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# MANIPULATION BY MERGING AND ANNEXATION IN WEIGHTED VOTING GAMES

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Abstract. The problem of manipulation in voting is fundamental and has received attention in recent research in game theory. In this paper, we consider two cases of manipulation in weighted voting games done by merging of coalitions into single players and by annexation of a part or all of the voting weights of another player viewed from two perspectives: of the effect of swings of players and of the role of the Banzhaf power index. We prove two theorems for manipulation by merging and annexation, and show several attractive properties in these two processes.

**1. Introduction.** The modern notion of a simple game was introduced by John von Neumann and Oscar Morgenstern in their monumental book *Theory of Games and Economic Behavior* in 1944 [15]. Previous works on this problem were fragmentary and did not attract much attention. This book provided some

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new important developments such as the introduction of information sets, the formal definitions, and the decision rules. According to Von Neumann and Morgenstern a simple game is a conflict in which the only objective is winning and the only rule is an algorithm to decide which coalitions of players are winning. It is also known that voting is a widely used method for decision making. In particular, manipulation has been studied intensely in social choice theory, starting with the classical works of Gibbard [5] and Satterthwaite [10]. The problem of coalitional manipulation was first explicitly introduced by Conitzer, Sandholm and Lang [3], where the authors initiated its analysis from a computational perspective.

The aim of this paper is to show two processes of manipulation in weighted voting games: by merging of two players into a single player and by annexation of a part or all of the voting weights of another player.

We start our study with a consideration of key terms. Let $N$ be a nonempty finite set of players in game $G$ and every subset $S \subset N$ be referred to as a coalition. The set $N$ is called the grand coalition and $\emptyset$ is called the empty coalition. We denote the collection of all coalitions by $2^N$ and the number of players of coalition $S \subset N$ by $|S|$. Let us label the players by $1, 2, \ldots, n$, i.e., $n = |N| \geq 2$.

**Definition 1.** A simple game in characteristic-function form is a pair $G = (N, v)$ where $N = \{1, 2, \ldots, n\}$ is the set of players and $v : 2^N \to \{0, 1\}$ is the characteristic function which satisfies the following three conditions:

(1) $v(\emptyset) = 0$;

(2) $v(N) = 1$;

(3) $v$ is monotonic, i.e., if $S \subset T \subset N$, then $v(S) \leq v(T)$.

Thus we formalize the idea of coalition decision making. From the above definition it follows that the characteristic function $v$ for a coalition $S \subset N$ indicates the value of $S$. This means that for each coalition $S \subset N$ we have either $v(S) = 0$ or $v(S) = 1$.

In this paper, we will consider a special class of simple games called weighted voting games (or weighted majority games) with a dichotomous voting rule—acceptance ("yes") or rejection ("no"). These games have been found to be well-suited to model economic or political bodies that exercise some kind of control. A weighted voting game is a type of simple cooperative game and a formal model of coalition decision making in which decisions are made by vote.

A weighted voting game $(N, v)$ is described by $G = [q; w_1, w_2, \ldots, w_n]$ where $q$ and $w_1, w_2, \ldots, w_n$ are nonnegative integer numbers such that

$$q \leq \sum_{k=1}^{n} w_k = \tau.$$

The set of weights $\{w_1, w_2, \ldots, w_n\}$ corresponds to the set of players $\{1, 2, \ldots, n\}$. For more information see [9], [12] and [14]. This game has the following properties:

(1) $1 \leq q \leq \tau$;

(2) $n = |N| \geq 2$ is the number of players;

(3) $w_i \geq 0$ is the number of votes of player $i \in N$;

(4) $q$ is the needed quota so that a coalition can win;

(5) the symbol $[q; w_1, w_2, \ldots, w_n]$ represents the weighted voting game $G$ defined by either $v(S) = 1$ when $\sum_{k \in S} w_k \geq q$ or $v(S) = 0$ when $\sum_{k \in S} w_k < q$ for coalition $S \subset N$.

