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R&D Dynamics with Asymmetric Efficiency

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Abstract We consider an R&D investment function in a Cournot duopoly competition model inspired in the logistic equation. We study the economical effects resulting from the firms having different R&D efficiencies. We present three cases: (1) both firms are efficient and have the same degree of efficiency; (2) both firms are less efficient and have the same degree of efficiency; (3) firms are asymmetric in terms of the efficiency of their R&D investment programs. We study the myopic dynamics on the production costs obtained from investing the Nash investment equilibria.

Keywords Strategic R&D · Cournot duopoly model · Patents

1 Introduction

We consider a Cournot duopoly competition model where two firms invest in R&D projects to reduce their production costs. This competition is modeled, as usual, by a two stages game (see [3, 6]). In the first subgame, two firms choose, simultaneously, R&D investment strategies to reduce their initial production costs. In the second

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subgame, the two firms are involved in a Cournot competition with production costs equal to the reduced cost determined by the R&D investment strategies chosen in the first stage. We use an R&D cost reduction function inspired in the logistic equation that was first introduced in [5].

We consider two firms that are identical except, at most, in their R&D investment programs efficiency. Concerning their R&D efficiency, we study the economical effects of three different scenarios: (1) both firms R&D investment programs are efficient and both firms hold the same degree of efficiency. We will refer to this case as the symmetric efficient (SE) case; (2) both firms R&D investment programs are less efficient and both have the same degree of efficiency. We will refer to this case as the symmetric inefficient (SI); (3) firms are asymmetric in terms of their R&D investment programs efficiency, i.e. one of the firms has a more efficient R&D investment program allowing bigger cost reduction under equal investment whereas the other firm has a less efficient R&D investment program forcing it to invest more in order to achieve the same cost reduction as the other firm. We will refer to this case as the Asymmetric (A) case.

We present the Perfect Nash equilibria of this two stages game and we study the economical effects of these equilibria. The second subgame, consisting of a Cournot competition, has a unique perfect Nash equilibrium. For the first subgame, consisting of an R&D cost reduction investment program, we exhibit four different regions of Nash investment equilibria that we characterize as follows: a competitive Nash investment region C where both firms invest, a single Nash investment region S_1 for firm F_1, where just firm F_1 invests, a single Nash investment region S_2 for firm F_2, where just firm F_2 invests, and a nil Nash investment region N, where neither of the firms invest (see [5, 6]).

The Nash investment equilibria are not necessarily unique. The non uniqueness leads to an economical complexity in the choice of the best R&D investment strategies by the firms. For high production costs, that can correspond to the production of new technologies, there are subregions of production costs where there are multiple Nash investment equilibria: a region $R_{S_i \cap C}$ where the intersection between the single Nash investment region S_i and the competitive Nash investment region C is non-empty; a region $R_{S_1 \cap S_2}$ where the intersection between the single Nash investment regions S_1 and S_2 is non-empty; a region $R_{S_1 \cap C \cap S_2}$ where the intersection between the single Nash investment regions S_1 and S_2 and the competitive Nash investment region C is non-empty. When we compare the cases symmetric efficient (SE) and symmetric inefficient (SI), we observe that, in the SI-scenario, the single Nash investment regions S_1 and S_2 increase in size and so the competitive Nash investment region C becomes smaller. In the asymmetric case (A), we observe that the single Nash investment region S_2 of firm F_2 is considerably bigger due to its advantage in the R&D cost reduction program efficiency.

We present the R&D deterministic dynamics on the production costs of the Cournot competition, based on the R&D investment strategies of the firms, as follows: at every period of time, the firms choose the investment corresponding to one of the Nash investment equilibria that determines the new production costs of the firms. Hence, the implicit equations determining the R&D deterministic dynamics are distinct in the competitive Nash investment region C and in the single Nash investment regions S_1 and S_2 (see Theorems 1 and 2 in [5]). The nil Nash investment region N determines the set of all production costs that are fixed by the dynamics. Depending upon the initial production costs of both firms and upon their R&D investment strategies, the nil Nash investment region N is the set of equilibria for the R&D deterministic dynamics. It is unusual in dynamical systems to have a non-isolated set of equilibrium points. This is due to the complex investment structure that we have to deal in these problems and to the lower bound in investment (economically, it must be non negative). The competitive Nash investment region determines the region where the production costs of both firms evolve, for both firms, along the time. The single Nash investment region S_1 (resp. S_2) determines the set of production costs where the production cost of firm F_2 (resp. F_1) is constant, along the time, and just the production costs of firm F_1 (resp. F_2) evolve that may lead Firm F_2 (resp. F_1) to bankruptcy or that may allow it to recover from an initial disadvantage.

