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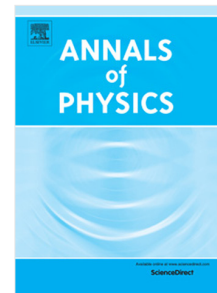
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Algebraic calculations for spectrum of superintegrable system from exceptional orthogonal polynomials

Md Fazlul Hoque^a, Ian Marquette^a, Sarah Post^b and Yao-Zhong Zhang^a

a. School of Mathematics and Physics, The University of Queensland, Brisbane, QLD 4072, Australia

b. Department of Mathematics, University of Hawaii at Manoa, Honolulu, HI 96822, USA

E-mail: m.hoque@uq.edu.au; i.marquette@uq.edu.au; spost@hawaii.edu; yzz@maths.uq.edu.au

Abstract

We introduce an extended Kepler-Coulomb quantum model in spherical coordinates. The Schrödinger equation of this Hamiltonian is solved in these coordinates and it is shown that the wave functions of the system can be expressed in terms of Laguerre, Legendre and exceptional Jacobi polynomials (of hypergeometric type). We construct ladder and shift operators based on the corresponding wave functions and obtain their recurrence formulas. These recurrence relations are used to construct higher-order, algebraically independent integrals of motion to prove superintegrability of the Hamiltonian. The integrals form a higher rank polynomial algebra. By constructing the structure functions of the associated deformed oscillator algebras we derive the degeneracy of energy spectrum of the superintegrable system.

1 Introduction

Many families of exceptional orthogonal polynomials have been successfully used to construct new superintegrable systems, higher order integrals of motion and higher order polynomial algebras [1, 2, 3, 4, 5]. In this paper, we use the recurrence approach to extend the three parameters Kepler-Coulomb system [6].

The exceptional orthogonal polynomials (EOP) were first explored in [7, 8]. These polynomials form complete, orthogonal systems extending the classical orthogonal poly-

nomials of Hermite, Laguerre and Jacobi. More recently much research has been done extending the theory of EOPs in various directions in mathematics and physics, in particular, exactly solvable quantum mechanical problems for describing bound states [9, 10, 11, 12, 13, 14, 15, 16, 17] and scattering states [18, 19, 20, 21], diffusion equations and random processes [22, 23, 24], quantum information entropy [25], exact solutions to Dirac equation [26], Darboux transformations [14, 15, 27, 28, 29, 30, 31] and finite-gap potentials [32]. Recent progress has been made constructing systems relating superintegrability and supersymmetric quantum mechanics with exceptional orthogonal polynomials [1, 33].

The research for superintegrable systems with second-order integrals in conformally flat spaces started in the mid sixties [34]. Over the last decade the topic of superintegrability has become an attractive area of research as these systems possess many desirable properties and can be found throughout various subjects in mathematical physics. For a detailed list of references on superintegrability, we refer the reader to the review paper [35]. One systematic approach to superintegrability is to derive spectra of 2D superintegrable systems based on quadratic and cubic algebras involving three generators [36, 37, 38]. In particular, the method of realization in the deformed oscillator algebras [39] has been effective for obtaining finite dimensional unitary representations [37, 40]. In fact, this approach was extended to classes of higher order polynomial algebras with three generators [41] as well as higher rank polynomial algebras of superintegrable systems in higher dimensional spaces [42, 43]. However, it is quite involved to apply the direct approach to obtain the corresponding polynomial algebras, Casimir operators and deformed oscillator algebras.

These difficulties can be overcome using a constructive approach based on eigenfunctions of the models. This approach is a useful tool to construct well-defined integrals of motion in classical and quantum mechanical problems. Many papers were devoted to construct integrals of motion and their corresponding higher order symmetry algebras based on lower-(first and second) ones [34, 44, 45, 46, 47] and higher-order ladder operators [4, 5, 48, 49, 50, 51, 52, 53, 54] in various aspects. In fact, the constructive approach has shown a close connection with special functions and (exceptional) orthogonal polynomials [1, 33, 55, 56, 57, 58, 59].

The search of exactly solvable generalizations of hydrogen like models took various directions [60, 61, 62, 63, 64, 65]. Such models can be used in context of ring shaped molecules [60] and in the study of quantum scattering [62]. A classification of superintegrable systems preserving rotational invariance and involving spin interaction was pursued [63, 64, 65]. In this paper, we introduce a new exactly solvable Hamiltonian

system in 3D, which is a singular deformation of the Coulomb potential. Its wave functions are given as products of Laguerre, Legendre and exceptional Jacobi polynomials. We show that the system is superintegrable by constructing higher order integrals of the motion using the recurrence relation approach. The symmetry algebra generated by these conserved quantities enables us to give an algebraic derivation for the energy spectrum.

The paper is organized as follows: in section 2, we present a new Hamiltonian system in 3D and show that its Schrödinger wave functions can be expressed in terms of Laguerre, Legendre and exceptional polynomials and obtain its physical spectra. In section 3, we construct a set of ladder and shift operators based on the wave functions and show that their suitable combination give the integrals of motion, thus proving the superintegrability of the model. We present the higher rank polynomial algebra generated by these integrals and the realization of this symmetry algebra in terms of the deformed oscillator algebra. By constructing finite-dimensional unitary representation of the symmetry algebra, we obtain the energy spectrum of superintegrable system.

