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Simulating quantum measurements and quantum correlations

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PhD thesis written under the cotutorship of Marcelo Terra Cunha and Antonio Acín as a partial requirement for the titles of PhD in Mathematics and PhD in Photonics.

January 2018

ali onde a perna bambeia
ali onde não há caminho

Chico Buarque - Massarandupió

Acknowledgements

Nascer numa família que prioriza acima de tudo a educação é um privilégio. Assim, agradeço de coração meus pais, Ângela e Olcei, que mesmo sem terem um diploma de curso superior veem agora seu terceiro filho concluir o doutorado. Agradeço às minha irmãs, Carolina e Débora, por serem exemplos de pessoa e estudante. São elas as responsáveis por eu desde sempre entender que estudar em uma universidade federal é apenas natural, e que o doutorado sempre foi uma realidade. Essa confiança faz toda a diferença.

Agradeço aos meu orientadores, Marcelo y Toni, por entenderem tan bien que la parte mais difícil do doutorado no es la investigación académica, e por lidarem tan bien con eso. A Marcelo, mais do que tudo pela preocupação com o fator humano que sempre demonstrou. A Toni, principalmente por la confianza y levedad con que gerencia su grupo.

Dentre muitos amigos de várias partes que fiz no doutorado, quatro tiverem uma influência muito grande na minha trajetória durante esse período. Gláucia[1] e Marco Túlio[2] foram uma condição suficiente para entrar para essa área, e Rafael[3] foi um grande amigo desde o t_0, enquanto que a Jessica[4] ativou nossas colaborações na hora certa.

Aos demais Terráqueos, Gabriel[5], Natália[6], Bárbara[7], Marcello, Roberto, Tassius, Junior, valeu pela parceria. Davi, Denise, Tchê[8], Rafael Gauchão, Andrades, Eliel e outros amigos da UFMG e da Casa Lama, obrigado pelas festas e discussões.

I thank the great friends I did in Barcelona, in special to the ex-Young PhDs

[1] Pessoa mais massa pra falar da vida e comer chocolate.
[2] Maximizador de tretas aleatórias e fonte infinita de motivação.
[3] Paga cerveja unbounded sempre que vira professor.
[4] Leva vinho, mas é ortogonal para qualquer série ou filme não-wesandersoniano.
[5] Anima tudo, e toca no violão até o que não sabe.
[6] Manja muito de sudoku.
[7] Te hospeda no lugar com a vista mais massa de Natal e ainda faz tapioca.
[8] Esses três são Sonserina até a alma.

Alexia[9], Boris[10], Flavio[11], and Ivan[12], and also Josep[13] and Dr. Biboundé for the friendship in so many nice moments. Also to Matteo, Florian, Dani, Peter Wittek, Alex Pozas, Dario Naharis, Sarah Keary, Martí, Michał, Senaida, Zahra, Markus, Christian, Janek, Mafalda, Karen, Paul, Jessica and all other Tonians I'm grateful for all the futbolín matches, coffees, hikes, football matches, and conferences. Also to Cher Antoine, Marco, Marcus, Pauli, and Claude the Cloud. Thank you, Bogna.

Ao grupo de capoeira ACESA em Belo Horizonte e a'Os Vald@s, principalmente Osvaldo y Borja, eu agradeço pela terapia mantenedora de sanidad.

Aos amigos deístas Lacra, Maurício e Leandro Lobo, valeu por fortalecer a raiz. À Ândrea, por insistir em trocar emails. À Thaiara, por ainda trocar mensagens.

Agradeço aos funcionários e mantedores da Cinemateca e Café Capitólio, onde a maior parte dessa tese foi escrita.

Finally, I acknowledge financial support from Capes, INCT, Spanish MINECO (QIBEQI FIS2016-80773-P and Severo Ochoa SEV-2015-0522), Fundaciò Cellex, Generalitat de Catalunya (SGR875 and CERCA Program), and ERC CoG QITBOX.

[9] Future president of the crazy cat ladies association.

[10] The countermeasure for the workaholic influences.

[11] Did you know that mozzarella is not a cheese?

[12] Very loud when playing futbolín. Also in the other moments.

[13] Best client of Café Solo.

Abstract

This PhD thesis is focused on the quantum measurement simulability problem, that is, deciding whether a given measurement can be simulated when only a restricted subset of measurements is accessible. We provide an operational framework for this problem based on classical manipulations over the set of simulators. Particular cases of interest are further investigated, in which the simulators are taken to be projective measurements, measurements of a fixed number of outcomes, and arbitrary sets of fixed cardinality. In each of these situations we derive either necessary or sufficient conditions for simulability, and full characterisations in terms of semidefinite programming for some specific cases. Since joint measurability is a particular case of simulability, we also present a natural generalisation for it.

Besides deciding whether a given measurement is simulable by some set of simulators, we also pose the question of what are the most robust measurements against simulability. We provide a strategy for approximating the set of quantum measurements based on relaxing the positivity constraint. This allows us to identify the most robust qubit measurement in terms of projective simulability, as well as the most incompatible sets of N measurements, for $N = 1, \ldots, 5$, which notably are found to be always projective.

By applying our simulability results in the context of Einstein-Podolsky-Rosen steering and Bell nonlocality we are able to construct improved and more general local models. Starting from models for a finite number of measurements we obtain the first general method for constructing local models for arbitrary families of quantum states. Similarly, our study on projective simulability yields a strategy for extending models for projective measurements to arbitrary ones, culminating in the most efficient local model for two-qubit Werner states and general measurements.

Resumo

Esta tese de doutorado é centrada no problema de simulação de medições quânticas, ou seja, em decidir se uma dada medição pode ser simulada quando temos acesso a apenas um subconjunto restrito de medições. Apresentamos um *framework* operacional para esse problema, baseado em manipulações clássicas sobre o conjunto de simuladores. Casos particulares de interesse são estudados em detalhe, nos quais o conjunto de simuladores é dado por medições projetivas, medições de um número fixo de *outcomes*, e conjuntos arbitrários de cardinalidade fixada. Em cada uma dessas situações, derivamos condições necessárias ou suficientes para simulabilidade, e uma caracterização completa em termos de programação semidefinida em alguns casos específicos. Como comensurabilidade é um caso particular de simulabilidade, apresentamos também uma generalização natural para esse conceito.

Além de decidir se uma dada medição é simulável ou não, também exploramos a questão de quais são as medições mais robustas contra simulabilidade. Apresentamos então uma estratégia para aproximar o conjunto das medições quânticas baseada em uma relaxação da condição de positividade. Isso nos permite identificar a medição mais robusta contra simulabilidade projetiva em dimensão 2, assim como os conjuntos de N medições mais incompatíveis, para $N = 1, \ldots, 5$, que notavelmente se revelam ser projetivas em todos esses casos.

Aplicando nossos resultados de simulabilidade no contexto de Einstein-Podolsky-Rosen *steering* e não-localidade de Bell, somos capazes de construir modelos locais melhores e mais gerais. Partindo de modelos para um número finito de medições, obtemos o primeiro método geral para construção de modelos locais para famílias arbitrárias de estados quânticos. De forma similar, nosso estudo de simulabilidade projetiva fornece uma estratégia para estender modelos locais para medições projetivas a medições arbitrárias, culminando no mais eficiente modelo local para estados de Werner de dois qubits e medições quaisquer.

Resum

Aquesta tesi doctoral se centra en el problema de la simulació de mesures quàntiques, és a dir, en decidir si es pot simular una determinada mesura quan només tenim accés a un subconjunt restringit de mesures diferents. Presentem un marc operacional per a aquest problema, basat en manipulacions clàssiques sobre el conjunt de simuladors. Casos particulars d'interès son estudiat en detall, on el conjunt de simuladors està donat per mesures projectius, mesures d'un nombre fix de resultats i conjunts arbitraris de cardinalitat fixa. En cadascuna d'aquestes situacions, derivem condicions necessaris o suficients per a la simulació, i una caracterització completa en termes de programació semi-definida en alguns casos específics. Com la mensurabilitat conjunta és un cas particular de simulació, presentem també una generalització natural per a aquest concepte.

A més de decidir si un mesura és simulable o no, també exploram la qüestió de quines son las mesures més robustes contra la simulabilitat. A continuació, presentem una estratègia per aproximar el conjunt de mesures quàntiques basat en una relaxació de la condició de positivitat. Això permet la identificació de la mesura més robusta envers la simulació projectiva en dimensió 2, així com els conjunts més incompatibles de N mesures, per $N = 1, \ldots, 5$, que notablement resulten ser projectivas en tots aquests casos.

Aplicant els nostres resultats de simulació en el context d'Einstein-Podolsky-Rosen *steering* i no-localitat de Bell, som capaços de construir models locals millors i més generals. A partir de models per a un nombre finit de mesures, obtenim el primer mètode general per a la construcció de models locals per a famílies arbitràries d'estats quàntics. De la mateixa manera, el nostre estudi de la simulació projectiva proporciona una estratégia per ampliar models locals per a mesures projectivas a mesures arbitraris, culminant en el model local més eficient per als estats de Werner de dos qubits i mesures generals.

List of papers

The content of this thesis is based on results developed in the following papers:

1. *General method for constructing local-hidden-variable models for entangled quantum states*
   D. Cavalcanti, **L. Guerini**, R. Rabelo, and P. Skrzypczyk
   Phys. Rev. Lett., **117**, 190401, (2016).

2. *Simulating general positive-operator-valued measures with projective measurements*
   M. Oszmaniec, **L. Guerini**, P. Wittek, and A. Acín
   Phys. Rev. Lett., **119**, 190501, (2017).

3. *Most incompatible measurements for robust steering tests*
   J. Bavaresco, M. T. Quintino, **L. Guerini**, T. O. Maciel, D. Cavalcanti, and M. Terra Cunha
   Phys. Rev. A, **96**, 022110, (2017).

4. *Operational framework for quantum measurement simulability*
   L. Guerini, J. Bavaresco, M. Terra Cunha, and A. Acín
   J. Math. Phys. **58**, 092102, (2017).

The author also contributed to the work:

- *Uniqueness of the joint measurement and the structure of set of compatible quantum measurements*
  L. Guerini, and M. Terra Cunha
  arxiv: 1711.04804.

Contents

Introduction

The present PhD thesis refers to two distinct graduate programs: the Graduate Program in Mathematics, at Department of Mathematics - Universidade Federal de Minas Gerais (Belo Horizonte, Brazil), and the Graduate Program in Photonics, at Institute of Photonic Sciences - Universitat Politècnica de Catalunya (Barcelona, Spain). The candidate spent two years in each institution, and this thesis represents the last requirement of each program left to be fulfilled in order to obtain the titles of PhD in Mathematics and PhD in Photonics, respectively.

The results presented here are placed in the intersection of foundations of quantum theory and mathematical physics. More specifically, we investigate simulability properties of sets of quantum measurements and apply our results to quantum correlations, making use of the intrinsic mathematical structure they present.

The set of quantum correlations is formed by families of probability distributions obtained upon performing measurements on a multipartite quantum system. Concepts like entanglement, Einstein-Podolsky-Rosen (EPR) steerability [WJD07], and Bell nonlocality [BCP+14] then appear as ways of classifying the state of such system, depending on how we are able to model the yielded correlations.

Entanglement is a necessary ingredient for several protocols that achieve the best known performance in many tasks within quantum cryptography [BBD08], quantum computation [NC11], and randomness expansation [BAK+17]. However, not all entangled states are steerable [Wer89], and not all unsteerable states are nonlocal [AGT06]. In order to understand the advantages that entanglement brings to many practical situations that involve nonlocality, it is also necessary to understand in which cases we can use classical resources (represented by shared randomness and local strategies) to simulate quantum phenomena. In this case, we say that we have a description in terms of a local model.

Another necessary feature for non-classical behaviour (Bell nonlocality, uncertainty relations [Hei27]), this time from the side of quantum measurements, is measurement incompatibility. Here we capture this idea in the concept of joint measurability [HMZ16], which is closely related to EPR steering [QVB14]. The first part of this thesis (Chapters 1-4) is devoted to formalise and present the mathematical background of the previous concepts[14].

There are local models tailored specifically to reproduce the statistics of certain highly symmetric families of entangled quantum states [Wer89, APB+07]. However, given an arbitrary and asymmetric state, deciding whether it admits a local model or not is a difficult task. The approach we propose to tackle this problem is based on the structure of the set of quantum measurements. More specifically, if we construct a local model for some specific subset of measurements, we conclude that such a model is also valid for any measurements that can be simulated via classical manipulations over the initial set.

This strategy was successfully implemented in our first contribution to the field [CS16], where we presented the first general method for constructing local models for arbitrary families of quantum states. A second instance of measurement simulability appears in our second contribution [OGWA17], where we introduced the study of projective simulability. Applying these ideas we constructed the currently most efficient local-hidden-variable model for two-qubit Werner states, the benchmark family of states in the area. The construction and extension of local models is the theme of Chapter 7.

Besides asking whether a given measurement is simulable by others, we can move the question one level above and ask which are the hardest measurements to simulate, according to the set of simulators. In Ref. [OGWA17] we presented a technique to approximate the set of quantum measurements by more tractable sets, over which optimisations turned out to be simpler. We then singled out the most robust qubit measurement regarding projective simulability. Since joint measurability can also be interpreted as a simulability task, we adapted our approximation technique to investigate the most incompatible sets of a fixed number of measurements [BQG+17]. In Chapter 6 we further detail these optimisations over sets of quantum measurements.

In Ref. [GBCA17] we generalized the previous works to the general problem of quantum measurement simulability, extending the idea of joint measurability and proving several connections between different types of simulability. The operational framework developed to study these questions is presented in Chapter 5.

[14] The reader that is familiar with the field is advised to start reading from the Chapter 5 on, and coming back to the first part to pinpoint specific definitions or results whenever needed.

Part I:

Preliminaries

Chapter 1

Basic notions of quantum theory

The goal of this chapter is simply to present the minimum of the mathematical framework related to the quantum operations and phenomena that we are interested in this text. Therefore, no physical motivation will be presented and various of basic and important topics will be completely ignored.

We will start directly making use of the density operator formalism. For an introduction to Quantum Theory and its formalisms we suggest Ref. [NC11].

1.1 Quantum states

In quantum theory, we postulate that each Hilbert space $\mathcal{H}$ is associated to a quantum system[1]. The canonical basis for $\mathcal{H}$ is denoted by $\{|0\rangle, |1\rangle, \ldots\}$.

Considering the set $\mathcal{L}(\mathcal{H})$ of linear operators acting on $\mathcal{H}$, a state is an element of $\mathcal{L}(\mathcal{H})$ that describes completely the system.

Definition 1. *A* ***state*** *of a quantum system associated to $\mathcal{H}$ is an operator $\rho \in \mathcal{L}(\mathcal{H})$ which is positive semi-definite and has unit trace, i.e.,*

$$\rho \geq 0, \tag{1.1a}$$

$$Tr(\rho) = 1. \tag{1.1b}$$

The subset of $\mathcal{L}(\mathcal{H})$ formed by quantum states, also called **density operators**, is denoted by $\mathcal{D}(\mathcal{H})$. $\mathcal{D}(\mathcal{H})$ is a convex set, that is, every convex combination of density operators is also a density operator. If the state ρ is a rank-1 projector (that is, if $\rho^2 = \rho$ and its image has dimension 1), then $\rho = |\psi\rangle\langle\psi|$ for some $|\psi\rangle \in \mathcal{H}$, and we can identify the density operator ρ with the vector $|\psi\rangle$.

[1] Since throughout this text we consider only Hilbert spaces of finite dimension, we can think that each quantum system is associated to $\mathbb{C}^d$, for some finite dimension d.

The unit trace condition implies that $|\psi\rangle$ is a unit vector in the Euclidean norm. Every other kind of density operator is called a **mixed state** and can be written as a non-trivial convex combination of projectors, *i.e.*,

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|, \tag{1.2}$$

with unit vectors $|\psi_i\rangle \in \mathcal{H}$ and weights $p_i \geq 0$ satisfying $\sum_i p_i = 1$. Notice that a rank-1 projector is a mixed state with only one term in the sum. Therefore, it is called a **pure state**.

The simplest quantum system that we can imagine is the one associated to the Hilbert space $\mathbb{C}^2$, called a **qubit** system. We can always decompose a qubit state $\rho \in \mathcal{D}(\mathbb{C}^2)$ in a basis of the real vector space $\mathrm{Herm}(\mathbb{C}^2)$ of Hermitian operators acting on $\mathbb{C}^2$, and perhaps the most natural such basis is given by the identity $\mathbb{I}$ together with the Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.3}$$

In this case we write

$$\rho = \frac{1}{2}\left(\alpha_1 \mathbb{I} + v_x\sigma_x + v_y\sigma_y + v_z\sigma_z\right), \tag{1.4}$$

where $\alpha, v_x, v_y, v_z \in \mathbb{R}$ and the $1/2$ factor is only for convenience. Since $\mathrm{Tr}(\rho) = 1$ and the Pauli matrices are traceless, we have that $\alpha = 1$. Calculating the eigenvalues of ρ in terms of the coefficients v_i, we find that

$$\rho \geq 0 \iff 1 \geq \sqrt{v_x^2 + v_y^2 + v_z^2} = ||(v_x, v_y, v_z)||. \tag{1.5}$$

Therefore, by defining the three-dimensional real vector $\vec{v} = (v_x, v_y, v_z)$ we have that every qubit state ρ can be represented as

$$\rho = \frac{\mathbb{I} + \vec{v}\cdot\vec{\sigma}}{2}, \tag{1.6}$$

where $\vec{v}$ is a vector in the closed unit ball centred at the origin $\mathbb{B}[0,1] \subset \mathbb{R}^3$ and

$$\vec{v}\cdot\vec{\sigma} \equiv v_x\sigma_x + v_y\sigma_y + v_z\sigma_z = \begin{pmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{pmatrix}. \tag{1.7}$$

With this characterisation, we see that pure states (equivalently, projective density operators) correspond to unit vectors in the sphere $\mathbb{S}_2 \subset \mathbb{B}[0,1]$.

Understanding each point of $\mathbb{B}[0,1]$ together with its density operator representation,

$$\mathbb{R}^3 \supset \mathbb{B}[0,1] \ni \vec{v} \leftrightarrow \rho \in \mathcal{D}(\mathbb{C}^2), \tag{1.8}$$

we obtain the so-called **Bloch ball**.

Example 1. The **maximally mixed state** in dimension 2 is given by

$$\frac{1}{2}\mathbb{I} = \frac{\mathbb{I} + \vec{0}\cdot\vec{\sigma}}{2}, \tag{1.9}$$

thus lying in the centre of the Bloch ball. The pure states associated to the canonical basis vectors $|0\rangle, |1\rangle$ are represented by the operators

$$|0\rangle\langle 0| = \frac{\mathbb{I} + \sigma_z}{2} = \frac{\mathbb{I} + (0,0,1)\cdot\vec{\sigma}}{2} \tag{1.10a}$$

$$|1\rangle\langle 1| = \frac{\mathbb{I} - \sigma_z}{2} = \frac{\mathbb{I} - (0,0,1)\cdot\vec{\sigma}}{2}, \tag{1.10b}$$

whose Bloch vectors are located in opposite poles of the ball.

Similarly to the qubit case, 3-dimensional systems are called qutrit systems, and in general d-dimensional systems are qudit systems.

1.2 Quantum measurements

After defining the mathematical objects that represent systems and states, we now define how an observer interacts with the system, or, in other words, how a measurement takes place.

Definition 2. *A* ***quantum measurement*** *on $\mathcal{H}$ is an ordered set of operators $\boldsymbol{M} = (M_i)$ acting on $\mathcal{H}$ that satisfy the completeness relation*

$$\sum_i M_i^\dagger M_i = \mathbb{I}. \tag{1.11}$$

The index i refers to the outcome obtained in the measurement, and each outcome occurs with probability given by the ***Born rule****,*

$$\Pr(i) = Tr(M_i \rho M_i^\dagger), \tag{1.12}$$

for any given quantum state $\rho \in \mathcal{D}(\mathcal{H})$. The state of the system after the measurement is performed and outcome i is obtained is dictated by the ***Lüders rule****,*

$$\frac{M_i \rho M_i^\dagger}{Tr(M_i \rho M_i^\dagger)}. \tag{1.13}$$

It is easy to see that the completeness relation implies that the probabilities of the outcomes sum up to one.

Here in this thesis we will not be concerned with the post-measurement resulting state of a system. Our focus will rather lie in the probabilities of obtaining each of the outcomes generated by a measurement. In this case, the simplest mathematical object to model quantum measurements are positive-operator-valued measures.

Definition 3. *A* ***positive-operator-valued measure (POVM)*** *on $\mathcal{H}$ is an ordered set of operators $\mathbf{A} = (A_i), A_I \in \mathcal{L}(\mathcal{H})$, that are positive semidefinite and sum up to the identity,*

$$A_i \geq 0, \ \forall i \tag{1.14a}$$

$$\sum_i A_i = \mathbb{I}. \tag{1.14b}$$

The probability of obtaining outcome i when a POVM is performed on a system in the state ρ is

$$\Pr(i) = Tr(A_i \rho). \tag{1.15}$$

We see that every quantum measurement $\mathbf{M}$ determines a POVM $\mathbf{A}$ by defining

$$M_i^\dagger M_i =: A_i, \tag{1.16}$$

which ensures that the probability of obtaining each outcome i is preserved when we represent $\mathbf{M}$ by the POVM $\{A_i\}$,

$$\mathrm{Tr}(M_i \rho M_i^\dagger) = \mathrm{Tr}(M_i^\dagger M_i \rho) = \mathrm{Tr}(A_i \rho), \tag{1.17}$$

for any quantum state $\rho \in \mathcal{D}(\mathcal{H})$. Hence, throughout the text we will treat both POVMs and quantum measurements as equivalent objects.

A POVM $\mathbf{A} = (A_1, \dots, A_n)$ that has n elements is called ***n*-outcome**, and each operator A_i is said to be an **effect** of the POVM. The label of an outcome is attached to each measurement operator in an arbitrary way; in general, they have no special meaning. The space of n-outcome POVMs acting on $\mathcal{H}$ is denoted by $\mathcal{P}(\mathcal{H}, n)$. In the case where only the dimension d of the system is relevant, we write $\mathcal{P}(d, n)$.

A simple and important type of measurement occurs when the measurement operators are projectors.

Definition 4. *A* ***projective measurement*** $\mathbf{P} = (P_1, \dots, P_n)$ *is a measurement whose effects are projectors, i.e.,*

$$P_i^2 = P_i, \ \forall i \tag{1.18a}$$

$$P_i P_j = \delta_{ij} P_i, \ \forall i, j. \tag{1.18b}$$

Projective measurements have a simpler mathematical structure that makes them more tractable than general measurements from an analytical point of view. They are also very useful approximations for many practical implementations, composing valuable set-ups for experiments. Together, these two features motivate us to study projective measurements in detail and to search for connections with more complex objects. A significant part of this thesis relates to such problems.

Example 2. Important examples of measurements on $\mathbb{C}^2$ are the projective measurements $\mathbf{A}^{(x)}, \mathbf{A}^{(y)}, \mathbf{A}^{(z)}$ corresponding to the Pauli matrices in Eq. (1.3), with effects given by

$$A_{\pm}^{(x)} = \frac{\mathbb{I} \pm \sigma_x}{2}, \quad A_{\pm}^{(y)} = \frac{\mathbb{I} \pm \sigma_y}{2}, \quad A_{\pm}^{(z)} = \frac{\mathbb{I} \pm \sigma_z}{2}, \tag{1.19}$$

where the outcomes are labelled by + and -. More generally, given a real three-dimensional unit vector $\vec{v} = (v_x, v_y, v_z)$, we can define a dichotomic qubit measurement by

$$A_{\pm} = \frac{\mathbb{I} \pm \vec{v} \cdot \vec{\sigma}}{2}, \tag{1.20}$$

where the notation follows the one of Eq. (1.7).

This shows that the Bloch sphere is also useful for visualising qubit measurements, represented by antipodal vectors such as $(\pm 1, 0, 0)$ in the case of $\mathbf{A}^{(x)}$.

Example 3. The non-projective trine measurement $\mathbf{A}^{\text{trine}}$ [JKL+03] on $\mathbb{C}^2$ is described by

$$A_i^{\text{trine}} = \frac{\mathbb{I} + \vec{u}_i \cdot \vec{\sigma}}{3}, \tag{1.21}$$

where $\vec{u}_i$ are unit vectors forming an equilateral triangle in $\mathbb{R}^3$. Hence, $\mathbf{A}^{\text{trine}}$ cannot be represented in the Bloch ball, since its effects are not trace-1. However, representations in Bloch-like balls addressing operators with a given fixed trace (in this case 2/3), via the Pauli vectors $\vec{u}_i$, are also valuable.

1.3 Composite systems

In many occasions, we will be considering two different quantum systems $\mathcal{H}_A$ and $\mathcal{H}_B$. The adequate way to describe the composition of both systems is through the tensor product of Hilbert spaces, which is a Hilbert space itself.

Definition 5. *The Hilbert space of a composite system is given by the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ of the component systems $\mathcal{H}_A, \mathcal{H}_B$.*

In the product $\mathcal{H}_A \otimes \mathcal{H}_B$, the first factor will be said to be Alice's system and the second, Bob's system, following the usual terminology used in the literature.

At this point it is important to emphasise that the set $\mathcal{D}(\mathcal{H}_A) \otimes \mathcal{D}(\mathcal{H}_B)$, called the set of **product states**, is strictly contained in the set of arbitrary states of $\mathcal{H}_A \otimes \mathcal{H}_B$, $\mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. This is illustrated with the following example.

Example 4. Considering $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$, it is simple to check that the state[2] $\rho = (|00\rangle\langle 00| + |11\rangle\langle 11|)/2$ cannot be written as a product of states of each subsystem,

$$\rho \neq \rho^A \otimes \rho^B \tag{1.22}$$

for any $\rho^A \in \mathcal{D}(\mathcal{H}_A), \rho^B \in \mathcal{D}(\mathcal{H}_B)$.

Example 5. In fact, we can show even more: some states cannot be written not even as convex combination of product states. The **singlet state** on $\mathbb{C}^2 \otimes \mathbb{C}^2$ given by $\Psi^- = |\psi^-\rangle\langle\psi^-|$, where

$$|\psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}, \tag{1.23}$$

is an example of that, satisfying

$$\Psi^- \neq \sum_i p_i \rho_A^i \otimes \rho_B^i, \tag{1.24}$$

for any choice of local states ρ_A^i, ρ_B^i and weights $p_i \geq 0$ satisfying $\sum_i p_i = 1$.

