On maximal independent sets in circulant digraphs

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Abstract

In this research note we introduce St-Nicolas graphs, i.e. circulant digraphs showing exactly n maximal independent sets, isomorph under the digraph's automorphisms group. This class of digraphs represent a generalisation of Andrásfai graphs with interesting links to finite group theory.

Keywords: Graph Theory, Maximum Independent Set, Graph Automorphism, Finite Groups.

1 Automorphism of digraphs

1.1 Graphs and digraphs

We consider a digraph (directed graph) G to consist of a set $V(G)$ of nodes and a set $A(G) \subseteq V(G) \times V(G)$ of arcs. The order $n(G) = |V(G)|$ of a digraph G is given by the number of its nodes. We shall only consider digraphs of finite order. The size $s(G)$ of the digraph G is given by the cardinality of its arc set, i.e. $s(G) = |A(G)|$. The fill rate (arc density) of G is defined as the ratio of $s(G)$ over $n(G)^2$.

The *out-degree*, i.e. the number of arcs leaving a node x is denoted $d^-(x)$. Similarly, the *in-degree*, i.e. the number of arcs entering a node x is denoted $d^+(x)$. A k-regular digraph G is such that $d^-(x) = d^+(x) = k$ for all $x \in V(G)$.

The complement $\overline{G}$ of a digraph G consists of the same set $V(G)$ of nodes as G and the complement arc set $\overline{A(G)} = \{(x, y) \in V(G) \times V(G) \,|\, (x, y) \notin A(G)\}$.

We call *graph*, a digraph G which shows symmetric arcs, i.e. G is such that $\forall x, y \in V(G)$ we have $(x, y) \in A(G) \Rightarrow (y, x) \in A(G)$. In the loopless (irreflexive) case, i.e. $(x, x) \notin A(G), \forall x \in V(G)$, both symmetric arcs between two nodes of G are called *edges* and we may replace the set $A(G)$ of oriented arcs with the set $E(G) = \{\{x, y\} : (xy) \in A(G)\}$ of unoriented edges to obtain the general concept of *simple* graph. In order to avoid the representational necessity to work with irreflexive relations, we shall always consider simple graphs to be symmetric and loopless digraphs in the sequel.

1.2 Unlabelled digraphs and automorphisms

Two digraphs G and H are said to be equal if they have the same node set and the same arc set. In this study, we are not directly interested in a precisely labelled node set, only the isomorph disposition of the arcs between the unlabelled nodes is of interest for us.

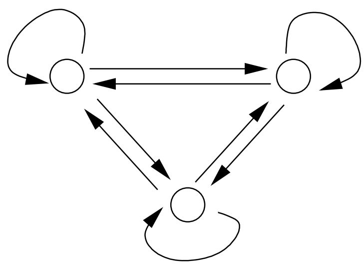

Figure 1: The unlabelled complete digraph on three nodes

Two digraphs G and H are called *isomorph*, denoted $G \cong H$, if there is a bijection, say Φ, from $V(G)$ to $V(H)$ such that $(x, y) \in A(G)$ if and only if $(\Phi(x), \Phi(y)) \in A(H)$. We say that Φ is an (digraph) *isomorphism* from G to H. The abstract class of all isomorph digraphs of a given digraph H is called the unlabelled digraph H. We denote K_n the unlabelled complete graph of order n where $A(G) = V(G) \times V(G)$. In figure 1 we show K_3, the unlabelled complete graph defined on three nodes's. The unlabelled graph G of order n where $A(G) = \emptyset$ is called the *empty graph* (of order n) and denoted $\mathbb{O}_n$.

An isomorphism of a digraph G to itself is called an (digraph) *automorphism*. An automorphism of a digraph is therefore a permutation of its nodes's that maps arcs to arcs and non arcs to non arcs. The set of all automorphisms of a digraph G form a group, denoted $Aut(G)$, where the identity permutation is denoted e. $Aut(G)$ is a subgroup of $Sym(V(G))$, the set of all permutation of the nodes's of G. As every permutation of $V(K_n)$ is an automorphism of K_n, we may notice that $Aut(K_n) = Sym(n)$, the permutation group of the list of integers $[0, 1, \ldots, n-1]$. A digraph which admits only the trivial identity permutation e will be called an *asymmetric* digraph. In general, for a digraph G, $1 \leqslant |Aut(G)| \leqslant |Sym(V(G))|$. A digraph admitting a non-trivial automorphism group will be called a *symmetric* digraph[1].