For any weighted voting game $G$, the form $[q; w_1, w_2, \ldots, w_n]$ is often called a weighted presentation of game $G$. Obviously, one weighted voting game has many representations. For example, the following three weighted voting games $G_1 = [51; 49, 49, 2]$, $G_2 = [9; 7, 6, 5]$, and $G_3 = [2; 1, 1, 1]$ represent the same voting rule—majority rule, that is, each coalition of 2 or 3 players is winning. It follows that they have the same characteristic functions.

**2. Definitions and concepts.** We first introduce a definition for two basic types of coalitions—winning and losing.

**Definition 2.** For any coalition $S \subset N$ in game $G$, $S$ is winning if and only if $v(S) = 1$, and $S$ is losing if and only if $v(S) = 0$. The collections of all winning and all losing coalitions in game $G$ are denoted by $W(G)$ and $L(G)$, respectively. If game $G$ is fixed, we simply write $W$ and $L$.

By definition, any simple game has winning and losing coalitions, and this game is determined by the set of all its winning or losing coalitions. We have that $N \in W$ and $\emptyset \in L$; therefore, $W$ and $L$ are nonempty, $W \bigcap L = \emptyset$ and

$W \bigcup L = 2^N$. Observe that a coalition having a winning sub-coalition is also winning and a sub-coalition of a losing coalition is also losing.

It is important to note that if the quota increases (decreases), then the set of all winning coalitions decreases (increases) and the set of all losing coalitions increases (decreases).

For any player $i \in N$, the collection of all winning coalitions including $i$ is denoted by $W_+^i$, the collection of all winning coalitions excluding $i$ is denoted by $W_-^i$, the collection of all losing coalitions including $i$ is denoted by $L_+^i$, and the collection of all losing coalitions excluding $i$ is denoted by $L_-^i$.

**Definition 3.** For any coalition $S \in W$, $S$ is called a minimal winning coalition if and only if $S \setminus \{i\}$ is not winning for all $i \in S$. The collection of all minimal winning coalitions is denoted by $MW$. For any player $i \in N$, the collection of all minimal winning coalitions including $i$ is denoted by $MW_+^i$ and the collection of all minimal winning coalitions excluding $i$ is denoted by $MW_-^i$.

It is easy to prove that $MW$ and $W$ are two finite sets, $MW \subset W$ and $MW$ is nonempty.

Thus, a simple game $(N, v)$ can alternatively be defined in winning-set form as $(N, W)$ or minimal-winning-set form as $(N, MW)$.

**Definition 4.** A player who does not belong to any minimal winning coalition is called a dummy, i. e., player $i \in N$ is a dummy if and only if $i \in N \setminus S$ for all $S \in MW$. A player who belongs to all minimal winning coalitions is called a veto player or vetoer, i. e., player $i \in N$ has the capacity to veto if and only if $i \in S$ for all $S \in MW$. A player $i \in N$ is a dictator if and only if $\{i\}$ is a winning coalition.

In voting power theory, a dummy player has no decision power, a veto player can block every decisions and a dictator has all of the decision power.

For any player $i \in N$, it is easy to show that $MW_+^i = \emptyset$ is equivalent to player $i$ being a dummy, and $MW_+^i = MW$ is equivalent to player $i$ being a veto player.

In the following example we illustrate that there exists a simple game that it is not a weighted voting game.

**Example 1.** [6, Example 5.12.4] Consider simple game $G$ given by $N = \{1, 2, 3, 4, 5\}$ and $MW = \{\{1, 2.3\}, \{4, 5\}\}$. Let us assume that there exist a quota $q$ and weights $w_1, w_2, w_3, w_4, w_5$ such that $G$ is a weighted voting game. This means that $w_1 + w_2 + w_3 \geq q$ and $w_4 + w_5 \geq q$. As a result we obtain

$q > w_1 + w_2 + w_4 \geq q - w_3 + w_4$; therefore, we have that $w_3 > w_4$. Thus, we find that $\{3, 5\} \in W$, but $\{3\}, \{5\} \in L$. It follows that $\{3, 5\} \in MW$. This leads to a contradiction; therefore, $G$ is not a weighted voting game.

