From classical absolute stability tests towards a comprehensive robustness analysis

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Vorgelegt von

Matthias Fetzer

aus Rottweil

Hauptberichter: Prof. Dr. Carsten W. Scherer
Mitberichter: Prof. Dr. Arjan van der Schaft

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Stuttgart, June 2017 *Matthias Fetzer*

Abstract

In this thesis, we are concerned with the stability and performance analysis of feedback interconnections comprising a linear (time-invariant) system and an uncertain component subject to external disturbances. Building on the framework of integral quadratic constraints (IQCs), we aim at verifying stability of the interconnection using only coarse information about the input-output behavior of the uncertainty.

In the first part of the thesis, we establish a comprehensive framework for global stability and performance analysis on general function spaces that significantly widens the range of applications if compared to standard IQC theory. Furthermore, our novel approach allows to flexibly combine and also improve on all multiplier based stability criteria available in the literature for the classical problem of absolute stability analysis, i.e., the case where the uncertain system is defined via a slope-restricted or sector-bounded nonlinearity.

By forging a strong and very general link to the theory of dissipation as developed by Willems, we demonstrate for the first time in the second part of the present thesis that general IQC theory can indeed be extended towards local analysis of feedback interconnections. This is achieved by a reformulation of IQC theory in a trajectory based setting that opens the way for the application of standard Lyapunov type arguments. Hence, we can now employ input-output descriptions of uncertainties in order to robustly verify and guarantee hard state and

output constraints on the linear part of the interconnection depending on the set of possible disturbances.

Zusammenfassung

DAS zentrale Thema dieser Arbeit ist die Stabilitäts- und Güteanalyse von Rückkopplungssystemen, die aus einem linearen (zeitinvarianten) Element und einer unsicheren Komponente bestehen und darüber hinaus externen Störungen ausgesetzt sind. Auf die Theorie der sogenannten integral quadratic constraints (IQCs) aufbauend, verfolgen wir das Ziel Stabilitätsaussagen lediglich auf Basis relativ grober Informationen über das Eingangs-Ausgangsverhalten der Unsicherheit zu treffen.

Der erste Teil dieser Arbeit ist der Entwicklung einer strukturierten und umfassenden Vorgehensweise zur globalen Stabilitäts- und Güteanalyse auf allgemeinen Funktionenräumen gewidmet. Hierdurch kann ein weitaus größeres Anwendungsgebiet als durch klassische IQC Theorie erschlossen werden. Für den konkreten Fall, dass die Unsicherheit durch eine sektoriell- oder steigungsbeschränkte Nichtlinearität definiert ist, ermöglicht es unsere neuartige Herangehensweise, alle in der Literatur verfügbaren und auf Multiplikatoren basierenden Stabilitätskriterien flexibel zu kombinieren und darüber hinaus zu verbessern.

Indem wir einen direkten und sehr allgemeinen Zusammenhang zur Willems'schen Dissipationstheorie herstellen, zeigen wir im zweiten Teil unserer Arbeit zum ersten Mal, dass allgemeine IQC Theorie sogar auf die lokale Analyse von Rückkopplungssystemen angewendet werden kann. Die Grundlage hierfür bildet eine Trajektorien basierte Formulierung der IQC Theorie, welche uns eine Kombination mit Standard Lyapunov

Argumenten erlaubt. Infolgedessen ist es uns nun möglich, lediglich auf Basis von Eingangs-Ausgangsbeschreibungen der Unsicherheiten, harte Zustands- oder Ausgangsbeschränkungen an den durch eine Zustandsraumdarstellung gegebenen linearen Teil des Rückkopplungssystems zu garantieren.

Contents

Chapter 1

Motivation and contributions

1.1 Some words on robustness analysis

One of the most important aspects of control theory is the understanding of the behavior of a dynamical system, which is usually defined by its processing properties that define the response or output to some external excitation or input (see Figure 1.1). In this setup, both the input and the output are functions of time and also the dynamic properties of the system may vary over time. A frequently taken approach to understanding the system characteristics relies on stimulating it with some input and observing the corresponding output, which motivates the abstract description of a dynamical system as an operator mapping input signals into outputs. This rather general viewpoint is one of the major reasons for the successful application of control theory concepts far beyond engineering applications, and in such diverse areas as, e.g., the social behavior of humans [30, 31], the manipulation of quantum states [114, 41, 99], and system theoretical understanding of biological processes [47, 179, 73].

Very often such abstract maps that transform an input signal into the output, can be described using differential (or difference) equations

Figure 1.1: Dynamical system

with fixed initial conditions. In a first step towards understanding and also predicting the dynamical behavior of a system, we, hence, stimulate the system and measure its corresponding response in order to model the underlying generating principles, i.e., we describe its input-output behavior using differential equations.

Another approach of obtaining a system model that is often used in engineering applications aims to deduce the overall behavior of the system from physical models of the individual components, which are again formulated as differential equations. Obviously, this approach relies on the a priori knowledge of the internal structure of the system that is not necessary for the above described input-output approach.

However, independent of the chosen path we will not be able to arrive at an exact quantification, which gives rise to discrepancies between the true dynamical behavior of the system and that of our approximate model. These intrinsic deviations are the fundamental reason behind the classical paradigm of robust control, the assumption of a so-called uncertain system; a nominal part of the system N, our best guess, and an uncertain component Δ, with which we take care of all effects that may have led to a wrong nominal system such as, e.g., modelling errors and measuring inaccuracies.

A further complication arises from the fact that highly accurate models are typically rather hard or even impossible to handle in terms of simulation, analysis or control. Thus we strive for models that describe the true behavior sufficiently well, while keeping the level of complexity as low as possible. Here, a natural first step is to assume a linear and also time-invariant, nominal model behavior, described by linear time-invariant differential equations. This allows us to rely on the large variety of tools developed in the well-established field of linear control theory.

However, there are no linear systems in real life. In addition, the nonlinear behavior of a system often critically influences the whole dynamics. As a consequence, the assumption that N is linear is paid for by subsuming nonlinearities like saturations, delays, parametric uncertainties or other unmodeled dynamics into the uncertain part of the system. In conclusion, Δ now comprises the intrinsic nonlinear behavior in addition to the mismatch between the real-life system and the model. Translated into the standard configuration of robust control, we approximate the setting depicted in Figure 1.1 by considering a nominal linear time-invariant (LTI) system N interconnected with an uncertainty Δ which is assumed to be contained in a set of possible systems $\boldsymbol{\Delta}$. This leads to the interconnection depicted in Figure 1.2.

Having obtained such a system model, we may now now proceed to the analysis of its behavior. Here the most important question are those of stability and performance; does the system operate in a safe way and does it respond to external disturbances in a desired way. In view of our uncertain system approach, stability and performance analysis for the system in Figure 1.1 now translate into robust stability and robust performance analysis of the interconnection in Figure 1.2, i.e., into the verification of stability and performance of the interconnection for all $\Delta \in \boldsymbol{\Delta}$.

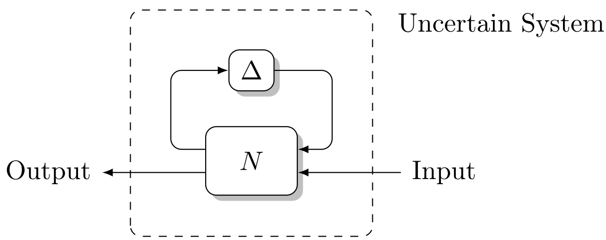

Figure 1.2: Approximated dynamical system

The input-output perspective of a dynamical system as a black box that somehow processes input data is now of pivotal importance as it allows to capture the nonlinear, uncertain, time-varying (or in any other way troubling) part of the system solely by means of their input-output relations. As a consequence, we do not need to have any information about the internal dynamics of the uncertainty Δ. The ability to capture the true and often complex system behavior by means of easier to handle yet possibly coarse input-output descriptions, is one of the reason for the successful application of robust analysis ideas in all areas of control. Furthermore, the obtained robustness guarantees are indispensable in many practical applications. A framework specifically designed for the analysis of systems containing an LTI part and a troubling one, that is only know to satisfy some constraints on its inputs and outputs, is that of integral quadratic constraints (IQCs) as briefly reviewed and put into its historical perspective in Chapter 2. Here, the exclusive focus lies on the so-called global robust stability and performance analysis, i.e., the analysis of the input-output behavior of the uncertain system without any restriction on the input signals.

However, this global perspective also comes at the expense of a major disadvantage. If we describe N and Δ only by their input-output relations it is intrinsically impossible to quantify the consequences for the internal dynamics in N due to external stimuli d and uncertain dynamics $\Delta \in \mathbf{\Delta}$. In many practical applications, the nominal system N is given by a (finite dimensional) state-space description, where the states have physical meaning. Furthermore, the inputs are often know to satisfy certain properties which allows to focus on a very particular set of signals. In conclusion, it is desirable to guarantee not only a certain input-output behavior but also the satisfaction of robust state constraints, i.e., the guaranty that the states do not exceed some limit value, under the assumption that the inputs are restricted to a given set. The extension of the framework of IQCs, which is exceptionally well suited for the previously described global analysis of feedback interconnections to this so-called local analysis is, however, an open problem [29].

1.2 Main goal of the thesis

Robust or absolute stability analysis of interconnections as depicted in Figure 1.2 is a widely and still actively researched field that has its roots in the 1960s. However, despite the early and also later efforts to achieve a comprehensive and unifying analysis theory, the last decades have witnessed the decoupling of this research area into various only loosely connected fractures. This is, at least in part, due to the fact that there does not seem to exist a framework that is, on the one hand, general enough to provide answers for a wide range of research problems while, at the same time, being powerful enough to be able to outperform specialized approaches. This motivates us to formulate the key objectives of the present thesis as follows:

> *The main goal of this thesis is to develop a comprehensive approach for robust stability and performance analysis. It is aimed for a framework that permits both global and local analysis of feedback interconnections, and furthermore provides the means for their efficient numerical verification.*

1.3 Outline and contribution

The above described goals are also reflected in the two parts of the present thesis where we focus on global and local analysis separately. In the first part, which is concerned with the verification of global properties, we focus on the extension and improvement of the classical framework of integral quadratic constraints. The presented generalizations comprise both a widening of the area of applications for IQC theory as well as an enhancing of the analysis tools themselves. In the second part, we consider local stability and performance analysis by establishing a connection between classical global IQC theory and local Lyapunov techniques by means of dissipation arguments. In the remainder of this section, we discuss the individual contributions in the respective chapters in some detail and finally point the reader to the already published or submitted contributions that, to some extent, constitute this thesis.

In Chapter 2 we discuss the historical roots and gradual development of the framework of IQCs as finally presented in [110]. This chapter is also designed to highlight the struggle for a unifying theory of stability and to discuss the milestone results achieved in this area.

In Chapter 3 we introduce a first generalization of classical IQC theory that significantly widens its range of applications. In particular, the following contributions are made:

- a generalization of the IQC framework that allows for the rigorous inclusion of uncertainties admitting sampling behavior;
- a lossless reduction of the occurring infinite dimensional FDI into a finite dimensional LMI that permits the verification of stability using standard solvers;
- a treatment of pulse-width modulators within the novel framework.

In Chapter 4 we extend the ideas outlined in Chapter 3 and present a comprehensive framework for global robust stability and performance analysis that extends the classical one in several respects as it

- allows for very general function spaces;
- permits the seamless incorporation of operators, constraints and performance measures on Sobolev spaces;
- provides the means for numerical verification of stability and performance in this more general setting.

In Chapter 5, we illustrate how the many advantages offered by our novel formulation come to flourish for the particular case of continuous-time feedback interconnections containing repeated slope-restricted nonlinearities. The contributions are

- a novel full-block Yakubovich criterion;

- an asymptotically exact parametrization of full-block Zames-Falb multipliers;
- a combined application of all multiplier based stability criteria available;
- an extension of the Popov and Yakubovich criteria to not strictly proper LTI systems.

With Chapter 6 we conclude the first part of this thesis by highlighting the simplicity with which the IQC framework, and also our novel extension thereof, allows to translate results from the continuous-time setting into the discrete-time one. The contributions of this chapter are

- a novel unstructured polytopic criterion that combines Yakubovich and circle criterion multipliers;
- a classification of various stability results in the literature;
- a combined application of all multiplier based stability criteria available similar to the continuous-time case.

In the second part of the thesis, we shift our focus towards local analysis of feedback interconnections by establishing in Chapter 7 a strong and very general link to classical dissipation theory which allows us to connect operator based IQC descriptions, as employed in the first part, to trajectory based Lyapunov arguments. This relies on the following contributions

- a local IQC stability theorem;
- a novel link between general dynamic multipliers and Lyapunov theory;
- a complete framework for local stability and performance analysis within IQC theory;
- several local performance criteria and their derivation from our local IQC theorem.

In Chapter 8 we conclude the contributions of this thesis by embedding the Zames-Falb multipliers discussed in Chapter 5 into the local analysis framework of Chapter 7. This is based on

- a simple and computationally effective hard IQC factorizations of causal and anti-causal Zames-Falb multipliers;
- a refined local analysis framework that allows for non-conservative incorporations of arbitrary hard IQCs.

As the topics discussed in the individual chapters are rather diverse, we aim at keeping the chapters reasonably self-contained such that each may be read independently from the others. Of course, the price to pay is some overlap between the chapters which is kept to a minimum.

Finally, we emphasize that the findings presented in the first part are the result of several papers that have already been published in conference proceedings and journals and some parts of the text overlap. In particular, this comprises the references [33, 55] for Chapter 3, [58, 56] for Chapters 4 and 5, as well as [57] for Chapter 6. Furthermore, the results presented in Chapters 7 and 8 have been submitted for publication in [60] and [59] and are currently under review.

Part I

A comprehensive framework for global analysis of feedback interconnections

Introduction to Part I

The first part of this thesis is devoted to the problem of global stability and performance analysis of feedback interconnections containing an LTI system M and an uncertain part Δ subject to external stimuli u, v (see Figure 1.3).

As already mentioned in the introduction, we do not require explicit knowledge about the system Δ but only rather coarse information concerning its input-output properties. These typically apply to a whole set of uncertain operators $\mathbf{\Delta}$. Moreover, we consider the external signals u, v as disturbances acting on the feedback interconnection and aim at concluding stability for all possible inputs and all $\Delta \in \mathbf{\Delta}$. Our main goal will be the development of a general analysis framework that still permits the application of standard optimization tools for the efficient numerical verification of stability and also of given performance specifications.

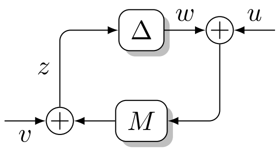

Figure 1.3: Uncertain feedback interconnection

As this problem has been the subject of a multitude of papers, their contributions ranging from the presentation of general analysis frameworks to the consideration of very particular settings, we illustrate the historical development of (robust) stability tests in some detail in Chapter 2. In order to concisely highlight the fundamental concepts common to all multiplier based stability results, we restrict our attention to the successive evolvement of the framework of integral quadratic constraints and, thus, provide the foundation for all subsequent chapters of this thesis.

