Characterization of total ill-posedness in linear semi-infinite optimization

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Abstract

This paper deals with the stability of linear semi-infinite programming (LSIP, for short) problems. We characterize those LSIP problems from which we can obtain, under small perturbations in the data, different types of problems, namely, inconsistent, consistent unsolvable, and solvable problems. The problems of this class are highly unstable and, for this reason, we say that they are totally ill-posed.

The characterization that we provide here is of geometrical nature, and it depends exclusively on the original data (i.e., on the coefficients of the nominal LSIP problem). Our results cover the case of linear programming problems, and they are mainly obtained via a new formula for the subdifferential mapping of the support function.

Key words. Linear semi-infinite programming, total ill-posedness, solvability, consistency.

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1 Introduction

Consider the linear optimization problem in the Euclidean space, $\mathbb{R}^n$,

$$\begin{array}{rl} \pi: & \operatorname{Inf} \langle c, x\rangle \\ & \text{s.t. } \langle a_t, x\rangle \leq b_t\,,\ t \in T, \end{array} \tag{1}$$

where c, x, $a_t \in \mathbb{R}^n$, $b_t \in \mathbb{R}$, and $\langle ., .\rangle$ denotes the scalar product in $\mathbb{R}^n$. The non-empty *index set*, T, whose elements identify the inequalities of the *constraint system*, $\sigma = \{\langle a_t, x\rangle \leq b_t\,,\ t \in T\}$, is arbitrary (possibly infinite). The problem π is alternatively represented by the pair (c, σ). When T is infinite the problem $\pi = (c, \sigma)$ is a *linear semi-infinite programming problem* (*LSIP*).

The *parameter space* of all the problems (1), with constraint systems having the same index set, is denoted by Π. By Π_c we represent the subset of all the *consistent problems* (i.e., those problems $\pi = (c, \sigma)$ whose feasible sets are non-empty), by $\Pi_i := \Pi \diagdown \Pi_c$ the class of *inconsistent problems*, and by Π_s the subset of the *solvable problems* (having at least an optimal solution). Obviously, $\Pi_s \subset \Pi_c$.

We introduce in Π the *extended distance* $\delta : \Pi \times \Pi \to [0, +\infty]$ given by

$$\delta(\pi_1, \pi) := \max\left\{\left\|c^1 - c\right\|,\ d(\sigma_1, \sigma)\right\}, \tag{2}$$

where

$$d(\sigma_1, \sigma) := \sup_{t\in T}\left\|\begin{pmatrix} a_t^1 \\ b_t^1\end{pmatrix} - \begin{pmatrix} a_t \\ b_t\end{pmatrix}\right\|,$$

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and $\|\cdot\|$ represents the Euclidean norm in both $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$. In this way Π is endowed with the topology of the uniform convergence of the coefficient vectors [11, Chapter 10]. Given $\pi \in \Pi$ and $\widetilde{\Pi} \subset \Pi$, we write, as usual, $\delta(\pi, \widetilde{\Pi}) := \inf\{\delta(\pi, \widetilde{\pi}), \widetilde{\pi} \in \widetilde{\Pi}\}$, but now $\delta(\pi, \widetilde{\Pi})$ can take the value $+\infty$.

In (Π, δ), and also in the Euclidean space, $\operatorname{int} X$, $\operatorname{cl} X$, $\operatorname{ext} X$, and $\operatorname{bd} X$ represent the *interior set*, the *closure*, the *exterior* (i.e., the complementary set of $\operatorname{cl} X$), and the *boundary* of X, respectively. In the Euclidean space, $\operatorname{rint} X$ denotes the *relative interior* of X (i.e., the interior of X in the topology relative to the affine manifold generated by X).

The stability of an optimization model is a key property, mainly when we deal with real-world problems, and it became a paradigmatic property. Many times users prefer to emphasize stability, instead of insisting in the optimal character of the chosen solution [20]. In [1] three notions of stability in linear programming are studied, and in [6] their equivalence in the LSIP setting is proven. A selection of different contibutions to the stability of the general LSIP problem, when all the coefficients in the problem can be perturbed (in the line of [18]), are [2],[8],[9],[10],[12],[13],[16], etc.

In [3] different notions of *well-posedness* in LSIP are proposed. Most of these concepts are closely related to the condition $\pi \in \operatorname{int} \Pi_s$. Generically, we say that a problem is *ill-posed with respect certain property* when arbitrarily small perturbations of the coefficients may yield problems for which this property is either kept or lost. In particular a problem is ill-posed with respect to the consistency (solvability) if small perturbations can produce either consistent or inconsistent problems (either solvable or unsolvable problems, respectively). Formally, these ill-posed problems are those in $\operatorname{bd} \Pi_c$ ($\operatorname{bd} \Pi_s$, respectively).

In the line of [17], the distance to ill-posedness with respect to the consistency will be $\delta(\pi, \operatorname{bd} \Pi_c)$. It turns out to have a great influence on the numerical complexity of certain feasibility algorithms. In our LSIP framework, $\delta(\pi, \operatorname{bd} \Pi_c)$ is measured in [4]. The set of ill-posed problems with respect to the solvability, $\operatorname{bd} \Pi_s$, is characterized in [5], and the problem of measuring the distance $\delta(\pi, \operatorname{bd} \Pi_s)$ is approached (either by means of an exact formula or through some lower/upper bounds) in [6].

It makes sense to call *totally ill-posed problems* to those problems in $(\operatorname{bd} \Pi_c) \cap (\operatorname{bd} \Pi_s)$. These problems are highly unstable since small perturbations may provide inconsistent, unsolvable consistent, and solvable problems. In [5] a characterization of the set $(\operatorname{bd} \Pi_c) \cap (\operatorname{bd} \Pi_s)$ is obtained, but this characterization involves some parameter subset which is not identified by means of the coefficients of π. Thus, the main objective of this paper is to characterize the set of totally ill-posed problems, $(\operatorname{bd} \Pi_c) \cap (\operatorname{bd} \Pi_s)$, in terms exclusively of c, a_t, and b_t, $t \in T$.

This is the summary of the paper. Section 2 is devoted to notation and preliminary results. Proposition 4 in Section 3, together with Proposition 2, leads us to the Proposition 5 which characterizes the set $(\operatorname{bd} \Pi_c) \cap (\operatorname{bd} \Pi_s)$ in terms exclusively of the coefficients of π. Finally in this section, Corollary 3 particularizes Proposition 5 for the ordinary linear programming problem (T finite). The last section, Section 4, includes some sufficient conditions for total ill-posedness, which add geometrical insight and make easier the detection of this property. Our sufficient conditions are related to the ones given in [7].

2 Preliminaries

Given a non-empty set $X \subset \mathbb{R}^k$, by $\operatorname{co} X$, $\overline{\operatorname{co}} X$, $\operatorname{cone} X$, and $\operatorname{aff} X$ we denote the *convex hull*, the *closed convex hull*, the *conical convex hull*, and the *affine hull*, respectively, of X. We also use the sets

$$X^\circ := \left\{ y \in \mathbb{R}^k \mid \langle y, x \rangle \leq 0 \text{ for all } x \in X \right\},$$

and

$$X^{\perp} := \left\{ y \in \mathbb{R}^k \mid \langle y, x \rangle = 0 \text{ for all } x \in X \right\};$$

i.e., the *dual cone* of X (or *polar cone* of $\operatorname{cone} X$) and the *orthogonal space* of X, respectively. If X is a closed convex set, X_∞ represents its *recession cone*

$$X_\infty := \left\{ y \in \mathbb{R}^k \mid x + \lambda y \in X \text{ for some } x \in X \text{ and all } \lambda \geq 0 \right\},$$

whereas

$$\operatorname{lin} X := X_\infty \cap (-X)_\infty$$

represents its *lineality space.*

If $\Lambda \subset \mathbb{R}$ is a non-empty set, we introduce the set

$$\Lambda X := \{\lambda x : \lambda \in \Lambda \text{ and } x \in X\}.$$

For any given $S \subset T$, the set $\mathbb{R}_+^{(S)}$ will denote the cone of the functions $\lambda : S \to \mathbb{R}_+$ taking positive values only at finitely many points of S. For $\lambda := \{\lambda_t, t \in S\} \in \mathbb{R}_+^{(S)}$,

$$\operatorname{supp} \lambda := \{t \in S : \lambda_t > 0\}$$

is the *support* of λ.

In any Euclidean space $\mathbb{R}^k$ involved in our analysis, with $\|\cdot\|$ representing the *Euclidian norm*, $\mathbb{B}$ denotes the associated *closed unit ball* centered at the origin 0_k. For the sake of simplicity, we write the vectors in $\mathbb{R}^{k+1}$ in the form (x, x_{k+1}); for instance $(0_k, 1)$ and (a_t, b_t), for $t \in T$.

