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GLOBAL ASYMPTOTIC STABILITY OF A FUNCTIONAL DIFFERENTIAL MODEL WITH TIME DELAY OF AN ANAEROBIC BIODEGRADATION PROCESS

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Abstract. We study a nonlinear functional differential model of an anaerobic digestion process of wastewater treatment with biogas production. The model equations of biomass include two different discrete time delays. A mathematical analysis of the model is completed including existence and local stability of nontrivial equilibrium points, existence and boundedness of the model solutions as well as global stabilizability towards an admissible equilibrium point. We propose and apply a numerical extremum seeking algorithm for maximizing the biogas flow rate in real time. Numerical simulation results are also included.

1. Introduction. We consider a well-known anaerobic digestion model for biological treatment of wastewater in a continuously stirred tank bioreactor described by four nonlinear ordinary differential equations (cf. for example [2, 3]). We modify the model by introducing discrete time delays in the equations to represent the delay in the conversion of the consumed substrate to viable biomass,

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i. e., by including delays in the phase variables of substrate and organisms concentrations.

Chemostat models involving time delays have been recently studied by many authors as an attempt to explain observed delay in the growth response of microorganisms due to environmental changes, see for example [17], [18] and the references therein.

The studied model is described by the following system of non-linear ordinary differential equations

$$
\begin{aligned}
\frac{d}{dt}s_1(t) &= u(s_1^i - s_1(t)) - k_1\mu_1(s_1(t))x_1(t) \\
\frac{d}{dt}x_1(t) &= e^{-\alpha u\tau_1}\mu_1(s_1(t-\tau_1))x_1(t-\tau_1) - \alpha u x_1(t) \\
\frac{d}{dt}s_2(t) &= u(s_2^i - s_2(t)) + k_2\mu_1(s_1(t))x_1(t) - k_3\mu_2(s_2(t))x_2(t) \\
\frac{d}{dt}x_2(t) &= e^{-\alpha u\tau_2}\mu_2(s_2(t-\tau_2))x_2(t-\tau_2) - \alpha u x_2(t)
\end{aligned} \tag{1}
$$

with gaseous output

$$Q = k_4\,\mu_2(s_2)\,x_2. \tag{2}$$

The phase variables s_1, s_2 and x_1, x_2 denote substrate and biomass concentrations, respectively: s_1 is the organic substrate, characterized by its chemical oxygen demand (COD), s_2 denotes the volatile fatty acids (VFA), x_1 and x_2 are the acidogenic and methanogenic bacteria respectively; s_1^i and s_2^i are the input substrate concentrations corresponding to s_1 and s_2. The parameter $\alpha \in (0, 1)$ represents the proportion of bacteria that are affected by the dilution rate u. The constants k_1, k_2 and k_3 are yield coefficients related to COD degradation, VFA production and VFA consumption respectively. The constants $\tau_j \geq 0$, $j = 1, 2$, stand for the time delay in conversion of the corresponding substrate to viable biomass of the j-th bacterial population. The terms $e^{-\alpha u\tau_j}x_j(t-\tau_j)$, $j = 1, 2$, represent the biomass of those microorganisms that consume nutrient τ_j units of time prior to time t and that survive in the chemostat the τ_j units of time necessary to complete the process of converting the substrate to viable biomass at time t. The output Q describes the methane (biogas) flow rate, where k_4 as a yield coefficient.

The functions $\mu_1(s_1)$ and $\mu_2(s_2)$ model the specific growth rates of the bacteria x_1 and x_2 respectively. Following [13] we impose the following general assumption on μ_1 and μ_2:

Assumption A1: For each $j = 1, 2$ the function $\mu_j(s_j)$ is defined for $s_j \in [0, +\infty)$, $\mu_j(0) = 0$, and $\mu_j(s_j) > 0$ for each $s_j > 0$; the function $\mu_j(s_j)$ is bounded and Lipschitz continuous for all $s_j \in [0, +\infty)$.

The model (1) with $\tau_1 = \tau_2 = 0$ is one of the "bench-mark" models describing anaerobic digestion and extensively investigated in the literature in recent years, see for example [1], [2], [3], [10], [13] and the references therein. The equations (1) with $\tau_1 = \tau_2 = 0$ have been already investigated by the authors: global stabilizability analysis via different feedback control laws is presented in [6], [7], whereas [8] considers the case of global stabilization of the solutions using a constant dilution rate u. The latter approach is now extended to model (1) involving the discrete delays $\tau_j > 0$, $j = 1, 2$. The paper [4] is devoted to the same problem and contains similar results as well as a sketch of the proof for global stabilizability of the model. Here we present a detailed and more precise proof of the global asymptotic stability of the solutions. The new moment in this paper is the development of an extremum seeking algorithm and its application to optimizing the output of the biogas (methane) production.

The paper is organized as follows. Section 2 is devoted to studying the local asymptotic stability of admissible equilibrium points with respect to the time delays τ_1 and τ_2. In Section 3 we prove existence and boundedness of the model solutions. The global stability of the dynamics (1) is proved in Section 4. The model-based extremum seeking algorithm is described shortly in Section 5 and further applied in Section 6 to maximize the methane flow rate in real time. Numerical experiments are finally reported to confirm the theoretical results.

2. Equilibrium points and their local stability.

Define

$$u_b = \max\left\{u : \ u\alpha e^{\alpha u \tau_1} \le \mu_1(s_1^i), \ u\alpha e^{\alpha u \tau_2} \le \mu_2(s_2^i)\right\}$$

and assume that $u \in (0, u_b)$.

Let the following assumption be satisfied:

Assumption A2. For each point $\bar{u} \in (0, u_b)$ there exist points $s_1(\bar{u}) = \bar{s}_1 \in \left(0, s_1^i\right)$ and $s_2(\bar{u}) = \bar{s}_2 \in \left(0, s_2^i\right)$, such that the following equalities hold true

$$\bar{u} = \frac{1}{\alpha} e^{-\alpha \bar{u} \tau_1} \mu_1(\bar{s}_1) = \frac{1}{\alpha} e^{-\alpha \bar{u} \tau_2} \mu_2\left(\bar{s}_2\right).$$

In accordance with [10] we shall call the above equality regulability of the system. It means that there exists at least one nontrivial (positive) equilibrium of (1).

Let $\bar{s}_1$ and $\bar{s}_2$ be determined according to Assumption A2. Compute further

$$(3) \qquad x_1(\bar{u}) = \bar{x}_1 = \frac{s_1^i - \bar{s}_1}{\alpha k_1 e^{\alpha \bar{u} \tau_1}}, \quad x_2(\bar{u}) = \bar{x}_2 = \frac{s_2^i - \bar{s}_2 + \alpha k_2 \bar{x}_1 e^{\alpha \bar{u} \tau_1}}{\alpha k_3 e^{\alpha \bar{u} \tau_2}}.$$

Then the point

$$p(\bar{u}) = \bar{p} = (\bar{s}_1, \bar{x}_1, \bar{s}_2, \bar{x}_2)$$

is a nontrivial equilibrium point for system (1).

