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STRATEGIC TRADE BETWEEN TWO COUNTRIES—EXPLORING THE CASE OF PARTIAL LOCAL CONSUMER PROTECTION*

Iordan Iordanov, Andrey Vassilev

ABSTRACT. The paper develops a dynamic model of trade between two countries where the trading entities interact in a strategic context. Consumers in both countries are endowed with certain incomes and try to acquire as much as possible of the quantities available on the markets. Consumers have privileged access to some of the good supplied locally, a form of partial local protection. Over time, prices are adjusted to respond to the outcomes of trading. For this setup, we prove the existence of Nash equilibria and simulate the model numerically in Python to illustrate the possibility of obtaining different types of price dynamics depending on the price adjustment rule used.

1. Introduction. In this paper we develop a trade model focusing on the case of two countries (regions), where the trading entities compete strategically

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*Some of the results were reported in abridged form in [4].

for the fixed supply of a good in each region. Depending on the outcome of the trade, prices are adjusted over time to correct for excess supply or demand on the respective markets. The setup is similar to our earlier works [3] and [8], with the exception that here local consumers enjoy only partial protection in terms of preferential access to the good on their home market. This stands in contrast to the cited works, which employ the assumption of full protection. Since the present model also covers the special case of full protection, it subsumes the previous formulations. In this work, we focus on discrete-time dynamics.

For our model numerical simulations implemented in IPython/Jupyter Notebook (see [7]) are carried out to explore the various types of price dynamics that can be obtained under different parameterizations and price adjustment rules. On the basis of our experiments we conclude that the model is capable of generating diverse types of price dynamics, including degenerate (zero price) outcomes and cyclicality.

Compared to our earlier work [4], here we expand the analysis in the following directions. First, we formally prove the existence of Nash equilibria for the model. Second, we provide details on the numerical methods and a software implementation in IPython. Finally, we demonstrate the robustness of our numerical findings by reporting the counterparts of the experiments in [4] for changed values of the model parameters which yield the same results in qualitative terms.

The paper is organized as follows. In section 2 we present our model. Section 3 deals with the issue of existence of Nash equilibria. Section 4 provides details of the numerical implementation and the results of the simulations performed. Section 5 presents our conclusions.

2. Model description. We study the interaction of two consumers from different countries (regions), labelled 1 and 2, who compete for a good supplied on both markets. Consumers are endowed with constant monetary income Y_i and the good is supplied in fixed quantities $q_i,\ i = 1, 2$. There are no savings instruments and therefore income cannot be accumulated but is available only in the current period.

The price of the good on the market of region 1 for the local consumer at time t is denoted $p_{1,t}$ and the local price of the good in region 2 is $p_{2,t}$. If consumer 1 wants to import from the other region, an additional cost of ρ_2 per unit of good is incurred. This cost may have various interpretations, including transportation costs, customs duties or other transaction costs associated with foreign trade. Analogously, consumer 2 pays a cost ρ_1 per unit of good imported from region 1. Thus, the total price of goods imported from region 2 for consumer 1 is $p'_{2,t} = p_{2,t} + \rho_2$ and the total price of imports from region 1 for consumer 2 is $p'_{1,t} = p_{1,t} + \rho_1$.

Consumer 1 can place orders for quantities α and β on the markets of region 1 and 2, respectively. The orders of consumer 2, placed in regions 1 and 2, are denoted γ and δ. The consumers are aware of each others' characteristics and therefore a game-theoretic situation occurs. The orders of the consumers form their strategy spaces (see [2], chapter 3, for details of the game-theoretic terminology and results), denoted $S_{1,t}$ and $S_{2,t}$, and assumed to be nonempty. They are defined as follows:

$$S_{1,t} = \left\{(\alpha, \beta) \in \mathbb{R}^2 \mid p_{1,t}\alpha + p'_{2,t}\beta \le Y_1,\ \alpha, \beta \ge 0\right\}, \tag{1}$$

$$S_{2,t} = \left\{(\gamma, \delta) \in \mathbb{R}^2 \mid p'_{1,t}\gamma + p_{2,t}\delta \le Y_2,\ \gamma, \delta \ge 0\right\}. \tag{2}$$

In the course of trading, each consumer enjoys partial protection on the local market. This means that a fixed share $\epsilon \in (0, 1]$ of the quantity q_1 is reserved for consumer 1 and, similarly, a share $\xi \in (0, 1]$ of the quantity q_2 is preferentially available to consumer 2. Consumers have the right to buy the respective quantities ϵq_1 and ξq_2 but are not obliged to do so. After the local consumer buys a part or all of the preferentially available quantity, the remaining quantity of the good is offered to the foreign consumer. In turn, the foreign consumer can purchase part or all of this remainder and, if there is anything left, it is again offered to the local consumer.