Our refinement of the standard IQC approach in Chapter 3 that allows to incorporate uncertainties exhibiting sampling behavior into the theory, ultimately leads to a general framework for stability and performance analysis presented in Chapter 4. The central characteristics of our novel framework is its ability to incorporate disturbances and operators defined on very general function spaces. As one of the main contributions of Part I, we subsequently present a unifying approach to the classical and very fundamental problem of absolute stability analysis in Chapter 5. Here Δ is defined via a nonlinearity $\varphi : \mathbb{R} \to \mathbb{R}$ that is either sector-bounded or slope-restricted. Our comprehensive framework allows to subsume all multiplier based stability criteria and thus presents the least conservative stability estimates available in the literature. Finally, we illustrate how the results developed in Chapter 5 carry over to discrete-time interconnections in Chapter 6 thus allowing to classify and outperform the multitude of classical and more recently proposed stability tests.

Chapter 2

The framework of integral quadratic constraints

BEFORE we present the main contributions of this thesis in the subsequent chapters, let us first take one step back and highlight some of the many earlier results this work is build upon. As the framework of integral quadratic constraints (IQCs) is at the very heart of this thesis, we devote this chapter to its historical roots in absolute stability analysis, the classical framework itself and also related approaches in the literature.

2.1 A historical perspective

The goal of robust stability analysis within the input-output framework is probably best described by Zames [195]:

> "It seems possible, from only coarse information about a system, and perhaps even without knowing details of internal structure, to make useful assessments of qualitative behavior."

Historically, robust or absolute stability analysis may be traced back to the paper of Lurye and Postnikov [106]. Due to the overwhelming number of contributions in this rather actively researched field, it is hopeless to give an exhaustive overview of the developments in the past 60 years. Consequently, we will only discuss the major milestones that sparked the development of the framework of IQCs as eventually formulated in [110]. We also recommend the insightfully written introduction to the problem given in the monograph [10]. Another attempt to give a comprehensive summary of the historical development was made by Liberzon [105] that, unfortunately, loses much of its readability by trying to do justice to all the contributors in the field. Nevertheless, it remains a valuable source of information.

The problem posed by Lurye and Postnikov [106] can be stated as follows (for a collection of early papers see [107] and also the excellent monograph [8]). Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous nonlinearity confined to the sector $\sec[0, \beta]$, i. e., there exist $\beta > 0$ such that

$$\varphi(x)(\beta x - \varphi(x)) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}. \tag{2.1}$$

With real matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$ and $C \in \mathbb{R}^{1 \times n}$ and Δ_φ defined through $\Delta_\varphi(z)(t) := \varphi(z(t))$ for all locally square integrable signals $z \in \mathscr{L}_{2e}$ and almost all $t \in [0, \infty)$, consider the feedback system

$$\begin{aligned} &\dot{x} = Ax + Bw, \qquad x(0) = x_0, \qquad\qquad w = \Delta_\varphi(z), \\ &z = Cx. \end{aligned} \tag{2.2}$$

The objective is to determine conditions on the linear part of the interconnection, i.e., A, B and C, that guarantee global asymptotic stability of the trivial solution $x = 0$ by exploiting the input-output properties of Δ_φ. If M denotes the linear part in (2.2) and we visualize the initial condition as an input to M, the interconnection (2.2) may be depicted as in Figure 2.1.

Popov obtained a sufficient condition, by only exploiting (2.1) and perfectly fitting with the characterization of Zames quoted above, for the stability of (2.2) that only involves the transfer function of the

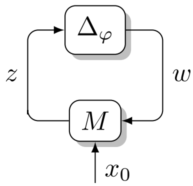

Figure 2.1: Lurye feedback interconnection

linear part, i.e., $M(i\omega) := C(i\omega I - A)^{-1}B$ for $\omega \in \mathbb{R}$. In fact, stability is guaranteed if there exists a positive ε and a real[1] λ such that

$$\begin{pmatrix} M(i\omega) \\ I \end{pmatrix}^* \begin{pmatrix} 0 & 1-\lambda i\omega \\ 1+\lambda i\omega & -\frac{2}{\beta} \end{pmatrix} \begin{pmatrix} M(i\omega) \\ I \end{pmatrix} \preccurlyeq -\varepsilon I \quad \text{for all} \quad \omega \in \mathbb{R}. \tag{2.3}$$

The solution proposed by Popov is remarkable in several respects. First, like the other celebrated frequency-domain criterion known at the time, the Nyquist criterion [119], it has a geometric counterpart that allows to verify stability based on a graphical test (see, e.g., [8, p. 53]). As the transfer function of a system can often be extracted from physical experiments, this provides the means for efficient stability analysis. From a more global perspective, the Popov criterion marked the first instance of the use of so-called multipliers for the analysis of feedback stability, thus paving the way for many central stability principles based on the so-called passivity theorem.

In order to proceed one step further towards a unifying theory of stability that comprises but also extends Popov's result, let us now discuss the contribution of Yakubovich. Shortly after Popov proposed his stability theorem, several others emerged among which are the circle criterion (see, e.g., [132, 18, 141, 142, 196]) and the small-gain theorem [194, 142]. These results raised the fundamental question whether there exists a general underlying principle that is common to

[1] Popov actually required $\lambda > 0$ in his seminal paper [126], but this was immediately recognized to impose an unnecessary restriction (see, e.g., [8]).

all stability criteria. One attempt to unify the aforementioned criteria was made by Yakubovich who developed a general framework [189] that allows to merge the circle, small-gain and Popov criterion in the case where M is LTI. The key aspect of this generalization is the collection of all information about the nonlinearity using quadratic forms. For example, the so-called sector constraint (2.1) implies the validity of the pointwise quadratic constraint

$$\begin{pmatrix} z(t) \\ \Delta_\varphi(z)(t) \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & -\frac{2}{\beta} \end{pmatrix} \begin{pmatrix} z(t) \\ \Delta_\varphi(z)(t) \end{pmatrix} \geq 0 \quad \text{for all} \quad z \in \mathscr{L}_2 \tag{2.4}$$

and almost all $t \in [0, \infty)$. Note that the middle matrix in (2.4) exactly matches the one in (2.3) for $\lambda = 0$, a key fact in the proof of Popov's stability result. Extending this idea, Yakubovich assumes that Δ_φ satisfies the (integral) quadratic constraint[2]

$$\int_0^T \begin{pmatrix} z(t) \\ \Delta_\varphi(z)(t) \end{pmatrix}^T \Pi \begin{pmatrix} z(t) \\ \Delta_\varphi(z)(t) \end{pmatrix} \geq 0 \quad \text{for all} \quad T \geq 0,\ z \in \mathscr{L}_2 \tag{2.5}$$

defined by a constant symmetric matrix Π. Simply put, Yakubovich proves that stability of (2.2) is then guaranteed if there exists $\varepsilon > 0$ such that the stable LTI system M satisfies the frequency-domain inequality (FDI)

$$\begin{pmatrix} M(i\omega)\hat{w}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix}^* \Pi \begin{pmatrix} M(i\omega)\hat{w}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix} \leq -\varepsilon \hat{w}(i\omega)^* \hat{w}(i\omega) \tag{2.6}$$

for all $\omega \in \mathbb{R}$ and all $w \in \mathscr{L}_2$. We emphasize that in the above formulation the matrix Π does not depend on time (or frequency) as is the case in (2.3). Hence, Popov's criterion is not contained so far. By a somewhat artificial addition of a frequency dependent term, Yakubovich was able to extend his criterion such that it also incorporates the

[2] The major stability theorem in [189] is formulated with pointwise quadratic constraints. Yet, as remarked by Yakubovich, the proof only requires the less restrictive integral constraint (2.5).

one proposed by Popov. Although the proof of stability within this framework relies on a Lyapunov argument, the ideas developed in [189] are essential for the later established framework of integral quadratic constraints. In particular, the approach by Yakubovich already allows to combine different individual stability criteria in order to enhance the resulting stability test.

Equally fundamental as the contributions by Popov and Yakubovich are those by Sandberg [141, 142, 143] and Zames [195, 196], who, in contrast to the predominantly pursued Lyapunov approach, applied functional analytic methods to the robust stability problem. As these papers are obviously closely related, and appeared at roughly the same time, it is now impossible to assign the individual contributions to the respective author. In the following, we discuss the general theory outlined in the seminal papers by Zames [195, 196].

The key problem considered by Zames and Sandberg can be summarized as follows. Given two dynamical systems M and Δ, defined as arbitrary maps from one function space into another[3] that are interconnected as

$$z = M(w) + v, \quad w = \Delta(z) + u, \tag{2.7}$$

with external disturbances u, v, find conditions on M, Δ such that the maps from $v \to z$ and $u \to w$ are bounded. The pivotal difference to the by then already well developed Lyapunov method lies in the mathematical formulation of this problem in the input-output setting which completely avoids details of the internal structure and considers the interconnection as an open system, able to interact with its surroundings. This point of view instrumentally relied on the novel concept of extended spaces, that allows to analyze situations where the signals in the loop are not a priori bounded (contained in a normed space) but are shown to have this property a posteriori; this is then interpreted as input-output stability.

[3] Zames actually defines systems more generally as relations, i.e., their graphs are subsets of the product of the input and the output space.

Most noteworthy among the many contributions and deep insight in [195, 196] are the formulation of three fundamental stability theorems: the small-gain theorem, the conicity (or conic relation) theorem and, formulated in [195] as a corollary of the latter, the passivity (or positivity) theorem. Intuitively, the small gain theorem states that if the loop gain is less than one, then the interconnection is stable. The conicity theorem generalizes this concept to the case where stability is guaranteed if M and Δ satisfy certain conic relations, while the passivity theorem is then obtained as a limit case of the conicity theorem.

It is exceptional that all three theorems sparked the development of entire fields within control theory. As already indicated in [196], the applicability of the passivity theorem is immensely enhanced by the idea of not only considering the systems M and Δ but by allowing a factorization of the loop into two parts that then have to satisfy the passivity conditions. This idea allowed Zames to merge his theory with the results of Popov [126] as well as Brockett and Willems [22, 23] and was further developed within multiplier theory (see, e.g., [197, 44]). In addition, the small gain theorem proved instrumental for the development of robust and H_∞ optimal control [62, 48, 49]. And finally, his result on conic relations was generalized by Safonov [138, 137] in his seminal work on topological separation that eventually lead to the formulation of the framework of integral quadratic constraints (see also [159, 70, 89, 88]).

It is also remarkable that many of the results following these early developments can be seen as natural extensions thereof. The celebrated theory of dissipativity as formulated by Willems in [182, 183], e.g., extends the results by Yakubovich (as well as those by Brockett and Willems [22, 23]) based on Lyapunov theory towards open systems. The pioneering contributions by Willems were made possible by merging Lyapunov theory with the concept of extended spaces as proposed by Zames and Sandberg. However, in contrast to Zames' work, knowledge about the interior dynamics, i.e., the assumption of a state-space description, is essential to the approach of Willems; thus it is a genuine extension of the work of Yakubovich [189]. The strength and general applicability of the ideas presented by Willems were recognized by many researches (see, e.g., [77, 78]) and they still play a fundamental role in the understanding of systems behavior [101, 12, 172]. Yet, similarly to the results by Yakubovich, and to a certain extent also those by Safonov that are discussed next, dissipation theory suffers from the drawback that it relies on quadratic constraints of the form (2.5) that are often limiting.

In contrast to the approach by Willems, the very elegant stability theorem proposed by Safonov completely avoids the concept of interior dynamics. In [138, 137] Safonov generalized the conic relation theorem of Zames by introducing separating functionals. Following the approach by Zames, the central result is formulated on general extended spaces and applicable to systems defined through relations. For simplicity, let us illustrate the key ideas of [138, 137] for the special case where M and Δ are operators defined on the extended space $\mathscr{L}_{2e}$. Then, stability of (2.7) is guaranteed if there exists a separating functional $\sigma : \mathscr{L}_2 \times \mathscr{L}_2 \to \mathbb{R}$ such that for all $T > 0$ the following two conditions hold[4]

a) $\sigma(z_T, (\Delta(z))_T) \geq 0$ for all $z \in \mathscr{L}_{2e}$;

b) there exists some $\varepsilon > 0$ such that

$$\sigma((Mw)_T, w_T) \leq -\varepsilon\big(\|w_T\|^2 + \|(Mw)_T\|^2\big) \quad \text{for all} \quad w \in \mathscr{L}_{2e}.$$

As a significant contribution, the general version of this result [137, Theorem 2.1] is not only applicable in the input-output setting proposed by Zames but also encompasses the Lyapunov based stability results derived by Yakubovich. Thus Safonov's contribution may be seen as the

[4] u_T denotes the truncation of $u \in \mathscr{L}_{2e}$, i.e., $u_T = u$ on $[0, T]$ and $u = 0$ on (T, ∞); see Definition 2.1.

unification (and also generalization) of both classical theories. Indeed, by choosing

$$\sigma(z,w) := \int_0^\infty \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}^T \Pi \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} dt \quad \text{for} \quad z, w \in \mathscr{L}_2,$$

we obtain the stability criterion by Yakubovich for LTI systems M. Moreover, the conditions formulated by Safonov reduce the stability theorem to its essence, a positivity constraint on the graph of one operator and a strict negativity constraint on the inverse graph of the other. It is this fundamental principle that all known stability theorems follow.

However, in terms of the formulation of novel stability criteria, the major issue concerning applicability lies of course in the choice of the functional σ. In order to recover most of the classical stability results, Safonov uses functionals defined via an inner product and globally Lipschitz functions $S_1,\ S_2$ as

$$\sigma(z_T, w_T) = \langle S_1(z_T), S_2(w_T)\rangle \quad \text{for all} \quad z,\ w \in \mathscr{L}_{2e}. \tag{2.8}$$

Yet, as we will see in the subsequent section, this definition (and also the more general stability criteria in [137, 159, 70]) does not allow to recover all classical results, where the one proposed by Zames and Falb [197] poses a well-known exception.

Let us now proceed one step further towards the framework established in this thesis by formulating a graph separation result, similar in spirit to the one by Safonov, that also allows for quadratic forms as in (2.8). To this end, consider the following scenario: With maps M, Δ on $\mathscr{L}_{2e}$ and an external disturbance $d \in \mathscr{L}_{2e}$ we study the feedback interconnection

$$z = M(w) + d, \quad w = \Delta(z) \tag{2.9}$$

as depicted in Figure 2.2. All results presented in this thesis funda-

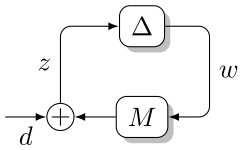

Figure 2.2: Feedback interconnection with external disturbance

mentally rely on the concept of causality that is typically defined using truncation operators.

Definition 2.1.
Let $T > 0$. Then the **truncation operator** (or past projection) $P_T : \mathscr{L}_{2e} \to \mathscr{L}_{2e}$ is defined as

$$(P_T u)(t) := u_T(t) := \begin{cases} u(t), & t \in [0, T], \\ 0, & t > T \end{cases}$$

for all $u \in \mathscr{L}_{2e}$ and almost all $t \in [0, \infty)$. For brevity of notation we write $u_T := P_T u$ for $u \in \mathscr{L}_{2e}$. An operator $S : \mathscr{L}_{2e} \to \mathscr{L}_{2e}$ is said to be **causal** if $P_T S = P_T S P_T$ holds for any $T > 0$ on $\mathscr{L}_{2e}$. ⋆

For the separating functionals $\Sigma : \mathscr{L}_2 \to \mathbb{R}$, we only assume the following property:

$$\exists c > 0 : \ \Sigma(u+v) - \Sigma(u) \leq 2c\|u\|\|v\| + c\|v\|^2 \ \text{ for all } \ u, v \in \mathscr{L}_2. \quad (2.10)$$

With these definitions, all prerequisites are assembled in order to state an intermediate stability result in the spirit of [137, 159], but already formulated it in such a way that it provides a transition to the integral quadratic constraints framework. This allows us to highlight the advantages and also the disadvantages if compared to the later presented classical IQC theory. We give a direct proof here (due to Scherer [146]) that will serve as a starting point for later more general IQC results.

