omega)\}-k^2(D\rho)\omega]-\frac{k^2}{n}(D\rho)(D\phi)\omega$$
$$-k^2(D\rho)\delta\phi+2ik\Omega(D\rho)\omega-\mu_0(D^2-k^2)^2\omega=0 \tag{14}$$

We assume that the plasma under consideration is semi-infinite i.e. infinitely extending along the two horizontal directions and contained between two boundaries, at $z=0$ and $z=d$. At the boundaries the normal components of velocity must vanish, we must, therefore, have

$$\omega=0 \text{ at } z=0 \text{ and } z=d \tag{15}$$

whether the boundaries are rigid or free. Also we must have

$$D\omega=0, \text{ on a rigid boundary}$$
$$D^2\omega=0, \text{ on a free boundary} \tag{16}$$

The condition to be satisfied by $\delta\phi$ can be obtained by matching the solutions on the boundaries and using the fact that ϕ and the normal component of grad ϕ must be continuous across the surface. This leads to the conditions

$$(D-k)\delta\phi=0 \text{ and } (D+k)\delta\phi=0 \text{ at } z=0 \text{ and } z=d \tag{17}$$

3. Discussion on equation (12)

Eq. (12) can be written as

$$[\eta D^2-(\eta k^2+n)](h_x, h_y, h_z)=0 \tag{18}$$

on solving this equation, we get
either

$$h=0 \text{ or } q_1=0 \text{ and } q_1-q_2=2im\pi, \tag{19}$$

where $q_{1,2}$ are the roots of the auxiliary equation obtained on replacing D by q in eq. (18) ;

$$q_1=0 \text{ gives } n=-\eta k^2, \tag{20}$$

which corresponds to a stable mode modified by the magnetic resistivity.
Again,

$$q_1-q_2=2im\pi \text{ gives } n=-\eta k_1^2, \tag{21}$$

where

$$k_1^2=k^2+m^2\pi^2, \tag{22}$$

which again corresponds to the viscous type of damped mode modified by magnetic resistivity, m being an integer.

4. Variational principle

Let us suppose that the solutions belonging to the characteristic value n_i are ω_i, $\delta\phi_i$ and the solutions corresponding to characteristic value n_j are ω_j and $\delta\phi_j$. Multiplying the i^{th} component of eq. (14) by ω_j, integrating over the vertical extent L of the plasma and intergrating by parts once or repeated we finally get,

$$\begin{aligned} &n_i^2\int_L \rho[D\omega_i D\omega_j+k^2\omega_i\omega_j]dz+n_i\int_L 2i\nu k(D\rho)[D\omega_i D\omega_j+k^2\omega_i\omega_j]dz \\ &+\int_L k^2(D\rho)(D\phi)\omega_i\omega_j dz-n_i^2\int_L \frac{k^2}{G}[D\delta\phi_i D\delta\phi_j+k^2\delta\phi_i\delta\phi_j]dz \\ &-n_i\int_L 2ik\Omega(D\rho)\omega_i\omega_j dz+n_i\int_L \mu_0[D^2\omega_i D^2\omega_j-2k^2D\omega_i D\omega_j \\ &+k^4\omega_i\omega_j]dz=0 \end{aligned} \tag{23}$$

The proof of this can be given by setting $i=j$ in eq. (23) and considering the arbitrary variations $\delta\omega$, $\delta(\delta\phi)$ in the corresponding physical quantities ω, $\delta\phi$ and showing, by proceeding along the usual lines, that the first order variation δn in n vanishes. Eq. (23) with $i = j$, therefore, provides the basis for obtaining the solution of the present problem.

5. Stratified layer of a self-gravitating plasma

Let us now obtain the solution for a continuously stratified plasma layer in which density varies exponentially along the vertical, i.e.,

$$\rho(z) = \rho_1 \exp(\beta z), \tag{24}$$

where ρ_1 and β are constants.