The following sets, associated with $\pi := (c, \sigma)$, are relevant in our analysis:

$$\begin{array}{ll} A := \operatorname{co}\{a_t,\ t \in T\}, & M := \operatorname{cone}\{a_t,\ t \in T\} = \mathbb{R}_+ A, \\ C := \operatorname{co}\{(a_t, b_t),\ t \in T\}, & N := \operatorname{cone}\{(a_t, b_t),\ t \in T\} = \mathbb{R}_+ C, \\ H := C + \mathbb{R}_+(0_n, 1), & K := N + \mathbb{R}_+\{(0_n, 1)\}, \\ Z^+ := \operatorname{co}\{-a_t,\ t \in T; c\}, & Z^- := \operatorname{co}\{-a_t,\ t \in T; -c\}, \end{array}$$

where $\mathbb{R}_+ := [0, +\infty[$.

Given a proper convex function $h : \mathbb{R}^k \longrightarrow \mathbb{R} \cup \{+\infty\}$, we denote by $\operatorname{dom} h$ its *effective domain*

$$\operatorname{dom} h := \{x \in \mathbb{R}^k : h(x) < +\infty\},$$

and by $\partial h(x)$, with $x \in \operatorname{dom} h$, the *subdifferential set* of h at x

$$\partial h(x) := \{u \in \mathbb{R}^k : h(y) - h(x) \geq \langle u, y - x \rangle \text{ for all } y \in \mathbb{R}^k\}.$$

If $\partial h(x) \neq \emptyset$, the point x is a global minimum of h if and only if $0_k \in \partial h(x)$.

Frequently we make use of the *support function* of $\operatorname{cl} C$, $f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R} \cup \{+\infty\}$, given by

$$f(x, \lambda) := \sup\{\langle a_t, x \rangle + b_t \lambda : t \in T\}. \tag{3}$$

f is a lower semicontinuous (lsc, for short) sublinear function, and its *effective domain* satisfies [15, Proposition V.2.2.4]

$$\operatorname{cl}(\operatorname{dom} f) = [(\operatorname{cl} C)_\infty]^\circ.$$

The subdifferential of f at $(x,\lambda) \in \operatorname{dom} f$ is ([15, Example VI.3.1])

$$\partial f(x,\lambda) = \{(u,\mu) \in \operatorname{cl} C : f(x,\lambda) = \langle u, x\rangle + \mu\lambda\}.$$

In particular

$$\partial f(0_{n+1}) = \operatorname{cl} C.$$

Given $\pi^r = (c^r, \sigma^r)$, with $\sigma_r := \{\langle a_t^r, x\rangle \leq b_t^r : t \in T\}$ and $f^r(x,\lambda) := \sup\{\langle a_t^r, x\rangle + b_t^r\lambda : t \in T\}$, for $r = 1, 2, \cdots$, such that

$$d(\pi^r, \pi) < +\infty, \; r = 1, 2, \cdots,$$

one has

$$f(x,\lambda) - \delta(\pi^r, \pi)\|(x,\lambda)\| \leq f^r(x,\lambda) \leq f(x,\lambda) + \delta(\pi^r,\pi)\|(x,\lambda)\|,$$

for every $(x,\lambda) \in \mathbb{R}^{n+1}$. Thus, $f^r \to f$ pointwisely and $\operatorname{dom} f^r = \operatorname{dom} f$, for $r = 1, 2, \cdots$. This yields

$$\operatorname{cl}(\operatorname{dom} f^r) = \operatorname{cl}(\operatorname{dom} f) = [(\operatorname{cl} C)_\infty]^\circ, \; r = 1, 2, \cdots.$$

When T is infinite, we can consider the following family of pathological problems

$$\Pi_\infty := \{\pi \in \Pi : \; \delta(\pi, \operatorname{bd} \Pi_c) = +\infty\},$$

which is characterized in the following proposition:

Proposition 1 *Given $\pi = (c, \sigma) \in \Pi$, the following statements are equivalent*:
(i) $\pi \in \Pi_\infty$,
(ii) $\sup\{\langle a_t, x\rangle - b_t : t \in T\} = +\infty$, *for all* $x \in \mathbb{R}^n$,
(iii) $(0_n, -1) \in (\operatorname{cl} C)_\infty$.

Proof. The equivalence of (i) and (iii) is given in [4, Proposition 1]. To show the equivalence of (ii) and (iii) we appeal to the function f defined in (3). Assume that (iii) holds, that is, for every fixed $(u,\mu) \in \operatorname{cl} C$, we have

$$(u,\mu) + \gamma(0_n \;, -1) \in \operatorname{cl} C = \partial f(0_{n+1}),$$

for all $\gamma \geq 0$. Thus, for each $x \in \mathbb{R}^n$,

$$f(x,-1) \geq \langle (u,\mu) + \gamma(0_n \;, -1), (x, -1)\rangle = \langle u, x\rangle - \mu + \gamma,$$

for all $\gamma \geq 0$, i.e. $f(x,-1) = \sup\{\langle a_t, x\rangle - b_t : t \in T\} = +\infty$.

Conversely, assume that (ii) holds but $(0_n \;, -1) \notin (\operatorname{cl} C)_\infty$. By the separation theorem, there will exist $(v,\alpha) \in \mathbb{R}^{n+1} \setminus \{0_{n+1}\}$ and $\beta \in \mathbb{R}$ such that

$$\langle (v,\alpha), (z,\mu)\rangle \leq \beta < -\alpha, \text{ for all } (z,\mu) \in (\operatorname{cl} C)_\infty.$$

Since $(\operatorname{cl} C)_\infty$ is a closed cone, we conclude, from the previous inequalities, that $\alpha < 0$ and $(v,\alpha) \in [(\operatorname{cl} C)_\infty]^\circ = \operatorname{cl}(\operatorname{dom} f)$. Consequently, applying Theorem 6.1 in [19] and taking into account the homogeneity of f, there would exist x satisfying $f(x,-1) < +\infty$, and this contradicts (ii). ∎

The following proposition gathers different results which are used throughout the paper.

Proposition 2 *Given $\pi = (c, \sigma) \in \Pi$, the following statements hold*:

(i) [11, Theorem 4.4] $\pi \in \Pi_c$ *if and only if* $(0_n, -1) \notin \operatorname{cl} N$,
(ii) [12, Theorem 3.1] *If* $\pi \in \Pi_c$, *then* $\pi \in \operatorname{int} \Pi_c$ *if and only if* $0_{n+1} \notin \operatorname{cl} C$,
(iii) [5, Lemma 1(ii)] *If* $\pi \in \Pi_i \cap \operatorname{bd} \Pi_c$, *then* $0_n \in \operatorname{bd} A$,
(iv) *If* $\pi \in \Pi_i \cap \operatorname{bd} \Pi_c$, *then* $0_{n+1} \in \operatorname{bd} C$ *if and only if* $0_{n+1} \in \operatorname{cl} C$,
(v) [4, Theorem 4, 5, and 6] *If* $\pi \in \Pi \setminus \Pi_\infty$, *then* $\pi \in \operatorname{ext} \Pi_c$, $\pi \in \operatorname{int} \Pi_c$, *or* $\pi \in \operatorname{bd} \Pi_c$ *if and only if* $0_{n+1} \in \operatorname{int} H$, $0_{n+1} \in \operatorname{ext} H$, *or* $0_{n+1} \in \operatorname{bd} H$, *respectively*,
(vi) [5, Theorem 2] *If* $\pi \in \operatorname{int} \Pi_c$, *then* $\pi \in \operatorname{ext} \Pi_s$, $\pi \in \operatorname{int} \Pi_s$, *or* $\pi \in \operatorname{bd} \Pi_s$ *if and only if* $0_{n+1} \in \operatorname{ext} Z^-$, $0_{n+1} \in \operatorname{int} Z^-$, *or* $0_{n+1} \in \operatorname{bd} Z^-$, *respectively,*
(vii) [5, Theorem 3] *If* $\pi \in \operatorname{bd} \Pi_c$, *then* $\pi \in \operatorname{bd} \Pi_s$ *if and only if either* $0_{n+1} \in \operatorname{bd} Z^+$ *or* $\pi \in \operatorname{cl}(\Pi_c \cap \operatorname{bd} \Pi_c)$.

Proof. (iv) The direct statement is trivial. For the converse, we have, by (iii), $0_n \in \operatorname{bd} A$, and this precludes $0_{n+1} \in \operatorname{int} C$. So, $0_{n+1} \in \operatorname{cl} C$ implies $0_{n+1} \in \operatorname{bd} C$ ■

Remark It is obvious that $(\operatorname{cl} C)_\infty \subset \operatorname{cl} N$, and Proposition 1(iii) together with Proposition 2(i) entail $\Pi_\infty \subset \Pi_i$. Accordingly, a problem $\pi \in \Pi_\infty$ can be called *totally inconsistent.*

Finally in this section we include an alternative characterization of the subdifferential set of the support function of an arbitrary set [14, Proposition 1].

