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QUASILINEAR STRUCTURES IN STOCHASTIC ARITHMETIC AND THEIR APPLICATION*

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Abstract. Stochastic arithmetic has been developed as a model for computing with imprecise numbers. In this model, numbers are represented by independent Gaussian variables with known mean value and standard deviation and are called stochastic numbers. The algebraic properties of stochastic numbers have already been studied by several authors. Anyhow, in most life problems the variables are not independent and a direct application of the model to estimate the standard deviation on the result of a numerical computation may lead to some overestimation of the correct value. In this work "quasilinear" algebraic structures based on standard stochastic arithmetic are studied and, from pure abstract algebraic considerations, new arithmetic operations called "inner stochastic addition and subtraction" are introduced. They appear to be stochastic analogues to the inner interval addition and subtraction used in interval arithmetic. The algebraic properties of these operations and the involved algebraic structures

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are then studied. Finally, the connection of these inner operations to the correlation coefficient of the variables is developed and it is shown that they allow the computation with non-independent variables. The corresponding methodology for the practical application of the new structures in relation to problems analogous to "dependency problems" in interval arithmetic is given and some numerical experiments showing the interest of these new operations are presented.

1. Introduction. Stochastic numbers and stochastic arithmetic have been developed as a model for computing with imprecise/uncertain data as a mean to control numerical computations [10]. In the case of stochastic numbers the uncertainty in the data is assumed to follow some known Gaussian distribution. This is an old idea [8] and has been formalized by J. Vignes and J.M. Chesneaux as a theoretical approach of the Cestac method [5], [9]. In the scope of stochastic arithmetic, imprecise data are interpreted as stochastic numbers and the computation on them is called stochastic arithmetic. Thus, stochastic arithmetic provides confidence intervals for the results of numerical computation in the same way that interval arithmetic provides bounds for these results but is achieved by means from probability theory.

The analogy between stochastic and interval (approximate) numbers begins already in their presentation. A stochastic number is usually presented as a pair of the form $(m; s)$, where the real number m is interpreted as a mean value and the non-negative real number s is a standard deviation; this form corresponds to the midpoint-radius presentation of intervals (approximate numbers). On the other side, stochastic numbers induce confidence intervals which correspond to the end-point (inf-sup) presentation of intervals [1]. For this reason stochastic arithmetic as algebraic structure is similar to interval arithmetic. In both cases the original structure involving addition is a commutative cancellative monoid and there are two possible ways to extend and study these structures: either to introduce improper elements in order to complete the monoid system up to a group structure, or to remain within the monoid structure and to introduce new useful arithmetic operations. Each approach has its merits and disadvantages.

In previous papers stochastic arithmetic structures focusing on the completion of the additive monoid into a group have been studied. These studies were inspired by analogous structures in interval arithmetic based on interval spaces of Kaucher type, see e. g. [2], [4]. Although algebraically very elegant, this latter approach leads to improper elements and needs rather specific interpretations. In this work the alternative approach is considered. The "quasilinear" algebraic structures based on stochastic numbers, using as basic operations addition and multiplication by scalar, are investigated. Moreover, to remain within the domain of proper stochastic elements with positive standard deviations, and to be able to solve simple algebraic equations, a new operation, here called "inner stochastic addition", is introduced. This is a stochastic analogue to the inner interval addition used in interval arithmetic and interval analysis. Then the algebraic properties of this operation and the involved algebraic structures are studied.

The corresponding methodology for the practical application of the new structures in relation to problems analogous to so-called "dependency problems" in interval arithmetic is also given and some numerical experiments in the lines of our previous investigations, see e. g. [4], [6], [3], [7] are presented.

2. Stochastic arithmetic: s-space. Let us denote by $\mathbb{R}$ the set of reals; the linearly ordered field of reals is denoted $\mathbb{R}_D = (\mathbb{R}, +, \cdot, \leq)$.