The above example shows not only that $\mathcal{D}(\mathcal{H}_A) \otimes \mathcal{D}(\mathcal{H}_B) \subsetneq \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, but also the convex hull of the former set is strictly contained in the latter,

$$\text{conv}\left((\mathcal{D}(\mathcal{H}_A) \otimes \mathcal{D}(\mathcal{H}_B)\right) \subsetneq \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B). \tag{1.25}$$

The fact that not all quantum states are combinations of product states implies that is not always possible to completely describe a composite system in terms of each subsystem, *i.e.*, there are global aspects that cannot be seen at the subsystem level. However, we often want to describe solely the information available to a particular system. In these cases we use the **partial trace**.

Definition 6. *Let $\mathcal{H}_A \otimes \mathcal{H}_B$ be a composite Hilbert space. We define the partial trace (with relation to Bob's system $\mathcal{H}_B$) by $Tr_B : \mathrm{Herm}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathrm{Herm}L(\mathcal{H}_A)$ by*

$$Tr_B(M_A \otimes M_B) = Tr(M_B)M_A \tag{1.26}$$

for product operators and extend it to non-product operators by linearity. Analogously, we define the partial trace Tr_A in relation to Alice's subsystem $\mathcal{H}_A$.

[2] To ease the notation, we will write $|a\rangle \otimes |b\rangle \equiv |ab\rangle$.

Given a state of a composite system $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$, we can find the states that best describe each subsystem via partial trace. Such states are called the **reduced density operators**.

Definition 7. *Let ρ_{AB} be the state that describes the composite system $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$. Then the reduced density operators*

$$\rho_A = Tr_B(\rho_{AB}),\ \rho_B = Tr_A(\rho_{AB}) \tag{1.27}$$

describe the subsystem $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_B$, respectively.

The description provided by the partial trace is important when we are dealing with local measurements: suppose that Alice shares the state ρ_{AB} with Bob and perform a measurement $\{M_i\}$. Then $\mathrm{Tr}_B(\rho_{AB}) = \rho_A$ is the only state that satisfies

$$\mathrm{Tr}(M_a \otimes \mathbb{I}_B \rho_{AB}) = \mathrm{Tr}(M_a \rho_A) \tag{1.28}$$

for any a and $\{M_i\}$, where $\mathbb{I}_B$ is the identity in $\mathcal{H}_B$. That is, ρ_A is the only state that provides the correct probability of obtaining outcome a when we consider a measurement only on Alice's system. Thus, concerning local measurements only on Alice's side, to say that Alice shares state $\rho - AB$ with Bob is the same as to say that Alice holds the state $\mathrm{Tr}_B(\rho_{AB})$.

Example 6. Suppose that Alice and Bob share the singlet state $|\Psi^-\rangle\langle\Psi^-|$. Then Alice's reduced state is

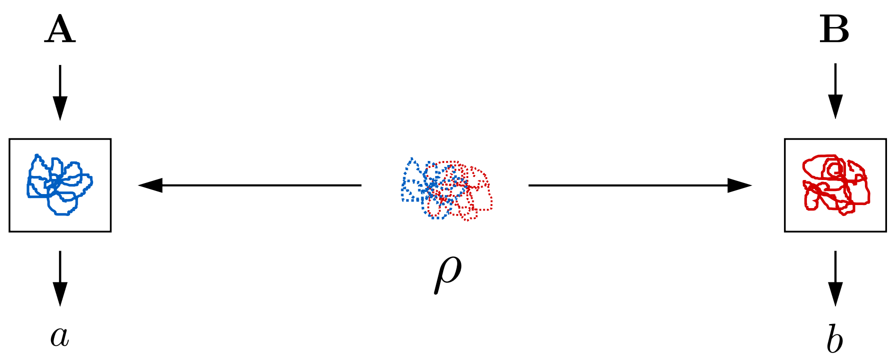
$$\rho_A = \mathrm{Tr}_B(\left|\Psi^-\right\rangle\left\langle\Psi^-\right|) = \frac{1}{2}\mathbb{I}_2. \tag{1.29}$$

Analogously to bipartite systems $\mathcal{H}_A \otimes \mathcal{H}_B$, we can define tripartite systems $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and, more generally, n-partite systems $\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$.

Chapter 2

Quantum correlations

One of the main advantages that quantum systems display in contrast with classical systems is the stronger type of correlations that distinct systems may share. We represent these correlations by families of joint probability distributions obtained by individually measuring each subsystem of a composite system, that is, applying local measurements.



$$p(a, b|\mathbf{A}, \mathbf{B}) = \mathrm{Tr}(\rho A_a \otimes B_b)$$

Figure 2.1: [Quantum correlations] A bipartite quantum system $\mathcal{H}_A \otimes \mathcal{H}_B$ is prepared in the state ρ. Local measurements are then performed in each subsystem, giving rise to outcomes according to the probabilities computed by the Borns rule. Correlations obtained in this way form the set of quantum correlations.

In the following, we characterise entanglement, Einstein-Podolsky-Rosen steering, and Bell nonlocality in terms of how we can model such distributions, and discuss forms of ensuring that a given distribution accepts each of these types of description.

2.1 Entanglement

Many properties of quantum states of composite systems can be defined according to the statistics obtained when local measurements are performed on each system. For instance, we say that a quantum state $\rho \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is **separable** if there is a set of labels Λ for two subsets of local quantum states $\{\rho_A^\lambda; \lambda \in \Lambda\} \subset \mathcal{D}(\mathcal{H}_A), \{\rho_B^\lambda; \lambda \in \Lambda\} \subset \mathcal{D}(\mathcal{H}_B)$ and some probability distribution π over it such that

$$\mathrm{Tr}(A_a \otimes B_b \rho) = \sum_{\lambda \in \Lambda} \pi(\lambda) \mathrm{Tr}(A_a \rho_A^\lambda) \mathrm{Tr}(B_b \rho_B^\lambda) \tag{2.1}$$

for any local measurements $\mathbf{A}, \mathbf{B}$. This is to say that if a variable λ is chosen according to the probability π and sent to both parties, they can prepare the local states $\rho_A^\lambda, \rho_B^\lambda$ and perform the local measurements $\mathbf{A}, \mathbf{B}$. Eq. (2.1) then says that the statistics obtained by this procedure match the statistics obtained by measuring the global state ρ.

The 4-tuple $(\Lambda, \pi, \{\rho_A^\lambda\}_\lambda, \{\rho_B^\lambda\}_\lambda)$ is called a **separable representation** of ρ, and λ can be interpreted as **shared randomness** between the parties, sometimes also called the **local hidden variable**.

Since Eq. (2.1) must hold for any pair of quantum measurements, the linearity of the trace implies that there must be an equivalence at the level of states, as depicted in the usual definition of separability below.

Definition 8. *A state ρ_{AB} acting on a composite system $\mathcal{H}_A \otimes \mathcal{H}_B$ is said to be separable if*

$$\rho_{AB} = \sum_\lambda p_\lambda \rho_A^\lambda \otimes \rho_B^\lambda \tag{2.2}$$

*for some $\{\rho_A^\lambda\} \subset \mathcal{D}(\mathcal{H}_A)$ and $\{\rho_B^\lambda\} \subset \mathcal{D}(\mathcal{H}_B)$, with $\sum_\lambda p_\lambda = 1$ and $p_\lambda \geq 0$. A state which is not separable is said to be **entangled**.*

Thus we see that the set of separable states is precisely the set $\mathrm{conv}(\mathcal{D}(\mathcal{H}_A) \otimes \mathcal{D}(\mathcal{H}_B))$ discussed in Section 1.3, which we denote by $\mathrm{Sep}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Example 5 then says that the singlet state Ψ^- is entangled.

Importantly, the shared variables λ allow us to factorize the joint probability in the left hand side of Eq. (2.1) as the convex combination of local probabilities that we see in the right side. Hence, we can say that a state is separable if and only if such local factorization is possible, given that both sides are using quantum local states and the Born rule to compute their probabilities.

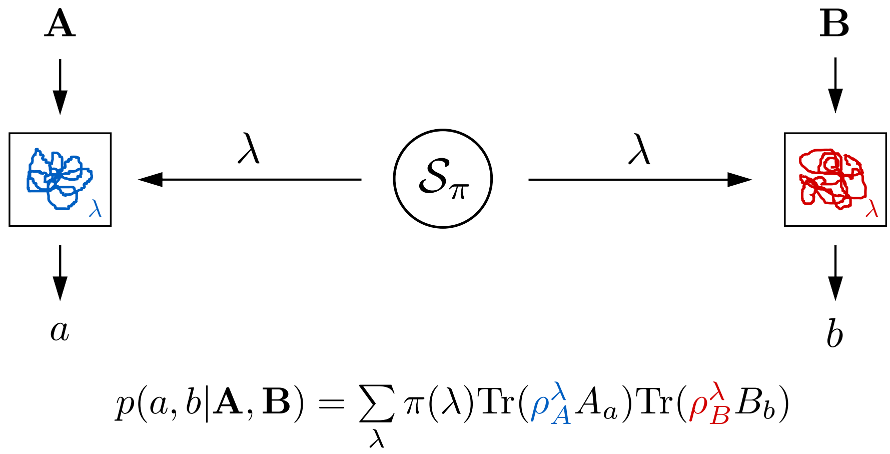


Figure 2.2: [Separable correlations] A random source sends a variable λ to each party according to the probability distribution π. Each party prepares a local quantum state depending on λ. Local measurements are then performed in each subsystem, giving rise to outcomes according to the probabilities computed by the Born rule. The correlations obtained in this way form the set of separable correlations.

It is straightforward to extend the concept of separability to a multipartite system $\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_k$, where the separable states are given by

$$\rho_{sep} = \sum_\lambda p_\lambda \rho_1^\lambda \otimes \ldots \otimes \ldots \rho_k^\lambda, \tag{2.3}$$

where (p_λ) are convex weights and $\rho_j^\lambda \in \mathcal{D}(\mathcal{H}_j)$ for each j. However, the idea of entanglement gets more complex, since we can have different subsets of systems correlated in different forms. For instance, for tripartite systems we can have pairs of systems entangled but uncorrelated with the third one, as denoted by states $\rho_A \otimes \rho_{BC}, \rho_B \otimes \rho_{AC}, \rho_C \otimes \rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$. Therefore a tripartite state ρ_{ABC} is said to be **genuinely tripartite entangled** if it satisfies

$$\rho_{ABC} \neq \sum_\lambda p_\lambda^A \rho_A^\lambda \otimes \rho_{BC}^\lambda + \sum_\lambda p_\lambda^B \rho_B^\lambda \otimes \rho_{AC}^\lambda + \sum_\lambda p_\lambda^C \rho_C^\lambda \otimes \rho_{AB}^\lambda, \tag{2.4}$$

where $p_\lambda^A, p_\lambda^B, p_\lambda^C \geq 0$ for all λ and $\sum_\lambda p_\lambda^A + p_\lambda^B + p_\lambda^C = 1$. Generalisations for a greater number of parties follow analogously.

For a review on entanglement we suggest Ref. [HHHH09].

2.1.1 Entanglement witnesses

The set of separable states $\mathrm{Sep}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is convex. The extremal points of this set are the product pure states $|\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B| \in \mathcal{D}(\mathcal{H}_A) \otimes \mathcal{D}(\mathcal{H}_B)$, and therefore to check whether a given state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is separable or not refers to searching for a separable representation of it. Since there is an infinite number of product states (in other words, the separable set is not a polytope), this is not a simple problem.

However, if ρ_{AB} is entangled, that is, if it belongs to the complement of the separable set, then due to the convexity and closeness of $\mathrm{Sep}(\mathcal{H}_A \otimes \mathcal{H}_B)$ there exists a hyperplane separating ρ_{AB} from this set [HHH96]. Such hyperplanes are related to the so-called **entanglement witnesses**. Entanglement witnesses are Hermitian operators W that have a positive overlap with all separable states of the system,

$$\mathrm{Tr}[W\rho_{AB}] \geq 0, \; \forall \rho_{AB} \in \mathrm{Sep}(\mathcal{H}_A \otimes \mathcal{H}_B). \tag{2.5}$$

Therefore, a negative overlap implies that the state is entangled, and we can use this as a strategy for entanglement detection.

In Ref. [HHH96] was proven the following theorem, based on the Hahn-Banach theorem [RS80], attesting the strength of entanglement witnesses.

Theorem 1. *For any entangled state, there exists an entanglement witness detecting it.*

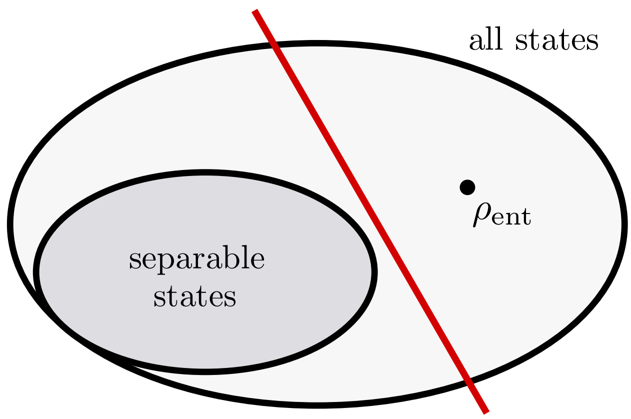


Figure 2.3: [Entanglement witness] There exists a hyperplane separating every convex, closed set from a point outside of it. Applying this reasoning to the set of separable states, for every entangled state ρ_{ent} there exists such a hyperplane (represented in red), corresponding to an entanglement witness.

Example 7. In the two-qubit space $\mathbb{C}^2 \otimes \mathbb{C}^2$, the **Werner states** ρ_W are given by combining the singlet with the maximally mixed state,

$$\rho_W(t) = t\Psi^- + (1-t)\frac{1}{4}\mathbb{I}_4, \tag{2.6}$$

for $t \in [0,1]$. As shown in Ref. [Wer89], the flip operator $F : |a\rangle\,|b\rangle \mapsto |b\rangle\,|a\rangle$ is the optimal entanglement witness for Werner states, in the sense that a negative overlap is not only a sufficient but also a necessary condition for entanglement,

$$\rho_W(t) \notin \mathrm{Sep}(\mathbb{C}^d \otimes \mathbb{C}^d) \iff \mathrm{Tr}[F\rho_W(t)] < 0. \tag{2.7}$$

With this we see that the range of parameters t for which $\rho_W(t)$ is entangled can be explicitly computed to be $t \in [0, 1/3]$.

As shown in Refs. [Bra05, EBA07], if the entanglement witness W has an additional appropriate structure, the absolute value of the negative overlap also provides a lower bound on the amount of entanglement of ρ_{AB}, *i.e.*,

$$E(\rho_{AB}) \geq -tr[W\rho_{AB}], \tag{2.8}$$

for some entanglement measure E. A particular case is the robustness-based quantifiers, which determine how much noise can be added to a quantum state before it becomes separable. We explore these concepts in the following example.

Example 8. Let $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be an entangled state. Consider now

$$\mu^* = \mu^*(\rho_{AB}) = \min\{\mu \geq 0;\ \frac{1}{1+\mu}\rho_{AB} + \frac{\mu}{1+\mu}\frac{\mathbb{I}_A}{d_A} \otimes \rho_B \in \mathrm{Sep}(\mathcal{H}_A \otimes \mathcal{H}_B)\}, \tag{2.9}$$

where $\rho_B = \mathrm{Tr}_A(\rho_{AB})$. That is, μ^* provides the minimum weight of the separable state $\mathbb{I}_A/d_A \otimes \rho_B$ such that its convex combination with ρ_{AB} is separable. The entanglement measure given by the coefficient $\mu^*/(1+\mu^*)$ is called **one-sided random robustness**.

Hence, for any entanglement witness W we have

$$\mathrm{Tr}\left[W\left(\frac{1}{1+\mu^*}\rho_{AB} + \frac{\mu^*}{1+\mu^*}\frac{\mathbb{I}_A}{d_A} \otimes \rho_B\right)\right] \geq 0, \tag{2.10}$$

which is equivalent to

$$\mu^*\mathrm{Tr}\left[W\frac{\mathbb{I}_A}{d_A} \otimes \rho_B\right] \geq -\mathrm{Tr}\left[W\rho_{AB}\right]. \tag{2.11}$$

If W has the additional property that

$$\mathrm{Tr}\left(W\frac{\mathbb{I}_A}{d_A}\otimes\rho_B\right)\leq 1, \tag{2.12}$$

then

$$\mu^*\geq -\mathrm{Tr}\left[W\rho_{AB}\right], \tag{2.13}$$

and we see that this entanglement witness provides a lower bound for the one-sided random robustness of ρ_{AB}.

In Section 4.1 we will discuss more on quantitative entanglement witnesses, including how to address these problems in practice.

2.1.2 The Peres-Horodecki criterion

Apart from entanglement witnesses, we can also use positive maps to derive sufficient conditions for a state to be entangled. Indeed, consider the map $T: M\mapsto M^T$ that takes a matrix to its transpose. Since the transposition preserves the spectrum of the operators associated to the matrix, we have that $M^T\geq 0$ for all $M\geq 0$. This implies that for every product state $\rho_A\otimes\rho_B$ we have that

$$\mathbb{I}_A\otimes T(\rho_A\otimes\rho_B):=\rho_A\otimes\rho_B^T \tag{2.14}$$

is also a valid state, since ρ_B^T is a valid state, and in particular it is positive semi-definite. Similarly to the partial trace, we call $\mathbb{I}_A\otimes T\in\mathcal{L}\left(\mathrm{Herm}(\mathcal{H}_A\otimes\mathcal{H}_B)\right)$ the **partial transpose** map.

Hence we can formulate the Peres-Horodecki criterion as the following [Per96, HHH96].

Theorem 2 (Peres-Horodecki criterion)**.** *Let $\rho_{AB}\in\mathcal{D}(\mathcal{H}_A\otimes\mathcal{H}_B)$. If $\mathbb{I}_A\otimes T(\rho_{AB})\not\geq 0$, then ρ_{AB} is entangled.*

The Peres-Horodecki criterion applies to composite quantum systems of any dimension, but for the systems $\mathbb{C}^2\otimes\mathbb{C}^2$ and $\mathbb{C}^2\otimes\mathbb{C}^3$ it is not only sufficient, but also a necessary condition for entanglement [Per96].

The entangled states ρ_{AB} identified by the above criterion, *i.e.*, such that $\mathbb{I}_A\otimes T(\rho_A B)\not\geq 0$, posses at least one negative eigenvalue. We can use this fact to define an entanglement measure: the **negativity** of ρ_{AB} is given by

$$N(\rho_{AB})=\sum_i\frac{|\lambda_i|-\lambda_i}{2}, \tag{2.15}$$

where the sum is over the eigenvalues λ_i of $\mathbb{I}_A\otimes T(\rho_{AB})$. Thus we see that the negativity of a state equals the sum of the negative eigenvalues of its partial transpose, and that $N(\rho_{AB})>0$ only if ρ_{AB} is entangled.

2.2 Einstein-Podolsky-Rosen steering

A natural way to weaken the notion of separability in Eq. (2.1) is to allow one of the systems, say Alice's, to calculate its probabilities using an arbitrary response function $f_A(a|\mathbf{A},\lambda)$, still depending on the measurement $\mathbf{A}$ being performed and the shared variables λ. This leads to the concept of **Einstein-Podolski-Rosen (EPR) steering** [WJD07].

Definition 9. *A state $\rho \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is* ***unsteerable*** *if there are variables $\lambda \in \Lambda$ distributed according to a probability π, a response function f_A and quantum states $\rho^\lambda \in \mathcal{D}(\mathcal{H}_B)$ such that*

$$tr(A_a \otimes B_b \rho) = \sum_\lambda \pi(\lambda) f_A(a|\boldsymbol{A},\lambda) tr(B_b \rho^\lambda) \tag{2.16}$$

for any measurements $\boldsymbol{A},\boldsymbol{B}$. Otherwise, the state is said to be ***steerable****.*

An equivalent formulation of Eq. (2.16) is

$$\sigma_{a|\mathbf{A}} = \sum_\lambda \pi(\lambda) f_A(a|\mathbf{A},\lambda)\rho^\lambda, \tag{2.17}$$

where $\sigma_{a|\mathbf{A}} = \mathrm{Tr}_A(A_a \otimes \mathbb{I}_{\mathbb{B}}\rho)$. The states ρ^λ are called **local hidden states**. If Eq. (2.17) holds, then we recover the definition and ρ is unsteerable.

If ρ is unsteerable, the 4-tuple $(\Lambda, \pi, f_A, \{\rho_\lambda\}^\lambda)$ compose a **local hidden state (LHS) model** for ρ. The set $\{\sigma_{a|\mathbf{A}}\}$ generated by ρ_{AB} and a particular choice of measurements $\{\mathbf{A}\}$ is called an **assemblage**. Hence ρ_{AB} is unsteerable if and only if every assemblage generated by it admits an LHS model.

A useful interpretation of the idea of steering takes place in a scenario where Alice wants to convince Bob that they are sharing a global state ρ_{AB} by sending him $\sigma_{a|\mathbf{A}}$, supposedly the reduced state of ρ_{AB} after Alice performed the local measurement $\mathbf{A}$. If ρ_{AB} is unsteerable, Eq. (2.17) tells us that Alice can simply prepare with probability $\pi(\lambda) \times f(a|\mathbf{A},\lambda)$ a completely uncorrelated hidden state ρ^λ and send to Bob, who will never be able to tell the difference, even if ρ_{AB} was entangled.

Example 9. In Ref. [Wer89], it was shown that for two-qubit Werner states $\rho_W(t) \in \mathcal{D}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, the assemblage generated by all projective measurements admits an LHS model given by

$$\Lambda = \mathbb{S}_2 \tag{2.18a}$$

$$\pi \equiv \mathrm{Unif}(\mathbb{S}_2) \tag{2.18b}$$

$$\rho^\lambda = \frac{\mathbb{I}_2 + \vec{\lambda}\cdot\vec{\sigma}}{2} \tag{2.18c}$$

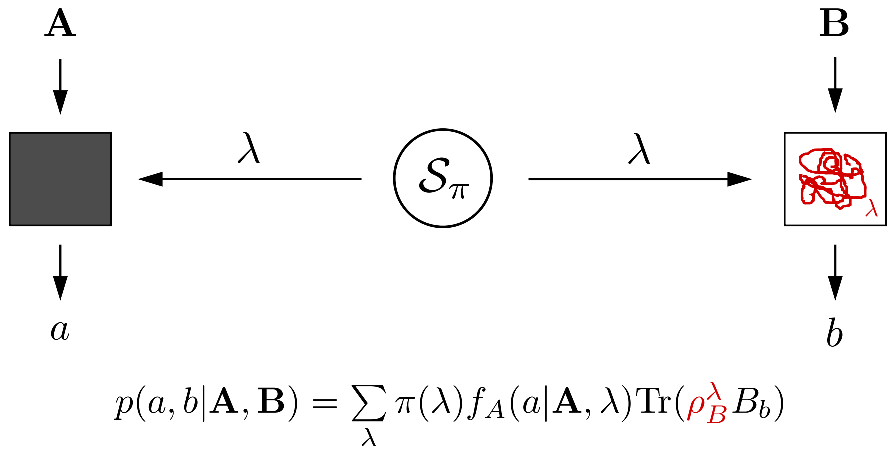


Figure 2.4: [Unsteerable correlations] A random source sends a variable λ to each party according to the probability distribution π. Alice's subsystem can be seen as a black box, that outputs an outcome a according to an arbitrary response function f_A, depending on the local measurement $\mathbf{A}$ and the shared variable λ. Bob prepares a local quantum state depending on λ, and obtains an outcome by implementing its local measurement (*i.e.*, according to the Born rule). The correlations obtained in this way form the set of unsteerable correlations.

$$f_A(a|\mathbf{A},\lambda) = \begin{cases} 1 \text{ if } \mathrm{Tr}(\rho^\lambda A_a) = \min_i\{\mathrm{Tr}(\rho^\lambda A_i)\} \\ 0 \text{ else} \end{cases} , \tag{2.18d}$$

for the range $t \in [0, 1/2]$, where $\mathbb{S}_2$ is the unit sphere in $\mathbb{R}^3$ and Unif denotes the uniform probability distribution. In this case, $\rho_W(t)$ is **projective-unsteerable**, or, equivalently, it admits a **projective LHS model**. The model was further improved in Ref. [Bar02], where another response function f_A was used to construct an LHS model for any general measurement for $t \in [0, 5/12]$.

Notice that the projective character of the model restrains only the uncharacterised party (Alice). Following Eq. (2.17), Bob is free to implement measurements of any type on its subsystem, independently of the type (or existence) of the LHS model.

The following theorem provides a method for, starting from a projective LHS model for a given state ρ_{AB}, constructing a general LHS model for a different state ρ'_{AB}, related to ρ_{AB} [HQBB13].

Theorem 3. *Let $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a projective-unsteerable state. Then the state*

$$\rho'_{AB} = \frac{1}{d_A}\rho_{AB} + \frac{d_A - 1}{d_A}\gamma_A \otimes \rho_B \tag{2.19}$$

has an LHS model for general POVMs, where d_A is dimension of $\mathcal{H}_A$, $\gamma_A \in \mathcal{D}(\mathcal{H}_A)$ is an arbitrary state and $\rho_B = Tr_A(\rho_{AB})$. [1]

One of the advantages of defining entanglement and EPR steering from the aspect of correlations is that it makes clear that every separable state is unsteerable, since the Born rule used in a separable representation is a particular choice for the general response function f_A allowed in an LHS model.

The analogous idea to entanglement witnesses in the context of EPR steering is provided by steering inequalities. Since this concept is out of the scope of this thesis, we refer to Ref. [SNC14] for more information.

2.3 Quantum Bell nonlocality

Making a further relaxation in the definition of unsteerability, allowing two completely arbitrary response functions in the modelling of the correlations $\mathrm{Tr}(A_a \otimes B_b \rho_{AB})$ generated by ρ_{AB} and general quantum measurements, we obtain the definition of **Bell locality** for quantum systems.