2 Extended Kepler-Coulomb system

Consider the generalization of the three parameter Kepler-Coulomb Hamiltonian [6] in the spherical coordinates

$$H = \frac{1}{2}\mathbf{p}^2 - \frac{\alpha}{2r} + \frac{1}{2r^2 \sin^2 \theta} \left[\frac{\gamma^2 - \frac{1}{4}}{4 \sin^2 \frac{\phi}{2}} + \frac{\delta^2 - \frac{1}{4}}{4 \cos^2 \frac{\phi}{2}} + \frac{2(1 - b \cos \phi)}{(b - \cos \phi)^2} \right], \quad (2.1)$$

where $p_i = -i\partial_i$, $b = \frac{\delta+\gamma}{\delta-\gamma}$, $\gamma \neq \delta$ and α, γ, δ are three real constants. We consider $\hbar = 1$ in the remaining of the paper and the model introduced is a purely quantum superintegrable model. The Schrödinger equation $H\Psi(r, \theta, \phi) = E\Psi(r, \theta, \phi)$ of (2.1) can be expressed as

$$\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{\alpha}{r} + 2E + \frac{1}{r^2} \left\{ \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} \right\} + \frac{1}{r^2 \sin^2 \theta} \left\{ \frac{\partial^2}{\partial \phi^2} - \frac{\gamma^2 - \frac{1}{4}}{4 \sin^2 \frac{\phi}{2}} - \frac{\delta^2 - \frac{1}{4}}{4 \cos^2 \frac{\phi}{2}} - \frac{2(1 - b \cos \phi)}{(b - \cos \phi)^2} \right\} \right] \Psi(r, \theta, \psi) = 0. \quad (2.2)$$

The separation of variable of the Hamiltonian (2.1) for the wave equation $H\Psi = E\Psi$ by the ansatz

$$\Psi(r, \theta, \phi) = R(r)\Theta(\theta)Z(\phi) \quad (2.3)$$

provides the following radial and angular ordinary differential equations

$$\left[\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \frac{\alpha}{r} + 2E - \frac{k_2}{r^2} \right] R(r) = 0, \quad (2.4)$$

$$\left[\frac{d^2}{d\theta^2} + \cot \theta \frac{d}{d\theta} - \frac{k_1}{\sin^2 \theta} + k_2 \right] \Theta(\theta) = 0, \quad (2.5)$$

$$\left[\frac{d^2}{d\phi^2} - \frac{\gamma^2 - \frac{1}{4}}{4 \sin^2 \frac{\phi}{2}} - \frac{\delta^2 - \frac{1}{4}}{4 \cos^2 \frac{\phi}{2}} - \frac{2(1 - b \cos \phi)}{(b - \cos \phi)^2} + k_1 \right] Z(\phi) = 0, \quad (2.6)$$

where k_1, k_2 are the associated separation constants.

We now turn to (2.6), which can be converted, by setting $z = \cos \phi$, $Z(z) = (z + 1)^{\frac{1}{4}(\delta+2)}(z - 1)^{\frac{1}{4}(\gamma+2)}(z - b)^{-1}f(z)$, to

$$\begin{aligned} & (z^2 - 1) \frac{d^2 f(z)}{dz^2} + \left\{ \gamma - \delta + (\gamma + \delta + 2)z - \frac{2(z^2 - 1)}{z - b} \right\} \frac{df(z)}{dz} \\ & + \left\{ \frac{1}{4}(\gamma + \delta + 1)^2 - k_1 + \frac{\gamma - \delta + (\gamma + \delta - 1)z}{(b - z)} \right\} f(z) = 0. \end{aligned} \quad (2.7)$$

Comparing (2.7) with exceptional Jacobi differential equation [7],

$$T^{(\eta, \xi)}(Y) = (n - 1)(n + \eta + \xi)Y, \quad n \in \mathbb{N}, \quad (2.8)$$

where

$$\begin{aligned} T^{(\eta, \xi)}(Y) &= (X^2 - 1)Y'' + 2A \left(\frac{1 - BX}{B - X} \right) \{(X - C)Y' - Y\}, \\ A &= \frac{1}{2}(\xi - \eta), \quad B = \frac{\xi + \eta}{\xi - \eta}, \quad C = B + \frac{1}{A}, \end{aligned} \quad (2.9)$$

we obtain $\gamma = \xi$, $\delta = \eta$ and the separation constant

$$k_1 = \left(n + \frac{\gamma + \delta - 1}{2} \right)^2. \quad (2.10)$$

Hence the solutions of (2.7) are given in terms of the exceptional Jacobi polynomials $\hat{P}_n^{(\delta, \gamma)}$ [1, 7] as

$$Z(\phi) \equiv F_n(\gamma, \delta) \frac{(\cos \phi + 1)^{\frac{1}{4}(2\delta+1)} (\cos \phi - 1)^{\frac{1}{4}(2\gamma+1)}}{(\cos \phi - b)} \hat{P}_n^{(\delta, \gamma)}(\cos \phi). \quad (2.11)$$

These polynomials admit generalizations to multi-indexed families of Jacobi exceptional orthogonal polynomials [30, 31]. These EOPs are related to the standard Jacobi polynomials $P_n^{(\delta, \gamma)}$ [66] via

$$\hat{P}_n^{(\delta, \gamma)} = -\frac{1}{2}(\cos \phi - b)P_{n-1}^{(\delta, \gamma)} + \frac{bP_{n-1}^{(\delta, \gamma)} - P_{n-2}^{(\delta, \gamma)}}{\delta + \gamma + 2n - 2}. \quad (2.12)$$

Using (2.10) in the angular part (2.5), we have

$$\left[\frac{d^2}{d\theta^2} + \cot \theta \frac{d}{d\theta} - \frac{(n + \frac{\gamma + \delta - 1}{2})^2}{\sin^2 \theta} + k_2 \right] \Theta(\theta) = 0. \quad (2.13)$$