2 R&D Investments on Costs

The Cournot duopoly competition with R&D investments on the reduction of the initial production costs consists of two subgames in one period of time. The first subgame is an R&D investment program, where both firms have initial production costs and choose, simultaneously, their R&D investment strategies to obtain new production costs. The second subgame is a Cournot competition with production costs equal to the reduced cost determined by the R&D investment program. As it is well known, the second subgame has a unique perfect Nash equilibrium.

2.1 The R&D Program

We consider an economy with a monopolistic sector with two firms, F_1 and F_2, each one producing a differentiated good. The inverse demands p_i are linear:

$$p_i = \alpha - \beta q_i - \gamma q_j, \tag{1}$$

with parameters $\alpha > 0$, $\beta > 0$ and γ. We assume that $\gamma > 0$ and thus the goods are substitutes. The firm F_i invests an amount v_i in an R&D program $a_i : \mathbb{R}_0^+ \to [b_i, c_i]$ that reduces its production cost to

$$a_i(v_i) = c_i - \frac{\varepsilon(c_i - c_L)v_i}{\lambda_i + v_i}. \tag{2}$$

Next, we explain the parameters of the R&D program: (i) the parameter c_i is the unitary production cost of firm F_i at the beginning of the period satisfying $c_L \leq c_i \leq$

α; (ii) the parameter c_L is the minimum attainable production cost; (iii) the parameter $0 < \varepsilon < 1$ has the following meaning: since $b_i = a_i(+\infty) = c_i - \varepsilon(c_i - c_L)$, the maximum reduction $\eta_i = \varepsilon(c_i - c_L)$ of the production cost is a percentage $0 < \varepsilon < 1$ of the difference between the current cost c_i and the lowest possible production cost c_L; (iv) the parameter $\lambda_i > 0$ can be seen as a measure of the inverse of the quality of the R&D investment program for firm F_i and is directly related to what we call efficiency of the R&D investment program that we define next (a smaller λ_i will result in a bigger reduction of the production costs for the same investment). The R&D investment program of firm F_1 is more efficient than the R&D investment program of firm F_2 if and only if with the same investment $v_1 = v_2 = v$, the new cost obtained by firm F_1, a_1, is smaller or equal to the new cost obtained by firm F_2, a_2, i.e. $a_1(v) \leq a_2(v)$. This R&D program was first introduced in [5].

2.2 Optimal Output Levels

The profit $\pi_i(q_i, q_j)$ of firm F_i is given by

$$\pi_i(q_i, q_j) = q_i(\alpha - \beta q_i - \gamma q_j - a_i) - v_i, \tag{3}$$

for $i, j \in \{1, 2\}$ and $i \neq j$. The Nash equilibrium output (q_1^*, q_2^*) is given by

$$q_i^* = \begin{cases} 0, \ if \ R_i \leq 0 \\ R_i, \ if \ 0 < R_i < \frac{\alpha - a_j}{\gamma} \\ \frac{\alpha - a_i}{2\beta}, \ if \ R_i \geq \frac{\alpha - a_j}{\gamma} \end{cases}, \tag{4}$$

where

$$R_i = \frac{2\beta(\alpha - a_i) - \gamma(\alpha - a_j)}{4\beta^2 - \gamma^2},$$

with $i, j \in \{1, 2\}$ and $i \neq j$. Hence, if $R_i \leq 0$ the firm F_j is at monopoly output level and, conversely, if $R_i \geq (\alpha - a_j)/\gamma$ the firm F_i is at monopoly output level and for intermediate values $0 \leq R_i < (\alpha - a_j)/\gamma$, both firms have positive optimal output levels and so we are in the presence of duopoly competition. From now on, we will always consider that both firms choose their Nash equilibrium output (q_1^*, q_2^*).