[1] Please notice that, contrary to the common usage in directed graph theory, we will call graph a digraph admitting only symmetric arcs and reserve the qualification symmetric for

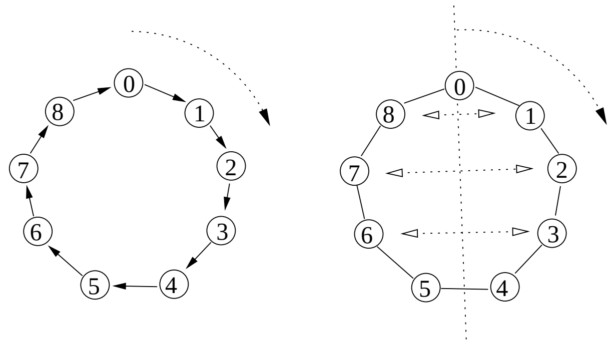

Figure 2: The automorphisms of the 9-circuit and the 9-cycle

In the sequel we are mainly interested in highly regular and symmetric digraphs such as the 1-regular circular digraph called n-circuit. The rotational permutations of the n-circuit give an automorphism group of order n as shown in figure 2. In the corresponding symmetric case we speak of the 1-regular n-cycle, denoted C_n. Its automorphism group's order is $2 \times n$. A lateral reflection along a central symmetry line through the n-cycle, replacing 1 with $n-1$, 2 with $n-2$, ..., $\lfloor \frac{n}{2} \rfloor$ with $n - \lfloor \frac{n}{2} \rfloor$, multiplies by two the rotational automorphisms (see figure 2) observed in the single directed case.

Digraphs supporting such circular automorphisms, all belong to the important class of circulant digraphs that we shall introduce in the next section.

2 The circulant digraphs

2.1 Introducing circulant digraphs

A *circulant digraph* of order n consists of a set of n nodes's enumerated $0, 1, 2 \ldots, n-1$ and denoted $[n]$, and a connection set $C = \{r, s, \ldots\}$, with $r, s, \ldots \in \mathbb{Z}_n$, indicating a corresponding set of circulant connected nodes's $(i, i+r \bmod n)$, $(i, i+s \bmod n), \ldots$, for each node i in $[n]$. For short we note $Circ([n], C)$.

The 1-regular n-circuit mentioned beforehand corresponds to the circulant graph $Circ([n], \{1\})$ and $C_n \cong Circ([n], \{1, -1\})$. The empty graph $\mathbb{O}_n$ corresponds to $Circ([n], \emptyset)$ and, similarly, $Circ([n], [n]) \cong K_n$.

Figure 3 shows for instance the circulant digraph $Circ([8], \{1, 4, 7\})$. which is obviously a 3-regular graph with $d^+(i) = d^-(i) = 3$ for all $i \in [n]$.

In general, all circulant digraphs are necessarily k-regular, where k equals the dimension of the connection set C.

The circulant digraph $Circ([8], \{1, 4, 7\})$ is furthermore a simple graph due to the symmetric distribution of its connection set C. In the circular embedding of nodes's, node $1 + 1 \bmod 8 = 2$ is symmetrically opposed to node $1 + 7 \bmod$

stating the presence of a non trivial automorphism group of the digraph.

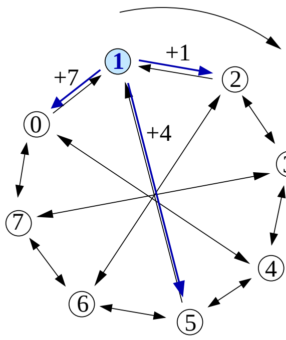

Figure 3: The circulant graph $Circ([8], \{1, 4, 7\})$

$8 = 0$, and similarly, node $1 + 4 \bmod 8 = 5$ is symmetrically opposed to node $5 + 4 \bmod 8 = 1$.