Poisson's equation, which must be satisfied by ϕ_0, gives the following distribution for ϕ_0,

$$\phi_0(z) = \frac{G\rho_1}{\beta^2}(-e^{\beta z} + \beta z + 1) \tag{25}$$

we assume that the plasma is confined between a rigid boundary at $z = 0$ and a free boundary at $z = 0$. Appropriate to the boundary conditions, let us take the trial solution for ω and $\delta\phi$ as

$$\omega(z) = A_1 (\cos lz - \cos 3lz) \tag{26}$$

and

$$(D-k)\delta\phi = A_2 e^{\beta z}(\cos lz - \cos 3lz) \tag{27}$$

where A_1 and A_2 are constants and $l = \frac{m\pi}{2d}$, m being an odd integer.

Substituting the trial solution in eq. (23) and evaluating the integrals contained therein, and writing

$$n^* = \frac{n}{\sqrt{G\rho_1}}, \quad \nu_0^* = \frac{\nu_0^2}{\sqrt{G\rho_1}}, \quad x = \frac{k}{l}, \tag{28}$$

$$\nu^* = \frac{\nu l^2}{\sqrt{G\rho_1}}, \quad \Omega^* = \frac{\Omega}{\sqrt{G\rho_1}}, \quad a = \frac{\beta}{l},$$

we obtain the dispersion relation

$$B_1 n^{*2} + B_2 n^* + B_3 = 0 \tag{29}$$

where,

$$B_1 = \left[(5+x^2) - a^2\left(\frac{(7+x^2)}{8P_2} + \frac{(9-x^2)}{72P_5} - \frac{(3-x^2)}{16P_4}\right)\right], \tag{30}$$

$$B_2 = \left[\nu_0^*(x^4 - 10x^2 + 41) + 2ixa\left\{\nu^*\left((5+x^2) - a^2\,\frac{(7+x^2)}{8P_2} + \frac{(9-x^2)}{72P_5}\right.\right.\right.$$

$$\left.\left.\left.-\frac{(3-x^2)}{16P_4}\right) - \Omega^*\left(1 + a^2\left(\frac{1}{72P_5} - \frac{1}{16P_4} - \frac{1}{8P_2}\right)\right)\right\} \cdot \frac{(e^{m\pi a} - 1)}{m\pi a}\right]. \tag{31}$$

$$B_3=\Big[x^2\Big\{1-a^2\Big(\frac{1}{8P_2}-\frac{1}{72P_5}+\frac{1}{16P_4}\Big)\Big\}\frac{(e^{m\pi a}-1)}{m\pi a}-x^2\Big\{1-a^2\Big(\frac{1}{2P_1}$$

$$-\frac{1}{18P_3}+\frac{1}{2P_2}\Big)\Big\}\frac{(e^{2m\pi a}-1)}{m\pi a}-\frac{a^2}{P_6}\Big\{2(a^4-2a^3x+2a^2(x^2+5)$$

$$-2a(x^3+5x)+x^4+10x-41)-\frac{a^2}{P_1}(a^4-2a^3x+2a^2(x^2+9)$$

$$-2a(x^3+9x)+x^4+14x^2+17)+\frac{a^2}{9P_3}(a^4-2a^3x+2a^2(x^2+9)$$

$$-2a(x^3+9x)+x^4-18x^2+81)-\frac{a^2}{4P_2}(2a^4-4a^3x+20a^2$$

$$-4a(x^3+5x)+2x^4-12x^2+18)+\frac{2a}{P_1}(a^2x-x^3-7x)$$

$$+\frac{a}{2P_2}(7a^2x-7x^3+15x)-\frac{2a}{P_3}(a^2x-x^3+9x)\Big\}$$

$$\frac{(e^{2m\pi a}-1)}{2m\pi a}\Big] \tag{32}$$

$$P_j=1+\Big(\frac{a}{j}\Big)^2,\ j=1,2,3,4,5,6, \tag{33}$$

$$P_7=(1-(a-x)^2)^2 \tag{34}$$

6. Discussion

To study the influence of the various physical effects on the growth rate of unstable

Table I. Values of growth rate (positive real part on $n^*\times10^2$) against the wave number, when $\nu_0^*=1$, $\nu^*=1$.

