SIAM J. OPTIM.
Vol. 20, No. 6, pp. 2788–2806

VARIATIONAL ANALYSIS IN SEMI-INFINITE AND INFINITE PROGRAMMING, II: NECESSARY OPTIMALITY CONDITIONS*

M. J. CÁNOVAS†, M. A. LÓPEZ‡, B. S. MORDUKHOVICH§, AND J. PARRA†

Abstract. This paper concerns applications of advanced techniques of variational analysis and generalized differentiation to problems of semi-infinite and infinite programming with feasible solution sets defined by parameterized systems of infinitely many linear inequalities of the type intensively studied in the preceding development [Cánovas et al., *SIAM J. Optim.*, 20 (2009), pp. 1504–1526] from the viewpoint of robust Lipschitzian stability. The main results establish necessary optimality conditions for broad classes of semi-infinite and infinite programs, where objectives are generally described by nonsmooth and nonconvex functions on Banach spaces and where infinite constraint inequality systems are indexed by arbitrary sets. The results obtained are new in both smooth and nonsmooth settings of semi-infinite and infinite programming. We illustrate our model and results by considering a practically meaningful model of water resource optimization via systems of reservoirs.

Key words. semi-infinite and infinite programming, parametric optimization, variational analysis, necessary optimality conditions, linear infinite inequality systems, generalized differentiation, coderivatives, lower and upper subdifferentials

AMS subject classifications. 90C34, 90C05, 49J52, 49J53, 65F22

DOI. 10.1137/09076595X

1. Introduction. The paper mainly deals with optimization problems formalized as

$$\text{minimize } \varphi(p,x) \text{ subject to } x\in\mathcal{F}(p), \tag{1.1}$$

where $\varphi\colon P\times X\to\overline{\mathbb{R}}:=(-\infty,\infty]$ is an extended-real-valued *cost* function defined on the product of Banach spaces and where $\mathcal{F}\colon P\rightrightarrows X$ is a set-valued mapping of *feasible solutions* given by

$$\mathcal{F}(p):=\big\{x\in X\,\big|\,\langle a_t^*,x\rangle\le b_t+\langle c_t^*,p\rangle,\ \ t\in T\big\} \tag{1.2}$$

with an *arbitrary* (possibly infinite) *index set* T and with fixed elements $a_t^*\in X^*$, $c_t^*\in P^*$, and $b_t\in\mathbb{R}$ for all $t\in T$. Optimization problems of this type relate to *semi-infinite programming*, provided that the space X is finite-dimensional, and to *infinite programming* if X is infinite-dimensional; see, e.g., [1, 10].

Note that a usual framework of sensitivity (or stability) analysis for semi-infinite/infinite programs corresponds to formalism (1.2) specified as

$$\mathcal{F}(p):=\big\{x\in X\,\big|\,\langle a_t^*,x\rangle\le b_t+p_t,\ \ t\in T\big\},\quad p=(p_t)_{t\in T}, \tag{1.3}$$

*Received by the editors July 22, 2009; accepted for publication (in revised form) June 8, 2010; published electronically August 19, 2010. This research was partially supported by grants MTM2008-06695-C03 (01-02) from MICINN (Spain), ACOMP/2009/047&133, and ACOMP/2010/269 from Generatitat Valenciana (Spain).
http://www.siam.org/journals/siopt/20-6/76595.html

†Operations Research Center, Miguel Hernández University of Elche, 03202 Elche (Alicante), Spain (canovas@umh.es, parra@umh.es).

‡Department of Statistics and Operations Research, University of Alicante, 03080 Alicante, Spain (marco.antonio@ua.es).

§Department of Mathematics, Wayne State University, Detroit, MI 48202 (boris@math.wayne.edu). The research of this author was partially supported by the National Science Foundation (USA) under grant DMS-0603846 and by the Australian Research Council under grant DP-12092508.

where $p \in P$ is treated as a *perturbation parameter*. This is a particular case of (1.2) when $c_t^* = \delta_t$ is the classical *Dirac measure* at $t \in T$. A natural choice of P in the latter case is the space $l_\infty(T)$ of all bounded functions on T with the supremum norm

$$\|p\|_\infty := \sup_{t \in T} |p_t| = \sup \left\{ |p(t)| \;\middle|\; t \in T \right\}.$$

When the index set T is compact (which is not assumed in what follows) and the perturbations $p(\cdot)$ are restricted to be continuous on T, the supremum above is realized, and thus $l_\infty(T)$ reduces to the classical space $\mathcal{C}(T)$ of continuous functions over a compact set.

In the preceding paper [5] we developed and applied advanced tools of variational analysis and generalized differentiation to fully characterize *robust Lipschitzian stability* of the feasible solution map in (1.3) expressed entirely via its initial data. The main goal of the present paper is to employ these and related techniques and some results from [5] to derive verifiable *necessary conditions* for *optimal solutions* to semi-infinite and infinite programs (1.1) with general nonsmooth and nonconvex cost functions and with feasible solution maps given by (1.2) in the case of arbitrary Banach spaces X and P and arbitrary index sets T. If (1.2) reduces to (1.3) in (1.1) with $P = l_\infty(T)$, the results obtained below recover those established in our preliminary research report [4].

Note that optimization in (1.1) is conducted with respect to both variables (p, x), which are interconnected through the infinite inequality system (1.2). This means in fact that we have *two groups of decision variables* represented by x and p. One player specifies p and the other solves (1.1) in x subject to (1.2) with the specified p as a parameter. The first one, having the same objective, varies his/her parameter p to get the best outcome via the so-called *optimistic approach*. We could treat this as a *two-level design*: optimizing the basic parameter p at the upper level, while at the lower level the cost function is optimized with respect to x for the given p. The reader is referred to, e.g., [15] and the bibliography therein for various tuning and tolerancing problems of such types arising in engineering design.

Other classes of optimization models that could be described in the two-variable form (1.1) with semi-infinite/infinite constraints of type (1.2) appear in optimal control and approximation theory; see, e.g., [1, 10, 11] for more details. There are various interesting problems of this type arising in electricity markets, multiobjective optimization, etc. In this paper we confine ourselves to presenting a valuable example related to a practical model of *of water resource optimization* via systems of reservoirs by formulating it in the two-variable form of infinite programming and illustrating the power of our new necessary optimality conditions and the assumptions made; see section 5 for mode details.

It is worth mentioning that the basic problem (1.1) under consideration is written in the format of the so-called *abstract mathematical programs with equilibrium constraints* (MPECs) [17, 22, 24], but the main emphasis there is the *generalized equation/variational condition* (in Robinson's sense [25]) structure of the set-valued mapping $\mathcal{F}$ in (1.1) given by

$$\mathcal{F}(p) := \left\{ x \in X \;\middle|\; 0 \in f(p, x) + Q(x) \right\}$$

with a single-valued mapping $f\colon P \times X \to Y$ and a set-valued mapping $Q\colon X \rightrightarrows Y$, which particularly encompasses solution maps to the classical variational inequalities and complementarity problems when $Q(x) = N(x; \Omega)$ is the normal cone mapping to a convex set $\Omega \subset X$. The underlying *infinite inequality* structure (1.2) of the mapping $\mathcal{F}$ in our framework is completely different from the MPEC case. The main goal and achievement of this paper is involving infinite inequality systems (1.2) and (1.3) into the general optimization framework (1.1) and deriving verifiable necessary optimality conditions for such problems entirely in terms of their initial data.

Note that the results obtained below are new not only in the general case of nonsmooth and nonconvex cost functions $\varphi(p,x)$ and arbitrary Banach spaces X and P as well as arbitrary index sets T in (1.1)–(1.3), but also for conventional classes of one-variable semi-infinite programs with compact or discretized index sets. We refer the reader to [1, 2, 3, 6, 8, 7, 10, 11, 13, 14, 16, 29, 30] and the bibliographies therein for a range of approaches and results concerning optimal solutions to various problems of semi-infinite and infinite programming; see more discussions in sections 3–5.

The rest of the paper is organized as follows. Section 2 is devoted to reviewing the basic tools of generalized differentiation widely used below for deriving the main results of the paper given in sections 3 and 4.

In section 3 we present necessary optimality conditions in the so-called *lower subdifferential* type, which are expressed via appropriate extensions of the subdifferential of convex analysis to the general class of lower semicontinuous (l.s.c.) cost functions. Besides well-developed calculus rules for the corresponding subdifferentials, a major role in deriving these conditions is played by the *coderivative* of the feasible solution maps under consideration, which is constructively computed in [5] entirely in terms of the initial data. The results obtained are generally given in a verifiable *qualified asymptotic* form introduced in this paper, while they are presented in an *extended Karush–Kuhn–Tucker* (KKT) form under some *closedness* (Farkas–Minkowski-type) *constraint qualification.*

Section 4 contains necessary optimality conditions for (1.1) established in a relatively new for minimization *upper subdifferential/superdifferential* form that has never been used before in semi-infinite/infinite programming. The upper subdifferential optimality conditions obtained here are generally independent of the lower subdifferential ones derived in section 3. In fact, both agree for smooth (i.e., continuously differentiable) objectives, while the upper conditions may be strictly better even in the case of cost functions that are merely Fréchet differentiable at optimal solutions. The main difference is as follows: the upper subdifferential conditions provide trivial information in the case of convex cost functions and the like when the lower subdifferential ones play a major role, but, on the other hand, the upper conditions give significantly stronger results for broad classes of "upper regular" functions, e.g., for minimization problems involving concave and semiconcave objectives; see more details and discussions in section 4. Similarly to the lower subdifferential conditions of section 3, the upper ones take advantages of precisely computing the coderivative of the feasible solution maps in (1.2) and (1.3) and thus express necessary optimality conditions entirely in terms of the initial data of the basic model (1.1).