The set of nonnegative real numbers is denoted $\mathbb{R}^+$. A *stochastic number* $X = (m; s)$ is a Gaussian random variable with mean value $m \in \mathbb{R}$ and (nonnegative) standard deviation $s \in \mathbb{R}^+$. The set of all stochastic numbers is $\mathbb{S} = \{(m; s) \mid m \in \mathbb{R}, s \in \mathbb{R}^+\}$.

Let $X_1 = (m_1; s_1)$, $X_2 = (m_2; s_2) \in \mathbb{S}$. (Outer) addition and multiplication by scalars are defined in ST by:

$$X_1 \oplus X_2 = (m_1; s_1) \oplus (m_2; s_2) \overset{def}{=} (m_1 + m_2;\ \sqrt{s_1^2 + s_2^2}), \tag{1}$$

$$\gamma * X = \gamma * (m; s) \overset{def}{=} (\gamma m;\ |\gamma| s),\ \gamma \in \mathbb{R}_D. \tag{2}$$

In particular multiplication by -1 (negation) is $\neg X = -1 * X = -1 * (m; s) = (-m;\ s)$. (Outer) subtraction is:

$$X_1 \ominus X_2 = X_1 \oplus (\neg X_2) = (m_1; s_1) \oplus \neg(m_2; s_2) = (m_1 - m_2;\ \sqrt{s_1^2 + s_2^2}). \tag{3}$$

In particular, $X \ominus X = (m; s) \oplus \neg(m; s) = (m - m;\ \sqrt{s^2 + s^2}) = (0; \sqrt{2}s)$.

The definition of addition implies $(m; 0) \oplus (0; s) = (m; s)$ showing that every stochastic number can be decomposed into two special stochastic numbers, called resp. degenerate and symmetric. A stochastic number of the form $(m; 0)$ is *degenerate*. The set of degenerate stochastic numbers can be identified with the set of real numbers and the above mentioned arithmetic operations coincide with the familiar real operations in the case of symmetric stochastic numbers. A stochastic number of the form $(0; s)$, $s \in \mathbb{R}^+$, is called *symmetric*. If X_1, X_2 are symmetric stochastic numbers, then $X_1 \oplus X_2$ and $\lambda * X_1$, $\lambda \in \mathbb{R}_D$, are also symmetric stochastic numbers. There is a one-to-one correspondence between the set of symmetric stochastic numbers and the set of standard deviations $\mathbb{R}^+$. Special symbols "$\oplus$", "$*$" for the arithmetic operations over standard deviations will be used, as these operations are different from the corresponding ones for real numbers. Operations "$\oplus$", "$*$" induce a special arithmetic on the set $\mathbb{R}^+$, namely the system $(\mathbb{R}^+, \oplus, \mathbb{R}_D, *)$, where:

$$\begin{aligned} s \oplus t &= \sqrt{s^2+t^2},\ s,t \in \mathbb{R}^+, \\ \gamma * s &= |\gamma| s,\ \gamma \in \mathbb{R}_D,\ s \in \mathbb{R}^+. \end{aligned}$$

Proposition 1. *The system $(\mathbb{R}^+, \oplus, \mathbb{R}_D, *)$ is an abelian additive monoid with cancellation, such that for $s,t \in \mathbb{R}^+$, $\alpha, \beta \in \mathbb{R}$:*

$$\begin{aligned} \alpha * (s \oplus t) &= \alpha * s \oplus \alpha * t, \\ \alpha * (\beta * s) &= (\alpha\beta) * s, \\ 1 * s &= s, \\ (-1) * s &= s, \\ \sqrt{\alpha^2+\beta^2} * s &= \alpha * s \oplus \beta * s,\ \alpha, \beta \geq 0. \end{aligned}$$

A system satisfying the conditions of Proposition 1 is called an s-space of monoid structure. Note that $s \oplus s = \sqrt{2}s \neq 2 * s = 2s$.

Remark. The sum $s \oplus s$ is different from $2 * s$ because addition supposes that the two variables are independent which is not the case for $s \oplus s$. Looking at the sum of two correlated variables, see Section , the variance is $s_x^2 + s_y^2 + 2\rho s_x s_y$ which in the case of $s_x = s_y = s$ (then $\rho = 1$) leads to $4s^2$ and thus the result is $2 * s$.