Circulant digraphs naturally correspond to a particular class of Cayley digraphs.

2.2 Cayley digraphs

Let Γ be a finite Abelian group of order $|\Gamma|$ where we denote $+$ the group operation with neutral element 0. Γ^+ will denote the positive elements of Γ. A subset $C \subseteq \Gamma$ is called a difference set for Γ.

Given the pair (Γ, C), we define a Cayley digraph G, denoted $Cay(\Gamma, C)$, as follows. Let $V(G) = \Gamma^+$ and $\forall x, y \in \Gamma^+$ we observe an arc $(x, y) \in A(G)$ as soon as $(y - x) \in C$.

The order of a Cayley digraph $Cay(\Gamma, C)$ is given by the dimension of Γ^+. In case the difference set C verifies the condition that $(x - y) \in C$ if $(y - x) \in C$, we get a Cayley *graph*.

We may notice that $Cay(\Gamma, \Gamma)$ gives a complete graph K_n of order $n = |\Gamma^+|$.

The correspondence with circulant digraphs becomes evident when associating Γ with $\mathbb{Z}_n$. A given connection set $C \subseteq \mathbb{Z}_n$ then consists of a set of distinguished positive and negative differences observed between elements of $\mathbb{Z}_n^+$.

Our example circulant digraph $Circ([8], \{1, 4, 7\})$, for instance, is isomorph to $Cay(\mathbb{Z}_8, \{1, -1, 4, -4\})$. We connect any two numbers $x, y \in \mathbb{Z}_8^+$ as soon as their difference $(x - y)$ takes one of the four values in $\{1, -1, 4, -4\}$. Indeed, the circulant connections $\{1, 7\}$ correspond to all differences of values 1 and -1 between the eight nodes's in [8]. Similarly, the circulant connection [4] correspond to the midpoint 4 in the even number of cyclic group elements in $\mathbb{Z}_8$ and therefore collects all differences of values 4 and -4 between the eight nodes's in [8].

This finite Abelian group embedding of circulant digraphs via the Cayley digraph correspondence will help us formulate and demonstrate essential properties of circulant digraphs later on.

2.3 Andrásfai graphs

In the sequel we shall distinguish a special class of Cayley graphs, namely the Andrásfai graphs. For any $k \geqslant 1$, let $And(k)$, the *Andrásfai graph of order* k, denote the Cayley graph $Cay(\mathbb{Z}_{3k-1}, C)$ with C containing all elements of $\mathbb{Z}_{3k-1}$ congruent to 1 modulo 3. $And(1)$ corresponds to K_2, and $And(2)$ corresponds to the 5-cycle C_5. Indeed, using the above introduced correspondence with

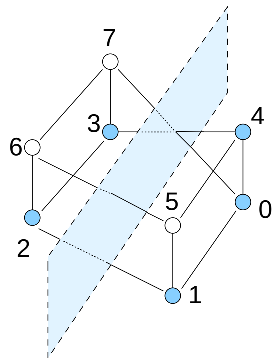

Figure 4: The Andrásfai graph of order 3

circulant digraphs, it easy to notice that $And(1)$ corresponds to $Circ([2], \{1\})$ and $And(2)$ to $Circ([5], \{1, 4\})$. The perspicacious reader may have furthermore noticed that our example $Circ([8], \{1, 4, 7\})$ is nothing else than $And(3)$, also called the (four level) Möbius ladder shown in figure 4. We have highlighted here a lateral reflection plane which illustrates the four transpositions: $(7, 6)(8, 5)(1, 4), (2, 3)$ that make up the central reflection symmetry besides the normal circular rotation. It becomes evident here that $Aut(And(3))$ is of order 16, i.e. 8 circular rotations times two lateral reflections.

Indeed, all Andrásfai graphs of order k support an automorphism group of order $2 \times (3k - 1)$ similar to the dihedral group.

We shall revisit the Andrásfai graphs later on. Let us conclude this section with presenting a special composition operator for circulant digraphs.