Section 5 is devoted to a practically meaningful model concerning *control and optimization of water resources* via systems of reservoirs, which can be formulated as an *infinite program* of the type studied in this paper. We pursue here a twofold goal. On one hand, this is a valuable example of a two-variable infinite programming model of a certain practical interest. On the other hand, we apply to this model the necessary optimality conditions derived above while fully clarifying the situation when the closedness/Farkas–Minkowski property holds—and so our necessary optimality conditions are satisfied in the extended KKT form—and when it does not, and thus we need to employ the general asymptotic form of the optimality conditions obtained.

The notation of this paper is basically standard and conventional in the areas of variational analysis and semi-infinite/infinite programming; see, e.g., [10, 21]. Unless otherwise stated, all the spaces under consideration are *Banach* with the corresponding norm $\|\cdot\|$. Recall that w^* indicates the weak* topology of a dual space, and we use the symbol w^*-lim for the weak* *topological* limit, which generally means the weak* convergence of *nets* denoted usually by $\{x^*_\nu\}_{\nu\in\mathcal{N}}$. In the case of sequences we use the standard notation $\mathbb{N} := \{1, 2, \dots\}$ for the collections of all natural numbers.

Given a subset $\Omega \subset Z$ of a Banach space, the symbols $\operatorname{int}\Omega$, $\operatorname{cl}\Omega$, $\operatorname{co}\Omega$, and $\operatorname{cone}\Omega$ stand, respectively, for the interior, closure, convex hull, and conic convex hull of Ω; the notation $\operatorname{cl}^*\Theta$ signifies the weak* closure of a subset $\Theta \subset Z^*$ in the dual space. Given a set-valued mapping $F\colon Z \rightrightarrows Y$, we denote its *domain* and *graph* by, respectively,

$$\operatorname{dom} F = \big\{z \in Z\big|\ F(z) \neq \emptyset\big\} \quad \text{and} \quad \operatorname{gph} F := \big\{(z, y) \in Z \times Y\big|\ y \in F(z)\big\}.$$

Considering finally an arbitrary index set T, let $\mathbb{R}^T$ be the *product space* of $\lambda = (\lambda_t|\ t \in T)$ with $\lambda_t \in \mathbb{R}$ for all $t \in T$, let $\mathbb{R}^{(T)}$ be the collection of $\lambda \in \mathbb{R}^T$ such that $\lambda_t \neq 0$ for *finitely many* $t \in T$, and let $\mathbb{R}^{(T)}_+$ be the *positive cone* in $\mathbb{R}^{(T)}$ defined by

$$\mathbb{R}^{(T)}_+ := \big\{\lambda \in \mathbb{R}^{(T)}\big|\ \lambda_t \geq 0 \ \text{ for all } \ t \in T\big\}. \tag{1.4}$$

2. Generalized differentiation.

In our model (1.2) the cost function $\varphi(p, x)$ is generally nonsmooth and nonconvex, which surely require the usage of appropriate tools of *generalized differentiation*. We also appeal to the *coderivative* construction for set-valued mappings and its full computation [5] in the case of the feasible solution maps $\mathcal{F}$ under consideration.

To proceed in this way, we briefly overview here some tools of generalized differentiation needed in what follows; the reader can find more details and discussions in [12, 21, 27, 28] and the references therein. Consider first (*lower*) *subdifferentials*, or collections of subgradients, for extended-real-valued functions that reduce to the classical subdifferential of convex analysis in the case of convex functions and are conventionally employed in *minimization* problems with "*less or equal* ($\leq$)" inequality constraints. Note that the adjective "lower" is usually taken for granted and is dropped in subdifferential studies and applications; see, however, the discussions and results in section 4.

Let Z be an arbitrary Banach space, let $\varphi\colon Z \to \overline{\mathbb{R}}$ be an extended-real-valued function finite at the reference point $\bar z$, and let $\varepsilon \geq 0$. The *ε-subdifferential* of φ at $\bar z$ is defined by

$$\widehat{\partial}_\varepsilon\varphi(\bar z) := \left\{z^* \in Z^*\Big|\ \liminf_{z\to\bar z} \frac{\varphi(z) - \varphi(\bar z) - \langle z^*, z - \bar z\rangle}{\|z - \bar z\|} \geq -\varepsilon\right\}. \tag{2.1}$$

For $\varepsilon = 0$ in (2.1), the construction $\widehat{\partial}\varphi(\bar z) := \widehat{\partial}_0\varphi(\bar z)$ is known as the *regular* (or *viscosity*, or *Fréchet*) *subdifferential* of φ at this point. It reduces to the classical subdifferential of convex analysis for convex functions φ while it may be empty in the absence of convexity (as, e.g., for $\varphi(x) = -|x|$ at $\bar x = 0$) and does not generally satisfy required calculus rules. Employing the sequential limiting procedure

$$\begin{aligned}\partial\varphi(\bar z) := \Big\{z^* \in Z^*\Big|&\exists \text{ sequences } (\varepsilon_k, z_k, z^*_k)_{k\in\mathbb{N}} \subset \mathbb{R}_+ \times Z \times Z^* \ \text{ with } \ z^*_k \in \widehat{\partial}_{\varepsilon_k}\varphi(z_k)\\ &\text{and } \ \varepsilon_k \downarrow 0,\ z_k \to \bar z,\ \varphi(z_k) \to \varphi(\bar z),\ z^*_k \overset{w^*}{\to} z^* \ \text{ as } \ k \to \infty\Big\},\end{aligned} \tag{2.2}$$

we arrive at the robust subdifferential construction known as the *limiting* (or *basic*, or *Mordukhovich*) *subdifferential* of φ at $\bar{z}$. Note that the limiting operation in (2.2) can be symbolically written in the *sequential* form of the Painlevé–Kuratowski outer/upper limit

$$\partial\varphi(\bar{z}) = \mathop{\mathrm{Lim\,sup}}_{\substack{z \xrightarrow{\varphi} \bar{z} \\ \varepsilon \downarrow 0}} \widehat{\partial}_\varepsilon \varphi(z),$$

where the symbol $z \xrightarrow{\varphi} \bar{z}$ signifies that $z \to \bar{z}$ with $\varphi(z) \to \varphi(\bar{z})$. If the function φ is l.s.c. around $\bar{z}$ and the space Z is *Asplund* (i.e., each of its separable subspaces has a separable dual; see [9, 21] for more details), then we can equivalently put $\varepsilon_k \equiv 0$ in (2.2) and get the representation

$$\partial\varphi(\bar{z}) = \mathop{\mathrm{Lim\,sup}}_{z \xrightarrow{\varphi} \bar{z}} \widehat{\partial} \varphi(z). \tag{2.3}$$

It turns out furthermore that, in spite of (actually due to) nonconvexity of the subgradient sets $\partial\varphi(\bar{x})$, the limiting subdifferential (2.3) admits *full calculus* in the Asplund space setting that is mainly based on *variational/extremal principles* of variational analysis; see [21] for the comprehensive study and references. On the other hand, the enlarged subdifferential construction (2.2), having many useful properties and applications in arbitrary Banach spaces (see, in particular, [21, Chapters 1 and 4] and [22, Chapters 5 and 6]), may fail to satisfy important calculus rules in general nonsmooth settings of non-Asplund spaces. This is the case of the space $P = \ell_\infty(T)$ naturally appeared in modeling infinite inequality constraints of type (1.3); see [5, Proposition 2.5].

In what follows we proceed with applications of the aforementioned sequential subdifferential constructions to deriving necessary optimality conditions for nonsmooth problems (1.1) with general infinite inequality constraints (1.2) in the case of Asplund spaces X and P. To cover simultaneously infinite programs with arbitrary Banach spaces of decision variables, we employ the approximate G-subdifferential by Ioffe [12], labeled in [12] as "the nucleus of the G-subdifferential," which provides another (more topologically complicated) infinite-dimensional extension of the original construction by Mordukhovich [18] while it turns out to be the most appropriate to work in the general Banach space settings including the underlying case of $P = l_\infty(T)$ as in (1.3).