We assume that under the above conditions, each consumer wants to maximize the quantity ot the good purchased. This implies the following payoff function for consumer 1:

$$\begin{aligned} P_1(\alpha, \beta, \gamma, \delta) = {} & \min(\alpha, \epsilon q_1) + \min(\beta, q_2 - \min(\delta, \xi q_2)) + \\ & \min(\alpha - \min(\alpha, \epsilon q_1), q_1 - \min(\alpha, \epsilon q_1) - \min(\gamma, q_1 - \min(\alpha, \epsilon q_1))) = \\ & \min(\beta, q_2 - \min(\delta, \xi q_2)) + \min(\alpha, q_1 - \min(\gamma, q_1 - \min(\alpha, \epsilon q_1))). \end{aligned} \tag{3}$$

Similarly, the payoff function for consumer 2 is

$$\begin{aligned} P_2(\alpha, \beta, \gamma, \delta) = {} & \min(\gamma, q_1 - \min(\alpha, \epsilon q_1)) + \\ & \min(\delta, q_2 - \min(\beta, q_2 - \min(\delta, \xi q_2))). \end{aligned} \tag{4}$$

For the above game, consumers are assumed to trade by placing orders in such a manner that a Nash equilibrium emerges as the outcome of the trade (see section 3 for details of the concept and the existence result). The elements of this equilibrium are the respective orders, denoted by (α^*, β^*) for consumer 1 and (γ^*, δ^*) for consumer 2. At the end of each period, after trading has been concluded and a Nash equilibrium has been reached, prices are decreased if the quantity available in the respective region has not been entirely consumed. Prices are increased if there is unspent income in the respective region. These two possibilities are mutually exclusive.

The above principle for the change in prices can be formalized through different price adjustment rules.

In discrete time, an example of price adjustment rules might be

$$p_{1,t+1} = p_{1,t}q_1^{\text{cons}}/q_1 + Y_1^{\text{res}}/q_1 - p_{1,t}, \tag{5}$$
$$p_{2,t+1} = p_{2,t}q_2^{\text{cons}}/q_2 + Y_2^{\text{res}}/q_2 - p_{2,t}, \tag{6}$$

where $Y_1^{\text{res}} = Y_1 - p'_{2,t}\beta^*$, $Y_2^{\text{res}} = Y_2 - p'_{1,t}\gamma^*$, $q_1^{\text{cons}} = \alpha^* + \gamma^*$ and $q_2^{\text{cons}} = \beta^* + \delta^*$.

As another example, the price adjustment rules can take the form

$$p_{1,t+1} = p_{1,t}q_1^{\text{cons}}/q_1 + (Y_1 - Y_1^{\text{cons}})/q_1, \tag{7}$$
$$p_{2,t+1} = p_{2,t}q_2^{\text{cons}}/q_2 + (Y_2 - Y_2^{\text{cons}})/q_2, \tag{8}$$

where $Y_1^{\text{cons}} = p_{1,t}\alpha^* + p'_{2,t}\beta^*$ and $Y_2^{\text{cons}} = p'_{1,t}\gamma^* + p_{2,t}\delta^*$.

The above rules are versions of the rules used in [3] and [8].

3. Nash equilibria.

Below we introduce several game-theoretic concepts that are necessary for the definition and proof of existence of Nash equilibria. The exposition follows chapter 3 in [2], adapted to our case and with minimal changes in notation. We conclude the section by providing a proof that a Nash equilibrium exists for our model.