**Definition 5.** A weighted voting game $G$ is called proper if and only if $v(S) + v(N \setminus S) \leq 1$ for all $S \subset N$.

Note that a weighted voting game to be proper is equivalent to the complement of a winning coalition to be not winning. This means that in a proper game coalitions $S$ and $N \setminus S$ cannot both be winning. In this context, if $S$ is winning, then $N \setminus S$ is losing, but the converse statement is not always true.

Clearly, the following statements are true:

(1) A proper game may have only one dictator and all other players are dummies. However, a proper game may have several veto players while a dictator is unique. A proper game with two or more veto players does not have a dictator.

(2) An improper game may have one pair of non-intersecting winning coalitions. In particular, an improper game may have more than one dictator.

In what follows, we will study proper games with $n \geq 3$ only.

**Definition 6.** A proper game $G$ is called decisive (or strong) if and only if $v(S) + v(N \setminus S) = 1$ for all $S \subset N$.

It is easy to show that weighted voting game $[5; 4, 3, 2, 1]$ is improper, game $[6, 4, 3, 2, 1]$ is proper but it is not decisive, and game $[3; 1, 1, 1, 1, 1]$ is decisive.

Now we will give the key definition in our paper.

**Definition 7.** For $i \in N$ and $S \in W^i_+$, player $i$ is called a negative swing (also critical or pivotal) member of $S$ if and only if $S \setminus \{i\}$ is not winning. The collection of all winning coalitions including $i$ as a negative swing member is denoted by $W^i_s$. For $i \in N$ and $S \in L^i_-$, player $i$ is called a positive swing (also critical or pivotal) member of $S$ if and only if $S \bigcup \{i\}$ is not losing. The collection of all losing coalitions including $i$ as a positive swing member is denoted by $L^i_s$.

Note that each member of a minimal winning coalition is a negative swing player, a winning coalition may have a negative swing member and a losing coalition may have a positive swing member.

It is often said that $|W^i_s|$ and $|L^i_s|$ are the number of swings of player $i \in N$.

**Remark 1.** In classical theory each positive swing for player $i \in N$ corresponds to a pair of coalitions $(S, S \cup \{i\}) \in L_-^i \times W_+^i$ such that $S$ is losing and $S \cup \{i\}$ is winning ($L \to W$ process), and each negative swing for player $i$ corresponds to a pair of coalitions $(S \setminus \{i\}, S) \in L_-^i \times W_+^i$ such that $S \setminus \{i\}$ is losing and $S$ is winning ($L \leftarrow W$ process). In the first case we say that player $i$ is a swing member of the pair $(S, S \cup \{i\})$, but in the second case we say that that player $i$ is a swing member of the pair $(S \setminus \{i\}, S)$.

It is easy to show that if a weighted voting game has a dictator, then he/she is the only swing player in this game.

**Theorem 1** [2, Corollary 4.1]**.** *For any proper game, $|W_s^i| = |L_s^i|$ for all $i \in N$.*

**Remark 2.** It is important to note that in the proof of Theorem 1 the authors construct a one-to-one mapping $m_i : W_s^i \to L_s^i$ such that coalition $S \in W_s^i$ only corresponds to coalition $S \setminus \{i\} \in L_s^i$ and conversely, coalition $S \setminus \{i\} \in L_s^i$ only corresponds to coalition $S \in W_s^i$. See also Remark 1.

**3. Decision powers of the players.** The concept of decision power of the players in weighted voting games is well-known. For example, let us consider a game $G = [51; 62, 27, 11]$. We may be tempted to say that player 1 has $\frac{62}{100}$ of the decision power, players 2 and 3 have $\frac{27}{100}$ and $\frac{11}{100}$, respectively. But this is not true because player 1 has $\frac{100}{100}$ of the power, and players 2 and 3 are powerless, i. e., player 1 is a dictator and players 2 and 3 are dummies. Weighted voting games use mathematical models to analyze the distribution of decision power of the players. These distributions of decision power are central in economics and political science.