The approximate subdifferential constructions on arbitrary Banach spaces are defined by the following multistep procedure. Given a function $\varphi\colon Z \to \overline{\mathbb{R}}$ finite at $\bar{z}$, consider first its *lower Dini* (or *Dini–Hadamard*) *directional derivative*

$$d^-\varphi(\bar{z}; v) := \liminf_{\substack{u \to v \\ t \downarrow 0}} \frac{\varphi(\bar{z} + tu) - \varphi(\bar{z})}{t}, \quad v \in Z,$$

and then define the *Dini ε-subdifferential* of φ at $\bar{z}$ by

$$\partial_\varepsilon^- \varphi(\bar{z}) := \big\{ z^* \in Z^* \big| \, \langle z^*, v \rangle \le d^-\varphi(\bar{z}; v) + \varepsilon\|v\| \ \text{ for all } \ v \in Z \big\}, \quad \varepsilon \ge 0.$$

As usual, put $\partial_\varepsilon^- \varphi(\bar{z}) := \emptyset$ if $\varphi(\bar{z}) = \infty$. The *A-subdifferential* of φ at $\bar{z}$ is defined via topological limits involving finite-dimensional reductions of ε-subgradients by

$$\partial_A \varphi(\bar{z}) := \bigcap_{\substack{L \in \mathcal{L} \\ \varepsilon > 0}} \overline{\mathop{\mathrm{Lim\,sup}}_{z \xrightarrow{\varphi} \bar{z}}} \, \partial_\varepsilon^- \big(\varphi + \delta(\cdot; L) \big)(z),$$

where $\mathcal{L}$ is the collection of all finite-dimensional subspaces of Z, where $\delta(\cdot;L)$ is the indicator function of L, and where $\overline{\mathrm{Lim\,sup}}$ stands for the *topological* Painlevé–Kuratowski upper/outer limit of a mapping $F\colon Z \rightrightarrows Z^*$ as $z \to \bar z$ defined by

$$\overline{\mathop{\mathrm{Lim\,sup}}_{z\to\bar z}} F(z) := \Big\{ z^* \in Z^* \Big| \exists \text{ a net } (z_\nu, z^*_\nu)_{\nu\in\mathcal{N}} \subset Z \times Z^* \text{ with } z^*_\nu \in F(z_\nu) \text{ and } (z_\nu, z^*_\nu) \to (\bar z, z^*) \text{ in the } \|\cdot\| \times w^* \text{ topology of } Z \times Z^* \Big\}.$$

Then the *approximate G-subdifferential* of φ at $\bar z$ is defined by

$$\partial_G\varphi(\bar z) := \left\{ z^* \in X^* \,\middle|\, (z^*, -1) \in \bigcup_{\lambda>0} \lambda\partial_A \mathrm{dist}\,((\bar z, \varphi(\bar z)); \mathrm{epi}\,\varphi) \right\}, \tag{2.4}$$

where $\mathrm{epi}\,\varphi := \{(z,\mu) \in Z \times \mathbb{R} |\ \mu \ge \varphi(z)\}$ and where $\mathrm{dist}(\cdot;\Omega)$ stands for the *distance function* associated with the set in question.

We have the following relationship between the constructions (2.2) and (2.4) for every l.s.c. function on a Banach space:

$$\partial\varphi(\bar z) \subset \partial_G\varphi(\bar z), \tag{2.5}$$

where the equality holds when φ is locally Lipschitzian around $\bar z$ and Z is Asplund and weakly compactly generated; see [23, Theorem 9.2] and [21, Theorem 3.59]. Observe that the inclusion in (2.5) may be proper for Lipschitz continuous functions on (nonseparable) Asplund spaces; see, e.g., [21, Example 3.61]. Both constructions (2.2) and (2.4) are always smaller than the Clarke subdifferential; they may be substantially smaller even for simple functions on $\mathbb{R}$. We refer the reader to [21, subsection 3.2.3] and [23, sections 8 and 9] for more results and discussions in this direction. Note that both constructions (2.2) and (2.4) reduce, in any Banach space, to the classical strict derivative in the case of smooth functions and to the classical subdifferential of convex analysis when φ is convex.

We also recall the *singular* counterparts of (2.2) and (2.4) defined, respectively, by

$$\partial^\infty\varphi(\bar z) := \mathop{\mathrm{Lim\,sup}}_{\substack{z \xrightarrow{\varphi} \bar z \\ \varepsilon,\lambda\downarrow 0}} \lambda\widehat{\partial}_\varepsilon\varphi(z), \tag{2.6}$$

$$\partial^\infty_G\varphi(\bar z) := \left\{ z^* \in X^* \,\middle|\, (z^*, 0) \in \bigcup_{\lambda>0} \lambda\partial_A \mathrm{dist}\,((\bar z, \varphi(\bar z)); \mathrm{epi}\,\varphi) \right\}. \tag{2.7}$$

As in (2.3), the singular subdifferential (2.6) can equivalently represented with $\varepsilon = 0$ on the right-hand side if Z is Asplund and if φ is l.s.c. around $\bar z$. Similarly to (2.5), we always have the inclusion $\partial^\infty\varphi(\bar z) \subset \partial^\infty_G\varphi(\bar z)$, where furthermore $\partial^\infty_G\varphi(\bar z) = \{0\}$ if φ is locally Lipschitzian around $\bar z$ on an arbitrary Banach space Z.

To deal with the set-valued term $\mathcal{F}$ in deriving necessary optimality conditions for infinite and semi-infinite programs (1.1), we use a generalized differential constructions for set-valued mappings known as coderivatives. Given a set-valued mapping $F\colon Z \rightrightarrows Y$ between Banach spaces and following the scheme of [19], define the *coderivative* of F at $(\bar z, \bar y) \in \mathrm{gph}\,F$ generated by the normal cone N to its graph as a positively homogeneous mapping $D^*F(\bar z,\bar y)\colon Y^* \rightrightarrows Z^*$ with the values

$$D^*F(\bar z,\bar y)(y^*) := \big\{ z^* \in Z^* \big|\ (z^*, -y^*) \in N\big((\bar z,\bar y); \mathrm{gph}\,F\big)\big\}, \tag{2.8}$$

where $N(\cdot;\Omega) := \partial\delta(\cdot;\Omega)$ is the normal cone corresponding to some subdifferential ∂ of extended-real-valued functions. In this way we get coderivatives corresponding to the subdifferentials (2.2) and (2.4). Since both these subdifferentials reduce to the subdifferential of convex analysis for convex functions and since they are used in sections 3 and 4 for the convex-graph mappings $\mathcal{F}\colon P \rightrightarrows X$ given by (1.2) and (1.3), the coderivatives of these mappings are the same for (2.2) and (2.4).

3. Lower subdifferential optimality conditions. This section concerns necessary optimality conditions of the lower subdifferential type in our model (1.1) constrained by (1.2) and (1.3). For these purposes we use the subdifferential (2.4) in the general Banach space setting and the smaller subdifferential (2.3) in the Asplund space framework, combining them with the precise computation of the coderivative (2.8) of the corresponding feasible solution maps $\mathcal{F}$. The general results obtained are given in the so-called *asymptotic form* involving the weak* closure of a set constructively built upon the initial data of the constraint systems (1.2) and (1.3). Furthermore, they are presented in the more conventional (while new) *KKT form* under additional constraint qualifications.

Following Definition 2.2 in [5] given for the case of system (1.3), we say that the *strong Slater condition* (SSC) holds for system (1.2) if there is a pair $(\widehat{p},\widehat{x}) \in P \times X$ such that

$$\sup_{t\in T}\big[\langle a_t^*,\widehat{x}\rangle - \langle c_t^*,\widehat{p}\rangle - b_t\big] < 0, \tag{3.1}$$

where $(\widehat{p},\widehat{x})$ is called the *strong Slater point* for (1.2). The reader can easily check the fulfillment of the equivalent descriptions of the SSC in (3.1) similar to [5, Lemma 2.3].

We say also that system (1.2) has the *Farkas–Minkowski property* if the convex cone

$$\operatorname{cone}\big\{(-c_t^*,a_t^*,b_t) \in P^* \times X^* \times \mathbb{R}\big|\ t \in T\big\}, \tag{3.2}$$

often called the *second moment cone*, is weak* closed in $P^* \times X^* \times \mathbb{R}$. The reader is referred to [2, 6, 7, 10, 16, 30] for sufficient conditions ensuring the validity of this property, its relationships with other constraint qualifications, and various applications to problems of semi-infinite and infinite programming.

Now we are ready to formulate and prove the main results of this section, which are presented in two similar while independent theorems. The first theorem gives necessary optimality conditions for problem (1.1) with infinite constraints (1.2) in arbitrary Banach spaces X and P employing the G-subdifferential (2.4) and its singular counterpart (2.7). The second theorem holds for Asplund spaces X and P while using the smaller subdifferential constructions (2.3), (2.6) and thus providing more selective necessary optimality conditions in the latter fairly general framework.