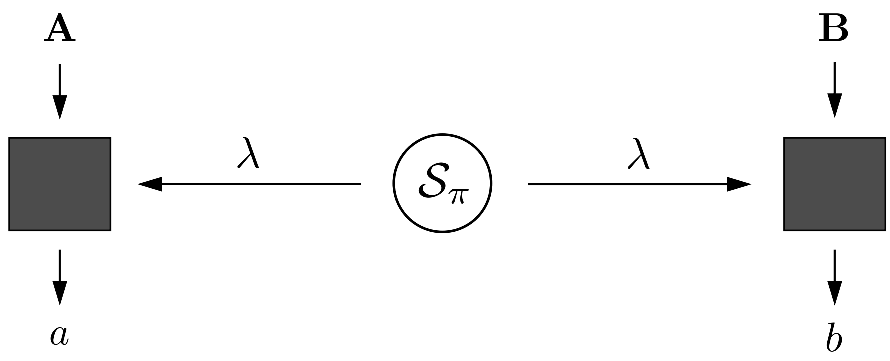
Definition 10. *A state ρ_{AB} is **local** if there are variables $\lambda \in \Lambda$ distributed according to a probability π and response functions p_A, p_B such that*

$$tr(A_a \otimes B_b \rho_{AB}) = \sum_{\lambda} \pi(\lambda) f_A(a|\mathbf{A}, \lambda) f_B(b|\mathbf{B}, \lambda) \tag{2.20}$$

*for any measurements $\mathbf{A}, \mathbf{B}$. The 4-tuple (Λ, π, f_A, f_B) compose a **local-hidden-variable (LHV) model** for ρ. In the case where there is no such 4-tuple, the state is said to be **nonlocal**.*

Example 10. The Werner states $\rho_W(t)$ were shown [AGT06] to admit an LHV model for projective measurements for the range $t \in [0, 1/K_G(3)]$, where $K_G(3)$ denotes the Grothendieck's constant of order 3. Thus for $t \leq 1/K_G(3)$ we say that $\rho_W(t)$ is **projective-local**, or equivalently that it admits a **projective LHV model**. (The best lower upper bound for $1/K_G(3)$ is currently 0.68 [HQV⁺17].) One of the main results that will be presented in this thesis is that we can use this projective LHV model to show that $\rho_W(t)$ is genuinely local (*i.e.*, local for general POVMs) for the range $t \leq 0.4533$ (see Section 7.2).

[1] In fact, Ref. [HQBB13] presented a more general construction which works for local-hidden-variable models. However, we will only need the weaker result stated, which is implicit in their construction.



$$p(a, b|\mathbf{A}, \mathbf{B}) = \sum_\lambda \pi(\lambda) f_A(a|\mathbf{A}, \lambda) f_B(b|\mathbf{B}, \lambda)$$

Figure 2.5: [Local correlations] A random source sends a variable λ to each party according to the probability distribution π. Each subsystem can be seen as a black box, that outputs an outcome according to an arbitrary response function, depending on the local measurement and the shared variable λ. The correlations obtained in this way form the set of local correlations.

Separable representations, LHS models and LHV models can be seen as three different ways of modelling the statistics of a global quantum state of a composite system, depending on how "quantum" the response functions are. Although separable states are unsteerable and unsteerable states are local, none of these implications are equivalences. Namely, there exist states that are entangled but unsteerable, and states that are steerable but local, as illustrated by Werner states (see Fig. 2.6).

In the case where the response functions are not characterized (as for LHS and LHV models) we say that the statistics are (partially or fully) **device-independent**, in the sense that we are not assuming they were provided by a measurement on a quantum system. Device-independence is a powerful paradigm that allows one to interpret experimental data apart from its (possibly imprecise) quantum theoretical formulation, as well as to move beyond the quantum scenario and investigate more general probability theories.

Similarly to EPR steering and separability, the nonlocality of a quantum state can be witnessed by the violation of so-called **Bell inequalities** [BCP+14]. Those are inequalities related to hyperplanes in the space of **behaviours**, families of joint probability distributions in a fixed scenario determined by the number of parties, the number of measurements performed by each party, and the number of outcomes of these measurements. A Bell inequality then represents a condition satisfied by all local behaviours, and only violated by nonlocal ones.

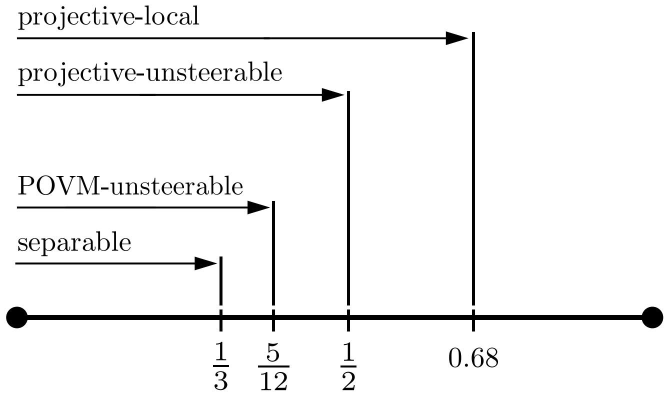


Figure 2.6: The optimal parameters for two-qubit Werner states currently known, concerning separability, POVM- and projective-unsteerability, and projective-locality. In Chapter 7 we update this figure (see Fig. 7.7).

For a review on Bell nonlocality we suggest Ref. [BCP+14], and for a review on LHV models we recommend Ref. [ADA14].

Chapter 3

Joint measurability

One of the most remarkable features of quantum theory is the existence of incompatible measurements. However, measurement incompatibility can present itself in various different aspects, and many different notions were created to describe this idea.

For instance, the famous Heisenberg's uncertainty relation was formulated initially for projective measurements, based on the fact that in general they do not commute [Hei27]. Later, in the context of sequential measurements, was studied the idea of measurements that disturb each other, generalising the case of non-commutative measurements [HW10]. A further generalisation takes place in the concept of joint measurability, centred in the fact that certain sets of measurements can be seen as a single measurement, therefore capturing the spirit of simultaneity [HMZ16]. Still, each of these phenomena can be interpreted as particular cases of the concept of co-existence of quantum measurements [BGL97]. In general, these incompatibility relations are inequivalent, each one being strictly stronger than the next one, with respect to the order we introduced them.

Here we will focus on joint measurability, partially due to the relevance of its formulation, and partially due to its important connection to EPR steering.

Definition 11. *A tuple*[1] *of d-dimensional, n-outcome measurements $\mathcal{A} = [\boldsymbol{A}^{(1)}, \ldots, \boldsymbol{A}^{(m)}] \in \mathcal{P}(d,n)^{\times m}$ is jointly measurable if there exists an n^m-outcome measurement $\boldsymbol{M} = (M_{a_1 \ldots a_m}) \in \mathcal{P}(d, n^m)$, where $a_i = 1, \ldots, n$ for each $i = 1, \ldots, m$, such that the statistics of the measurements in $\mathcal{A}$ can be recovered via coarse-graining upon having*

[1] We refer to tuples (ordered sets) of measurements in order to determine an unambiguous correspondence between the l-th POVM of the tuple and the l-th marginal of the joint measurement.

*implemented the single measurement **M**, that is,*

$$\sum_{r\neq l}\sum_{a_r} Tr(M_{a_1\ldots(a_l=i)\ldots a_m}\rho) = Tr(A_i^{(l)}\rho), \tag{3.1}$$

*for any $l = 1,\ldots,m$, any $i = 1,\ldots,n$, and any quantum state ρ. In this case, **M** is called a **joint measurement** for $\mathcal{A}$.*

Since Eq. (3.1) should hold regardless of the quantum state being measured, we obtain an equivalent description in terms of the effects of the measurements,

$$\sum_{r\neq l}\sum_{a_r} M_{a_1\ldots(a_l=i)\ldots a_m} = A_i^{(l)}. \tag{3.2}$$

For the joint measurability of two POVMs $[\mathbf{A}^{(1)}, \mathbf{A}^{(2)}]$, there is a very visual way of interpreting the joint POVM **M**. If we organise the effects of **M** in an $n \times n$ table, where the effect $M_{a_1 a_2}$ occupies position (a_1, a_2), then the marginals correspond to summing over the rows and columns, and Eq. (3.2) can be represented by

$$\begin{array}{ccc|c} M_{11} & \cdots & M_{1n} & A_1^{(1)} \\ \vdots & \ddots & \vdots & \vdots \\ M_{n1} & \cdots & M_{nn} & A_n^{(1)} \\ \hline A_1^{(2)} & \cdots & A_n^{(2)} & \end{array}. \tag{3.3}$$

3.1 White-noise robustness and the depolarising map

By applying the **depolarising map**

$$\Phi_t : A \mapsto tA + (1-t)\frac{\mathrm{Tr}(A)}{d}\mathbb{I}_d \tag{3.4}$$

for some $t \in [0,1]$, to each effect of **A**, we obtain a depolarised version of the measurement,

$$\Phi_t(\mathbf{A}) := (\Phi_t(A_1), \ldots, \Phi_t(A_n)). \tag{3.5}$$

The parameter t is called the **visibility** of **A** in $\Phi_t(\mathbf{A})$. The depolarising map can be physically interpreted as the presence of white noise in the implementation of **A**, and therefore its consideration is natural from an experimental point of view.

Notice that the completely depolarised version of $\mathbf{A}$,

$$\Phi_0(\mathbf{A}) = \left(\frac{\mathrm{Tr}(A_1)}{d} \mathbb{I}_d, \ldots, \frac{\mathrm{Tr}(A_n)}{d} \mathbb{I}_d \right), \tag{3.6}$$

have all effects proportional to the identity. Measurements with this property are called **trivial POVMs**, and it is straightforward to check that any tuple of trivial POVMs is jointly measurable. This leads us to the study of the **white-noite robustness** of the incompatibility of a given tuple of POVMs $\mathcal{A}$. By depolarising each POVM in a tuple $\mathcal{A} = [\mathbf{A}^{(j)}]$ we can define its depolarised version,

$$\Phi_t(\mathcal{A}) := [\Phi_t(\mathbf{A}^{(j)})], \tag{3.7}$$

and then its white-noise robustness regarding joint measurability,[2]

$$t_{\mathrm{JM}}^{\mathcal{A}} = \max\{t;\ \Phi_t(\mathcal{A}) \text{ is jointly measurable}\}, \tag{3.8}$$

that is, the critical noise parameter that turns the tuple jointly measurable.

At the level of quantum states, notice that the completely depolarised version of any state ρ is the maximally mixed state,

$$\Phi_0(\rho) = \frac{1}{d} \mathbb{I}_d. \tag{3.9}$$

Since this state is separable (hence unsteerable and local), we can define the white-noise robustness of ρ regarding entanglement, steerability and nonlocality analogously to Eq. (3.8).

3.2 Joint measurability and EPR steering

Joint measurability represents a sufficient condition for locality. Indeed, if one of the parties of a bipartite system has only one option of measurement to implement, then the resulting statistics obtained by locally measuring the global system will always admit a description in terms of local hidden variables [BCP+14]. Therefore, if all measurements available for one of the parties compose a jointly measurable tuple, then they can be interpreted as a single measurement, and the statistics obtained will be local. Recently, it was proved that the converse does not hold, meaning that there are sets of measurements that, regardless of not being jointly measurable, cannot violate any Bell inequality [EV17].

[2] The fact that the set of jointly measurable tuples is closed ensures that $t_{\mathrm{JM}}^{\mathcal{A}}$ is a maximum, and not just a supremum.

In spite of Bell noncality being too broad to imply that the involved POVMs are not jointly measurable, we can address the same question to the particular class of nonlocal correlations described in EPR steering. In Refs. [QVB14, UMG14] it was shown the following close connection between joint measurability and EPR steering.

Theorem 4. *Let* $\mathcal{A} = [\boldsymbol{A}] \in \mathcal{P}(\mathcal{H}_A, n)^{\times m}$ *by an arbitrary tuple of measurements and* $\Psi^+ \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ *be the maximally entangled state of this system. Then the assemblage* $\{\sigma_{a|\boldsymbol{A}} = Tr_A(A_a \otimes \mathbb{I}_B \Psi^+)\}$ *admits an LHS model if and only if* $\mathcal{A}$ *is jointly measurable.*

The theorem provides a recipe to construct a joint measurement for the tuple starting from the LHV model for the assemblage, and vice-versa. Therefore, when we investigate the most incompatible tuples of measurements (in the sense of joint measurability) in Section 6.3, we will also be studying the "most steerable" assemblages, since these objects are in one-to-one correspondence.

Chapter 4

Semidefinite programming

Semidefinite programming (SDP) is a class of convex optimisation problems particularly useful in the context of quantum theory [Wat]. On the one hand, both quantum states and measurements are associated to positive semi-definite matrices, and many problems of interest regarding these objects can be solved efficiently via SDP [VB96]. On the other, it brings techniques to analytically investigate certain questions, in particular the duality aspect of some pairs of problems. In this technical chapter, we overview well-known results on the SDP approach to some specific problems that will be later addressed.

One possible way of defining an SDP is the following.

Definition 12. *Let $\mathcal{H}_1, \mathcal{H}_2$ be two complex Euclidean spaces. A **semidefinite program** is a triple (Φ, A, B), where $\Phi : \mathrm{Herm}(\mathcal{H}_1) \to \mathrm{Herm}(\mathcal{H}_2)$, $A \in \mathrm{Herm}(\mathcal{H}_1)$ and $B \in \mathrm{Herm}(\mathcal{H}_2)$, associated to a **primal optimisation problem***

$$
\begin{aligned}
(P): \quad & \max_{X \in \mathrm{Herm}(\mathcal{H}_1)} && Tr(AX) \\
& \text{s.t.} && \Phi(X) = B, \\
& && X \geq 0,
\end{aligned}
\tag{4.1}
$$

*and a **dual optimisation problem***

$$
\begin{aligned}
(D): \quad & \min_{Y \in \mathrm{Herm}(\mathcal{H}_2)} && Tr(BY) \\
& \text{s.t.} && \Phi^\dagger(Y) \geq A,
\end{aligned}
\tag{4.2}
$$

where $\Phi^\dagger : \mathrm{Herm}(\mathcal{H}_2) \to \mathrm{Herm}(\mathcal{H}_1)$ is the dual map of Φ.

Other equivalent forms can be derived from the above definition [], and we often need to implement small modifications in the way we phrase a problem in order to meet the appropriate description aforementioned. In practice, the constraints represented by Φ, A, B refer to any linear or positive semi-definite restriction imposed by our problem, while the objective function $\mathrm{Tr}(AX)$ of (P) (and analogously for (D)) is simply any functional being applied to X, which may encode many different free variables. Also, whenever there exists an operator X satisfying the constraints $\Phi(X) = B, X \geq 0$ of (P) we say that (P) is **feasible**, and the analogous holds for (D).

The relation between the primal and dual problems (P) and (D) rely on the fact that the dual is constructed from the primal with the help of Lagrangian multipliers [VB96], in such a way that the optimal value β of (D) is an upper bound to the optimal value α of (P)[1]

$$\alpha = \mathrm{Tr}[AX^*] \leq \mathrm{Tr}[\Phi^\dagger(Y^*)X^*] = \mathrm{Tr}[Y^*\Phi(X^*)] = \mathrm{Tr}[Y^*B] = \beta, \tag{4.3}$$

where X^* and Y^* are the primal and dual optimal arguments that achieve the optimal value of each corresponding problem.

Relation (4.3) is known as **weak duality**. However it is common and much more useful when we have **strong duality**, in which case $\alpha = \beta$. The following result says that strong duality is attained whenever anyof the SDP problems are strictly feasible.

Theorem 5 (Slater's condition)**.** *Let α and β be the optimal values for the SDPs (P) and (D) of Problems (4.1) and (4.2), respectively. Then*

(i) If there exists $X \in \mathrm{Herm}(\mathcal{H}_1)$ such that $\Phi(X) = B$ and $X > 0$, then there exists $Y^* \in \mathrm{Herm}(\mathcal{H}_2)$ such that $\Phi^\dagger(Y^*) \geq A$, and $Tr(BY^*) = \beta = \alpha$.

(ii) If there exists $Y \in \mathrm{Herm}(\mathcal{H}_2)$ such that $\Phi^\dagger(Y) > A$, then there exists $X^* \in \mathrm{Herm}(\mathcal{H}_1)$ such that $\Phi(X^*) = B$, $X^* \geq 0$, and $Tr(AX^*) = \alpha = \beta$.

In the next sections we will present some concepts of the previous chapters that can be calculated via SDP. All the SDPs appearing there posses the strong duality property.

4.1 Entanglement witnesses as an SDP

As discussed in Subsection 2.1.1, entanglement witnesses are operators W such that $\mathrm{Tr}(W\rho_{AB}) < 0$ certifies that the quantum state ρ_{AB} is entangled. For a fixed bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$, we denote by $\mathcal{W}$ the set of entanglement witnesses for that system.

[1]Here we must consider the possibility of (P) being unbounded, in which case the max does not exist and we define $\alpha = \infty$; similarly, we allow β to be $-\infty$. By convention, we define $\alpha = -\infty$ whenever (P) is infeasible, and $\beta = \infty$ whenever (D) is infeasible.

In Ref. [Bra05] it was shown that many entanglement quantifiers can be written as

$$E(\rho) = -\min_{W \in \mathcal{W}_E} \operatorname{Tr}(W\rho), \tag{4.4}$$

where $\mathcal{W}_E$ is the set of entanglement witnesses satisfying some additional properties, dependent on the choice of quantifier $E(\cdot)$.

In some cases, the above minimisation problem (4.4) is natively an SDP, or can be approximated by one. This is the case, for instance, for robustness-based quantifiers, such as the one-sided random robustness presented in Example 8. The corresponding set $\mathcal{W}_{E_{1\mathrm{SRR}}}$ to this quantifier, which also depends on the state ρ_{AB} under consideration, is

$$\mathcal{W}_{E_{1\mathrm{SRR}}} = \{W \in \mathcal{W};\ \operatorname{Tr}[W(\mathbb{I}/d_A \otimes \rho_B)] \leq 1\}. \tag{4.5}$$

The main point is that solving (4.4) for a given state provides the optimal entanglement witness in that class for that state, according to the given quantifier.

In the present study we have chosen 7 such quantifiers, summarised in Table 4.1.

Note that in general the condition in the primal problem $\sigma_{AB} \in \operatorname{Sep}(\mathcal{H}_A \otimes \mathcal{H}_B)$, that σ_{AB} should be a separable state, and similarly the condition in the dual problem $W \in \mathcal{W}$, that W should be a valid entanglement witness, are complicated constraints. More precisely, there is no efficient way to strictly enforce that a state is separable or that an operator is an entanglement witness, apart from the simple case of qubit-qubit or qubit-qutrit systems where we can use the Peres-Horodecki criterion (Theorem 2). In these cases separability is equivalent to positivity under partial transposition (PPT), and the above optimisation problems become SDPs, which are then readily solved using standard software packages[2].

In the general case, instead of calculating directly a given entanglement quantifier, we find lower bounds by relaxing the separability constraint. In particular, we can relax this to PPT (positive partial trace), or more generally to the set of states that admit a k-symmetric PPT extension [DPS02, DPS04]. Both constraints can be implemented efficiently via SDP, and are weaker than imposing separability. Hence in both cases we can efficiently obtain lower bounds on the quantifiers. Consequently, the entanglement witness that is obtained from the dual, which is still a valid entanglement witness, is not only positive on separable states, but also positive on all PPT states, or on all states that admit an extension, respectively.

[2] All the SDPs implemented in this work used MATLAB and the packages CVX [GB14, GB08] and QETLAB [Joh15], or PYTHON and the interface PICOS.

Quantifier	Given by	Primal	Dual
One-sided random robustness	$E_{1SRR} = \frac{\mu^*}{1+\mu^*}$	$\mu^* = \min \mu$ s.t $\mu \geq 0$, $(\rho_{AB} + \mu \mathbb{I}_A/d_A \otimes \rho_B) \in \text{Sep}$	$\mu^* = -\min \text{Tr}[W\rho_{AB}]$ s.t. $W \in \mathcal{W}$, $\text{Tr}[W(\mathbb{I}_A/d_A \otimes \rho_B)] \leq 1$
One-sided fixed robustness	$E_{1SFR} = \frac{\mu^*}{1+\mu^*}$	$\mu^* = \min \mu$ s.t. $\mu \geq 0$, $(\rho_{AB} + \mu \|0\rangle\langle 0\| \otimes \rho_B) \in \text{Sep}$	$\mu^* = -\min \text{Tr}[W\rho_{AB}]$ s.t. $W \in \mathcal{W}$, $\text{Tr}[W(\|0\rangle\langle 0\| \otimes \rho_B)] \leq 1$
One-sided generalised robustness	$E_{1SGR} = \frac{\mu^*}{1+\mu^*}$	$\mu^* = \min \text{Tr}[\omega_A]$ s.t. $\omega_A \geq 0$, $(\rho_{AB} + \omega_A \otimes \rho_B) \in \text{Sep}$	$\mu^* = -\min \text{Tr}[W\rho_{AB}]$ s.t. $W \in \mathcal{W}$, $\text{Tr}_B[W(\mathbb{I}_A \otimes \rho_B)] \leq \mathbb{I}_A$
Two-qubit random robustness	$E_{RR} = \frac{\mu^*}{1+\mu^*}$	$\mu^* = \min \mu$ s.t. $\mu \geq 0$, $(\rho_{AB} + \mu \mathbb{I}_{AB}/4) \in \text{Sep}$	$\mu^* = -\min \text{Tr}[W\rho_{AB}]$ s.t. $W \in \mathcal{W}$, $\text{Tr}[W] \leq 4$
Generalised Robustness	$E_{GR} = \frac{\mu^*}{1+\mu^*}$	$\mu^* = \min \text{Tr}[\omega_{AB}]$ s.t. $\omega_{AB} \geq 0$, $(\rho_{AB} + \omega_{AB}) \in \text{Sep}$	$\mu^* = -\min \text{Tr}[W\rho_{AB}]$ s.t. $W \in \mathcal{W}$, $W \leq \mathbb{I}_{AB}$
Best separable approximation	$E_{BSA} = \mu^*$	$\mu^* = 1 - \max \text{Tr}[\omega_{AB}]$ s.t. $\omega_{AB} \in \text{Sep}$, $\rho_{AB} \geq \omega_{AB}$	$\mu^* = -\min \text{Tr}[W\rho_{AB}]$ s.t. $W \in \mathcal{W}$, $-W \leq \mathbb{I}_{AB}$
Negativity	$E_{Neg} = \mu^*$	$\mu^* = \min \text{Tr}[\omega_{AB}]$ s.t. $\omega_{AB} \geq 0$, $\rho_{AB}^{\Gamma_A} + \omega_{AB} \geq 0$	$\mu^* = -\min \text{Tr}[W\rho_{AB}]$ s.t. $W^{\Gamma_A} \geq 0$, $W^{\Gamma_A} \leq \mathbb{I}_{AB}$

Table 4.1: The seven quantifiers of entanglement used in this thesis. All quantifiers are such that $E = 0$ for separable states, while $E > 0$ quantifies entanglement. The first five are robustness-type quantifiers, with the specific definition described in the primal representation. The last two are the best separable approximation – the minimal admixture of entangled state necessary in any decomposition of the state – and the negativity.

4.1.1 Genuine multipartite entanglement

In Ref. [JMG11], it was shown that a given state ρ can be proven to be genuinely multipartite entangled if there exists a witness W which is **fully decomposable**. W is fully decomposable if there exists positive semidefinite operators P_M and Q_M such that $W = P_M + Q_M^{T_M}$, for all partitions M of the system, where T_M denotes partial transposition with respect to M.

This test can be implemented as the following SDP:

$$\begin{aligned} \text{given} \;\; & \rho \\ \min_{W} \;\; & \mathrm{Tr}[W\rho] \\ \text{s.t.} \;\; & \mathrm{Tr}[W] = 1, \\ & W = P_M + Q_M^{T_M},\ \forall M, \\ & P_M \geq 0,\ \forall M, \\ & Q_M \geq 0,\ \forall M. \end{aligned} \tag{4.6}$$

4.2 LHS models for finitely many measurements as an SDP

Although deciding whether a given bipartite state is unsteerable or not (regarding all quantum measurements) is a difficult problem, if we narrow the question to address only a finite set of measurements $\{\mathbf{A}^{(l)}\}$ with a finite number of outcomes, then we can answer it be means of SDP.

In this case we have exactly n^m equations

$$\mathrm{Tr}_A(A_a^{(l)} \otimes \mathbb{I}_B \rho) = \sum_{\lambda} \pi(\lambda) f_A(a|\mathbf{A}^{(l)}, \lambda)\rho^{\lambda}, \tag{4.7}$$

indexed by the outcomes $i = 1, \ldots, n$ of each measurement $\mathbf{A}^{(l)}$, with $l = 1, \ldots, m$. Recall that we assume that π is a probability distribution over $\Lambda = \{\lambda\}$, $f_A(a|l, \lambda)$ is a valid response function, and ρ^{λ} are quantum states.

Without loss of generality, we can assume that the hidden variables determine completely the outcome of each measurement, thus requiring only deterministic response functions in the model but distributed according to a different probability π',

$$\mathrm{Tr}_A(A_a^{(l)} \otimes \mathbb{I}_B \rho) = \sum_{\lambda} \pi'(\lambda) D(a|\mathbf{A}^{(l)}, \lambda)\rho^{\lambda}, \tag{4.8}$$

where $\lambda = (\lambda_1, \ldots, \lambda_m)$, and $D(i|\mathbf{A}^{(l)}, \lambda) = \delta_{i,\lambda_l}$ are deterministic response functions. Notice that there are only n^m such functions.

By grouping $\pi'(\lambda)$ and ρ^{λ} under a single variable

$$\tilde{\rho}^{\lambda} := \pi'(\lambda)\rho^{\lambda}, \tag{4.9}$$

we have that $\tilde{\rho}^{\lambda} \geq 0$ and $\mathrm{Tr}(\tilde{\rho}^{\lambda}) = \pi'(\lambda)$.

Thus the following SDP tests the existence of an LHS model for measurements $\{\mathbf{A}^{(l)}; l = 1\ldots, m\}$ performed on a state ρ_{AB}:

$$\begin{aligned} \text{given} \quad & \rho_{AB}, \{\mathbf{A}^{(l)}\} \\ \text{find} \quad & \{\tilde{\rho}^{\lambda}\} \\ \text{s.t.} \quad & \mathrm{Tr}_A\left[\left(A_i^{(l)} \otimes \mathbb{I}_B\right)\rho_{AB}\right] = \sum_{\lambda} \tilde{\rho}^{\lambda} D(i|\mathbf{A}^{(l)}, \lambda),\ \forall a, x \\ & \tilde{\rho}^{\lambda} \geq 0,\ \forall \lambda \\ & \sum_{\lambda} \mathrm{Tr}(\tilde{\rho}^{\lambda}) = 1. \end{aligned} \tag{4.10}$$

The above SDP extends to multipartite states in a rather straightforward way. In particular, extending $B \to B_1 \otimes \cdots \otimes B_k$, we demand in addition that each ρ_λ (now an operator on $\mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_k}$) is a fully separable state.

For a survey on the SDP approach to EPR steering we suggest Ref. [CS16].