Definition 1 (Best reply). *The best reply for consumer 1, for given strategies $\bar{\gamma}, \bar{\delta}$ played by consumer 2, is defined as*

$$BR_1(\bar{\gamma}, \bar{\delta}) = \underset{(\alpha,\beta)\in S_{1,t}}{\text{argmax}}\, P_1(\alpha, \beta, \bar{\gamma}, \bar{\delta}).$$

Analogously, the best reply for consumer 2 is defined as

$$BR_2(\bar{\alpha}, \bar{\beta}) = \underset{(\gamma,\delta)\in S_{2,t}}{\text{argmax}}\, P_2(\bar{\alpha}, \bar{\beta}, \gamma, \delta).$$

Since P_i are continuous and $S_{i,t}$ are compact, there exist solutions to the best-reply problems, i. e., the best-reply sets are nonempty.

Definition 2 (Best-reply mapping)**.** *The best-reply mapping BR for a game is a correspondence associating each strategy profile $(\alpha, \beta, \gamma, \delta)$ with the set $BR_1(\gamma, \delta) \times BR_2(\alpha, \beta)$, i. e.,*

$$BR : (\alpha, \beta, \gamma, \delta) \to BR_1(\gamma, \delta) \times BR_2(\alpha, \beta).$$

The best-reply mapping can be interpreted as showing how each player would like to change their strategy after observing the strategy played by the other player in the game.

Definition 3 (Nash equilibrium)**.** *A Nash equilibrium for a game is a strategy profile $(\alpha^*, \beta^*, \gamma^*, \delta^*)$ for which $(\alpha^*, \beta^*, \gamma^*, \delta^*) \in BR(\alpha^*, \beta^*, \gamma^*, \delta^*)$.*

According to Definition 3, a Nash equilibrium is a fixed point of the best-reply mapping. It can be interpreted as a situation in which none of the players finds it profitable to deviate from their current strategy given the choice of the other player. Thus, the concept of Nash equilibrium captures the idea of a stable outcome in a game.

A Nash equilibrium is guaranteed to exist under certain circumstances. To formulate the existence result precisely, we provide the following definition of quasiconcavity ([2], p. 66).

Definition 4 (Quasiconcavity)**.** *A function $y = f(x)$, defined on $D \in \mathbb{R}^n$, is called quasiconcave if, for any choice of $x^1, x^2 \in D$ for which $f(x^1) = f(x^2)$, and for any $\lambda \in [0, 1]$, we have $f(\lambda x^1 + (1 - \lambda)x^2) \geq f(x^1)$.*

Quasiconcavity is obviously a more general concept than concavity.

The existence of Nash equilibria is given by the next result ([2], p. 70).

Theorem 1. *Let the following hold true:*

1. *The sets $S_{1,t}, S_{2,t} \subset \mathbb{R}^2$ are compact and convex;*

2. *The functions $P_1(s)$ and $P_2(s)$ are bounded and continuous for all $s \in S_t$, where $S_t := S_{1,t} \times S_{2,t}$;*

3. *The function $P_1(\alpha, \beta, \gamma, \delta)$ is quasiconcave with respect to $(\alpha, \beta) \in S_{1,t}$ and the function $P_2(\alpha, \beta, \gamma, \delta)$ is quasiconcave with respect to $(\gamma, \delta) \in S_{2,t}$.*

   Then the noncooperative game defined by $S_{1,t},\ S_{2,t}, P_1$ and P_2 has at least one Nash equilibrium.

To proceed with showing the existence of Nash equilibria for the game in our model, we first prove the following

Lemma 1. *The functions* P_1 *and* P_2 *defined respectively by* (3) *and* (4) *are quasiconcave.*

Proof. We shall verify this claim for P_1. Let $(\alpha_1, \beta_1, \gamma, \delta)$ and $(\alpha_2, \beta_2, \gamma, \delta)$ be such that

$$P_1(\alpha_1, \beta_1, \gamma, \delta) = P_1(\alpha_2, \beta_2, \gamma, \delta).$$

We shall check that, for all $\lambda \in [0, 1]$, we have

$$P_1(\lambda\alpha_1 + (1 - \lambda)\alpha_2, \lambda\beta_1 + (1 - \lambda)\beta_2, \gamma, \delta) \geq P_1(\alpha_1, \beta_1, \gamma, \delta).$$

Without loss of generality, let $0 \leq \alpha_1 < \alpha_2$. There are two cases:

Case 1. $0 \leq \beta_1 < \beta_2$,

Case 2. $0 \leq \beta_2 < \beta_1$.