Proposition 3 *Consider a non-empty set $A \subset \mathbb{R}^p$ and its associated support function $h : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$*

$$h(z) := \sup\{\langle a, z\rangle : a \in A\}.$$

Then, for every $z \in \operatorname{dom} h$ we have

$$\partial h(z) = \bigcap_{\delta>0} \operatorname{cl}\left((\operatorname{co} A_\delta) + (\overline{\operatorname{co}} A)_\infty \cap \{z\}^\perp\right), \tag{4}$$

where

$$A_\delta := \{a \in A : \langle a, z\rangle \geq h(z) - \delta\},$$

for $\delta > 0$. If $z \in \operatorname{rint}(\operatorname{dom} h)$, then one has

$$\partial h(z) = \bigcap_{\delta>0} \operatorname{cl}\left((\operatorname{co} A_\delta) + \operatorname{lin}(\overline{\operatorname{co}} A)\right),$$

whereas $z \in \operatorname{int}(\operatorname{dom} h)$ entails

$$\partial h(z) = \bigcap_{\delta>0} \overline{\operatorname{co}} A_\delta.$$

3 Characterization of the total ill-posedness

The main objective of this section is to characterize the set $\operatorname{cl}(\Pi_c \cap \operatorname{bd} \Pi_c)$, and this is achieved in Proposition 4 by means of conditions relying exclusively on the position of 0_{n+1} with respect to $\operatorname{cl} C$. In this way, our Proposition 4, together with some results in [5] (gathered in Proposition 2), lead us to Proposition 5 in this section, which provides a characterization of the class of totally ill-posed problems, $(\operatorname{bd} \Pi_c) \cap (\operatorname{bd} \Pi_s)$, in terms exclusively of the coefficients of σ and the vector c.

Let us denote, for $x \in \mathbb{R}^n$ and $\varepsilon > 0$,

$$T_\varepsilon(x) := \{t \in T : |\langle (a_t, b_t), (x, -1)\rangle| \leq \varepsilon \|(x, -1)\|\}. \tag{5}$$

Proposition 4 *Let* $\pi \in \Pi_i \cap \operatorname{bd} \Pi_c$. *Then* $\pi \in \operatorname{cl}(\Pi_c \cap \operatorname{bd} \Pi_c)$ *if and only if for every* $\varepsilon > 0$ *there exists* $x^\varepsilon \in \mathbb{R}^n$ *such that the following statements hold*:

(i) $\langle (a_t, b_t), (x^\varepsilon, -1) \rangle \leq \varepsilon \|(x^\varepsilon, -1)\|$, *for every* $t \in T$,

(ii) $0_{n+1} \in \operatorname{co}\{(a_t, b_t) : t \in T_\varepsilon(x^\varepsilon)\} + (\operatorname{cl} C)_\infty \cap \{(x^\varepsilon, -1)\}^\perp + \varepsilon \mathbb{B}$.

Proof. Assume that $\pi \in \operatorname{cl}(\Pi_c \cap \operatorname{bd} \Pi_c)$ and fix $\varepsilon > 0$. Let $\pi^\varepsilon := (c^\varepsilon, \sigma^\varepsilon) \in \Pi_c \cap \operatorname{bd} \Pi_c$, with $\sigma^\varepsilon := \{\langle a_t^\varepsilon, x \rangle \leq b_t^\varepsilon,\, t \in T\}$, be such that $\delta(\pi, \pi^\varepsilon) \leq \varepsilon/2$.

If f^ε is the support function of $C^\varepsilon := \operatorname{co}\{(a_t^\varepsilon, b_t^\varepsilon) : t \in T\}$, and x^ε is feasible for π^ε, we have $f^\varepsilon(x^\varepsilon, -1) \leq 0$. Moreover, Proposition 2(ii) provides $0_{n+1} \in \operatorname{cl} C^\varepsilon = \partial f^\varepsilon(0_{n+1})$, entailing that 0_{n+1} is a global minimum of f^ε, as well as $(x^\varepsilon, -1)$. Hence $f^\varepsilon(x^\varepsilon, -1) = f^\varepsilon(0_{n+1}) = 0$, and Proposition 3 yields

$$\partial f^\varepsilon(x^\varepsilon, -1) = \bigcap_{\delta > 0} \operatorname{cl}\left(\operatorname{co}\{(a_t^\varepsilon, b_t^\varepsilon) : \langle (a_t^\varepsilon, b_t^\varepsilon), (x^\varepsilon, -1) \rangle \geq -\delta\} + (\operatorname{cl} C^\varepsilon)_\infty \cap \{(x^\varepsilon, -1)\}^\perp\right). \tag{6}$$

We proceed by showing that x^ε satisfies (i). Certainly, for every $t \in T$, $\langle (a_t^\varepsilon, b_t^\varepsilon), (x^\varepsilon, -1) \rangle \leq 0$, so that

$$\langle (a_t, b_t), (x^\varepsilon, -1) \rangle \leq \langle (a_t^\varepsilon, b_t^\varepsilon), (x^\varepsilon, -1) \rangle + \frac{\varepsilon}{2} \|(x^\varepsilon, -1)\| \leq \varepsilon \|(x^\varepsilon, -1)\|.$$

Now we prove that x^ε also satisfies (ii). From (6), with $\delta = \varepsilon/2$, and the fact that $(x^\varepsilon, -1)$ is a global minimum of f^ε, we get

$$\begin{aligned}
0_{n+1} &\in \partial f^\varepsilon(x^\varepsilon, -1) \\
&\subset \operatorname{cl}\left(\operatorname{co}\{(a_t^\varepsilon, b_t^\varepsilon) : \langle (a_t^\varepsilon, b_t^\varepsilon), (x^\varepsilon, -1) \rangle \geq -\varepsilon/2\} + (\operatorname{cl} C^\varepsilon)_\infty \cap \{(x^\varepsilon, -1)\}^\perp\right) \\
&\subset \operatorname{co}\{(a_t^\varepsilon, b_t^\varepsilon) : \langle (a_t^\varepsilon, b_t^\varepsilon), (x^\varepsilon, -1) \rangle \geq -\varepsilon/2\} + (\operatorname{cl} C^\varepsilon)_\infty \cap \{(x^\varepsilon, -1)\}^\perp + \frac{\varepsilon}{2}\mathbb{B} \\
&\subset \operatorname{co}\{(a_t, b_t) : \langle (a_t^\varepsilon, b_t^\varepsilon), (x^\varepsilon, -1) \rangle \geq -\varepsilon/2\} + (\operatorname{cl} C)_\infty \cap \{(x^\varepsilon, -1)\}^\perp + \varepsilon \mathbb{B},
\end{aligned} \tag{7}$$

the last inclusion being a consequence of $\sup_{t \in T} \|(a_t^\varepsilon, b_t^\varepsilon) - (a_t, b_t)\| \leq \delta(\pi, \pi^\varepsilon) \leq \varepsilon/2$, $(\operatorname{cl} C^\varepsilon)_\infty = (\operatorname{cl} C)_\infty$, and the Cauchy-Swartz inequality.

Taking into account that $\|(x^\varepsilon, -1)\| \geq 1$, that $\delta(\pi, \pi^\varepsilon) \leq \varepsilon/2$, and applying again the Cauchy-Swartz inequality, we write

$$\begin{aligned}
&\{t \in T : \langle (a_t^\varepsilon, b_t^\varepsilon), (x^\varepsilon, -1) \rangle \geq -\varepsilon/2\} \\
&\quad \subset \{t \in T : \langle (a_t^\varepsilon, b_t^\varepsilon), (x^\varepsilon, -1) \rangle \geq -\varepsilon/2 \|(x^\varepsilon, -1)\|\} \\
&\quad = \left\{t \in T : 0 \geq \left\langle (a_t^\varepsilon, b_t^\varepsilon), \frac{(x^\varepsilon, -1)}{\|(x^\varepsilon, -1)\|} \right\rangle \geq -\varepsilon/2\right\} \\
&\quad \subset \left\{t \in T : \varepsilon/2 \geq \left\langle (a_t, b_t), \frac{(x^\varepsilon, -1)}{\|(x^\varepsilon, -1)\|} \right\rangle \geq -\varepsilon\right\} \subset T_\varepsilon(x^\varepsilon).
\end{aligned}$$

This inclusion, together with (7), give rise to

$$0_{n+1} \in \operatorname{co}\{(a_t, b_t) : t \in T_\varepsilon(x^\varepsilon)\} + (\operatorname{cl} C)_\infty \cap \{(x^\varepsilon, -1)\}^\perp + \varepsilon \mathbb{B};$$

i.e. x^ε also satisfies (ii).