THEOREM 3.1 (lower subdifferential optimality conditions for nonsmooth infinite programs in arbitrary Banach spaces). *Let $(\bar{p},\bar{x}) \in \operatorname{gph}\mathcal{F}$ be a local minimizer for problem* (1.1) *with the general linear constraint system $\mathcal{F}$ given by infinite inequalities* (1.2). *Assume that both spaces X and P are Banach and that the cost function $\varphi\colon P \times X \to \overline{\mathbb{R}}$ is l.s.c. around $(\bar{p},\bar{x})$ with $\varphi(\bar{p},\bar{x}) < \infty$. Suppose also that*

(a) *either φ is locally Lipschitzian around $(\bar{p},\bar{x})$;*

(b) *or* $\operatorname{int}(\operatorname{gph}\mathcal{F}) \neq \emptyset$ (*which holds, in particular, when $\mathcal{F}$ satisfies the SSC in* (3.1) *and the set $\{(a_t^*,c_t^*)|\ t \in T\}$ is bounded in $X^* \times P^*$*), *and the system*

$$\begin{gathered}(p^*,x^*) \in \partial_G^\infty\varphi(\bar{p},\bar{x}),\\ -\big(p^*,x^*,\langle(p^*,x^*),(\bar{p},\bar{x})\rangle\big) \in \operatorname{cl}{}^*\operatorname{cone}\big\{(-c_t^*,a_t^*,b_t)\big|\ t \in T\big\}\end{gathered} \tag{3.3}$$

admits only the trivial solution $(p^*,x^*) = (0,0)$.

Then there exists a G-subgradient pair $(p^*,x^*)\in\partial_G\varphi(\bar p,\bar x)$ *such that*

$$-\big(p^*,x^*,\langle p^*,\bar p\rangle+\langle x^*,\bar x\rangle\big)\in \mathrm{cl}^*\mathrm{cone}\big\{(-c_t^*,a_t^*,b_t)\big|\ t\in T\big\}. \tag{3.4}$$

If furthermore the constraint system (1.2) *satisfies the Farkas–Minkowski property* (3.2)*, then the asymptotic condition* (3.4) *can be equivalently written in KKT form: there are* $(p^*,x^*)\in\partial_G\varphi(\bar p,\bar x)$ *and* $\lambda=(\lambda_t)_{t\in T}\in\mathbb{R}_+^{(T)}$ *for which we have*

$$(p^*,x^*)+\sum_{t\in T(\bar p,\bar x)}\lambda_t(-c_t^*,a_t^*)=0, \tag{3.5}$$

where $T(\bar p,\bar x):=\{t\in T|\ \langle a_t^*,\bar x\rangle-\langle c_t^*,\bar p\rangle=b_t\}$ *and where* $\mathbb{R}_+^{(T)}$ *is defined in* (1.4)*.*

THEOREM 3.2 (lower subdifferential optimality conditions for nonsmooth infinite programs in Asplund spaces). *In the framework of Theorem* 3.1*, suppose that the spaces* X *and* P *are Asplund and that assumption* (b) *is replaced by the weaker one on the triviality of solutions to the system*

$$\begin{gathered}(p^*,x^*)\in\partial^\infty\varphi(\bar p,\bar x),\\ -\big(p^*,x^*,\langle(p^*,x^*),(\bar p,\bar x)\rangle\big)\in\mathrm{cl}^*\mathrm{cone}\big\{(-c_t^*,a_t^*,b_t)\big|\ t\in T\big\}\end{gathered} \tag{3.6}$$

with the singular subdifferential of φ *defined in* (2.6)*. Then we have the stronger necessary optimality conditions for the given solution* $(\bar p,\bar x)$ *with the replacement* $(p^*,x^*)\in\partial_G\varphi(\bar p,\bar x)$ *in* (3.4) *and* (3.5) *by* $(p^*,x^*)\in\partial\varphi(\bar p,\bar x)$ *from the limiting subdifferential* (2.3)*.*

We prove Theorems 3.1 and 3.2 simultaneously by using the corresponding calculus rules for the subdifferentials (2.4) and (2.3) in Banach and Asplund spaces, respectively.

Proofs of Theorems 3.1 *and* 3.2*.* The original infinite programming problem (1.1) can be obviously rewritten as a mathematical program with geometric constraints:

$$\text{minimize }\ \varphi(p,x)\ \text{ subject to }\ (p,x)\in\mathrm{gph}\,\mathcal{F}, \tag{3.7}$$

which is equivalently described by unconstrained minimization with "infinite penalties"

$$\text{minimize }\ \varphi(p,x)+\delta\big((p,x);\mathrm{gph}\,\mathcal{F}\big)$$

via the indicator function of the graph of the feasible map $\mathcal{F}$ given in (1.2). Considering the general Banach space setting of Theorem 3.1 and applying the generalized Fermat rule (see, e.g., [21, Proposition 1.14]) to the latter problem at its local minimizer $(\bar p,\bar x)$, we have

$$(0,0)\in\partial_G\big[\varphi+\delta(\cdot;\mathrm{gph}\,\mathcal{F})\big](\bar p,\bar x). \tag{3.8}$$

Employing further the G-subdifferential sum rule to (3.8), formulated in [12, Theorem 7.4] for the "nuclei," we obtained from (3.8) that

$$(0,0)\in\partial_G\varphi(\bar p,\bar x)+N\big((\bar p,\bar x);\mathrm{gph}\,\mathcal{F}\big), \tag{3.9}$$

provided that either φ is locally Lipschitzian around $(\bar p,\bar x)$ or the interior of $\mathrm{gph}\,\mathcal{F}$ is nonempty and the qualification condition

$$\partial_G^\infty\varphi(\bar p,\bar x)\cap\big[-N\big((\bar p,\bar x);\mathrm{gph}\,\mathcal{F}\big)\big]=\{(0,0)\} \tag{3.10}$$

is satisfied. It is easy to check that the strong Slater condition (3.1) and the boundedness of $\{(a_t^*, c_t^*) |\, t \in T\}$ surely imply that the interior of $\operatorname{gph} \mathcal{F}$ is nonempty; cf. [5, Remark 2.4].

Observe that by the coderivative definition (2.8) we get

$$(p^*, x^*) \in -N\big((\bar{p}, \bar{x}); \operatorname{gph} \mathcal{F}\big) \text{ if and only if } -p^* \in D^*\mathcal{F}(\bar{p}, \bar{x})(x^*).$$

Following the proofs of [5, Proposition 3.1] and [5, Theorem 3.2] in the case of the feasible map $\mathcal{F}$ from (1.2), we see that the previous conditions are equivalent to

$$\big(-p^*, -x^*, -(\langle p^*, \bar{p}\rangle + \langle x^*, \bar{x}\rangle)\big) \in \operatorname{cl}^*\operatorname{cone}\big\{(-c_t^*, a_t^*, b_t)\big|\, t \in T\big\}. \tag{3.11}$$

Note to this end that the result of [5, Theorem 3.2] characterizes the coderivative condition $p^* \in D^*\mathcal{F}(0, \bar{x})(x^*)$ for $\mathcal{F}$ in (1.3). However, we can easily adapt the proof of the latter result to the current setting by replacing there p^* with $-p^*$, δ_t with c_t^*, $-\langle x^*, \bar{x}\rangle$ with $:-(\langle p^*, \bar{p}\rangle + \langle x^*, \bar{x}\rangle)$, and $\langle a_t^*, \bar{x}\rangle$ with $-\langle c_t^*, \bar{p}\rangle + \langle a_t^*, \bar{x}\rangle$, since now we do not assume that $\bar{p} = 0$. Employing then the above characterization of $-N((\bar{p}, \bar{x}); \operatorname{gph} \mathcal{F})$ in (3.9) and (3.10), we, respectively, arrive at the necessary optimality condition (3.4) under the qualification condition (3.3). If furthermore the Farkas–Minkowski property (3.2) is satisfied, then the operation cl^* in (3.4) can be omitted, and the latter qualification condition easily reduces to (3.5). This completes the proof of Theorem 3.1.

To prove now Theorem 3.2, we observe first that the assumed Asplund property of the spaces X and P implies that their product $P \times X$ is also Asplund; see, e.g., [9]. Proceeding further as in the above proof of Theorem 3.1, we arrive at the generalized Fermat rule

$$(0, 0) \in \partial\big[\varphi + \delta(\cdot; \operatorname{gph} \mathcal{F})\big](\bar{p}, \bar{x}) \tag{3.12}$$

in terms of the limiting subdifferential (2.3) in Asplund spaces and then apply to the sum in (3.12) the subdifferential sum rule from [21, Theorem 3.36] by taking into account the results of [21, Proposition 1.25 and Theorem 1.26] and recalling that the indicator function $\delta(\cdot; \operatorname{gph} \mathcal{F})$ is l.s.c. on $P \times X$, since the graph $\operatorname{gph} \mathcal{F}$ is a closed set. The aforementioned sum rule ensures the fulfillment of the inclusion

$$(0, 0) \in \partial\varphi(\bar{p}, \bar{x}) + N\big((\bar{p}, \bar{x}); \operatorname{gph} \mathcal{F}\big),$$

provided that either φ is locally Lipschitzian around $(\bar{p}, \bar{x})$ or the interior of $\operatorname{gph} \mathcal{F}$ is nonempty and the qualification condition

$$\partial^\infty\varphi(\bar{p}, \bar{x}) \cap \big[-N\big((\bar{p}, \bar{x}); \operatorname{gph} \mathcal{F}\big)\big] = \{(0, 0)\}$$

is satisfied via the singular subdifferential (2.6) of φ at $(\bar{p}, \bar{x})$. The rest of the proof follows the lines in the above proof of Theorem 3.1. ∎

Next we present several consequences of Theorems 3.1 and 3.2, which are formulated as remarks. These specifications seem to be new for the classes of semi-infinite and infinite programs under consideration.