Theorem 2.2 ([146])
Suppose that $M : \mathscr{L}_{2e} \to \mathscr{L}_{2e}$ is causal and bounded, $\Delta : \mathscr{L}_{2e} \to \mathscr{L}_{2e}$ is causal, $\Sigma : \mathscr{L}_2 \to \mathbb{R}$ satisfies (2.10), $\mathscr{D} \subset \mathscr{L}_{2e}$, and that

a) there exist $\epsilon > 0$ and m_0 such that

$$\Sigma \begin{pmatrix} M(w)_T \\ w_T \end{pmatrix} \leq -\epsilon \|w_T\|^2 + m_0 \quad \textit{for all} \quad T > 0, \; w \in \mathscr{L}_2; \tag{2.11}$$

b) there exists $\delta_0 \geq 0$ with

$$\Sigma \begin{pmatrix} z_T \\ \Delta(z)_T \end{pmatrix} \geq -\delta_0 \quad \textit{for all} \quad T > 0, \; z \in M(\mathscr{L}_2) + \mathscr{D}. \tag{2.12}$$

Then there exist $\gamma > 0$ and $\gamma_0 \in \mathbb{R}$ such that for any $d \in \mathscr{D}$ and any response $z \in \mathscr{L}_{2e}$ satisfying (2.9), we have

$$\|z_T\|^2 \leq \gamma^2 \|d_T\|^2 + \gamma\gamma_0 \quad \textit{for all} \quad T > 0. \tag{2.13}$$

If M is linear one can choose $\gamma_0 = m_0 + \delta_0$.

Proof. A proof is found in Appendix C.1.1. □

Remark 2.3.
In accordance with [195, 137, 159] neither existence nor uniqueness of a solution to (2.9) is assumed. Instead, the conclusion is formulated for all those disturbances $d \in \mathscr{D}$ for which the feedback interconnection does have a response. ⋆

Remark 2.4.
In the terminology of [110], Theorem 2.2 can be interpreted as a hard IQC stability result. For the particular choices $\Sigma(x_T) := \int_0^T x(t)\Pi(t)x(t)\,dt$ (with some appropriately chosen Hermitian valued and essentially bounded function $\Pi : [0, \infty) \to \mathbb{R}^{n \times n}$), $\mathscr{D} = \mathscr{L}_2$ and $\delta_0 = 0$, the constraint (2.12) reads as

$$\int_0^T \begin{pmatrix} z(t) \\ \Delta(z)(t) \end{pmatrix}^T \Pi(t) \begin{pmatrix} z(t) \\ \Delta(z)(t) \end{pmatrix} \geq 0 \quad \text{for all} \quad T > 0, \; z \in \mathscr{L}_2 \tag{2.14}$$

which is a generalization of (2.5) to time varying Π and coincides with the **hard IQC** constraint defined in [110]. The term *hard* IQC originates from the fact that (2.14) is required to hold for all $T > 0$ and not just for $T = \infty$ (as is the case for soft (time-domain) IQCs). $\star$

It is important to note that the general formulation of Theorem 2.2 for arbitrary Δ and bounded M typically renders the (numerical) verification of stability for a given separating functional Σ and two nonlinear systems M and Δ impossible. This problem is circumvented in the subsequent section by assuming a particular structure of Σ and restricting the attention to LTI systems M.

2.2 Integral quadratic constraints

Let us now introduce the framework of IQCs as established in the seminal papers [129, 110, 130, 93] by Megretski, Rantzer and Jönsson. The subsequent sections are devoted to the description of the underlying setting, the highlighting of their major contributions and also the numerical verification of stability using linear matrix inequalitys (LMIs).

2.2.1 The setting of Megretski and Rantzer

As already mentioned, the results by Megretski and Rantzer were largely motivated by the general framework developed by Yakubovich in the 1960s. By the 1990s it was very well-known (see, e.g., [182, 74, 137, 78]) that hard IQC constraints could be exploited in order to guarantee the existence of a Lyapunov function, thus providing an alternative to the functional analytic approach taken by Zames and Sandberg. Yet, there remained annoying exceptions, such as the celebrated Zames-Falb stability criterion [197], for which no hard IQC representation is known. Moreover, the application of the corresponding, typically non-causal, multipliers for stability analysis required factorizations [197, 44]. This inspired the formulation of a more general stability criterion that covers all hard IQC results as special cases. Aiming at a framework that allows for efficient numerical verification of stability, Megretski and Rantzer restricted their setting to the case where M is an LTI system. The underlying setting can then be formulated as follows.

Given an stable LTI system M as realized by

$$\dot{x} = Ax + Bw, \qquad x(0) = 0,$$
$$z = Cx + Dw$$

with A being Hurwitz, and a causal and bounded operator $\Delta : \mathscr{L}_{2e} \to \mathscr{L}_{2e}$, we consider the feedback interconnection

$$z = Mw + Mu + v, \qquad w = \Delta(z) \tag{2.15}$$

with external disturbances $u, v \in \mathscr{L}_2$. Due to the linearity of M, (2.15) is equivalent to the canonical configuration (2.7) and may be depicted as in Figure 2.3.

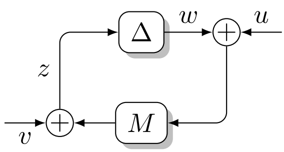

Figure 2.3: Feedback interconnection in the input-output framework

In accordance with the classical papers [195, 196] we split the questions of existence and uniqueness of solutions from the one of stability of the interconnection. Yet, in contrast to these papers which completely avoid the subject of well-posedness, we will assume it as a prerequisite. The necessary assumptions were most concisely already formulated by Zames [195] who stated in view of his work with relations instead of operators that:

> "For the results to be practically significant, it must usually be known from some other source that solutions exist and are unique (and have infinite escape times)."

All this will be reflected in the following definition of well-posedness of (2.15).

Definition 2.5 ([110]).
The interconnection (2.15) is **well-posed** if for each $(u, v) \in \mathscr{L}_{2e} \times \mathscr{L}_{2e}$ and for each $\tau \in [0, 1]$ there exists a unique $z \in \mathscr{L}_{2e}$ satisfying $z - M\tau\Delta(z) = Mu + v$ and such that the correspondingly defined response map $(u, v) \mapsto z = R_\tau(u, v)$ is causal:

$$R_\tau(u, v)_T = R_\tau(u_T, v)_T \quad \text{for all} \quad T \geq 0,\ \tau \in [0, 1],\ (u, v) \in \mathscr{L}_{2e} \times \mathscr{L}_{2e}.$$

The feedback system (2.15) is **stable** if, in addition, $R_1 : \mathscr{L}_2 \times \mathscr{L}_2 \to \mathscr{L}_2$ is bounded. ⋆

Thus, well-posedness not only requires existence and uniqueness of a solution of (2.15), but for all interconnections (2.15) where Δ is replaced by $\tau\Delta$ and $\tau \in [0, 1]$. This stronger assumption is introduced in order to deal with soft IQCs that are defined as follows: Two signals $z, w \in \mathscr{L}_2$ with Fourier transforms $\hat{z}, \hat{w}$ are said to satisfy the IQC defined by a **multiplier** $\Pi = \Pi^* \in RL_\infty$, if

$$\Sigma_\Pi \begin{pmatrix} z \\ w \end{pmatrix} = \int_{-\infty}^{\infty} \begin{pmatrix} \hat{z}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix}^* \Pi(i\omega) \begin{pmatrix} \hat{z}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix} d\omega \geq 0. \tag{2.16}$$

A causal operator $\Delta : \mathscr{L}_2 \to \mathscr{L}_2$ satisfies the IQC imposed by Π, in short $\Delta \in \mathrm{IQC}(\Pi)$, in case that

$$\Sigma_\Pi \begin{pmatrix} z \\ \Delta(z) \end{pmatrix} \geq 0 \quad \text{for all} \quad z \in \mathscr{L}_2. \tag{2.17}$$

With these preparations, we can formulate the central result in [110].

Theorem 2.6 ([110, Theorem 1])
Assume that the interconnection (2.15) *is well-posed. Then it is also stable if*

a) $\tau\Delta$ *satisfies the IQC defined by* Π *for all* $\tau \in [0, 1]$*;*

b) there exists some $\varepsilon > 0$ such that the following IQC holds:

$$\Sigma_\Pi \begin{pmatrix} Mw \\ w \end{pmatrix} \leq -\varepsilon \|w\|^2 \quad \textit{for all} \quad w \in \mathscr{L}_2. \tag{2.18}$$

Remark 2.7.
In [110] the constraint in Theorem 2.6 b) is rewritten in terms of the following FDI. There exists $\varepsilon > 0$ such that

$$\begin{pmatrix} M(i\omega) \\ I \end{pmatrix}^* \Pi(i\omega) \begin{pmatrix} M(i\omega) \\ I \end{pmatrix} \preccurlyeq -\varepsilon I \quad \text{for all} \quad \omega \in \mathbb{R}. \tag{2.19}$$

Both conditions can be shown to be equivalent if the left-hand side in (2.19) is continuous and bounded as is the case here. In the sequel, we will typically work with (2.18) since it is more suitable for generalizations and also nicely displays the symmetry in the treatment of M and Δ. Yet, as will be demonstrated in Section 2.3, (2.19) allows for the immediate translation of standard IQCs into LMIs and, thus, the verification of stability using standard optimization solvers. ⋆

Remark 2.8.
The choice of the separating function in (2.16) is of course rather particular. However, it is explicitly tailored for numerical verification and thus the practical application of Theorem 2.6. Moreover, as another contribution in [110], it contains a rather long list of relevant choices of Π, thus enabling the application of IQC theory for a wide range of problems. ⋆

If we compare Theorems 2.6 and 2.2 it emerges that the generalization toward soft IQCs is paid for by requiring both the boundedness of Δ and the well-posedness of (2.15) for Δ replaced by $\tau\Delta$ and all $\tau \in [0, 1]$. The latter is an essential ingredient to the proof that relies on a homotopy argument connecting the stable (trivial) interconnection for $\tau = 0$ with the interconnection under consideration for $\tau = 1$. There exists several generalizations that weaken both the assumptions on boundedness of Δ [152] and also the well-posedness of (2.15) [109].

One of the major drawbacks of Theorem 2.6 is that the restriction to multipliers $\Pi \in RL_\infty$ does not allow for Popov multipliers as in (2.3) since they are obviously not bounded on $i\mathbb{R}$. The remedy proposed by Jönsson is addressed in the following section.

2.2.2 The Popov criterion in the IQC framework

As in the case of the framework proposed by Yakubovich (but due to different reasons), also the IQC framework does not allow for the immediate inclusion of Popov's stability criterion. A first, rather obvious reason is the fact that the middle matrix, i.e., the multiplier, in (2.3) is unbounded on the imaginary axis (for $\lambda \neq 0$). In addition, the frequency dependent part of this multiplier given by

$$\Pi_\lambda(i\omega) = \begin{pmatrix} 0 & -i\omega\lambda \\ i\omega\lambda & 0 \end{pmatrix} \quad \text{for some real} \quad \lambda = \lambda^T \quad \text{and all} \quad \omega \in \mathbb{R} \tag{2.20}$$

was introduced by Popov in order to exploit constraints of the from

$$\int_0^\infty w(t)^T \lambda \dot{z}(t)\, dt \geq 0 \quad \text{where} \quad w = \Delta(z). \tag{2.21}$$

However, in order for this integral to make sense, the signal z has to be differentiable. Thus, if we want to capture the operation of Δ using constraints as in (2.21), the formulation (2.17), that requires non-negativity for all $z \in \mathscr{L}_2$, is too restrictive.

A solution to both problems was proposed by Jönsson. We discuss the extension of the IQC framework outlined in [93, 92] in some detail since these represent, in fact, a first step towards our subsequently derived general framework. In order to render z in (2.15) differentiable, Jönsson restricted his attention to interconnections of a strictly proper and stable LTI system M with some causal and bounded uncertainty Δ

$$\begin{aligned} &\dot{x} = Ax + Bw, \qquad x(0) = x_0, \quad w = \Delta(z) + u, \qquad u \in \mathscr{L}_2 \\ &z = Cx. \end{aligned} \tag{2.22}$$

Note that, due to the nonzero initial condition, M is not linear and hence we cannot consider this interconnection within the IQC framework directly. However, by setting $\mathscr{V} = \left\{Ce^{A\bullet}x_0 \,\middle|\, x_0 \in \mathbb{R}^n\right\}$, (2.22) is equivalent to

$$\begin{aligned} \dot{x} &= Ax + Bw, \qquad x(0) = 0, \quad w = \Delta(z) + u, \qquad (u,v) \in \mathscr{L}_2 \times \mathscr{V} \\ z &= Cx + v. \end{aligned} \tag{2.23}$$

Thus, if compared to the setting of Megretski and Rantzer, Jönsson, on the one hand, allowed for non-zero initial conditions, but on the other hand, confines the originally free input v to the set $\mathscr{V}$ containing the response due to x_0. By further exploiting the filtering property of the strictly proper plant M, the signal z in the loop (2.15) is now indeed differentiable. This leads to the following definition of well-posedness.

Definition 2.9 ([93, Definition 2]).
The interconnection (2.22) is **well-posed** if for any $\tau \in [0,1]$, any initial condition x_0, and for any input $u \in \mathscr{L}_{2e}$ there exists a solution (x,z) such that $(x,\dot{x},z) \in \mathscr{L}_{2e} \times \mathscr{L}_{2e} \times \mathscr{L}_{2e}$, where Δ is replaced by $\tau\Delta$. Furthermore, the map from u to (x,z) should be causal. ⋆

In order to circumvent the second stumbling block in applying Theorem 2.6, the fact that the multiplier (2.20) is unbounded, Jönsson added a second (bounded) multiplier $\Pi_b = \Pi_b^* \in RL_\infty$, which allowed him to prove the following theorem.

Theorem 2.10 ([93, Theorem 1])
Assume that the interconnection (2.15) *is well-posed. Further let*

a) there exist $\delta > 0$ such that for all $z \in \mathscr{L}_2$ with $\dot{z} \in \mathscr{L}_2$ the following constraint is satisfied:

$$\Sigma_{\Pi_\lambda}\begin{pmatrix} z \\ \Delta(z) \end{pmatrix} + \Sigma_{\Pi_b}\begin{pmatrix} z \\ \Delta(z) \end{pmatrix} \geq -\delta\|Cx(0)\|^2; \tag{2.24}$$

b) there exist $\varepsilon > 0$ such that (2.19) *holds with $\Pi = \Pi_\lambda + \Pi_b$.*

Then the interconnection (2.15) *is stable, i.e., there exist positive constants* γ, γ_0 *such that*

$$\|z\|^2 \leq \gamma\|u\|^2 + \gamma_0\|x_0\|^2 \quad \textit{for arbitrary} \quad x_0 \in \mathbb{R}^n \quad \textit{and} \quad u \in \mathscr{L}_2.$$

In conclusion, Jönsson's approach allows for constraints that are not valid on the full space $\mathscr{L}_2$ but on a subspace that is somehow compatible with the set of multipliers and the feedback interconnection under consideration. We will exploit this idea in much greater generality in the subsequent chapters.