In what follows we shall also need the following assumption:

Assumption A3. There exist positive numbers ν_1 and ν_2 such that the following inequalities hold true

$$\mu_1(s_1^-) < \mu_1(\bar{s}_1) < \mu_1(s_1^+), \quad \mu_2(s_2^-) < \mu_2(\bar{s}_2) < \mu_2(s_2^+)$$

for each $s_1^- \in (0, \bar{s}_1)$, $s_1^+ \in (\bar{s}_1, s_1^i + \nu_1]$, $s_2^- \in (0, \bar{s}_2)$ and $s_2^+ \in (\bar{s}_2, s_2^i + \nu_2]$.

Assumption A3 is always fulfilled when the functions $\mu_j(\cdot)$, $j = 1, 2$, are monotone increasing (like the Monod specific growth rate, see Section 6). If at least one of the functions $\mu_j(\cdot)$ is not monotone increasing (like the Haldane law, see Section 6) then the points $\bar{s}_j$ (or equivalently $\bar{u}$) should be chosen sufficiently small in order to satisfy Assumption A3.

Proposition 1. *The equilibrium point $\bar{p}$ is locally asymptotically stable for all values of the delays $\tau_1 \geq 0$ and $\tau_2 \geq 0$.*

Proof. Denote for simplicity

$$a = k_1 \mu_1'(\bar{s}_1) \bar{x}_1, \quad b = k_3 \mu_2'(\bar{s}_2) \bar{x}_2,$$

where μ_1' and μ_2' mean $\frac{d}{ds_1}\mu_1$ and $\frac{d}{ds_2}\mu_2$ respectively. It follows from Assumption A3 that $a > 0$ and $b > 0$ hold true.

The characteristic equation of the system corresponding to the equilibrium point $\bar{p}$ has the form

$$\begin{aligned} 0 &= P(\lambda; \tau_1, \tau_2) \\ &= \begin{vmatrix} -(\bar{u} + a) - \lambda & -k_1 \alpha \bar{u} e^{\alpha \bar{u} \tau_1} \\ \mu_1'(\bar{s}_1) \bar{x}_1 e^{-(\lambda + \alpha \bar{u}) \tau_1} & \alpha \bar{u}(e^{-\lambda \tau_1} - 1) - \lambda \end{vmatrix} \\ &\times \begin{vmatrix} -(\bar{u} + b) - \lambda & -k_3 \alpha \bar{u} e^{\alpha \bar{u} \tau_2} \\ \mu_2'(\bar{s}_2) \bar{x}_2 e^{-(\lambda + \alpha \bar{u}) \tau_2} & \alpha \bar{u}(e^{-\lambda \tau_2} - 1) - \lambda \end{vmatrix} \end{aligned}$$

$$= P_1(\lambda;\tau_1) \times P_2(\lambda;\tau_2),$$

where λ is a complex number and

$$\begin{aligned} P_1(\lambda;\tau_1) &= \lambda^2 + (\bar{u} + a + \alpha\bar{u})\lambda + \alpha\bar{u}(\bar{u} + a) - \alpha\bar{u}(\bar{u} + \lambda)e^{-\lambda\tau_1}, \\ P_2(\lambda;\tau_2) &= \lambda^2 + (\bar{u} + b + \alpha\bar{u})\lambda + \alpha\bar{u}(\bar{u} + b) - \alpha\bar{u}(\bar{u} + \lambda)e^{-\lambda\tau_2}. \end{aligned}$$

First we shall show that if $\tau_1 = \tau_2 = 0$ then there exist no roots λ of $P(\lambda;\tau_1,\tau_2) = 0$ with $Re(\lambda) \geq 0$. The equation $P_1(\lambda;0) = 0$ is equivalent with $\lambda^2 + (\bar{u} + a)\lambda + \alpha\bar{u}a = 0$. Obviously, the latter quadratic equation has no roots λ with $Re(\lambda) \geq 0$. The same is true for $P_2(\lambda;0) = 0$.

Let $\tau_1 > 0$ and $\tau_2 > 0$. We shall show that there are no roots of $P(\lambda;\tau_1,\tau_2) = 0$ on the imaginary axis. Let $\lambda = i\omega$ with $\omega > 0$. For $P_1(i\omega;\tau_1) = 0$ we obtain consecutively:

$$\begin{aligned} &-\omega^2 + (\bar{u} + a + \alpha\bar{u})i\omega + \alpha\bar{u}(\bar{u} + a) - \alpha\bar{u}(\bar{u} + i\omega)e^{-i\omega\tau_1} = 0, \\ &-\omega^2 + (\bar{u} + a + \alpha\bar{u})i\omega + \alpha\bar{u}(\bar{u} + a) - \alpha\bar{u}(\bar{u} + i\omega)(\cos(\tau_1\omega) - i\sin(\tau_1\omega)) = 0. \end{aligned}$$

Separating the real and the imaginary parts of the last equation we obtain the system

$$\begin{aligned} &-\omega^2 + \alpha\bar{u}(\bar{u} + a) = \alpha\bar{u}^2\cos(\tau_1\omega) + \alpha\bar{u}\omega\sin(\omega\tau_1) \\ &(\bar{u} + a + \alpha\bar{u})\omega = -\alpha\bar{u}^2\sin(\tau_1\omega) + \alpha\bar{u}\omega\cos(\omega\tau_1). \end{aligned} \tag{4}$$

Squaring both sides of the equations (4) and adding them together leads to

$$\omega^4 + (\bar{u} + a)^2\omega^2 + \alpha^2\bar{u}^2 a(2\bar{u} + a) = 0.$$

With $v := \omega^2$ we obtain the quadratic equation $v^2 + (\bar{u} + a)^2 v + \alpha^2\bar{u}^2 a(2\bar{u} + a) = 0$ which does not possess positive real roots since $a > 0$ holds according to Assumption A3.

The same conclusion is valid for $P_2(i\omega;\tau_2) = 0$. Therefore, $P(\lambda;\tau_1,\tau_2) = 0$ does not possess purely imaginary roots for any $\tau_1 > 0$ and $\tau_2 > 0$.

Applying Lemma 2 from [14] (see also Theorem 3 and Corollary 4 from [14] as well as [15], [16] for similar results) to the exponential polynomial $P(\lambda;\tau_1,\tau_2)$ we obtain that the characteristic equation $P(\lambda;\tau_1,\tau_2) = 0$ does not have roots with nonnegative real parts. This means that for any $\tau_1 \geq 0$ and $\tau_2 \geq 0$ the positive equilibrium $\bar{p}$ is locally asymptotically stable. □

3. Existence and boundedness of the model solutions. Denote by R^+ the set of all non negative real numbers and by C_τ^+ the cone of continuous functions $\varphi : [-\tau, 0] \to R^+$, where $\tau = \max\{\tau_1, \tau_2\}$, and set

$$C_\tau^4 := \{\varphi = (\varphi_{s_1}, \varphi_{x_1}, \varphi_{s_2}, \varphi_{x_2}) \in C_\tau^+ \times C_\tau^+ \times C_\tau^+ \times C_\tau^+\}.$$

Let $\bar{u} \in (0,\, u_b)$ be chosen in such a way that Assumptions A2 and A3 are satisfied. Denote by Σ the system obtained from (1) by substituting the control variable u by $\bar{u}$. Using the Schauder fixed-point theorem it is easy to prove that for each $\varphi \in C_\tau^4$ there exists $\varrho > 0$ and a unique solution

$$\Phi(t, \varphi) = (s_1(t, \varphi_{s_1}), x_1(t, \varphi_{x_1}), s_2(t, \varphi_{s_2}), x_2(t, \varphi_{x_2}))$$

of (1) defined on $[-\tau, \varrho)$ such that $\Phi(t, \varphi) = \varphi(t)$ for each $t \in [-\tau, 0]$ (cf. Theorem 2.1 in Chapter 2 of [11]).