Remark 3.3 (necessary optimality conditions for smooth infinite programs in Banach spaces). Recall that a function $\varphi\colon Z \to \mathbb{R}$ is *strictly differentiable* at $\bar{z}$, with its gradient at this point denoted by $\nabla\varphi(\bar{z}) \in Z^*$, if

$$\lim_{z, u \to \bar{z}} \frac{\varphi(z) - \varphi(u) - \langle \nabla\varphi(\bar{z}), z - u\rangle}{\|z - u\|} = 0, \tag{3.13}$$

which surely holds if φ is continuously differentiable around $\bar{z}$. Assuming now that the cost function $\varphi\colon P\times X\to\mathbb{R}$ in (1.1) with constraints (1.2) is strictly differentiable at a local minimizers $(\bar{p},\bar{x})\in\operatorname{gph}\mathcal{F}$, we get that assumption (a) of Theorem 3.1 is satisfied and condition (3.4) reduces to

$$\Big(\nabla_p\varphi(\bar{p},\bar{x}),\nabla_x\varphi(\bar{p},\bar{x}),\langle\nabla_p\varphi(\bar{p},\bar{x}),\bar{p}\rangle+\langle\nabla_x\varphi(\bar{p},\bar{x}),\bar{x}\rangle\Big)\in\operatorname{cl}^*\operatorname{cone}\Big\{(c_t^*,-a_t^*,-b_t)\Big|\ t\in T\Big\}. \tag{3.14}$$

Furthermore, the KKT condition (3.5) under the fulfillment of the Farkas–Minkowski property is formulated in this case as follows: there are multipliers $\lambda=(\lambda_t)_{t\in T}\in\mathbb{R}_+^{(T)}$ with

$$\nabla_p\varphi(\bar{p},\bar{x})=\sum_{t\in T(\bar{p},\bar{x})}\lambda_t c_t^*,\quad -\nabla_x\varphi(\bar{p},\bar{x})=\sum_{t\in T(\bar{p},\bar{x})}\lambda_t a_t^*. \tag{3.15}$$

Remark 3.4 (lower subdifferential optimality conditions for l_∞-programs). In the case of l_∞-*programs*, i.e., problems (1.1) constrained by (1.3) with $P=l_\infty(T)$, we specify the conditions of Theorem 3.1 by putting $\bar{p}=0$ and $c_t^*=\delta_t$ therein. Observe that for system (1.3) the presence of some feasible point yields the fulfillment of the SSC. Indeed, the inclusion $(0,\bar{x})\in\operatorname{gph}\mathcal{F}$ implies that $(1_T,\bar{x})$ is a strong Slater point of (1.3), where the function $1_T\in l_\infty(T)$ is defined by $1_T(t):=1$ for all $t\in T$.

As mentioned at the beginning of this section, there are various qualification conditions implying the fulfillment of the Farkas–Minkowski property for infinite inequality systems (1.2) and (1.3); see the references and discussions above. By Theorems 3.1 and 3.2, all such assumptions ensure the validity of necessary optimality conditions of the KKT type (3.5) for the nonsmooth problems of semi-infinite and infinite programming under consideration. The next corollary establishes one of the results of this type for semi-infinite programs with constraints (1.2) over compact index sets. For simplicity we present optimality conditions only for locally Lipschitzian cost functions, while the reader can similarly extract from case (b) of the above theorems the corresponding result for l.s.c. objectives.

Corollary 3.5 (necessary optimality conditions of the KKT type for nonsmooth semi-infinite programs). *Suppose that in the setting of Theorem* 3.2 *we have that T is a compact Hausdorff space, $X=\mathbb{R}^n$, $P=\mathbb{R}^m$, and $\varphi\colon\mathbb{R}^m\times\mathbb{R}^n\to\overline{\mathbb{R}}$ is locally Lipschitzian around $(\bar{p},\bar{x})$. Assume in addition that the mappings $t\in T\mapsto a_t^*\in\mathbb{R}^n$, $t\in T\mapsto c_t^*\in\mathbb{R}^m$, and $t\in T\mapsto b_t\in\mathbb{R}$ are continuous on T and that the SSC* (3.1) *holds for* (1.2). *Then there are subgradients $(p^*,x^*)\in\partial\varphi(\bar{p},\bar{x})$ and multipliers $\lambda=(\lambda_t)_{t\in T}\in\mathbb{R}_+^{(T)}$ such that the KKT condition* (3.5) *is satisfied.*

Proof. Employing Theorem 3.2 in this framework, it remains to check that the Farkas–Minkowski property (3.2) holds for (1.2) under the assumptions made. Indeed, we directly get the boundedness of the set $\{(c_t^*,a_t^*,b_t)|\ t\in T\}$ in $\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}$ due to the assumed continuity of (c_t^*,a_t^*,b_t) and the compactness of T. Further, the equivalent description of the SSC from [5, Lemma 2.3(ii)] ensures that

$$(0,0,0)\notin\operatorname{co}\big\{(-c_t^*,a_t^*,b_t)\big|\ t\in T\big\},$$

which implies by [26, Corollary 9.6.1] that the conic hull cone $\{(-c_t^*,a_t^*,b_t)|\ t\in T\}$ is closed in $\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}$. The latter signifies the fulfillment of the Farkas–Minkowski property (3.2) for system (1.2) and thus completes the proof of the corollary. ∎

Let us finally discuss some significant features of the necessary optimality conditions obtained in this section and compare them with known results in this direction.

Remark 3.6 (discussions on lower subdifferential optimality conditions). Observe first that the general necessary optimality conditions obtained above in Theorem 3.1, Theorem 3.2, and their consequences are given in the *normal form* involving nonzero multipliers for the cost function. Regarding constraints, these conditions are generally expressed in the *asymptotic form* that involves the weak* closure of the sets on the right-hand sides in (3.4) and (3.14). The latter new feature partly relates to arbitrary index sets in the semi-infinite and infinite models considered in the paper but may also be exhibited in infinite programs on compact intervals for practically realistic models; see section 5. Under the Farkas–Minkowski property/constraint qualification (3.2) we get necessary optimality conditions in the nonasymptotic *KKT form* (3.5), which reduce to those recently obtained in [7, 8] for one-variable convex and difference of convex (DC) objectives in semi-infinite and infinite programs with arbitrary index sets. Results of such a KKT type have been mainly developed for smooth and convex semi-infinite and infinite programs with compact index sets; cf. Corollary 3.5 for a rather broad nonsmooth extension. We refer the reader to the recent paper [30], probably the first one on nonsmooth and nonconvex semi-infinite optimization, containing necessary optimality conditions of a Lagrangian type for nonsmooth and nonconvex semi-infinite programs with compact index sets and $\mathcal{C}(T)$ data. The necessary conditions obtained in [30] are expressed in terms of Clarke's generalized gradient, which can be significantly larger than the subdifferentials used above.

4. Upper subdifferential optimality conditions. This section is devoted to deriving a new type of *upper subdifferential* necessary optimality conditions for the class of semi-infinite/infinite programs (1.1) with infinitely many linear inequality constraints (1.2) and (1.3). Optimality conditions of this type were initiated in [20] for other classes of nonsmooth minimization problems with finitely many constraints, while in fact they have their roots in the study of maximization (versus minimization) problems for concave functions over convex sets; see, e.g., [26].

The main difference of the results derived in this section from those in section 3 is the usage of upper subgradients (or supergradients) of *minimizing* cost functions instead of the conventional use of (lower) subgradients in minimization. In this way we obtain independent sets of necessary optimality conditions for the problems under consideration in general Banach spaces; see Remark 4.5 for more details and discussions.

To proceed, we recall the notion of the *Fréchet upper subdifferential* (known also as the Fréchet or viscosity superdifferential) of $\varphi\colon Z \to \overline{\mathbb{R}}$ at $\bar z$ defined by

$$\widehat{\partial}^+\varphi(\bar z) := \left\{ z^* \in Z^* \,\middle|\, \limsup_{z\to\bar z} \frac{\varphi(z)-\varphi(\bar z)-\langle z^*, z-\bar z\rangle}{\|z-\bar z\|} \le 0 \right\}, \tag{4.1}$$

which reduces to the classical gradient $\nabla\varphi(\bar z)$ if φ is Fréchet differentiable at $\bar z$ and to the (upper) subdifferential of *concave* functions in the framework of convex analysis. Note that we always have the relationship $\widehat{\partial}^+\varphi(\bar z) = -\widehat{\partial}(-\varphi)(\bar z)$ between the upper subdifferential (4.1) and its *lower* Fréchet counterpart defined in (2.1) with $\varepsilon = 0$.

We have the following upper subdifferential necessary optimality conditions for the infinite and semi-infinite programs (1.1) with constraints (1.2) under consideration.