2.3 A note on the application of Theorem 2.6

Let us now focus on the application of Theorem 2.6 with particular emphasis on the numerical verification of stability. Readers interested in a more thorough presentation are referred to the elaborate tutorial [176] on IQCs.

Assume that we are given an uncertainty Δ and a class of multipliers $\mathbf{\Pi} \subset RL_\infty$ such that (2.17) holds for all $\Pi \in \mathbf{\Pi}$ and all $\tau\Delta$ with $\tau \in [0,1]$. Moreover, we assume that (2.15) is well-posed. The approach outlined in this section immediately extend to the Popov criterion formulated above, even though the requirement $\mathbf{\Pi} \subset RL_\infty$ is not satisfied, as discussed in detail in Section 5.3.4.

First note that any $\Pi = \Pi^* \in RL_\infty$ can be factorized as

$$\Pi = \Psi^* P \Psi \quad \text{with a real} \quad P = P^T \quad \text{and} \quad \Psi \in RH_\infty. \tag{2.25}$$

Indeed, we can choose some (large) η such that $\Pi + \eta I \succ 0$ on $\mathbb{C}_0^\infty$; if ψ is a spectral factor with $\Pi + \eta I = \psi^*\psi$ [62], we get (2.25) for $\Psi := \mathrm{col}(\psi, I)$ and $P := \mathrm{diag}(I, -\eta I)$. This insight motivates to parameterize the class $\mathbf{\Pi}$ with fixed $\Psi \in RH_\infty$ as

$$\mathbf{\Pi} = \{\Psi^* P \Psi \mid \Psi \in RH_\infty \text{ and } P \in \mathbf{P}\}$$

for some subset $\mathbf{P}$ of the real symmetric matrices. Then, then interconnection (2.15) is stable, if there exits some $P \in \mathbf{P}$ such that

$$\exists \varepsilon > 0 : \qquad \begin{pmatrix} M \\ I \end{pmatrix}^* \Psi^* P \Psi \begin{pmatrix} M \\ I \end{pmatrix} \preccurlyeq -\varepsilon I \quad \text{on} \quad \mathbb{C}^0. \tag{2.26}$$

The celebrated KYP lemma [187, 180, 128, 14] equivalently characterises (2.26) as an LMI feasibility problem. A very general version is derived in [14] and stated below.

Lemma 2.11 (Generalized KYP Lemma [14])
Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and suppose that the real symmetric matrix K is structured as $K = \begin{pmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{pmatrix} \in \mathbb{S}^{n+m}$. Then the following statements are equivalent:

a) There exists $X \in \mathbb{S}^n$ such that

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^T \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + K \prec 0;$$

b) $K_{22} \prec 0$ and for all $\omega \in \mathbb{R}$ the following implication holds:

$$(i\omega I - A \;\; B) \underbrace{\begin{pmatrix} x \\ w \end{pmatrix}}_{\neq 0} = 0 \quad \Longrightarrow \quad \begin{pmatrix} x \\ w \end{pmatrix}^T K \begin{pmatrix} x \\ w \end{pmatrix} < 0.$$

Typically, it will suffice to work with the following particular version, that is an easy corollary of Lemma 2.11 under the additional assumption that A has no eigenvalues on the imaginary axis.

Lemma 2.12 (KYP Lemma [128])
Let $P \in \mathbb{S}^m$ and assume that the realization (A, B, C, D) of the LTI system G satisfies $\operatorname{eig}(A) \cap \mathbb{C}^0 = \emptyset$. Then the following statements are equivalent:

a) There exists $X \in \mathbb{S}^n$ *such that*

$$\begin{pmatrix} I & 0 \\ A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} 0 & X & 0 \\ X & 0 & 0 \\ 0 & 0 & P \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \\ C & D \end{pmatrix} \prec 0;$$

b) There exists $\varepsilon > 0$ *such that* $G^* P G \preccurlyeq -\varepsilon I$ *on* $\mathbb{C}^0$.

In conclusion, if $\mathbf{P}$ admits an LMI description, the search for $P \in \mathbf{P}$ satisfying (2.26) is characterized through Lemma 2.12 with $G = \Psi\,\mathrm{col}(M, I)$ as an LMI feasibility problem and is thus verifiable with standard optimization tools. Note that all LMIs in this thesis are solved using MATLAB's LMI Lab.

2.4 Connection to other approaches

Let us emphasize that this chapter is solely devoted to a very brief exposition of the evolution of the framework of integral quadratic constraints and its practical application. For brevity of display, many related topics were only touched upon or even completely omitted in the presentation.

In particular, we did not elaborate on the connection between classical multiplier theory (see, e.g., [44, 197, 181, 22, 23]) and the framework of IQCs as this is already the subject of several papers (see, e.g., [63] and also [72, 71]) and is also discussed by Megretski and Rantzer [110]. Moreover, Jönsson [92] devoted an insightfully written section in his thesis on this subject (see also [94]) that highlights all major links between the two fields.

Furthermore, the links between dissipation theory, the IQC framework and the stated KYP results are completely omitted. Yet, as one of the major contributions of this thesis is the merging of dissipation theory with integral quadratic constraints, we postpone the discussion on their precise connection to Chapter 7.

Chapter 3

A first generalization of the IQC theorem with applications to a class of sampled-data systems

3.1 Introduction

INSPIRED by the classical IQC framework as outlined in Section 2.2, we show in the present chapter that it is still possible to prove stability under weaker assumptions on the multiplier and the IQC description of the uncertainty. By taking the interconnection structure into account, we can relax the assumptions on the uncertainty such that they are no longer required to hold on the full signal space. Our generalizations are largely stimulated by system interconnections containing uncertainties Δ that exhibit sampling behavior, which means in our context that they satisfy $\Delta = \Delta S_h$ with the sample and hold operator S_h. By means of our extension of Theorem 2.6 (and also its discrete-time counterpart [97, Theorem 1]), we show how to handle such uncertainties within the IQC framework. Moreover, we present an exact reduction of the resulting infinite dimensional frequency-domain inequality (FDI) to a finite dimensional linear matrix inequality (LMI) feasibility problem, thus rendering the stability test computational. As an example of an uncertainty exhibiting sampling behavior, we consider a pulse-width modulator (PWM). Applications of such interconnections are manifold and vary from power converters (see, e.g., [80, 98, 64] and references therein) and biological models (see, e.g., [38, 43, 113, 34]) to attitude control of satellites as considered, for example, in [1, 108]. For background information on the physical modeling of satellite thrusters using pulse-width modulation and a discussion on the resulting stability issues, we refer the reader to our preliminary work in [33]. The advantages of the generalized framework presented in this chapter is demonstrated by comparing our results with the analysis techniques introduced in [69] and [82].

The chapter is structured into three parts. In Section 3.2 our IQC stability result is proven in a general setting. We then illustrate in Section 3.3 how to incorporate systems into the proposed framework where the uncertainty shows sampling behavior. Furthermore, we reduce our stability test to deciding the feasibility of a standard finite dimensional LMI. In Section 3.4 the derived results are applied to a feedback interconnection including a PWM, and some numerical illustration is provided. Finally, we emphasize that the results in this chapter have already appeared in [55] and large portions of the text overlap.

3.2 A generalized IQC theorem

Let us first prepare the stage for our generalized discrete-time version of Theorem 2.6 by discussing the underlying setup.

3.2.1 Basic definitions

Let $\mathscr{H}$ be a Hilbert space, $k \in \mathbb{N}$, and $\mathscr{H}^k$ the k-fold Cartesian product. Then $\ell(\mathscr{H}^k) := (\mathscr{H}^k)^{\mathbb{N}_0}$ is the set of all $\mathscr{H}^k$-valued sequences. With the standard norm on $\mathscr{H}^k$ we set

$$\|u\|^2 := \sum_{n=0}^{\infty} \|u(n)\|^2_{\mathscr{H}^k} \quad \text{for} \quad u \in \ell(\mathscr{H}^k)$$

and $\ell_2(\mathscr{H}^k) := \left\{ u \in \ell(\mathscr{H}^k) \;\middle|\; \|u\| < \infty \right\}$. For $T \in \mathbb{N}_0$ let P_T denote the (discrete-time) truncation operator on $\ell(\mathscr{H}^k)$, i.e.,

$$u_T := P_T u := (u(0), u(1), \ldots, u(T), 0, \ldots) \quad \text{for} \quad u \in \ell(\mathscr{H}^k).$$

We infer that $P_T u \in \ell_2(\mathscr{H}^k)$ for all $u \in \ell(\mathscr{H}^k)$ and $T \in \mathbb{N}_0$. With a subspace $\mathscr{E}_e \subset \ell(\mathscr{H}^k)$ satisfying

$$(\mathscr{E}_e)_T := \{ z_T \mid z \in \mathscr{E}_e \} \subset \mathscr{E}_e \quad \text{for all} \quad T \in \mathbb{N}_0, \tag{3.1}$$

a dynamical system S on $\mathscr{E}_e$ is a mapping $S : \mathscr{E}_e \subset \ell(\mathscr{H}^k) \to \ell(\mathscr{H}^l)$ which takes any input $u \in \mathscr{E}_e$ into the output $y = S(u) \in \ell(\mathscr{H}^l)$. The system S is said to be linear if the map is; it is causal if $S(u)_T = S(u_T)_T$ for all $T \in \mathbb{N}_0$ and $u \in \mathscr{E}_e$. The ℓ_2-gain $\|S\|$ of the system S is the infimal real number $\gamma \geq 0$ for which there exists some $\gamma_0 \in \mathbb{R}$ with

$$\|S(u)_T\| \leq \gamma \|u_T\| + \gamma_0 \quad \text{for all} \quad T \in \mathbb{N}_0,\ u \in \mathscr{E}_e.$$

For linear systems one can take $\gamma_0 = 0$, and if $\|S\|$ is finite we say that S is bounded.

Concerning the separating map Σ we further generalize the property (2.10) in order to allow for a broader class of constraints. As visible from the proof of the following theorem, it is sufficient to only require the existence of $\sigma_{ij} \in \mathbb{R}$ with

$$\Sigma \begin{pmatrix} w \\ Mw + Mu + Nv \end{pmatrix} - \Sigma \begin{pmatrix} w \\ Mw \end{pmatrix} \leq (\star)^T \begin{pmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \begin{pmatrix} \|w\| \\ \|u\| \\ \|v\| \end{pmatrix} \tag{3.2}$$

for all $w, u \in \ell_2(\mathscr{H}^k)$, $v \in \mathscr{V}$. This is indeed true if (2.10) holds.

3.2.2 Stability theorem

The interconnection structure for our problem is depicted in Figure 3.1. Here we assume that $M : \ell(\mathscr{H}^k) \to \ell(\mathscr{H}^l)$ and $\Delta : \mathscr{E}_e \subset \ell(\mathscr{H}^l) \to \ell(\mathscr{H}^k)$ are causal and bounded, while M is linear. The two systems are interconnected as

$$z = Mq + d_2 \quad \text{and} \quad q = \Delta(z) + d_1$$

with $d_1 \in \ell(\mathscr{H}^k)$, $d_2 \in \ell(\mathscr{H}^l)$. Since M is linear, this reduces to $z = Mw + (Md_1 + d_2)$ and $w = \Delta(z)$ with the single external input $d = Md_1 + d_2$. In order to adapt this setting to sampled-data applications,

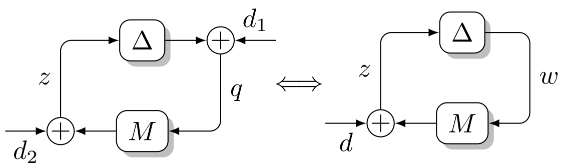

Figure 3.1: External disturbances in the interconnection

we further introduce a bounded linear filter $N : \ell_2(\mathscr{H}^m) \to \ell_2(\mathscr{H}^l)$ and consider the case where the disturbance d is confined to $M\ell(\mathscr{H}^k) + N\mathscr{V}$ with some subset $\mathscr{V} \subset \ell_2(\mathscr{H}^m)$. In our application, M, N will be strictly proper and stable LTI systems acting as prefilters on the external disturbances and allowing for subsequent sampling. The feedback system under consideration in this chapter is hence described by

$$z - M\Delta(z) = Mu + Nv \quad \text{with} \quad (u, v) \in \ell(\mathscr{H}^k) \times \mathscr{V} \tag{3.3}$$

and depicted in Figure 3.2. Since Δ is only defined on $\mathscr{E}_e \subset \ell(\mathscr{H}^l)$, we need to ensure that $z \in \mathscr{E}_e$ for all possible inputs (u, v). Note that we

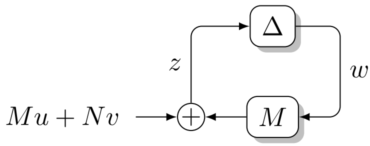

Figure 3.2: General interconnection

treat u and v differently, as we consider extended spaces only for u and z but not for v. Yet, in our definition of well-posedness and the later proof of our stability result, it suffices to consider finite energy signals $v \in \mathscr{V}$.

Definition 3.1.
The interconnection (3.3) is **well-posed** if for each $(u, v) \in \ell(\mathscr{H}^k) \times \mathscr{V}$ and for each $\tau \in [0, 1]$ there exists a unique $z \in \mathscr{E}_e$ satisfying $z - M\tau\Delta(z) = Mu + Nv$ and such that the map $(u, v) \mapsto z = R_\tau(u, v)$ is causal in the first argument:

$$R_\tau(u, v)_T = R_\tau(u_T, v)_T \text{ for all } T \in \mathbb{N}_0,\ \tau \in [0, 1],\ (u, v) \in \ell(\mathscr{H}^k) \times \mathscr{V}.$$

The feedback system (3.3) is **stable** if, in addition, $R_1 : \ell_2(\mathscr{H}^k) \times \mathscr{V} \to \ell_2(\mathscr{H}^l)$ is bounded.

Remark 3.2.
Classically (see Definition 2.5), well-posedness is defined through existence and causality of the inverse $(I - \tau M\Delta)^{-1}$ for all $\tau \in [0, 1]$ [110]. We recover this condition as a special case, by removing the distinction between u and v, i.e., we set $\mathscr{V} = \ell_2(\mathscr{H}^m)$, $N = I$, and require causality in both arguments of the response map; moreover, we assume that Δ is defined on the full signal space $\ell(\mathscr{H}^l)$. Jönsson [93] already relaxed the aforementioned concept of well-posedness (Definition 2.9). The further generalization in Definition 3.1 is mostly due to our definition of Δ only on the subspace $\mathscr{E}_e$. ⋆

Remark 3.3.
The introduction of $\mathscr{V}$ offers flexibility beyond sampled-data appli-

cations. Indeed, for $\mathscr{H} = \mathbb{R}$ and by setting $N = C$ as well as $\mathscr{V} := \{v \in \ell_2(\mathbb{R}^n) \mid v(t) = A^t x_0\}$ we may treat initial conditions of an LTI system M as discussed in Section 2.2.2. ⋆

Let us now state the central stability result of this chapter.