These notes allow us to discuss the Banzhaf power index. This index was introduced by the American legal activist and law professor John Banzhaf III in 1965 as a measure of the real power of players in a cooperative game [1]. It depends on the number of ways in which each player can affect a negative swing. The absolute Banzhaf index concerns the number of times each player $i \in N$ could change a coalition from winning to losing and it requires that we know the number of negative swings for each player $i$. For each player $i \in N$, the absolute Banzhaf index is denoted by $\eta_i$ and it equals the number of negative swings for this player.

**Theorem 2** [4] [7, Lemma 1]. For any proper game $\eta_i = |W_s^i| = |W_+^i| - |W_-^i|$ for all $i \in N$.

As a corollary of Theorem 2 we also obtain $\eta_i = |W_+^i| - (|W| - |W_+^i|) = 2|W_+^i| - |W|$ for all $i \in N$ and $\eta_i - \eta_j = 2(|W_+^i| - |W_+^j|)$ for all $i, j \in N$. Thus, we also have that $\eta_i \equiv \eta_j$ modulo 2 for all $i, j \in N$.

**Example 2.** Consider a weighted voting game $[7; 3, 3, 1, 1]$. The collections of all winning and all minimal winning coalitions are $W = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$ and $MW = \{\{1, 2, 3\}, \{1, 2, 4\}\}$, respectively. The swings of player 1 correspond to three pairs of coalitions $(\{2, 3\}, \{1, 2, 3\})$, $(\{2, 4\}, \{1, 2, 4\})$ and $(\{2, 3, 4\}, \{1, 2, 3, 4\})$, i. e., we get that $\eta_1 = 3$, see Remark 1. Similarly, we obtain $\eta_2 = 3$, $\eta_3 = 1$ and $\eta_4 = 1$.

**Remark 3.** From Theorems 1 and 2 it follows that $\eta_i = |W_s^i| = |L_s^i|$ for all $i \in N$, i. e., $\eta_i$ is either the number of negative swings or the number of positive swings of player $i$.

The normalized Banzhaf power index is the vector $\overrightarrow{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$, given by $\beta_i = \dfrac{\eta_i}{\sum_{k=1}^n \eta_k}$ for $i = 1, 2, \ldots, n$.

The Banzhaf index is similar to the Penrose–Banzhaf (or Banzhaf–Coleman) index which is defined by $b_i = \dfrac{\eta_i}{2^{n-1}}$ for $i = 1, 2, \ldots, n$. The Banzhaf index was originally created in 1946 by Leonel Penrose, but was reintroduced by John Banzhaf in 1965.

**Theorem 3** [11, Theorem 2]. For any proper game, player $i \in N$ being a dummy is equivalent to $\eta_i = 0$.

**Remark 4.** It is important to note that the Banzhaf power index is monotonic with respect to the weights when we are evaluating the power, i. e., for two different players $i, j \in N$, $\eta_i = \eta_j$ when $w_i = w_j$ and $\eta_i \geq \eta_j$ when $w_i > w_j$, see Theorem 3 and [4].

**Theorem 4** [11, Theorem 2(b)]. For any proper game if $i, j \in N$, $i \neq j$, player $i$ is a dummy and $w_i \geq w_j$, then player $j$ is also a dummy.

**4. Concepts for manipulation by merging, splitting and annexation.** Weighted voting games are cooperative games; therefore, the analysis of manipulation is natural. The study of methods of manipulation has practical applications as well. Decision rules in voting games can be manipulated by coalitions merging into single players and players splitting into a number of smaller units [8]. Decision rules can also be manipulated by annexation of a part or all of the weights of other players.

First, we will focus our attention on manipulation by merging, that is, two or more different players merge into a single player. Consider a proper weighted voting game $G = [q; w_1, w_2, \dots, w_n]$. We construct a new game $G' = [q; w'_1, w'_2, \dots, w'_{n'}]$ such that $n > n'$, each player $i'$ in game $G'$ is a fixed coalition of one or more players in game $G$ and its weight $w'_{i'}$ is the sum of the weights of this fixed coalition. The quota and the total sum of the weights remain the same. We denote the set of all players in games $G$ and $G'$ by $N$ and $N'$, respectively. Game $G'$ is called a derivative game of the original game $G$. For more information see [13].