Now we prove the converse. Fix $\varepsilon > 0$ and let x^ε be the associated vector verifying (i) and (ii). Because of (ii), there will exist $\lambda \in \mathbb{R}_+^{(T)}$, with $\operatorname{supp} \lambda \subset T_\varepsilon(x^\varepsilon)$ and $\sum_{t \in T} \lambda_t = 1$, $(v, \alpha) \in (\operatorname{cl} C)_\infty \cap \{(x^\varepsilon, -1)\}^\perp$, and $(w, \gamma) \in \mathbb{B}$ such that

$$0_{n+1} = \sum_{t \in T_\varepsilon(x^\varepsilon)} \lambda_t (a_t, b_t) + (v, \alpha) + \varepsilon (w, \gamma). \tag{8}$$

Set $\sigma^{\varepsilon} := \{\langle a_t^{\varepsilon}, x\rangle \leq b_t^{\varepsilon} : t \in T\}$, where $(a_t^{\varepsilon}, b_t^{\varepsilon})$ is defined as follows

$$(a_t^{\varepsilon}, b_t^{\varepsilon}) := \begin{cases} (a_t, b_t) + \varepsilon\,(w, \gamma) - \frac{\langle (a_t,b_t)+\varepsilon(w,\gamma),(x^{\varepsilon},-1)\rangle}{\|(x^{\varepsilon},-1)\|^2}(x^{\varepsilon}, -1), & \text{if } t \in T_{\varepsilon}(x^{\varepsilon}), \\ (a_t, b_t)\,, & \text{otherwise.} \end{cases} \tag{9}$$

We shall show that $\pi^{\varepsilon} := (c, \sigma^{\varepsilon})$ satisfies $\delta(\pi, \pi^{\varepsilon}) \leq 3\varepsilon$ and $\pi^{\varepsilon} \in \Pi_c \cap \operatorname{bd}\Pi_c$. To this aim we proceed as follows:
First, for all $t \in T_{\varepsilon}(x^{\varepsilon})$, we have

$$\begin{aligned} \|(a_t^{\varepsilon}, b_t^{\varepsilon}) - (a_t, b_t)\| &\leq \varepsilon\,\|(w, \gamma)\| + \frac{|\langle (a_t, b_t) + \varepsilon(w, \gamma), (x^{\varepsilon}, -1)\rangle|}{\|(x^{\varepsilon}, -1)\|} \\ &\leq \varepsilon + \frac{|\langle (a_t, b_t)\,, (x^{\varepsilon}, -1)\rangle|}{\|(x^{\varepsilon}, -1)\|} + \frac{|\langle \varepsilon(w, \gamma), (x^{\varepsilon}, -1)\rangle|}{\|(x^{\varepsilon}, -1)\|} \leq 3\varepsilon, \end{aligned}$$

so that

$$\delta(\pi^{\varepsilon}, \pi) = d(\sigma^{\varepsilon}, \sigma) = \sup_{t\in T} \|(a_t^{\varepsilon}, b_t^{\varepsilon}) - (a_t, b_t)\| \equiv \sup_{t\in T_{\varepsilon}(x^{\varepsilon})} \|(a_t^{\varepsilon}, b_t^{\varepsilon}) - (a_t, b_t)\| \leq 3\varepsilon.$$

Second, we check that x_{ε} is feasible for π^{ε}.
According to (9) we have $\langle (a_t^{\varepsilon}, b_t^{\varepsilon})\,, (x^{\varepsilon}, -1)\rangle = 0$ for all $t \in T_{\varepsilon}(x^{\varepsilon})$, whereas condition (i) gives us

$$\langle (a_t^{\varepsilon}, b_t^{\varepsilon})\,, (x^{\varepsilon}, -1)\rangle \equiv \langle (a_t, b_t)\,, (x^{\varepsilon}, -1)\rangle < -\varepsilon\,\|(x^{\varepsilon}, -1)\| \leq 0,$$

for all $t \in T \setminus T_{\varepsilon}(x^{\varepsilon})$. In this way the feasibility of x_{ε} for π^{ε} follows; i.e. $\pi^{\varepsilon} \in \Pi_c$.
The last point to be checked is that $\pi^{\varepsilon} \in \operatorname{bd}\Pi_c$. To this aim, and because $\pi^{\varepsilon} \in \Pi_c$, it suffices to establish that $0_{n+1} \in \operatorname{cl} C^{\varepsilon} := \overline{\operatorname{co}}\,\{(a_t^{\varepsilon}, b_t^{\varepsilon}) : t \in T\}$, according to Proposition 2(ii).

Reasoning by contradiction, suppose that $0_{n+1} \notin \operatorname{cl} C^{\varepsilon}$. Then, by the separation theorem, there would exist $(u, \mu) \in \mathbb{R}^{n+1}\backslash\{0_{n+1}\}$ and $\beta < 0$ such that

$$\langle (a_t^{\varepsilon}, b_t^{\varepsilon})\,, (u, \mu)\rangle \leq \beta,$$

for all $t \in T$. This inequality entails $(u, \mu) \in \operatorname{dom} f^{\varepsilon}$, domain of the support function f^{ε} of C^{ε}. In particular, for $t \in T_{\varepsilon}(x^{\varepsilon})$,

$$\left\langle (a_t, b_t) + \varepsilon(w, \gamma) - \frac{\langle (a_t, b_t) + \varepsilon(w, \gamma), (x^{\varepsilon}, -1)\rangle}{\|(x^{\varepsilon}, -1)\|^2}(x^{\varepsilon}, -1), (u, \mu)\right\rangle \leq \beta. \tag{10}$$

Multiplying both sides of the inequality (10) above by λ_t, for $t \in T_{\varepsilon}(x^{\varepsilon})$, and summing up over $T_{\varepsilon}(x^{\varepsilon})$, we obtain

$$\left\langle \textstyle\sum_{t\in T_{\varepsilon}(x^{\varepsilon})} \lambda_t\,(a_t, b_t) + \varepsilon(w, \gamma) - \frac{\langle \sum_{t\in T_{\varepsilon}(x^{\varepsilon})} \lambda_t(a_t,b_t)+\varepsilon(w,\gamma),(x^{\varepsilon},-1)\rangle}{\|(x^{\varepsilon},-1)\|^2}(x^{\varepsilon}, -1), (u, \mu)\right\rangle \leq \beta,$$

so that, making use of (8) and the condition $(v, \alpha) \in \{(x^{\varepsilon}, -1)\}^{\perp}$,

$$\left\langle -(v, \alpha) + \frac{\langle (v, \alpha), (x^{\varepsilon}, -1)\rangle}{\|(x^{\varepsilon}, -1)\|^2}(x^{\varepsilon}, -1), (u, \mu)\right\rangle = \langle -(v, \alpha), (u, \mu)\rangle \leq \beta.$$

In this way one gets $\langle (v, \alpha), (u, \mu)\rangle \geq -\beta > 0$, which constitutes a contradiction because $(v, \alpha) \in (\operatorname{cl} C)_{\infty}$ and

$$(u, \mu) \in \operatorname{dom} f^{\varepsilon} \subset [(\operatorname{cl} C^{\varepsilon})_{\infty}]^{\circ} = [(\operatorname{cl} C)_{\infty}]^{\circ}.$$

(The last equality $[(\mathrm{cl}\, C^\varepsilon)_\infty]^\circ = [(\mathrm{cl}\, C)_\infty]^\circ$ holds because $\delta(\pi^\varepsilon, \pi)$ is finite.)
Summarizing, we have proven that for every $\varepsilon > 0$ there exists $\pi^\varepsilon \in \Pi_c \cap \mathrm{bd}\, \Pi_c$ such that $\delta(\pi^\varepsilon, \pi) \leq 3\varepsilon$, thus we conclude that $\pi \in \mathrm{cl}(\Pi_c \cap \mathrm{bd}\, \Pi_c)$. ■

The following corollary is used in the sequel.

Corollary 1 *Let $\pi \in \Pi_i$. If $\pi \in \mathrm{cl}\,(\Pi_c \cap \mathrm{bd}\, \Pi_c)$, then we have:*

(i) $0_{n+1} \in \mathrm{bd}\, C$.
(ii) *Condition (ii) in Proposition 4 can be expressed in the alternative form*

$$0_{n+1} \in \mathrm{bd}\left(\bigcap_{\varepsilon>0} \left(\mathrm{co}\, \{(a_t, b_t) : t \in T_\varepsilon(x^\varepsilon)\} + (\mathrm{cl}\, C)_\infty \cap \{(x^\varepsilon, -1)\}^\perp + \varepsilon\mathbb{B}\right)\right).$$

Proof. (i) is already known [5, Theorem 4], and here we provide a straightforward alternative proof.
Statement (ii) in Proposition 4 implies

$$0_{n+1} \in \mathrm{cl}\, C + (\mathrm{cl}\, C)_\infty + \varepsilon\mathbb{B} = \mathrm{cl}\, C + \varepsilon\mathbb{B}, \text{ for all } \varepsilon > 0.$$

Hence $0_{n+1} \in \mathrm{cl}\, C$, or equivalently $0_{n+1} \in \mathrm{bd}\, C$, according to Proposition 2(iv).
(ii) Otherwise, we will have

$$0_{n+1} \in \mathrm{int}\left(\bigcap_{\varepsilon>0} (\mathrm{cl}\, C + (\mathrm{cl}\, C)_\infty + \varepsilon B)\right) = \mathrm{int}\, C,$$

a contradiction with (i). ■

We proceed by analyzing systems whose coefficients $\{(a_t, b_t) : t \in T\}$ are bounded. The following corollary is the counterpart of Proposition 4 under this boundedness assumption and, so, it applies to the case of ordinary linear programming.

Corollary 2 *Let $\pi \in \Pi_i \cap \mathrm{bd}\, \Pi_c$ and assume that $\sup_{t\in T} \|(a_t, b_t)\| \leq M$, for some $M > 0$. Then, the following statements are equivalent:*

(i) $\pi \in \mathrm{cl}\,(\Pi_c \cap \mathrm{bd}\, \Pi_c)$,
(ii) $0_{n+1} \in \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, \{(a_t, b_t) : \langle a_t, x\rangle \geq -\varepsilon\}$, *for every $x \in A^\circ \setminus \{0_n\}$.*

Proof. $[(i) \Rightarrow (ii)]$ From Corollary 1(i), $0_{n+1} \in \mathrm{bd}\, C \subset \mathrm{cl}\, C = \partial f(0_{n+1})$, so that 0_{n+1} is a global minimum of f.