Theorem 3.4
Assume that $M : \ell(\mathscr{H}^k) \to \ell(\mathscr{H}^l)$ *and* $\Delta : \mathscr{E}_e \subset \ell(\mathscr{H}^l) \to \ell(\mathscr{H}^k)$ *are causal and bounded, while* M *is linear, and that* Σ *satisfies* (3.2). *Suppose, in addition, that*

a) the feedback system (3.3) *is well-posed;*

b) there exists $\varepsilon > 0$ *such that*

$$\Sigma \begin{pmatrix} Mw \\ w \end{pmatrix} \leq -\epsilon \|w\|^2 \quad \textit{for all} \quad w \in \ell_2(\mathscr{H}^k);$$

c) there exists some function $\delta_0 : \mathscr{V} \to [0, \infty)$ *with*

$$\Sigma \begin{pmatrix} z \\ \tau\Delta(z) \end{pmatrix} \geq -\delta_0(v) \quad \textit{for all} \quad \tau \in [0, 1]$$

and all $z = Mu + Nv$ *with* $(u, v) \in \ell_2(\mathscr{H}^k) \times \mathscr{V}$.

Then there exists some $\gamma > 0$ *(only depending on* M, N, *and* Σ*) such that*

$$\|R_1(u, v)\|^2 \leq \gamma^2 \left(\|u\|^2 + \|v\|^2\right) + \gamma\delta_0(v) \quad \textit{for all} \quad (u, v) \in \ell_2(\mathscr{H}^k) \times \mathscr{V}. \tag{3.4}$$

Proof. A proof is found in Appendix C.2.1. □

In contrast to Theorem 2.6 (and also its discrete-time version [97, Theorem 1]), all assumptions on Δ are only required to hold on a subspace $\mathscr{E}_e \subset \ell(\mathscr{H}^l)$ with the key property $R_\tau(\ell(\mathscr{H}^k) \times \mathscr{V}) \subset \mathscr{E}_e$ for $\tau \in [0, 1]$ that ensures well-posedness. In our application the uncertainty will be a PWM. As we will see, this operator is not well-defined on $\mathscr{L}_2$ and unbounded even on $C[0,\infty)\cap\mathscr{L}_2$. However, we can still choose a suitable subspace $\mathscr{E}$ such that it becomes bounded and even passive. Hence, only by considering the PWM on a smaller set of signals, IQC theory becomes applicable.

Remark 3.5.
The function $\delta_0(.)$ is introduced in order to cover the Popov criterion as discussed in Section 2.2.2 and generalizes the right hand side in (2.24). ⋆

Remark 3.6.
The derived framework is easily translated to the continuous-time setting. With the continuous-time truncation operator (Definition 2.1), we can define well-posedness and stability as in Definition 3.1 by exchanging $\mathscr{L}_{2e}^k$ and $\mathscr{L}_2^k$ for $\ell(\mathscr{H}^k)$ and $\ell_2(\mathscr{H}^k)$, respectively. Apart from these modifications, the statement of Theorem 3.4 stays unchanged for systems on $\mathscr{L}_{2e}$ and the proof proceeds in an analogous fashion. However, we will present in Chapter 4 a further generalization for continuous-time interconnections that allows for operators and constraints on much more general function spaces. ⋆

3.3 A class of sampled-data systems

Let us now illustrate how the generalizations in Theorem 3.4 are employed to incorporate uncertainties exhibiting sampling behavior into our framework. As we will see, this heavily relies on a loop transformation, which renders the system time varying. By using a lifting formalism we can then return to an LTI description that allows for a straightforward transformation to the frequency domain. Verification of the IQC in the frequency domain finally leads to an infinite dimensional FDI, which we reduce without any loss to a finite dimensional LMI that can be checked efficiently using standard techniques.

3.3.1 Motivation

Consider the feedback interconnection of a strictly proper and stable LTI system M realized by $(A, B, C, 0)$ with $\operatorname{eig}(A) \subset \mathbb{C}^-$ and an uncertainty $\Delta : \mathscr{E}_e \subset \mathscr{L}_{2e} \to \mathscr{L}_{2e}$ as described by

$$\begin{aligned} \dot{x} &= Ax + Bw, \quad x(0) = 0, \qquad w = \Delta(z), \\ z &= Cx + d. \end{aligned}$$

In the context of Theorem 3.4, we confine the external disturbance d to $M\mathscr{L}_{2e} + N\mathscr{V}$, where N is a strictly proper, stable and finite dimensional LTI system and $\mathscr{V} \subset \mathscr{L}_2$ (see left-hand side of Figure 3.3). Moreover, let an equidistant time grid on $[0, \infty)$ be given by a fixed sampling period $h > 0$, i.e., $t_n := nh$ for $n \in \mathbb{N}_0$. We then set $I_n := [t_n, t_{n+1})$ and define the sample and hold operator S_h by

$$S_h(z)(t) = z(t_n) \quad \text{for} \quad z \in PC[0, \infty),\ t \in I_n. \tag{3.5}$$

Since the sampling period is assumed to be fixed, we drop the subscript h in the sequel. For the uncertainty, we assume the following key property to hold on $\mathscr{E}_e \subset PC[0, \infty)$:

$$\Delta = \Delta S. \tag{3.6}$$

Here the filter N is essential since it ensures that $d \in PC[0, \infty)$ and hence $\Delta(z) = \Delta(Sz)$ is well defined.

In IQC theory we are interested in describing the input-output behavior of uncertainties as accurately as possible. Property (3.6) states that the output of Δ over the time interval I_n is completely defined by the value of the input at the time instance t_n. Hence all inputs coinciding at t_n lead to the same output $\Delta(z)(t)$ for all $t \in I_n$ and thus to the same square integral on I_n. On the other hand, the square integral of the input signal can change arbitrarily on this interval. Consequently, IQC relations and especially gain bounds for uncertainties with (3.6) are either hard or even impossible to derive, or very conservative.

An elegant way to bypass this problem relies on two key ideas. First we exploit (3.6) and move the sample and hold operator from $\Delta = \Delta S$ to M, i.e., we consider Sz as an input to the uncertainty (Figure 3.3). Then we take advantage of the freedom offered by Theorem 3.4 and choose as the domain $\mathscr{E}_e$ of Δ the space of functions that are constant in each time interval. We will demonstrate in the examples that these ideas significantly simplify the derivation of suitable IQCs and open the way for the application of IQC theory.

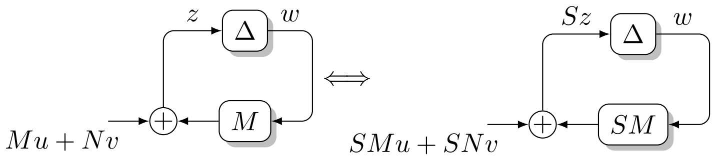

Figure 3.3: Equivalent interconnections

Note that this approach, in principle, amounts to interpreting S as a classical multiplier that enables the application of some stability criterion to the transformed loop. In order to conclude stability of the original interconnection, it suffices to prove stability for the system interconnection on the right in Figure 3.3. Indeed, assume we used Theorem 3.4 to show that the interconnection on the right is stable (with $\delta_0(.) = 0$), i.e., there exists $\gamma > 0$ such that $\|Sz\| \leq \gamma(\|u\| + \|v\|)$ for all $(u, v) \in \mathscr{L}_2 \times \mathscr{V}$. Since $w = \Delta(Sz) = \Delta(z)$ is the same signal in both configurations and Δ, M are bounded, we infer

$$\|z\| = \|M\Delta(z)+Mu+Nv\| = \|M\Delta(Sz)+Mu+Nv\| \leq \widetilde{\gamma}_u\|u\|+\widetilde{\gamma}_v\|v\|$$

for all $(u, v) \in \mathscr{L}_2 \times \mathscr{V}$ and some positive constants $\widetilde{\gamma}_u, \widetilde{\gamma}_v$. Hence the interconnection on the left is stable.

3.3.2 Lifting and frequency domain

In order to deal with the time-varying system SM in the IQC framework, we rely on the lifting formalism. Lifting procedures, have originally been introduced for sampled-data systems (see, e.g., [191, 192, 161, 15, 16]) as they provide a means of transforming the sampled system SM into an LTI system on a lifted space. The cited approaches mainly differ in the way they capture the inter-sampling behavior and define the state space of the lifted system. For our purpose, it will be essential that the state space is finite dimensional, which is a distinguishing feature of the approach presented in [16]. Since lifting as well as transformation to the frequency domain are standard in the theory of sampled-data systems [36], we state only the required results. Both topics are discussed in more detail in [16, 50, 111].

Lifting

Let

$$\widetilde{\mathscr{L}_2} := \left\{ \widetilde{u} : \mathbb{N}_0 \to \mathscr{L}_2[0,h) \,\middle|\, \|\widetilde{u}\|^2_{\widetilde{\mathscr{L}_2}} := \sum_{n=0}^{\infty} \|\widetilde{u}(n)\|^2_{\mathscr{L}_2[0,h)} < \infty \right\}.$$

Then the lifting operator $\widetilde{L} : \mathscr{L}_2 \to \widetilde{\mathscr{L}_2}$, given by

$$(\widetilde{L}u)(n,\tau) := \widetilde{u}(n,\tau) := u(\tau + nh) \quad \text{for} \quad \tau \in [0,h) \quad \text{and} \quad n \in \mathbb{N}_0,$$

is an isometric isomorphism between the spaces $\mathscr{L}_2$ and $\widetilde{\mathscr{L}_2}$. We state the following result, which is a combination of those derived in [35] and [15] for a stable LTI operator M represented as

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad x(0) = 0, \\ y &= Cx \end{aligned} \tag{3.7}$$

with A being Hurwitz.

Lemma 3.7
Let M be given by (3.7)*. Then SM is bounded and $\widetilde{SM} := \widetilde{L}SM\widetilde{L}^{-1}$ can be described as*

$$\begin{aligned} x_d(n+1) &= \check{A}x_d(n) + \check{B}\widetilde{u}(n), \qquad x_d(0) = 0 \in \mathbb{R}^n, \\ \widetilde{z}(n) &= \check{C}x_d(n) \end{aligned} \tag{3.8}$$

for $\widetilde{u} \in \widetilde{\mathscr{L}_2}$, $\widetilde{z} = \widetilde{SM}\widetilde{u}$, and

$$\begin{aligned} &\check{A} \in \mathbb{R}^{n\times n}, \quad \check{A} := e^{Ah}, \\ &\check{B} : \mathscr{L}_2[0,h) \to \mathbb{R}^n, \quad \check{B}\psi := \int_0^h e^{A(h-\tau)}B\psi(\tau)d\tau, \\ &\check{C} : \mathbb{R}^n \to \mathscr{L}_2[0,h), \quad (\check{C}\xi)(\tau) := C\xi, \quad \textit{for all} \quad \tau \in [0,h). \end{aligned}$$

Note that the input and output spaces are infinite dimensional, while the state dimension is invariant under lifting.

Frequency domain

In order to exploit the fact that the lifted system is time invariant, we now employ the z-transform to obtain a frequency-domain description. Let $\mathcal{H}_2$ denote the class of analytic functions $\hat{u}$ mapping the open unit disc $\mathbb{D}$ into $\mathscr{L}_2[0,h)$ such that

$$\|\hat{u}\|^2_{\mathcal{H}_2} := \sup_{0\le r<1} \frac{1}{2\pi}\int_0^{2\pi} \|\hat{u}(re^{i\omega})\|^2_{\mathscr{L}_2[0,h)}d\omega < \infty.$$

This space is complete and can be associated with $\widetilde{\mathscr{L}_2}$ via the z-transform. For $\widetilde{u} \in \widetilde{\mathscr{L}_2}$ the one-sided z-transform $Z : \widetilde{\mathscr{L}_2} \to \mathcal{H}_2$ is defined by

$$\hat{u}(\mathrm{z}) := (Z\widetilde{u})(\mathrm{z}) := \sum_{n=0}^{\infty} \widetilde{u}(n)\,\mathrm{z}^n \quad \text{for} \quad |\mathrm{z}| < 1,$$

where we used the symbol z to distinguish the frequency-domain variable from the signal z. For Banach space valued sequences many of the familiar results persist to hold, such as that the spaces $\widetilde{\mathscr{L}_2}$ and $\mathcal{H}_2$ are isomorphic via the z-transform [50, Prop. 2.9] or that the nontangential limit for $r \to 1$ exists pointwise almost everywhere [154]. With the pointwise limit, the inner product on $\mathcal{H}_2$ as given by

$$\langle \hat{u}, \hat{v} \rangle_{\mathcal{H}_2} := \frac{1}{2\pi} \int_0^{2\pi} \left\langle \hat{u}(e^{i\omega}), \hat{v}(e^{i\omega}) \right\rangle_{\mathscr{L}_2[0,h)} d\omega$$

is well defined (see, e.g., [135]). Moreover, by Parseval's theorem (see, e.g., [111]), we infer $\langle \hat{u}, \hat{v} \rangle_{\mathcal{H}_2} = \langle \widetilde{u}, \widetilde{v} \rangle_{\widetilde{\mathscr{L}_2}}$. In [50] it is shown that the transfer function $\hat{T} \in \mathcal{H}_\infty$ associated with $\widetilde{SM}$ as represented in (3.8) is given by

$$\hat{T}(\mathrm{z}) = \check{C}\,\mathrm{z}(I - \mathrm{z}\,\check{A})^{-1}\check{B} \quad \text{for} \quad \mathrm{z} \in \mathbb{D}.$$

Hence the time-domain equation $\widetilde{z} = \widetilde{SM}\widetilde{w}$ corresponds in the frequency domain to $\hat{z} = \hat{T}\hat{w}$ for $\hat{w}, \hat{z} \in \mathcal{H}_2$. Moreover, if A is Hurwitz, then $\check{A}$ is Schur stable and, consequently, the definition of $\hat{T}(\mathrm{z})$ can be extended to $\overline{\mathbb{D}} := \mathbb{D} \cup \mathbb{T}$.

3.3.3 IQC description and reduction to finite dimensions

From IQCs to FDIs

In the following application of Theorem 3.4 we restrict our attention to uncertainties $\Delta : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^k$ and bounded quadratic forms Σ defined by a multiplier $P \in \mathbb{R}^{2k \times 2k}$ as

$$\Sigma_P \begin{pmatrix} \widetilde{z} \\ \widetilde{w} \end{pmatrix} := \left\langle \begin{pmatrix} \widetilde{z} \\ \widetilde{w} \end{pmatrix}, P \begin{pmatrix} \widetilde{z} \\ \widetilde{w} \end{pmatrix} \right\rangle_{\widetilde{\mathscr{L}_2}} := \left\langle \begin{pmatrix} \widetilde{z} \\ \widetilde{w} \end{pmatrix}, \begin{pmatrix} p_{11}I & p_{12}I \\ p_{12}I & p_{22}I \end{pmatrix} \begin{pmatrix} \widetilde{z} \\ \widetilde{w} \end{pmatrix} \right\rangle_{\widetilde{\mathscr{L}_2}}. \tag{3.9}$$

This particular structure of P covers the standard static multipliers corresponding to small gain, passivity, or circle criteria. By Parseval's theorem, Theorem 3.4.b) is then equivalent to

$$\left\langle \begin{pmatrix} \hat{T}\hat{w} \\ \hat{w} \end{pmatrix}, P \begin{pmatrix} \hat{T}\hat{w} \\ \hat{w} \end{pmatrix} \right\rangle_{\mathcal{H}_2} \leq -\epsilon \|\hat{w}\|^2 \quad \text{for some } \varepsilon > 0 \text{ and all } \hat{w} \in \mathcal{H}_2,$$

which is guaranteed by the infinite dimensional FDI

$$\begin{pmatrix} \hat{T}(\mathrm{z}) \\ I \end{pmatrix}^* P \begin{pmatrix} \hat{T}(\mathrm{z}) \\ I \end{pmatrix} \preccurlyeq -\varepsilon I \quad \text{for some } \varepsilon > 0 \text{ and all } \mathrm{z} \in \mathbb{T}. \tag{3.10}$$

To simplify the notation in this chapter, for an arbitrary operator X we sometimes use $X \prec_\varepsilon 0$ if there exists $\varepsilon > 0$ such that $X \preccurlyeq -\varepsilon I$.