2.3 New Production Costs

The sets of possible new production costs for firms F_1 and F_2, given initial production costs c_1 and c_2 are, respectively,

$$A_1 = A_1(c_1, c_2) = [b_1, c_1] \quad \text{and} \quad A_2 = A_2(c_1, c_2) = [b_2, c_2],$$

where $b_i = c_i - \varepsilon(c_i - c_L)$, for $i \in \{1, 2\}$. The R&D programs a_1 and a_2 of the firms determine a bijection between the investment region $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ of both firms and the new production costs region $A_1 \times A_2$, given by the map

$$\begin{aligned} \mathbf{a} = (a_1, a_2) : \mathbb{R}_0^+ \times \mathbb{R}_0^+ &\to \quad A_1 \times A_2 \\ (v_1, v_2) &\mapsto (a_1(v_1), a_2(v_2)) \end{aligned}$$

where

$$a_i(v_i) = c_i - \frac{\eta_i v_i}{\lambda_i + v_i}.$$

We denote by $W = (W_1, W_2) : \mathbf{a}(\mathbb{R}_0^+ \times \mathbb{R}_0^+) \to \mathbb{R}_0^+ \times \mathbb{R}_0^+$

$$W_i(a_i) = \frac{\lambda_i (c_i - a_i)}{\eta_i - (c_i - a_i)}$$

the inverse map of **a**.

The new production costs region can be decomposed, at most, in three disconnected economical regions characterized by the optimal output level of the firms:

M_1 The *monopoly region* M_1 of firm F_1 that is characterized by the optimal output level of firm F_1 being the monopoly output and, so, the optimal output level of firm F_2 is zero;

D The *duopoly region* D that is characterized by the optimal output levels of both firms being non-zero and, so, below their monopoly output levels;

M_2 The *monopoly region* M_2 of firm F_2 that is characterized by the optimal output level of firm F_2 being the monopoly output and, so, the optimal output level of firm F_1 is zero.

The boundaries between the duopoly region D and the monopoly region M_i are l_{M_i} with $i \in \{1, 2\}$ and are presented, explicitly in [5].

In equilibrium, i.e. when both firms choose their optimal output levels, the profit function $\pi_i : A_i \times A_j \to \mathbb{R}$ of firm F_i, in terms of its new production costs (a_1, a_2), is a piecewise smooth continuous function given by

$$\pi_i(a_1, a_2) = \begin{cases} \pi_{i,M_i}, & \text{if } (a_1, a_2) \in M_i \\ \pi_{i,D}, & \text{if } (a_1, a_2) \in D \\ -W_i(a_1, a_2), & \text{if } (a_1, a_2) \in M_j \end{cases},$$

where

$$\pi_{i,M_i} = \pi_{i,M_i}(a_1, a_2; c_1, c_2) = \frac{(\alpha - a_i)^2}{4\beta} - W_i(a_1, a_2),$$

$$\pi_{i,D} = \pi_{i,D}(a_1, a_2; c_1, c_2) = \beta \left(\frac{2\beta(\alpha - a_i) - \gamma(\alpha - a_j)}{4\beta^2 - \gamma^2} \right)^2 - W_i(a_1, a_2).$$

2.4 Nash Investment Regions

Let $V_i(v_j)$ be the best investment response function of firm F_i to a given investment v_j of firm F_j. The best investment response function $V_i : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ of firm F_i is explicitly computed in [5]. Note that the best investment response function $V_i : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ can be a multi-valued function.

Let c_L be the minimum attainable production cost and α the value to buyers. Given production costs $(c_1, c_2) \in [c_L, \alpha] \times [c_L, \alpha]$, the *Nash investment equilibria* $(v_1, v_2) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ are the solutions of the system

$$\begin{cases} v_1 = V_1(v_2) \\ v_2 = V_2(v_1) \end{cases}$$

where V_1 and V_2 are the best investment response functions computed in the previous sections.

All the results presented, hold in an open region of parameters $(c_L, \varepsilon, \alpha, \beta, \gamma)$ containing the point $(4, 0.2, 10, 0.013, 0.013)$. The parameter λ_i that measures the efficiency of the R&D investment program, i.e. the smaller the λ_i, the more efficient the R&D investment program, is the parameter we are interested in studying. In the case we referred to as symmetric efficient, λ_i is equal to 10; in symmetric inefficient case, λ_i is equal to 20; in the asymmetric case, λ_1 is equal to 30 and λ_2 is equal to 10.

We observe that the Nash investment equilibria consists of a unique, or two, or three points depending upon the pair of initial production costs, as we will explain throughout the chapter. The set of all Nash investment equilibria form the *Nash investment equilibrium set*. We discuss the Nash investment equilibria by considering the following three regions of production costs:

- C the *competitive Nash investment region* C that is characterized by both firms investing;
- S_i the *single Nash investment region* S_i that is characterized by only one of the firms investing;
- N the *nil Nash investment region* N that is characterized by neither of the firms investing.