THEOREM 4.1 (upper subdifferential optimality conditions for nonsmooth infinite programming in Banach spaces). *Let* $(\bar p, \bar x) \in \operatorname{gph} \mathcal{F}$ *be a local minimizer for problem* (1.1) *with the infinite inequality constraints* (1.2) *in Banach spaces* X *and* P. *Then* every *upper subgradient* $(p^*,x^*)\in\widehat{\partial}^+\varphi(\bar p,\bar x)$ *satisfies inclusion* (3.4) *of Theorem* 3.1. *If furthermore the constraint system* (1.2) *has the Farkas–Minkowski property* (3.2)*, then the asymptotic condition* (3.4) *can be equivalently written in the following upper subdifferential KKT form: for* every $(p^*,x^*)\in\widehat{\partial}^+\varphi(\bar p,\bar x)$ *there are multipliers* $\lambda=(\lambda_t)_{t\in T}\in\mathbb{R}^{(T)}_+$ *such that the optimality condition* (3.5) *is satisfied.*

Proof. Pick any $(p^*,x^*)\in\widehat{\partial}^+\varphi(\bar p,\bar x)$ and, employing [21, Theorem 1.88(i)] held in arbitrary Banach spaces, construct a function $s\colon P\times X\to\mathbb{R}$ such that

$$s(\bar p,\bar x)=\varphi(\bar p,\bar x),\quad \varphi(p,x)\le s(p,x)\ \text{ for all }\ (p,x)\in P\times X, \tag{4.2}$$

and $s(\cdot)$ is Fréchet differentiable at $(\bar p,\bar x)$ with the gradient $\nabla s(\bar p,\bar x)=(p^*,x^*)$. Taking into account that $(\bar p,\bar x)$ is a local minimizer for (1.1) with constraints (1.2) and that

$$s(\bar p,\bar x)=\varphi(\bar p,\bar x)\le\varphi(p,x)\le s(p,x)\ \text{ for all }\ (p,x)\in\operatorname{gph}\mathcal{F}\ \text{ near }\ (\bar p,\bar x)$$

by (4.2), we conclude that $(\bar p,\bar x)$ is a *local minimizer* for the auxiliary problem

$$\text{minimize }\ s(p,x)\ \text{ subject to }\ (p,x)\in\operatorname{gph}\mathcal{F} \tag{4.3}$$

with the objective $s(\cdot)$ that is Fréchet differentiable at $(\bar p,\bar x)$. Rewriting (4.3) in the infinite-penalty unconstrained form

$$\text{minimize }\ s(p,x)+\delta\big((p,x);\operatorname{gph}\mathcal{F}\big)$$

via the indicator function of $\operatorname{gph}\mathcal{F}$, observe directly from definition (2.1) of the Fréchet subdifferential at a local minimizer that

$$(0,0)\in\widehat{\partial}\big[s+\delta(\cdot;\operatorname{gph}\mathcal{F})\big](\bar p,\bar x). \tag{4.4}$$

Since $s(\cdot)$ is Fréchet differentiable at $(\bar p,\bar x)$, we easily get from (4.4) that

$$(0,0)\in\nabla s(\bar p,\bar x)+N\big((\bar p,\bar x);\operatorname{gph}\mathcal{F}\big),$$

which implies by $\nabla s(\bar p,\bar x)=(p^*,x^*)$ and the coderivative definition (2.8) that

$$-p^*\in D^*\mathcal{F}(\bar p,\bar x)(x^*). \tag{4.5}$$

It follows from the proof of Theorem 3.1 that the coderivative condition (4.5) can be constructively described via (3.11) in terms of the initial data of the problem under consideration. The latter justifies (3.4) for the given upper subgradient $(p^*,x^*)\in\widehat{\partial}^+\varphi(\bar p,\bar x)$. The KKT conclusion of the theorem is proved similarly to Theorem 3.1. $\square$

Remark 4.2 (necessary optimality conditions for infinite programs with Fréchet differentiable objectives). Similarly to Remark 3.3 we get specifications (3.14) and (3.15) of the necessary optimality conditions of Theorem 4.1, provided that the cost function φ is merely *Fréchet differentiable* at the optimal point $(\bar p,\bar x)$ in the classical case with $u=\bar z=(\bar p,\bar x)$ and $z=(p,x)$ in (3.13). This follows Theorem 4.1 due to the fact that $\widehat{\partial}^+\varphi(\bar p,\bar x)=\{\nabla\varphi(\bar p,\bar x)\}$ when φ is Fréchet differentiable at $(\bar p,\bar x)$.

Remark 4.3 (upper subdifferential optimality conditions for l_∞-programs). Similarly to Remark 3.4 we have the specifications of Theorem 4.1 for l_∞-programs by putting $\bar p=0$ and $c_t^*=\delta_t$ in the conditions therein.

Based on Theorem 4.1 and proceeding similarly to the proof of Corollary 3.5, we ensure the validity of the upper subdifferential necessary optimality conditions of the KKT type under the following assumptions.

COROLLARY 4.4 (upper subdifferential optimality conditions of the KKT type for nonsmooth semi-infinite programs). *Let in the framework of Corollary* 3.5 *the cost function* φ *be just finite in* $(\bar{p},\bar{x})$. *Then for* every $(p^*,x^*)\in\widehat{\partial}^+\varphi(\bar{p},\bar{x})$ *there are multipliers* $\lambda=(\lambda_t)_{t\in T}\in\mathbb{R}^{(T)}_+$ *from* (1.4) *such that the KKT condition* (3.5) *is satisfied.*

Finally, we discuss the major relationships between the lower and upper subdifferential optimality conditions obtained in this paper, focusing mainly on comparison between the corresponding conditions of Theorems 3.1, 3.2, and 4.1. Note that there is no particular counterpart of Theorem 4.1 in the Asplund space setting.

Remark 4.5 (comparison between lower and upper subdifferential optimality conditions for infinite and semi-infinite programs). We can see that the necessary optimality conditions in Theorems 3.1, 3.2, and 4.1 are formulated in the similar formats with two visible distinctions:

(i) The upper subdifferential conditions in the asymptotic form hold in Theorem 4.1 with *no assumptions* imposed on φ and $\mathcal{F}$, in contrast to those in Theorems 3.1 and 3.2.

(ii) The resulting inclusion (3.4) is proved to hold for *every* Fréchet upper subgradient $(p^*,x^*)\in\widehat{\partial}^+\varphi(\bar{p},\bar{x})$ in Theorem 4.1 in comparison with just *some* (lower) subgradients $(p^*,x^*)\in\partial_G\varphi(\bar{p},\bar{x})$ and $(p^*,x^*)\in\partial\varphi(\bar{p},\bar{x})$ in the lower subdifferential result of Theorems 3.1 and 3.2, respectively.

The underlying issue to draw the reader's attention is that the Fréchet upper subdifferential $\widehat{\partial}^+\varphi(\bar{p},\bar{x})$ may be *empty* in many important situations (e.g., for convex cost functions), while the lower subdifferentials $\partial_G\varphi(\bar{p},\bar{x})$ and $\partial\varphi(\bar{p},\bar{x})$ are surely *nonempty* at least for any locally Lipschitzian functions on Banach and Asplund spaces, respectively. Note that the optimality condition of Theorem 4.1 holds trivially if $\widehat{\partial}^+\varphi(\bar{p},\bar{x})=\emptyset$, while even in this case it provides some easily checkable information on optimality without taking constraints into account. Of course, a real strength of upper subdifferential optimality conditions as in Theorem 4.1 should be exhibited for *nonsmooth* cost functions admitting Fréchet upper subgradients at the point in question.

There are remarkable classes of nonsmooth functions enjoying the latter property. First we mention *concave* continuous functions on arbitrary Banach spaces and also *DC* functions whose minimization can be reduced to minimizing concave functions subject to convex constraints. Another important class of functions admitting a nonempty set of Fréchet upper subgradients consists of the so-called *semiconcave* functions, known also under various other names (e.g., upper subsmooth, paraconcave, approximately concave, etc.) and being particularly important for applications to optimization, viscosity solutions of the Hamilton–Jacobi partial differential equations, optimal control, and differential games; see more discussions and references in [22, Commentary 5.5.4, pp. 135–136].

Since the subdifferentials $\partial_G\varphi(\bar{p},\bar{x})$ and $\partial\varphi(\bar{p},\bar{x})$ used above are smaller than the Clarke generalized gradient $\partial_C\varphi(\bar{p},\bar{x})$ for every l.s.c. function in any Banach space, Theorems 3.1 and 3.2 immediately imply their counterparts with some C-subgradient $(p^*,x^*)\in\partial_C\varphi(\bar{p},\bar{x})$ therein. It is worth emphasizing that the latter lower subdifferential optimality condition is *significantly weaker* than the upper subdifferential one in Theorem 4.1 for concave and other "upper regular" functions (see [21]) including those mentioned above. Considering for simplicity the case of concave continuous functions, we have

$$\widehat{\partial}^{+}\varphi(\bar{z}) = -\widehat{\partial}(-\varphi)(\bar{z}) = -\partial_C(-\varphi)(\bar{z}) = \partial_C\varphi(\bar{z}) \neq \emptyset$$

due to the plus-minus symmetry of the generalized gradient for locally Lipschitzian functions. Thus Theorem 4.1 dramatically strengthens the C-counterpart of Theorems 3.1 and 3.2 in such cases justifying the necessary optimality condition held for *every* $(p^*, x^*) \in \partial_C\varphi(\bar{p}, \bar{x})$ instead of just *one* element from this set.