We fix an arbitrary $\varphi^0 \in C_\tau^4$ with $\varphi^0(0) > 0$. Then there exists $\varrho > 0$ such that the corresponding solution $\Phi(t, \varphi^0)$ of Σ is defined on $[-\tau, \varrho)$. Denote for simplicity $\Phi(t, \varphi^0) = \Phi(t) = (s_1(t), x_1(t), s_2(t), x_2(t))$.

Proposition 2. *The components of* $\Phi(t)$ *take positive values for each* $t \in [-\tau, \varrho)$.

P r o o f. If $s_1(t) = 0$ for some $t \in [0, \varrho)$, then $\dot{s}_1(t) > 0$. This implies that $s_1(t) > 0$ for each $t \in [-\tau, \varrho)$. Analogously one can obtain that $s_2(t) > 0$ for each $t \in [-\tau, \varrho)$. The presentation

$$x_j(t) = \varphi_{x_j}^0(0) e^{-\alpha \bar{u} t} + \int_0^t e^{-\alpha \bar{u}(t-\sigma)} \mu_j(s_j(\sigma - \tau_j)) x_j(\sigma - \tau_j) d\sigma,\ j = 1, 2,$$

implies that $x_j(t) > 0$ for each $t \in [-\tau, \varrho)$. This completes the proof. □

Proposition 3. *The solution* $\Phi(t)$ *of* Σ *is defined for each* $t \in [-\tau, +\infty)$ *and is bounded.*

P r o o f. Denote

(5) $$s(t) := e^{-\alpha \bar{u} \tau_1} \left(k_2 s_1(t) + k_1 s_2(t)\right) \quad \text{and} \quad s^i := e^{-\alpha \bar{u} \tau_1} \left(k_2 s_1^i + k_1 s_2^i\right).$$

Then $s(t)$ satisfies the differential equation

(6) $$\dot{s}(t) = \bar{u}(s^i - s(t)) - k_1 k_3 e^{-\alpha \bar{u} \tau_1} \mu_2(s_2(t)) x_2(t).$$

We set

$$q_1(t) := s(t) + k_1k_3e^{-\alpha\bar{u}(\tau_1-\tau_2)}x_2(t+\tau_2) - \frac{s^i}{\alpha}$$
$$q_2(t) := s(t) + k_1k_3e^{-\alpha\bar{u}(\tau_1-\tau_2)}x_2(t+\tau_2) - s^i.$$

Then

$$\begin{aligned}\dot{q}_1(t) &= \bar{u}[s^i - s(t) - \alpha k_1k_3e^{-\alpha\bar{u}(\tau_1-\tau_2)}x_2(t+\tau_2)] \\ &\leq \bar{u}[s^i - \alpha\left(s(t) + k_1k_3e^{-\alpha\bar{u}(\tau_1-\tau_2)}x_2(t+\tau_2)\right)] = -\alpha\bar{u}q_1(t),\end{aligned}$$

and hence

$$q_1(t) \leq q_1(0)\cdot e^{-\alpha\bar{u}t}. \tag{7}$$

The latter inequality shows that $q_1(t)$ is bounded. Using the fact that the values of $s_1(t)$, $s_2(t)$ and $x_2(t)$ are positive, it follows that $s_1(t)$, $s_2(t)$ and $x_2(t)$ are bounded as well. Analogously one can obtain that

$$q_2(t) \geq q_2(0)\cdot e^{-\bar{u}t}. \tag{8}$$

The estimates (7), (8) and the definition of $s(\cdot)$ imply that for each $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that for each $t \geq T_\varepsilon$ the following inequalities hold true

$$s^i - \varepsilon < s(t) + k_1k_3e^{-\alpha\bar{u}(\tau_1-\tau_2)}x_2(t+\tau_2) < \frac{s^i}{\alpha} + \varepsilon. \tag{9}$$

In the same way as the estimates (9) were obtained one can show that for each $\varepsilon > 0$ there exists a finite time $T_\varepsilon > 0$ such that for all $t \geq T_\varepsilon$ the following inequalities hold

$$s_1^i - \varepsilon < s_1(t) + k_1e^{\alpha\bar{u}\tau_1}x_1(t+\tau_1) < \frac{s_1^i}{\alpha} + \varepsilon. \tag{10}$$

The inequalities (10) imply that $x_1(t)$ is also bounded. Thus the trajectory $\Phi(t)$ of Σ is well defined and bounded for all $t \geq -\tau$ (cf. also Theorem 3.1 in Chapter 2 of [11]). This completes the proof. □

4. Global stability of the model solutions. The main result of the paper is presented in the next Theorem 1, which states that the equilibrium point $\bar{p}$ is globally asymptotically stable for system Σ for all values of the delays $\tau_1 \geq 0$ and $\tau_2 \geq 0$.

Theorem 1. *Let Assumptions A1, A2 and A3 be fulfilled and let φ^0 be an arbitrary element of C^4_τ with $\varphi^0(0) > 0$. Then the corresponding solution $\Phi(t, \varphi^0)$ converges asymptotically towards $\bar{p}$.*

The proof of Theorem 1 is based on Propositions 1, 2 and 3 as well as on the following Lemmas.

Barbălat's Lemma (cf. [9]). If $f : (0, \infty) \to R$ is Riemann integrable and uniformly continuous, then $\lim_{t\to\infty} f(t) = 0$.

Fluctuation Lemma (cf. [12]). Let $f : [0, +\infty) \to R$ be a differentiable function. If $\liminf_{t\to\infty} f(t) < \limsup_{t\to\infty} f(t)$ then there exist sequences $\{t_m\} \uparrow \infty$ and $\{s_m\} \uparrow \infty$ such that for all m the following relations hold true:

$$f(t_m) \to \limsup_{t\to\infty} f(t) \text{ as } m \uparrow \infty \text{ and } f'(t_m) = 0,$$
$$f(s_m) \to \liminf_{t\to\infty} f(t) \text{ as } m \uparrow \infty \text{ and } f'(s_m) = 0.$$

Lemma 1. *There exists $T_0 > 0$ such that $s_1(t) < s_1^i$ and $s_2(t) < s_2^i + \dfrac{k_2}{k_1} s_1^i$ for each $t \geq T_0$.*

Proof. Let us assume that there exists $\bar{t} > 0$ such that $s_1(t) \geq s_1^i$ for all $t \geq \bar{t}$. Then

$$\dot{s}_1(t) = \bar{u}(s_1^i - s_1(t)) - k_1\mu_1(s_1(t))x_1(t) < 0,$$

and hence $s_1(\cdot)$ is a strictly decreasing function. It follows from Proposition 3 that $s_1(\cdot)$ and $x_1(\cdot)$ are bounded differentiable functions defined on $[-\tau, +\infty)$, thus $\dot{s}_1(\cdot)$ is a uniformly continuous function. Applying Barbălat's Lemma we obtain

$$0 = \lim_{t\to\infty} \dot{s}_1(t) = \lim_{t\to\infty} [\bar{u}(s_1^i - s_1(t)) - k_1\mu_1(s_1(t))x_1(t)].$$