3 Nash Investment Equilibria Under Asymmetric Efficiency

In this chapter we compare the Nash investment equilibria choices of the two firms under three different scenarios: (i) symmetric efficient (SE) scenario where both firms R&D investment programs' are efficient and both firms hold the same degree of efficiency; (ii) symmetric inefficient (SI) scenario where both firms R&D investment programs' are less efficient and both firms hold the same degree of efficiency;

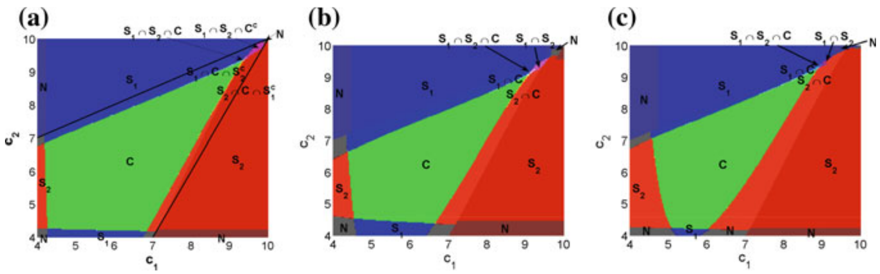

Fig. 1 Full characterization of the Nash investment regions in terms of the firms' initial production costs (c_1, c_2). The monopoly lines l_{M_i} are colored *black*. The nil Nash investment region N is colored *grey*. The single Nash investment regions S_1 and S_2 are colored *blue* and *red*, respectively. The competitive Nash investment region C is colored *green*. The region where S_1 and S_2 intersect are colored *pink*, the region where S_1 and C intersect are colored light *blue* and the region where S_2 and C intersect are colored *yellow*. The region where the regions S_1, S_2 and C intersect are colored light *grey*. (**a**) Symmetric efficient case. (**b**) Symmetric inefficient case. (**c**) Asymmetric case

asymmetric (A) scenario where one of the firms possesses a more efficient R&D program and the other firm possesses a less efficient R&D investment program.

We observe the existence, in the three distinct cases, of four different regions of Nash investment equilibria: a competitive Nash investment region C where both firms invest, a single Nash investment region S_1 for firm F_1, where just firm F_1 invests, a single Nash investment region S_2 for firm F_2, where just firm F_2 invests, and a nil Nash investment region N, where neither of the firms invest.

Let $R = [c_L, \alpha] \times [c_L, \alpha]$ be the region of all possible pairs of productions costs (c_1, c_2). Let $A^c = R - A$ be the complementary of A in R. The intersection between different Nash investment regions can be non-empty: (i) the intersection $R_{S_1 \cap S_2} = S_1 \cap S_2 \cap C^c$ between the *single Nash investments regions* S_1 and S_2 can be non empty; (ii) the intersection $R_{C \cap S_i} = C \cap S_i \cap S_j^c$ with $i \neq j$ between the *competitive Nash investment region* C and the *single Nash investment region* S_i can be non-empty; (iii) the intersection $R_{S_1 \cap C \cap S_2} = S_1 \cap C \cap S_2$ between the *competitive Nash investment region* C and the *single Nash investment regions* S_1 and S_2 can be non-empty (Fig. 1).

Let us consider the region of high production costs, that can correspond to the production of new technologies, where there are multiple Nash investment equilibria. In this section, we exhibit the production costs that correspond to the existence of multiple Nash investment equilibria.

We observe that the intersection $R_{S_1 \cap S_2} = S_1 \cap S_2 \cap C^c$ between the *single Nash investments regions* S_1 and S_2 is non empty. Thus, in this region we have two equilibria: a single Nash investment equilibrium to firm F_1 and a single Nash investment equilibrium to firm F_2. We also observe that the intersection $R_{C \cap S_i} = C \cap S_i \cap S_j^c$ with $i \neq j$ between the *competitive Nash investment region* C and the *single Nash investment region* S_i is non-empty. Therefore, in this region we have two Nash investment equilibria, one single Nash investment equilibrium for firm F_1 and a competitive Nash investment equilibrium. Finally, we see that the intersection