**Example 3.** Let $G = [5; 4, 3, 1, 1]$ be an original game. For the derivative game $G'$, let the coalition of players 2 and 4 in game $G$ be a new player in game $G'$ (players 2 and 4 merge into a single player and it is player $2'$) and the other players remain the same. Thus, we get the derivative game $G' = [5; 4, 4, 1]$.

Second, we will consider manipulation by splitting, that is, a player splits into a number of smaller different players. Consider a proper weighted voting game $G = [q; w_1, w_2, \dots, w_n]$. We construct a new game $G' = [q; w'_1, w'_2, \dots, w'_{n'}]$ such that $n < n'$, each player $i$ in game $G$ splits into one or more players in game $G'$ and the weight $w_i$ of player $i$ in game $G$ is the sum of the weights of the split players in game $G'$. The quota and the total sum of the weights remain the same. Game $G'$ is called a derivative game of the original game $G$. In the other words, game $G$ transforms to game $G'$ by the splitting of player $i$ into players $i'$ and $j'$, $w_i = w'_{i'} + w'_{j'}$ while the other players remain the same.

**Example 4.** Let $G = [5; 4, 4, 1]$ be an original game. For the derivative game $G'$, let player 2 in game $G$ split into two players with weights 3 and 1, while the other players remain the same. Thus, we get the derivative game $G' = [5; 4, 3, 1, 1]$.

It is necessary to note that the converse process of manipulation by merging is the process of manipulation by splitting, see Examples 3 and 4.

Finally, we will discuss manipulation by annexation, that is, a player annexes a part or all of the voting weights of other players. Consider a proper weighted voting game $G = [q; w_1, w_2, \dots, w_n]$. We construct a new game $G' =$

$[q; w'_1, w'_2, \ldots, w'_{n'}]$ such that $n = n'$, fix two different players $i$ and $j$ in game $G$, player $i$ annexes a part or all of the voting weights of player $j$, i. e., player $i$'s new weight is $w'_i = w_i + t$ and player $j$'s new weight is $w'_j = w_j - t$ for $w_j \geq t > 0$. In other words, player $i$ takes $t$ votes from player $j$ and the other players remain the same. Game $G'$ is called a derivative game of the original game $G$.

**Example 5.** Let $G = [6; 4, 3, 2, 1]$ be an original game. For the derivative game $G'$, let player 1 with weight 4 in game $G$ annex one vote of player 3 with weight 2. As a result we get that player 1 has weight 5, player 3 has weight 1 and the other players remain the same. Thus, we get the derivative game $G' = [6; 5, 3, 1, 1]$.

It is interesting to note that one player can consecutively annex a part or all of the voting weights of several players.

**5. Main result.** In this section, we present two basic theorems; first, for manipulation by merging of two players into a single player, and second, for manipulation by annexation of a part or all of the voting weights of another player.

The original version of the first theorem can be seen in [13]. Here we give the two theorems with their proofs because they are connected and logically complementary.

**Theorem 5.** Transform game $G$ to game $G'$ by merging two different players $i$ and $j$ into player $i'$, and the other players remaining the same. The following statements are true.

(a) $\eta_i + \eta_j = 2\eta'_{i'}$, $|W_s^i(G))| + |W_s^j(G)| = 2|W_s^{i'}(G')|$ and $|L_s^i(G))| + |L_s^j(G)| = 2|L_s^{i'}(G')|$.

(b) If player $j$ is a dummy in game $G$, then $\eta_i = 2\eta'_{i'}$, $|W_s^i(G))| = 2|W_s^{i'}(G')|$ and $|L_s^i(G))| = 2|L_s^{i'}(G')|$.

(c) Players $i$ and $j$ being dummies in game $G$ is equivalent to player $i'$ being a dummy in game $G'$.