Since $\Pi_i \cap \mathrm{bd}\, \Pi_c$, Proposition 2(iii) provides $0_n \in \mathrm{bd}\, A$. By the separation theorem, there must exist $x \in \mathbb{R}^n \setminus \{0_n\}$ such that $\langle a_t, x\rangle \leq 0$, for all $t \in T$; i.e. $x \in A^\circ \setminus \{0_n\}$.
Moreover, since 0_{n+1} is a global minimum of f, one has $f(x, \lambda) \geq 0$ for every $(x, \lambda) \in \mathbb{R}^{n+1}$. In particular, $f(x, 0) = 0$ for all $x \in A^\circ = \{x \in \mathbb{R}^n : \langle a_t, x\rangle \leq 0, \text{ for all } t \in T\}$.
Thus, taking into account that the support function f of the bounded set C is finite everywhere, Proposition 3 yields

$$0_{n+1} \in \partial f(x, 0) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, \{(a_t, b_t) : \langle a_t, x\rangle \geq -\varepsilon\},$$

for every $x \in A^\circ \setminus \{0_n\}$; that is, (ii) holds.

$[(ii) \Rightarrow (i)]$ Since $\pi \in \mathrm{bd}\,\Pi_c$ there exists a sequence of consistent problems $\pi^r := (c^r, \sigma^r)$, with $\sigma^r := \{\langle a_t^r, x\rangle \leq b_t^r : t \in T\}$, $r = 1, 2, \cdots$, converging to π. Letting x^r be a feasible point of π^r, for $r = 1, 2, \cdots$, we assume, without loss of generality, that the sequence $\frac{(x^r,-1)}{\|(x^r,-1)\|}$, $r = 1, 2, \cdots$, converges to some $(z, 0) \in \mathbb{R}^{n+1} \diagdown \{0_{n+1}\}$. Otherwise the sequence would converge to (z, α), with $\alpha < 0$, and then $z/|\alpha|$ would be a feasible point of π (contradicting its inconsistency). Moreover, the sequence $\|x^r\|$, $r = 1, 2, \cdots$, must converge to $+\infty$, because otherwise there will exist a subsequence of $(x^r)_{r=1}^{\infty}$ converging to a feasible point of π, and this constitutes a contradiction, again with the inconsistency of π.

Fixed $t \in T$, we have $\langle a_t^r, x^r\rangle \leq b_t^r$, $r = 1, 2, ...$, which yields $z \in A^{\circ} \setminus \{0_n\}$ due to the boundedness of $\{b_t,\ t \in T\}$ and $\lim_{r\to+\infty} \|x^r\| = +\infty$. By the current assumption

$$0_{n+1} \in \bigcap_{\delta>0} \mathrm{cl}\left(\mathrm{co}\left\{(a_t, b_t) : \langle a_t, z\rangle \geq -\delta\right\}\right),$$

and for a fixed $\varepsilon > 0$ one has

$$0_{n+1} \in \mathrm{cl}\left(\mathrm{co}\left\{(a_t, b_t) : \langle a_t, z\rangle \geq -\frac{\varepsilon}{2}\right\}\right). \tag{11}$$

Assume that r_ε is big enough to guarantee that

$$\left\| \frac{(x^{r_\varepsilon}, -1)}{\|(x^{r_\varepsilon}, -1)\|} - (z, 0) \right\| \leq \frac{\varepsilon}{2M} \quad \text{and} \quad \delta(\pi^{r_\varepsilon}, \pi) \leq \varepsilon.$$

Then, $\langle (a_t, b_t), (z, 0)\rangle \geq -\frac{\varepsilon}{2}$ implies

$$\begin{aligned}\left|\left\langle (a_t, b_t), \frac{(x^{r_\varepsilon}, -1)}{\|(x^{r_\varepsilon}, -1)\|}\right\rangle\right| &\leq |\langle (a_t, b_t), (z, 0)\rangle| + \left|\left\langle (a_t, b_t), \frac{(x^{r_\varepsilon}, -1)}{\|(x^{r_\varepsilon}, -1)\|} - (z, 0)\right\rangle\right| \\ &\leq \frac{\varepsilon}{2} + \|(a_t, b_t)\| \left\| \frac{(x^{r_\varepsilon}, -1)}{\|(x^{r_\varepsilon}, -1)\|} - (z, 0)\right\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Now, (11) allows us to write

$$\begin{aligned}0_{n+1} &\in \mathrm{cl}\left(\mathrm{co}\left\{(a_t, b_t) : |\langle (a_t, b_t), (x^{r_\varepsilon}, -1)\rangle| \leq \varepsilon \|(x^{r_\varepsilon}, -1)\|\right\}\right) \\ &\subset \mathrm{co}\left\{(a_t, b_t) : |\langle (a_t, b_t), (x^{r_\varepsilon}, -1)\rangle| \leq \varepsilon \|(x^{r_\varepsilon}, -1)\|\right\} + \varepsilon\mathbb{B} \\ &\subset \mathrm{co}\left\{(a_t, b_t) : t \in T_\varepsilon(x^{r_\varepsilon})\right\} + (\mathrm{cl}\, C)_\infty \cap \{(x^{r_\varepsilon}, -1)\}^{\perp} + \varepsilon\mathbb{B};\end{aligned}$$

that is, condition (ii) in Proposition 4 holds taking $x^\varepsilon := x^{r_\varepsilon}$. Moreover, one has for all $t \in T$,

$$\begin{aligned}\langle (a_t, b_t), (x^{r_\varepsilon}, -1)\rangle &\leq \langle (a_t, b_t) - (a_t^{r_\varepsilon}, b_t^{r_\varepsilon}), (x^{r_\varepsilon}, -1)\rangle \\ &\leq \|(a_t, b_t) - (a_t^{r_\varepsilon}, b_t^{r_\varepsilon})\| \, \|(x^{r_\varepsilon}, -1)\| \\ &\leq \delta(\pi^{r_\varepsilon}, \pi) \, \|(x^{r_\varepsilon}, -1)\| \leq \varepsilon \, \|(x^{r_\varepsilon}, -1)\|,\end{aligned}$$

so that condition (i) in Proposition 4 also holds taking $x^\varepsilon := x^{r_\varepsilon}$. Applying Proposition 4, we conclude $\pi \in \mathrm{cl}\,(\Pi_c \cap \mathrm{bd}\,\Pi_c)$ and, so, $[(ii) \Rightarrow (i)]$. ■

In the following example we give an application of Proposition 4.

Example 1 Let us consider, in $\mathbb{R}^2$, the system of inequalities σ, given by

$$\begin{cases} \langle (-1, s), (x, y)\rangle \leq 0, & s > 0, \\ \langle (0, 0), (x, y)\rangle \leq -1, & \\ \langle (t, 1), (x, y)\rangle \leq 0, & t > 0, \end{cases}$$

and the LSIP problem $\pi := (c, \sigma)$, with $c \in \mathbb{R}^2$. In order to apply Proposition 4 we shall verify first the following points:
1) $\pi \in \Pi_i$: This follows from the inequatlity $\langle (0,0), (x,y) \rangle \leq -1$.
2) $\pi \in \operatorname{bd} \Pi_c$: It is not difficult to graphically check that $0_3 \in \operatorname{bd} H$ and $(0,0,-1) \notin (\operatorname{cl} C)_\infty$. Hence $\pi \notin \Pi_\infty$ and Proposition 2(v) leads us to $\pi \in \operatorname{bd} \Pi_c$.
3) π satisfies both conditions (i) and (ii) in Proposition 4. In order to verify this statement, let $\varepsilon > 0$ and set $x^\varepsilon := (0, -1/\varepsilon)$. We have, for all $s, t > 0$,

$$\begin{aligned}
\langle (-1,s,0), (0,-1/\varepsilon,-1) \rangle &= -s/\varepsilon \leq \varepsilon \left\| (0,-1/\varepsilon,-1) \right\|, \\
\langle (0,0,-1), (0,-1/\varepsilon,-1) \rangle &= 1 \leq \varepsilon \left\| (0,-1/\varepsilon,-1) \right\|, \\
\langle (t,1,0), (0,-1/\varepsilon,-1) \rangle &= -1/\varepsilon \leq \varepsilon \left\| (0,-1/\varepsilon,-1) \right\|,
\end{aligned}$$

thus condition (i) in Proposition 4 follows.
In order to check condition (ii), we observe first that $(1,0,0) \in (\operatorname{cl} C)_\infty \cap \{(x^\varepsilon, -1)\}^\perp$. Next, for $s \in]0, \varepsilon]$ fixed, we have

$$|\langle (-1,s,0), (0,-1/\varepsilon,-1) \rangle| = s/\varepsilon \leq 1 < \varepsilon \left\| (0,-1/\varepsilon,-1) \right\|,$$

thus $s \in T_\varepsilon(x^\varepsilon)$, and

$$\begin{aligned}
0_3 &= (-1,s,0) + (1,0,0) - (0,s,0) \\
&\in \{(-1,s,0)\} + (\operatorname{cl} C)_\infty \cap \{(x^\varepsilon,-1)\}^\perp + \varepsilon \mathbb{B} \\
&\in \operatorname{co}\{(a_t,b_t) : t \in T_\varepsilon(x^\varepsilon)\} + (\operatorname{cl} C)_\infty \cap \{(x^\varepsilon,-1)\}^\perp + \varepsilon \mathbb{B},
\end{aligned}$$

that is, condition (ii) in Proposition 4 also holds, and so, $\pi \in \operatorname{cl}(\Pi_c \cap \operatorname{bd} \Pi_c)$.