Ultimately, we would like to employ Lemma 2.12 in order to represent (3.10) as an LMI. Since KYP results for general Hilbert spaces [190] do not allow for the reduction to finite dimensional (computationally tractable) LMIs, we first reduce (3.10) to a finite dimensional FDI. In order to keep this reduction lossless, it is crucial that the state space dimension is finite after lifting, as this translates into $\check{B}, \check{C}$ having finite rank. This property was first exploited in [16] and used to calculate the $\mathcal{H}_\infty$-norm of a sampled-data system by decomposing $\check{B}, \check{C}$. We will now extend this approach to the IQC setting with static multipliers.

Decomposition of the input and the output space

Following [16] we decompose $\mathscr{L}_2[0,h)$ into an infinite and a finite dimensional part as

$$\mathscr{L}_2[0,h) = \mathrm{Ker}(\check{B}) \oplus \mathrm{Ker}(\check{B})^\perp =: \mathscr{U}_B \oplus \mathscr{V}_B \quad \text{with} \quad \dim(\mathscr{V}_B) < \infty.$$

This naturally induces the isometry $T_B : \mathscr{L}_2[0,h) \to \mathscr{U}_B \times \mathscr{V}_B$,

$$T_B w := \begin{pmatrix} T_B^1 w \\ T_B^2 w \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{with} \quad w = u + v,\ u \in \mathscr{U}_B,\ v \in \mathscr{V}_B,$$

and the embedding

$$J_B : \mathscr{U}_B \times \mathscr{V}_B \to \mathscr{L}_2[0,h), \quad J_B \begin{pmatrix} u \\ v \end{pmatrix} := u + v = J_B^1 u + J_B^2 v,$$

where $\mathscr{U}_B \times \mathscr{V}_B$ is equipped with the usual inner product. Throughout this chapter we write elements of such product spaces as column vectors. By standard computations we infer that $T_B^* = J_B$ and

$$\check{B}\check{B}^* = \overline{B}\,\overline{B}^* = \int_0^h e^{At} B B^T e^{A^T t} dt \quad \text{with} \quad \overline{B} := \check{B}|_{\mathscr{V}_B}.$$

Consequently, $\check{B}\check{B}^*$ has a representation as an $n \times n$ matrix which is invertible if the pair (A, B) is controllable. Figure 3.4 illustrates the relation between the appearing spaces and operators.

$$\begin{tikzcd}
\mathscr{L}_2[0,h) \arrow[d, "T_B"'] \arrow[dr, "\check{B}"] & \\
\mathscr{U}_B \times \mathscr{V}_B \arrow[r, "(0\ \overline{B})"'] & \mathbb{R}^n \cong \mathscr{V}_B
\end{tikzcd}$$

Figure 3.4: Decomposition of $\check{B}$

In complete analogy we now define the subspaces and operators for the decomposition of the output space as

$$\mathscr{L}_2[0,h) = \operatorname{Ran}(\check{C})^\perp \oplus \operatorname{Ran}(\check{C}) =: \mathscr{U}_C \oplus \mathscr{V}_C \quad \text{with} \quad \dim(\mathscr{V}_C) < \infty.$$

Again, this gives rise to the isometry $T_C : \mathscr{L}_2[0,h) \to \mathscr{U}_C \times \mathscr{V}_C$ and the embedding $J_C : \mathscr{U}_C \times \mathscr{V}_C \to \mathscr{L}_2[0,h)$. All results derived for T_B hold in an analogous fashion for T_C and we infer

$$\check{C}^*\check{C} = \overline{C}^*\overline{C} = hC^TC. \tag{3.11}$$

Effect of the decomposition on z-transformed signals

The decomposition of $\check{B}$ as in Figure 3.4 induces a decomposition of the input signal $\hat{w}$ as

$$T_B\hat{w}(\mathrm{z}) = \begin{pmatrix} \hat{w}_i(\mathrm{z}) \\ \hat{w}_f(\mathrm{z}) \end{pmatrix} \quad \text{for} \quad |\mathrm{z}| < 1$$

since $\hat{w}(\mathrm{z}) \in \mathscr{L}_2[0,h)$ for $\mathrm{z} \in \mathbb{D}$. Note that $\hat{w}_f$ takes finite dimensional values. The analogue holds true for the output space, thus inducing the decomposition of the transfer function as

$$T_C \hat{T}(\mathrm{z}) T_B^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \overline{T}(\mathrm{z}) \end{pmatrix} : \begin{matrix} \mathscr{U}_B \\ \times \\ \mathscr{V}_B \end{matrix} \longrightarrow \begin{matrix} \mathscr{U}_C \\ \times \\ \mathscr{V}_C \end{matrix} \quad \text{for} \quad \mathrm{z} \in \mathbb{T}, \tag{3.12}$$

where $\overline{T}(\mathrm{z}) := \overline{C}\,\mathrm{z}(I - \mathrm{z}\,\check{A})^{-1}\overline{B}$ is a finite dimensional linear operator for any $\mathrm{z} \in \mathbb{T}$. We can now take advantage of this decomposition for the reduction of the FDI (3.10).

Reduction of the FDI

For a better illustration of the reduction process, we regard P as a multiplication operator on a function space and express this by using $\mathbb{1}$ for the identity operator on an infinite dimensional space and I for that on a finite dimensional space. Then, (3.10) takes the form

$$\begin{pmatrix} \hat{T}(\mathrm{z}) \\ \mathbb{1} \end{pmatrix}^* \begin{pmatrix} p_{11}\mathbb{1} & p_{12}\mathbb{1} \\ p_{12}\mathbb{1} & p_{22}\mathbb{1} \end{pmatrix} \begin{pmatrix} \hat{T}(\mathrm{z}) \\ \mathbb{1} \end{pmatrix} \prec_\varepsilon 0 \qquad \text{for all} \quad \mathrm{z} \in \mathbb{T} \tag{3.13}$$

and can be reformulated as follows.

Theorem 3.8
With the previous definitions, the following statements are equivalent:

a) (3.13) *holds.*

b) $p_{22} < 0$ *and for all* $\mathrm{z} \in \mathbb{T}$, *the FDI*

$$\begin{pmatrix} \overline{T}(\mathrm{z}) \\ I \end{pmatrix}^* \overline{P} \begin{pmatrix} \overline{T}(\mathrm{z}) \\ I \end{pmatrix} \prec_\varepsilon 0 \tag{3.14}$$

holds with

$$\overline{P} := \begin{pmatrix} p_{11}I - p_{12}^2/p_{22}(J_C^2)^*(T_B^1)^* T_B^1 J_C^2 & p_{12}T_B^2 J_C^2 \\ p_{12}T_C^2 J_B^2 & p_{22}I \end{pmatrix}. \tag{3.15}$$

Remark 3.9.
We emphasize that no approximation is needed; both criteria are equivalent. This key step clears the way for efficient computations by LMI techniques, as we can now verify Theorem 3.4.b) by checking the finite dimensional FDI (3.14). The term $p_{12}^2/p_{22}(J_C^2)^*(T_B^1)^*T_B^1J_C^2$, resulting from a Schur complement, acts as a correction to the finite dimensional part of the multiplier. In [16] (see also [115]) this correction is zero since the multiplier for the $\mathcal{H}_\infty$-norm bound is diagonal, i.e., $p_{12} = 0$. ⋆

Proof. Fix $\mathrm{z} \in \mathbb{T}$. Then

$$
\begin{aligned}
&\begin{pmatrix} \hat{T}(\mathrm{z}) \\ \mathbb{1} \end{pmatrix}^* \begin{pmatrix} p_{11}\mathbb{1} & p_{12}\mathbb{1} \\ p_{12}\mathbb{1} & p_{22}\mathbb{1} \end{pmatrix} \begin{pmatrix} \hat{T}(\mathrm{z}) \\ \mathbb{1} \end{pmatrix} \prec_\varepsilon 0 \\
\iff & (\star)^* \begin{pmatrix} p_{11}\mathbb{1} & p_{12}\mathbb{1} \\ p_{21}\mathbb{1} & p_{22}\mathbb{1} \end{pmatrix} \begin{pmatrix} T_C^{-1}T_C\check{C}\,\mathrm{z}(I - \mathrm{z}\,\check{A})^{-1}\check{B}T_B^{-1}T_B \\ T_B^{-1}T_B \end{pmatrix} \prec_\varepsilon 0 \\
\iff & (\star)^* \begin{pmatrix} p_{11}\mathbb{1} & p_{12}\mathbb{1} \\ p_{12}\mathbb{1} & p_{22}\mathbb{1} \end{pmatrix} \begin{pmatrix} T_C^{-1} & 0 \\ 0 & T_B^{-1} \end{pmatrix} \left(\begin{array}{cc} 0 & 0 \\ 0 & \overline{T}(\mathrm{z}) \\ \hline \mathbb{1} & 0 \\ 0 & I \end{array}\right) \prec_\varepsilon 0 \\
\iff & (\star)^* \begin{pmatrix} p_{11}T_CT_C^{-1} & p_{12}T_CT_B^{-1} \\ p_{12}T_BT_C^{-1} & p_{22}T_BT_B^{-1} \end{pmatrix} \left(\begin{array}{cc} 0 & 0 \\ 0 & \overline{T}(\mathrm{z}) \\ \hline \mathbb{1} & 0 \\ 0 & I \end{array}\right) \prec_\varepsilon 0 \\
\iff & (\star)^* \left(\begin{array}{cc|cc} p_{11}\mathbb{1} & 0 & p_{12}T_C^1J_B^1 & p_{12}T_C^1J_B^2 \\ 0 & p_{11}I & p_{12}T_C^2J_B^1 & p_{12}T_C^2J_B^2 \\ \hline p_{12}T_B^1J_C^1 & p_{12}T_B^1J_C^2 & p_{22}\mathbb{1} & 0 \\ p_{12}T_B^2J_C^1 & p_{12}T_B^2J_C^2 & 0 & p_{22}I \end{array}\right) \left(\begin{array}{cc} 0 & 0 \\ 0 & \overline{T}(\mathrm{z}) \\ \hline \mathbb{1} & 0 \\ 0 & I \end{array}\right) \prec_\varepsilon 0 \\
\iff & (\star)^* \left(\begin{array}{c|cc} p_{22}\mathbb{1} & p_{12}T_B^1J_C^2 & 0 \\ \hline p_{12}T_C^2J_B^1 & p_{11}I & p_{12}T_C^2J_B^2 \\ 0 & p_{12}T_B^2J_C^2 & p_{22}I \end{array}\right) \left(\begin{array}{cc} \mathbb{1} & 0 \\ \hline 0 & \begin{pmatrix} \overline{T}(\mathrm{z}) \\ I \end{pmatrix} \end{array}\right) \prec_\varepsilon 0.
\end{aligned}
\tag{3.16}
$$

To eliminate the infinite dimensional part, we now employ the Schur complement and obtain that (3.16) is equivalent to $p_{22} < 0$ and

$$\begin{pmatrix} \overline{T}(\mathrm{z}) \\ I \end{pmatrix}^* \overline{P} \begin{pmatrix} \overline{T}(\mathrm{z}) \\ I \end{pmatrix} \prec_\varepsilon 0$$

with $\overline{P}$ given in (3.15). Since $\mathrm{z} \in \mathbb{T}$ was arbitrary, this proves the claim. □

Matrix representations

Let us now compute matrix representations S_{22} and $S_{12}^* S_{12}$ of the operators $T_B^2 J_C^2 : \mathscr{V}_C \to \mathscr{V}_B$ and $(J_C^2)^*(T_B^1)^* T_B^1 J_C^2 : \mathscr{V}_C \to \mathscr{V}_C$ occurring in (3.15).

In order to determine S_{22} we solve

$$\begin{pmatrix} T_B^2 J_C^2 y_1 & \cdots & T_B^2 J_C^2 y_k \end{pmatrix} = \begin{pmatrix} u_1 & \cdots & u_m \end{pmatrix} S_{22} \tag{3.17}$$

with bases y_j of $\mathscr{V}_C$ and u_j of $\mathscr{V}_B$. First, to construct y_j, choose vectors $\xi_1, \ldots, \xi_k \in \mathbb{R}^n$ such that $C\xi_1, \ldots, C\xi_k$ forms a basis of $\operatorname{Ran}(C)$. Then, $y_j = \check{C}\xi_j \in \mathscr{L}_2[0,h)$ and $(y_j)_{j=1}^k$ is a basis of $\mathscr{V}_C$. In addition, we have $J_C^2 y_j = y_j$ for all j.

Now let $(u_j)_{j=1}^m$ be a basis of $\mathscr{V}_B$. Since $\check{B}$ is a bijective map between the spaces $\mathscr{V}_B$ and $\operatorname{Ran}(\check{B})$, it follows that $(\check{B}u_j)_{j=1}^m$ is a basis of

$$\operatorname{Ran}(\check{B}) = \operatorname{Ran}(B, AB, \ldots, A^{n-1}B) \subset \mathbb{R}^n$$

and thus easy to compute. Together with $\check{B}T_B^2 = \check{B}$, (3.17) implies

$$\begin{pmatrix} \check{B}\check{C}\xi_1 & \cdots & \check{B}\check{C}\xi_k \end{pmatrix} = \begin{pmatrix} \check{B}u_1 & \cdots & \check{B}u_m \end{pmatrix} S_{22}. \tag{3.18}$$

As a basis matrix, $F := \begin{pmatrix} \check{B}u_1 & \cdots & \check{B}u_m \end{pmatrix}$ has full column rank. By recalling

$$\check{B}\check{C}\xi = \int_0^h e^{A(h-\tau)} BC \, d\tau \; \xi$$

we can thus compute S_{22} from (3.18).

