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Prime power divisors of binomial coefficients: Reprise

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In [1], we proved an old conjecture of P. Erdös by showing that, given a positive integer j, for each sufficiently large n, there is a prime p such that

$$p^j \,\Big|\, \binom{2n}{n}.$$

As K. Ramachandra recently pointed out to me, Erdös once asked him in a letter if the following holds.

Proposition. *Let j be a positive integer. For sufficiently large n, there is a prime p such that*

$$p^j \,\Big\|\, \binom{2n}{n},$$

i.e. p^j is the highest power of p dividing $\binom{2n}{n}$.

This can be dealt with by the method introduced in section 5 of [1]. In particular, we consider

$$K_j(n) := \operatorname{card}\left\{n^{\frac{1}{j+1}} < p < \left(\frac{3}{2}n\right)^{\frac{1}{j+1}} : \left\{\frac{n}{p^i}\right\} > \frac{2}{3}\ (2 \leqq i < j), \left\{\frac{n}{p}\right\} < \frac{1}{3}\right\},$$

where $\{x\}$ denotes the fractional part of the real number x. By Theorem 3 of [1], $K_j(n) > 0$ for sufficiently large n. Hence

$$n = n_j p^j + \cdots + n_1 p + n_0$$

for some prime p with $\frac{1}{2}p < n_i < p$ $(1 \leqq i \leqq j)$ and $0 \leqq n_0 < \frac{1}{3}p$. By Kummer's lemma ([1], Lemma 9), this implies

$$(1) \qquad p^j \,\Big\|\, \binom{2n}{n}.$$

In fact, we can show much more by applying the following lemma which is a corollary to a recent upper bound for exponential sums of the type

$$\sum_{p \leqq P} e\left(x\left(\frac{h_1}{p^{j_1}} + \cdots + \frac{h_r}{p^{j_r}}\right)\right)$$

with $e(x) := e^{2\pi i x}$ for real x.

Lemma (= Proposition 2 in [2]). *For* $0 < \sigma_i \leqq 1$ $(1 \leqq i \leqq j)$,

$$\exp(c_1 j (\log 2j)^3) \leqq P \leqq x^{1/j},$$

and $0 < \varepsilon \leqq 1/12$, *we have*

$$\left|\operatorname{card}\left\{p \leqq P : \left\{\frac{x}{p^i}\right\} < \sigma_i (1 \leqq i \leqq j)\right\} - \sigma_1 \dots \sigma_j \pi(P)\right|$$
$$\leqq c_2^j \left(P^{1 - c_3 \varepsilon v_6(j)(\log P/\log x)^2} + P^{\frac{j+2}{2} + \varepsilon} x^{-\frac{1}{2}}\right)(\log x)^{4j},$$

where c_2 *and* c_3 *are some positive absolute constants, and*

$$v_i(j) := \frac{1}{j^i (\log 2j)^2}.$$

Theorem. *For sufficiently large* n, *there are* $s = s(n)$ *primes* $p_1, \dots, p_s$ *with*

$$s(n) \gg \left(\frac{\log n}{(\log\log n)^3}\right)^{\frac{1}{10}}$$

such that

$$p_j^j \,\Big\|\, \binom{2n}{n} \quad (1 \leqq j \leqq s(n)).$$

Proof. We follow the proof of Proposition 3 in [2]. Let $j > C_0$,

$$n \geqq N_0 := C^{j^{10}(\log j)^3},$$

where C_0 and C are some absolute positive constants. Applying the lemma for $x := n \geqq N_0$, $\varepsilon := 1/12$, and $P := n^{1/(j+1)}$ resp. $P := (3n/2)^{1/(j+1)}$, we obtain

$$\left|K_j(n) - \left(\frac{1}{3}\right)^j \left(\pi\left(\left(\frac{3}{2}n\right)^{\frac{1}{j+1}}\right) - \pi\left(n^{\frac{1}{j+1}}\right)\right)\right| \leqq c_4^j n^{\frac{1}{j+1} - c_5 v_9(j)} (\log n)^{4j},$$

where c_4 and c_5 are some absolute positive constants.

Clearly,

$$\pi\left(\left(\frac{3}{2}n\right)^{\frac{1}{j+1}}\right) - \pi\left(n^{\frac{1}{j+1}}\right) > c_6 \frac{n^{\frac{1}{j+1}}}{\log n}$$

for some absolute positive constant c_6. Consequently,

$$K_j(n) \geqq c_7^j \frac{n^{1/(j+1)}}{\log n} - c_4^j n^{\frac{1}{j+1} - c_5 v_9(j)} (\log n)^{4j}$$

for yet another positive absolute constant c_7.

As in the proof of Proposition 3 in [2], we conclude that $K_j(n) > 0$. Hence, by the same argument as the one leading to (1),

$$p_j^j \,\Big\|\, \binom{2n}{n} \tag{2}$$

for some prime p_j.

In order to prove the theorem, we may assume without loss of generality that

$$n > C^{(C_0+1)^{10}(\log(C_0+1))^3}.$$

By the above reasoning, (2) holds for any $j > C_0$ satisfying

$$C^{j^{10}(\log j)^3} \leqq n.$$

In other words, (2) holds for all j with

$$C_0 < j \ll \left(\frac{\log n}{(\log j)^3}\right)^{\frac{1}{10}}.$$

Sufficient for this is

$$C_0 < j \ll \left(\frac{\log n}{(\log\log n)^3}\right)^{\frac{1}{10}}.$$

For $j \leqq C_0$, (2) is rather easy to verify if n is large enough. This completes the proof of the theorem.

Remark. We end this article by pointing out that the results obtained in the Proposition and the Theorem can be extended to binomial coefficients

$$\binom{m}{k}$$

by the above methods, if m is sufficiently large and $|m-k|$ is "small" in comparison with m (see [1]).

References

[1] *J. W. Sander*, Prime power divisors of binomial coefficients, J. reine angew. Math. **430** (1992), 1–20.

[2] *J. W. Sander*, On the order of prime powers dividing $\binom{2n}{n}$, to appear.

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