x	$\Omega^*=5$	$\Omega^*=10$	$\Omega^*=15$	$\Omega^*=20$
0.0	5.63993	5.63993	5.63993	5.63993
0.1	5.43486	5.43464	5.43434	5.43395
0.2	5.31412	5.31325	5.31204	5.31049
0.3	5.30287	5.30087	5.29808	5.29452
0.4	5.49310	5.48928	5.48397	5.47719
0.5	6 09766	6.09073	6.08114	6.06891
0.6	7.59733	7.58429	7.56633	7.54350
0.7	11.1696	11.1425	11.1053	11.0582
0.8	20.1786	20.1148	20.0275	19.9173
0.9	47.0432	46 8816	46.6661	46.3830
1.0	166.280	165.960	165.523	164.970

mode, we take the numerical values of the physical parameters, which correspond to the conditions in the galaxies :

$$\rho_1 = 1.7 \times 10^{-21} \text{Kg.m}^{-3}$$

$$G = 6.658 \times 10^{-11} (\text{Kg})^{-1} \text{m}^3.\text{s}^{-2}$$

Table 2. Values of growth rate (positive real part of $n^* \times 10^2$) against the wave number, when $\Omega^* = 1$, $\nu^* = 1$.

x	$\nu_0^* = 5$	$\nu_0^* = 10$	$\nu_0^* = 15$	$\nu_0^* = 20$
0.0	1.11367	0.56849	0.37901	0.28426
0.1	1.09515	0.54770	0.36515	0.27386
0.2	1.07085	0.53555	0.35705	0.26779
0.3	1.06893	0.53459	0.35641	0.26731
0.4	1.11081	0.55421	0.36949	0.27712
0.5	1.23209	0.61623	0.41084	0.30814
0.6	1.54025	0.77043	0.51366	0.38525
0.7	2.28109	1.14126	0.76093	0.57072
0.8	4.19485	2.10000	1.40031	1.05032
0.9	10.2982	5.16551	3.44457	2.58482
1.0	44.6381	22.6462	15.1392	11.3655

The values of the critical wavelength for the gravitational instability of a homogeneous infinitely extending plasma are of the order of 10^{20} metres. Therefore, we

Table 3. Values of growth rate (positive real part of $n^* \times 10^2$) against the wave number, when $\Omega^* = 1$, $\nu_0^* = 1$.

x	$\nu^* = 5$	$\nu^* = 10$	$\nu^* = 15$	$\nu^* = 20$
0.0	5.63992	5.63992	5.63991	5.63992
0.1	5.43339	5.42844	5.42021	5.40873
0.2	5.30822	5.28848	5.25590	5.21093
0.3	5.28912	5.24341	5.16892	5.06802
0.4	5.46645	4.37880	5.23860	5.05384
0.5	6.04848	5.88902	5.64055	5.32499
0.6	7.30285	7.20218	6.74929	6.20026
0.7	10.9681	10.3412	9.43374	8.39119
0.8	19 6902	18.2000	16.1199	13.8483
0.9	45.7585	41.8218	36.3031	30.3305
1.0	163.597	154.827	140.805	122.717

have calculated the roots of the dispersion relation (29) for different values of the parameters ν_0^*, ν^* and Ω^*, characterizing viscosity, FLR and rotation respectively.

These calculations are presented in Tables 1–3, where we have given the growth rate (the positive real part of n^* after multiplying by 10^4) against wave number x (after multiplying by 10^{20}) for ν_0^*(after multiplying by 10^4)$=5$, 10, 15, 20, ν^* (after multiplying by 10^4)$=5$, 10, 15, 20, and Ω^* (after multiplying by 10^4)$=5$, 10, 15, 20. In all these calculations we have taken $a=0.1$, $m=1.0$, $l=1$. It is seen from Tables 1–3 that the growth rate decreases for same x as the values of the parameters ν_0^*, ν^* and Ω increase. Thus the effects of viscosity, FLR, and coriolis forces are stabilizing.

Thus we may conclude that viscosity, FLR and rotation have stabilizing influence on the instability of a stratified layer of a self-gravitating plasma.

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