(d) If $w_i = w_j$, then $2w_i = w'_{i'}$, $\eta_i = \eta'_{i'}$, $|W_s^i(G))| = |W_s^{i'}(G')|$ and $|L_s^i(G))| = |L_s^{i'}(G')|$.

(e) If $k \neq i, j$ and player $k$ in game $G$ transforms to player $k'$ in game $G'$, then $\eta_k \geq \eta'_{k'}$, $|W_s^k(G))| \geq |W_s^{k'}(G')|$ and $|L_s^k(G))| \geq |L_s^{k'}(G')|$.

(f) If player $i'$ is a dummy in game $G'$, then $\beta_i + \beta_j = \beta'_{i'} = 0$.

(g) If player $i'$ is not a dummy in game $G'$, then $\beta_i + \beta_j < 2\beta'_{i'}$.

(h) If $w_j \leq w_i$ and player $i'$ is not a dummy in game $G'$, then $\beta_j < \beta'_{i'}$.

(i) If player $i$ is a dictator in game $G$, then player $i'$ is a dictator in game $G'$ and $\beta_i + \beta_j = \beta'_{i'} = 1$.

Proof.

(a) Let $S \in L_s^{i'}(G')$, $w_i + w_j = w'_{i'}$ and let us assume that $w_j \leq w_i$. This means that $0 < q - \sum_{h \in S} w_h \leq w'_{i'} = w_j + w_i$ and $S \cup \{i'\} \in W(G')$.

There are three cases for positive swings of each player $i$, $j$ or $i'$.

Case 1. If $0 < q - \sum_{h \in S} w_h \leq w_j$, then $q \leq \sum_{h \in S} w_h + w_j$ and $q \leq \sum_{h \in S} w_h + w_i$. As a result we see that player $i'$ is a swing member of the pair $(S, S \cup \{i'\})$ in game $G'$, and players $j$ and $i$ are swing members of the pairs $(S, S \cup \{j\})$ and $(S, S \cup \{i\})$ in game $G$, respectively. Hence, in this case we have $\eta_i + \eta_j = 2\eta'_{i'}$.

Case 2. If $w_j < q - \sum_{h \in S} w_h \leq w_i$, then $w_j + \sum_{h \in S} w_h < q \leq \sum_{h \in S} w_h + w_i$. Here we get that player $i'$ is a swing member of the pair $(S, S \cup \{i'\})$ in game $G'$, and players $j$ and $i$ are swing members of the pairs $((S \cup \{i\}) \setminus \{j\}, S \cup \{i\})$ and $(S, S \cup \{i\})$ in game $G$, respectively. We also obtain $\eta_i + \eta_j = 2\eta'_{i'}$.

Case 3. If $w_i < q - \sum_{h \in S} w_h \leq w'_{i'}$, then $w_i + \sum_{h \in S} w_h < q \leq \sum_{h \in S} w_h + w_i + w_j$. So we have that player $i'$ is a swing member of the pair $(S, S \cup \{i'\})$ in game $G'$, and players $j$ and $i$ are swing members of the pairs $((S \cup \{i\}) \setminus \{j\}, S \cup \{i\})$ and $((S \cup \{j\}) \setminus \{i\}, S \cup \{j\})$ in game $G$, respectively. Here we get $\eta_i + \eta_j = 2\eta'_{i'}$ too.

In summary, we obtain $\eta_i + \eta_j = 2\eta'_{i'}$. From $\eta_i = |W_s^i(G)| = |L_s^i(G)|$, $\eta_j = |W_s^j(G)| = |L_s^j(G)|$ and $\eta'_{i'} = |W_s^{i'}(G')| = |L_s^{i'}(G')|$ it follows that $|W_s^i(G))| + |W_s^j(G)| = 2|W_s^{i'}(G')|$ and $|L_s^i(G))| + |L_s^j(G)| = 2|L_s^{i'}(G')|$.