Because $s_1^i - s_1(t) \leq 0$ and $x_1(t) > 0$, the above equalities imply that $s_1(t) \downarrow s_1^i$ and $x_1(t) \downarrow 0$ as $t \uparrow \infty$. Define (cf. Lemma 2.2 in [18])

$$z_1(t) := x_1(t) + \int_{t-\tau_1}^{t} e^{-\alpha\bar{u}\tau_1} \mu_1(s_1(\sigma))x_1(\sigma)d\sigma.$$

It follows then from Assumption A3 (because $s_1^i + \nu_1 > s_1(t) \geq s_1^i > \bar{s}_1$ for sufficiently large t) that

$$\dot{z}_1(t) = x_1(t)\left(e^{-\alpha\bar{u}\tau_1}\mu_1(s_1(t)) - \alpha\bar{u}\right) = x_1(t)\left(e^{-\alpha\bar{u}\tau_1}\mu_1(s_1(t)) - e^{-\alpha\bar{u}\tau_1}\mu_1(\bar{s}_1)\right)$$

$$= \quad x_1(t)e^{-\alpha\bar{u}\tau_1}(\mu_1(s_1(t)) - \mu_1(\bar{s}_1)) > 0 \text{ for all sufficiently large } t \geq \bar{t},$$

and so there exists $z_1^* > 0$ such that $z_1(t) \uparrow z_1^*$ as $t \uparrow \infty$. But this is impossible according to the definition of $z_1(\cdot)$ and because we have already shown that $x_1(t) \downarrow 0$ as $t \uparrow \infty$.

Hence, there exists a sufficiently large $T_0 > 0$ with $s_1(T_0) \leq s_1^i$. Moreover, if the equality $s_1(\bar{t}) = s_1^i$ holds true for some $\bar{t} \geq T_0$, then we have

$$\dot{s}_1(\bar{t}) = \bar{u}(s_1^i - s_1(\bar{t})) - k_1\mu_1(s_1(\bar{t}))x_1(\bar{t}) = -k_1\mu_1(s_1(\bar{t}))x_1(\bar{t}) < 0.$$

The last inequality shows that $s_1(t) < s_1^i$ for each $t \geq T_0$.

Further with $s(t)$ and s^i from (5), and $\dot{s}(t)$ from (6) one can show in the same way as above that $s(t) < s^i$ for each $t \geq T_0$ (if necessary T_0 can be enlarged), i. e.,

$$e^{-\alpha\bar{u}\tau_1}\left(k_2 s_1(t) + k_1 s_2(t)\right) \leq e^{-\alpha\bar{u}\tau_1}\left(k_2 s_1^i + k_1 s_2^i\right).$$

Since $0 < s_1(t) < s_1^i$, it follows that $s_2(t) < s_2^i + \dfrac{k_2}{k_1}s_1^i$ is fulfilled. The proof of Lemma 1 is completed. $\square$

Lemma 2. *Denote*

$$\begin{aligned}
&\gamma_1 := \limsup_{t\uparrow\infty} x_1(t), \quad \delta_1 := \liminf_{t\uparrow\infty} x_1(t),\\
&v_1(t) := e^{-\alpha\bar{u}\tau_1}s_1(t) + k_1x_1(t+\tau_1),\\
&\alpha_1 := \limsup_{t\uparrow\infty} v_1(t), \quad \beta_1 := \liminf_{t\uparrow\infty} v_1(t).
\end{aligned}$$

Then the following relations hold true: $\gamma_1 = \delta_1 > 0$ *and* $\alpha_1 = \beta_1$.

Proof. Let us assume that $\delta_1 = 0$. Choose an arbitrary

$$\varepsilon \in (0, (s_1^i - \bar{s}_1)/(1 + k_1e^{\alpha\bar{u}\tau_1})).$$

According to Proposition 3 (see (10)) there exists $T_\varepsilon > 0$ such that for all $t \geq T_\varepsilon$ the following inequalities hold true

$$s_1^i - \varepsilon < s_1(t-\tau_1) + k_1e^{\alpha\bar{u}\tau_1}x_1(t) < \frac{s_1^i}{\alpha} + \varepsilon. \tag{11}$$

Since $\delta_1 = 0$ there exists $t_0 > \max(T_\varepsilon, T_0)$ such that $x_1(t_0) < \varepsilon$. We set (cf. Lemma 3.5 in [18])

$$\begin{aligned}
&\sigma := \min\{x_1(t) : \ t \in [t_0 - \tau_1, t_0]\}\\
&\bar{t} := \sup\{t \geq t_0 - \tau_1 : \ x_1(\tau) \geq \sigma \text{ for all } \tau \in [t_0 - \tau_1, t]\}.
\end{aligned}$$

Clearly $\sigma \in (0, \varepsilon]$, $\bar{t} \in [t_0, +\infty)$, $x_1(t) \geq \sigma$ for all $t \in [t_0 - \tau_1, \bar{t}]$ and

$$x_1(\bar{t}) = \sigma \text{ and } \dot{x}_1(\bar{t}) \leq 0. \tag{12}$$

Taking into account Assumption A3, (11) and the choice of ε, we obtain consecutively

$$s_1^i > s_1(\bar{t} - \tau_1) \geq s_1^i - k_1 e^{\alpha \bar{u} \tau_1} x_1(\bar{t}) - \varepsilon \geq s_1^i - (1 + e^{\alpha \bar{u} \tau_1} k_1)\varepsilon > \bar{s}_1,$$
$$\dot{x}_1(\bar{t}) = e^{-\alpha \bar{u} \tau_1} \mu_1(s_1(\bar{t} - \tau_1)) x_1(\bar{t} - \tau_1) - \alpha \bar{u} x_1(\bar{t}) > \alpha \bar{u} \sigma - \alpha \bar{u} \sigma = 0.$$

The last inequality contradicts (12) which means that $\delta_1 > 0$.

Let us assume that $\gamma = \gamma_1 = \delta_1 > 0$, i. e., the limit of $x_1(t)$ exists as $t \to \infty$. We shall show that $\alpha_1 = \beta_1$ holds true. Applying Barbălat's Lemma, we obtain that $\lim_{t \to \infty} \dot{x}_1(t) = 0$, i. e.,

$$e^{-\alpha \bar{u} \tau_1} \mu_1(s_1(t - \tau_1)) x_1(t - \tau_1) - \alpha \bar{u} x_1(t) \to 0 \text{ as } t \to \infty.$$

From here it follows that $\mu_1(s_1(t)) \to \alpha \bar{u}$ as $t \to \infty$. Applying Assumption A3 leads to $s_1(t) \to \bar{s}_1$ as $t \to \infty$. But then

$$v_1(t) = e^{-\alpha \bar{u} \tau_1} s_1(t) + k_1 x(t + \tau_1) \to e^{-\alpha \bar{u} \tau_1} \bar{s}_1 + k_1 \gamma \quad \text{for} \quad t \to \infty.$$

Hence $\alpha_1 = \beta_1$ in this case.