Then (2.13) can be converted, by setting $z = \cos \theta$, to

$$\left[(1 - z^2) \frac{d^2}{dz^2} - 2z \frac{d}{dz} + k_2 - \frac{(n + \frac{\gamma + \delta - 1}{2})^2}{1 - z^2} \right] f(z) = 0. \quad (2.14)$$

Comparing (2.14) with the Legendre differential equation

$$(1 - x^2)y'' - 2xy' + \left[m(m + 1) - \frac{\mu^2}{1 - x^2} \right] y = 0, \quad (2.15)$$

we obtain the constants

$$k_2 = m(m + 1), \quad \mu = n + \frac{\gamma + \delta - 1}{2}. \quad (2.16)$$

Hence the solutions of (2.5) are given in terms of the Legendre polynomials P_m^μ [66] as

$$\Theta(\theta) \equiv F_m(\mu) P_m^\mu(\cos \theta), \quad (2.17)$$

where $F_m(\mu)$ is a normalization constant and $m, \mu \in \mathbb{Z}$.

Using (2.16), the radial part (2.4) becomes

$$\left[\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \frac{\alpha}{r} + 2E - \frac{m(m + 1)}{r^2} \right] R(r) = 0. \quad (2.18)$$

(2.18) can be converted, by setting $z = \varepsilon r$, $R(z) = z^m e^{-\frac{1}{2}z} f_1(z)$ and $\varepsilon^2 = -8E$, to

$$\left[z \frac{d^2}{dz^2} + (2m + 2 - z) \frac{d}{dz} + \frac{\alpha}{\varepsilon} - m - 1 \right] f_1(z) = 0. \quad (2.19)$$

Set

$$N = \frac{\alpha}{\varepsilon} - m - 1. \quad (2.20)$$

Then (2.19) can be identified with the Laguerre differential equation. Hence the solutions of (2.4) are given in terms of the N -th order Laguerre polynomial functions L_N^β [66] as

$$R(r) \equiv e^{-\frac{\varepsilon r}{2}} (\varepsilon r)^m L_N^{2m+1}(\varepsilon r). \quad (2.21)$$

Hence the energy spectrum of the model (2.1), $E = \frac{-\varepsilon^2}{8}$, is given by

$$E = -\frac{\alpha^2}{8(N + m + 1)^2}, \quad N = 1, 2, 3, \dots \quad (2.22)$$

Here N represents the principal quantum number.

3 Algebraic calculation to the extended Kepler-Coulomb system

We can rewrite the Hamiltonian of the three parameter Kepler-Coulomb system in the standard way as a sequence of operators corresponding to separation in spherical coordinates (2.1),

$$H = \frac{1}{2} \left[\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{\alpha}{r} + \frac{L_\theta}{r^2} \right], \quad (3.1)$$

where

$$L_\theta = \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{L_\phi}{\sin^2 \theta}, \quad (3.2)$$

$$L_\phi = \frac{\partial^2}{\partial \phi^2} - \frac{\gamma^2 - \frac{1}{4}}{4 \sin^2 \frac{\phi}{2}} - \frac{\delta^2 - \frac{1}{4}}{4 \cos^2 \frac{\phi}{2}} - \frac{2(1 - b \cos \phi)}{(b - \cos \phi)^2}. \quad (3.3)$$

Making a slight change in the definition of these operators,

$$H_\theta = 1 - 4L_\theta, \quad H_\phi = -L_\phi, \quad (3.4)$$

leads to the following system of eigenvalue equations, from the previous section,

$$H\Psi = E\Psi, \quad H_\theta\Psi = \rho^2\Psi, \quad H_\phi\Psi = \mu^2\Psi. \quad (3.5)$$

Moreover, these three operators mutually commute, i.e. $[H_\theta, H] = [H_\phi, H] = [H_\theta, H_\phi] = 0$. The wave functions found in the previous section are then

$$\Psi(r, \theta, \phi) = \psi_N^\rho \Theta_{\frac{\rho-1}{2}}^\mu \hat{Z}_n, \quad (3.6)$$

$$\begin{aligned} \text{where } \psi_N^\rho &= e^{-\frac{\varepsilon r}{2}} (\varepsilon r)^{\frac{\rho-1}{2}} L_N^\rho(\varepsilon r), \quad \Theta_{\frac{\rho-1}{2}}^\mu = P_{\frac{\rho-1}{2}}^\mu(\cos \theta), \\ \hat{Z}_n &= (\cos \phi - b)^{-1} \sin^{\delta+\frac{1}{2}} \frac{\phi}{2} \cos^{\gamma+\frac{1}{2}} \frac{\phi}{2} \hat{P}_n^{(\delta, \gamma)}(\cos \phi). \end{aligned} \quad (3.7)$$

Here $\varepsilon = 2\alpha/(2N + \rho + 1)$ and

$$\begin{aligned} \rho &= 2m + 1, \quad m = 0, 1, 2, \dots, \\ \mu &= n + \frac{\gamma + \delta - 1}{2}, \quad n = 1, 2, 3, \dots \end{aligned}$$

As in the previous section, the relation among E , N and ρ is the quantization condition (2.22)

$$E = -\frac{\alpha^2}{2(2N + \rho + 1)^2}. \quad (3.8)$$

In the following we will construct additional integrals of motion to prove the superintegrability of the Hamiltonian (3.1).