4.3 Joint measurability as an SDP

Since the joint measurability of a tuple of POVMs $\{\mathbf{A}^{(l)}\}$ is a feature that concerns the existence of a joint measurement, and the only requirements for this are positive semidefinitiveness and the linear contraints in Eq. (3.2), this problem can easily be casted as an SDP [WPGF09]:

$$\begin{aligned} \text{given} \quad & \{\mathbf{A}^{(l)}\} \in \mathcal{P}(d, n)^{\times m} \\ \max_{\mathbf{M} \in \mathcal{P}(d, n^m)} \quad & t \qquad (4.11) \\ \text{s.t.} \quad & \Phi_t(A_i^{(l)}) = \sum_{r \neq l} \sum_{a_r} M_{a_1 \ldots (a_l = i) \ldots a_m}\ \forall l, i \qquad (4.12) \\ & M_{a_1 \ldots a_m} \geq 0,\ \forall a_1 \ldots a_m, \end{aligned}$$

where Φ_t is the depolarising channel. Hence, if the optimal value t^* obtained is 1, then $\{\mathbf{A}^{(l)}\}$ is jointly measurable. If not, t^* represents the white-noise robustness of the tuple regarding joint measurability (see Section 3.1).

Part II:

Results

Chapter 5

Framework for quantum measurement simulability

The main topic of this thesis is the problem of simulating a given quantum measurement while having access only to a restricted subset of measurements. In order to make this a well-defined question, we first should explain what we mean by "simulation", that is, what class of operations we are allowed to implement over the set of simulators. Therefore, we start by defining our framework, introduced in Ref. [GBCA17].

5.1 The simulability framework

We say that a measurement $\mathbf{A}$ can be simulated by a subset of POVMs $\mathcal{B} = \{\mathbf{B}^{(j)}\}_j$ if there is a protocol based on classical manipulations of the measurements in $\mathcal{B}$ that yields the same statistics as $\mathbf{A}$ when performed on any quantum state,

$$\Pr_{\text{prot}}(i|\rho) = \text{Tr}(A_i\rho), \tag{5.1}$$

for any outcome i and any state ρ.

Quantum measurements can be classically manipulated in two ways [HHP12]: as a *pre-processing* (mixing) and as a *post-processing* (relabeling). Here we restrict ourselves to operations only on the level of the measurements, although pre-processing operations involving the preparation of quantum states could also be defined [BKD+05]. Therefore, the most general protocol for simulating $\mathbf{A}$ with $\mathcal{B}$ consists in three steps:

(i) Choose a measurement $\mathbf{B}^{(j)} \in \mathcal{B}$ with probability $p(j|\mathbf{A})$;

(ii) Perform $\mathbf{B}^{(j)}$;

(iii) Upon obtaining outcome i', output i according to some probability $q(i|\mathbf{A}, j, i')$.

In the above protocol, step (i) represents a pre-processing and step (iii) represents a post-processing. In the latter, the final output i is produced with a probability $q(i|\mathbf{A}, j, i')$ conditioned on the POVM $\mathbf{A}$ to be simulated, on the performed measurement $\mathbf{B}^{(j)}$, and on the obtained outcome i'. This can be understood as a new measurement $\tilde{\mathbf{B}}^{(j)}$ given by effects[1]

$$\tilde{B}_i^{(j)} = \sum_{i'} q(i|\mathbf{A}, j, i') B_{i'}^{(j)}. \tag{5.2}$$

Notice that $\tilde{\mathbf{B}}^{(j)}$ may have a different number of outcomes than $\mathbf{B}^{(j)}$ (either more or less).

Step (i) allows for probabilistic mixing of the post-processed POVMs $\tilde{\mathbf{B}}^{(j)}$. Therefore, we say that an n-outcome POVM $\mathbf{A}$ is $\mathcal{B}$-*simulable* if there are probability distributions $p(\cdot|\mathbf{A}), q(\cdot|\mathbf{A}, j, i')$ such that for any state ρ,

$$\begin{aligned} \operatorname{Tr}(A_i\rho) &= \Pr_{\text{prot}}(i|\rho) \\ &= \operatorname{Tr}\left(\left[\sum_j p(j|\mathbf{A}) \sum_{i'} q(i|\mathbf{A}, j, i') B_{i'}^{(j)}\right]\rho\right), \end{aligned} \tag{5.3}$$

or, equivalently,

$$A_i = \sum_j p(j|\mathbf{A}) \sum_{i'} q(i|\mathbf{A}, j, i') B_{i'}^{(j)}, \tag{5.4}$$

for $i \in \{1, \ldots, n\}$. In this case, we say that the particular choice of measurements $\mathbf{B}^{(j)}$ involved in the above decomposition are the $\mathcal{B}$-*simulators* of $\mathbf{A}$.

It is straightforward to see that any trivial POVM $\mathbf{A} = (a_1\mathbb{I}, \ldots, a_n\mathbb{I})$ can be simulated only with classical post-processing, simply by taking $q(i|j, i') = a_i$, for all j, i'. Therefore trivial measurements are simulable by any set $\mathcal{B}$ of simulators. This leads us to the study of the robustness of a given POVM $\mathbf{A}$ regarding simulability, in analogy to robustness regarding joint measurability (see Section 3.1).

We define the white noise robustness of a POVM $\mathbf{A}$ regarding its simulation by $\mathcal{B}$ as

$$t_{\mathcal{B}}^{\mathbf{A}} = \max\{t;\ \Phi_t(\mathbf{A}) \text{ is } \mathcal{B}\text{-simulable}\}, \tag{5.5}$$

and extend it to sets of POVMs $\mathcal{A} = \{\mathbf{A}\}$,

$$t_{\mathcal{B}}^{\mathcal{A}} = \max\{t;\ \Phi_t(\mathbf{A}) \text{ is } \mathcal{B}\text{-simulable for all } \mathbf{A} \in \mathcal{A}\}. \tag{5.6}$$

[1] The set $\mathcal{B}$ needs not to be countable, as it happens, *e.g.* , for projective simulable measurements. However, given a measurement $\mathbf{A}$ and a set of simulators $\mathcal{B}$, Caratheodory's Theorem guarantees that a finite subset of $\mathcal{B}$ is enough. This justifies our use of sums instead of integrals.

We are now in position to present our first results. The main goal of the next sections is to study the $\mathcal{B}$ simulability of general POVMs, under different sets of simulators, depending on the number of simulators, type of simulators, or the number of their outcomes. In the case where a POVM (or a set of POVMs) is not simulable, a secondary question we address refers to the white noise robustness of the non-simulability of such a POVM, that is, how much we need to depolarise it in order to make it simulable.

5.2 Many-POVMs simulability

We start by considering completely general accessible measurements, restricting solely the number of simulators.

5.2.1 Single-POVM simulability

Perhaps the simplest form of simulation refers to the case where the subset $\mathcal{B}$ of measurements to which one has access contains a single POVM $\mathbf{B}$ of n_B outcomes. In this case, step (i) of the general protocol is trivial, and the only relevant operation is the post-processing. Therefore, the $\mathcal{B}$-simulable POVMs are the ones described in Eq. (5.2).

When we consider a set of m measurements $\{\mathbf{A}^{(l)}\}_{l=1}^{m}$ that are simulable by the same (arbitrary) POVM $\mathbf{B}$ we recover the usual definition of joint measurability, as already pointed in Ref. [ACHT09]. Indeed, consider that

$$A_i^{(l)} = \sum_{i'} q(i|l,i')B_{i'}, \tag{5.7}$$

for all i, l and some post-processings $q(\cdot|l,i')$, $i' \in \{1,\dots,n_B\}$. Then define a *joint measurement* $\mathbf{M}$ by

$$M_{a_1\dots a_m} = \sum_{i=1}^{n_B}\prod_{l=1}^{m} q(a_l|l,i)B_i, \tag{5.8}$$

for $a_l \in \{1,\dots,n_l\}$, where n_l is the number of outcomes of $\mathbf{A}^{(l)}$. Hence

$$\sum_{r\neq l}\sum_{a_r=1}^{n_r} M_{a_1\dots(a_l=i)\dots a_m} = A_i^{(l)}, \tag{5.9}$$

for all l, i, and we obtain the usual definition of joint measurability: the set $\{\mathbf{A}^{(j)}\}$ is jointly measurable, since all POVM elements $A_i^{(j)}$ can be recovered by (deterministically) coarse-graining over the joint measurement $\mathbf{M}$ (see Chapter 3). This proves the following lemma.

Lemma 1. *A set of POVMs is jointly measurable if and only if it can be simulated by a single measurement.*

Joint measurability thus appears as a particular instance of measurement simulability where only one simulator is considered. The joint measurement $\mathbf{M}$ derived from $\mathbf{B}$ in Eq. (5.8) simplifies the post-processing at the cost of typically increasing the number of outcomes of the simulator.

If we can simulate a set of POVMs using only one POVM we will say that the set is *single-POVM-simulable*, as an easily generalisable synonymous of jointly measurable. One can efficiently decide on the single-POVM simulability of a given set of measurements via a feasibility SDP, as discussed in Section 4.3.

5.2.2 J-POVM simulability

The natural next step is now to consider a set of simulators containing two POVMs, $\mathcal{B} = \{\mathbf{B}^{(1)}, \mathbf{B}^{(2)}\}$. Again we look at sets of POVMs $\mathcal{A} = \{\mathbf{A}^{(l)}\}$ that can be simulated by the same simulators, *i.e.*, for every effect $A_i^{(l)}$ we have

$$A_i^{(l)} = p(1|l)\sum_{i'} q(i|l,1,i')B_{i'}^{(1)} + p(2|l)\sum_{i'} q(i|l,2,i')B_{i'}^{(2)}. \tag{5.10}$$

These will be called 2-POVM-simulable sets. Following Eq. (5.8), using deterministic post-processing, this is equivalent to

$$\begin{aligned} A_i^{(l)} &= p(1|l)\sum_{r\neq l}\sum_{a_r=1}^{n_r} M^{(1)}_{a_1\ldots(a_l=i)\ldots a_m} \\ &\quad + p(2|l)\sum_{r\neq l}\sum_{a_r=1}^{n_r} M^{(2)}_{a_1\ldots(a_l=i)\ldots a_m}. \end{aligned} \tag{5.11}$$

Hence, in terms of joint measurability, now we can combine the marginals of two joint measurements $\mathbf{M}^{(1)}, \mathbf{M}^{(2)}$.

In contrast with the previous case, we were unable to cast the problem of deciding whether a given set of measurements is 2-POVM-simulable as an SDP. Since the free variables are the pre-processing, the simulators, and the post-processing, Eq. (5.12) represents apparently unavoidable non-linear constraints.

Since single-POVM simulability is equivalent to joint measurability, by increasing the number of simulators in the accessible set $\mathcal{B}$ we create a hierarchy of simulability protocols where each case strictly contains the previous one and whose first level is joint measurability. Namely, if the set $\mathcal{A}$ can be simulated by some set $\{\mathcal{B}^{(j)}\}$ containing J elements,

$$A_i^{(l)} = \sum_{j=1}^{J} p(j|l) \sum_{i'} q(i|l,j,i') B_{i'}^{(j)}, \tag{5.12}$$

for all i, l, we say that $\mathcal{A}$ is **J-POVM-simulable**.

However, we now show that if the POVMs $\{\mathbf{A}^{(j)}\}$ can be simulated by $\mathcal{B}$ using always the same weights $p(j|l) = p(j)$ in Eq. (5.12), independently of l, then this set is jointly measurable (and therefore simulable by a single POVM). This is a general feature of the framework, valid for any set of simulators $\mathcal{B}$.

Proposition 1. *If every measurement in $\{\mathbf{A}^{(j)}\}$ is $\mathcal{B}$-simulable with the same pre-processing step, then $\{\mathbf{A}^{(j)}\}$ is jointly measurable.*

Proof. The proof is analogous to the one of Lemma 1. If a $\mathcal{B}$-simulable set shares the same pre-processing, then $p(j|l) = p(j)$ and we can describe its elements by

$$A_i^{(l)} = \sum_{j} p(j) \sum_{i'} q(i|l,j,i') B_{i'}^{(j)}. \tag{5.13}$$

Hence a joint measurement $\mathbf{M}$ is defined by

$$M_{a_1 \dots a_m} = \sum_{j} p(j) \sum_{i=1}^{n_j} \prod_{l=1}^{m} q(a_l|l,j,i) B_i^{(j)}. \tag{5.14}$$

□

Similarly to Lemma 1, under the conditions of Proposition 1 we can exchange many simulators by a single one, generally with a greater number of outcomes. In spite of its simplicity, in Section 5.3.4 we provide a valuable application of Proposition 1.

Considering simulability with more than one simulator we can refine our notion of incompatibility, as illustrated by the following example.

Example 11. Consider the set $\mathcal{A} = \{\mathbf{A}^{(x)}, \mathbf{A}^{(y)}, \mathbf{A}^{(z)}, \mathbf{A}^{(\Sigma)}\}$, where $\mathbf{A}^{(x)}, \mathbf{A}^{(y)}, \mathbf{A}^{(z)}$ are the projective qubit measurements associated to the Pauli observables σ_x, σ_y, σ_z and $\mathbf{A}^{(\Sigma)}$ is the projective measurement described by

$$A_{\pm}^{(\Sigma)} = (\mathbb{I} \pm \vec{v} \cdot \vec{\sigma})/2, \tag{5.15}$$

with $\vec{v} = (1,1,1)/\sqrt{3}$. Now, our goal is to understand for which values of the visibility t the set $\Phi_t(\mathcal{A})$ becomes single-, 2- and 3-POVM-simulable. Let us start by the latter.

For 3-POVM simulability, a straightforward protocol can be obtained for visibilities in which a pair of POVMs of $\Phi_t(\mathcal{A})$ becomes jointly measurable. This happens at $t_{\text{PI}} = 0.7420$, where $\mathbf{A}^{(\Sigma)}$ becomes jointly measurable with any of the other three measurements in the set. For visibilities larger than t_{PI} the set is pairwise incompatible, as there is no pair of POVMs in $\Phi_t(\mathcal{A})$ which is jointly measurable. However, we next show that this protocol is not optimal for 3-POVM simulability.

Since one of the three-element subsets of $\mathcal{A}$ is clearly more incompatible than the others (namely, $\{\mathbf{A}^{(x)}, \mathbf{A}^{(y)}, \mathbf{A}^{(z)}\}$), a better strategy to simulate $\mathcal{A}$ with 3 simulators is to assign each element of this subset to an exclusive simulator. This means that for these measurements each pre-processing is deterministic,

$$p(j|\mathbf{A}^{(w)}) = \delta_{j,1}\delta_{x,w} + \delta_{j,2}\delta_{y,w} + \delta_{j,3}\delta_{z,w}, \tag{5.16}$$

where $w = x, y, z$, and each $\mathbf{A}^{(w)}$ is simulated by a single simulator $\mathbf{B}^{(j)}$, while $\mathbf{A}^{(\Sigma)}$ uniformly combines all three simulators,

$$p(j|\mathbf{A}^{(\Sigma)}) = \frac{1}{3},\ j = 1, 2, 3. \tag{5.17}$$

By fixing this pre-processing, we can now write an SDP to calculate the best post-processing steps corresponding to it and the best parameter t such that $\Phi_t(\mathcal{A})$ is simulated by this protocol. With this strategy, we find that the set is 3-POVM-simulable at visibility $t_{\text{3-POVM}} = 0.7746$. Note that for this value of the visibility we have constructed a particular simulation protocol employing three measurements. It is in principle conceivable that a better simulation protocol exists, which would imply a larger range for 3-POVM simulation. Yet this protocol was enough to show a gap with the value required to observe pairwise joint measurability.

At visibility $t_{\text{2-POVM}} = 1/\sqrt{2} \approx 0.7071$, $\Phi_{t_{\text{2-POVM}}}(\mathcal{A})$ becomes 2-POVM-simulable. This coincides with the visibility t_{PC} needed to make $\mathcal{A}$ pairwise compatible, identifying it as a "hollow tetrahedron", that is, a set of four incompatible POVMs from which every pair of elements is compatible. Indeed, since any pair of POVMs of $\mathcal{A}$ is compatible, we can use the joint measurements $\mathbf{M}^{(xy)}$ (for depolarised versions of $\mathbf{A}^{(x)}$ and $\mathbf{A}^{(y)}$), and $\mathbf{M}^{(z\Sigma)}$ (for depolarised versions of $\mathbf{A}^{(z)}$ and $\mathbf{A}^{(\Sigma)}$) as simulators, each one simulating its corresponding pair.

$\mathcal{A}$ is triplewise incompatible for visibilities $t \geq t_{TI} = 0.6236$. The set becomes triplewise compatible at visibility $t_{\text{TC}} = 1/\sqrt{3} \approx 0.5774$, and, finally, fully compatible when depolarised by a parameter of $t_{\text{1-POVM}} = 0.5730$. Recall that these values t_{TI}, t_{TC}, and $t_{\text{1-POVM}}$ are obtained via SDP.

A brute force numerical search supports the claim that $t_{\text{2-POVM}}, t_{\text{3-POVM}}$ are the optimal parameters for 2- and 3-POVM simulability of $\mathcal{A}$, respectively. On Figure 5.1 we organise all optimal visibilities for the simulability of $\mathcal{A}$.

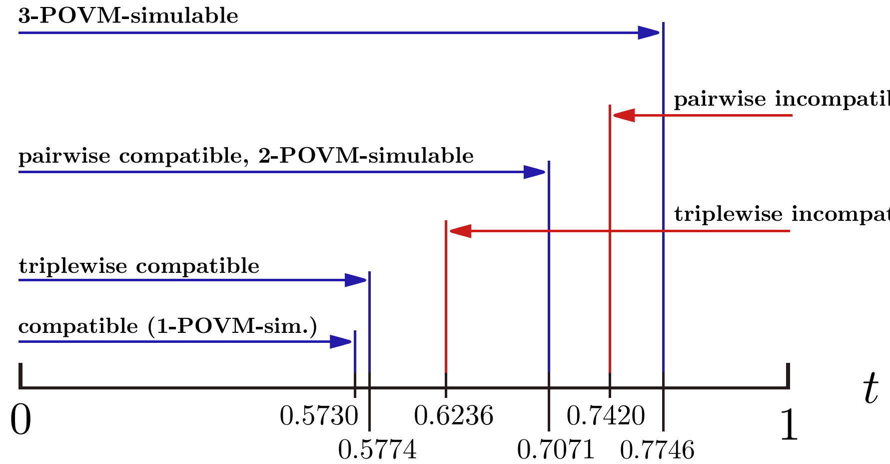


Figure 5.1: The optimal visibilities for the single-, 2-, and 3-POVM simulability of $\mathcal{A} = \{\mathbf{A}^{(x)}, \mathbf{A}^{(y)}, \mathbf{A}^{(z)}, \mathbf{A}^{(\Sigma)}\}$. Note that the intervals where the set is, say, pairwise compatible and pairwise incompatible are not complementary because these concepts address every possible pair of the set, and different pairs present different degrees of robustness.

On the one hand, the above example shows that the number of simulators available yields genuinely different forms of simulability. On the other, it makes clear that internal compatibility relations between the POVMs of the set provide lower bounds for its J-POVM simulability. For instance, for any set $\{\mathbf{A}^{(l)}\}$ of m measurements we have

$$t_{\text{PI}} \leq t_{\text{(m-1)-POVM}}, \tag{5.18}$$

where t_{PI} defines open interval of visibilities for which the set is pairwise incompatible, and $t_{\text{(m-1)-POVM}}$ is the critical depolarising parameter for which the set becomes $(m-1)$-POVM-simulable. Indeed, at $t = t_{PI}$ some pair of POVMs is compatible, say $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$, and thus we can use the simulators $\mathbf{B}^{(1)} = \mathbf{M}^{(12)}$ (the joint measurement for $\Phi_t(\mathbf{A}^{(1)}), \Phi_t(\mathbf{A}^{(2)})$), $\mathbf{B}^{(2)} = \mathbf{A}^{(3)}, \ldots, \mathbf{B}^{(m-1)} = \mathbf{A}^{(m)}$. Similarly, we can derive other bounds related to t_{TI}, t_{PC} and so on.

More generally, for a set $\{\mathbf{A}^{(l)}\}$ of m incompatible POVMs we can consider its robustness regarding simulability with any number $J < m$ of simulators. For the particular case $J = 1$, the noise robustness of joint measurability was extensively studied already [HKR15, CS16, BQG+17], but for $J > 1$ this is a new question to be investigated.

5.3 k-outcome simulability

Another form of simulability we investigate is by POVMs of less outcomes. In this case, we do not limit the number of accessible measurements employed for the simulation, but only their number of outcomes. In other words, now our set of simulators $\mathcal{B}$ is the set of k-outcome POVMs on dimension d, and the $\mathcal{B}$-simulable measurements will be called **k-outcome-simulable**. This topic arises naturally as another variant of the general simulation problem, and this sort of limitation plays a key role in Bell nonlocality scenarios [BCP+14, KC16, Mas05].

Note that by applying Lemma 1 and Proposition 1 one reduces the number of simulators but raises the number of outcomes of the simulators; now we want to improve on the other direction and reduce the number of outcomes, possibly by increasing the number of involved measurements.

5.3.1 Sufficient condition for 2-outcome simulability

We start presenting a sufficient condition for the simplest form of simulability in this context, given by 2-outcome, or dichotomic, POVMs. In this case, there are only two effects B_1, B_2 in each simulator, which means that one completely defines the other due to the normalisation constraint, $B_2 = \mathbb{I} - B_1$. As a consequence, we are able to characterise a particular case of 2-outcome simulability based on the greatest eigenvalue of each effect of the measurement.

Proposition 2. *Let $\boldsymbol{A} \in \mathcal{P}(d,n)$ be an n-output POVM on the Hilbert space $\mathcal{H}$ of dimension d, λ_i^{max} be the maximal eigenvalue of the i-th effect A_i and $j_0 \in \{1, \dots, n\}$. Then $\boldsymbol{A}$ can be written as a convex combination*

$$\boldsymbol{A} = \sum_{j \neq j_0} p_j \boldsymbol{B}^{(j)}, \tag{5.19}$$

where $\boldsymbol{B}^{(j)}$ is a dichotomic POVM having outcomes j and j_0, if and only if it holds that

$$\sum_{j \neq j_0} \lambda_j^{\max} \leq 1. \tag{5.20}$$

Proof. Without loss of generality, lets say that $j_0 = 1$. By assuming (5.20), we have

$$\sum_{i=2}^{N} \lambda_i^{\max} + \delta = 1, \tag{5.21}$$

for some $\delta \geq 0$. Hence, the normalisation of $\mathbf{A}$ says that

$$M_1 = \mathbb{I} - \sum_{i=2}^{n} M_i \tag{5.22}$$

$$= \left(\sum_{i=2}^{n} \lambda_i^{\max} + \delta \right) \mathbb{I} - \sum_{i=2}^{N} M_i \tag{5.23}$$

$$= \delta \mathbb{I} + \sum_{i=2}^{n} (\lambda_i^{\max} \mathbb{I} - M_i). \tag{5.24}$$

The definition of $\lambda_i^{\max}$ ensures $\lambda_i^{\max}\mathbb{I} - M_i \geq 0$, and therefore the dichotomic POVMs $\mathbf{N}^{(i)}$, whose effects are

$$N_k^{(i)} = \begin{cases} \mathbb{I} - M_k/\lambda_i^{\max} & \text{if } k = 1 \\ M_k/\lambda_i^{\max} & \text{if } k = i \\ 0 & \text{otherwise} \end{cases} \tag{5.25}$$

are well-defined. Then, we have the convex decomposition

$$\mathbf{M} = \delta\,(\mathbb{I}, 0, \ldots, 0) + \sum_{i=2}^{n} \lambda_i^{\max} \mathbf{N}^{(i)}. \tag{5.26}$$

On the other hand, if Eq. (5.19) holds for POVMs $\mathbf{B}^{(j)}$ which have outcomes j and j_0, than for $i \neq j_0$ we have $A_i = p_i B_i^{(i)}$ and since $B_i^{(i)} \leq \mathbb{I}$, we see that the largest eigenvalue of A_i must satisfy $\lambda_i^{max} \leq p_i$, yielding Eq. (5.20). □

Consequently, we have that this kind of 2-outcome-simulable POVMs is a robust property, in the sense that small perturbations applied to the target measurement cause minor changes in the eigenvalues of its effects, and therefore preserve this property.

5.3.2 SDP characterisation of k-outcome simulability

We now show that for k-outcome simulability the post-processing step can be implemented in a quite simple way [OGWA17].

Consider a protocol in which we perform an k-outcome measurement $\mathbf{B}$ and upon obtaining outcome i', we output i_0 with probability p and i_1 with probability $(1-p)$. This is equivalent to the protocol in which with probability p we perform $\mathbf{B}^{(0)} = \mathbf{B}$ and always relabel outcome i' by i_0, and with probability $(1-p)$ we perform $\mathbf{B}^{(1)} = \mathbf{B}$ and always relabel i' by i_1. By doing this for each outcome i', we artificially increase the number of simulators but restrict the post-processing to be deterministic; since we want to simulate an n-outcome POVM via k-outcome POVMs ($k < n$), we are left with $\frac{n!}{(n-k)!}$ possibilities of post-processing.

Now we notice that post-processing operations that shuffle the order of effects can also be mapped to the pre-processing, in the sense that for each of the $k!$ post-processing permutations on the non-null outcomes we associate a different simulator with permuted effects, which is also a k-outcome POVM. Hence we do not lose generality by considering only the $\frac{n!}{k!(n-k)!} = \binom{n}{k}$ deterministic post-processing strategies that carries k-outcome POVMs to n-outcome ones while preserving the relative order of effects.

Finally, we can group the simulators that share the same post-processing. Indeed, imagine that $\mathbf{A}$ is simulated by the k-outcome POVMs $\{\mathbf{B}^{(j)} = (B_1^{(j)}, \ldots, B_k^{(j)})\}$ and $\mathbf{B}^{(1)}, \mathbf{B}^{(2)}$ after being post-processed have the form $\tilde{\mathbf{B}}^{(j)} = (B_1^{(j)}, \ldots, B_k^{(j)}, 0, \ldots, 0), j = 1, 2$. Then

$$\begin{aligned}\mathbf{A} =& p(1)\tilde{\mathbf{B}}^{(1)} + p(2)\tilde{\mathbf{B}}^{(2)} + \sum_{j>3} p(j)\tilde{\mathbf{B}}^{(j)} \\ =& (p(1) + p(2))\tilde{\mathbf{B}}' + \sum_{j>3} p(j)\mathbf{B}^{(j)},\end{aligned} \tag{5.27}$$

where the measurement $\tilde{\mathbf{B}}'$ given by

$$\tilde{B}_i = \frac{p(1)B_i^{(1)} + p(2)B_i^{(2)}}{p(1) + p(2)} \tag{5.28}$$

also has the form $\tilde{\mathbf{B}} = (\tilde{B}_1, \ldots, \tilde{B}_k, 0, \ldots, 0)$. We conclude that we can consider only one representant of each post-processing class, arriving at the following result.