Assume now that $\alpha_1 = \beta_1$, i. e., the limit of $v_1(t)$ exists as $t \to \infty$. We shall show that $\gamma_1 = \delta_1$ is fulfilled. Applying again Barbălat's lemma, we obtain that $\lim_{t \to \infty} \dot{v}_1(t) = 0$, i. e.,

$$\dot{v}_1(t) = \bar{u} e^{-\alpha \bar{u} \tau_1} s_1^i - \bar{u} v_1(t) + (1 - \alpha) k_1 \bar{u} x_1(t + \tau_1) \to 0 \text{ as } t \to \infty.$$

From here it follows that there exists the limit of $x_1(t)$ as $t \to \infty$ and we can continue as in the previous case to obtain that $\gamma_1 = \delta_1$.

Assume that $\alpha_1 > \beta_1$ and $\gamma_1 > \delta_1$ hold true. We study this case using some ideas from the proofs of Lemma 4.3 of [18] and Theorem 3.1 of [17].

Let $\varepsilon > 0$ be an arbitrary fixed number. Applying the Fluctuation Lemma, there exists a sequence $\{t_m\}_{m=1}^{\infty} \to \infty$ such that for each m we have

$$\lim_{m \uparrow \infty} x_1(t_m) = \gamma_1, \;\; \dot{x}_1(t_m) = 0 \text{ and } x_1(t_m - \tau_1) \leq \gamma_1 + \varepsilon.$$

The equality $\dot{x}_1(t_m) = 0$ implies

$$e^{-\alpha\bar{u}\tau_1}\mu_1(s_1(t_m-\tau_1)) = e^{-\alpha\bar{u}\tau_1}\mu_1\left(\frac{v_1(t_m-\tau_1)-k_1x_1(t_m)}{e^{-\alpha\bar{u}\tau_1}}\right)$$
$$= \frac{\alpha\bar{u}x_1(t_m)}{x_1(t_m-\tau_1)} \geq \frac{\alpha\bar{u}x_1(t_m)}{\gamma_1+\varepsilon},$$
$$\liminf_{m\uparrow\infty} e^{-\alpha\bar{u}\tau_1}\mu_1\left(\frac{v_1(t_m-\tau_1)-k_1x_1(t_m)}{e^{-\alpha\bar{u}\tau_1}}\right) \geq \frac{\alpha\bar{u}\gamma_1}{\gamma_1+\varepsilon},$$

and since $\varepsilon > 0$ can be arbitrarily small, it follows that

$$\liminf_{m\uparrow\infty} e^{-\alpha\bar{u}\tau_1}\mu_1\left(\frac{v_1(t_m-\tau_1)-k_1x_1(t_m)}{e^{-\alpha\bar{u}\tau_1}}\right) \geq \alpha\bar{u}.$$

Using Assumption A2 this inequality implies that

$$\liminf_{m\uparrow\infty} \frac{v_1(t_m-\tau_1)-k_1x_1(t_m)}{e^{-\alpha\bar{u}\tau_1}} \in [\bar{s}_1, s_1^i),$$

and hence

$$\alpha_1 \geq e^{-\alpha\bar{u}\tau_1}\bar{s}_1 + k_1\gamma_1. \tag{13}$$

Similarly one can show that $\beta_1 \leq e^{-\alpha\bar{u}\tau_1}\bar{s}_1 + k_1\delta_1$. This and (13) imply

$$\alpha_1 - \beta_1 \geq k_1(\gamma_1 - \delta_1) \geq 0. \tag{14}$$

Let $\varepsilon > 0$ be an arbitrary fixed number. Applying the Fluctuation Lemma, there exists a sequence $\{t_k\}_{k=1}^{\infty} \to \infty$ such that for each k we have

$$\lim_{k\uparrow\infty} v_1(t_k) = \alpha_1,\ \dot{v}_1(t_k) = 0 \text{ and } x_1(t_k+\tau_1) \leq \gamma_1 + \varepsilon. \tag{15}$$

The equality $\dot{v}_1(t_k) = 0$ implies that

$$0 = \dot{v}_1(t_k) = \bar{u}e^{-\alpha\bar{u}\tau_1}s_1^i - \bar{u}v_1(t_k) + (1-\alpha)k_1\bar{u}x_1(t_k+\tau_1).$$

From here and from (15) we obtain

$$\bar{u}e^{-\alpha\bar{u}\tau_1}s_1^i - \bar{u}v_1(t_k) + (1-\alpha)k_1\bar{u}(\gamma_1+\varepsilon) \geq 0,$$
$$e^{-\alpha\bar{u}\tau_1}s_1^i \geq \alpha_1 - (1-\alpha)k_1(\gamma_1+\varepsilon).$$

Because $\varepsilon > 0$ can be arbitrarily small, it follows that

$$e^{-\alpha\bar{u}\tau_1}s_1^i \geq \alpha_1 - (1-\alpha)k_1\gamma_1.$$

In the same way one can show that $e^{-\alpha\bar{u}\tau_1}s_1^i \le \beta_1 - (1-\alpha)k_1\delta_1$, and thus $\alpha_1 - \beta_1 \le (1-\alpha)k_1(\gamma_1 - \delta_1)$. Using (14), we obtain the inequalities

$$\alpha_1 - \beta_1 \le (1-\alpha)k_1(\gamma_1 - \delta_1) \le (1-\alpha)(\alpha_1 - \beta_1),$$

which are impossible because $\alpha \in (0,1)$. The contradiction shows that $\alpha_1 = \beta_1$. Using again (14), we obtain that $\gamma_1 = \delta_1$. This completes the proof. $\square$

Lemma 3. *Denote*

$$\begin{aligned} &\gamma_2 := \limsup_{t\uparrow\infty} x_2(t), \quad \delta_2 := \liminf_{t\uparrow\infty} x_2(t),\\ &v_2(t) := s(t) + k_1k_3e^{-\alpha\bar{u}(\tau_1-\tau_2)}x_2(t+\tau_2),\\ &\alpha_2 := \limsup_{t\uparrow\infty} v_2(t), \quad \beta_2 := \liminf_{t\uparrow\infty} v_2(t). \end{aligned}$$

Then the following relations hold true: $\alpha_2 = \beta_2$ *and* $\gamma_2 = \delta_2$.