3.1 Ladder and shift operators for the associated Laguerre and Legendre polynomials

We now search for recurrence operators which preserve the energy E . Equation (3.8) shows that E is preserved under either

$$N \rightarrow N + 1, \quad \rho \rightarrow \rho - 2 \quad \text{or} \quad N \rightarrow N - 1, \quad \rho \rightarrow \rho + 2,$$

as well as arbitrary shifts in n (equivalently μ). We now construct the ladder operators from the associated Laguerre functions, as in [1, 55],

$$L_N = (\rho + 1) \frac{\partial}{\partial r} + \alpha - \frac{1}{2r}(\rho^2 - 1), \quad R_N = (-\rho + 1) \frac{\partial}{\partial r} + \alpha - \frac{1}{2r}(\rho^2 - 1), \quad (3.9)$$

whose action on the corresponding wave functions are given by

$$L_N \psi_N^\rho = -\frac{2\alpha}{2N + \rho + 1} \psi_{N-1}^{\rho+2}, \quad R_N \psi_N^\rho = -\frac{2\alpha(N+1)(N+\rho)}{2N + \rho + 1} \psi_{N+1}^{\rho-2}. \quad (3.10)$$

We can also construct lowering and rising differential operators of the θ related part of separated solution for the associated Legendre functions

$$L_\rho = (1 - z^2) \frac{\partial}{\partial z} + \frac{\rho - 1}{2} z, \quad R_\rho = (1 - z^2) \frac{\partial}{\partial z} - \frac{\rho + 1}{2} z, \quad (3.11)$$

where $z = \cos \theta$, and obtain their action on the corresponding wave functions

$$L_\rho \Theta_{\frac{\rho-1}{2}}^\mu = \left(\mu + \frac{\rho-1}{2}\right) \Theta_{\frac{\rho-3}{2}}^\mu, \quad R_\rho \Theta_{\frac{\rho-1}{2}}^\mu = \left(\mu - \frac{\rho+1}{2}\right) \Theta_{\frac{\rho+1}{2}}^\mu. \quad (3.12)$$

Both of these pairs of ladder operators are obtained by taking the standard ladder operators of the special functions [66] and conjugating by the ground state.

3.2 Ladder and shift operators for the exceptional Jacobi polynomials and associated Legendre polynomials

Ladder operators for the exceptional Jacobi polynomials can be constructed from ladder operators for the Jacobi polynomials [66]

$$\begin{aligned} \mathcal{L}_n &= \frac{1}{2}(2n + \gamma + \delta)(1 - y^2) \frac{\partial}{\partial y} - \frac{1}{2}n\{\gamma - \delta - (2n + \gamma + \delta)y\}, \\ \mathcal{R}_n &= -\frac{1}{2}(2n + \gamma + \delta + 2)(1 - y^2) \frac{\partial}{\partial y} + \frac{1}{2}(n + \gamma + \delta + 1) \\ &\quad \times \{\gamma - \delta + (2n + \gamma + \delta + 2)y\}. \end{aligned} \quad (3.13)$$

Their action is as

$$\begin{aligned}\mathcal{L}_n P_n^{(\delta, \gamma)}(y) &= (n + \gamma)(n + \delta) P_{n-1}^{(\delta, \gamma)}(y), \\ \mathcal{R}_n P_n^{(\delta, \gamma)}(y) &= (n + 1)(n + \gamma + \delta + 1) P_{n+1}^{(\delta, \gamma)}(y), \quad y = \cos \phi.\end{aligned}\quad (3.14)$$

To extend these operators to the EOP case, we make use of forward and backward operators [1, 31] for the exceptional Jacobi polynomials

$$\begin{aligned}\mathcal{F} &= (y - 1)\left(y + \frac{\gamma + \delta}{\delta - \gamma}\right) \frac{\partial}{\partial y} + \delta\left(y + \frac{2 + \gamma + \delta}{\delta - \gamma}\right), \\ \mathcal{B} &= \frac{\gamma - \delta}{\gamma + \delta - (\gamma - \delta)y} \left\{ (1 + y) \frac{\partial}{\partial y} + \delta \right\}, \quad y = \cos \phi,\end{aligned}\quad (3.15)$$

whose actions are

$$\mathcal{F} P_n^{(\delta+1, \gamma-1)}(y) = -2(n + \delta - 1) \hat{P}_{n+1}^{(\delta, \gamma)}(y), \quad (3.16)$$

$$\mathcal{B} \hat{P}_n^{(\delta, \gamma)}(y) = \frac{1}{2}(n + \gamma) P_{n-1}^{(\delta+1, \gamma-1)}(y). \quad (3.17)$$

We can then define, as in [1], the corresponding ladder operators for the exceptional Jacobi polynomials via

$$L_n = \mathcal{F} \circ \mathcal{L}_n^- \circ \mathcal{B}, \quad R_n = \mathcal{F} \circ \mathcal{R}_n^+ \circ \mathcal{B}. \quad (3.18)$$

The final step is to conjugate these ladder operators by the ground state $y_0 = (y + 1)^{\frac{1}{4}(\delta+2)}(y - 1)^{\frac{1}{4}(\gamma+2)}(y - b)^{-1}$ for the angular component of the eigenfunction

$$L_n = y_0 L_n y_0^-, \quad R_n = y_0 R_n y_0^-, \quad (3.19)$$

so that their actions on the ϕ -components of the wave function are as follows,

$$\begin{aligned}L_n \hat{\mathcal{Z}}_n &= -(n + \delta)(n + \gamma)(n + \delta - 2)(n + \gamma - 2) \hat{\mathcal{Z}}_{n-1}, \\ R_n \hat{\mathcal{Z}}_n &= -n(n + \delta)(n + \gamma)(n + \delta + \gamma - 1) \hat{\mathcal{Z}}_{n+1}.\end{aligned}\quad (3.20)$$