Proposition 3. *An n-outcome POVM $\boldsymbol{A}$ is k-outcome-simulable if and only if there is a set of at most $\binom{n}{k}$ POVMs $\{\boldsymbol{B}^{(j)}\}$ with at most k non-null effects satisfying*

$$\boldsymbol{A} = \sum_{j=1}^{\binom{n}{k}} p_j \boldsymbol{B}^{(j)}, \tag{5.29}$$

one for each possible distribution of the k non-null outcomes among the n possibilities.

This Proposition allows one to efficiently decide on the k-outcome simulability of a given POVM and to compute the amount of depolarisation the POVM endures before becoming k-outcome-simulable by means of SDP.

For instance, a 4-outcome POVM $\mathbf{A} = (A_1, \ldots, A_4)$ is 3-outcome-simulable if and only if we can find $\binom{4}{3} = 4$ simulators $\mathbf{B}^{(ijk)}$ and convex weights (p_{ijk})

such that

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = p_{123} \begin{pmatrix} B_1^{(123)} \\ B_2^{(123)} \\ B_3^{(123)} \\ 0 \end{pmatrix} + \cdots + p_{234} \begin{pmatrix} 0 \\ B_2^{(234)} \\ B_3^{(234)} \\ B_4^{(234)} \end{pmatrix}, \tag{5.30}$$

which can be phrased as the feasibility SDP

$$\begin{aligned} \text{given } & \mathbf{A} = (A_1, \ldots, A_4) \\ \max \ & t \\ \text{s.t. } & \Phi_t(A_1) = N_1^{(123)} + N_1^{(124)} + N_1^{(134)} \\ & \Phi_t(A_2) = N_2^{(123)} + N_2^{(124)} + N_2^{(234)} \\ & \Phi_t(A_3) = N_3^{(123)} + N_3^{(134)} + N_3^{(234)} \\ & \Phi_t(A_4) = N_4^{(124)} + N_4^{(134)} + N_4^{(234)} \\ & N_i^{(ijk)}, N_j^{(ijk)}, N_k^{(ijk)} \geq 0, \ \ \forall i < j < k \\ & N_i^{(ijk)} + N_j^{(ijk)} + N_k^{(ijk)} = p_{ijk}\mathbb{I}, \ \ \forall i < j < k \\ & p_{ijk} \geq 0, \ \ \forall i < j < k \\ & \sum_{i<j<k} p_{ijk} = 1 \end{aligned} \tag{5.31}$$

Since $\Phi_1(\mathbf{A}) = \mathbf{A}$, if the optimal value t^* found in the above maximisation equals 1, then $\mathbf{A}$ is simulable. Otherwise, the above optimisation provides the white-noise robustness of the target POVM $\mathbf{A}$. Notice that simple modifications can adapt SDP (5.31) for any number of outcomes both in the target POVM $\mathbf{A}$ and in the simulators $\mathbf{B}^{(ijk)}$.

Example 12. Consider a tetrahedral qubit measurement $\mathbf{A}^{\text{tetra}}$ given by

$$A_i^{\text{tetra}} = \frac{1}{4}(\mathbb{I} + \vec{v}_i \cdot \vec{\sigma}), \ i \in \{1, \ldots, 4\}, \tag{5.32}$$

where the unit vectors $\vec{v}_i \in \mathbb{R}^3$ form the vertices of a regular tetrahedron. This 4-outcome POVM is not 3-outcome-simulable, but when depolarised by $t^{\text{tetra}}_{\text{3-out}} = 2\sqrt{2}/3$, we see that the resulting POVM, $\Phi_{t^{\text{tetra}}_{\text{3-out}}}(\mathbf{A}^{\text{tetra}})$, can be decomposed into $\binom{4}{3} = 4$ trine POVMs, $\mathbf{B}^{\text{trine},r}$, $r \in \{1, \ldots, 4\}$, each one with effects whose Bloch vectors form an equilateral triangle on the plane perpendicular to $\vec{v}_r$ (Figure 5.2).

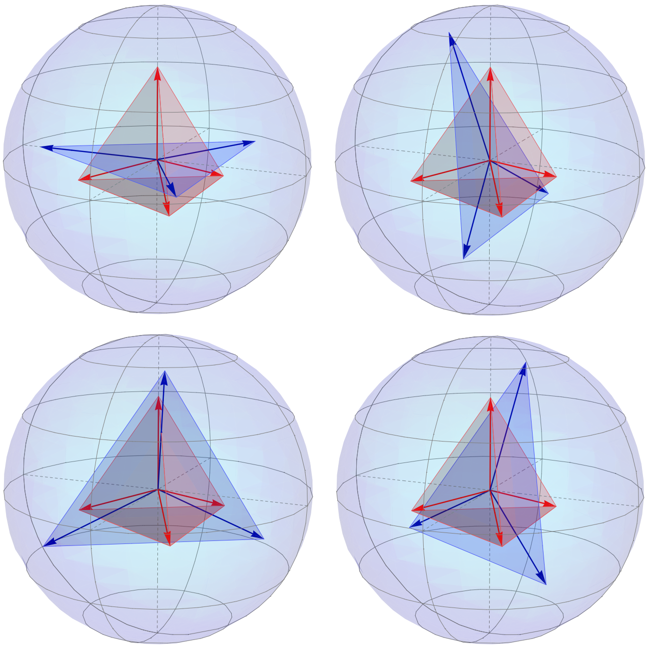


Figure 5.2: The 4-outcome measurement $\mathbf{A}^{\text{tetra}}$ becomes 3-outcome-simulable when depolarised by a parameter $t^{\text{tetra}}_{3\text{-out}} = 2\sqrt{2}/3$. Its optimal 3-outcome simulators are regular trines measurements, each one lying on a plane parallel to a facet of the tetrahedron.

The trine POVMs $\mathbf{B}^{\text{trine},r}$ are not 2-outcome-simulable, but this can be achieved by depolarising them by $t^{\text{trine}}_{2\text{-out}} = \sqrt{3}/2$. The critical visibility to make $\mathbf{A}^{\text{tetra}}$ 2-outcome-simulable is $t^{\text{tetra}}_{2\text{-out}} = \sqrt{2/3}$ [OGWA17, HQV+17], and therefore in this case we have

$$t^{\text{tetra}}_{3\text{-out}} \cdot t^{\text{trine}}_{2\text{-out}} = t^{\text{tetra}}_{2\text{-out}}. \tag{5.33}$$

However, in general one value is only a lower bound for the other,

$$t^{\mathbf{A}}_{k\text{-out}} \cdot \min\{t^{\mathbf{B}}_{(k-1)\text{-out}}; \mathbf{B} \text{ is a } k\text{-outcome simulator of } \mathbf{A}\} \leq t^{\mathbf{A}}_{(k-1)\text{-out}}, \tag{5.34}$$

meaning that decomposing $\mathbf{A}$ into k-outcome simulators and subsequently each simulator into $(k-1)$-outcome measurements may not be the optimal $(k-1)$-outcome simulation protocol for $\mathbf{A}$.

5.3.3 k-outcome simulability and joint measurability

Recall from Chapter 3 that we can represent a joint measurement $\mathbf{M}$ for a pair $\{\mathbf{A}^{(1)}, \mathbf{A}^{(2)}\}$ of POVMs as a table of effects

$$\begin{array}{ccc|c} M_{11} & \cdots & M_{1n} & A_1^{(1)} \\ \vdots & \ddots & \vdots & \vdots \\ M_{n1} & \cdots & M_{nn} & A_n^{(1)} \\ \hline A_1^{(2)} & \cdots & A_n^{(2)} & \end{array} . \tag{5.35}$$

Our next result shows that we can construct such tables by reorganising the effects of k-outcome simulators, which leads to an equivalent condition to k-outcome simulability in terms of joint measurability.

Proposition 4. *A qudit measurement $\mathbf{A}$ is k-outcome-simulable if and only if there is a joint POVM $\mathbf{M}$ for the pair $\{\mathbf{A}, \vec{p}\cdot\mathbb{I}\}$ with at least $n-k$ null effects in each column $(M_{1j}, \ldots, M_{nj})$, where $\vec{p}\cdot\mathbb{I} = (p_1\mathbb{I}, \ldots, p_{\binom{n}{k}}\mathbb{I})$.*

Proof. Suppose we have a decomposition of a POVM $\mathbf{A}$ into k-outcome POVMs $\mathbf{B}^{(j)}$,

$$p_1 \begin{pmatrix} B_1^{(1)} \\ \vdots \\ B_n^{(1)} \end{pmatrix} + \cdots + p_m \begin{pmatrix} B_1^{(m)} \\ \vdots \\ B_n^{(m)} \end{pmatrix} = \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}, \tag{5.36}$$

where at least $n-k$ effects are null in each $\mathbf{B}^{(j)}$, and $m = \binom{n}{k}$ according to Proposition 3. We can now group the weights p_j with each effect $B_{b_j}^{(j)}$ and organise these effects in a table

$$\begin{array}{ccc|c} p_1 B_1^{(1)} & \cdots & p_m B_1^{(m)} & A_1 \\ \vdots & \ddots & \vdots & \vdots \\ p_1 B_n^{(1)} & \cdots & p_m B_n^{(m)} & A_n \\ \hline \end{array} . \tag{5.37}$$

Due to the normalisation of the $\mathbf{B}^{(j)}$, summing over each column we obtain $p_j\mathbb{I}$, and analogously to Table (5.35), we can see the table as a joint POVM for $\mathbf{A}$ and $\vec{p}\cdot\mathbb{I}$.

On the other hand, every joint POVM for $\mathbf{A}$ and $\vec{p}\cdot\mathbb{I}$ with $n-k$ null effects in each column can generate a decomposition like Eq. (5.36), where each column represents one of the k-outcome simulators. □

Although any POVM is jointly measurable with a trivial POVM $\vec{p} \cdot \mathbb{I}$ having all effects proportional to the identity, Proposition 4 is a criterion that requires this compatibility to be given in an optimised way where the joint measurement has many null effects, in order to ensure k-outcome simulability.

5.3.4 k-outcome simulability and the antipodal measurement

One of the main advantages of studying different forms of simulation on the same framework is the possibility of devising connections between them. In this subsection we present relations between k-outcome simulability and joint measurability (single-POVM simulability). Our starting point is to check the consequences of Proposition 1 for k-outcome simulability.

Consider the simple case of an n-outcome POVM $\mathbf{A}$ which is 2-outcome-simulable. Then, according to Lemma 3, there are $\binom{n}{2}$ convex weights (p_{ij}) and dichotomic POVMs $\mathbf{B}^{(ij)} = (B_{ij}, \mathbb{I} - B_{ij})$, $(i,j) \in \{(1,2), \ldots, (n-1,n)\}$, that can be embedded in the set of n-outcome POVMs via post-processing, such that B_{ij} takes place on the i-th entry of the tuple and $\mathbb{I} - B_{ij}$ on the j-th. Thus we can write

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_n \end{pmatrix} = p_{12} \begin{pmatrix} B_{12} \\ \mathbb{I} - B_{12} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + p_{13} \begin{pmatrix} B_{13} \\ 0 \\ \mathbb{I} - B_{13} \\ \vdots \\ 0 \end{pmatrix} + \ldots + p_{(n-1)n} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ B_{(n-1)n} \\ \mathbb{I} - B_{(n-1)n} \end{pmatrix}. \tag{5.38}$$

Notice that this is equivalent to write each effect of $\mathbf{A}$ as

$$A_i = \sum_{j;j<i} p_{ji}(\mathbb{I} - B_{ji}) + \sum_{j;j>i} p_{ij} B_{ij}, \tag{5.39}$$

and that each effect A_i is the sum of only $\binom{n-1}{2-1} = n - 1$ non-null operators $p_{ij} B_{ij}$ or $p_{ji}(\mathbb{I} - B_{ji})$.

According to Proposition 1, if we maintain the same pre-processing (p_{ij}) on the right-hand side of Eq. (5.38) but change the post-processing that embeds the dichotomic measurements, the resulting POVM will be jointly measurable with $\mathbf{A}$. Now consider the post-processing of $\mathbf{B}^{(ij)}$ that takes B_{ij} to the j-th position and $\mathbb{I} - B_{ij}$ to the i-th position. This way, we construct another 2-outcome-simulable POVM $\tilde{\mathbf{A}}$ given by

$$\tilde{A}_i = \sum_{j;j<i} p_{ji} B_{ji} + \sum_{j;j>i} p_{ij}(\mathbb{I} - B_{ij}), \tag{5.40}$$

in contrast with Eq. (5.39). Proposition 1 says that

$$M_{a_1a_2} = \begin{cases} p_{a_1a_2}B_{a_1a_2}, & \text{if } a_1 < a_2 \\ 0, & \text{if } a_1 = a_2 \\ p_{a_1a_2}(\mathbb{I} - B_{a_1a_2}), & \text{if } a_1 > a_2 \end{cases}. \tag{5.41}$$

defines a joint measurement for $\{\mathbf{A}, \tilde{\mathbf{A}}\}$. For example, for $n = 4$ this joint measurement reads

$$\begin{array}{cccc|c} 0 & p_{12}B_{12} & p_{13}B_{13} & p_{14}B_{14} & A_1 \\ p_{12}(\mathbb{I}-B_{12}) & 0 & p_{23}B_{23} & p_{24}B_{24} & A_2 \\ p_{13}(\mathbb{I}-B_{13}) & p_{23}(\mathbb{I}-B_{23}) & 0 & p_{34}B_{34} & A_3 \\ p_{14}(\mathbb{I}-B_{14}) & p_{24}(\mathbb{I}-B_{24}) & p_{34}(\mathbb{I}-B_{34}) & 0 & A_4 \\ \hline \tilde{A}_1 & \tilde{A}_2 & \tilde{A}_3 & \tilde{A}_4 & \end{array}. \tag{5.42}$$

A drawback in the definition of $\tilde{\mathbf{A}}$ is that we cannot construct it directly from $\mathbf{A}$, since it depends on the simulators $\mathbf{B}^{(ij)}$ and the pre-processing (p_{ij}). We can avoid this by restricting more the simulation and imposing that the simulators $\mathbf{B}_{ij}$ are *unbiased* 2-outcome POVMs [Bus09], meaning that each effect has the same parcel of identity when decomposed into a Hermitian operator basis,

$$B_{ij} = \frac{1}{2}\mathbb{I} + \vec{v}_{ij} \cdot \vec{\lambda} \tag{5.43a}$$

$$\mathbb{I} - B_{ij} = \frac{1}{2}\mathbb{I} - \vec{v}_{ij} \cdot \vec{\lambda} \tag{5.43b}$$

where $\vec{v}_{ij} \in \mathbb{R}^{d^2-1}$. Here, $\vec{\lambda}$ is a vector of $d^2 - 1$ Hermitian traceless operators that, together with $\mathbb{I}$, form an orthogonal basis for the real vector space of Hermitian operators in dimension d [BK08] (*e.g.* the Pauli matrices for $d = 2$, and the Gell-Mann matrices for $d = 3$). We call $\vec{\lambda}$ the **generalised Pauli vector**.

If that is the case and $A_i = a_i\mathbb{I} + \vec{u}_i \cdot \vec{\lambda}$, then Eq. (5.39) yields

$$a_i = \sum_{j;j<i} \frac{p_{ji}}{2} + \sum_{j;j>i} \frac{p_{ij}}{2}, \tag{5.44}$$

$$\vec{u}_i = \sum_{j;j<i} p_{ji}(-\vec{v}_{ji}) + \sum_{j;j>i} p_{ij}\vec{v}_{ij}, \tag{5.45}$$

and from Eq. (5.40) we have that $\tilde{A}_i = a_i\mathbb{I} - \vec{u}_i \cdot \vec{\lambda}$. In other words, $\tilde{\mathbf{A}}$ can be defined directly from $\mathbf{A}$ by flipping the sign of the generalised Pauli vector of each effect, when the latter is simulable via unbiased dichotomic POVMs. This motivates the definition of antipodal operator: given an Hermitian operator $A = a\mathbb{I} + \vec{v}\cdot\vec{\lambda}$, its **antipodal operator** is $\bar{A} = a\mathbb{I} - \vec{v}\cdot\vec{\lambda}$.

Since the antipodal POVM $\bar{\mathbf{A}}$ can be constructed from the simulators of $\mathbf{A}$ (Eq. (5.40)), the proof of Proposition 1 ensures that $\bar{A}_i \geq 0$, as it writes it as a sum of positive semi-definite operators. However, the antipodal of a positive semidefinite operator is not always positive semidefinite, this will generally depend on the eigenvalues of the traceless operator $v\cdot\vec{\lambda}$. An exception is the qubit case, where $d = 2$; in this case $\vec{\lambda} = \vec{\sigma}$ is the usual vector of Pauli matrices and it holds that $a\mathbb{I} + \vec{v}\cdot\vec{\lambda} \geq 0$ if and only if $a \geq ||\vec{v}||$, which implies that $A = a\mathbb{I} + \vec{v}\cdot\vec{\lambda} \geq 0$ if and only if $\bar{A} = a\mathbb{I} - \vec{v}\cdot\vec{\lambda} \geq 0$.

The above reasoning proves the following particular case of Proposition 1.

Proposition 5. *If a qudit measurement $\mathbf{A}$, given by $A_i = a_i\mathbb{I} + \vec{u}_i\cdot\vec{\lambda}$, is simulable via unbiased 2-outcome POVMs, then the antipodal operators $\bar{A}_i = a_i\mathbb{I} - \vec{u}_i\cdot\vec{\lambda}$ are positive semidefinite, $\bar{\mathbf{A}}$ is a valid POVM, and $\{\mathbf{A},\bar{\mathbf{A}}\}$ is jointly measurable.*

Proposition 5 is an example of the power of Proposition 1 that has a clear geometrical interpretation. For the particular case of qubit measurements, in Section 5.4.4 we are able to show its converse (see Theorem 9).

5.4 Projective simulability

We now investigate the case where the simulating set $\mathcal{B}$ is constrained to have only projective POVMs. This automatically limits the number of outcomes to be at most equal to the dimension of the system ($k \leq d$). In this case, a $\mathcal{B}$-simulable measurement is said to be **projective-simulable** [OGWA17]. Apart from their fundamental importance, projective measurements are often much easier to be physically implemented, as they do not require any ancilla system.

We start by recalling the following well-known result (Lemma 2.3 of Ref. [Dav76]), due to the fact that extremal dichotomic POVMs are projective.

Lemma 2. *For any dimension d, any 2-outcome POVM is projective-simulable.*

Proof. Let $\mathbf{A} = (A, \mathbb{I} - A)$ be a dichotomic measurement acting in an arbitrary dimension d. Then the spectral decomposition of A and $\mathbb{I} - A$ involve the same projectors,

$$A = \sum_{i=1}^{d} \lambda_i \Pi_i, \quad \mathbb{I} - A = \sum_{i=1}^{d} (1 - \lambda_i)\Pi_i. \tag{5.46}$$

Consider now the following protocol:

(i) implement the projective measurement $\Pi = (\Pi_1, \ldots, \Pi_d)$, and

(ii) upon obtaining outcome i, output 0 with probability λ_i and 1 with probability $1 - \lambda_i$.

Implementing it over any quantum state ρ, we have

$$\begin{aligned} \Pr_{\text{prot}}(0) &= \sum_i \text{Tr}(\rho \Pi_i)\lambda_i && (5.47) \\ &= \text{Tr}(\rho A), && (5.48) \end{aligned}$$

and the analogous holds for the probability of outputting 1. Hence **A** is projective-simulable.

□

5.4.1 Sufficient conditions for projective simulability

In this section we present two sufficient (but not necessary) conditions for ensuring projective simulability that holds for arbitrary dimension. The first result is a direct consequence of the previous Lemma 2 and Proposition 2, reducing projective simulability to 2-outcome simulability.

Proposition 6. *Let* $\mathbf{A} = (A_1, \ldots, A_n) \in \mathcal{P}(d, n)$ *and* λ_i^{max} *the maximum eigenvalue of* A_i. *If*

$$\sum_{i \neq i_0} \lambda_i^{max} < 1 \tag{5.49}$$

for some $i_0 \in \{1, \ldots, n\}$, *then* $\mathbf{A}$ *is projective-simulable.*

Proof. Under this conditions, Proposition 2 says that **A** is 2-outcome simulable, which implies projective simulability. □

We now proceed to show a general bound for the white-noise robustness of qudit measurements regarding projective simulability that depends only on the dimension of the system. In the same spirit of the proof of Lemma 2, we associate to each measurement a family of projective measurements via the spectral decomposition of its effects, and check that the simulation is effective when a certain level of depolarisation is considered.

Theorem 6. *For every* $\mathbf{A} \in \mathcal{P}(d, n)$, *its depolarised version* $\Phi_{\frac{1}{d}}(\mathbf{A})$ *is projective-simulable.*

Proof. Let $\mathbf{A} \in \mathcal{P}(d,n)$. For simplicity, lets assume that $\mathbf{A}$ has rank-1 effects, *i.e.*$A_i = \alpha_i \Pi_i$, where Π_i are rank-1 projectors and the normalisation of $\mathbf{A}$ yields $\sum_{i=1}^{n} \alpha_i = d$. Otherwise, we could simply consider the spectral decomposition of A_i and add an extra step to the protocol below, comprehending a coarse-graining.

Consider the following protocol:

1. Choose $i \in \{1, \dots, n\}$ with probability α_i / d;
2. Perform the projective measurement $(\Pi_i, \mathbb{I} - \Pi_i)$;
3. If the outcome corresponds to Π_i, output i; if the outcome corresponds to $\mathbb{I} - \Pi_i$, output any $j \in \{1, \dots, n\}$ with probability α_j / d.

Implementing the above protocol over any quantum state ρ, we obtain output i with probability

$$\begin{aligned}
\Pr_{\text{prot}}(i) &= \frac{\alpha_i}{d}\left[\mathrm{Tr}[\rho\Pi_i] + \mathrm{Tr}[\rho(\mathbb{I}-\Pi_i)]\frac{\alpha_i}{d}\right] + \left[\sum_{j\neq i}\frac{\alpha_j}{d}\mathrm{Tr}[\rho(\mathbb{I}-\Pi_j)]\right]\frac{\alpha_i}{d} \\
&= \mathrm{Tr}\left[\rho\left(\frac{1}{d}\alpha_i\Pi_i + (1-\frac{1}{d})\alpha_i\frac{\mathbb{I}}{d}\right)\right] \qquad (5.50)\\
&= \mathrm{Tr}\left[\rho\Phi_{\frac{1}{d}}(A_i)\right].
\end{aligned}$$

Hence, we see that $\Phi_{\frac{1}{d}}(\mathbf{A})$ is projective simulable. □

Theorem 6 presents a uniform bound for the white-noise robustness of any qudit measurement, showing that no POVM has to be completely depolarised in order to become projective-simulable in whatsoever dimension. Nevertheless, in Section 6.2 we show that this bound is not tight, since any measurement in dimension $d = 2$ is already projective-simulable when depolarised by $t = \sqrt{2/3} - \epsilon > 1/2$, for a given small ϵ.

5.4.2 Characterisation of projective simulability for $d = 2$

We now turn our attention to the simplest scenario of projective simulability where the dimension is two or three, corresponding to qubit and qutrit measurements. We start by pointing out that for $d = 2$ projective simulability is equivalent to 2-outcome simulability, and therefore it is an efficiently solvable problem to decide whether a given qubit POVM is simulable or not.

Theorem 7. *For $d = 2$, projective simulability is equivalent to dichotomic simulability.*

Proof. For $d = 2$ every projective measurement is either trivial or dichotomic, and therefore projective simulability implies 2-outcome-simulability.

On the other hand, Lemma 2 says that every 2-outcome simulation can be further extended to a projective simulation,what completes the proof. □

Therefore, we can use SDP (5.31) to calculate the robustness of a target qubit POVM regarding projective simulability and find its optimal projective simulators.

Example 13. Coming back to the tetrahedral measurement $\mathbf{A}^{\text{tetra}}$ of Example 12, SDP (5.31) yields that $\Phi_{\sqrt{\frac{2}{3}}}(\mathbf{A}^{\text{tetra}})$ is projective simulable, and the optimal projective simulators in this case point in the direction of the bissectrices of the angles between the vertices, given by $v_i - v_j$, where v_i, v_j are the Bloch vectors of the effects A_i, A_j (see Figure 5.3). In Section 6.2 we show that $\mathbf{A}^{\text{tetra}}$ is the most robust qubit measurement regarding projective simulability.

5.4.3 Characterisation of projective simulability for $d = 3$

It is natural to suspect that we can generalise Theorem 7 for arbitrary d, that is, that projective simulability is equivalent to d-outcome simulability in dimension d. However, this is already false for $d = 3$, as shown in the following example.

Example 14. For $d = 3$, consider a modified trine measurement

$$\mathbf{A} := \left(\frac{2}{3} |\psi_1\rangle\langle\psi_1| , \frac{2}{3} |\psi_2\rangle\langle\psi_2| , \frac{2}{3} |\psi_3\rangle\langle\psi_3| + |2\rangle\langle 2| \right) \in \mathcal{P}(3,3), \tag{5.51}$$

where $|\psi_j\rangle = \cos(\pi j/3) |0\rangle + \sin(\pi j/3) |1\rangle$ correspond to the effects of the regular trine qubit measurement $\mathbf{A}^{\text{trine}}$ (see Example **??**). The extremality of $\mathbf{A}$ follows from the extremality of $\mathbf{A}^{\text{trine}}$, since projecting a decomposition of the former onto the subspace spanned by $\{|0\rangle , |1\rangle\}$ would provide a decomposition for the latter.

Fortunately, there is still an alternative description of the projective-simulable qutrit measurements, convenient for SDP. We prove that these are exactly the extremal points of the set that comprehends 2-outcome POVMs (embedded in $\mathcal{P}(3,3)$ by adding an extra null effect) and 3-outcome POVMs with trace-1 effects. We achieve this with an analogue of the "method of perturbations" [DPP05] that allows one to check whether a given POVM $\mathbf{A} \in \mathcal{P}(d,n)$ is extremal or not. For sake of completeness, let us introduce the basics of this method.