5. Applications to optimization of water resources. The concluding section of the paper concerns a *water resource model*, which is of a certain practical meaning. We formulate this problem and reduce it to a *two-variable infinite program* of the type investigated in the previous sections. The cost function in this problem may be either smooth or nonsmooth, and we study the possibility of applying to its solution the necessary optimality conditions derived in sections 3 and 4. On one hand, we explore the outcome of the optimality conditions obtained in the *extended KKT form* under the *Farkas–Minkowski property* (3.2) and discuss qualitative and quantitative consequences of these conditions for optimal strategies in our model. On the other hand, we *fully characterize* the situation when the Farkas–Minkowski property holds and show that it is *not* the case of the model considered on the *compact continuous-time* interval/index set. In the latter case the *asymptotic* necessary optimality conditions of Theorems 3.1 and 4.1 hold, and we discuss their impact by using reasonable and practically realistic approximations.

The *water resource problem* under consideration is inspired by a continuous-time network flow model (see [1], section 1.2.2). Consider a system of n reservoirs $R_1, R_2, \ldots, R_n$ from which a time-varying water demand is required during a fixed continuous-time period $T = [\underline{t}, \overline{t}]$. Let c_i be the capacity of the reservoir R_i, and let water flow into R_i at rate $r_i(t)$ for each $i = 1, \ldots, n$ and $t \in T$. Denote by $D(t)$ the rate of water demand at t, and suppose that all these nonnegative functions $r_1, \ldots, r_n$ and D are piecewise continuous on the closed and bounded interval T and are known in advance.

If there is enough water to fill all the reservoir capacity, then the rest can be sold to a neighboring dry area, provided that the demand is satisfied. Conversely, if the inflows are short and the reservoirs have free capability for holding additional water, then some water can be bought from outside to meet the inner demand in the region.

Denote by $x_i(t)$ the rate at which water is fed from the reservoir R_i at time $t \in T$. It is natural to assume in our basic model that $x_i \in \mathcal{C}(T)$ for all $i = 1, \ldots, n$. The *feeder constraints* can be expressed by

$$0 \le x_i(t) \le \eta_i, \quad i = 1, \ldots, n, \tag{5.1}$$

with fixed bounds $\eta_i \ge 0$. The *selling rate* of water from the reservoir R_i at time t is given by $dp_i(t)$, which means that $p_i(t)$ is the quantity of water sold until instant t and depending on t continuously on the the time interval T. Without loss of generality, we assume that $p_i(\underline{t}) = 0$ for all i. Note that we are actually buying water at time $t \in T$ if the selling rate $dp_i(t)$ is negative. Denoting by $s_i \ge 0$ the amount of water initially stored in R_i, we formulate the *storage constraints* by

$$\begin{aligned} 0 &\le \int_{\underline{t}}^{t} \big[r_i(\tau) - x_i(\tau)\big]\, d\tau - \int_{\underline{t}}^{t} dp_i(\tau) + s_i \\ &= \int_{\underline{t}}^{t} \big[r_i(\tau) - x_i(\tau)\big]\, d\tau - p_i(t) + s_i \\ &\le c_i \ \text{ for all } \ t \in T \ \text{ and } \ i = 1, \ldots, n \end{aligned} \tag{5.2}$$

and finally arrive at the following problem of *water resource optimization*:

$$\begin{cases} \text{minimize } \ \varphi(p,x) \ \text{ subject to} \\ (5.1), \quad (5.2), \quad \text{and} \\ \displaystyle\sum_{i=1}^{n} x_i(t) \geq D(t), \quad t \in T, \end{cases} \tag{5.3}$$

where the cost/objective function $\varphi(p,x)$ is determined by the cost of water, environmental requirements in the region, and the technology of reservoir processes in the water resource problem (WR). It is clear that we should impose the relationship

$$D(t) \leq \sum_{i=1}^{n} \eta_i, \quad t \in T,$$

in order to ensure the consistency of the constraints in (5.3).

Let us show that problem (5.3) can be reduced to the form of *infinite programming* (1.1) with two groups of variables $(p,x) \in \mathcal{C}(T)^n \times \mathcal{C}(T)^n$ and infinitely many linear inequality constraints (1.2). To proceed, define the following t-parametric families of functions on T:

$$\delta_t(\tau) := \begin{cases} 0 & \text{if } \ \underline{t} \leq \tau < t, \\ 1 & \text{otherwise;} \end{cases}$$

$$\alpha_t(\tau) := \begin{cases} \tau & \text{if } \ \underline{t} \leq \tau < t, \\ t & \text{otherwise.} \end{cases}$$

Both families $\{\delta_t |\ t \in T\}$ and $\{\alpha_t |\ t \in T\}$ can be seen as subsets of the dual space $\mathcal{C}(T)^*$. In fact, the Riesz representation theorem (see, e.g., [9, Proposition 2.19]) ensures that each function $\gamma \colon T \to \mathbb{R}$ of bounded variation on T determines a linear functional on $\mathcal{C}(T)$ by

$$z \mapsto \langle \gamma, z \rangle := \int_{\underline{t}}^{\overline{t}} z(\tau)\, d\gamma(\tau), \quad z \in \mathcal{C}(T),$$

via the Stieltjes integral; all elements of $\mathcal{C}(T)^*$ are of this type. We can easily check that

$$\int_{\underline{t}}^{t} x_i(\tau)\, d\tau = \langle \alpha_t, x_i \rangle, \quad t \in T,$$

and observe the relationship

$$d\alpha_t(\tau) = \chi_{[\underline{t},t]}(\tau)\, d\tau, \quad t \in T,$$

where $\chi_{[\underline{t},t]}$ is the standard characteristic function of the interval $[\underline{t}, t]$. Moreover, it holds for each element $z \in \mathcal{C}(T)$ that

$$\langle \delta_t, z \rangle = z\,(t)\,, \quad t \in T,$$

and thus δ_t can be identified in this context with the classical *Dirac measure* at t.

Denote further the functions

$$\beta_i(t) := \int_{\underline{t}}^{t} r_i(\tau)\, d\tau \ \text{ for } \ i = 1, \ldots, n, \quad t \in T,$$

and notice that the constraints in (5.2) can be rewritten as

$$\begin{cases} \langle \delta_t, p_i \rangle + \langle \alpha_t, x_i \rangle \le \beta_i(t) + s_i, \\ -\langle \delta_t, p_i \rangle - \langle \alpha_t, x_i \rangle \le c_i - s_i - \beta_i(t), \end{cases} \tag{5.4}$$

while the one in (5.3) admits the form

$$\sum_{i=1}^{n} \langle \delta_t, x_i \rangle \ge D(t), \quad t \in T. \tag{5.5}$$

Observing finally that the constraints in (5.1) can be equivalently written as

$$0 \le \langle \delta_t, x_i \rangle \le \eta_i, \quad i = 1, \ldots, n, \quad t \in T, \tag{5.6}$$

we arrive at the desired reduction result.

Proposition 5.1 (water resource problem as infinite programming). *The problem of water resource optimization* (5.3) *is equivalent to the following two-variable infinite program of type* (1.1), (1.2) *in the space* $\mathcal{C}(T) \times \mathcal{C}(T)$*:*

$$\begin{cases} \textit{minimize} \;\; \varphi(p, x) \;\; \textit{subject to} \\ (5.4), \;\; (5.5), \;\; \textit{and} \;\; (5.6) \end{cases} \tag{5.7}$$

with the given data δ_t*,* α_t*,* β_t*,*c_i*,* s_i*,* η_i*, and* D *defined above.*

Next we examine the possibility of applying necessary optimality conditions obtained in sections 3 and 4 to the case of the water resource model (5.7). Since the space $\mathcal{C}(T)$ for both variables x and p in our model is not Asplund, we consider applications of Theorems 3.1 and 4.1 to infinite program (5.7). For simplicity of notation, suppose in what follows that $n = 1$ in our model and write $(p, x, \beta, c, s, \eta)$ instead of $(p_1, x_1, \beta_1, c_1, s_1, \eta_1)$.

Involving the initial data of problem (5.7), construct the conic hull in the dual space $\mathcal{C}(T)^* \times \mathcal{C}(T)^* \times \mathbb{R}$ by

$$K(T) := \text{cone} \left\{ \begin{array}{l} \Big[\big(\delta_t, \alpha_t, \beta(t) + s\big), \big(-\delta_t, -\alpha_t, c - s - \beta(t)\big), \\ \big(0, -\delta_t, -D(t)\big), \big(0, \delta_t, \eta\big) \;\; \text{over all} \;\; t \in T \Big] \end{array} \right\}, \tag{5.8}$$

which is a specification of the general second moment cone (3.2) for our problem (5.7). It is convenient for us to indicate the explicit dependence of the cone (5.8) on the time/index interval T. Given a solution pair $(\bar{p}, \bar{x})$, define the set of *active indices* corresponding to all four inequality constraints in (5.7):

$$T_1(\bar{p}, \bar{x}) := \big\{ t \in T \big| \; \langle \delta_t, \bar{p} \rangle + \langle \alpha_t, \bar{x} \rangle = \beta(t) + s \big\}, \tag{5.9}$$

$$T_2(\bar{p}, \bar{x}) := \big\{ t \in T \big| \; -\langle \delta_t, \bar{p} \rangle - \langle \alpha_t, x \rangle = c - s - \beta(t) \big\}, \tag{5.10}$$

$$T_3(\bar{p}, \bar{x}) := \big\{ t \in T \big| \; -\langle \delta_t, x \rangle = -D(t) \big\}, \tag{5.11}$$

$$T_4(\bar{p}, \bar{x}) := \big\{ t \in T \big| \; \langle \delta_t, \bar{x} \rangle = \eta \big\}. \tag{5.12}$$

Now we are ready to formulate the main results for problem (5.7), which are consequences of Theorems 3.1 and 4.1. Regarding applications of Theorem 3.1, we confine ourselves for simplicity to the case (a) therein.