While these operators shift the parameter n (equivalently μ) in the ϕ -factor $\hat{\mathcal{Z}}_n$, we must account for this shift in the $\Theta_{\frac{\rho-1}{2}}^\mu(z)$ component as well. To do so, we now construct a pair of operators from associated Legendre polynomials that can lower and rise μ while fixing ρ ,

$$L_\mu = \sqrt{1 - z^2} \frac{\partial}{\partial z} - \frac{\mu z}{\sqrt{1 - z^2}}, \quad R_\mu = \sqrt{1 - z^2} \frac{\partial}{\partial z} + \frac{\mu z}{\sqrt{1 - z^2}}, \quad (3.21)$$

where $z = \cos \theta$, and their actions on the corresponding wave functions are given by

$$L_\mu \Theta_{\frac{\rho-1}{2}}^\mu(z) = \left(\frac{\rho-1}{2} + \mu\right) \left(\frac{\rho+1}{2} - \mu\right) \Theta_{\frac{\rho-1}{2}}^{\mu-1}(z), \quad R_\mu \Theta_{\frac{\rho-1}{2}}^\mu(z) = -\Theta_{\frac{\rho-1}{2}}^{\mu+1}(z). \quad (3.22)$$

3.3 Integrals of motion and algebraic structure

Let us now consider the following suitable combinations of the operators

$$D_1^- = L_N R_\rho, \quad D_1^+ = R_N L_\rho, \quad D_2^- = R_\mu L_n, \quad D_2^+ = L_\mu R_n. \quad (3.23)$$

The action of the operators D_i^\pm , $i = 1, 2$ fixes our complete basis of eigenfunctions, thus providing higher order integrals of the motion. Their explicitly action on the eigenfunctions are given by

$$\begin{aligned} D_1^- \Psi(r, \theta, \phi) &= \frac{(\rho - 2\mu + 1)\alpha}{(2N + \rho + 1)} \psi_{N-1}^{\rho+2} \Theta_{\frac{\rho+1}{2}}^\mu \hat{Z}_n, \\ D_1^+ \Psi(r, \theta, \phi) &= -\frac{\alpha(N+1)(N+\rho)(\rho+2\mu-1)}{2N+\rho+1} \psi_{N+1}^{\rho-2} \Theta_{\frac{\rho-3}{2}}^\mu \hat{Z}_n, \\ D_2^- \Psi(r, \theta, \phi) &= (n+\delta)(n+\gamma)(n+\delta-2)(n+\gamma-2) \psi_N^\rho \Theta_{\frac{\rho+1}{2}}^{\mu+1} \hat{Z}_{n-1}, \\ D_2^+ \Psi(r, \theta, \phi) &= \frac{-1}{4} n(n+\delta)(n+\gamma)(n+\delta+\gamma-1)(\rho+2\mu-1) \\ &\quad \times (\rho-2\mu+1) \psi_N^\rho \Theta_{\frac{\rho-1}{2}}^{\mu-1} \hat{Z}_{n+1}. \end{aligned} \quad (3.24)$$

$$\quad \quad \quad (3.25)$$

The following commutation relations of the operators can be easily verified via the action on the eigenfunctions (3.1),

$$\begin{aligned} [D_1^-, H] = 0 = [D_1^+, H], \quad [D_1^-, H_\phi] = 0 = [D_1^+, H_\phi], \\ [D_2^-, H] = 0 = [D_2^+, H], \quad [D_2^-, H_\theta] = 0 = [D_2^+, H_\theta]. \end{aligned} \quad (3.26)$$

For the convenience we present a diagram representation of the above commutation relations

$$\begin{array}{ccc} H & & H \\ \diagdown & & \diagdown \\ D_1^- & \text{---} & H_\phi & \text{---} & D_1^+ \\ \diagup & & \diagup \end{array} \quad \begin{array}{ccc} H & & H \\ \diagdown & & \diagdown \\ D_2^- & \text{---} & H_\theta & \text{---} & D_2^+ \\ \diagup & & \diagup \end{array} \quad (3.27)$$

Moreover, we obtain

$$\begin{aligned} [H_\theta, D_1^-] &= \frac{1}{4}(\rho+1)D_1^-, & [H_\phi, D_2^-] &= (2\mu+1)D_2^-, \\ [H_\theta, D_1^+] &= \frac{-1}{4}(\rho-1)D_1^+, & [H_\phi, D_2^+] &= -(2\mu-1)D_2^+. \end{aligned} \quad (3.28)$$