(b) If player $j$ is a dummy, then set $W_s^j(G)$ is empty, see Theorem 3. According to (a) we get that $\eta_i = 2\eta'_{i'}$, $|W_s^i(G))| = 2|W_s^{i'}(G')|$ and $|L_s^i(G))| = 2|L_s^{i'}(G')|$.

(c) This is immediate from (a) and Theorem 3.

(d) The proof follows from (a).

(e) Let us consider a player $k' \in N'$ in game $G'$ such that $k' \neq i'$ and coalition $S \in L_s^{k'}(G')$. This means that $S \in L_s^k(G)$; therefore, we obtain $|L_s^k(G)| \geq |L_s^{k'}(G')|$. It also follows that $\eta_k \geq \eta'_{k'}$ and $|W_s^k(G))| \geq |W_s^{k'}(G')|$.

(f) If player $i'$ is a dummy, then set $W_s^{i'}(G')$ is empty, see Theorem 3. This means that $|W_s^{i'}(G')| = 0$, i. e., $\beta'_{i'} = 0$. Thus we find that $\beta_i = \beta_j = 0$; therefore, $\beta_i + \beta_j = \beta'_{i'} = 0$.

(g) If player $i'$ is not a dummy, then $\eta'_{i'} > 0$. Clearly, we have that $\eta_i + \eta_j = 2\eta'_{i'} > 0$, $2\eta'_{i'} > \eta'_{i'}$ and $\eta_i > 0$ or $\eta_j > 0$, see (a) and (f).

Applying now (a) and (e) we calculate that

$$\beta_i + \beta_j = \frac{\eta_i + \eta_j}{\sum_{h\in N} \eta_h} = \frac{\eta_i + \eta_j}{\eta_i + \eta_j + \sum_{h\in N\setminus\{i,j\}} \eta_h}$$

$$\leq \frac{2\eta'_{i'}}{2\eta'_{i'} + \sum_{h\in N\setminus\{i'\}} \eta'_h} < \frac{2\eta'_{i'}}{\eta'_{i'} + \sum_{h\in N'} \eta'_h} < 2\beta'_{i'}.$$

Finally, we obtain $\beta_i + \beta_j < 2\beta'_{i'}$.

(h) Remark 4 implies $\beta_j \leq \beta_i$; therefore, we find that $2\beta_j \leq \beta_i + \beta_j < 2\beta'_{i'}$, i. e., $\beta_j < \beta'_{i'}$.

(i) If player $i$ is a dictator in game $G$, then the other players in game $G$ are dummies. These players are dummies in game $G'$ too, see (a), (e) and (f). Hence, player $i'$ is a dictator in game $G'$. As a result we obtain $\beta_j = 0$, $\beta_i = \beta'_{i'} = 1$ and $\beta_i + \beta_j = \beta'_{i'} = 1$. □

Now we show the following two examples for merging a coalition of three players into one player.

**Example 6.** [13] Let $G = [17; 8, 7, 4, 4, 2, 1, 1]$ be an original game with $n = 7$, $q = 17$ and $\tau = 27$. For the derivative game $G'$, let the coalition of players 2, 3 and 5 in game $G$ be a new player in game $G'$ and the other players remain the same. So we get $G' = [17; 13, 8, 4, 1, 1]$ where $n' = 5$, $q' = 17$, $\tau' = 27$, $1' = \{2, 3, 5\}$, $2' = \{1\}$, $3' = \{4\}$, $4' = \{6\}$ and $5' = \{7\}$. The sum of the Banzhaf power indices of players 2, 3 and 5 in game $G$ is $\beta_2 + \beta_3 + \beta_5 = 0,4851$ but the index of player $1'$ in game $G'$ is $\beta'_{1'} = 0,6000$. As a result we obtain $\beta_2 + \beta_3 + \beta_5 < \beta'_{1'}$.