3. Inner addition and subtraction: definition. Denote by $\mathbb{S}$ the set of all stochastic numbers. Let $X_1 = (m_1; s_1)$, $X_2 = (m_2; s_2) \in \mathbb{S}$ be two stochastic numbers. The inner addition and inner subtraction operations are defined by

$$\begin{aligned} X_1 \oplus^- X_2 &= (m_1; s_1) \oplus^- (m_2; s_2) \stackrel{def}{=} (m_1 + m_2;\ \sqrt{|s_1^2 - s_2^2|}\ ), \\ X_1 \ominus^- X_2 &= (m_1; s_1) \ominus^- (m_2; s_2) \stackrel{def}{=} (m_1 - m_2;\ \sqrt{|s_1^2 - s_2^2|}\ ). \end{aligned}$$

The *inner* operations for addition/subtraction of (one-dimensional) stochastic numbers can be introduced using an "algebraic approach". Indeed, for $A = (m_A; s_A)$, $B = (m_B; s_B) \in \mathbb{S}$, we have

$$A \oplus^- B = \begin{cases} Y|_{Y \ominus B = A}, & \text{if } s_B \leq s_A, \\ X|_{X \ominus A = B}, & \text{if } s_A \leq s_B. \end{cases} \tag{4}$$

Similarly inner subtraction can be introduced:

$$A \ominus^- B = \begin{cases} Y|_{B \oplus Y = A}, & \text{if } s_B \leq s_A; \\ X|_{A \ominus X = B}, & \text{if } s_A \leq s_B. \end{cases} \tag{5}$$

Note that the relations $A \oplus^- B = A \ominus^- (\neg B)$, $A \ominus^- B = A \oplus^- (\neg B)$ can be deduced from (4), (5).

Remark. Operation "$\oplus^-$" is defined similarly to inner addition (or subtraction), cf. the literature on interval analysis and, to so-called "generalized Hukuhara difference" in fuzzy sets analysis.

The stochastic number $X = [0, 0] = 0$ is the unique identity (neutral element) with respect to inner addition $\oplus^-$, that is for every $A \in \mathbb{S}$: $A = 0 \oplus^- A = A \oplus^- 0$. Note that $A \oplus^- (\neg A) = 0$, thus every element $A \in \mathbb{S}$ has unique inverse w. r. t. "$\oplus^-$", and this is the element $\neg A = (-m_A; s_A)$. Relations (4), (5) show that inner operations serve to solve equations of the simple form $A \oplus X = B, A \ominus X = B$.

4. Symmetric stochastic number. Symmetric stochastic numbers, the set of which can be identified with the set $\mathbb{R}^+$ of nonnegative reals, are considered here. Their algebraic properties are those of the system of standard deviations $(\mathbb{R}^+, \oplus, \leq)$. Hence for $s, t \in \mathbb{R}^+$, $s \oplus t = \sqrt{s^2 + t^2}$, and "$\leq$" is the usual order relation (following, preceeding) on the set of (nonnegative) reals. As mentioned in Proposition 1 $(\mathbb{R}^+, \oplus)$ is a commutative monoid, satisfying the cancellation property $A \oplus X = B \oplus X \Longrightarrow A = B$. System $(\mathbb{R}^+, \oplus)$ is not a group as equation $A \oplus X = 0$ has no solution when $A \neq 0$.

Thus there exists no additive inverse (opposite) in the cancellative monoid $(\mathbb{R}^+, \oplus)$. The subtractability property does not hold either, as $A \oplus X = B$ does not possess a solution in general. Our attention is now focused on inner addition "$\oplus^-$". For $s, t \in \mathbb{R}^+$, $s \oplus^- t$ can be defined as

$$s \oplus^- t = \sqrt{|s^2 - t^2|} = \begin{cases} y|_{t \oplus y = s} & \text{if } t \leq s; \\ x|_{s \oplus x = t} & \text{if } s \leq t. \end{cases} \tag{6}$$

In words, $s \oplus^- t$ is either the solution y of $t \oplus y = s$ or is the solution x of $s \oplus x = t$ depending on which one exists; note that if both solutions x, y exist (which only happens when $s = t$), then they coincide $x = y = 0$.