The following proposition gives rise to a characterization of the class of totally ill-posed problems in Π.

Proposition 5 *In relation to a problem $\pi \in \Pi \diagdown \Pi_\infty$, let us consider the following conditions*:
(a) $0_{n+1} \in \operatorname{bd} H$,
(b) $0_{n+1} \in \operatorname{cl} C$,
(c) $0_n \in \operatorname{bd} Z^+$,
(d) *for every $\varepsilon > 0$ there exists x^ε such that* (i) *and* (ii) *in Proposition 4 are satisfied.*
Then π is totally ill-posed, i.e. $\pi \in (\operatorname{bd} \Pi_c) \cap (\operatorname{bd} \Pi_s)$, if and only if at least one of these pairs of conditions holds:

$$\{(a),(b)\}, \ \{(a),(c)\}, \ or \ \{(a),(d)\}.$$

Proof. This characterization result is a straightforward consequence of some statements in Proposition 2 and Proposition 4. More precisely, if $\pi \in (\operatorname{bd} \Pi_c) \cap (\operatorname{bd} \Pi_s)$ the discussion is:
1) If $\pi \in \Pi_c \cap \operatorname{bd} \Pi_c$ then conditions (a) and (b) hold, according to Proposition 2(ii) and (v). In this case, trivially, $\pi \in \operatorname{cl}(\Pi_c \cap \operatorname{bd} \Pi_c)$ and Proposition 2(vii) applies.
2) If $\pi \in \Pi_i \cap \operatorname{cl}(\Pi_c \cap \operatorname{bd} \Pi_c)$ then (a) and (d) hold according to Proposition 2(v) and Proposition 4. Now Proposition 2(vii) applies again.
3) Finally, (a) and (c) also yield $\pi \in (\operatorname{bd} \Pi_c) \cap (\operatorname{bd} \Pi_s)$ as a consequence of Proposition 2(v) and (vii).
(It is evident that 1), 2) and 3) cover all the possibilities for the total ill-posedness of π, according to Proposition 2(v) and (vii).) ■

The following corollary is the counterpart of Proposition 5 in the context of ordinary linear programming (T finite).

Corollary 3 *Assuming T finite, we consider the following conditions*:

(a) $0_{n+1} \in \operatorname{bd} H$,
(b) $0_{n+1} \in C$,
(c) $0_n \in \operatorname{bd} Z^+$,
(d) $0_{n+1} \in \bigcap_{\varepsilon>0} \operatorname{co}\{(a_t, b_t) : \langle a_t, x\rangle \geq -\varepsilon\}$, *for every* $x \in A^\circ \setminus \{0_n\}$.

Then π is totally ill-posed, i.e. $\pi \in (\operatorname{bd}\Pi_c) \cap (\operatorname{bd}\Pi_s)$, *if and only if at least one of these pairs of conditions holds*:

$$\{(a),(b)\},\ \{(a),(c)\},\ \textit{or}\ \{(a),(d)\}.$$

Remark 1 For $\pi \in \Pi_i \cap \operatorname{bd}\Pi_c$, conditions (b) and (d) in Corollary 3 are equivalent, according to Corollary 2. In fact the implication (d)⇒(b) is obvious, and (b)⇒(d) follows from the proof of (i)⇒(ii) in Corollary 2.

4 Sufficient conditions for the total ill-posedness

In this section we establish some conditions guarantying that $\pi \in \operatorname{cl}(\Pi_c \cap \operatorname{bd}\Pi_c)$, which provide sufficient conditions for the total ill-posedness according to Proposition 2(vii). We shall need a pair of technical lemmas, where the following notation is used:

$$\widetilde{T}_\varepsilon(x) := \{t \in T : |\langle a_t, x\rangle| \leq \varepsilon \|x\|\}, \tag{12}$$

with $x \in \mathbb{R}^n$ and $\varepsilon > 0$.

Lemma 1 *Let us assume that, for some* $(\lambda^k)_{k=1}^\infty \subset \mathbb{R}_+^{(T)}$ *with* $\sum_{t\in T}\lambda_t^k = 1$, $k = 1, 2, \cdots$, *we have*

$$0_{n+1} = \lim_{k\to\infty}\sum_{t\in T}\lambda_t^k (a_t, b_t).$$

Then, for every x such that $\sup\{\langle a_t, x\rangle : t \in T\} = 0$ *and all* $\varepsilon > 0$, *there exists a subsequence of* $(\lambda^k)_{k=1}^\infty$, *that we denote in the same way, such that:*

(i) $\lim_{k\to\infty}\sum_{t\in\widetilde{T}_\varepsilon(x)}\lambda_t^k\langle a_t, x\rangle = \lim_{k\to\infty}\sum_{t\in T\setminus\widetilde{T}_\varepsilon(x)}\lambda_t^k\langle a_t, x\rangle = 0$,
(ii) $\lim_{k\to\infty}\sum_{t\in T\setminus\widetilde{T}_\varepsilon(x)}\lambda_t^k = 0$ *and so*, $\lim_{k\to\infty}\sum_{t\in\widetilde{T}_\varepsilon(x)}\lambda_t^k = 1$.

Proof. Take fixed x such that $\sup\{\langle a_t, x\rangle : t \in T\} = 0$ and $\varepsilon > 0$.

(i) As $-\varepsilon\|x\| \leq \sum_{t\in\widetilde{T}_\varepsilon(x)}\lambda_t^k\langle a_t, x\rangle \leq 0$ for $k = 1, 2, \cdots$, we assume w.l.o.g. that $\lim_{k\to\infty}\sum_{t\in\widetilde{T}_\varepsilon(x)}\lambda_t^k\langle a_t, x\rangle$ exists. Since $\langle a_t, x\rangle \leq 0$, for all $t \in T$, (i) follows from the equalities

$$0 = \lim_{k\to\infty}\sum_{t\in T}\lambda_t^k\langle a_t, x\rangle = \lim_{k\to\infty}\sum_{t\in\widetilde{T}_\varepsilon(x)}\lambda_t^k\langle a_t, x\rangle + \lim_{k\to\infty}\sum_{t\in T\setminus\widetilde{T}_\varepsilon(x)}\lambda_t^k\langle a_t, x\rangle.$$

(ii) As $0 \leq \sum_{t\in T\setminus\widetilde{T}_\varepsilon(x)}\lambda_t^k \leq 1$, for $k = 1, 2, \cdots$, we assume without loss of generality that $\lim_{k\to\infty}\sum_{t\in T\setminus\widetilde{T}_\varepsilon(x)}\lambda_t^k$ exists. Then (ii) is a consequence of condition (i) and

$$0 = \lim_{k\to\infty}\sum_{t\in T\setminus\widetilde{T}_\varepsilon(x)}\lambda_t^k\langle a_t, x\rangle \leq -\varepsilon\|x\|\lim_{k\to\infty}\sum_{t\in T\setminus\widetilde{T}_\varepsilon(x)}\lambda_t^k \leq 0.$$

■

Lemma 2 *Given x such that $\sup\{\langle a_t, x\rangle : t \in T\} = 0$, $M > 0$, and $\varepsilon > 0$, the following statements are equivalent*:

(i) $0_{n+1} \in \overline{\mathrm{co}}\left\{(a_t, b_t) : t \in \widetilde{T}_\varepsilon(x)\right\}$,

(ii) $0_{n+1} \in \overline{\mathrm{co}}\left\{(a_t, b_t) : t \in \widetilde{T}_\varepsilon(x) \text{ or } \|(a_t, b_t)\| \leq M\right\}$.

Proof. Let us prove the nontrivial implication. Assume that (ii) holds, and let $(\mu^k)_{k=1}^\infty \subset \mathbb{R}_+^{(T)}$ be such that

$$\operatorname{supp}\mu^k \subset \widetilde{T}_\varepsilon(x) \cup \{t \in T : \|(a_t, b_t)\| \leq M\},$$

$\sum_{t\in T} \mu_t^k = 1$, for $k = 1, 2, \cdots$, and

$$0_{n+1} = \lim_{k\to\infty} \sum_{t\in T} \mu_t^k (a_t, b_t).$$

By Lemma 1(ii), we have $\lim_{k\to\infty} \sum_{t\in T\setminus \widetilde{T}_\varepsilon(x)} \mu_t^k = 0$, so that

$$\lim_{k\to\infty} \sum_{t\in T\setminus \widetilde{T}_\varepsilon(x)} \mu_t^k (a_t, b_t) = 0_{n+1}, \tag{13}$$

because, for $t \in T \setminus \widetilde{T}_\varepsilon(x)$, one has $\mu_t^k = 0$ or $\|(a_t, b_t)\| \leq M$.

