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Hamiltonians for magnetic fields in simple geometries

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Hamiltonian representations of magnetic fields using magnetic coordinate systems have recently been used by some authors in plasma physics (Boozer 1982a, b, Carey and Littlejohn 1983, Channel 1984). Magnetic coordinates are used in the study of magnetic surfaces (Miyamoto 1980) which have a relevance in toroidal fusion devices. As a first step towards an actual stability analysis, we would write the magnetic field line equations in the Hamiltonian canonical form. This requires that we find a Hamiltonian for the system in terms of suitably defined 'canonical coordinates and momenta'. Once this is done, the KAM theorem or the Chirikov criteria can be applied to study the stability of the system.

We consider two relatively simple cases : (1) a helical magnetic field, (2) the field due to a long straight steady current.

A directed line element in orthogonal curvilinear coordinates u_1, u_2, u_3 is

$$\boldsymbol{ds} = \hat{e}_1 h_1 du_1 + \hat{e}_2 h_2 du_2 + \hat{e}_3 h_3 du_3$$

and the magnetic field $\boldsymbol{B} = \hat{e}_1 B_1 + \hat{e}_2 B_2 + \hat{e}_3 B_3$, $\hat{e}_i$ are unit vectors in the directions of increasing u_i.

The field line equation is

$$\boldsymbol{ds} \times \boldsymbol{B} = 0 \tag{1}$$

This leads to

$$\frac{h_1 du_1}{B_1} = \frac{h_2 du_2}{B_2} = \frac{h_3 du_3}{B_3} \tag{2}$$

We define the 'canonical coordinates and momenta' as

$$q = q(u_1, u_2)$$
$$p = p(u_1, u_2) \tag{3}$$

and the 'canonical time' $\tau = u_3$

Requirement of Local single valuedness of H together with the condition $\nabla . \boldsymbol{B}=0$ leads to the following possible sets of q, p and H, these sets being connected to each other by canonical transformations (Janaki and Ghosh 1987)

$$\text{i) } q_1=u_1,\ p_1=\int^{u_2} h_1 h_2 B_3 du_2,\ H_1=\int^{u_2} h_2 h_3 B_1 du_2 \tag{4}$$

$$\text{ii) } q_2=u_2,\ p_2=-\int^{u_1} h_1 h_2 B_3 du_1,\ H_2=-\int^{u_1} h_1 h_3 B_2 du_1 \tag{5}$$

$$\text{iii) } p_3=u_1,\ q_3=-\int^{u_2} h_1 h_2 B_3 du_2,\ H_3=\int^{u_2} h_2 h_3 B_1 du_2 \tag{6}$$

$$\text{iv) } p_4=u_2,\ q_4=\int^{u_1} h_1 h_2 B_3 du_1,\ H_4=-\int^{u_1} h_1 h_3 B_2 du_1 \tag{7}$$

The Hamiltonian is such that the equations of the field lines are consistent with the canonical equations

$$\frac{\delta H}{\delta q}=-\frac{dp}{d\tau}\ ;\ \frac{\delta H}{\delta p}=\frac{dq}{d\tau} \tag{8}$$

We use cylindrical polar coordinates r, z, θ with $\hat{e}_r \times \hat{e}_z = \hat{e}_\theta$ etc ; this convention is somewhat different from that usually used. But this has a definite advantage in the case of toroidal geometry (Woods 1987). Here

$$u_1,\ u_2,\ u_3 = r,\ z,\ \theta\ ;\ h_1=1,\ h_2=1, h_3=r \text{ and } B_1=B_r,$$
$$B_2=B_z,\ B_3=B_\theta . \tag{9}$$

The field components for a helical magnetic field are (Blewett and Chasman 1977)

$$B_r=2B_0[I_0(kr)-\frac{1}{kr}I_1(kr)] \sin (\theta - kz) \tag{10a}$$

$$B_z=-2B_0 I_1(kr) \cos (\theta - kz) \tag{10b}$$

$$B_\theta=\frac{2B_0}{kr} I_1(kr) \cos (\theta - kz) \tag{10c}$$

where B_0 is the transverse on-axis field amplitudes and I_0, I_1 are zero-th order and 1st order Bessel functions of imaginary arguments. In the case of a helical magnetic field we have the first choice

$$q_1=r,\ p_1=\int B_\theta dz,\ H_1=\int r B_r dz \tag{11}$$

In view of eqn. (10), this leads to

$$p_1 = -\frac{2B_0}{r} I_1(kr) \sin(\theta - kz) \tag{12a}$$

$$H_1 - 2B_0[kr\, I_0(kr) - I_1(kr)] \cos(\theta - kz) \tag{12b}$$

From the field line eqs. (2) it follows that

$$\frac{dr}{d\theta} = \frac{rB_r}{B_\theta} = kr^2 \left[\frac{I_0}{I_1} - \frac{1}{kr}\right] \frac{\sin(\theta - kz)}{\cos(\theta - kz)} \tag{13a}$$

$$\frac{dz}{d\theta} = \frac{rB_z}{B_\theta} = -kr^2 \tag{13b}$$

$$\dot{p}_1 - \frac{dp_1}{d\tau} \equiv \frac{dp_1}{d\theta} = -\frac{2B_0 k}{r} I_1'(kr) \frac{dr}{d\theta} \sin(\theta - kz)$$

$$-\frac{2B_0}{r} I_1(kr) \cos(\theta - kz) \left(1 - k\frac{dz}{d\theta}\right) \tag{14}$$