Let us now define the higher order operators $D_i^\mp D_i^\pm$, $i = 1, 2$. We can also obtain the action of the operators $D_i^\mp D_i^\pm$, $i = 1, 2$ on the wave functions. It follows from the

construction that they are algebraically independent sets of differential operators and hence the system is superintegrable. The system also evidences a common feature of superintegrable systems in that it admits higher-order algebraic structure. A direct computation of the action of the operators D_1^\pm on the basis leads to

$$[H_\theta, D_1^-] = \frac{1}{4}(\sqrt{H_\theta} + 1)D_1^-, \quad [H_\theta, D_1^+] = -\frac{1}{4}(\sqrt{H_\theta} - 1)D_1^+, \quad (3.29)$$

$$\begin{aligned} D_1^- D_1^+ &= \frac{1}{4}[\sqrt{-\alpha^2} - \sqrt{2H}\sqrt{H_\theta} + \sqrt{2H}][\sqrt{-\alpha^2} + \sqrt{2H}\sqrt{H_\theta} - \sqrt{2H}] \\ &\quad \times [\sqrt{H_\theta} + 2\sqrt{-H_\phi} - 1][\sqrt{H_\theta} - 2\sqrt{H_\phi} - 1], \\ D_1^+ D_1^- &= \frac{1}{4}[\sqrt{-\alpha^2} - \sqrt{2H}\sqrt{H_\theta} - \sqrt{2H}][\sqrt{-\alpha^2} + \sqrt{2H}\sqrt{H_\theta} + \sqrt{2H}] \\ &\quad \times [\sqrt{H_\theta} + 2\sqrt{H_\phi} + 1][\sqrt{H_\theta} - 2\sqrt{H_\phi} + 1]. \end{aligned} \quad (3.30)$$

Similarly for the D_2^\pm operators

$$[H_\phi, D_2^-] = (2\sqrt{H_\phi} + 1)D_2^-, \quad [H_\phi, D_2^+] = -(2\sqrt{H_\phi} - 1)D_2^+, \quad (3.31)$$

$$\begin{aligned} D_2^- D_2^+ &= -\frac{1}{1024}[1 + \sqrt{H_\theta} + 2\sqrt{H_\phi}][-1 + \sqrt{H_\theta} - 2\sqrt{H_\phi}] \\ &\quad \times [1 + 2\sqrt{H_\phi} - \gamma - \delta][-1 + 2\sqrt{H_\phi} + \gamma - \delta][1 + 2\sqrt{H_\phi} + \gamma - \delta] \\ &\quad \times [3 + 2\sqrt{H_\phi} + \gamma - \delta][-1 + 2\sqrt{H_\phi} - \gamma + \delta][1 + 2\sqrt{H_\phi} - \gamma + \delta] \\ &\quad \times [3 + 2\sqrt{H_\phi} - \gamma + \delta][-1 + 2\sqrt{H_\phi} + \gamma + \delta], \\ D_2^+ D_2^- &= -\frac{1}{1024}[-1 + \sqrt{H_\theta} + 2\sqrt{H_\phi}][1 + \sqrt{H_\theta} - 2\sqrt{H_\phi}] \\ &\quad \times [-1 + 2\sqrt{H_\phi} - \gamma - \delta][-3 + 2\sqrt{H_\phi} + \gamma - \delta][-1 + 2\sqrt{H_\phi} + \gamma - \delta] \\ &\quad \times [1 + 2\sqrt{H_\phi} + \gamma - \delta][-3 + 2\sqrt{H_\phi} - \gamma + \delta][-1 + 2\sqrt{H_\phi} - \gamma + \delta] \\ &\quad \times [1 + 2\sqrt{H_\phi} - \gamma + \delta][-3 + 2\sqrt{H_\phi} + \gamma + \delta]. \end{aligned} \quad (3.32)$$

So the above higher order algebraic structure is the full symmetry algebra for the superintegrable system (3.1).

3.4 Higher rank polynomial algebra

In this subsection we will redefine the operators in sense of [6] and show that they form a well-defined higher rank polynomial algebra. We now define the following operators

as

$$\begin{aligned} J_1 &= \frac{D_1^- - D_1^+}{\rho}, & J_2 &= D_1^- + D_1^+, \\ K_1 &= \frac{D_2^+ - D_2^-}{2\mu}, & K_2 &= D_2^- + D_2^+. \end{aligned} \quad (3.33)$$

It is easily verified that

$$\begin{aligned} [J_1, H] &= 0 = [J_2, H], & [J_1, H_\phi] &= 0 = [J_2, H_\phi], \\ [K_1, H] &= 0 = [K_2, H], & [K_1, H_\theta] &= 0 = [K_2, H_\theta]. \end{aligned} \quad (3.34)$$

The commutation relations also can be represented by the following diagrams

$$\begin{array}{ccc} & H & \\ & \vdots & \\ J_1 & \text{---} H_\phi \text{---} & J_2 \end{array} \quad \begin{array}{ccc} & H & \\ & \vdots & \\ K_1 & \text{---} H_\theta \text{---} & K_2 \end{array} \quad (3.35)$$

We obtain the following, still quantum-number dependent, commutation relations

$$\begin{aligned} [H_\theta, J_1] &= \frac{1}{4}(J_1 + J_2), & [H_\theta, J_2] &= \frac{1}{4}(\rho^2 J_1 + J_2), \\ [H_\phi, K_1] &= K_1 + K_2, & [H_\phi, K_2] &= (2n + \delta + \gamma - 1)^2 K_1 + K_2. \end{aligned} \quad (3.36)$$

These can be expressed back in terms of the algebra generators as

$$\begin{aligned} [H_\theta, J_1] &= \frac{1}{4}(J_1 + J_2), & [H_\theta, J_2] &= \frac{1}{4}(H_\theta J_1 + J_2), \\ [H_\phi, K_1] &= K_1 + K_2, & [H_\phi, K_2] &= 4H_\phi K_1 + K_2. \end{aligned} \quad (3.37)$$

The last set of algebra relations to recover are the commutators $[J_1, J_2]$ and $[K_1, K_2]$. A first step is to see the following relations from the action on the eigenfunctions

$$[J_1, J_2] = \frac{2}{\sqrt{H_\theta}}[D_1^-, D_1^+], \quad [K_1, K_2] = \frac{1}{\sqrt{H_\phi}}[D_2^+, D_2^-]. \quad (3.38)$$