Since $\lim_{k\to\infty} \sum_{t\in \widetilde{T}_\varepsilon(x)} \mu_t^k = 1$, according to Lemma 1(ii), we can suppose w.l.o.g. that $\sum_{t\in \widetilde{T}_\varepsilon(x)} \mu_t^k > 0$, for $k = 1, 2, \cdots$. Let us define the sequence $(\lambda^k)_{k=1}^\infty \subset \mathbb{R}_+^{(T)}$ as follows

$$\lambda_t^k := \begin{cases} \frac{\mu_t^k}{\sum_{s\in \widetilde{T}_\varepsilon(x)} \mu_s^k}, & \text{if } t \in \widetilde{T}_\varepsilon(x), \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\sum_{t\in \widetilde{T}_\varepsilon(x)} \lambda_t^k = 1$, for $k = 1, 2, \cdots$, and thanks to (13) we obtain

$$0_{n+1} = \lim_{k\to\infty} \sum_{t\in \widetilde{T}_\varepsilon(x)} \lambda_t^k (a_t, b_t);$$

that is, condition (i) holds. ■

Proposition 6 *Let $\pi \in \Pi_i \cap \operatorname{bd}\Pi_c$ and $x \in \mathbb{R}^n \setminus \{0_n\}$ such that $\sup\{\langle a_t, x\rangle : t \in T\} = 0$. Assume, for every $\varepsilon > 0$, the existence of $M_\varepsilon > 0$ such that*

$$0_{n+1} \in \overline{\mathrm{co}}\left\{(a_t, b_t) : t \in \widetilde{T}_\varepsilon(x) \text{ or } \|(a_t, b_t)\| \leq M_\varepsilon\right\}. \tag{14}$$

Then, $\pi \in \operatorname{cl}(\Pi_c \cap \operatorname{bd}\Pi_c)$.

Remark (before the proof). The existence of such a point $x \in \mathbb{R}^n\setminus\{0_n\}$ for which $\sup\{\langle a_t, x\rangle : t \in T\} = 0$ is a consequence of the fact $0_n \in \operatorname{bd} A$ (Proposition 2(iii)).

Proof. Fix $\varepsilon > 0$. Thanks to Lemma 2, for $\varepsilon/4$ (14) is equivalent to

$$0_{n+1} \in \overline{\mathrm{co}}\left\{(a_t, b_t) : t \in \widetilde{T}_{\varepsilon/4}(x)\right\}.$$

Let $(\lambda^k)_{k=1}^\infty \subset \mathbb{R}_+^{(T)}$ be such that $\operatorname{supp}\lambda^k \subset \widetilde{T}_{\varepsilon/4}(x)$, $\sum_{t\in T} \lambda_t^k = 1$, for $k = 1, 2, \cdots$, and

$$0_{n+1} = \lim_{k\to\infty} \sum_{t\in T} \lambda_t^k (a_t, b_t),$$

and take k_ε big enough in order to guarantee that

$$(u_\varepsilon, \mu_\varepsilon) := \sum_{t\in T} \lambda_t^{k_\varepsilon}(a_t, b_t) \in (\varepsilon/4)\mathbb{B}.$$

Since π is not totally inconsistent, we apply Proposition 1 to conclude the existence of $w \in \mathbb{R}^n$ and $\rho > 0$ such that

$$\langle a_t, w\rangle + b_t \geq -\rho, \tag{15}$$

for all $t \in T$. Choose $\gamma > 0$ small enough to guarantee that

$$\frac{\|x\|}{2} \leq \|(x - \gamma w, -\gamma)\| \text{ and } \gamma\rho \leq \frac{\varepsilon\,\|x\|}{2}, \tag{16}$$

and

$$\max\left\{ \frac{|\langle (a_t, b_t), (\gamma w - x, \gamma)\rangle|}{\|(\gamma w - x, \gamma)\|} : t \in \operatorname{supp}\lambda^{k_\varepsilon} \right\} \leq \frac{\varepsilon}{2}. \tag{17}$$

Such γ exists because $\lim_{\gamma\to 0}\|(\gamma w - x, \gamma)\| = \|x\|$ and, for all $t \in \operatorname{supp}\lambda^{k_\varepsilon}$,

$$\lim_{\gamma\to 0} |\langle (a_t, b_t), (\gamma w - x, \gamma)\rangle| \in \left[0, \frac{\varepsilon\,\|x\|}{4}\right].$$

We proceed by proving that $x^\varepsilon := (x - \gamma w)/\gamma$, where γ satisfies (16) and (17), verifies (i) and (ii) in Proposition 4. First observe that, by taking into account that $\sup\{\langle a_t, x\rangle : t \in T\} = 0$, (15), and (16), we have, for all $t \in T$,

$$\langle (a_t, b_t), (x - \gamma w, -\gamma)\rangle \leq \langle (a_t, b_t), (-\gamma w, -\gamma)\rangle \leq \gamma\rho \leq \frac{\varepsilon\,\|x\|}{2} \leq \varepsilon\,\|(x - \gamma w, -\gamma)\|,$$

so that

$$\langle (a_t, b_t), (x^\varepsilon, -1)\rangle \leq \varepsilon\,\|(x^\varepsilon, -1)\| \tag{18}$$

and hence, condition (i) in Proposition 4 holds.

In order to prove condition (ii) in Proposition 4, we define, for $t \in \operatorname{supp}\lambda^{k_\varepsilon}$,

$$(a_t^\varepsilon, b_t^\varepsilon) := (a_t, b_t) - (u_\varepsilon, \mu_\varepsilon) - \frac{\langle (a_t, b_t) - (u_\varepsilon, \mu_\varepsilon), (\gamma w - x, \gamma)\rangle}{\|(\gamma w - x, \gamma)\|^2}(\gamma w - x, \gamma).$$

Let us show first that $\operatorname{supp}\lambda^{k_\varepsilon} \subset T_\varepsilon(x^\varepsilon)$. Given $t \in \operatorname{supp}\lambda^{k_\varepsilon}$, we have, from the definition of $(a_t^\varepsilon, b_t^\varepsilon)$,

$$\|(a_t^\varepsilon, b_t^\varepsilon) - (a_t, b_t)\| \leq 2\,\|(u_\varepsilon, \mu_\varepsilon)\| + \frac{|\langle (a_t, b_t), (\gamma w - x, \gamma)\rangle|}{\|(\gamma w - x, \gamma)\|} \leq 2\frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon,$$

so that, because $\langle (a_t^\varepsilon, b_t^\varepsilon), (x^\varepsilon, -1)\rangle = 0$, we can write

$$\begin{aligned} |\langle (a_t, b_t), (x^\varepsilon, -1)\rangle| &= |\langle (a_t, b_t) - (a_t^\varepsilon, b_t^\varepsilon), (x^\varepsilon, -1)\rangle| \\ &\leq \|(a_t^\varepsilon, b_t^\varepsilon) - (a_t, b_t)\|\,\|(x^\varepsilon, -1)\| \\ &\leq \varepsilon\,\|(x^\varepsilon, -1)\|, \end{aligned}$$

thus $\operatorname{supp}\lambda^{k_\varepsilon} \subset T_\varepsilon(x^\varepsilon)$.

Now, observing that

$$0_{n+1} = \sum_{t\in\operatorname{supp}\lambda^{k_\varepsilon}} \lambda_t^{k_\varepsilon}(a_t^\varepsilon, b_t^\varepsilon),$$

we deduce

$$0_{n+1} \in \operatorname{co}\left\{(a_t, b_t) : t \in \operatorname{supp} \lambda^{k_\varepsilon}\right\} + \varepsilon\mathbb{B} \subset \operatorname{co}\{(a_t, b_t) : t \in T_\varepsilon(x^\varepsilon)\} + \varepsilon\mathbb{B},$$

that is, condition (ii) in Proposition 4 also holds, and so, $\pi \in \operatorname{cl}(\Pi_c \cap \operatorname{bd}\Pi_c)$. ■

As a direct consequence of Proposition 6 we obtain (see, also [7, Theorem 2]):

Corollary 4 *Let $\pi \in \Pi_i \cap \operatorname{bd}\Pi_c$. Suppose that there exists $M > 0$ such that*

$$0_{n+1} \in \overline{\operatorname{co}}\{(a_t, b_t) : t \in T \text{ such that } \|(a_t, b_t)\| \leq M\}.$$

Then, $\pi \in \operatorname{cl}(\Pi_c \cap \operatorname{bd}\Pi_c)$.

Proof. It is enough to apply Proposition 6 with $M_\varepsilon = M$, for every $\varepsilon > 0$. ■

The following proposition slightly relaxes the assumptions in [7, Theorem 3]:

Proposition 7 *Let $\pi \in \Pi_i \cap \operatorname{bd}\Pi_c$, $x \in A^\circ \setminus \{0_n\}$, and assume the existence of $M > 0$ such that the following statements hold:*

(i) $\langle a_t, x\rangle - b_t \leq M$, *for all $t \in T$,*
(ii) $0_{n+1} \in \overline{\operatorname{co}}\{(a_t, b_t) : b_t \leq M\}$.

Then, $\pi \in \operatorname{cl}(\Pi_c \cap \operatorname{bd}\Pi_c)$.