2.4 Lower closed composition of circulant digraphs

Let G and H denote two circulant digraphs $Circ([n], C_G)$ and $Circ([n], C_H)$ of same order n. We denote $G \otimes H$, called *lower closed composition* of G and H, the circulant digraph $Circ([n], C_T)$ resulting from the union and the standard composition of both the graphs G and H with connection set $C_T = C_G \cup C_H \cup (C_G \circ C_H)$, where $C_G \circ C_H$ represents the transitive closure of all circulant connections in G and H.

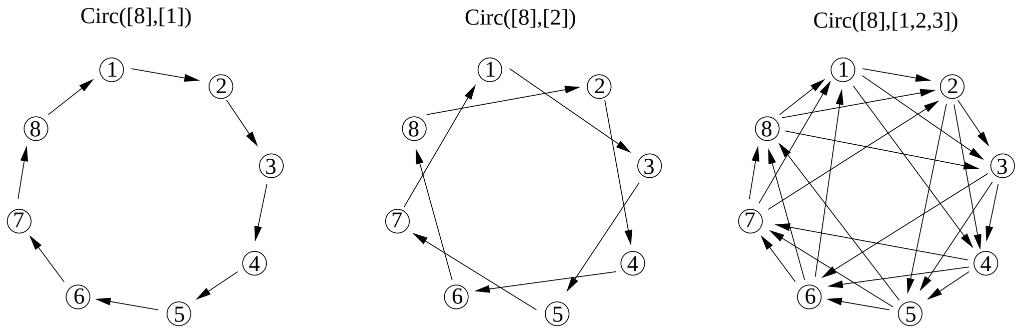

Figure 5: The lower closed composition illustrated

Figure 5 shows $Circ([8],[1,2,3])$, the lower closed composition of $Circ([8],[1])$ and $Circ([8],[2])$.

We shall mainly be concerned with the repeated compositions of the n-cycle C_n with itself, which we denote with the help of a standard exponential notation as $C_n^{\mathbf{p}} = C_n^{\mathbf{p}-1} \otimes C_n$, where $p \geqslant 1$. By convention, C_n^0 denotes the empty graph $\mathbb{O}_n$.

With this exponential notation we may rewrite for instance the circulant digraph $Circ([8],[1,-1,2,-2,3,-3])$ as C_8^3.

It is easy to verify that all C_n^p with $p \geqslant \lfloor \frac{n}{2} \rfloor$ are isomorph to K_n. Indeed, all the differences $(\lfloor \frac{n}{2} \rfloor + r \bmod n)$ equal the differences $-(\lfloor \frac{n}{2} \rfloor - r \bmod n)$ for $r \geqslant 1$ in the corresponding Cayley graph defined on $\mathbb{Z}_n$.

A first result will relate now the lower closed composed circulant graphs with the previously introduced Andrásfai graphs.

Proposition 1
The Andrásfai graph $And(k)$ for $k \geqslant 1$ are isomorph to the complement digraph of $C_{3k-1}^{\lceil \frac{3k-1}{3} \rceil - 1}$.

We proceed by induction. For $k = 1$, $\lceil \frac{3k-1}{3} \rceil = 1$, so that proposition 1 is true. Let us now suppose that the proposition is true for $k > 1$. To see that, in this case, proposition 1 is true for $k+1$, we may rely on the fact that Andrásfai graphs are all constructive, i.e. the subset of nodes's enumerated $[3k-1]$ in $And(k+1)$ induce the Andrásfai graph $And(k)$ (see (1)). When adding nodes's $\{3k, 3k+1, 3k+2\}$ with the respective new difference $3k-1$ to C_{3k-1} we get $C_{3(k+1)-1}$.

Complete digraphs and Andrásfai graphs share a same interesting property. Both graphs admit a unique unlabelled maximal independent set associated with an orbit of isomorphĥic labelled instances of cardinality equal to the order of the graph. Such graphs will be discussed in the next section.

3 St-Nicolas graphs

3.1 Choices and induced subgraphs

A subdigraph H of G consists of a subset $V(H) \subseteq V(G)$ of nodes and a subset $A(H) \subseteq A(G)$ of arcs from G. A subgraph H is called *induced* if $A(H)$ contains all arcs from $A(G)$ such that $x, y \in V(H)$.