We can rewrite the expressions (3.30) as

$$\begin{aligned} D_1^- D_1^+ &= P_1(H, H_\theta, H_\phi) + P_2(H, H_\theta, H_\phi)\sqrt{H_\theta}, \\ D_1^+ D_1^- &= P_1(H, H_\theta, H_\phi) - P_2(H, H_\theta, H_\phi)\sqrt{H_\theta}, \end{aligned} \quad (3.39)$$

where

$$\begin{aligned} P_1(H, H_\theta, H_\phi) &= -\frac{1}{4}[16H - 64HH_\theta + 32HH_\theta^2 + 16HH_\phi - 32HH_\theta H_\phi \\ &\quad + 2\alpha^2 - 4H_\theta\alpha^2 + 4H_\phi\alpha^2], \\ P_2(H, H_\theta, H_\phi) &= \frac{1}{4}[16H - 32HH_\theta + 16HH_\phi + 2\alpha^2]. \end{aligned} \quad (3.40)$$

Hence we have

$$[D_1^-, D_1^+] = 2P_2(H, H_\theta, H_\phi)\sqrt{H_\theta}, \quad (3.41)$$

$$\{D_1^-, D_1^+\} = 2P_1(H, H_\theta, H_\phi). \quad (3.42)$$

Also, the expressions (3.32) can be written as

$$\begin{aligned} D_2^- D_2^+ &= P_3(H_\theta, H_\phi) + P_4(H_\theta, H_\phi)\sqrt{H_\phi}, \\ D_2^+ D_2^- &= P_3(H_\theta, H_\phi) - P_4(H_\theta, H_\phi)\sqrt{H_\phi}, \end{aligned} \quad (3.43)$$

where

$$\begin{aligned} P_3(H_\theta, H_\phi) &= -\frac{1}{1024}[(\gamma + \delta - 1)^2 - 4H_\phi][\{(\gamma - \delta)^2 - 9\}(H_\theta - 1) \\ &\quad - 4\{(\gamma - \delta)^2 + H_\theta - 22\}H_\phi + 16H_\phi^2][(\gamma - \delta)^4 + (4H_\phi - 1)^2 \\ &\quad - 2(\gamma - \delta)^2(4H_\phi + 1)], \\ P_4(H_\theta, H_\phi) &= \frac{1}{256}[(\gamma + \delta - 1)^2 - 4H_\phi][(\gamma - \delta)^4 + (1 - 4H_\phi)^2 - 2(\gamma - \delta)^2(1 + 4H_\phi)] \\ &\quad \times [(\gamma - \delta)^2 + 3H_\theta - 4(3 + 4H_\phi)]. \end{aligned} \quad (3.44)$$

Hence we also have

$$[D_2^-, D_2^+] = 2P_4(H_\theta, H_\phi)\sqrt{H_\phi}, \quad (3.45)$$

$$\{D_2^-, D_2^+\} = 2P_3(H_\theta, H_\phi). \quad (3.46)$$

Thus, we realize the final set of algebra relations as

$$[J_1, J_2] = 4P_2(H, H_\theta, H_\phi), \quad [K_1, K_2] = 2P_4(H_\theta, H_\phi). \quad (3.47)$$

Thus, we have shown that the operators H , H_θ , H_ϕ , K_1 , K_2 , J_1 and J_2 close to form a higher-rank polynomial algebra. Finally, we mention that it is possible to show that these operators are well-defined and can be expressed without recourse to the action on the wave-functions. This is accomplished via the usual observation that the operators constructed are polynomial in ρ^2 and μ^2 and so these can be replaced with the appropriate operators and the algebra relations will still hold.

3.5 Deformed oscillators, structure functions and spectrum

In order to derive the spectrum using the algebraic structure, we realize the substructure (3.29) and (3.30) as well as the substructure (3.31) and (3.32), respectively, in terms of deformed oscillator algebra [36, 39] $\{\aleph, b^\dagger, b\}$ of the form

$$[\aleph, b^\dagger] = b^\dagger, \quad [\aleph, b] = -b, \quad bb^\dagger = \Phi(\aleph + 1), \quad b^\dagger b = \Phi(\aleph), \quad (3.48)$$

where \aleph is the number operator and $\Phi(x)$ is well behaved real function satisfying

$$\Phi(0) = 0, \quad \Phi(x) > 0, \quad \forall x > 0. \quad (3.49)$$

We recall (3.29) and (3.31) in the following forms

$$[\sqrt{H_\theta}, D_1^\mp] = \pm 2D_1^\mp, \quad [\sqrt{H_\phi}, D_2^\pm] = \pm D_2^\pm. \quad (3.50)$$

Setting

$$\sqrt{H_\theta} = 2(\aleph_1 + u_1), \quad b_1 = D_1^+, \quad b_1^\dagger = D_1^-, \quad (3.51)$$

$$\sqrt{H_\phi} = (\aleph_2 + u_2), \quad b_2 = D_2^-, \quad b_2^\dagger = D_2^+, \quad (3.52)$$

where $u_1 > 0$ and $u_2 > 0$ are some representation dependent constants, we obtain from (3.30) and (3.32),

$$\begin{aligned} [\aleph_i, b_i^\dagger] &= b_i^\dagger, & [\aleph_i, b_i] &= -b_i, \quad i = 1, 2, \\ b_1 b_1^\dagger &= \Phi_1(\aleph_1 + 1, \aleph_2 + 1, H, u_1, u_2), & b_1^\dagger b_1 &= \Phi_1(\aleph_1, \aleph_2, H, u_1, u_2), \\ b_2 b_2^\dagger &= \Phi_2(\aleph_1 + 1, \aleph_2 + 1, u_1, u_2), & b_2^\dagger b_2 &= \Phi_1(\aleph_1, \aleph_2, u_1, u_2). \end{aligned} \quad (3.53)$$