Let us consider the algebraic properties of inner addition "$\oplus^-$" in some detail.

— inner addition "$\oplus^-$" is a closed (total) operation.

— associativity $(A \oplus^- B) \oplus^- C = A \oplus^- (B \oplus^- C)$ fails, indeed, e. g. $(7 \oplus^- 5) \oplus^- 3 \neq 7 \oplus^- (5 \oplus^- 3)$, as $(7 \oplus^- 5) \oplus^- 3 = \sqrt{15}$ and $7 \oplus^- (5 \oplus^- 3) = \sqrt{33}$;

— the property of the existence of an identity (neutral, null) element, such that $A \oplus^- 0 = A$ for all $A \in \mathbb{R}^+$, holds true;

— the property of the existence of an inverse (opposite) element holds true as well. Indeed, the inverse of A is the element A itself, since $A \oplus^- A = 0$ for all A.

— cancellation property: $A \oplus^- X = B \oplus^- X \Longrightarrow A = B$ fails, e. g. take $A = 3, B = 5, X = \sqrt{17}$. Then $3 \oplus^- \sqrt{17} = 5 \oplus^- \sqrt{17}$, but $3 \neq 5$;

— commutativity property: $A \oplus^- B = B \oplus^- A$ holds true.

Thus system $(\mathbb{R}^+, \oplus^-)$ is a commutative unital magma. Associativity and cancellation can be replaced by certain weaker properties, that is properties satisfied under specific conditions, as follows.

Proposition 2 (Conditional associativity or C-associativity)**.** *Let A, B, $C \in \mathbb{R}^+$ be such that $B \geq A, B \geq C$. Then $(A \oplus^- B) \oplus^- C = A \oplus^- (B \oplus^- C)$.*

Indeed, it is easy to see that for $B \geq A$, $B \geq C$ we have $(A \oplus^- B) \oplus^- C = A \oplus^- (B \oplus^- C) = \sqrt{|A^2 - B^2 + C^2|}$. C-associativity is practically important, since "summing up" three elements (in the sense of inner addition) does not depend on the order of summation unless we start summation from the largest element. In practice this requirement is not too restrictive.

The failure of the cancellation law $A \oplus^- X = B \oplus^- X \Longrightarrow A = B$ in $(\mathbb{R}^+, \oplus^-)$ has already been mentioned. However, cancellation is valid under certain conditions. Indeed, consider relation $A \oplus^- X = B \oplus^- X$ as an equation for X, then it has a unique solution $X = (A \oplus B)/\sqrt{2}$, that is $X^2 = (A^2 + B^2)/2$. Thus in fact cancellation of X holds true unless $X \neq (A \oplus B)/\sqrt{2}$. This property can be formulated as follows:

Proposition 3 (Conditional cancellation or C-cancellation)**.** *Let $A, B \in \mathbb{R}^+$. Equation $A \oplus^- X = B \oplus^- X$ is satisfied for $X = (A \oplus B)/\sqrt{2}$. If $X \neq (A \oplus B)/\sqrt{2}$, then cancellation of X holds true: $A \oplus^- X = B \oplus^- X \Longrightarrow A = B$.*

Note that the value $(A \oplus B)/\sqrt{2}$ lies between A and B. Hence, if X is not between A and B, then cancellation holds true.

The summary is then: The set of standard deviations with inner addition $(\mathbb{R}^+, \oplus^-)$, is a C-associative and C-cancellative commutative unital magma. Note that equations $A \oplus X = B$ and $A \oplus^- X = B$ cannot be solved directly. However, the operation inner addition "$\oplus^-$" allows solving equations of the form $A \oplus X = B$ in certain cases. Namely, using "$\oplus^-$" the equation $A \oplus X = B$ can be solved when $A \leq B$ and the equation $B \oplus X = A$ can be solved when $A \geq B$. Thus inner addition plays a role in $(\mathbb{R}^+, \oplus)$ analogous to the role of subtraction in the group $(\mathbb{R}, +)$.