A choice Y in a digraph G is a non empty subset of nodes of $V(G)$. A choice inducing an empty subgraph is called an *independent* (or stable) choice. The choice inducing a subgraph which is complete is called a *clique*.

A choice Y verifying a property P is called minimal (resp. maximal) for property P is no proper subset (resp. superset) of Y verifies the same property P

3.2 Maximal independent sets and cliques

Let G be any digraph. For a node $x \in V(G)$, we denote $N(x) = \{y \in V(G) | \{x, y\} \in E(G)$ its neighbourhood in G. The (open) neighbourhood $N(Y)$ of a choice Y consists of the union of the (open) neighbourhoods of its members.

A maximal independent set (MIS) in G is a choice in G such that $Y \cap N(Y) = \emptyset$ and $Y \cup N(Y) = V(G)$.

A maximal clique in G is a MIS in the complement graph $\overline{G}$

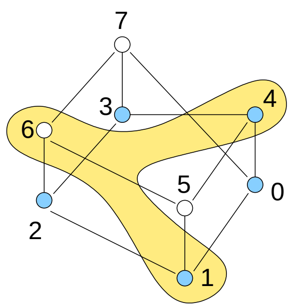

Figure 6: The choice $\{2, 5, 7\}$ gives a MIS in *And*(3)

In K_n, each singleton $\{x\}$ gives a MIS, and the complete graph K_n is evidently itself its maximal clique. Conversely, in $\overline{K_n} = \mathbb{O}_n$, the maximal independent set is K_n and each singleton is a maximal clique. In figure 6 we show for instance a maximal independent set in *And*(3).

3.3 Unlabelled maximal independent sets and cliques

Let G be a given digraph with a non trivial automorphism group $Aut(G)$. The image of a node $x \in V(G)$ under an automorphism $g \in Aut(G)$ will be denoted x^g. If H is an induced subgraph of G, then H^g will denote a digraph with node set $V(H^g) = \{x^g \ : \ x \in V(H)\}$ and arc set $A(H^g) = \{(x^g, y^g) \ : \ (x, y) \in A(H)$.

From the fact that g is an automorphism of G, it follows immediately, first, that H^g is isomorph to H, and secondly, that H^g is also a subgraph of G. In such a way an isomorph copy of a given maximal independent set or clique in G remains a maximal independent set and clique of G.

The orbit of isomorph maximal independent sets or cliques in a digraph G following the automorphism group $Aut(G)$ is called an unlabelled MIS, respectively an unlabelled clique of G.

For instance K_n admits a single (unlabelled) node as unlabelled MIS, and $\mathbb{O}_n$ admits a single unlabelled node as unlabelled maximal clique. Each time the given digraph supports a single MIS orbit (resp. maximal clique orbit) of dimension n.

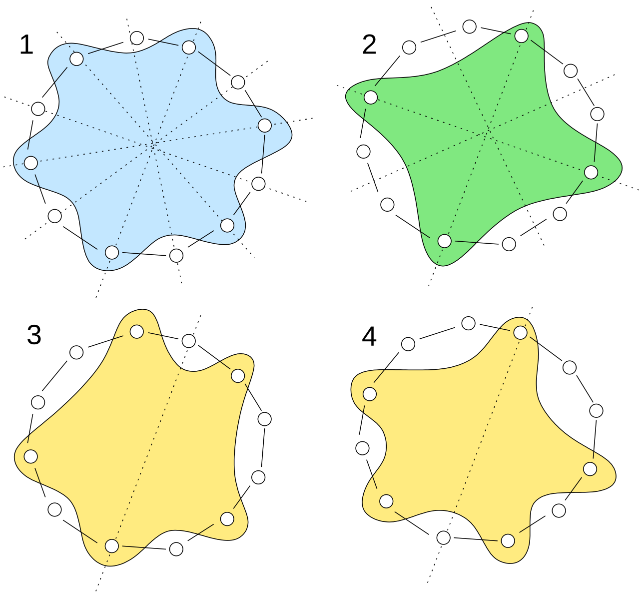

Figure 7: The four unlabelled MIS of C_{12}

Not all symmetric digraphs admits a single MIS or maximal clique orbit. In figure 7 we show for instance the four MIS orbits of the 12-cycle.