In Case 1 we set $\tilde{\alpha} = \lambda\alpha_1 + (1 - \lambda)\alpha_2$, $\lambda \in [0, 1]$ and $\tilde{\beta} = \lambda\beta_1 + (1 - \lambda)\beta_2$. Since $\tilde{\alpha} \in [\alpha_1, \alpha_2]$, i. e., $\tilde{\alpha} \geq \alpha_1$ (and also $\alpha_2 > \alpha_1$), it follows that

$$\min(\tilde{\alpha}, \epsilon q_1) \geq \min(\alpha_1, \epsilon q_1),$$

$$\min(\alpha_2, \epsilon q_1) \geq \min(\alpha_1, \epsilon q_1).$$

Consequently, we have

$$\max(q_1 - \gamma, \min(\tilde{\alpha}, \epsilon q_1)) \geq \max(q_1 - \gamma, \min(\alpha_1, \epsilon q_1))$$

and

$$\max(q_1 - \gamma, \min(\alpha_2, \epsilon q_1)) \geq \max(q_1 - \gamma, \min(\alpha_1, \epsilon q_1)).$$

The latter, combined with the fact that $\tilde{\alpha}$ and α_2 are greater than α_1, imply that

$$\min(\tilde{\alpha}, \max(q_1 - \gamma, \min(\tilde{\alpha}, \epsilon q_1))) \geq \min(\alpha_1, \max(q_1 - \gamma, \min(\alpha_1, \epsilon q_1)))$$

and

$$\min(\alpha_2, \max(q_1 - \gamma, \min(\alpha_2, \epsilon q_1))) \geq \min(\alpha_1, \max(q_1 - \gamma, \min(\alpha_1, \epsilon q_1))).$$

Considering the specific form of P_1 as given in (3) and the fact that the term containing β is concave in β, we have the required

$$P_1(\tilde{\alpha}, \tilde{\beta}, \gamma, \delta) \geq P_1(\alpha_1, \beta_1, \gamma, \delta) = P_1(\alpha_2, \beta_2, \gamma, \delta).$$

In Case 2 we change the multipliers in front of α_1 and α_2 from λ to $(1 - \lambda)$ and vice versa. The claim follows immediately. □

We can now readily prove the following

Proposition 1. *For the game defined by $S_{1,t}, S_{2,t}, P_1$ and P_2 as given in* (1), (2), (3) *and* (4)*, there exists a Nash equilibrium.*

P r o o f. Conditions 1) and 2) of Theorem 1 are obviously fulfilled. Condition 3) of the theorem follows from Lemma 1. □

4. Numerical implementation and results. The model described above was implemented in Python, using the infrastructure provided by IPython in the context of the Jupyter Notebook (see [7] and [5], respectively). Our choice to work with IPython within the Jupyter Notebook was motivated by several considerations. First, Python itself was selected as the implementation language due to its clean and transparent syntax, rich ecosystem of scientific computing packages (notably Numpy, Scipy and the plotting library Matplotlib) and suitability for applications in exploratory computation (REPL-type environment). Our second consideration builds on the last observation, as IPython further expands the native Python capabilities in terms of interactive computing, ease of debugging and speed of issuing and editing commands. Third, we chose to work with the IPython kernel within the Jupyter Notebook because the notebook provides a complete environment that bundles together code, output and documenting texts in a transparent and reproducible manner.

To find a solution to the best-reply problem of the respective consumers, functionality for constrained optimization from the library `scipy.optimize` was employed. More specifically, we used the function `minimize` (with the required change in the sign of the objective function to account for the fact that the problem is one of maximization), which is a wrapper around several optimization algorithms that are automatically selected based on the type of optimization problem under consideration (with equality or inequality constraints, bounds on the variables etc.). For our formulation the `minimize` function employs the SLSQP method. As per the documentation, this method uses sequential least squares programming to minimize a function of several variables with any combination of bounds, equality and inequality constraints. The method wraps the SLSQP Optimization subroutine presented in [6].