The next proposition shows that operations "$+$" and "$\oplus^-$" are closely related in the solution of equations.

Proposition 4. *i) For $A, B \in \mathbb{R}^+$, such that $A \leq B$, the unique solution of $A \oplus X = B$ is $X = B \oplus^- A$. ii) The equation $A \oplus^- X = B$ has a solution $X = A \oplus B$ for $A, B \in \mathbb{R}^+$. If $A, B \in \mathbb{R}^+$ are such that $A \geq B > 0$, then equation $A \oplus^- X = B$ has one more solution $X = A \oplus^- B$.*

The above formal definitions and properties of outer and inner addition and subtractions have a simple statistical interpretation.

5. Inner addition and sum of correlated variables. Outer addition (2) and subtraction (3) correspond to addition and subtraction of independent variables. To interpret inner operations, let us recall here that the sum of two random variables $X = (m_X; s_X)$ and $Y = (m_Y; s_Y)$ with correlation coefficient ρ is classically:

$$Z = X + Y = (m_X; s_X) + (m_Y; s_Y) = (m_X + m_Y; s_Z) \tag{7}$$

where $s_Z = s_{X+Y} = \sqrt{s_X^2 + s_Y^2 + 2\rho s_X s_Y}$ with ρ in $[-1, +1]$.

Note that, when $\rho = 1$ then $s_Z = \sqrt{(s_X + s_Y)^2} = s_X + s_Y$ and whenever $\rho < 1$, the standard deviation is less than the sum of the standard deviations of X and Y. The classical particular cases are:

• $\rho = \rho_1 = 1$ (positive linear dependency)

$$s_Z^1 = \sqrt{(s_X + s_Y)^2} = s_X + s_Y, \tag{8}$$

• $\rho = \rho_{-1} = -1$ (negative linear dependency)

$$s_Z^{-1} = \sqrt{(s_X - s_Y)^2} = |s_X - s_Y|, \tag{9}$$

• $\rho = \rho_0 = 0$ (no correlation)

$$s_Z^0 = \sqrt{s_X^2 + s_Y^2}. \tag{10}$$

The extreme cases (8) and (9) correspond to outer and inner addition in interval analysis. These two operations complement each other algebraically.

The symmetry of the extreme cases and the form of case for $\rho_0 = 0$ suggests that there should be a symmetrical case to this latter one. We can derive it as follows.

• Denote $\rho^* = -\min\{s_X/s_Y, s_Y/s_X\}$. This value $\rho = \rho^*$ is now called "mean negative correlation" or "weak correlation". We have $-1 \le \rho^* \le 0$. For this value ρ^* a simple calculation leads to:

$$s_Z^* = \sqrt{|s_X^2 - s_Y^2|}, \tag{11}$$

which retrieves the standard deviation of inner operations of Section 3. Clearly operation (11) is the algebraic counterpart of operation (10).

Example 1. If $s_X = s_Y = s$, then s_Z obtains the following values depending on the value of ρ in increasing order $\rho \in (-1, \rho^*, 0, 1)$: $s_Z \in (0, 0, \sqrt{2}s, 2s)$ (note that in this case $\rho^* = \rho^{-1}$).

Example 2. If $s_X = s, s_Y = s/2$, then s_Z obtains the following values $s_Z \in (s/2, (\sqrt{3}/2)s, (\sqrt{5}/2)s, (3/2)s)$.

Remark. A functional similar to ρ^*, the so-called Ratschek's functional, plays an important role in interval arithmetic.

Let us now see that the use of inner operations in the model of stochastic arithmetic leads to better estimation of the standard deviation on numerical results in the case of non-independent variables than standard stochastic operations. To fit the developed theory only linear problems are considered here.

6. Numerical experiments. The experiments have been done as follows. An experimental software has been written in Fortran 90 allowing the computation of all operations and most standard functions on stochastic numbers. Moreover for addition and subtraction the standard deviation can be computed using any one of the formulae (8–11).