However, Andrásfai graphs $And(k)$ are in this sense similar to K_n, as they admit surprisingly a single MIS orbit of dimension equal to the order $3k-1$ of $And(k)$. This property of Andrásfai graphs may be verified by noticing that they are precisely characterised by the fact each node's neighbourhood corresponds bijectively each to precisely one of the $3k-1$ isomorph MIS. Indeed, as the connection sets C of Andrásfai graphs only contain numbers congruent to 1 mod 3, all differences between the elements of C are congruent to 0 mod 3 and thus may never be connected. Furthermore, one may verify that adding any available further element to the actual neighbourhood of a given node will necessarily violate (see (1)) the independence property.

We shall call *St-Nicolas graphs*, the digraphs G admitting a singe MIS-orbit of size $n(G)$. Similarly, we call St-Nicolas a MIS-orbit of size $n(G)$.

From the neighbourhood preservation property of the graph isomorphism definition, it immediately follows that a digraph G and its complement $\overline{G}$ necessarily share the same automorphism group $Aut(G) = Aut(\overline{G})$. The complement of St-Nicolas graphs G of order $n(G)$ will therefore admit a single maximal k-clique orbit of same dimension $n(\overline{G})$ as the MIS-orbit, i.e. the order of $n(G)$.

The last part, concerned with the general study of MIS-orbits of circulant digraphs in general, will allow us to precisely delimit the class of St-Nicolas graphs.

3.4 The MIS-symmetry signature of circulant graphs

From figure 7, we may notice that the cycle C_{12} has four MIS-orbits. The first orbit contains two isomorph MIS of cardinality 6, the second orbit contains 3 MIS of cardinality 4, the third and fourth orbit contain each 12 MIS of cardinality 5.

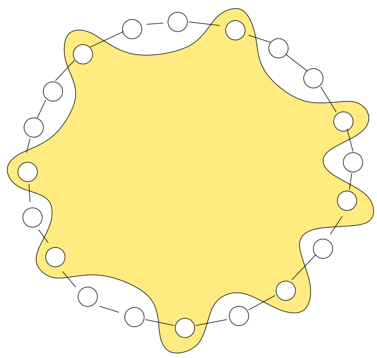

Figure 8: An irregular MIS from C_{20}

The cycle C_{20} shows 14 kernel orbits: two orbits of respectively 40 isomorph MIS of cardinality 8. One of these MISs is shown in figure 8. We may notice that, contrary to the St-Nicolas graphs, this MIS shows an irregular pattern as it doesn't organise its nodes's along any central reflection axis. Therefore we obtain in fact 40 different isomorph copies following from the actual order 40 of the automorphism group of C_{20}.

C_{20} shows furthermore 9 St-Nicolas orbits with MIS of cardinalities between 7 and 9; a single MIS orbit of size 10 (cardinality 8), of size 5 (cardinality 8), and, of size 2 (cardinality 10). This last orbit contains in fact the two trivial

MIS you may get by choosing every second node on the circular embedding of C_{20}.

It is worthwhile noticing that the different orbit sizes observed for C_{12} and C_{20} respectively, reveal the presence of a variable number of central symmetry axes which may divide the order $2n$ of the underlying automorphism group of the n-cycles (see figure 7). In order to capture this symmetry characteristic in a synthetic way, we will denote $k\mathcal{S}_s$ (with $s \geq 0$ and $k \geq 1$), k MIS orbits based on s central symmetry axes. With this notation, a St-Nicolas digraph is qualified as $\mathcal{S}_1$.

digraph	MIS-orbit characterisation
C_k for $k = 3, 5, 7$	$\mathcal{S}_1$ (St-Nicolas)
C_{11}	$2\mathcal{S}_1$ (2 St-Nicolas)
C_{13}	$4\mathcal{S}_1$ (4 St-Nicolas)
C_{17}	$\mathcal{S}_0 + 5\mathcal{S}_1$
C_{19}	$2\mathcal{S}_0 + 7\mathcal{S}_1$
C_{23}	$8\mathcal{S}_0 + 12\mathcal{S}_1$