A measurement $\mathbf{A} \in \mathcal{P}(d,n)$ is not extremal if and only if there exits a non-null vector of Hermitian operators $\mathbf{D} = (D_1, \ldots, D_n) \in \mathrm{Herm}(\mathbb{C}^d)^{\times n}$ such that for every output i we have

$$\mathrm{supp}(D_i) \subset \mathrm{supp}(A_i), \tag{5.52}$$

where $\mathrm{supp}(D)$ denotes the support of the Hermitian operator D, and

$$\sum_{i=1}^{n} D_i = 0. \tag{5.53}$$

The existence of a perturbation $\mathbf{D} \neq 0$ allows to construct POVMs

$$\mathbf{A}^+ = \mathbf{A} + \epsilon\mathbf{D},\ \mathbf{A}^- = \mathbf{A} - \epsilon\mathbf{D}, \tag{5.54}$$

meaning that $\pm\epsilon\mathbf{D}$ preserves the "POVMness" of $\mathbf{A}$ for sufficiently small $\epsilon > 0$. From (5.54) follows that $\mathbf{A} = \frac{1}{2}\left(\mathbf{A}^+ + \mathbf{A}^-\right)$, explicitly showing the non-extremality of $\mathbf{A}$.

Now we are able to proceed with the theorem.

Theorem 8. *A measurement $\boldsymbol{A} \in \mathcal{P}(3,n)$ is projective-simulable if and only if it can be simulated by 2-outcome measurements together with 3-outcome measurements with trace-1 effects.*

Proof. The "if" part of the theorem follows from the fact that every projective measurement in dimension three has either one outcome (and therefore is trivial), two outcomes (and therefore is trivially 2-outcome simulable) or three outcomes (and therefore each effect is trace-1).

For the converse, Lemma 2 already showed that any 2-outcome POVM is projective-simulable, what leaves us only to prove that 3-outcome, trace-1 measurements are projective-simulable. Denoting the set of such measurements by $\mathcal{P}_1(3,3)$, we apply a method of perturbations similar to the one presented above to show that the extremal points of $\mathcal{P}_1(3,3)$ are projective POVMs. Namely, we search for non-zero perturbations $\mathbf{D} = (D_1, D_2, D_3)$ that preserve both the POVMness and the trace of the effects of a given $\mathbf{A} \in \mathcal{P}_1(3,3)$, that is, a vector of operators $\mathbf{D}$ satisfying, for each outcome $i \in \{1, \ldots, 3\}$,

(i) $\mathrm{supp}(D_i) \subset \mathrm{supp}(A_i)$,

(ii) $\mathrm{Tr}(D_i) = 0$, and

(iii) $\sum_{i=1}^{3} D_i = 0$.

Thus $\mathbf{A}$ is extremal in $\mathcal{P}_1(d,n)$ if and only if there is no perturbation satisfying (i)-(iii) other than the null one.

Let now $r(\mathbf{A}) = (r_1, r_2, r_3)$ be the list of ranks of the effects of $\mathbf{A}$, where $r_i = \text{rank}(A_i) \in \{1, \ldots, 3\}$ (permutations in the order of the ranks correspond to relabelling, which do not influence the simulability of such a measurement). In what follows we show that whenever $r(\mathbf{A}) \neq (1,1,1)$ there always exist a perturbation $\mathbf{D} \neq 0$ satisfying the above conditions (i)-(iii), what completes the proof.

- The case $r(\mathbf{A}) = (a,b,0)$ is not possible, as $\text{Tr}(M_3) = 1 \neq 0$;
- The case $r(\mathbf{A}) = (3,3,1)$ is also impossible, since if A_3 has rank one and $\text{Tr}(A_3) = 1$ we necessarily have $A_3 = |\psi\rangle\langle\psi|$, for some unit vector $|\psi\rangle$. Therefore, the remaining effects must be supported in orthogonal spaces to this direction, and their ranks are at most 2;
- Case $r(\mathbf{A}) = (a,1,1)$, with $a > 1$, is also impossible for analogous reasons;
- Case $r(\mathbf{A}) = (3,b,c)$, with $b = 2,3$, we can take a perturbation $\mathbf{D}$ of the form

$$\mathbf{D} = (|\psi_1\rangle\langle\psi_2| + |\psi_2\rangle\langle\psi_1|\,, -|\psi_1\rangle\langle\psi_2| - |\psi_2\rangle\langle\psi_1|\,, 0)\,, \tag{5.55}$$

  where $|\psi_1\rangle\,, |\psi_2\rangle$ are eigenvectors of M_2 corresponding to non-zero eigenvalues. This ensures that D_2 is supported in the support of A_2; A_3 is full-rank so it accepts any traceless perturbation, and A_3 remains untouched;
- Case $r(\mathbf{A}) = (2,2,2)$, for each $j = 1, \ldots, 3$ we consider the traceless operators

$$X^j = |\psi_1^j\rangle\langle\psi_2^j| + |\psi_2^j\rangle\langle\psi_1^j| \tag{5.56}$$

$$Y^j = i|\psi_1^j\rangle\langle\psi_2^j| - i|\psi_2^j\rangle\langle\psi_1^j| \tag{5.57}$$

$$Z^j = |\psi_1^j\rangle\langle\psi_1^j| - |\psi_2^j\rangle\langle\psi_2^i|, \tag{5.58}$$

  analogous to the Pauli matrices but constructed from the eigenvectors $|\psi_1^j\rangle, |\psi_2^j\rangle$ associated to non-null eigenvalues of A_j. Therefore,

$$D_j = \alpha_X^j X^j + \alpha_Y^j Y^j + \alpha_Z^j Z^j, \tag{5.59}$$

  $j = 1, \ldots, 3$, satisfies conditions (i) and (ii) for any coefficients. Since there are 9 such operators and the space of traceless Hermitian operators of $\mathbb{C}^3$ is 8-dimensional, we see that at least one of them can be written as a linear combination of the remaining ones. Hence, for suitable choices of coefficients $\alpha_X^j, \alpha_Y^j, \alpha_Z^j$ the operators D_j in Eq. (5.59) satisfy (iii) as well and compose a proper perturbation $\mathbf{D}$.

- The remaining case $r(\mathbf{A}) = (2,2,1)$ is easy to analyse, as $\mathrm{Tr}(A_3) = 1$ implies $A_3 = |\psi\rangle\langle\psi|$, for some unit vector $|\psi\rangle$. Then the operators A_1, A_2 necessarily commute and have the same support. Consequently, the perturbation given in (5.55) works.

□

Similarly to the qubit case, the above characterisation of projective-simulable qutrit POVMs reduces deciding whether a measurement $\mathbf{A}$ is projective-simulable to an SDP:

$$
\begin{aligned}
\text{Given } & \mathbf{A} = (A_1, \dots, A_n) \in \mathcal{P}(3,n) \\
\max \ & t \\
\text{s.t. } & \Phi_t(\mathbf{A}) = \sum_{i=1}^{\binom{n}{2}} \mathbf{N}^{(i)} + \sum_{j=1}^{\binom{n}{3}} \mathbf{M}^{(j)} \\
& N_1^{(1)}, N_2^{(1)} \geq 0 \\
& N_1^{(1)} + N_2^{(1)} = p_1 \mathbb{I} \\
& N_i^{(1)} = 0, \ \forall i \notin \{1,2\} \\
& \vdots \\
& N_{\binom{n}{2}-1}^{\binom{n}{2}}, N_{\binom{n}{2}}^{\binom{n}{2}} \geq 0 \\
& N_{\binom{n}{2}-1}^{\binom{n}{2}} + N_{\binom{n}{2}}^{\binom{n}{2}} = p_{\binom{n}{2}} \mathbb{I} \\
& N_i^{\binom{n}{2}} = 0, \ \forall i \neq \binom{n}{2} - 1, \binom{n}{2} \\
& M_1^{(1)}, M_2^{(1)}, M_3^{(1)} \geq 0 \\
& M_1^{(1)} + M_2^{(1)} + M_3^{(1)} = q_1 \mathbb{I} \\
& M_j^{(1)} = 0, \ \forall j \neq 1,2,3 \\
& \vdots \\
& M_{\binom{n}{3}-2}^{\binom{n}{3}}, M_{\binom{n}{3}-1}^{\binom{n}{3}}, M_{\binom{n}{3}}^{\binom{n}{3}} \geq 0
\end{aligned}
\tag{5.60}
$$

$$
\begin{aligned}
&M^{\binom{n}{3}}_{\binom{n}{3}-2} + M^{\binom{n}{3}}_{\binom{n}{3}-1} + M^{\binom{n}{3}}_{\binom{n}{3}} = q_{\binom{n}{3}}\mathbb{I} \\
&M^{\binom{n}{3}}_{j} = 0,\ \forall j \neq \binom{n}{3} - 2, \binom{n}{3} - 1, \binom{n}{3} \\
&\mathrm{Tr}(M^{(j)}_{l}) = q_j,\ \forall j, l \\
&p_i \geq 0,\ \forall i \\
&q_j \geq 0,\ \forall i \\
&\sum_i p_i + \sum_j q_j = 1,
\end{aligned}
$$

where the variable $\mathbf{N}^{(i)}$ encodes a 2-outcome measurement together with its weight p_i and $\mathbf{M}^{(j)}$ encodes a 3-outcome, trace-1 measurement together with its weight q_j.

At this point we are inclined to try to generalise Theorem 8, conjecturing that the extremal trace-one d-outcome qudit POVMs are projective also for $d > 3$. However, already for $d = 4$ we can find a counterexample.

Example 15. Consider the 4-dimensional, trace-1 POVM $\mathbf{T} = \mathbf{A}^{(12)} + \mathbf{A}^{(34)} \in \mathcal{P}(4,4)$ defined by the sum of two copies of the tetrahedral POVM $\mathbf{A}^{\text{tetra}}$ (see Eq. (5.32)) supported in orthogonal two-dimensional subspaces of $\mathbb{C}^4$. Again, $\mathbf{T}$ is extremal since projections of its decomposition would provide a decomposition for $\mathbf{A}^{\text{tetra}}$.

5.4.4 Projective simulability and joint measurability for $d = 2$

In Section 5.3.4, we showed that if $\mathbf{A}$ is simulable by dichotomic measurements of the form

$$B_i = \frac{1}{2}\mathbb{I} + \vec{v}_i \cdot \vec{\lambda},\ i = 1, 2, \tag{5.61}$$

called unbiased 2-outcome measurements, then $\mathbf{A}$ is jointly measurable with its antipodal measurement $\bar{\mathbf{A}}$. In dimension $d = 2$, we see that unbiased 2-outcome measurements are exactly the projective POVMs and their depolarised versions.

In this particular case, where 2-outcome and projective-simulability coincide (Lemma 2), we can prove the converse of Proposition 5, which completely characterises the projective-simulable qubit POVMs.

Theorem 9. *A qubit POVM is projective-simulable if and only if the pair $\{\mathbf{A}, \bar{\mathbf{A}}\}$ is jointly measurable, where $\bar{\mathbf{A}}$ is the antipodal measurement of $\mathbf{A}$.*

Proof. The "only if" part is a particular case of Proposition 5, so we need only to show the "if" part.

Assume that $\mathbf{A}$ and $\bar{\mathbf{A}}$ are jointly measurable and $\mathbf{M}$ is a joint measurement for the pair, described by $M_{ab} = m_{ab}\mathbb{I} + \vec{w}_{ab} \cdot \vec{\sigma}$. Consider now $N_{ab} = (M_{ab} + \bar{M}_{ba})/2$, where $\bar{M}_{ba}$ represents the antipodal operator of M_{ba} (which is also non-negative since $d = 2$). We have that $\mathbf{N}$ is also a joint POVM for the pair, since $N_{ab} \geq 0$ for all a, b and

$$\sum_b N_{ab} = \frac{1}{2}\left(\sum_b M_{ab} + \sum_b \bar{M}_{ba}\right) = A_a \tag{5.62a}$$

$$\sum_a N_{ab} = \frac{1}{2}\left(\sum_a M_{ab} + \sum_a \bar{M}_{ba}\right) = \bar{A}_b, \tag{5.62b}$$

with the feature that symmetric effects sum up to a multiple of the identity,

$$N_{ab} + N_{ba} = (m_{ab} + m_{ba})\mathbb{I}. \tag{5.63}$$

Thus Eqs. (5.62) guarantee the decomposition

$$\mathbf{A} = \sum_{a \leq b} (m_{ab} + m_{ba})\mathbf{B}^{(ab)}, \tag{5.64}$$

where the POVMs $\mathbf{B}^{(ab)}$ are defined by

$$B_s^{ab} = \begin{cases} N_{ab}/(m_{ab} + m_{ba}), & \text{if } s = a \\ N_{ba}/(m_{ab} + m_{ba}), & \text{if } s = b \\ 0, & \text{otherwise} \end{cases}, \tag{5.65}$$

and therefore can be interpreted as 2-outcome measurements embedded in the space of n-outcome POVMs. The normalization of $\mathbf{N}$ implies that $\sum_{a,b} m_{ab} = 1$, which ensures that the decomposition is convex. Finally, since every 2-outcome measurement is projective-simulable (Lemma 2), we conclude that $\mathbf{A}$ is projective-simulable. □

Example 16. As seen in Example 13, $\Phi_t(\mathbf{A}^{\text{tetra}})$ is projective-simulable if and only if $t \leq \sqrt{2/3}$. Thus, according to Theorem 9, for the same range of visibilities t the pair of POVMs $\{\mathbf{A}^{\text{tetra}}, \bar{\mathbf{A}}^{\text{tetra}}\}$ is jointly measurable (see Figure 5.3).

A direct consequence of Theorem 9 is given by the close connection between joint measurability and EPR steering presented in Theorem 4, namely that a set of POVMs is jointly measurable if and only if it cannot demonstrate steering when applied to any quantum state. Hence we see that projective simulability is also connected to EPR steering.

Corollary 1. *A qubit measurement $\boldsymbol{A}$ is projective-simulable if and only if the pair $\{\boldsymbol{A}, \bar{\boldsymbol{A}}\}$ cannot demonstrate steering when applied to any quantum state of local dimension 2.*

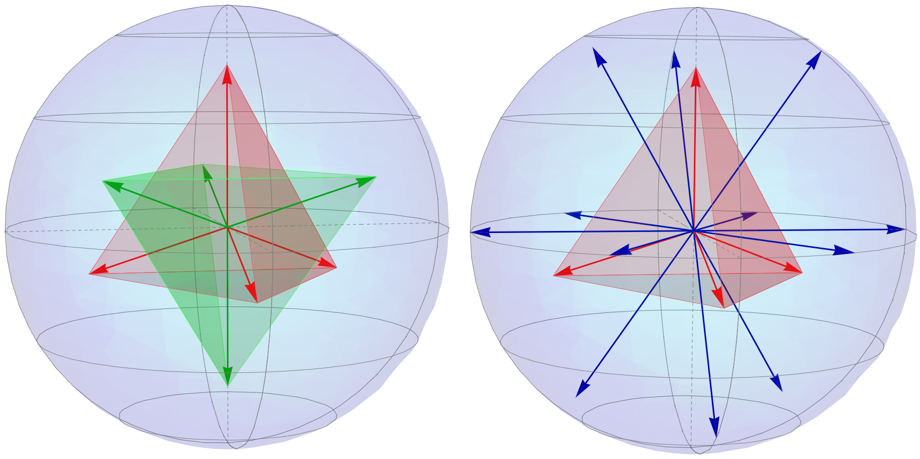


Figure 5.3: Two antipodal tetrahedral measurements and their optimal projective simulators, which define a joint measurement for the tetrahedral pair. Both joint measurability with the antipodal and projective simulability are achieved at the same critical visibility $t = \sqrt{2/3}$.

5.5 Quantum measurement simulability as a resource theory

The approach to measurement simulability we use here is close to a resource theory in many aspects. A resource theory is a formal framework to study a given property of a class of objects, which plays the role of *resource*. The framework is defined by a subset of operations called *free operations*, that has the key feature of not being able to generate the resource. This means that when a free operation is applied to a *free object*, *i.e.*to an object without the property of interest, the resulting object is also free. This approach was succesfully used to investigate properties such as entanglement [VPRK97, PV07], thermal equilibrium [BaHO$^+$13], asymmetry [AJR13], reference frames [GS08], and nonlocality [GWAN12].

In our case, for every type of simulators $\mathcal{B}$ we can define a resource theory where the resource is the non-$\mathcal{B}$-simulability. In the case of J-POVM simulability (Section 5.2.2), the objects are sets of quantum measurements, the free operations are classical processing, and sets of J measurements are free objects, implying that every simulable set is also free. Analogously, in the case of k-outcome and projective simulability (Sections 5.3 and 5.4), the objects are single measurements, and the free operations and objects are again classical processing and simulable measurements, respectively.

To formalise these notions, we prove now the invariance of the set of simulable POVMs by classical processing. We show that the simulability relation is transitive, namely that if a set of measurements is $\mathcal{B}$-simulable, then any classical manipulation of it is $\mathcal{B}$-simulable as well. This encompasses J-POVM simulability of sets of POVMs as a particular case, as well as k-outcome and projective simulability of single POVMs.

Proposition 7. *Let $\mathcal{B} \subset \mathcal{P}(d,n)$ be a subset of measurements. If a set of measurements $\mathcal{A} = \{\mathbf{A}^{(l)}\}$ is $\mathcal{B}$-simulable, then any set $\tilde{\mathcal{A}}$ obtained by classically processing $\mathcal{A}$ is $\mathcal{B}$-simulable as well.*

Proof. Suppose $\tilde{\mathcal{A}}$ contains POVMs $\tilde{\mathbf{A}}^{(l)}$, constructed by pre- and post-processing the elements of $\mathcal{A}$,

$$\tilde{A}^{(k)}_{a_k} = \sum_l p'(l|k) \sum_{a_l} q'(a_k|k,l,a_l) A^{(l)}_{a_l}, \tag{5.66}$$

for all outcomes a_k and for some probability distributions $p'(\cdot|k), q'(\cdot|k,l,a_l)$, where k runs over the number of elements of $\mathcal{A}'$, l runs over the number of elements of $\mathcal{A}$, and a_l runs over the outcomes of $\mathbf{A}^{(l)}$. Since we can simulate $\mathcal{A}$ using $\mathcal{B}$, there are probability distributions $p(\cdot|l), q(\cdot|l,j,b_j)$, where j labels a POVM $\mathbf{B}^{(j)} \in \mathcal{B}$ and b_j its outcomes, satisfying Eq. (5.4). Thus we can substitute it in the above equation, yielding

$$\tilde{A}^{(k)}_{a_k} = \sum_j \tilde{p}(j|k) \sum_{b_j} \tilde{q}(a_k|k,j,b_j) B^{(j)}_{b_j}, \tag{5.67}$$

where

$$\tilde{p}(\cdot|k) := \sum_l p'(l|k) p(\cdot|l) \tag{5.68}$$

$$\tilde{q}(\cdot|k,j,b_j) := \sum_{a_l} q'(\cdot|k,l,a_l) q(a_l|l,j,b_j) \tag{5.69}$$

define pre- and post-processings that simulate $\tilde{\mathcal{A}}$ with $\mathcal{B}$. □

A secondary but still important element of a resource theory is a way of quantifying the resource. A quantifier function must be monotonic with respect to the free operations, meaning that by performing a free operation one should not be able to increase the measured quantity of resource of the initial object.

Usually the same theory allows many different quantifiers. We finish this section showing that, for measurement simulability, the white noise robustness of a set of measurements,

$$t^{\mathcal{A}}_{\mathcal{B}} = \max\{t;\ \Phi_t(\mathcal{A}) \text{ is } \mathcal{B}\text{-simulable}\}, \tag{5.70}$$

is a suitable measure of non-simulability.

Proposition 8. *The white noise robustness of a set of POVMs regarding $\mathcal{B}$ simulability is monotonic with respect to classical processings.*

Proof. Suppose $\tilde{\mathcal{A}}$ is obtained by classical processing $\mathcal{A}$. Following Eq. (5.66), we have

$$\Phi_t(\tilde{A}^{(k)}_{a_k}) = \sum_l p'(l|k) \sum_{a_l} q'(a_k|k,l,a_l) \Phi_t(A^{(l)}_{a_l}). \tag{5.71}$$

This implies that at the critical visibility $t^{\mathcal{A}}_{\mathcal{B}}$ that makes $\mathcal{A}$ $\mathcal{B}$-simulable we can write each effect $\Phi_t(A^{(l)}_{a_l})$ as an appropriate combination of effects of the simulators, and then substitute in the previous equation to find that $\tilde{\mathcal{A}}$ is also $\mathcal{B}$-simulable. Therefore, $t^{\tilde{\mathcal{A}}}_{\mathcal{B}} \geq t^{\mathcal{A}}_{\mathcal{B}}$. $\square$

Chapter 6

Optimising over the set of quantum measurements

In the previous chapters we showed that for a given d-dimensional quantum measurement $\mathbf{A} \in \mathcal{P}(d,n)$ with n outcomes it is possible to decide whether it is projective simulable by running an SDP, as long as $d \in \{2,3\}$. Similarly, for a fixed set of measurements $\mathcal{A} \subset \mathcal{P}(d,n)$, it is possible to check whether $\mathcal{A}$ is jointly measurable or not by means of an SDP. Hence both problems can be computationally solved efficiently, and both cases can be adapted to yield the white-noise robustness of such POVM/set of POVMs regarding that property.

Since we can solve individual instances of the mentioned properties, we move the question a level above: What is the measurement that has to be depolarised the most in order to become projective simulable? Which is the most incompatible set of measurements?

Clearly the answer to these questions has to be an extremal object, since the set of measurements $\mathcal{P}(d,n)$ (and therefore the set of tuples of measurements $\mathcal{P}(d,n)^m$) is convex and the depolarising map is linear. However, both sets have an infinite number of extremal points, thus being impractical to test each of them.

The strategy we developed to investigate these problems is to construct polytopes (convex sets with a finite number of extremal points) that contain each set of interest. Therefore, since each point of $\mathcal{P}(d,n)$ can be expressed as a convex combination of these finitely many extremes, the optimal extreme of such polytope provides a lower bound for the critical robustness of any POVM in $\mathcal{P}(d,n)$. Furthermore, the constructed polytopes are able to approximate arbitrarily well the convex sets, thus yielding arbitrarily tight lower bounds. On the other hand, any point of the set provides an upper bound for the general critical depolarisation, since each point is at least as robust as the most robust point.

We first depict the construction of outer polytopes containing the set $\mathcal{P}(d,n)$, based on a relaxation of the positive semidefinite constraint. Then we apply it first to investigate the most robust measurements regarding projective-simulability [OGWA17], and later to find the most incompatible sets of measurements [BQG+17].

Our method is general and can in principle be used in any situation. However, regarding computational implementations, here we were limited by the available computational power to work only in dimension $d = 2$ and tuples with a restricted number of POVMs.

6.1 Approximating the set of quantum measurements by outer polytopes

Recall that a quantum measurement is a collection of positive semi-definite operators that sum to the identity. We can characterise a positive semi-definite (PSD) operator $A \in \text{Herm}(\mathbb{C}^d)$ as an operator for which

$$\text{Tr}(A\,|\psi\rangle\langle\psi|) \geq 0,\ \forall\,|\psi\rangle \in \mathbb{C}^d. \tag{6.1}$$

This condition represents an infinite number of constraints, as there are infinite pure states $|\psi\rangle$ in $\mathbb{C}^d$. This is a reflection of the fact that the cone of PSD operators has infinitely many extremal points.

A possible relaxation of the PSD property is to require the above inequality to hold only for a finite number of pure states, that is,

$$\text{Tr}(A\,|\psi_i\rangle\langle\psi_i|) \geq 0,\ |\psi_i\rangle \in \mathbb{C}^d,\ i = 1,\dots,N. \tag{6.2}$$

Writing

$$A = \alpha\mathbb{I}_d + \vec{a}\cdot\vec{\lambda}, \tag{6.3}$$

$$|\psi_i\rangle\langle\psi_i| = \frac{1}{d}\mathbb{I}_d + \vec{v}_i\cdot\vec{\lambda}, \tag{6.4}$$

where $\vec{\lambda}$ is the vector of generalised Pauli matrices and $\vec{v}_i$ is a unit vector of $\mathbb{R}^{d^2-1}$, we can rewrite the inequalities (6.2) in terms of the basis $\{\mathbb{I}, \lambda_1, \dots, \lambda_{d^2-1}\}$. Every PSD operator satisfies these conditions, thus the set of operators satisfying (6.2) contains the set of PSD operators. Since the above inequalities are linear in A, we can interpret them as a finite set of facets tangent to the PSD set and delimiting a larger set of so-called **quasi-positive** operators, which therefore correspond to an object with a finite number of extremal points.

This relaxation holds for operators acting on any dimension d, and the greater the number N of constraints, the better the approximation.

As a POVM is a normalised collection of positive operators, we say that a d-dimensional n-outcome **quasi-POVM** is a collection of quasi-positive operators acting on $\mathbb{C}^d$ that sum up to the identity, and denote the set of quasi-POVMs by

$$\Delta(d,n) := \{(Q_1,\ldots,Q_n); \mathrm{Tr}(Q_i\,|\psi_i\rangle\langle\psi_i|) \geq 0 \;\; i=1,\ldots,N, \; \sum_i Q_i = \mathbb{I}\}, \quad (6.5)$$

for some fixed pure states $|\psi_i\rangle \in \mathbb{C}^d$. Since the normalisation constraint is also linear, it represents another facet in the larger space $\mathrm{Herm}(\mathbb{C}^d)^{\times n}$ that contains $\Delta(d,n)$. Therefore $\Delta(d,n)$ is a polytope.

6.1.1 Approximating $\mathcal{P}(2,4)$ by outer polytopes

We now discuss in detail the construction of outer polytopes to approximate the set $\mathcal{P}(2,4)$ of 4-outcome qubit measurements and explore several symmetries in order to optimise their description. Recall that every extremal qudit measurement has at most d^2 outcomes [DPP05], hence all extremal qubit POVMs belong to $\mathcal{P}(2,4)$ or are embedded copies of some POVM belonging to this space.

For $d=2$, we can use the Bloch sphere to have a geometrical visualisation of the relaxation previously described. Using Eq. (6.3) and calculating the eigenvalues μ of A we get

$$\mu_{\pm} = \alpha \pm ||\vec{a}||. \quad (6.6)$$

Hence A is positive if and only if $\alpha \geq ||\vec{a}||$ and the 3-dimensional real vector $\vec{a}$ lies inside the sphere with radius α. The Bloch sphere corresponds to the geometric place of trace-1 operators, *i.e.*, where $\alpha = 1/2$. In this context, we see that condition (6.2) depicts an external polyhedron tangent to the α-sphere exactly in the points represented by the Bloch vectors $\vec{v}_i$ (see Fig. 6.1).