Using this software, many samples have been tested. To illustrate the theory, the two first examples reported here concern the computation of the norm of the solution of linear systems obtained by Gaussian elimination. One is a Van der Monde system obtained with a 4-degree polynomial interpolation $P(x_i) = y_i$, $i = 1, \ldots, 5$ with uncertain values y_i, and the second is a system with a random matrix. Both systems have stochastic right-hand sides. In these examples the norm of the solution has been computed using the outer and inner formulas for additions and subtractions and also using a statistical simulation providing an empirical mean-value and empirical standard deviation. These empirical values are considered very close to the exact values and are compared to the theoretical values provided by outer and inner operations.

The third example is the scalar product of two vectors: $q = \sum_{i=0}^{i=k} x_i y_i$, whose components have been randomly generated with a constant distribution in a chosen range. The components have also been blurred with a Gaussian white noise with standard deviation $\sigma_x = 0.001$. Then as for the linear systems, the result has been computed with the inner and outer formulas and compared to a statistical simulation providing an empirical mean-value and empirical standard deviation. In all examples the experimental empirical standard deviation and the standard deviations obtained on the result with inner and outer standard operations have been plotted.

Numerical example 1. Norm of the solution of a Van der Monde linear system $A\ X = B$ obtained for the four-degree interpolation with the following values of x and $P(x)$:

x_i	-1	1	3.5	5	7
y_i	1	3	5	9	5

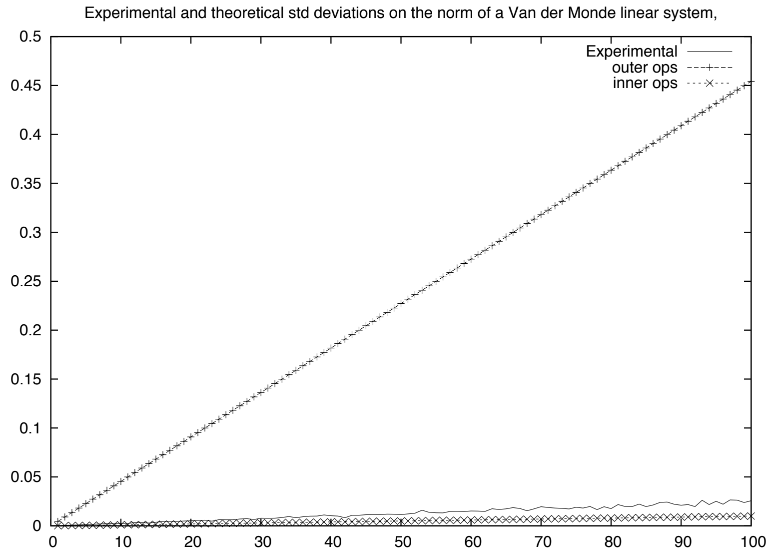

Fig. 1. Norm of the solution of Example 1 with increasing error on the right-hand side: theoretical and experimental standard deviations

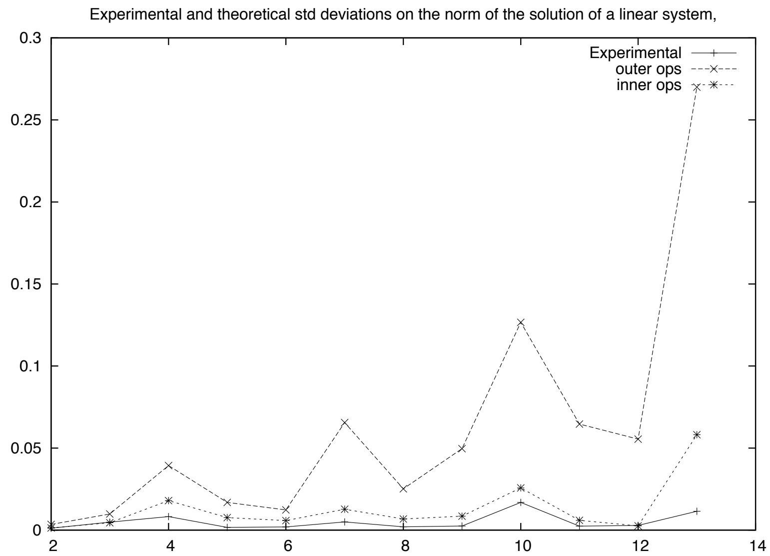

Fig. 2. Norm of the solution of a random linear system of increasing size: theoretical and experimental standard deviations