P r o o f. The proof of Lemma 3 is similar to the proof of the previous Lemma 2. We consider only the case when $\alpha_2 > \beta_2$ and $\gamma_2 > \delta_2$. Using (5) and (6) it is straightforward to see that

$$\begin{aligned} \dot{v}_2(t) &= \dot{s}(t) + k_1k_3e^{-\alpha\bar{u}(\tau_1-\tau_2)}x_2(t+\tau_2)\\ &= \bar{u}(s^i - s(t)) - k_1k_3\alpha\bar{u}e^{-\alpha\bar{u}(\tau_1-\tau_2)}x_2(t+\tau_2)\\ &= \bar{u}\left(e^{-\alpha\bar{u}\tau_1}s^i - v_2(t) + (1-\alpha)k_1k_3x_2e^{-\alpha\bar{u}(\tau_1-\tau_2)}(t+\tau_2)\right). \end{aligned}$$

Let $\varepsilon > 0$ be an arbitrary fixed number. Applying the Fluctuation Lemma, there exists a sequence $\{t_m\}_{m=1}^{\infty}$ tending to ∞ such that for each m we have

$$\lim_{m\uparrow\infty} x_2(t_m) = \gamma_2, \quad \dot{x}_2(t_m) = 0 \ \text{ and } \ x_2(t_m - \tau_2) \le \gamma_2 + \varepsilon.$$

The equality $\dot{x}_2(t_m) = 0$ implies

$$\begin{aligned} &e^{-\alpha\bar{u}\tau_2}\mu_2(s_2(t_m-\tau_2))\\ &\quad = e^{-\alpha\bar{u}\tau_2}\mu_2\left(\frac{v_2(t_m-\tau_2) - k_2e^{-\alpha\bar{u}\tau_1}s_1(t_m) - k_1k_3e^{-\alpha\bar{u}(\tau_1-\tau_2)}x_2(t_m)}{k_1e^{-\alpha\bar{u}\tau_1}}\right)\\ &\quad = \frac{\alpha\bar{u}x_2(t_m)}{x_2(t_m-\tau_2)} \ge \frac{\alpha\bar{u}x_2(t_m)}{\gamma_2+\varepsilon}. \end{aligned}$$

From here we obtain

$$\liminf_{m\uparrow\infty} e^{-\alpha\bar{u}\tau_2}\mu_2\left(\frac{v_2(t_m-\tau_2)-k_2e^{-\alpha\bar{u}\tau_1}s_1(t_m)-k_1k_3e^{-\alpha\bar{u}(\tau_1-\tau_2)}x_2(t_m)}{k_1e^{-\alpha\bar{u}\tau_1}}\right)$$
$$\geq \frac{\alpha\bar{u}\gamma_2}{\gamma_2+\varepsilon},$$

and since $\varepsilon>0$ can be arbitrarily small, it follows that

$$\liminf_{m\uparrow\infty} e^{-\alpha\bar{u}\tau_2}\mu_2\left(\frac{v_2(t_m-\tau_2)-k_2e^{-\alpha\bar{u}\tau_1}s_1(t_m)-k_1k_3e^{-\alpha\bar{u}(\tau_1-\tau_2)}x_2(t_m)}{k_1e^{-\alpha\bar{u}\tau_1}}\right)\geq\alpha\bar{u}.$$

This inequality implies

$$\liminf_{m\uparrow\infty}\frac{v_2(t_m-\tau_2)-k_2e^{-\alpha\bar{u}\tau_1}s_1(t_m)-k_1k_3e^{-\alpha\bar{u}(\tau_1-\tau_2)}x_2(t_m)}{k_1e^{-\alpha\bar{u}\tau_1}}\geq\bar{s}_2,$$

and hence

$$\alpha_2\geq e^{-\alpha\bar{u}\tau_1}\left(k_2\tilde{s}_1+k_1\bar{s}_2\right)+k_1k_3e^{-\alpha\bar{u}(\tau_1-\tau_2)}\gamma_2,$$

where $\tilde{s}_1=\lim_{t\to\infty}s_1(t)$ according to Lemma 2. In the same way one can obtain that

$$\beta_2\leq e^{-\alpha\bar{u}\tau_1}\left(k_2\tilde{s}_1+k_1\bar{s}_2\right)+k_1k_3e^{-\alpha\bar{u}(\tau_1-\tau_2)}\delta_2.$$

The last two inequalities imply

$$\alpha_2-\beta_2\geq k_1k_3e^{-\alpha\bar{u}(\tau_1-\tau_2)}(\gamma_2-\delta_2). \tag{16}$$

Let $\varepsilon>0$ be an arbitrary fixed number. Applying the Fluctuation Lemma, one can conclude that there exists a sequence $\{t_k\}_{k=1}^{\infty}$ tending to ∞ such that for each k we have

$$\lim_{k\uparrow\infty}v_2(t_k)=\alpha_2,\ \ \dot{v}_2(t_k)=0 \text{ and } x_2(t_k+\tau_2)\leq\gamma_2+\varepsilon.$$

The equality $\dot{v}_2(t_k)=0$ leads to

$$0=\dot{v}_2(t_k)=\bar{u}e^{-\alpha\bar{u}\tau_1}s^i-\bar{u}v_2(t_k)+(1-\alpha)\bar{u}k_1k_3e^{-\alpha\bar{u}(\tau_1-\tau_2)}x_2(t_k+\tau_2).$$

From here we obtain that

$$\bar{u}e^{-\alpha\bar{u}\tau_1}s^i-\bar{u}v_2(t)+(1-\alpha)\bar{u}k_1k_3e^{-\alpha\bar{u}(\tau_1-\tau_2)}(\gamma_2+\varepsilon)\geq 0,$$

and hence

$$e^{-\alpha\bar{u}\tau_1} s^i \geq \alpha_2 - (1-\alpha)k_1 k_3 e^{-\alpha\bar{u}(\tau_1-\tau_2)}(\gamma_2 + \varepsilon).$$

Since $\varepsilon > 0$ can be arbitrarily small, it follows that

$$e^{-\alpha\bar{u}\tau_1} s^i \geq \alpha_2 - (1-\alpha)k_1 k_3 e^{-\alpha\bar{u}(\tau_1-\tau_2)}\gamma_2.$$

In the same way one can obtain that

$$e^{-\alpha\bar{u}\tau_1} s^i \leq \beta_2 - (1-\alpha)k_1 k_3 e^{-\alpha\bar{u}(\tau_1-\tau_2)}\delta_2,$$

and hence

$$\alpha_2 - \beta_2 \leq (1-\alpha)k_1 k_3 e^{-\alpha\bar{u}(\tau_1-\tau_2)}(\gamma_2 - \delta_2).$$

From here and (16) we get the inequalities

$$\alpha_2 - \beta_2 \leq (1-\alpha)k_1 k_3 e^{-\alpha\bar{u}(\tau_1-\tau_2)}(\gamma_2 - \delta_2) \leq (1-\alpha)(\alpha_2 - \beta_2),$$

which are impossible because $\alpha \in (0,1)$. The contradiction shows that $\alpha_2 = \beta_2$. Using again (16) it follows that $\delta_2 = \gamma_2$. This completes the proof. □

Proof of Theorem 1. Lemmas 2 and 3 imply that the solution $\Phi(t,\varphi^0) = \Phi(t) = (s_1(t), x_1(t), s_2(t), x_2(t))$ is convergent as $t \uparrow \infty$. Let

$$\lim_{t\uparrow\infty} s_1(t) = \tilde{s}_1,\ \lim_{t\uparrow\infty} x_1(t) = \tilde{x}_1,\ \lim_{t\uparrow\infty} s_2(t) = \tilde{s}_2,\ \lim_{t\uparrow\infty} x_2(t) = \tilde{x}_2.$$