**Example 7.** [13] Let $G = [30; 9, 8, 5, 5, 4, 3, 1]$ be an original game with $n = 7$, $q = 30$ and $\tau = 35$. For the derivative game $G'$, let the coalition of players 1, 2 and 3 in game $G$ be a new player in game $G'$ and the other players remain the same. In this case we get $G' = [30; 22, 5, 4, 3, 1]$ where $n' = 5$, $q' = 30$, $\tau' = 35$, $1' = \{1, 2, 3\}$, $2' = \{4\}$, $3' = \{5\}$, $4' = \{6\}$ and $5' = \{7\}$. Here the sum of the Banzhaf power indices of players 1, 2 and 3 in game $G$ is $\beta_1+\beta_2+\beta_3 = 0,5789$ but the index of player $1'$ in game $G'$ is $\beta'_{1'} = 0,3684$. Now we obtain $\beta_1+\beta_2+\beta_3 > \beta'_{1'}$.

Note that in Example 6 decision power increases but in Example 7 it decreases.

**Theorem 6.** Transform game $G$ to game $G'$ such that player $i$ annexes $t \in \{1, 2, \ldots, w_j\}$ voters of player $j$ and the other players remain the same. The following statements are true.

(a) $\eta_i(G') \geq \eta_i(G)$ and $\eta_j(G') \leq \eta_j(G)$.

(b) $\eta_i(G') - \eta_i(G) = \eta_j(G) - \eta_j(G')$.

(c) If player $j$ is a dummy in game $G$ and $w_j > 0$, then player $j$ is a dummy in game $G'$ and $\eta_i(G') = \eta_i(G)$.

(d) Players $i$ and $j$ being dummies in game $G$ is equivalent to these two players $i$ and $j$ being dummies in game $G'$.

(e) If player $i$ is a vetoer in game $G$, then player $i$ is a vetoer in game $G'$.

(f) If player $i$ is a dictator in game $G$, then player $i$ is a dictator in game $G'$.

Proof.

(a) Let $S \in L^i_s(G)$, i.e., $S \in L^i_-(G)$ and $S \cup \{i\} \in W^i_+(G)$. This means that $0 < q - \sum_{h \in S} w_h \leq w_i$. From $w'_i = w_i + t > w_i$ it follows $0 < q - \sum_{h \in S} w_h < w'_i$; therefore, $S \in L^i_-(G')$ and $S \cup \{i\} \in W^i_+(G')$. As a result we obtain $S \in L^i_s(G')$ and $\eta_i(G') \geq \eta_i(G)$.

By analogy with the above, let $S \in L^j_s(G')$, i.e., $S \in L^j_-(G')$ and $S \cup \{j\} \in W^j_+(G')$. So, $0 < q - \sum_{h \in S} w_h \leq w'_j < w_j$ implies $S \in L^j_-(G)$ and $S \cup \{j\} \in W^i_+(G)$. In this case we find that $\eta_j(G') \leq \eta_j(G)$.

(b) Let us transform game $G$ to game $G_1$ by merging players $i$ and $j$ into player $i'$ and the other players be the same. Now, if we transform game $G'$ to game $G_2$ by merging players $i$ and $j$ into player $i'$ and the other players remain the same, then we obtain $G_1 = G_2$. According to Theorem 5(a) we obtain $\eta_i(G) + \eta_j(G) = 2\eta_{i'}(G_1) = 2\eta_{i'}(G_2) = \eta_i(G') + \eta_j(G')$. Finally, we get that $\eta_i(G') - \eta_i(G) = \eta_j(G) - \eta_j(G')$.

(c) Let player $j$ be a dummy in game $G$ and $w_j > 0$. From (a) and (b) it follows $\eta_i(G') - \eta_i(G) = \eta_j(G) - \eta_j(G') \geq 0$. This means that player $j$ is a dummy in game $G'$ and $\eta_i(G') = \eta_i(G)$.

(d) The proof follows from (c).

(e) If player $i$ is a vetoer in game $G$, then $\tau - q < w_i$. From $w'_i = w_i + t$ and $t > 0$ it follows that player $i$ is a vetoer in game $G'$.

(f) If player $i$ is a dictator in game $G$, then $q \leq w_i$. By analogy, $w'_i = w_i + t$ and $t > 0$ imply player $i$ is a dictator in game $G'$. □

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