The corresponding structure functions are given by

$$\begin{aligned} \Phi_1(\aleph_1, \aleph_2, H, u_1, u_2) &= b_1^\dagger b_1 = D_1^- D_1^+ \\ &= P_1(\aleph_1, \aleph_2, H, u_1, u_2) + 2P_2(\aleph_1, \aleph_2, H, u_1, u_2)(\aleph_1 + u_1), \end{aligned} \quad (3.54)$$

$$\begin{aligned} \Phi_2(\aleph_1, \aleph_2, u_1, u_2) &= b_2^\dagger b_2 = D_2^+ D_2^- \\ &= P_3(\aleph_1, \aleph_2, u_1, u_2) + 2P_4(\aleph_1, \aleph_2, u_1, u_2)(\aleph_2 + u_2). \end{aligned} \quad (3.55)$$

At this stage, the explicit expressions of the corresponding structure functions are as follows

$$\begin{aligned} \Phi_1(\aleph_1, \aleph_2, E, u_1, u_2) &= \frac{1}{4} \left[\sqrt{-\alpha^2} - 2(\aleph_1 + u_1)\sqrt{2E} + \sqrt{2E} \right] \\ &\quad \left[\sqrt{-\alpha^2} + 2(\aleph_1 + u_1)\sqrt{2E} - \sqrt{2E} \right] \\ &\quad [2(\aleph_1 + u_1) + 2(\aleph_2 + u_2) - 1] \\ &\quad [2(\aleph_1 + u_1) - 2(\aleph_2 + u_2) - 1], \end{aligned} \quad (3.56)$$

$$\begin{aligned}
\Phi_2(\aleph_1, \aleph_2, u_1, u_2) = & -\frac{1}{1024} [-1 + 2(\aleph_1 + u_1) + 2(\aleph_2 + u_2)] \\
& [1 + 2(\aleph_1 + u_1) - 2(\aleph_2 + u_2)] [-1 + 2(\aleph_2 + u_2) - \gamma - \delta] \\
& [-3 + 2(\aleph_2 + u_2) + \gamma - \delta] [-1 + 2(\aleph_2 + u_2) + \gamma - \delta] \\
& [1 + 2(\aleph_2 + u_2) + \gamma - \delta] [-3 + 2(\aleph_2 + u_2) - \gamma + \delta] \\
& [-1 + 2(\aleph_2 + u_2) - \gamma + \delta] [1 + 2(\aleph_2 + u_2) - \gamma + \delta] \\
& [-3 - 2(\aleph_2 + u_2) + \gamma + \delta]. \tag{3.57}
\end{aligned}$$

Note that only Φ_1 contains the energy parameter E . To determine the energy spectrum, we need to construct the finite-dimensional unitary representations of (3.53). We thus impose the following constraints on the structure functions:

$$\Phi_1(p_1 + 1, p_2 + 1, E, u_1, u_2) = 0, \quad \Phi_1(0, 0, E, u_1, u_2) = 0, \tag{3.58}$$

$$\Phi_2(p_1 + 1, p_2 + 1, u_1, u_2) = 0, \quad \Phi_2(0, 0, u_1, u_2) = 0, \tag{3.59}$$

where $p_i, i = 1, 2$, are positive integers. These constraints give rise to finite-dimensional unitary presentations. We now solve the constraints (3.58) and (3.59) simultaneously. First of all, it can be readily verified that the only solution for the constraints ((3.59) is given by

$$u_2 = u_1 + \frac{1}{2}, \quad p_1 = p_2 = p, \tag{3.60}$$

where p is a positive integer. It follows that these constraints (3.58) and (3.59) lead to $(p + 1)$ -dimensional unitary representations of (3.53). Now we find solutions to the constraints (3.58) which satisfy (3.60). This will provide us the energy spectrum E of the system as well as the allowed values of the parameters u_1 and u_2 . After some computations, we obtain all allowed values of the energy E and the parameters u_1, u_2 as follows.

$$E = -\frac{\alpha^2}{2(p+1)^2}, \quad u_1 = \frac{1}{2} + \frac{p+1}{2}, \quad u_2 = 1 + \frac{p+1}{2}. \tag{3.61}$$

Here $p = 0, 1, \dots$, is any positive integer. Making the identification $p = 2(N + m) + 1$, the energy spectrum becomes (2.22).

4 Conclusion

In this paper, we see the construction of a new, exactly-solvable system with wave functions comprised of products of Laguerre, Legendre and exceptional Jacobi polynomials.

This system is a perturbation of a superintegrable system, the singular Coulomb system. We show that the system is superintegrable by constructing 7 integrals of motion and show that the algebra closes to form a polynomial algebra. The construction of the higher order integrals of the motion is a systematic constructive approach from ladder operators which based on (exceptional) orthogonal polynomials. We also discuss the representations of this new algebra via the deformed oscillator method to derive spectra and degeneracies of unitary representations.

It would be of interest to further investigate this system further, in particular to understand how the singular term behaves in the classical limit. Here we have normalized $\hbar = 1$, if it is reintroduced the system will depend non-trivially on this parameter. Also of interest could be to understand the scattering states and how the exception perturbation affects those.

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