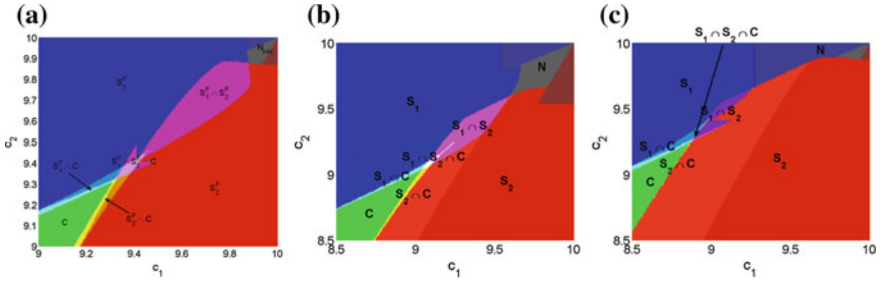

Fig. 2 Nash investment regions in the high production costs region, $c_i \in [9, 10]$, with $i \neq j$; (**a**) Symmetric efficient case. (**b**) Symmetric inefficient case. (**c**) Asymmetric case

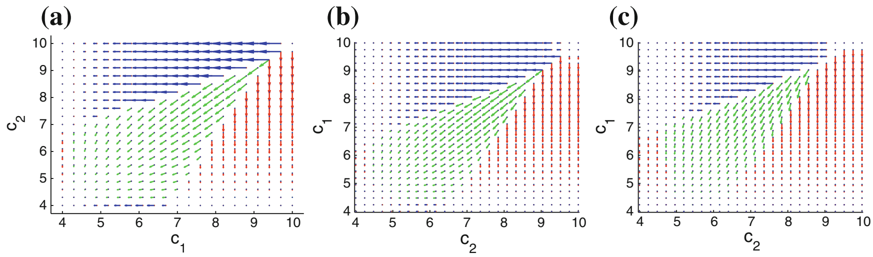

Fig. 3 Dynamics on the production costs in terms of the initial production costs (c_1, c_2): in *blue*, the dynamics in the single Nash investment region for firm F_1, S_1 where just firm F_1 invests; in *red* the dynamics in the single Nash investment region for firm F_2, S_2 where just firm F_2 invests; and in *green* the dynamics in the competitive Nash investment region C where both firms invest. (**a**) Symmetric efficient case. (**b**) Symmetric inefficient case. (**c**) Asymmetric case

$R_{S_1 \cap C \cap S_2} = S_1 \cap C \cap S_2$ between the *competitive Nash investment region* C and the *single Nash investment regions* S_1 and S_2 is non-empty. Thus, we have, simultaneously, a competitive equilibrium, a single favorable Nash investment equilibrium for firm F_1 and a single Nash investment equilibrium for firm F_2. This aspect enhances the high complexity of the R&D strategies of the firms, for high values of initial production costs (Figs. 2 and 3).

4 R&D Deterministic Dynamics

The R&D deterministic dynamics on the production costs of the duopoly competition appear from the firms deciding to play a perfect Nash equilibrium in the Cournot competition with R&D investment programs, period after period. The nil Nash investment region is the set of equilibria for these dynamics (Figs. 4 and 5).

The R&D deterministic dynamics in the single Nash investment region are implicitly determined by Theorems 1 and 2 in [5]. Let $S_1 = S_1^F \cup S_1^R$ be the single Nash

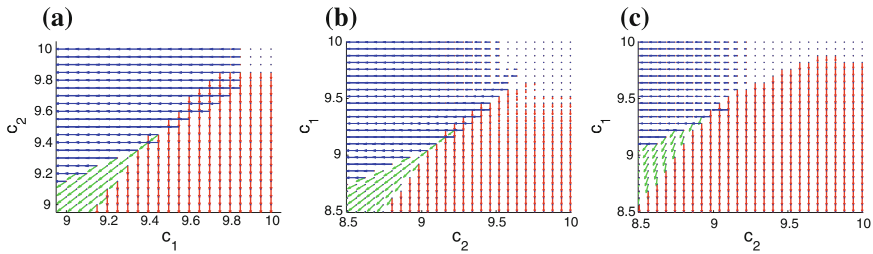

Fig. 4 Dynamics on the production costs in the high production costs region, $c_i \in [9, 10]$, with $i \neq j$: in *blue*, the dynamics in the single Nash investment region for firm F_1, S_1 where just firm F_1 invests; in *red* the dynamics in the single Nash investment region for firm F_2, S_2 where just firm F_2 invests; and in *green* the dynamics in the competitive Nash investment region C where both firms invest. (**a**) Symmetric efficient case. (**b**) Symmetric inefficient case. (**c**) Asymmetric case