The Nash equilibrium points for the game were computed on the basis of the best reply solutions, using a fixed point routine, again from `scipy.optimize`. The specific function used was `fixed_point`. This function by default finds a fixed point through an algorithm using Steffensen's method with Aitken's Δ^2

convergence acceleration (see [1], p. 88). The essence of our approach was to construct the best reply mapping for the game by combining the individual best replies obtained via `minimize` and then pass it to `fixed_point` to compute a fixed point of this mapping, which is by definition a Nash equilibrium.

In order for the `minimize` and `fixed_point` to complete successfully, one also needs to specify the initial conditions and tolerance parameters for the respective solvers. In our case the initial conditions for the SLSQP solver for consumer 1 were chosen to be $Y_1/(2p_{1,t})$ and $Y_1/(2p'_{2,t})$. The initial conditions for consumer 2 were respectively set at $Y_2/(2p'_{1,t})$ and $Y_2/(2p_{2,t})$. The same initial conditions were provided to the fixed point solver implementing Steffensen's method. The default tolerance and iteration options were used for the computations using `minimize`. For `fixed_point` the convergence tolerance parameter `xtol`, which by default is set to 10^{-8}, was relaxed to 10^{-4}, while the number of iterations was increased to 1000 from the default of 500.

Our simulations explored the types of price dynamics that can be obtained under the chosen price rules by varying incomes, quantities and transportation costs. Below we report several representative outcomes, shown graphically in Figures 1–5.

The simulations were parametrized as follows. The common parameters across all simulations are $\epsilon = 0.2$, $\xi = 0.15$, $p_{1,0} = 4$ and $p_{2,0} = 5$. Table 1 presents the subset of model parameters that changes across simulations. The numbering of the simulations corresponds to the numbering of the figures presenting price dynamics.

Sim. №	Rules	Y_1	Y_2	q_1	q_2	ρ_1	ρ_2
1	(5),(6)	105	125	32	42	2.5	2.5
2	(5),(6)	203	325	31	42	2.1	2.1
3	(7),(8)	98	322	510	44	1.9	1.9
4	(7),(8)	103	322	49	39	3	210
5	(7),(8)	55	5	32	42	2.2	2.2

Table 1. Parametrization of the model simulations

The simulation results make it evident that different price adjustment rules used in conjunction with different parameter sets can produce very diverse outcomes. These can range from trivial one-period "jumps" of prices to a steady state value to much less regular behaviour such as transitional dynamics settling down on a steady state value or cyclical behaviour.

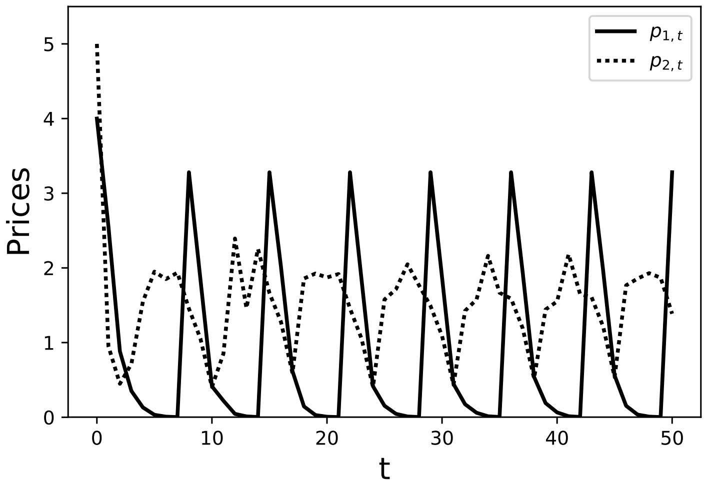

Fig. 1. Complex price dynamics with multi-period transition to the cyclical orbit (Simulation 1)