Proof. Fix $\varepsilon > 0$ and let $\gamma \geq 1$ big enough to guarantee that

$$2M \leq \varepsilon\|(\gamma x, -1)\|. \tag{19}$$

This γ exists since $\lim_{\gamma\to+\infty}\|(\gamma x, -1)\| = +\infty$. We are going to prove that the vector $x^\varepsilon := \gamma x$ verifies both conditions (i) and (ii) in Proposition 4.
Since $x \in A^\circ$, and using (19) and the current assumption (i), we can write

$$\langle (a_t, b_t), (x^\varepsilon, -1)\rangle \equiv \langle (a_t, b_t), (\gamma x, -1)\rangle \leq \langle (a_t, b_t), (x, -1)\rangle \leq M \leq \varepsilon\|(x^\varepsilon, -1)\|, \tag{20}$$

for all $t \in T$, and condition (i) in Proposition 4 follows.
In order to verify that x^ε also satisfies (ii) in Proposition 4 we proceed as follows. Set

$$T_M := \{t \in T : b_t \leq M\}, \quad C_M := \operatorname{co}\{(a_t, b_t) : t \in T_M\},$$

and denote by f_M the support function of C_M, so that $f_M(x, 0) \leq 0$. Then our current assumption (ii) entails $0_{n+1} \in \operatorname{cl} C_M = \partial f_M(0_{n+1})$. In this way, $(x, 0)$ is a global minimum of f_M, $f_M(x, 0) = 0$, and $0_{n+1} \in \partial f_M(x, 0)$. Appealing to Proposition 3, we obtain

$$0_{n+1} \in \bigcap_{\delta>0} \operatorname{cl}(\operatorname{co}\{(a_t, b_t) : t \in T_M \text{ s.t. } \langle a_t, x\rangle \geq -\delta\} + (\operatorname{cl} C_M)_\infty \cap \{(x, 0)\}^\perp). \tag{21}$$

Next we shall show that

$$(\operatorname{cl} C_M)_\infty \cap \{(x, 0)\}^\perp = \{(v, 0) \in (\operatorname{cl} C_M)_\infty : \langle v, x\rangle = 0\}. \tag{22}$$

In fact, our current assumption (i) entails $(x, -1) \in \operatorname{dom} f_M \subset [(\operatorname{cl} C_M)_\infty]^\circ$. So, for every $(v, \alpha) \in (\operatorname{cl} C_M)_\infty \cap \{(x, 0)\}^\perp$, one has

$$-\alpha = \langle v, x\rangle - \alpha = \langle (v, \alpha), (x, -1)\rangle \leq 0,$$

and this implies $(\operatorname{cl} C_M)_\infty \cap \{(x,0)\}^\perp \subset \mathbb{R}^n \times \mathbb{R}_+$. On the other hand, the condition $\sup_{t\in T_M} b_t \leq M$ yields $(\operatorname{cl} C_M)_\infty \subset \mathbb{R}^n \times (-\mathbb{R}_+)$, and consequently,

$$(\operatorname{cl} C_M)_\infty \cap \{(x,0)\}^\perp \subset \mathbb{R}^n \times \{0\},$$

which leads us to (22).
Now we are going to establish the inclusion

$$\left\{t \in T_M : \langle a_t, x\rangle \geq -\frac{M}{\gamma}\right\} \subset T_\varepsilon(x^\varepsilon). \tag{23}$$

Let $t \in \left\{t \in T_M : \langle a_t, x\rangle \geq -\frac{M}{\gamma}\right\}$. We have, by the current assumptions (i) and (ii) and the condition $x \in A^\circ \setminus \{0_n\}$,

$$M \geq \langle (a_t, b_t), (x, -1)\rangle \geq \langle (a_t, b_t), (\gamma x, -1)\rangle \geq -M - b_t \geq -2M,$$

thus

$$|\langle (a_t, b_t), (\gamma x, -1)\rangle| \leq 2M,$$

and hence, taking into account (19),

$$\begin{aligned}\left\{t \in T_M : \langle a_t, x\rangle \geq -\frac{M}{\gamma}\right\} &\subset \{t \in T : |\langle (a_t, b_t), (\gamma x, -1)\rangle| \leq 2M\} \\ &\subset \{t \in T : |\langle (a_t, b_t), (\gamma x, -1)\rangle| \leq \varepsilon \|(\gamma x, -1)\|\} \\ &= T_\varepsilon(\gamma x) \equiv T_\varepsilon(x^\varepsilon).\end{aligned}$$

Consequently, using (22) and (23), (21) for $\delta = \frac{M}{\gamma}$ yields

$$\begin{aligned}0_{n+1} \in\ & \operatorname{cl}\left(\operatorname{co}\left\{(a_t, b_t) : t \in T_M \text{ s.t. } \langle a_t, x\rangle \geq -\tfrac{M}{\gamma}\right\} + \{(v,0) \in (\operatorname{cl} C_M)_\infty : \langle v, x\rangle = 0\}\right) \\ &\subset \operatorname{cl}(\operatorname{co}\{(a_t, b_t) : t \in T_\varepsilon(x^\varepsilon)\} + \{(v,0) \in (\operatorname{cl} C_M)_\infty : \langle v, x\rangle = 0\}) \\ &= \operatorname{cl}(\operatorname{co}\{(a_t, b_t) : t \in T_\varepsilon(x^\varepsilon)\} + \{(v,0) \in (\operatorname{cl} C_M)_\infty : \langle (v,0), (\gamma x, -1)\rangle = 0\}) \\ &\subset \operatorname{cl}\left(\operatorname{co}\{(a_t, b_t) : t \in T_\varepsilon(x^\varepsilon)\} + (\operatorname{cl} C)_\infty \cap \{(x^\varepsilon, -1)\}^\perp\right) \\ &\subset \operatorname{co}\{(a_t, b_t) : t \in T_\varepsilon(x^\varepsilon)\} + (\operatorname{cl} C)_\infty \cap \{(x^\varepsilon, -1)\}^\perp + \varepsilon\mathbb{B}.\end{aligned}$$

In this way we have established that condition (ii) in Proposition 4 also holds (with $x^\varepsilon := \gamma x$).

■

The conditions used in Propositions 6 and 7 are not necessary in general as show the following examples.

Example 2 Let us consider the system σ, in $\mathbb{R}^2$,

$$\begin{cases} \langle (-1, s), (x, y)\rangle \leq 0, & s > 0, \\ \langle (0, 0), (x, y)\rangle \leq -1, & \\ \langle (t, 1), (x, y)\rangle \leq 0, & t > 0, \end{cases}$$

and set $\pi := (c, \sigma)$, with $c \in \mathbb{R}^2$. We have $\pi \in \Pi_i \cap \operatorname{cl}(\Pi_c \cap \operatorname{bd} \Pi_c)$, according to Example 1, and we check that $A^\circ = \{0\} \times (-\mathbb{R}_+)$. Setting $z := (0, -y) \in A^\circ$ with $y > 0$, it can be seen that

$$0_3 \notin \overline{\operatorname{co}}\left\{(-1, s, 0),\ s \leq \frac{1}{2}; (0, 0, -1)\right\} = \overline{\operatorname{co}}\left\{(a_t, b_t) : t \in \widetilde{T}_{\frac{1}{2}}(z)\right\}.$$

According to Lemma 2, (14) in Proposition 6 does not hold with $\varepsilon = \frac{1}{2}$, for any $M_\varepsilon > 0$.

Example 3 Let σ be the system, in $\mathbb{R}$, given by

$$\begin{cases} \langle s, x\rangle \leq \frac{1}{s}, & s > 0, \\ \langle 0, x\rangle \leq -1, & \end{cases}$$

and set $\pi := (c, \sigma)$, with $c \in \mathbb{R}$. It can be seen that $(0, -1) \in \operatorname{cl} C \subset \operatorname{cl} N$ and $0_3 \in \operatorname{bd} H$, so that $\pi \in \Pi_i \cap \operatorname{bd} \Pi_c$. Given $z \in A^\circ \setminus \{0\} = (-\mathbb{R}_+) \setminus \{0\}$ we have, for every $\varepsilon > 0$,

$$0_2 = \lim_{\substack{k\to+\infty \\ k \geq \frac{1}{\varepsilon}}} \left[\frac{1}{k}\left(\frac{1}{k}, k\right) + \frac{k-1}{k}(0, -1)\right] \in \overline{\operatorname{co}}\left\{(a_t, b_t) : t \in \widetilde{T}_\varepsilon(z)\right\},$$

thus $\pi \in \operatorname{cl}(\Pi_c \cap \operatorname{bd} \Pi_c)$, according to Proposition 6. On the other hand, because $sz - \frac{1}{s} \leq 0$, for every $s > 0$, condition (i) in Proposition 7 is satisfied for any $M \geq 1$, whereas

$$0_2 \notin \overline{\operatorname{co}}\left\{(0, -1), \left(s, \frac{1}{s}\right) : \frac{1}{s} \leq M\right\} = \overline{\operatorname{co}}\left\{(a_t, b_t) : b_t \leq M\right\},$$

for every $M \geq 1$. Hence condition (ii) in Proposition 7 never holds.

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