For a helix of constant pitch ($k =$ constant, as we have assumed above) we can take $\theta - 0$ for $z = 0$ which makes $(\theta - kz) - 0$ for all z. In such a case eq. (14) reduces to

$$\dot{p}_1 = \frac{dp_1}{d\theta} = -\frac{2B_0}{r} I_1(kr)[1 + k^2 r^2] \tag{15}$$

Also under the same boundary condition,

$$\frac{\delta H_1}{\delta q_1} \equiv \frac{\delta H_1}{\delta r} = 2B_0[kI_0(kr) + k^2 r I_0(kr) - kI_1(kr)] \tag{16}$$

Using the recursion relations (Watson 1958)

$$I_0'(x) - I_1(x) \qquad\qquad I_{-n}(x) = I_n(x)$$

$$xI_n'(x) + nI_n(x) - xI_{n-1}(x)$$

we can write

$$\frac{\delta H_1}{\delta r} = 2B_0[kI_0(kr) + k^2 r I_1(kr) - \frac{1}{r}\{krI_0(kr) - I_1(kr)\}]$$

$$= \frac{2B_0}{r}(1 + k^2 r^2) I_1(kr) \tag{17}$$

From eq. (15) and (17) we get

$$\frac{\delta H_1}{\delta r} = -\dot{p}_1 \tag{18}$$

Now, in the Hamiltonian theory, p and q are regarded as independent variables, both being (implicit) functions of the 'time'. Therefore in dealing with the other canonical equation, we have to regard p_1 and r as independent.

From eq. (12a) and (12b) we then get

$$\frac{\delta H_1}{\delta p_1}=\frac{r[krI_0(kr)-I_1(kr)]}{I_1(kr)}\cdot\frac{\sin(\theta-kz)}{\cos(\theta-kz)}$$

$$=\frac{rB_r}{B_\theta}=\frac{dr}{d\theta}=\dot{q}_1 \tag{19}$$

We note that this eq. (19) holds independently of the boundary condition $(\theta\ \ kz)=0$. When this boundary condition is taken into consideration, eq. (19) is trivially satisfied because both $\frac{\delta H_1}{\delta p_1}$ and q_1 are then zero.

Linear current :

With our choice of

$$u_1=r,\ u_2=z \text{ and } u_3=\theta$$

$$B=\hat{e}_\theta B_\theta=-\hat{e}_\theta\frac{\mu_0 I}{2\pi r} \tag{20}$$

$$p_1=\int^z B_\theta dz=-\frac{\mu_0 I}{2\pi}\left(\frac{z}{r}\right),\qquad q_1=r \tag{21}$$

$$\theta=t$$

$$H_1=\int rB_r dz$$

$$=f_1(r,\theta)$$

[Since $B_r=0$, the integral should be a function only of r and θ]

$$\frac{dp_1}{dt}=\frac{dp_1}{d\theta}=-\frac{\mu_0 I}{2\pi}\left[\frac{1}{r}\frac{dz}{d\theta}-\frac{z}{r^2}\frac{dr}{d\theta}\right]=0$$

Since $\frac{dz}{d\theta}=0=\frac{dr}{d\theta}$ for the field lines

$$\therefore\ \frac{\delta H_1}{\delta p_1}=\frac{\delta H_1}{\delta r}=\frac{\delta f_1}{\delta r}$$

If the canonical equation $\frac{dp_1}{d\theta}=-\frac{\delta H_1}{\delta r}$ is to hold,

we should have $\frac{\delta f_1}{\delta r}=0$ so that f_1 is a function only of θ.

We may take $H_1 = \alpha\theta$, where α is a constant.

Then $\frac{\delta H_1}{\delta p_1} = \frac{\delta f_1}{\delta p_1} = 0$, because p_1 is function of z and r while f_1 is a function only of θ and $\frac{dq_1}{d\theta} = \frac{dr}{d\theta} = 0$ for the field lines. So the other canonical equation, namely, $\frac{dq_1}{d\theta} = \frac{\delta H_1}{\delta p_1}$ is also trivially satisfied.

By a canonical transformation, we may obtain

$$q = q_2 = z \;;\; p = p_2 = -\int h_1 h_2 B_3 du_1$$

$$= -\int B_\theta dr' = \frac{\mu_0 I}{2\pi} \int^{r} \frac{dr'}{r'}$$

$$= \frac{\mu_0 I}{2\pi} \ln r + \text{constant}$$

$$H_2 = -\int h_1 h_3 B_2 du_1$$

$$= -\int r B_z dr = f_2(z, \theta), \text{ because } B_z = 0$$

This set of quantities also leads to the canonical equations

$$\frac{dp_2}{d\theta} = -\frac{\delta H_2}{\delta p_2}, \text{ both sides being zero}$$

and

$$\frac{dq_2}{d\theta} = \frac{\delta H_2}{\delta p_2}, \text{ both sides being again zero.}$$

We can obtain two more alternative forms by similar canonical transformations.

We have shown that (i) in the case of a helical magnetic field (given by eq. 10.), the canonical coordinate q and momentum p and the Hamiltonian H given by eqs. (11) and (12) and (ii) in the case of the magnetic field due to a long steady current, q, p and H given by eq. (21) lead via the field line equations to the correct canonical equation. We may, therefore, conclude that the canonical equations themselves represent the field line equations in the respective cases.

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