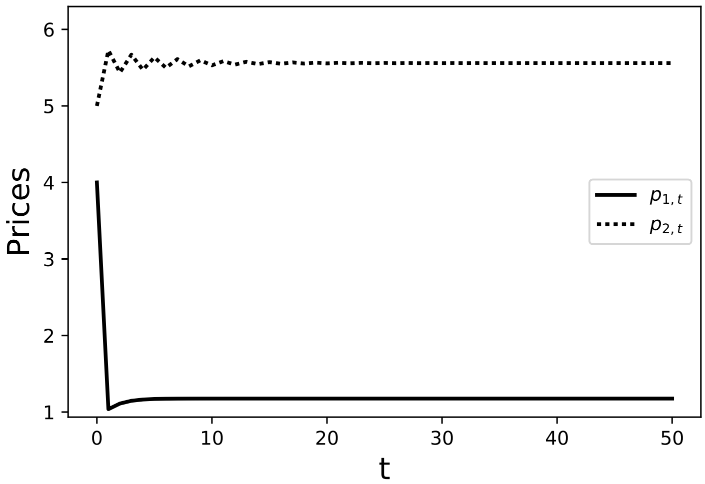

Fig. 2. Convergence to steady state prices with dampened oscillations in the transition dynamics (Simulation 2)

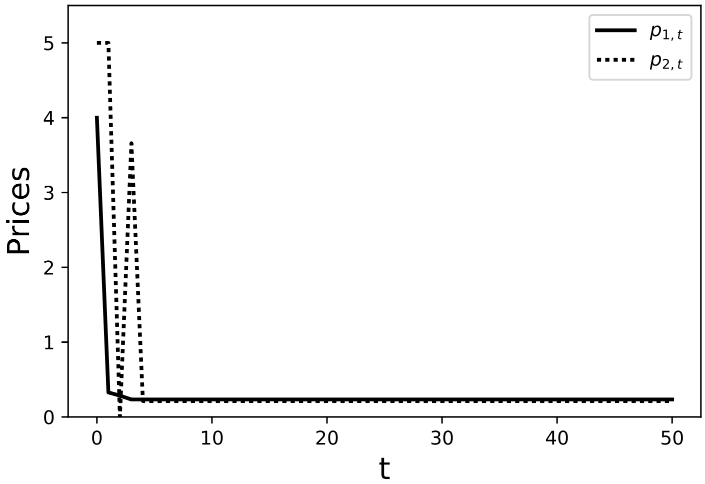

Fig. 3. Simple transition to steady state prices (Simulation 3)

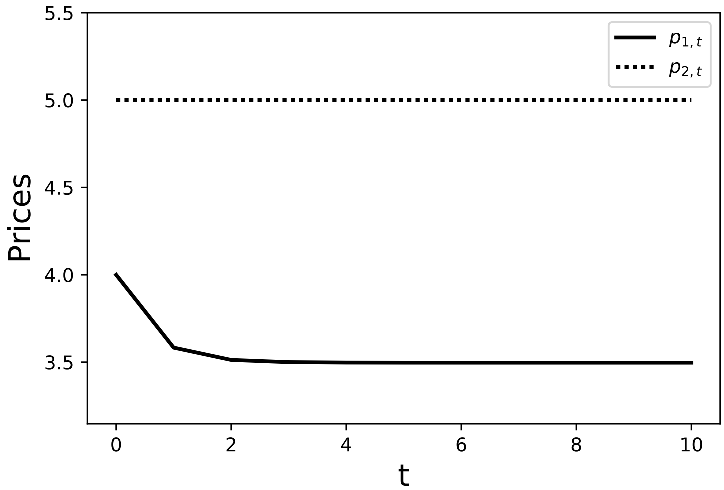

Fig. 4. Monotonic convergence to a steady state price (Simulation 4)

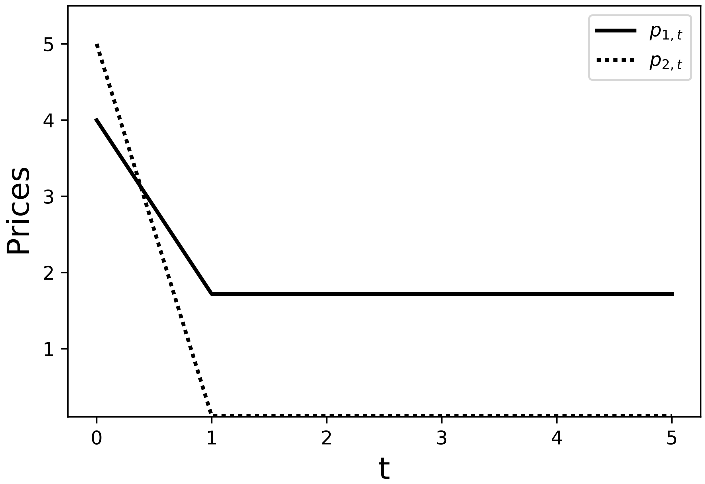

Fig. 5. Convergence to a steady state in one period (Simulation 5)