Proposition 5.2 (necessary optimality conditions for water resource optimization). *Let* $(\bar{p}, \bar{x})$ *be a local optimal solution to problem* (5.7) *with some cost function* $\varphi\colon \mathcal{C}(T) \times \mathcal{C}(T) \to \overline{\mathbb{R}}$ *and be* φ *finite at* $(\bar{p}, \bar{x})$*. The following assertions hold:*

(i) *Every upper subgradient* $(p^*,x^*)\in\widehat{\partial}^+\varphi(\bar p,\bar x)$ *satisfies the inclusion*

$$-\big(p^*,x^*,\langle p^*,\bar p\rangle+\langle x^*,\bar x\rangle\big)\in \mathrm{cl}^*K(T), \tag{5.13}$$

where the cone $K(T)$ *is defined in* (5.8). *If furthermore* φ *is locally Lipschitzian around* $(\bar p,\bar x)$, *then there is a G-subgradient* $(p^*,x^*)\in\partial_G\varphi(\bar p,\bar x)$ *satisfying* (5.13).

(ii) *Assume that the cone* $K(T)$ *in* (5.8) *is weak* closed. Then for every* $(p^*,x^*)\in\widehat{\partial}^+\varphi(\bar p,\bar x)$ *there are generalized multipliers* $\lambda=(\lambda_t)_{t\in T}$, $\mu=(\mu_t)_{t\in T}$, $\gamma=(\gamma_t)_{t\in T}$, *and* $\rho=(\rho_t)_{t\in T}\in\mathbb{R}_+^{(T)}$ *satisfying the condition*

$$\begin{cases} -(p^*,x^*)=\displaystyle\sum_{t\in T_1(\bar p,\bar x)}\lambda_t(\delta_t,\alpha_t)+\sum_{t\in T_2(\bar p,\bar x)}\mu_t(-\delta_t,-\alpha_t)\\[2ex] \qquad\qquad\qquad +\displaystyle\sum_{t\in T_3(\bar p,\bar x)}\gamma_t(0,-\delta_t)+\sum_{t\in T_4(\bar p,\bar x)}\rho_t(0,\delta_t),\end{cases} \tag{5.14}$$

where the sets of active indices $T_i(\bar p,\bar x)$, $i=1,2,3,4$, *are built in* (5.9)–(5.12), *respectively. If in addition* φ *is locally Lipschitzian around* $(\bar p,\bar x)$, *then there are* $(p^*,x^*)\in\partial_G\varphi(\bar p,\bar x)$ *and generalized multipliers* $\lambda=(\lambda_t)_{t\in T}$, $\mu=(\mu_t)_{t\in T}$, $\gamma=(\gamma_t)_{t\in T}$, *and* $\rho=(\rho_t)_{t\in T}\in\mathbb{R}_+^{(T)}$ *satisfying condition* (5.14).

Proof. It follows from the necessary optimality conditions in Theorems 4.1 and 3.1(a), respectively, applied to problem (5.7), with taking into account the derived formula (5.8) for the second moment cone $K(T)$ and the form of the active index sets (5.9)–(5.12) corresponding to the infinite inequality constraints in (5.4)–(5.6). ∎

Observe that the necessary optimality conditions obtained in Proposition 5.2 provide a valuable contribution to our understanding of *optimal strategies* in the water resource problem. Indeed, it follows from the structures of constraints in (5.7) and their active index sets that the time inclusion $t\in T_1(\bar p,\bar x)$ means that at this moment t the reservoir is empty, while the one of $t\in T_2(\bar p,\bar x)$ means that at this time the quantity of water inside the reservoir (given by $\langle\delta_t,p\rangle+\langle\alpha_t,x\rangle-s-\beta(t)$) attains its maximum level c; i.e., the reservoir is full. Similarly the inclusions $t\in T_i(\bar p,\bar x)$ for $i=3,4$ signify, respectively, that the water is flowing at its minimum rate to satisfy the demand or at its maximum rate technically possible. The KKT relationship (5.14), valid under the Farkas–Minkowski qualification condition, reflects therefore that the "dual action" (p^*,x^*) is a linear combination of these "bang-bang" strategies with the corresponding weights $(\lambda,\mu,\gamma,\rho)$. Our general *asymptotic* optimality condition (5.13) indicates from this viewpoint that, in the absence of the Farkas–Minkowski property, the optimal impulse can be approximated by such combinations.

Finally, we fully characterize the setting of Proposition 5.2(ii) when the Farkas–Minkowski property is satisfied in problem (5.7).

PROPOSITION 5.3 (Farkas–Minkowski property in water resource optimization). *Let* $\widetilde T$ *be a nonempty subset of the time interval* $T=[\underline t,\overline t]$ *in* (5.7). *Then the second moment cone* $K(\widetilde T)$ *in* (5.8) *is weak* closed in* $\mathcal{C}(T)^*\times\mathcal{C}(T)^*\times\mathbb{R}$ *if and only if* $\widetilde T$ *is finite.*

Proof. The "if" part is standard. Let us justify the "only if" part arguing by contradiction and taking into account that the space $\mathcal{C}(T)$ is separable. Assume that the set $\widetilde T$ is infinite and pick for simplicity a strictly increasing or decreasing sequence $\{t_k\}_{k\in\mathbb{N}}$ in $\widetilde T$, which is therefore converging to a certain element of T. The reader can easily check that the sequence in $\mathcal{C}(T)^* \times \mathcal{C}(T)^* \times \mathbb{R}$ given by

$$\left\{ \sum_{j=1}^{k} \frac{1}{j^2} \left(\delta_{t_j}, \alpha_{t_j}, \beta(t_j) + s \right) \right\}_{k \in \mathbb{N}}$$

weak* converges to some (δ, α, b) defined as $\langle (\delta, \alpha, b), (p, x, q) \rangle := \langle \delta, p \rangle + \langle \alpha, x \rangle + bq$ with

$$\langle \delta, p \rangle := \sum_{j=1}^{\infty} \frac{1}{j^2} p(t_j), \quad \langle \alpha, x \rangle := \sum_{j=1}^{\infty} \frac{1}{j^2} \int_{\underline{t}}^{t_j} x(t)\, dt, \quad \text{and} \;\; b := \sum_{j=1}^{\infty} \frac{1}{j^2} \left(\beta(t_j) + s \right).$$

The weak* convergence of the above sequence follows from the boundedness of the set $\left\{ \left(\delta_{t_j}, \alpha_{t_j}, \beta(t_j) + s \right) \right\}_{k \in \mathbb{N}}$ in $\mathcal{C}(T)^* \times \mathcal{C}(T)^* \times \mathbb{R}$ and the convergence of the series $\sum_{j=1}^{\infty} \frac{1}{j^2}$.

Let us show now that $(\delta, \alpha, b) \notin K(\widetilde{T})$, and thus the cone $K(\widetilde{T})$ is not weak* closed. Indeed, the inclusion $(\delta, \alpha, b) \in K(\widetilde{T})$ implies that

$$\delta = \sum_{t \in \widetilde{T}} \lambda_t \delta_t \;\; \text{for some} \;\; \lambda \in \mathbb{R}_+^{(\widetilde{T})},$$

which is discontinuous only on a finite subset of T. It is easy to check at the same time that δ is the weak* limit of the functions $\sum_{j=1}^{k} \frac{1}{j^2} \delta_{t_j}$ as $k \to \infty$, and hence it is discontinuous on the infinite set $\{t_k\}_{k \in \mathbb{N}}$. This contradiction completes the proof of the proposition. $\square$

One of the remarkable consequences of Proposition 5.3 is that the Farkas–Minkowski qualification condition does *not* hold for the water resource problem stated in (5.7) on the compact continuous-time interval $T = [\underline{t}, \overline{t}]$. On the other hand, this result justifies yet another interpretation of the optimality conditions of Proposition 5.2 corresponding to the practical realization of control strategies for reservoirs. Since in practice the measuring and control processes for the water resource model under consideration are implemented only at discrete instants of time, we can consider a *discretization* $\widetilde{T}$ of the time interval T and apply the simplified optimality conditions of Proposition 5.2(ii) on $\widetilde{T}$.

Acknowledgment. We are gratefully indebted to two anonymous referees and associate editor Jiří Outrata for their valuable suggestions and remarks that allowed us to essentially improve the original presentation.

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