Applying Barbălat's Lemma, we obtain from Σ that

$$\begin{aligned}
0 = \lim_{t\to\infty} \frac{d}{dt} s_1(t) &= \lim_{t\to\infty} \left(\bar{u}(s_1^i - s_1(t)) - k_1\mu_1(s_1(t))x_1(t)\right)\\
0 = \lim_{t\to\infty} \frac{d}{dt} x_1(t) &= \lim_{t\to\infty} \left(e^{-\alpha\bar{u}\tau_1}\mu_1(s_1(t-\tau_1))x_1(t-\tau_1) - \alpha\bar{u}x_1(t)\right)\\
0 = \lim_{t\to\infty} \frac{d}{dt} s_2(t) &= \lim_{t\to\infty} \left(\bar{u}(s_2^i - s_2(t)) + k_2\mu_1(s_1(t))x_1(t) - k_3\mu_2(s_2(t))x_2(t)\right)\\
0 = \lim_{t\to\infty} \frac{d}{dt} x_2(t) &= \lim_{t\to\infty} \left(e^{-\alpha\bar{u}\tau_2}\mu_2(s_2(t-\tau_2))x_2(t-\tau_2) - \alpha\bar{u}x_2(t)\right),
\end{aligned}$$

and hence

$$\begin{aligned}
0 &= \left(\bar{u}(s_1^i - \tilde{s}_1) - k_1\mu_1(\tilde{s}_1)\tilde{x}_1\right)\\
0 &= \left(e^{-2\alpha\bar{u}\tau_1}\mu_1(\tilde{s}_1)\tilde{x}_1 - \alpha\bar{u}\tilde{x}_1\right)\\
0 &= \left(\bar{u}(s_2^i - \tilde{s}_2) + k_2\mu_1(s_1(t))\tilde{x}_1 - k_3\mu_2(\tilde{s}_2)\tilde{x}_2\right)\\
0 &= \left(e^{-2\alpha\bar{u}\tau_2}\mu_2(\tilde{s}_2)\tilde{x}_2 - \alpha\bar{u}\tilde{x}_2\right).
\end{aligned}$$

From here it follows that

$$\bar{p} = (\bar{s}_1, \bar{x}_1, \bar{s}_2, \bar{x}_2) = (\tilde{s}_1, \tilde{x}_1, \tilde{s}_2, \tilde{x}_2).$$

Finally, the attractivity of the positive equilibrium $\bar{p}$ together with the local stability of $\bar{p}$, means that $\bar{p}$ is globally asymptotically stable for all values of $\tau_1 \geq 0$ and $\tau_2 \geq 0$. The proof of Theorem 1 is completed. □

5. Maximizing the biogas production via extremum seeking. Consider the model (1) with the output (2). Let Assumptions A1, A2 and A3 hold true for the dynamics (1). Denote by $u \in (0, u_b)$ some value of the dilution rate and consider the equilibrium point $p(u) = (s_1(u), x_1(u), s_2(u), x_2(u))$ where $s_1(u)$, $x_1(u)$, $s_2(u)$ and $x_2(u)$ are computed according to Assumption A2 and to (3). Denote further by

$$Q(u) = Q(p(u)) = k_4\, \mu_2(s_2(u))\, x_2(u)$$

the output, computed on the set of all steady states $p(u)$, parameterized on the input u. $Q(u)$ is called input–output static characteristic of the model.

Assumption A4. The function $u \mapsto Q(u)$, $u \in (0, u_b)$, is strictly unimodal, i. e., there exists a unique point $u_{\max} \in (0, u_b)$ where $Q(u)$ takes a maximum, $Q_{\max} = Q(u_{\max}) = Q(p(u_{\max}))$, the function strictly increases in the interval $(0, u_{\max})$ and strictly decreases in $(u_{\max}, u_b)$.

Denote further

$$p(u_{\max}) = (s_1(u_{\max}), x_1(u_{\max}), s_2(u_{\max}), x_2(u_{\max})) = (s_1^{\max}, x_1^{\max}, s_2^{\max}, x_2^{\max}).$$

Our goal is to stabilize the system (1) towards the (unknown) equilibrium point $p(u_{\max})$ and therefore to the maximum methane flow rate $Q_{\max}$. This is realized by applying a numerical model-based extremum seeking algorithm (ESA). The algorithm is presented in details in [5] for the model (1) within $\tau_1 = \tau_2 = 0$ and used in [6], [7] and [8]. The ESA is now adapted to system (1) with discrete delays $\tau_1 > 0$, $\tau_2 > 0$ and in accordance with the requirements of Theorem 1.

The main idea of the algorithm is the following: we construct in a proper way a sequence of points $u^{(1)}, u^{(2)}, \ldots, u^{(n)}, \ldots$ from the interval $(0, u_b)$ which approaches $u_{\max}$; Theorem 1 guarantees that the dynamics is globally asymptotically stabilizable for each $u = u^{(j)}$, i. e., the solution approaches the equilibrium $p(u^{(j)})$, $j = 1, 2, \ldots$. Then by computing and comparing the values $Q(p(u^{(1)})), Q(p(u^{(2)})), \ldots, Q(p(u^{(n)})), \ldots$, the desired equilibrium point $p(u_{\max})$ and thus $Q_{\max}$ are achieved.

In the computer implementation the algorithm is carried out in two stages. In the first stage, "rough" intervals $[U]$ and $[Q]$ are found which enclose $u_{\max}$ and $Q_{\max}$ respectively; in the second stage, the interval $[U]$ is refined using an elimination procedure based on the golden mean value strategy. The second stage produces the final intervals $[u_{\max}] = [u^-_{\max}, u^+_{\max}]$ and $[Q_{\max}]$ such that $u_{\max} \in [u_{\max}]$, $Q_{\max} \in [Q_{\max}]$ and $u^+_{\max} - u^-_{\max} \le \epsilon$, where the tolerance $\epsilon > 0$ is specified by the user.

ESA is implemented in a web-based software environment, see [19].

6. Computer simulation. The following specific growth rate functions are considered in the model (1), taken from [1], [2] and [3]:

$$\mu_1(s_1) = \frac{m_1 s_1}{k_{s_1} + s_1} \text{ (Monod law)}, \quad \mu_2(s_2) = \frac{m_2 s_2}{k_{s_2} + s_2 + (s_2/k_I)^2} \text{ (Haldane law)}.$$

In the simulation process we use the following numerical values for the model coefficients, which are obtained by laboratory experiments and given in [1]:

$k_1 = 10.53$	$k_2 = 28.6$	$k_3 = 1074$	$k_4 = 675$	$s_1^i = 7.5$	$s_2^i = 75$
$m_1 = 1.2$	$k_{s_1} = 7.1$	$m_2 = 0.74$	$k_{s_2} = 9.28$	$k_I = 16$	$\alpha = 0.5$

To demonstrate the theoretical results and the facilities of ESA we consider two numerical examples corresponding to different values of the delays. Let the initial conditions be

$$\varphi_{s_1}(t) = 2, \ \varphi_{x_1}(t) = 0.1, \ \varphi_{s_2}(t) = 10, \ \varphi_{x_2}(t) = 0.05.$$

Example 1. $\tau_1 = 2$, $\tau_2 = 7$.

For these values of the delays the input–output static characteristic $u \mapsto Q(u)$, $u \in (0, u_b)$, satisfies Assumption A4, see Figure 1 (left).

ESA produces the following numerical values:

$$u_{\max} \approx 0.299019, \quad Q_{\max} \approx 14.646,$$
$$s_1^{\max} \approx 1.434, \ x_1^{\max} \approx 0.854, \ s_2^{\max} \approx 13.546, \ x_2^{\max} \approx 0.051.$$

Figure 2 (left) and Figure 3 visualize the numerical outputs resulting from ESA.