For the second matrix representation we solve

$$\big((J_C^2)^*(T_B^1)^*T_B^1J_C^2y_1 \ \dots \ (J_C^2)^*(T_B^1)^*T_B^1J_C^2y_k\big) = \big(y_1 \ \dots \ y_k\big)\, S_{12}^*S_{12} \tag{3.19}$$

for $S_{12}^*S_{12}$. Since $J_C^2\check{C} = \check{C}$ and hence $\check{C}^* = \check{C}^*(J_C^2)^*$, we get

$$\big(\check{C}^*(T_B^1)^*T_B^1\check{C}\xi_1 \ \dots \ \check{C}^*(T_B^1)^*T_B^1\check{C}\xi_k\big) = \big(\check{C}^*\check{C}\xi_1 \ \dots \ \check{C}^*\check{C}\xi_k\big)\, S_{12}^*S_{12}. \tag{3.20}$$

To understand the operator on the left, let $\xi \in \mathbb{R}^n$. Then $T_B^1\check{C}\xi$ is the projection of $\check{C}\xi$ onto $\mathscr{U}_B$. Hence, using our basis of $\mathscr{V}_B$, this projection may be written as

$$T_B^1\check{C}\xi = \check{C}\xi - \sum_{j=1}^{m} u_j\alpha_j \quad \text{with} \quad \alpha_j \in \mathbb{R} \quad \text{for} \quad j \in \{1,\dots,m\}. \tag{3.21}$$

The coefficients vector $\alpha = \big(\alpha_1 \ \dots \ \alpha_m\big)^T$ can be computed from the equation

$$0 = \check{B}\Big(\check{C}\xi - \sum_{j=1}^{m} u_j\alpha_j\Big) = \check{B}\check{C}\xi - \big(\check{B}u_1 \ \dots \ \check{B}u_m\big)\,\alpha = \check{B}\check{C}\xi - F\alpha$$

as $\alpha = F^+\check{B}\check{C}\xi$ with the Moore-Penrose inverse F^+. Finally, using (3.21), we get

$$\check{C}^*(T_B^1)^*T_B^1\check{C}\xi = \check{C}^*J_B^1\Big(\check{C}\xi - \sum_{j=1}^{m} u_j\alpha_j\Big) = \check{C}^*\check{C}\xi - \big(\check{C}^*u_1 \ \dots \ \check{C}^*u_1\big)\,\alpha.$$

Hence, with (3.11), we can compute $S_{12}^*S_{12}$ from (3.20).

Affine dependence on decision variables

The multiplier $\overline{P}$ in (3.15) depends fractionally on the original coefficients p_{ij}. For later computations, we show how to render the dependence affine. Since $S_{12}^*S_{12}$ is positive semidefinite, it has a positive semidefinite square root and

$$\begin{pmatrix}\overline{T}(\mathrm{z})\\ I\end{pmatrix}^* \overline{P} \begin{pmatrix}\overline{T}(\mathrm{z})\\ I\end{pmatrix} \prec 0 \quad \text{for all} \quad \mathrm{z} \in \mathbb{T} \tag{3.22}$$

is equivalent to

$$\begin{pmatrix} \overline{T}(\mathrm{z}) & 0 \\ \hline I & 0 \\ 0 & I \end{pmatrix}^* \underbrace{\left(\begin{array}{c|cc} p_{11} I & p_{12} S_{22}^* & p_{12}\sqrt{S_{12}^* S_{12}} \\ \hline p_{12} S_{22} & p_{22} I & 0 \\ p_{12}\sqrt{S_{12}^* S_{12}} & 0 & p_{22} I \end{array}\right)}_{=:\overline{P}_e} \begin{pmatrix} \overline{T}(\mathrm{z}) & 0 \\ \hline I & 0 \\ 0 & I \end{pmatrix} \prec 0 \tag{3.23}$$

for all $\mathrm{z} \in \mathbb{T}$. Equation (3.23) can be written more compactly as

$$\begin{pmatrix} \overline{T}_e(\mathrm{z}) \\ I \end{pmatrix}^* \overline{P}_e \begin{pmatrix} \overline{T}_e(\mathrm{z}) \\ I \end{pmatrix} \prec 0 \quad \text{for} \quad \overline{T}_e(\mathrm{z}) := \begin{pmatrix} \overline{T}(\mathrm{z}) & 0 \end{pmatrix}. \tag{3.24}$$

The transformation to an LMI and, hence, to a convex problem is now standard (see Section 2.3).

3.4 Application to PWM feedback systems

Let us now apply the previously derived results to the stability analysis of a PWM feedback interconnection. After having defined the overall setup, we will illustrate two possibilities of incorporating PWM systems into our framework. In Section 3.4.4 we compare these approaches to the ones in [82] and [33] and show numerical results.

3.4.1 Definition of a PWM

The interconnection to be studied consists of a strictly proper and stable plant M, realized by $(A, B, C, 0)$, a PWM denoted as Δ, and an external disturbance $d \in M\mathscr{L}_2 + N\mathscr{L}_2$ (where N is a strictly proper and stable transfer function such that $\Delta(Mw + d)$ is well defined). It is described by the set of equations

$$z = Mw + d, \quad w = \Delta(z). \tag{3.25}$$

Given $z_{\max}, z_{\min} \geq 0$ and the time span

$$\tau_n = \begin{cases} 0, & |z(t_n)| < z_{\min}, \\ \frac{|z(t_n)|}{z_{\max}} h, & z_{\min} \leq |z(t_n)| \leq z_{\max}, \\ h, & |z(t_n)| > z_{\max}, \end{cases} \tag{3.26}$$

Δ is defined with $\mu_n := \operatorname{sgn}(z(t_n)) z_{\max}$ (see [69]) as

$$w(t) = \Delta(z)(t) = \begin{cases} \mu_n, & t \in [t_n, t_n + \tau_n), \\ 0, & t \in [t_n + \tau_n, t_{n+1}). \end{cases} \tag{3.27}$$

Here $z_{\min} \geq 0$ is a constant threshold under which the PWM generates no output pulse. We consider the cases $z_{\min} > 0$ and $z_{\min} = 0$ separately.

3.4.2 Direct approach to PWM analysis

The approach taken in this section heavily relies on $z_{\min} > 0$, which is required to derive a bound on Δ. In modeling physical processes $z_{\min} > 0$ is in fact a natural requirement, since it provides a positive lower bound on the pulse length τ_n as

$$\tau_n \geq h \frac{z_{\min}}{z_{\max}}. \tag{3.28}$$

Embedding into our framework

By definition, the output of the PWM on the time interval I_n only depends on the input at time instance $t_n = nh$; hence $\Delta = \Delta S$ with $S = S_h$. Following Section 3.3.1, we consider the interconnection of Δ with SM (see Figure 3.3) represented by

$$\begin{aligned} SM: \quad & \dot{x} = Ax + Bw, \qquad x(0) = 0, \qquad w = \Delta(z), \\ & z = S(Cx) + Sd. \end{aligned} \tag{3.29}$$

At this point we apply the lifting formalism to the whole interconnection. Well-posedness of the lifted interconnection (Figure 3.5) depends on the domain $\widetilde{\mathscr{E}}_e$ of $\widetilde{\Delta}$. We choose $\mathscr{E}_e \subset \mathscr{L}_{2e}$ as the subspace of all functions that are constant on each time interval I_n. Then $\mathscr{E}_e \subset PC[0,\infty)$ and $(\widetilde{\mathscr{E}}_e)_N \subset \widetilde{\mathscr{E}}_e$ for all $N \in \mathbb{N}_0$. Existence and uniqueness of a solution to

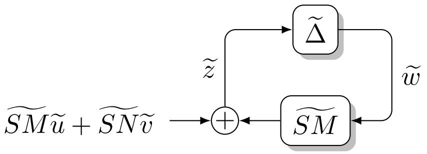

Figure 3.5: Lifted interconnection

(3.29) with Δ replaced by $\tau\Delta$, $\tau \in [0,1]$ can be verified by existence conditions for the defining differential equation: For $n \in \mathbb{N}_0$, $t \in I_n$ we have

$$\begin{aligned} z(t) &= Cx(t_n) + d(t_n) \\ &= Ce^{At_n}x_0 + C\int_0^{t_n} e^{A(t_n - s)}Bw(s)ds + d(t_n). \end{aligned} \tag{3.30}$$

The initial condition x_0 leads to the initial output Cx_0 that, together with $d(0)$, completely defines the output of $\tau\Delta$ on I_0. Since this output is piecewise constant we get a unique solution on the interval I_0. By induction, we acquire a unique solution of (3.29) for $t \in [0,\infty)$ and Δ replaced by $\tau\Delta$. As we constructed this solution on each time interval, existence and uniqueness are preserved during lifting. Moreover, z as in (3.30) is constant on each time interval and hence $\widetilde{z} \in \widetilde{\mathscr{E}}_e$ with the above defined domain $\widetilde{\mathscr{E}}_e$. Consequently the feedback interconnection of $\widetilde{SM}$ with $\widetilde{\Delta}$ is well-posed.

Now we show boundedness of $\widetilde{\Delta}$ on $\widetilde{\mathscr{E}}_e$: Fix $n \in \mathbb{N}_0$, $z \in \mathscr{E}$ and assume that $|z(t_n)| \geq z_{\min}$ (otherwise $\Delta(z)(.) = 0$ on I_n). Then, with $\gamma = z_{\max}/z_{\min}$ and by recalling the definition of τ_n, we have (since z is constant on I_n) that

$$\int_{I_n} [\Delta(z)(t)]^2 dt = \int_{t_n}^{t_n+\tau_n} z_{\max}^2 dt = z_{\max}|z(t_n)|h \leq \gamma |z(t_n)|^2 h$$

$$= \gamma \int_{I_n} z(t)^2 dt. \tag{3.31}$$

As lifting is isometric, this proves $\|\widetilde{\Delta}(\widetilde{z})\|^2 \leq \gamma \|\widetilde{z}\|^2$ for all $\widetilde{z} \in \widetilde{\mathscr{E}}$. In view of the boundedness of $\widetilde{SM}$ it remains to verify items b) and c) in order to apply Theorem 3.4.

IQC description of the PWM

Since $\widetilde{SM}\widetilde{\mathscr{L}_2} + \widetilde{SN}\widetilde{\mathscr{L}_2} \subset \widetilde{\mathscr{E}}$ it suffices to derive IQC relations for signals in $\widetilde{\mathscr{E}}$. The output of the PWM does not change its sign in any time interval (since it is either μ_n or zero) and thus $0 \leq z(t_n)\Delta(z)(t_n)$. For signals $z \in \mathscr{E}$ we hence have $0 \leq z(t)\Delta(z)(t)$ for all $t \in I_n$, $n \in \mathbb{N}_0$. This passivity property readily translates into the IQC

$$\int_0^h (\star)^* \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} z(t_n+\tau) \\ \Delta(z)(t_n+\tau) \end{pmatrix} d\tau \geq 0 \quad \text{for all} \quad z \in \mathscr{E},\, n \in \mathbb{N}_0. \tag{3.32}$$

Again, since (3.32) holds on each time interval, the IQC persists to hold after lifting:

$$\left\langle (\star), \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \widetilde{z} \\ \widetilde{\Delta}(\widetilde{z}) \end{pmatrix} \right\rangle_{\widetilde{\mathscr{L}_2}} = \sum_{n=0}^{\infty} \int_0^h (\star)^* \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} z(t_n+\tau) \\ \Delta(z)(t_n+\tau) \end{pmatrix} d\tau$$
$$\geq 0.$$

In combination, the gain bound in (3.31) and the passivity property above lead to the following set of valid multipliers:

$$P_\kappa := \begin{pmatrix} \gamma I & \kappa I \\ \kappa I & -I \end{pmatrix} \quad \text{with} \quad \gamma = \frac{z_{\max}}{z_{\min}} \quad \text{and} \quad \kappa \geq 0. \tag{3.33}$$

Following our approach, we now define the extended multiplier P_e according to (3.23) and arrive at the FDI (3.24). The validity of this

FDI for all z $\in \mathbb{T}$ is equivalent, via the KYP lemma (Lemma 2.12, [128]), to the existence of some $X = X^T$ and $\kappa \in [0, \infty)$ for which

$$\begin{pmatrix} I & 0 \\ \check{A} & \overline{B}_e \end{pmatrix}^T \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} \begin{pmatrix} I & 0 \\ \check{A} & \overline{B}_e \end{pmatrix} + \begin{pmatrix} \overline{C} & 0 \\ 0 & I \end{pmatrix}^T \overline{P}_e \begin{pmatrix} \overline{C} & 0 \\ 0 & I \end{pmatrix} \prec 0 \quad (3.34)$$

is feasible with $\overline{B}_e := (0, \overline{B})$. We show numerical results in Section 3.4.4.

3.4.3 Averaging approach to PWM

General setup

For $z_{\min} = 0$ the PWM is unbounded even on $\mathscr{E}_e$, which prevents us from incorporating the PWM directly into our framework. To overcome this difficulty, we follow the approach in [67, 69], based on averaging of the modulator output. The underlying idea is related to the equivalent area principle and works for general modulation laws; it relies on the introduction of a so-called equivalent nonlinearity $\phi : \mathbb{R} \to \mathbb{R}$ with the property

$$\int_{I_n} \phi(Sz)(t)\, dt = \int_{I_n} \Delta(z)(t)\, dt \quad \text{for all} \quad n \in \mathbb{N}_0. \quad (3.35)$$

For our definition of the PWM, ϕ takes the form of a saturation [69] with saturation level $z_{\max}$:

$$\phi(x) = \begin{cases} x, & |x| \le z_{\max}, \\ z_{\max} \operatorname{sgn}(x), & |x| > z_{\max}. \end{cases} \quad (3.36)$$

The key idea is to substitute the PWM by the sampled equivalent nonlinearity $\Delta_1 := S\phi = \phi S$ and absorb the resulting error, in integrated form, into the second uncertainty as

$$\Delta_2 = \int (\Delta - \Delta_1), \quad (3.37)$$

where $\int$ denotes the map $z(.) \mapsto \int_0^t z(s) ds$ for $z \in \mathscr{L}_1[0, \infty)$. Hence, we split the PWM into two uncertainties that, as we will see, both fit into our framework. If M has a relative degree of at least two, the interconnection of M realized as $(A, B, C, 0)$ and Δ is equivalent to the one of H with realization $(A, \begin{pmatrix} B & AB \end{pmatrix}, \begin{pmatrix} C^T & C^T \end{pmatrix}^T, 0)$ and Δ_1 as well as Δ_2 [68] (Figure 3.6). Note that the assumption on the relative

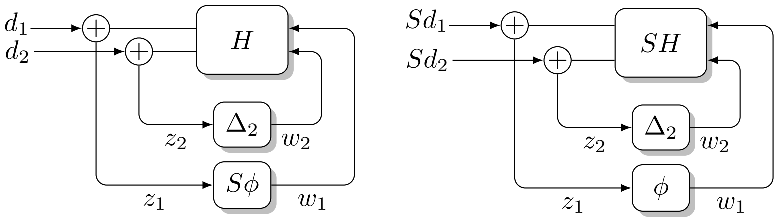

Figure 3.6: Averaging approach interconnections

degree of M is not necessary for the transformation but ensures that the transformed plant H is strictly proper and hence fits into our framework [68, 33].

Averaging in the IQC framework

The method described in [68, 69] uses multipliers to capture the non-linearities but relies on a Lyapunov argument for stability. Due to the generalizations in Theorem 3.4 we are now able to incorporate this approach into the IQC framework. Again it will be crucial to define the uncertainties on an appropriate subspace $\mathscr{E}_e \subset \mathscr{L}_{2e}$. For Δ_2 as defined in (3.37) the required property (3.6) holds: Since $\Delta_2(z_2) = \Delta_2(Sz_2)$ as shown above and $\phi S = S\phi$, we get

$$\Delta_2(Sz_2) = \int (\Delta - S\phi)(Sz_2) = \int \Big(\Delta(Sz) - S^2\phi(z_2)\Big) = \Delta_2(z_2).$$

Again, if we incorporate and move the sample-and-hold operator around in the loop to obtain the interconnection on the right in Figure 3.6, the first uncertainty channel now only contains the saturation ϕ, which allows for standard descriptions with IQCs. By redefining $\Delta_1 = \phi$, the new system equations can be written as

$$\begin{aligned} SH: \quad \dot{x} &= Ax + Bw_1 + ABw_2, \quad x(0) = 0, & w_1 &= \Delta_1(z_1), \\ z_1 &= S(Cx + d_1), & w_2 &= \Delta_2(z_2), \\ z_2 &= S(Cx + d_2). \end{aligned} \tag{3.38}$$

As before, we choose the space $\mathscr{E}_e$ of functions that are constant on each time interval as the domain of definition for the uncertainties. Well-posedness of interconnection (3.38) can then be verified with the very same arguments as in Section 3.4.2.