We start writing a quasi-POVM $\mathbf{Q}$ as a 16-dimensional real vector

$$\mathbf{Q} \equiv (\alpha_1, x_1, y_1, z_1, ..., \alpha_4, x_4, y_4, z_4) \in \mathbb{R}^{16}, \quad (6.7)$$

where each four entries representing one quasi-positive operator $Q_i = \alpha_i\mathbb{I} + x_i\sigma_x + y_i\sigma_y + z_i\sigma_z$ in the basis of Pauli operators. An initial description of $\Delta(2,4)$ is given by $4N$ inequalities defined by (6.2) plus the 8 "global" constraints

$$\alpha_i \geq 0, \; i = 1, ..., 4 \quad (6.8)$$

$$\sum_i \alpha_i = 1 \quad (6.9)$$

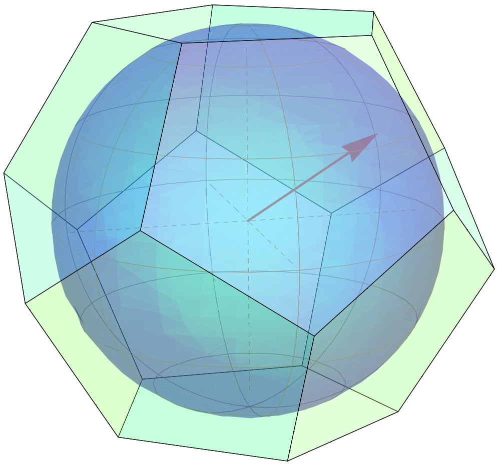


Figure 6.1: Example of an approximation of the α-Bloch sphere by an outer polytope, given by unit vectors at the tangency points.

$$\sum_i x_i = \sum_i y_i = \sum_i z_i = 0, \tag{6.10}$$

where the last two lines are equivalent to $\sum_i Q_i = \mathbb{I}$.

We now list a series of simplifications, with the goal of minimising the number of parameters needed to describe $\Delta(2,4)$. One extra assumption we make is that the feature we investigate is invariant under unitary rotations (such as projective simulability).

- The normalisation of the quasi-POVMs allows us to write $Q_4 = \mathbb{I} - Q_1 - Q_2 - Q_3$, and thus we can drop the 4 parameters describing Q_4;
- Since we can consider a single representative of each class of unitarily equivalent measurements, $bA \equiv U.A := (UA_1U^\dagger, \ldots, UA_4U^\dagger)$, we can restrict our construction to POVMs where (i) the Bloch vector of the first effect points in the direction of the x axis; and (ii) the Bloch vector of the second effect lies on the xy plane;
- By doing this, we describe A_1 with only two parameters $A_1 = \alpha_1 + x_1\sigma_x$, and this description is exact: A_1 is a genuine effect (as opposed to a quasi-effect). Since extremal POVMs with d^2 outcomes have rank-one effects [DPP05], we can set $\alpha_1 = x_1$ and dismiss one more parameter.
- With M_2 lying in the xy plane, we can drop the parameter z_2;

- Since M_2 lies in the plane (more specifically in a circle), we can always assume that this vector lies in the y-positive semi-plane, and take a polygon approximating the semi-circle (rather than a polyhedron approximating the sphere, as in the general case). In what follows, we refer to the Bloch vectorss representing the extremal points of this polygon by $\{\vec{w}_i\}_{i=1}^{N'}$;

By implementing these settings, we obtain a description of $\mathbf{Q}$ with only 8 parameters,

$$\mathbf{Q} \equiv (\alpha_1, \alpha_2, x_2, y_2, \alpha_3, x_3, y_3, z_3). \tag{6.11}$$

In case we wish to improve the approximation in the vicinity of a given POVM, say $\mathbf{A}^{\text{tetra}}$, we can adapt the construction of $\Delta(d, n)$ to be tangent to a rotated $\mathbf{A}^{\text{tetra}}$ having $A_1^{\text{tetra}} = (\mathbb{I} + \sigma_x)/4$. We thus add to $\{\vec{v}\}_{j=1}^{N}$ the vector $(\cos\frac{2\pi}{3}, \sin\frac{2\pi}{3}, 0)$, which corresponds to the second effect A_2^{tetra}, and add to vectors $\{\vec{v}_j\}_{j=1}^{N}$ the vertices $(-\frac{1}{2}, -\frac{\sqrt{3}}{4}, \pm\frac{3}{4})$ corresponding to the third and fourth effects of $\mathbf{A}^{\text{tetra}}$.

6.2 Most robust measurements regarding simulability

In Section 5.4 we showed how to calculate the white-noise robustness of qubit measurements regarding projective simulability. We now address the problem of finding the most robust qubit POVM regarding projective simulability.

Notice that if $\mathbf{A}^* \in \mathcal{P}(2, 4)$ is the most robust POVM and t^*_{proj} is its robustness, then $\Phi_{t^*_{\text{proj}}}(\mathbf{A})$ is projective-simulable for any POVM $\mathbf{A} \in \mathcal{P}(2, n)$. We will denote such optimal robustness by $t_{\text{proj}}(2)$, making explicit the dimension of the POVM. Then Theorem 6 says that $t_{\text{proj}}(2) \geq 1/2$, and more generally, $t_{\text{proj}}(d) \geq 1/d$. Now we will apply the polytopes constructed in the previous section to show that $t_{\text{proj}}(2)$ is very close to $t^{\text{tetra}}_{\text{proj}}$, the robustness of $\mathbf{A}^{\text{tetra}}$.

Recall that our strategy to lower bound $t_{\text{proj}}(2)$ was to construct a sequence of polytopes $\Delta(2, 4) \subset \text{Herm}(\mathbb{C}^2)^{\times 4}$ containing the set of four-outcome qubit POVMs $\mathcal{P}(2, 4)$. Then we check using the SDP (5.31) the minimum amount of depolarisation needed for the projective simulation of each extremal point of $\Delta(d, n)$, and find the most robust one among the finitely many of them. The value t^{Δ}_{proj} yielded this way is a lower bound for the minimum amount of depolarisation $t_{\text{proj}}(2)$ for which any qubit POVM becomes simulable by projective measurements.

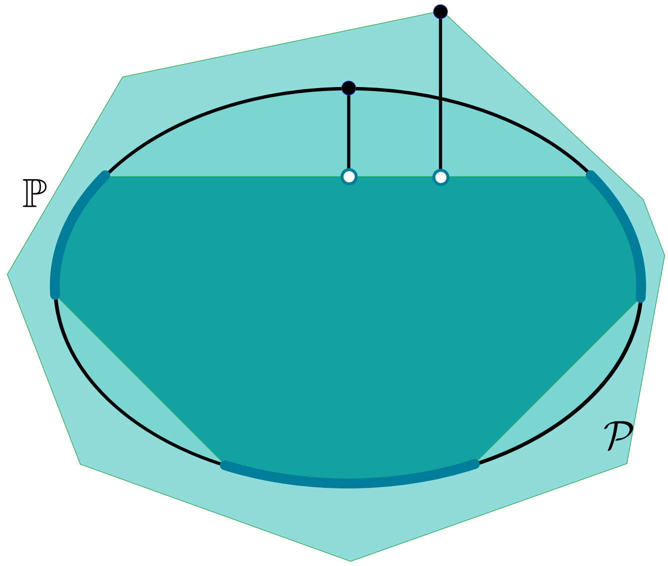


Figure 6.2: Representing the set $\mathcal{P}$ of POVMs by an ellipse, projective measurements form the subset $\mathbb{P}$ of the extremal points of $\mathcal{P}$ (in bold green), and in dark green we have the set of projective-simulable POVMs. As we depolarise a non-simulable point, we move it in direction of the projective-simulable set; thus, in some sense, we want to find the farthest point from this set. By constructing a polytope containing $\mathcal{P}$ we are able to lower bound the robustness of the most robust POVM in $\mathcal{P}$ by testing only the finitely many extremal points of the polytope. In the qubit case, this approximation can be computationally constructed to be very close to the actual set $\mathcal{P}$.

Setting $\{\vec{w}_i\}_{i=1}^{N'}$ to be the vertices of the "regular half-polygon" of 100 sides and the remaining $\vec{v}_i$ to be the vertices of the Archimedean solid called truncated icosahedron together with the vertices of its dual polyhedron, we obtain a polytope $\Delta(2,4)$ with $\approx 850,000$ extremal points and $t^{\Delta}_{\text{proj}} \approx 0.8160$, quite close to $t^{\text{tetra}}_{\text{proj}} = \sqrt{2/3} \approx 0.8165$.

Theorem 10. *For any qubit measurement $\boldsymbol{A} \in \mathcal{P}(2,n)$, the depolarised measurement $\Phi_{t^*}(\boldsymbol{A})$ is projective-simulable, where $t^* = \sqrt{2/3} - \epsilon$ and $\epsilon \approx 10^{-4}$.*

This provides strong evidence that $\mathbf{A}^{\text{tetra}}$ is the most robust qubit measurement, which was analytically proven in Ref. [HQV+17].

As a corollary, we see that the pair of antipodal tetrahedrons is the most robust pair of antipodal qubit measurements regarding joint measurability, since joint measurability with the antipodal is equivalent to projective simulability for qubit POVMs (Theorem 9).

Corollary 2. *$\{A^{tetra}, \bar{A}^{tetra}\}$ is the most robust pair of antipodal qubit measurements regarding joint measurability.*

6.3 Most incompatible sets of measurements

In the previous section we approximated the optimal robustness of qubit POVMS regarding projective simulability. Notice that we could use the very same strategy to find the optimal measurement regarding any other feature that depends only on the individual measurement (such as k-simulability, for instance).

In the same fashion that we could find the amount of depolarisation needed to make a target POVM simulable, we can calculate the amount of depolarisation needed to make a target tuple of measurements jointly measurable (see Chapter 3). In this section our goal is to implement the outer polytope approach to find the most incompatible set of m measurements. Due to the close relation between joint measurability and EPR steering (Theorem 4), this set will also be the most robust regarding the steerability of the maximally entangled state Φ_+.

The adaptation of the previous idea is simple. Given a tuple of m measurements

$$\mathcal{A} = \{\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(m)}\} \in \mathcal{P}(d,n)^{\times m} \tag{6.12}$$

and a polytope $\Delta(d,n)$ containing $\mathcal{P}(d,n)$ as constructed in Section 6.1, we have that

$$\mathcal{A} \in \Delta(d,n)^{\times m}. \tag{6.13}$$

Therefore, every tuple $\mathcal{A}$ can be decomposed into the extremal points of $\Delta(d,n)^{\times m}$, which is just the product of m extremal points of $\Delta(d,n)$,

$$\text{ext}[\Delta(d,n)^{\times m}] = \text{ext}[\Delta(d,n)]^{\times m}. \tag{6.14}$$

Hence, given $\Delta(d,n)$ with e extremal points, we obtain a larger polytope $\Delta(d,n)^{\times m}$ with e^m extremal points, from which only $\binom{e}{m}$ are relevant for the study of joint measurability (since we are interested in tuples of m distinct POVMs).

Again due to computational limitations, we could investigate only sets of at most five 4-outcome measurements in dimension 2, that is, the sets $\mathcal{P}(2,4)^m$, with $m = 2,\ldots,5$. For all these cases, numerical searches indicated that sets of projective measurement were more incompatible than any set of general POVMs. Translating this to the context of EPR steering via Theorem 4, our numerical findings reinforce the conjecture that states that are projective-unsteerable are also unsteerable for general measurements.

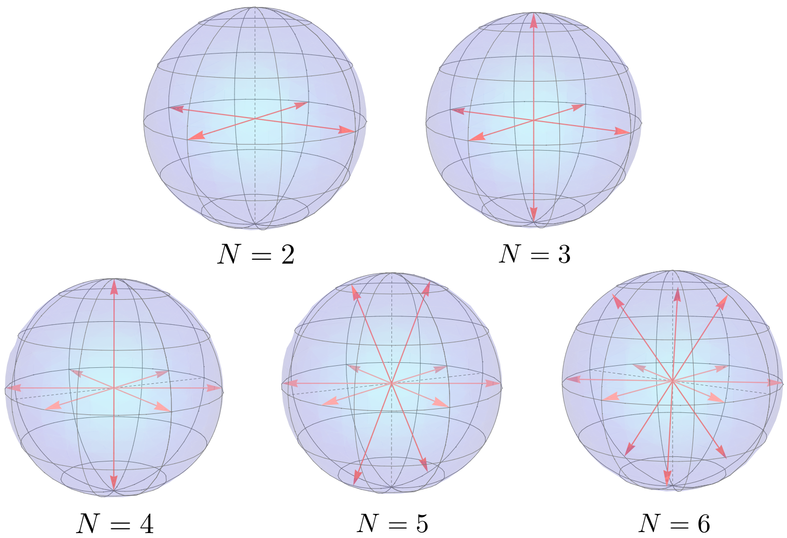


Figure 6.3: Candidates for the most incompatible set of $N \in \{2, \ldots, 6\}$ qubit measurements. Equivalently, these sets are also candidates for the optimal set of $N \in \{2, \ldots, 6\}$ qubit measurements for steering the two-qubit Werner states. Each par of antipodal vectors represents a projective measurement.

The Bloch-vectors of the optimal sets we found are distributed in a particular way: for 2 and 3 measurements, we have sets of orthogonal vectors; for 4 measurements we have 3 coplanar and equally distributed vectors and 1 vector orthogonal to the other 3; for 5 and 6 measurements, the structure of 3 coplanar equally spaced vectors is maintained and the other vectors are agglomerated in the poles of the sphere with the same z-projection (see Fig. 6.3).

m	Upper bound	Lower bound
2	0.7071	0.7071
3	0.5774	0.5755
4	0.5547	0.5437
5	0.5422	0.5283
6	0.5270	–

Table 6.1: Summary of numerical results for the critical white-noise robustness of m qubit measurements regarding joint measurability. The upper bounds were obtained via the polytope approximation method, while the lower bounds are given by particular examples of sets of POVMs, found via "see-saw" algorithms and parametric searches [BQG+17].

For $m \leq 5$ we used the outer polytope technique to produce lower bounds for the optimal robustness. Unfortunately for greater number of measurements this strategy becomes prohibitively expensive from the computational point of view. In Table 6.1 we present the lower bounds obtained together with the upper bounds generated by the projective measurements described above.

Chapter 7

Applications to local-hidden-variable models

In this chapter we apply results developed in the previous chapters to the context of Bell nonlocality, more specifically for constructing LHV models for entangled quantum states. Starting from LHS models for a given state and a restricted subset of quantum measurements, we investigate how to extend it in order to reproduce the statistics of any measurement. The connection we will use is the following link between simulability and LHV models.

Lemma 3. *Let $\mathcal{B} \subset \mathcal{P}(d,n)$ be a subset of quantum measurements and ρ a quantum state of a bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$ which admits an LHS model for $\mathcal{B}$,*

$$Tr(B_i \otimes B'_j \rho) = \int d\Pi(\lambda) f_A(i|\boldsymbol{B},\lambda) f_B(j|\boldsymbol{B}',\lambda), \tag{7.1}$$

for any outcomes i,j of measurements $\boldsymbol{B}, \boldsymbol{B}' \in \mathcal{B}$. Then ρ admits an LHV model for any $\mathcal{B}$-simulable measurement.

Proof. Suppose Eq. (7.1) holds for any pair of measurements in $\mathcal{B} = \{\mathbf{B}^{(j)}\}$ and that $\mathbf{A}, \mathbf{A}'$ are $\mathcal{B}$-simulable,

$$A_i = \sum_l p(l|\mathbf{A}) \sum_k q(i|\mathbf{A}, l, k) B_k^{(l)} \tag{7.2}$$

$$A'_i = \sum_l p(l|\mathbf{A}') \sum_k q(i|\mathbf{A}', l, k) B_k^{(l)}. \tag{7.3}$$

Then we can adapt the simulation protocol to the response functions f_A, f_B and define

$$\tilde{f}_A(i|\mathbf{A},\lambda) = \sum_l p(l|\mathbf{A}) \sum_k q(i|\mathbf{A}, l, k) f_A(k|\mathbf{B}^{(l)},\lambda) \tag{7.4}$$

$$\tilde{f}_B(j|\mathbf{A}',\lambda) = \sum_l p(l|\mathbf{A}') \sum_k q(j|\mathbf{A}',l,k) f_B(k|\mathbf{B}^{(l)},\lambda), \tag{7.5}$$

which satisfy

$$\mathrm{Tr}(A_i \otimes A'_j \rho) = \int d\Pi(\lambda) \tilde{f}_A(i|\mathbf{A},\lambda) \tilde{f}_B(j|\mathbf{A}',\lambda). \tag{7.6}$$

□

In the following sections, we first consider the case where we have an LHS model for a finite number of projective measurements, and the particular instance of the quantum measurement simulability problem of simulating every other measurement via the initially given POVMs. From this we derive sufficient conditions to generalise the model for any measurement, to the cost of considering a noisier version of the state. Since this strategy is focused on the measurements, we can use it as a criterion to decide on the locality of arbitrary states, as well as for generating random local states different from the ones previously known [CGRS16].

Secondly, we address the case where we assume the existence of an LHS model for any projective measurement, and use projective-simulability robustness bounds to find conditions to extend the model to any general measurement. With this strategy we are able to construct the best LHV model for two-qubit Werner states regarding general POVMs so far [OGWA17].

7.1 A general method for constructing LHS models for arbitrary quantum states

As shown in Section 4.2, the problem of checking whether a state ρ admits an LHV model for the finite set of measurements $\{\mathbf{B}^{(j)}\}$ can be phrased as the SDP (4.10). Nevertheless, it remains a difficult task to decide whether ρ is unsteerable for *any* set of measurements. Crucially, all previous constructions make use of the symmetries present in the quantum states under scrutiny, and consequently they cannot be readily applied to different states. In fact, apart from a sufficient condition for the special case of two-qubits (and one-sided projective measurements) [**?**], there is no general criterion to test whether a given quantum state is local.

We now present sufficient conditions for a general quantum state to admit an LHV model, either for projective measurements or general POVMs, that can be tested via SDP. The main insight behind the following theorems is to use the geometry of the set of projective measurements to extend the model found by

SDP (4.10) to any measurement. Then the second step is to map the depolarisation from the measurements to the state. Therefore, we define the initial set of measurements to be projective, which ensures extremality in $\mathcal{P}(d,n)$ and are easier to describe, since they are always associated to unit vectors.

Theorem 11. *Let $\mathcal{B} \subset \mathcal{P}(d_A, n)$ be a finite collection of projective measurements in $\mathbb{C}^{d_A}$. A state ρ_{AB} acting on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ admits an LHS model for all projective measurements if there exists a unit-trace operator O_{AB} acting on the same Hilbert space, such that O_{AB} admits a LHS model for the measurements in $\mathcal{B}$, and*

$$\rho_{AB} = rO_{AB} + (1-r)\frac{\mathbb{I}_A}{d_A} \otimes O_B, \tag{7.7}$$

where r is the radius of the insphere[1] of the polytope generated by $\mathcal{B}$.

Proof. We first address the particular case where $d = 2$. Let $\mathcal{B}$ define a finite set of measurements for Alice given by rank-1 projective operators

$$B_i^{(j)} = \frac{\mathbb{I} + (-1)^i \vec{v}^{(j)} \cdot \vec{\sigma}}{2}, \tag{7.8}$$

where $j = 1, \ldots, m$, $i \in \{0, 1\}$, and $\vec{v}^{(j)} \in \mathbb{R}^3$. This measurement set can be chosen arbitrarily – for example in a regular fashion (along the vertices or faces of a regular solid), or at random. Suppose that these measurements, when applied to a given operator O_{AB}, have an LHS description (see Chapter 2) of the form

$$\mathrm{Tr}_A\left[\left(B_i^{(j)} \otimes \mathbb{I}_B\right) O_{AB}\right] = \int dq(\lambda) p_A(i|\mathbf{B}^{(j)}, \lambda)\rho_\lambda, \ \forall a, x. \tag{7.9}$$

According to Lemma 3, any measurement that is $\mathcal{B}$-simulable also has an LHS description. This is valid, in particular, for depolarised projective measurements whose elements are contained within a shrunken Bloch sphere completely contained inside the convex hull of $\mathcal{A}$ (see Fig. 7.1). Indeed, this r-sphere is given by operators

$$\Phi_r(B) = rB + (1-r)\frac{1}{2}\mathbb{I}_A. \tag{7.10}$$

Finally, notice that the depolarising channel is self-dual, *i.e.*,

$$\mathrm{Tr}_A[(\Phi_r(\Pi_i^{(j)}) \otimes \mathbb{I}_B)O_{AB}] = \mathrm{Tr}_A[\Pi_i^{(j)} \otimes \mathbb{I}_B(\Phi_r \otimes \mathbb{I}_B)(O_{AB})] \tag{7.11}$$

$$= \mathrm{Tr}_A[(\Pi_i^{(j)} \otimes \mathbb{I}_B)\rho_{AB}], \tag{7.12}$$

[1]The insphere of a polytope is the largest centered sphere contained in it.

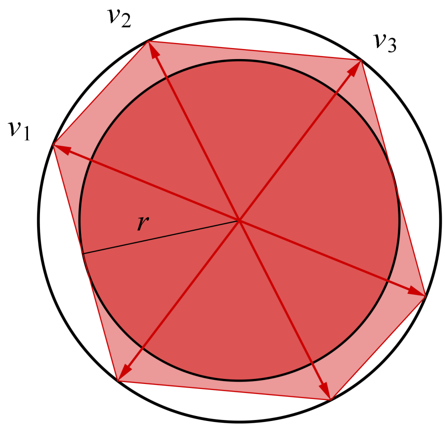


Figure 7.1: The Bloch vectors v_j, associated to projective measurements $\mathbf{B}^{(j)}$, define a polytope contained in the Bloch sphere. The r-sphere contained in it represents all r-depolarised operators. Hence, every measurement that can be represented inside the sphere, *i.e.*, is the r-depolarised version of some projective measurement, is $\{B^{(j)}\}$-simulable.

assuming that $\rho_{AB} = rO_{AB} + (1-r)\frac{\mathbb{I}_A}{2} \otimes O_B$. That is, applying depolarised measurements on an operator O_{AB} is equivalent, at the level of the states prepared for Bob, to applying depolarisation-free measurements on a depolarised version of O_{AB}, denoted here as ρ_{AB}. Therefore, if O_{AB} admits an LHS model for the set $\mathcal{A}$, then it also does for the set $\{\Phi_r(\Pi_i^{(j)})\}$, which implies that ρ_{AB} admits an LHS model for all projective measurements.

Lets now move to the case $d > 2$, where we use the generalised Pauli vector $\vec{v} \in \mathbb{R}^{d^2-1}$ to denote d-dimensional, trace-1 Hermitian operators. In this basis, every rank-1 projector acting on $\mathbb{C}^d$ can be associated to a unit vector, and therefore every rank-1 projective measurement is associated to a set of unit vectors. Notice that we can restrict to rank-1 projective measurements (*i.e.*with d outcomes), making use of the Spectral Theorem and coarse graining.

A difference from the qubit case is that for arbitrary d not all unit vectors correspond to a projector [BK08]. However, the converse implication is enough for our purposes, namely that all projector are mapped to some unit vector of $\mathbb{R}^{d^2-1}$. Indeed, if a shrunken version of the unit sphere fits inside a polytope, then a depolarised version of all projective measurements does also, and we can decompose them as convex combinations of the extremal points of such a polytope. $\square$

Note that the operator O_{AB} need not to be a valid density operator (it can have negative eigenvalues). The requirements on O_{AB} are that it admits an LHS model for the measurements in $\mathcal{B}$, and that it equals ρ_{AB} when depolarized. Note also that in the case that $\mathcal{B}$ is the (infinite) set of all projective measurements, then this is precisely a brute force test for the existence of an LHS model. Thus, our method can be seen to provide a sequence of tests (sufficient conditions), in terms of the set $\mathcal{B}$, for a state to have an LHS model, which in the limit converges to the brute force test.

Uniting Theorem 11 with SDP (4.10), we obtain an SDP test for the projective simulability of ρ_{AB}, having as input a finite set of projective measurements $\mathcal{B}$ and its insphere radius r:

$$\begin{aligned} \text{given } & \rho_{AB}, \mathcal{B}, r \\ \text{find } & O_{AB},\ \{\rho_\lambda\}_\lambda \\ \text{s.t. } & \mathrm{Tr}_A\left[\left(B_i^{(j)} \otimes \mathbb{I}_B\right) O_{AB}\right] = \sum_\lambda D_\lambda(i|j)\rho_\lambda,\ \forall i, j \\ & \rho_\lambda \geq 0, \quad \forall\, \lambda \\ & \sum_\lambda \mathrm{Tr}(\rho_\lambda) = 1, \\ & \rho_{AB} = rO_{AB} + (1-r)\frac{\mathbb{I}_A}{d_A} \otimes O_B, \end{aligned} \tag{7.13}$$

where the $D_\lambda(i|j) = \delta_{i,\lambda_j}$ are deterministic response functions.

Example 17. As an illustration of the technique we first investigate the Bell diagonal states, given by

$$\rho_{\text{Bell}} = \sum_i p_i \left|\Psi_i\right\rangle \left\langle\Psi_i\right|, \tag{7.14}$$

where $|\Psi_i\rangle$ are the four Bell states, $p_i \geq 0$, and $\sum_i p_i = 1$ have LHS models. In this case we adapted the SDP 7.13 to maximise p_1 provided the same constraints. We find $p_1 \approx 0.4454$, and $p_2 = p_3 = p_4 = (1 - p_1)/3$, which is a Werner state, using $\mathcal{B}$ along the vertices of the rhombicuboctahedron, an Archimedian solid with 24 vertices. Notice that Werner states admit an LHS model for $p_1 \leq 1/2$ (see Section 2.2), thus with 12 measurements our method already recaptures $\approx$ 89% of unsteerable Werner states. We also looked at rank-3 Bell diagonal states, by setting $p_4 = 0$, and found the largest p_1 equal to 0.5664, with the same $\mathcal{B}$.

Example 18. We also considered noisy 3-qubit GHZ and W states given by

$$\rho(t) = t\,|\psi\rangle\langle\psi| + (1-t)\mathbb{I}/8, \tag{7.15}$$

where $|\psi\rangle$ equals

$$|\text{GHZ}\rangle := \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \tag{7.16}$$

or

$$|W\rangle := \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle). \tag{7.17}$$

These states are fully separable for $p \leq 0.2$ and $p \leq 0.2096$ respectively. With $\mathcal{B}$ corresponding to the rhombicuboctahedron we found that these states are unsteerable for projective measurements for $p \leq 0.232$ and $p \leq 0.228$ respectively.