The specification of the price adjustment rules is crucial for the type of dynamics obtained and the interpretation of the results. Parameter configurations are also important but they can be justified in terms of the economic context the model is trying to capture, e. g., trade between a relatively rich country and a poor economy, or a situation of abundance of the supply of the good on one of the markets vs. scarcity on the other market. In contrast, the specification of the price rules may be more difficult to defend, especially in situations where there is insufficient data to corroborate the chosen form. One commonly used approach in the economics literature is to derive the price adjustment rules from explicitly formulated problems for a new type of agent, for instance a profit maximization problem for the good supplier. This is a fruitful direction for future research.

5. Conclusions. Our simulation results illustrate the possibility of obtaining diverse types of price dynamics in discrete time, depending on the specific parametrization and price adjustment rules used. As mentioned, this opens up the issue of the formulation and implications of using a specific adjustment rule.

Experience with similar models from [3] and [8] shows that the analytic classification of Nash equilibria can be a daunting task even in the case of two regions. This is confirmed by our initial attempts to classify the Nash equilibria for this model. As a consequence, for a more general version of the model, for instance one involving several countries or more than one good, numerical simulation appears to be the most promising direction for establishing the properties of the model.

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REFERENCES

[1] Burden R.L., J. D. Faires. Numerical Analysis. Brooks/Cole, Cengage Learning, Boston, 2010.

[2] Friedman J.W. Game Theory with Applications to Economics. Oxford University Press, New York, 1990.

[3] Iordanov I., S. Stoyanov, A. Vassilev. Price dynamics in a two region model with strategic interaction. In: Proceedings of the 33 Spring Conference of the Union of Bulgarian Mathematicians, 2004, 144–149.

[4] Iordanov I. V., A. A. Vassilev. Strategic trade with partial local consumer protection, In: Proceedings of the 46th Spring Conference of the Union of Bulgarian Mathematicians, 2017, 181–186.

[5] Kluyver T., B. Ragan-Kelley, F. Pérez, B. Granger, M. Bussonnier, J. Frederic, K. Kelley, J. Hamrick, J. Grout, S. Corlay, P. Ivanov, D. Avila, S. Abdalla, C. Willing, Jupyter Development Team. Jupyter Notebooks — a publishing format for reproducible computational workflows. In: Positioning and Power in Academic Publishing: Players, Agents and Agendas, IOS Press Ebooks, 2016, 87–90.

[6] Kraft D. A software package for sequential quadratic programming. 1988. Tech. Rep. DFVLR-FB 88-28, DLR German Aerospace Center – Institute for Flight Mechanics, Köln, Germany.

[7] Pérez F., B. E. Granger. IPython: A System for Interactive Scientific Computing. *Computing in Science and Engineering*, **9** (2007), No 3, 21–29. doi:10.1109/MCSE.2007.53.

[8] VASSILEV A., I. IORDANOV, S. STOYANOV. A strategic model of trade between two regions with continuous-time price dynamics. *Comptes rendus de l'Acad. bul. Sci.*, **58** (2005), No 4, 361–366.

Iordan Iordanov
International Business School
14, Gurko Str.
2140 Botevgrad, Bulgaria
e-mail: `iiordanov@ibsedu.bg`
and
Faculty of Mathematics and Informatics
St. Kliment Ohridski University of Sofia
5, James Bourchier Blvd.
1164 Sofia, Bulgaria

Andrey Vassilev
Bulgarian National Bank
1, Knyaz Alexander I Sq.
1000 Sofia, Bulgaria
and
Faculty of Mathematics and Informatics
Faculty of Economics and Business Administration
St. Kliment Ohridski University of Sofia
125, Tsarigradsko Shose Blvd, bl. 3
1113 Sofia, Bulgaria
e-mail: `avassilev@fmi.uni-sofia.bg`

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