Example 2. $\tau_1 = 5$, $\tau_2 = 3$

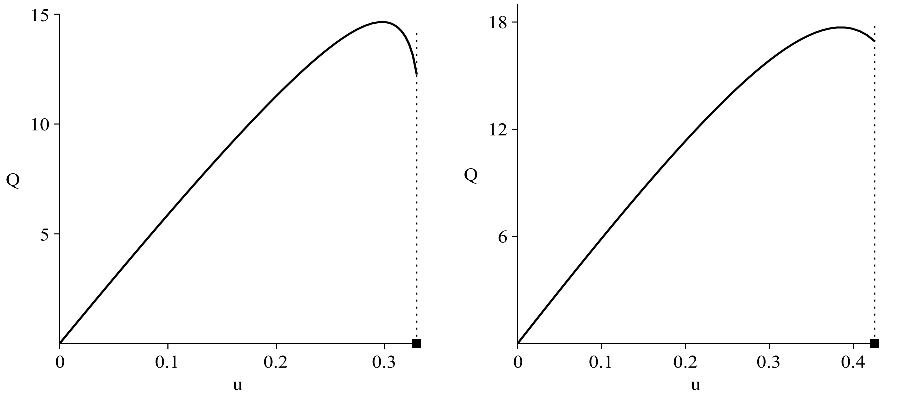

Fig. 1. The input–output static characteristic $Q(u)$ with $\tau_1 = 2$, $\tau_2 = 7$ (left), and $\tau_1 = 5$, $\tau_2 = 3$ (right). The vertical dot lines pass through u_b

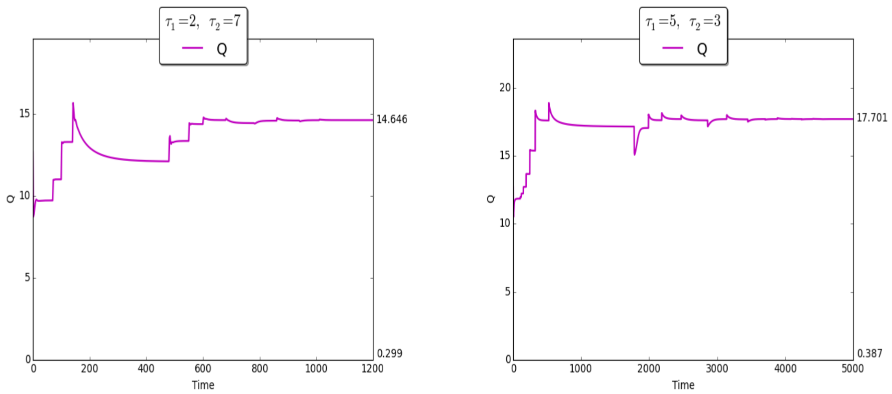

Fig. 2. Time evolution of $Q(t)$ towards $Q_{\max}$ corresponding to Example 1 (left) and Example 2 (right)

For these values of the delays the input–output static characteristic $u \mapsto Q(u)$, $u \in (0, u_b)$, satisfies Assumption A4, see Figure 1 (right).

In this case we obtain the following numerical values:

$$u_{\max} \approx 0.386966, \quad Q_{\max} \approx 17.701,$$
$$s_1^{\max} \approx 5.231, \; x_1^{\max} \approx 0.164, \; s_2^{\max} \approx 8.378, \; x_2^{\max} \approx 0.076.$$

The numerical outputs resulting from ESA are visualized in Figure 2 (right) and Figure 4.

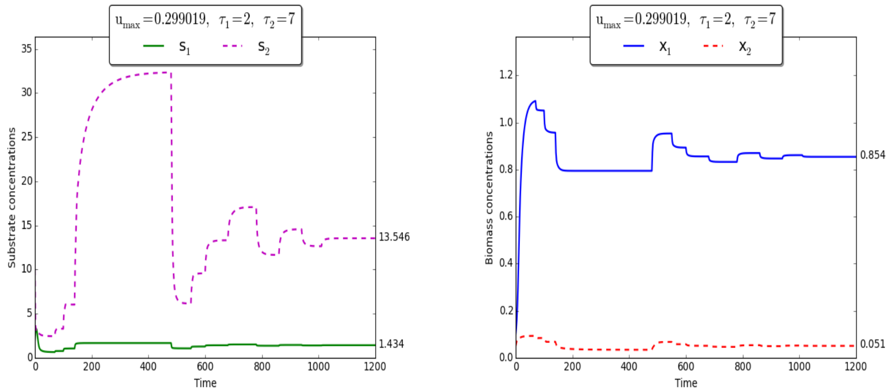

Fig. 3. Example 1. Time evolution of $s_1(t)$, $s_2(t)$ (left), $x_1(t)$ and $x_2(t)$ (right) towards $s_1^{\max}$, $s_2^{\max}$, $x_1^{\max}$ and $x_2^{\max}$ respectively

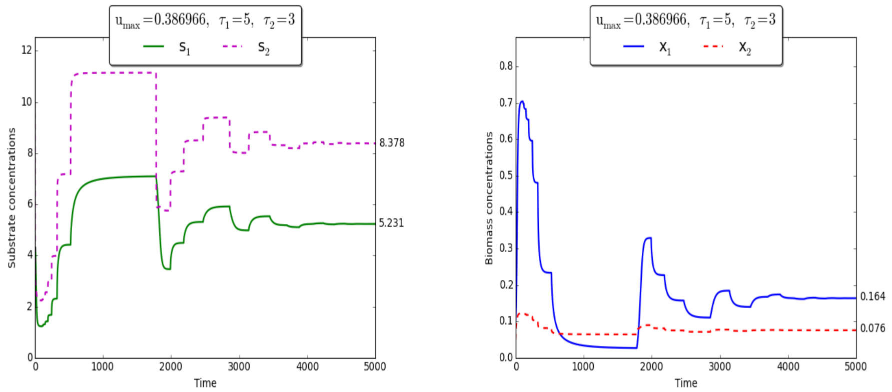

Fig. 4. Example 2. Time evolution of $s_1(t)$, $s_2(t)$ (left), $x_1(t)$ and $x_2(t)$ (right) towards $s_1^{\max}$, $s_2^{\max}$, $x_1^{\max}$ and $x_2^{\max}$ respectively

7. Conclusion. In this paper we investigate a bioreactor model for wastewater treatment by anaerobic digestion involving two different discrete delays to describe the time delay in substrate conversion to viable biomass. To the authors' knowledge, such investigations have not been yet performed for this bioreactor model. Using a properly chosen admissible value for the dilution rate $\bar{u}$ we prove in Theorem 1 the global asymptotic convergence of the solutions towards an equilibrium point, corresponding to $\bar{u}$. Moreover, the global stability is proved under general assumptions of the specific growth rates, without knowing the particular forms of these functions. Analytic expressions for the latter are introduced later to carry out the computer simulations. It is interesting to note that the qualitative behavior of the solutions of (1) with $\tau_1 > 0$, $\tau_2 > 0$ is the same as the solutions behavior of the model without delays, i. e., when $\tau_1 = \tau_2 = 0$, see [8]. A model-based extremum seeking algorithm is applied to stabilize the dynamics towards the equilibrium point where maximum production of methane is achieved. Numerical simulation is included to illustrate the theoretical results.

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