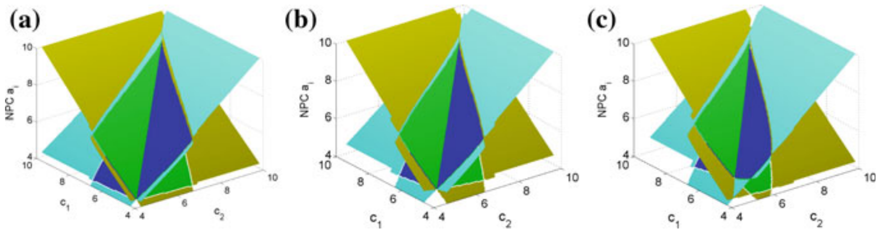

Fig. 5 New Production costs (NPC) for both firms F_1 and F_2 (a_1, a_2) in terms of the initial production costs (c_1, c_2): in *green* the NPC for firm F_2 and *blue* the NPC for firm F_1; in strong *green* the NPC in the competitive Nash investment region C and lighter *green*, the NPC in the single Nash investment region S_2; in strong blue the NPC in the competitive Nash investment region C and lighter *blue*, the NPC in the single Nash investment region S_1; (**a**) Symmetric efficient case. (**b**) Symmetric inefficient case. (**c**) Asymmetric case

investment region of firm F_1. If $(c_1, c_2) \in S_1^F$, then just firm F_1 invests along the time. Furthermore, at some period of time, the pair of new production costs falls in the monopoly region and so firm F_2 is driven out of the market by firm F_1. The production costs approach, along the time, the region N_{LH}. Hence, the production costs of firm F_1 approach low costs of production but the production costs of firm F_2 are always fixed at high production costs. If $(c_1, c_2) \in S_1^R$ then just firm F_1 invests along the time. So, firm F_1 will recover, along the time, from its disadvantageous position. The production costs approach, along the time, the region N_{LL}. Hence firm F_1 is able to recover, along the time, to the region where both firms have low production costs. The R&D deterministic dynamics in the competitive Nash investment region are implicitly determined by Theorems 1 and 2 in [5]. In the competitive Nash investment region both firms invest along the time and the production costs converge to the nil equilibrium region N_{LL}. Hence, the production costs of both firms are driven by the R&D deterministic dynamics to low production costs.

5 Conclusions

We presented R&D deterministic dynamics on the production costs of Cournot competitions based on perfect Nash equilibria of R&D investment strategies of the firms at every period. The following conclusions are valid in some parameter region of our model. We used an R&D investment function inspired in the logistic equation introduced in [5] and found all Perfect Nash investment equilibria of the Cournot competition model with R&D programs.

We described four main economic regions for the R&D deterministic dynamics corresponding to distinct perfect Nash equilibria: a competitive Nash investment region C where both firms invest, a single Nash investment region for firm F_1, S_1, where just firm F_1 invests, a single Nash investment region for firm F_2, S_2, where just firm F_2 invests, and a nil Nash investment region N where neither of the firms invest.

We considered three different scenarios in terms of the firms' R&D investment program efficiency: a first scenario, corresponding to the one studied in detail in [5], denominated symmetric efficient, where both firms possess an efficient R&D investment program; a second scenario, denominated symmetric inefficient, where both firms have an inefficient R&D program; and a third scenario, denominated asymmetric, where one of the firms has an efficient R&D program and the other one possesses and R&D program that is less efficient.

We showed, following [5], the existence of regions where the Nash investment equilibrium are not unique: the intersection $R_{S_1 \cap S_2}$ between the single Nash investment region S_1 and the single Nash investment region S_2 is non empty; the intersection $R_{S_i \cap C}$, with between the single Nash investment region S_i and the competitive Nash investment region C is non empty; the intersection $R_{S_1 \cap C \cap S_2}$ between the single Nash investment region S_1, the single Nash investment region S_2 and the competitive Nash investment region C is non empty. In this chapter we observed the persistence of these regions and described how these regions change as we change the efficiency of the R&D programs of both firms.

We presented the R&D deterministic dynamics on the production costs of Cournot competitions based on R&D investment strategies of the firms and we illustrated the transients and the asymptotic limits of the R&D deterministic dynamics on the production costs.

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