To further generalise Theorem 11 to accommodate general POVMs we can make use of Theorem 3, which starting from an LHV model for projective measurements for a state ρ constructs an LHV model for general measurements for a state ρ'. Combining these two results we obtain the following theorem.

Theorem 12. *A state ρ_{AB} acting in $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ admits an LHS model for all measurements if there exists an operator O_{AB} that admits an LHS model for $\mathcal{B} \subset \mathcal{P}(d,n)$ such that*

$$\rho_{AB} = \frac{1}{d_A}\left[rO_{AB} + (1-r)\frac{\mathbb{I}_A}{d_A} \otimes O_B\right] + \frac{d_A - 1}{d_A}\gamma_A \otimes O_B, \tag{7.18}$$

where γ_A is an arbitrary state acting on $\mathbb{C}^{d_A}$.

Note however, that unlike the previous case, which becomes a brute force search for the existence of an LHS model for all projective measurements in an appropriate limit, this test provides only a sufficient criteria (in the same way that Theorem 3 is a sufficient condition for general locality, but not necessary).

Both theorems can be easily adapted to the case of LHV models by applying the same ideas also to Bob's system. That is, one can also (i) choose a set of measurements to Bob, (ii) compute the corresponding radius r_B, (iii) impose that Alice's and Bob's measurements generate local probability distributions and (iv) locally depolarise according to Alice and Bob's shrinking factors.

7.1.1 Generating bipartite entangled states admitting LHS models

A complementary problem to the one of deciding if a target state is local, is to generate local entangled states. Furthermore, it is also interesting generating local states which contain as much entanglement as possible. To this end, we make use of the concept of entanglement witnesses, introduced in Subsection 2.1.1 and further explored in Section 4.1 in the context of SDP.

We now propose a method to generate entangled states with LHS models and high entanglement. We start with a given witness W (obtained via an SDP). As before, we choose a set of measurements $\mathcal{B}$ and compute the radius of the insphere r. We now search for the state which maximally violates the witness and has an LHS model for projective measurements by solving the following SDP:

$$\begin{aligned}
\text{given } & W, \mathcal{B}, r \\
\min_{O_{AB}, \rho_{\mathbf{A}}} & \operatorname{Tr}[W(rO_{AB} + (1-r)\frac{\mathbb{I}_A}{d_A} \otimes O_B)] \\
\text{s.t. } & \operatorname{Tr}_A\left[\left(B_i^{(j)} \otimes \mathbb{I}_B\right) O_{AB}\right] = \sum_\lambda D_\lambda(i|j)\rho_\lambda,\ \forall i,j \\
& \rho_\lambda \geq 0,\ \forall \lambda \\
& \operatorname{Tr}[O_{AB}] = 1 \\
& rO_{AB} + (1-r)\frac{\mathbb{I}_A}{d_A} \otimes O_B \geq 0.
\end{aligned} \tag{7.19}$$

If the solution of this SDP is negative, then the minimising operator $\rho^*_{AB} = rO^*_{AB} + (1-r)\frac{\mathbb{I}_A}{d_A} \otimes O^*_B$ is an entangled state which has a LHS model: entanglement is guaranteed by the violation of the witness and the fact it is has an LHS model is imposed by the constraints of the SDP.

Once we find an example of an LHS entangled state ρ^*_{AB}, we can iterate this procedure and find new examples with more entanglement: we find the entanglement witness W^* which is optimal for the state ρ^*_{AB} and use W^* in the SDP (7.19) to find a new state ρ^{**}_{AB}, which is generally more entangled according to the chosen quantifier.

This procedure can then be iterated until it converges[2]. Note that each entanglement quantifier has specific properties, and thus exploring a number of different quantifiers can provide LHS states with different properties. Another possibility is to modify the construction for searching specific features of local states, as long as they represent valid SDP constraints to be added to (7.19) (for instance, we could search for the local state with the largest singlet fidelity). Finally, as before, we can adapt (7.19) accordingly to Eq. (7.18) to find examples of entangled states with LHS models for all POVM measurements.

Using this method we generated a large list of bipartite entangled states which have LHS models for projective and POVM measurements[3]. The below analysis is focused mostly in the amount of entanglement in the obtained examples (measured by their negativity), and the trace distance between any two examples, to ensure that they were different and that the set of states was being reasonably probed.

[2]Typically this is convergence up to numerical precision. Note also that this can occur at a local maximum.

[3]The data is available at https://git.io/vV7Bu

Polyhedron	(Vertices, Radius)	Werner states $\rho_{\mathrm{W}}(w)$		Bell diagonal states $\rho_{\mathrm{Bell}}(p_1, p_2, p_3)$		Witnessed states
		w^*	$\mathcal{N}$	p_1^*	$\mathcal{N}$	$\mathcal{N}$
Icosahedron	(12, 0.79)	0.4285	0.0714	0.5390	0.0390	0.0754
Dodecahedron	(20, 0.79)	0.4160	0.0620	0.5296	0.0296	0.0647
Truncated cube	(24, 0.67)	0.3553	0.0164	0.500	0	0.0181
Truncated octahedron	(24, 0.77)	0.4082	0.0561	0.5071	0.0071	0.0601
Truncated tetrahedron[4]	(24, 0.85)	0.4404	0.0803	0.5581	0.0581	0.0839
Rhombicuboctahedron	(24, 0.86)	0.4454	0.0840	0.5664	0.0664	0.0883

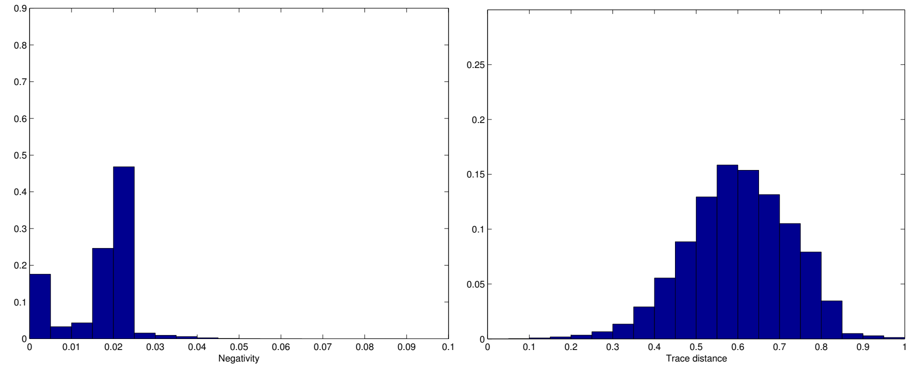
Table 7.1: Examples of optimal entangled LHS states within specific families – Werner states, rank 3 Bell diagonal states, and witnessed states – with measurements performed along the vertices of Platonic and Archimedian solids. The table presents the values of the optimized parameters of merit and the negativity $\mathcal{N}$ of the corresponding state. The entanglement witness considered for the witnessed state is the one-sided generalised robustness.

- **States of specific families**
  First, in Table 7.1 we give the optimal parameters found for the classes of states studied, along with the amount of entanglement in the optimal example. In each case we studied 6 different sets of measurements, such that the measurement directions are aligned along the vertices of different regular solids.

- **States with LHS models for projective measurements, obtained with uniformly chosen witnesses and 6 uniformly chosen measurements on the Bloch sphere.**

  We collected 450,000 such states. In Fig. 7.2a, we present a normalized histogram of their negativities, and in Fig. 7.2b we present a normalized histogram of trace distances between every two states in a random sample of 9,000 such states. The mean negativity of all 450,000 states is $\langle E_{\mathrm{Neg}} \rangle = 0.016$, with standard deviation $\sigma_{\mathrm{Neg}} = 0.008$. The mean trace distance of the states in the sample is $\langle T \rangle = 0.602$, with standard deviation $\sigma_T = 0.124$.

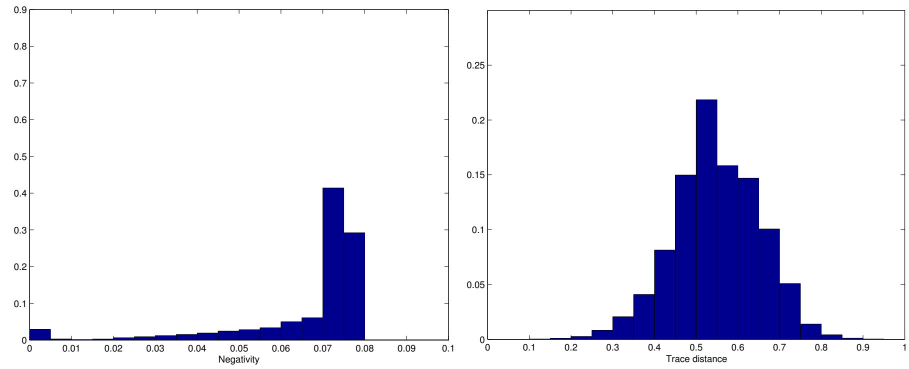
- **States with LHS models for projective measurements, obtained with uniformly chosen witnesses and measurements along the vertices of the icosahedron.**
  We collected 300,000 such states. In Fig. 7.3a, we present a normalized histogram of their negativities, and in Fig. 7.3b we present a normalized



(a) Negativity histogram (b) Trace distance histogram

Figure 7.2: Normalized histograms of the states with LHS models for projective measurements, obtained with uniformly chosen entanglement witnesses and 6 uniformly chosen measurements on the Bloch sphere.

histogram of trace distances between every two states in a random sample of 6,000 such states. The mean negativity of all 300,000 states is $\langle E_{\mathrm{Neg}} \rangle = 0.066$, with standard deviation $\sigma_{\mathrm{Neg}} = 0.016$. The mean trace distance of the states in the sample is $\langle T \rangle = 0.549$, with standard deviation $\sigma_T = 0.103$.

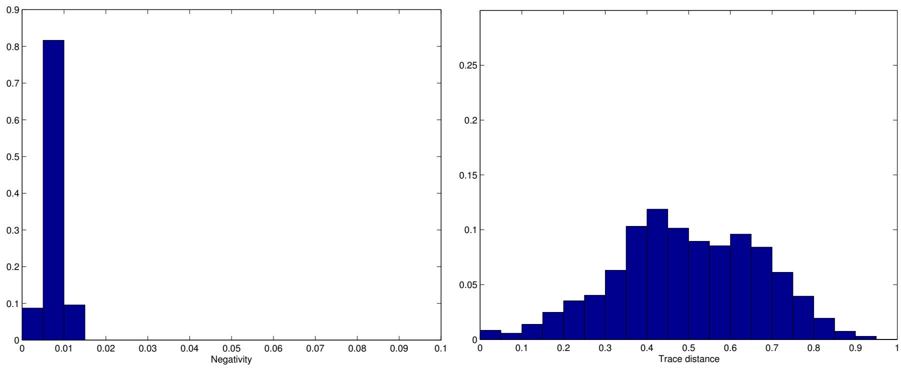


(a) Negativity histogram (b) Trace distance histogram

Figure 7.3: Normalized histograms for the states with LHS models for projective measurements, obtained with uniformly chosen witnesses and measurements along the vertices of the icosahedron.

- **States with LHS models for general POVMs, obtained with uniformly chosen witnesses and measurements along the vertices of the rhombicuboctahedron.**

We collected 1,500 such states. In Fig. 7.4a we present a normalized histogram of their negativities, and in Fig. 7.4b we present a normalized histogram of trace distances between every two states. The mean negativity of all 1500 states is $\langle E_{\text{Neg}} \rangle = 0.008$, with standard deviation $\sigma_{\text{Neg}} = 0.002$. The mean trace distance of the states is $\langle T \rangle = 0.496$, with standard deviation $\sigma_T = 0.175$.



(a) Negativity histogram (b) Trace distance histogram

Figure 7.4: Normalized histograms for the states with LHS models for general POVMs, obtained with uniformly chosen witnesses and measurements along the vertices of the rhombicuboctahedron.

7.1.2 Genuinely multipartite entangled states with LHS models

By using entanglement witnesses that detect genuine multipartite entanglement (see Section 4.1) we were also able to obtain new examples of genuine tripartite entangled three-qubit states with LHS models for projective measurements. To the best of our knowledge, only two examples of such states were previously known [TA06, BHQB16].

The search for genuine tripartite entangled states with LHS models follows the same method used in the previous section. We start with a random pure three-qubit state, and use SDP (4.6) to find the best witness W that detects its genuine tripartite entanglement. Fixing now the witness W, we apply SDP (7.19) to obtain the LHS state ρ_{ABC} that minimizes the expected value of W – the collection of measurements $\mathcal{B}$ we chose are the projective measurements along the vertices of the rhombicuboctahedron. If $\text{Tr}(W\rho_{ABC}) < 0$, we obtain a genuine tripartite entangled state with a LHS model for all projective measurements. We then iterate the process, obtaining a better witness based on ρ_{ABC}, and so on. By this procedure we were able to obtain 150 genuine tripartite entangled states with LHS models for all projective measurements. We were not able, though, to find any genuine tripartite entangled state with LHS model for general POVMs.

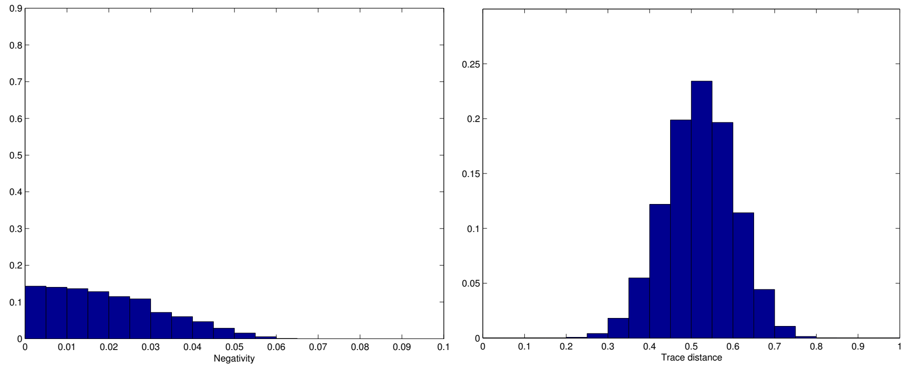
7.1.3 Estimating the volume of the set of local states

To show the power of the above techniques, we provide a lower-bound estimate on the volume of the set of entangled two-qubit states that possess LHV models for projective and general measurements. We uniformly sampled 2×10^4 two-qubit states according to the Hilbert-Schmidt and Bures measures, for which we obtained, $\approx 23\%$ and $\approx 7\%$ separable states, respectively, in good accordance with the values 24.2% and 7.3%, obtained from geometrical arguments [Sla07]. We then applied the above SDPs to estimate how many of the entangled states admit LHS models.

With the measurements $\mathcal{B}$ chosen to be the vertices of the icosahedron ($r \approx 0.79$), we obtain that $\gtrsim$25% of the entangled states sampled according to the Hilbert-Schmidt measure are LHS, while $\gtrsim$7% are LHS using the Bures measure. We were not able to obtain any entangled state admiting LHS models for POVMs by applying the same technique with measurements given by the icosahedron. A better estimation of the volume of the set of local states could be obtained, both for projective measurements and POVMs, by considering more measurements in the set $\mathcal{B}$.

- **States with LHS models for projective measurements, obtained from uniformly chosen two-qubit density matrices according to the Hilbert-Schmidt measure.**
  Of $20,000$ two-qubit states we drawed according to the Hilbert-Schmidt measure, we obtained $15,228$ entangled states ($\approx 76\%$), certified by means of the Peres-Horodecki criterion. Among all entangled states, we were able to certify that $2,961$ states ($\approx 19\%$) admit LHS models for projective measurements by performing SDP (), considering projective measurements along the vertices of the icosahedron.

  In Fig. 7.5a we present a normalized histogram of the of the $2,961$ entangled states with LHS models we obtained, and in Fig. 7.5b we present a normalized histogram of trace distances between every two such states. The mean negativity of all $2,961$ states is $\langle E_{\mathrm{Neg}} \rangle = 0.020$, with standard deviation $\sigma_{\mathrm{Neg}} = 0.013$. The mean trace distance of the states is $\langle T \rangle = 0.520$, with standard deviation $\sigma_T = 0.083$.



(a) Negativity histogram (b) Trace distance histogram

Figure 7.5: Normalized histograms for the states with LHS models for projective measurements, obtained from uniformly chosen two-qubit density matrices according to the Hilbert-Schmidt measure.

- **States with LHS models for projective measurements, obtained from uniformly chosen two-qubit density matrices according to the Bures measure.**

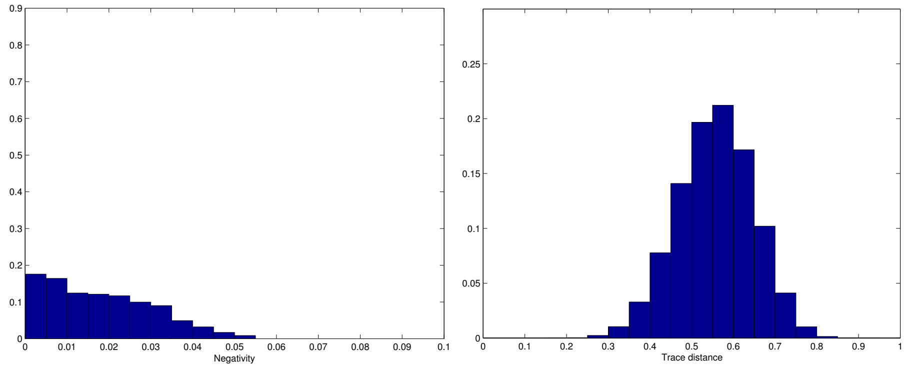
  Of $20,000$ two-qubit states we drew according to the Bures measure, we obtained $18,447$ entangled states ($\approx 92\%$), certified by means of the Peres-Horodecki criterion. Among all entangled states, we were able to certify that 932 states ($\approx 5\%$) admit LHS models for projective measurements by performing the SDP test we present in the main text, considering projective measurements along the vertices of the icosahedron.

  In Fig. 7.6a we present a normalized histogram of the of the 932 entangled states with LHS models we obtained, and in Fig. 7.6b we present a normalized histogram of trace distances between every two such states. The mean negativity of all 932 states is $\langle E_{\text{Neg}} \rangle = 0.018$, with standard deviation $\sigma_{\text{Neg}} = 0.012$. The mean trace distance of the states is $\langle T \rangle = 0.556$, with standard deviation $\sigma_T = 0.090$.

7.2 Extending LHV models via projective simulability

Regardless of some general approaches [CGRS16, HQV+16], the most studied family of states regarding LHV models are the two-qubit Werner states

$$\rho_W(t) = t\Psi_- + (1-t)\frac{\mathbb{I}_4}{4}, \tag{7.20}$$



(a) Negativity histogram (b) Trace distance histogram

Figure 7.6: Normalized histograms for the states with LHS models for projective measurements, obtained from uniformly chosen two-qubit density matrices according to the Bures measure.

$t \in [0, 1]$.

As discussed in Section 2.3, Werner states are known to admit a projective LHS model for the range $t \leq 0.68$ and a general LHS model for $t \leq 5/12 \approx 0.416$. We now provide an improvement over the latter result based on projective simulability.

In Section 6.2, we showed that any depolarised qubit POVM $\Phi_{t^*}(\mathbf{A})$ is projective simulable for $t^* \approx \sqrt{2/3}$ (Theorem 10). We now directly apply this result to extend models for projective measurements to address any quantum measurement.

Notice that here the class of the model plays a central role: local-hidden-state models already allow the quantum-characterised party to implement general POVMs (see Section 2.2 for details), and therefore only the measurements of the uncharacterised party will have to be simulated, in contrast to the case of local-hidden-variable models.

Theorem 13. *Let ρ be a quantum state acting on $\mathbb{C}^d \otimes \mathbb{C}^d$ and $t \in [0, 1]$ be such that $\Phi_t(\boldsymbol{A})$ is projective-simulable for any given $\boldsymbol{A} \in \mathcal{P}(d, n)$. Then*

(i) if ρ admits an LHS model for projective measurements, then $\Phi_t(\rho)$ admits an LHS model for general measurements;

(ii) if ρ admits an LHV model for projective measurements, then $\Phi_{t^2}(\rho)$ admits an LHV model for general measurements.

Proof. Let ρ admit an LHS model for projective measurements (and therefore for all projective-simulable measurements, as stated in Lemma 3). Hence we can use it to describe statistics yielded by the projective-simulable POVM $\Phi_t(\mathbf{A})$,

$$\mathrm{Tr}(\rho\Phi_t(A_i)\otimes\mathbb{I}_d) = \int d\Pi(\lambda)\rho_\lambda f_A(i|\mathbf{A},\lambda). \tag{7.21}$$

Notice that the channel $\Phi_t \otimes \mathbb{I}_d$ being applied to the measurements is self-dual, and satisfy[5]

$$\Phi_t(A_i)\otimes\mathbb{I}_d = \Phi_t(A_I\otimes\mathbb{I}_d), \tag{7.22}$$

thus we can use once more the self-duality of the depolarising channel to map the noise from the measurements to the state, and obtain

$$\mathrm{Tr}(\Phi_t(\rho)A_i\otimes\mathbb{I}) = \int d\Pi(\lambda)\rho_\lambda f_A(i|\mathbf{A},\lambda). \tag{7.23}$$

Since this holds for any POVM $\mathbf{A}$, affirmation (i) is proved.

For (ii), it is simple to check that

$$\mathrm{Tr}\left(\rho\Phi_t(A_i)\otimes\Phi_t(B_j)\right) = \mathrm{Tr}\left(\Phi_{t^2}(\rho)A_i\otimes B_j)\right), \tag{7.24}$$

and apply the same reasoning as above. □

Very much in the same spirit as Theorems 11 and 12, Theorem 13 provides a recipe that starting from a projective model for a mixed state, one constructs a model for general measurements on a noisier state.

As a direct application of (i) to Werner states, we have that $\rho_W(1/2)$ admits a projective-LHS model and we can take $t = \sqrt{2/3}$, therefore concluding that the Werner state with parameter

$$t^* = \frac{1}{2}\times\sqrt{\frac{2}{3}} \approx 0.4082 \tag{7.25}$$

has a POVM-LHS model. Since this value is still below $5/12 \approx 0.4166$, this model is still outperformed by Barrett's model.

However, the payback comes from applying (ii). Indeed, $\rho_W(0.68)$ admits a projective-LHV model, and for the same simulability parameter $t = \sqrt{2/3}$ of before we have that the Werner state of parameter

$$t^{**} = 0.68\times\left(\sqrt{\frac{2}{3}}\right)^2 \approx 0.4533 \tag{7.26}$$

has a POVM-LHV model. Currently, this is the most efficient model for Werner states, in the sense that Eq. (7.26) yields the most entangled Werner state that admits such a model (see Figure 7.7). This illustrates the power of projective simulability applied to Bell nonlocality.

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[5] Here we abuse the notation to denote by Φ_t the depolarising channel acting both on $\mathrm{Herm}(\mathbb{C}^d)$ and $\mathrm{Herm}(\mathbb{C}^d\otimes\mathbb{C}^d)$.

Corollary 3. *The Werner state $\rho_W(0.4533)$ admits an LHV model for general measurements.*

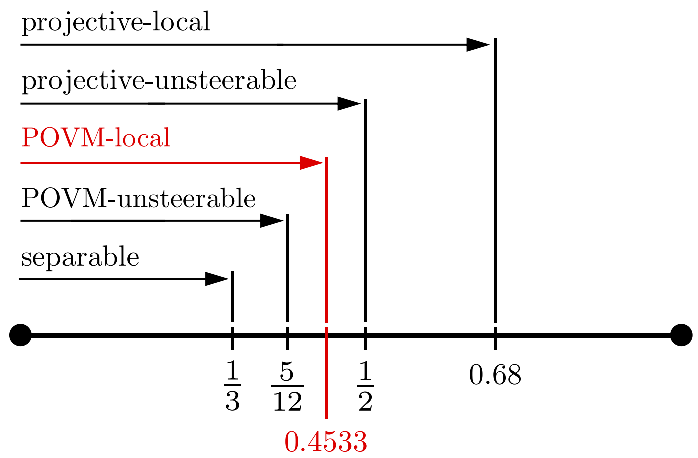

Figure 7.7: The optimal parameters for two-qubit Werner states currently known, concerning separability, POVM- and projective-unsteerability, and POVM- and projective-locality. In red, one of the contributions presented in this thesis.

Summary and open questions

In this thesis we addressed many different instances of the quantum measurement simulability problem. We developed a framework to study these questions, investigating in detail which measurements can be simulated when the simulators are the set of projective measurements, the set of k-outcome measurements, and arbitrary sets of J measurements. We derived sufficient conditions for simulability in any dimension for each of the former two cases. While k-outcome simulability can always be phrased as an SDP, for projective simulability we could provide such characterisations only for dimensions 2 and 3.

From the fundamental point of view, it would be interesting to find alternative characterisations for the set of projective-simulable POVMs for $d > 3$, even if not by means of SDP. Another open questions is to understand the asymptotic behaviour of the critical depolarisation parameter needed to make any d-dimensional POVM projective-simulable, and compare it with the general bound of $1/d$ proved here. This could be useful for extending projective local models for higher-dimensional states.

As for J-POVM simulability, we show it to be a non-trivial generalisation of joint measurability, that provides a hierarchy for the incompatibility of sets of measurements. One can investigate the relation between this broader concept and generalisations of EPR steering, namely the advantages of considering a state that admits a mixture of different LHS models that depends on the measurement performed.

The sequences of polytopes approximating the quantum states we provided has its own independent interest, as we proved by applying it to different types of simulability problems. In practice, we can use this approach to derive bounds for any property of POVMs that is efficiently quantified on a single set of measurements. This technique is also applicable for approximating the set of quantum states.

We also presented the first method for constructing local models for arbitrary families of states, based on extending an initial model valid only for finitely many measurements. In principle, this method requires the initial measurements to be projective, so that we know how to calculate the critical depolarisation factor needed for extending the model. Hence, it would be important to develop a better understanding of the set of POVMs, in order to find the equivalent factor for non-projective measurements and generalise this strategy.

Finally, we applied our results from projective simulability to extend projective local models for two-qubit states to any POVM, obtaining the currently best local model for Werner states and general measurements. The model works for $\rho_W(t)$, with $t \leq 0.4533$, which is still considerably far from $\rho_W(1/2)$, the most entangled projective-unsteerable Werner state. Since we conjecture that the latter is also unsteerable for general measurements, we still are left to derive different techniques for the construction of general local models.

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