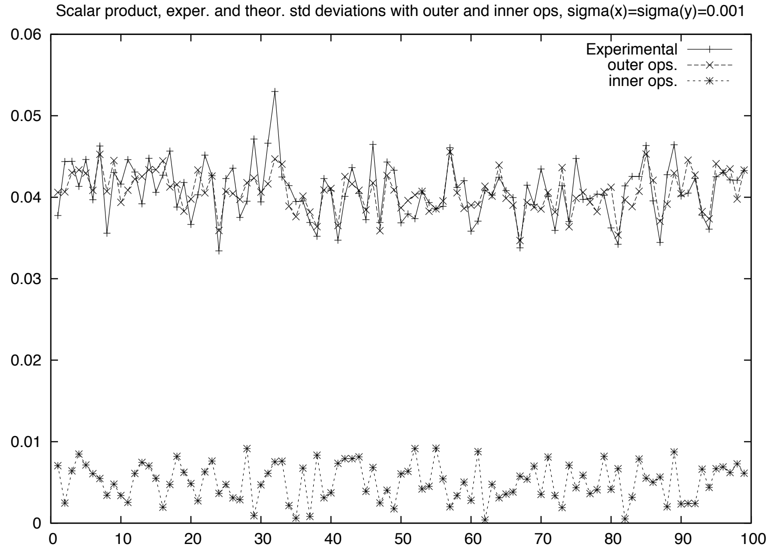

Fig. 3. Theoretical and experimental standard deviations on inner products

In this example, the values of y_i are imprecise with a common standard deviation σ which is chosen to increase from 0.0001 to 0.01 with step 0.0001.

Numerical example 2. Gaussian elimination and computation of the norm of the solution of a linear system whose matrix is real with random coefficients in $[-1, +1]$ and stochastic right-hand side with random mean value and standard deviation $\sigma = 0.01$.

Numerical example 3. Scalar product $q = \sum_{i=0}^{i=k} x_i y_i$ with $x_i, y_i \in [-1, 1]$. 100 random scalar products of size 50 have been reported numbered from 1 to 100.

The standard deviation of the results computed with the above inner and outer formulae and with the empirical simulation are reported in Figures 1, 2, 3.

It can be seen in these figures that in the two first examples where the variables involved in the sum are correlated, the closest case to the experimental standard deviation is always the one using inner operations that is formula (11). Conversely, Figure 3 shows that the formula leading to the closest result to the experimental is the one with outer operations.

This confirms the theory that inner operations must be used for non-independent variables and outer operations for independent variables.

7. Conclusion. In this paper the main properties of standard (outer) stochastic arithmetic have been recalled. Then, from pure theoretical algebraic considerations, a new addition and a new subtraction called inner operations have been introduced in the same manner as was done in interval arithmetic. Many properties of the algebraic structure thus induced by these new operations have been studied as well as the relationship between these two types of operations. Finally, a statistical interpretation of these inner operations has been given and numerical examples have been provided showing their utility and when to use them. When the variables are not independent, and this is the common rule, inner arithmetic provides theoretical standard deviations of the results very close to the empirical one. Anyhow, in the case of independent variables, the best arithmetic is clearly the one with standard operations, i. e. outer stochastic arithmetic. In fact the theory of stochastic arithmetic had been formerly developed for independent variables and is now being enlarged to the case of non-independent variables. Hence stochastic arithmetic is a tool as easy to use as before but is now more powerful and precise for the estimation of errors on numerical results provided that outer or inner operations are used correctly.

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