Table 1: MIS-orbit characterisation for n-cycles of prime order

In table 1 we show the symmetry characteristics of the MIS-orbits of n-cycles of prime order n. We see that only asymmetric and St-Nicolas MISs may appear.

digraph	MIS-orbit characterisation
C_9	$\mathcal{S}_1 + \mathcal{S}_3$
$Circ([9], \{1, -1, 3, -3\})$	$\mathcal{S}_1$ (St-Nicolas)
C_{10}	$\mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_5$
$Circ([10], \{1, -1, 2, -2, 5, -5\})$	$\mathcal{S}_1$ (St-Nicolas)
C_{12}	$2\mathcal{S}_1 + \mathcal{S}_4 + \mathcal{S}_6$
$Circ([12], \{1, -1, 4, -4, 6, -6\})$	$\mathcal{S}_1$ (St-Nicolas)
C_{20}	$2\mathcal{S}_0 + 9\mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_4 + \mathcal{S}_{10}$
$Circ([20], \{1, -1, 2, -2, 4, -4, 10, -10\})$	$3\mathcal{S}_1$ (St-Nicolas)

Table 2: MIS-orbit characterisation for non prime cycles and regular circulant graphs

Table 2 shows furthermore the MIS-orbit characterisations for non prime n-cycles. It is worthwhile noticing that there appears a clear link between the MIS-orbit characterisations of the n-cycles and the corresponding circulant graphs giving a St-Nicolas graph.

Conjecture 1

1. *All cycles C_n of prime order $n \geq 3$ admit unlabelled kernels only of class $p\mathcal{S}_0 + q\mathcal{S}_1$, with $p \geq 0$ and $q \geq 1$;*

2. *The circulant graph $Circ([n], \{r, s, \ldots\})$ corresponding to a cycle C_n supporting a positive number of kernel orbits with $r, s, \ldots$ symmetry axes, admit unlabelled kernels only of class $p\mathcal{S}_0 + q\mathcal{S}_1$, with $p \geq 0$ and $q \geq 1$.*

We will close this study with the partial characterisation of St-Nicolas graphs.

Theorem 1
A digraph G of order n is a St-Nicolas graph if G is isomorph to the complement of the lower closed composition C_n^p, with $0 \leqslant p \leqslant (\lceil \frac{n}{3} \rceil - 1)$, of the n-cycle.

Theorem 1 readily follows from the observation that the progressive lower closed p-composition of the n-cycles produces n circulant maximal cliques of increasing order p which may each be bijectively related with a particular node. For $p = 0$, we obtain indeed the empty clique whose complement is K_n, i.e. a St-Nicolas graph. Similarly, from property 1 we know that the complement of the maximal lower-closed p-composition of an n-cycle, with $n = 3k - 1$ and $p = (\lceil \frac{n}{3} \rceil - 1))$, is isomorph to $And(k)$, i.e. again a St-Nicolas graph. In between, it is possible to show that the connection set of the corresponding circulant graph, presents a regular disposition along a unique central symmetry axis defined by the finite group $\mathbb{Z}_n$ structure along 0 and $\lfloor \frac{n}{2} \rfloor$. Raising p to a value higher than $(\lceil \frac{n}{3} \rceil - 1)$ will always produce, following the transitive closure of the lower closed composition, maximal cliques of higher order than p. The complements of these dense digraphs get quickly very sparse and consequently admit a much larger number of MIS than a St-Nicolas graph.

4 Conclusion

Open question: Are the complement of lower-closed p-compositions for $0 \leqslant p \leqslant (\lceil \frac{n}{3} \rceil - 1)$ of the n-cycle the only possible St-Nicolas graphs? To answer eventually the question, it will be necessary to investigate more deeply the algebraic properties of circulant digraphs and particularly of St-Nicolas graphs.

Knowing more about St-Nicolas graphs will also give hints for eventually proving both sentences of conjecture 1.

References

[1] Chr. Godsil and G. Royle, *Algebraic Graph Theory*. Springer-Verlag (2001).