Lifting of uncertainties

Since Δ_1 is static, lifting is trivial. For $\widetilde{\Delta}_2 := \widetilde{L}\Delta_2\widetilde{L}^{-1}$ we derive an explicit expression: With (3.35), the definition of Δ_2 reduces to

$$\Delta_2(z)(t) = \int_{t_n}^{t} (\Delta - S\phi)(z)(\tau)\, d\tau \quad \text{for} \quad t \in I_n \quad \text{and all} \quad z \in \mathscr{E}.$$

Consequently, for $t \in I_n$ and with $\tau := t - nh \in [0, h)$, we have

$$\begin{aligned} \big(\widetilde{\Delta}_2(\widetilde{z})\big)(n)(\tau) = \big(\widetilde{L}\Delta_2(z)\big)(n)(\tau) &= \Delta_2(z)(nh + \tau) \\ &= \int_{nh}^{nh+\tau} (\Delta - S\phi)(z)(t)\, dt \\ &= \int_0^{\tau} (\Delta - S\phi)(z)(s + nh)\, ds \\ &= \int_0^{\tau} (\Delta - S\phi)\big(\widetilde{z}(n)\big)(s)\, ds. \end{aligned} \tag{3.39}$$

This can now be used to derive IQCs in the lifted domain, as will be discussed next.

IQC description

The uncertainty Δ_1 is a so-called sector-bounded nonlinearity. We capture this property by the multiplier corresponding to the circle criterion[1] (with parameters $\alpha \le 0 \le \beta$); the IQC in the lifted domain then reads as

$$\left\langle \begin{pmatrix} \widetilde{z} \\ \widetilde{\Delta}_1(\widetilde{z}) \end{pmatrix}, \begin{pmatrix} -\alpha\beta\mathbb{1} & \frac{\alpha+\beta}{2}\mathbb{1} \\ \frac{\alpha+\beta}{2}\mathbb{1} & -\mathbb{1} \end{pmatrix} \begin{pmatrix} \widetilde{z} \\ \widetilde{\Delta}_1(\widetilde{z}) \end{pmatrix} \right\rangle_{\widetilde{\mathscr{L}_2}} \ge 0 \quad \text{for} \quad \widetilde{z} \in \widetilde{\mathscr{E}}. \tag{3.40}$$

With ϕ given in (3.36) we can take $\alpha = 0$, $\beta = 1$. Now consider the second uncertainty. For $z \in \mathscr{E}$ and $n \in \mathbb{N}_0$, $\widetilde{z}(n)$ is constant and, with (3.39), we immediately infer

$$\widetilde{z}(n)(\tau)\widetilde{\Delta}_2(\widetilde{z})(n)(\tau) = z(nh) \int_{nh}^{nh+\tau} (\Delta - S\phi)(z)(s)\, ds \ge 0$$

for $\tau \in [0, h)$ and $n \in \mathbb{N}_0$. Nonnegativity follows directly with (3.35) and is proven in [33]. This implies $\int_0^h \widetilde{z}(n)(\tau)\widetilde{\Delta}_2(\widetilde{z})(n)(\tau) d\tau \ge 0$ and, accordingly,

$$\left\langle \begin{pmatrix} \widetilde{z}(n) \\ \widetilde{\Delta}_2(\widetilde{z})(n) \end{pmatrix}, \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \widetilde{z}(n) \\ \widetilde{\Delta}_2(\widetilde{z})(n) \end{pmatrix} \right\rangle_{\mathscr{L}_2[0,h)} \ge 0 \;\; \text{for all} \;\; n \in \mathbb{N}_0,\ \widetilde{z} \in \widetilde{\mathscr{E}}.$$

By taking the sum over all n, we arrive at the IQC

$$\left\langle \begin{pmatrix} \widetilde{z} \\ \widetilde{\Delta}_2(\widetilde{z}) \end{pmatrix}, \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \widetilde{z} \\ \widetilde{\Delta}_2(\widetilde{z}) \end{pmatrix} \right\rangle_{\widetilde{\mathscr{L}_2}} \ge 0 \quad \text{for all} \quad \widetilde{z} \in \widetilde{\mathscr{E}},$$

showing passivity of $\widetilde{\Delta}_2$ on $\widetilde{\mathscr{E}}$. In addition, from [69] we have the following gain bound for $z \in \mathscr{E}$ and all $n \in \mathbb{N}_0$:

$$\int_{nh}^{(n+1)h} |(\Delta - S\phi)(z)(\tau)|^2 \, d\tau \le \frac{Lh^2}{3} \int_{nh}^{(n+1)h} |z(\tau)|^2 \, d\tau, \tag{3.41}$$

where L denotes the Lipschitz constant of ϕ as defined in (3.36). Hence, $L = 1$ and this trivially extends to a gain bound on $\mathscr{L}_2[0, h)$ and, in turn, on $\widetilde{\mathscr{L}_2}$:

$$\|\widetilde{\Delta}_2(\widetilde{z})\|_{\widetilde{\mathscr{L}_2}} \le \frac{h}{\sqrt{3}} \|\widetilde{z}\|_{\widetilde{\mathscr{L}_2}} \quad \text{for all} \quad \widetilde{z} \in \widetilde{\mathscr{E}}.$$

[1] We discuss this and other criteria for sector-bounded nonlinearities in detail in Chapter 5.

Conic combination of both IQCs for $\widetilde{\Delta}_2$ leads to

$$\left\langle \begin{pmatrix} \widetilde{z} \\ \widetilde{\Delta}_2(\widetilde{z}) \end{pmatrix}, \begin{pmatrix} \frac{h^2}{3}\mathbb{1} & \kappa\mathbb{1} \\ \kappa\mathbb{1} & -\mathbb{1} \end{pmatrix} \begin{pmatrix} \widetilde{z} \\ \widetilde{\Delta}_2(\widetilde{z}) \end{pmatrix} \right\rangle_{\widetilde{\mathscr{L}_2}} \geq 0 \quad \text{for all} \quad \widetilde{z} \in \widetilde{\mathscr{E}},\ \kappa \geq 0.$$

With $\gamma = h^2/3$ this implies in the frequency domain that

$$\left\langle (\star), \left(\begin{array}{cc|cc} -\alpha\beta\mathbb{1} & 0 & \frac{\alpha+\beta}{2}\mathbb{1} & 0 \\ 0 & \kappa_1\gamma\mathbb{1} & 0 & \kappa_2\mathbb{1} \\ \hline \frac{\alpha+\beta}{2}\mathbb{1} & 0 & -\mathbb{1} & 0 \\ 0 & \kappa_2\mathbb{1} & 0 & -\kappa_1\mathbb{1} \end{array}\right) \begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \\ \hline \widehat{\widetilde{\Delta}_1(\widetilde{z}_1)} \\ \widehat{\widetilde{\Delta}_2(\widetilde{z}_2)} \end{pmatrix} \right\rangle_{\mathcal{H}_2} \geq 0 \quad (3.42)$$

for all $\kappa_1, \kappa_2 \geq 0$ and $\widetilde{z}_1, \widetilde{z}_2 \in \widetilde{\mathscr{E}}$. In complete analogy to the case of one uncertainty channel, the diagonally augmented structure of this multiplier allows for a reduction to finite dimensions, as shown next.

Reduction to finite dimensional problem

If we apply Lemma 3.7 to SH given in (3.38), the operators defining the lifted system are $\check{A} = e^{Ah}$,

$$\check{B}\psi = \begin{pmatrix} \check{B}_1 & \check{B}_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \int_0^h e^{A(h-\tau)} B\psi_1(\tau)\, d\tau + \int_0^h e^{A(h-\tau)} AB\psi_2(\tau)\, d\tau$$

and

$$(\check{C}\xi)(\tau) = \begin{pmatrix} (\check{C}_1\xi) \\ (\check{C}_1\xi) \end{pmatrix}(\tau) = \begin{pmatrix} C \\ C \end{pmatrix}\xi \quad \text{for all} \quad \tau \in [0, h).$$

Decomposition of the corresponding transfer function according to the uncertainty channels results in

$$\hat{T}(\mathrm{z}) = \begin{pmatrix} \check{C}_1 \\ \check{C}_2 \end{pmatrix} \mathrm{z}(I - \mathrm{z}\,\check{A})^{-1} \begin{pmatrix} \check{B}_1 & \check{B}_2 \end{pmatrix} =: \begin{pmatrix} \hat{T}_{11}(\mathrm{z}) & \hat{T}_{12}(\mathrm{z}) \\ \hat{T}_{21}(\mathrm{z}) & \hat{T}_{22}(\mathrm{z}) \end{pmatrix} \quad \text{for} \quad \mathrm{z} \in \mathbb{T}.$$

Here the partitioning of $\hat{T}$ again induces the usual decomposition of input and output spaces with the operators and embeddings for $j \in \{1, 2\}$:

$$T_{Bj} : \mathscr{L}_2[0, h) \to \mathscr{U}_{Bj} \times \mathscr{V}_{Bj}, \quad T_{Cj} : \mathscr{L}_2[0, h) \to \mathscr{U}_{Cj} \times \mathscr{V}_{Cj},$$

$$J_{Bj} : \mathscr{U}_{Bj} \times \mathscr{V}_{Bj} \to \mathscr{L}_2[0,h), \quad J_{Cj} : \mathscr{U}_{Cj} \times \mathscr{V}_{Cj} \to \mathscr{L}_2[0,h).$$

Then, the FDI constraint on the LTI system corresponding to (3.42) reads as

$$(\star)\left(\begin{array}{cc|cc} -\alpha\beta\mathbb{1} & 0 & \frac{\alpha+\beta}{2}\mathbb{1} & 0 \\ 0 & \kappa_1\gamma\mathbb{1} & 0 & \kappa_2\mathbb{1} \\ \hline \frac{\alpha+\beta}{2}\mathbb{1} & 0 & -\mathbb{1} & 0 \\ 0 & \kappa_2\mathbb{1} & 0 & -\kappa_1\mathbb{1} \end{array}\right)\left(\begin{array}{cc} \hat{T}_{11}(\mathrm{z}) & \hat{T}_{12}(\mathrm{z}) \\ \hat{T}_{21}(\mathrm{z}) & \hat{T}_{22}(\mathrm{z}) \\ \hline \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{array}\right) \prec_\varepsilon 0$$

for all $\mathrm{z} \in \mathbb{T}$, $\kappa_1, \kappa_2 \geq 0$. Now fix $\mathrm{z} \in \mathbb{T}$. With

$$T_{Cj}\hat{T}_{ij}(\mathrm{z})T_{Bi}^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \overline{T}_{ij}(\mathrm{z}) \end{pmatrix},$$

this leads first to

$$(\star)^*\left(\begin{array}{c:c|cc:cc} -\alpha\beta I & 0 & \frac{\alpha+\beta}{2}T_{C1}^2J_{B1}^1 & \frac{\alpha+\beta}{2}T_{C1}^2J_{B1}^2 & 0 & 0 \\ \hdashline \star & \kappa_1\gamma I & 0 & 0 & \kappa_2T_{C2}^2J_{B2}^1 & \kappa_2T_{C2}^2J_{B2}^2 \\ \hline \star & \star & -\mathbb{1} & 0 & 0 & 0 \\ \star & \star & 0 & -I & 0 & 0 \\ \hdashline \star & \star & \star & \star & -\kappa_1\mathbb{1} & \\ \star & \star & \star & \star & 0 & -\kappa_1 I \end{array}\right)\times$$

$$\times\left(\begin{array}{cc|cc} 0 & \overline{T}_{11}(\mathrm{z}) & 0 & \overline{T}_{12}(\mathrm{z}) \\ \hdashline 0 & \overline{T}_{21}(\mathrm{z}) & 0 & \overline{T}_{22}(\mathrm{z}) \\ \hline \mathbb{1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hdashline 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & I \end{array}\right) \prec_\varepsilon 0$$

and then, by applying the Schur complement once again, to the finite dimensional FDI

$$
(\star)^* \left(\begin{array}{cc|cc|cc}
-\alpha\beta I & 0 & \frac{\alpha+\beta}{2}\sqrt{S_{12}^{1*}S_{12}^1} & \frac{\alpha+\beta}{2}S_{22}^1 & 0 & 0 \\
\star & \kappa_1\gamma I & 0 & 0 & \kappa_2\sqrt{S_{12}^{2*}S_{12}^2} & \kappa_2 S_{22}^2 \\
\hline
\star & \star & -\mathbb{1} & 0 & 0 & 0 \\
\star & \star & 0 & -I & 0 & 0 \\
\hline
\star & \star & \star & \star & -\kappa_1 I & \\
\star & \star & \star & \star & 0 & -\kappa_1 I
\end{array}\right) \times
$$

$$
\times \left(\begin{array}{cc|cc}
0 & \overline{T}_{11}(\mathrm{z}) & 0 & \overline{T}_{12}(\mathrm{z}) \\
\hline
0 & \overline{T}_{21}(\mathrm{z}) & 0 & \overline{T}_{22}(\mathrm{z}) \\
\hline
I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
\hline
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{array}\right) \prec 0,
$$

where S_{22}^j and $S_{12}^{j*}S_{12}^j$ denote the matrix representations of $T_{B_j}^2 J_{C_j}^2$ and $(T_{B_j}^1 J_{C_j}^2)^* T_{B_j}^1 J_{C_j}^2$, respectively, for $j = 1, 2$. For computations, we can equivalently represent this FDI as an LMI according to (3.34).

3.4.4 Computational results

In order to compare our results with the analysis techniques in [82] and [69, 33], we compute the largest sampling time h for which the respective approaches still guarantee stability of the feedback interconnection (3.25).

Direct approach

We first contrast our approach to the preliminary results in [33], where a model for attitude control is considered based on the averaging approach in [69]. For $z_{\min} > 0$ and $z_{\max} = 1$ the equivalent nonlinearity (3.36) takes the form

$$\phi(x) = \begin{cases} 0, & |x| < z_{\min}, \\ x, & z_{\min} \leq |x| \leq 1, \\ \operatorname{sgn}(x), & |x| > 1. \end{cases}$$

By (3.41) the jump of $\phi(.)$ at $x = z_{\min}$ renders Δ_2 unbounded. To model this behavior approximately, large Lipschitz constants L and hence, with (3.41), large gain bounds are considered in [33]. This only guarantees stability of the approximated system, while our multiplier (3.33) does not depend on such a heuristic argument and hence guarantees stability for the original problem. We consider the two values $z_{\min,1} = 0.05$ and $z_{\min,2} = 0.01$, corresponding to lower bounds on τ_n as in (3.28) as well as gain bounds γ_1, γ_2 as in (3.33). Table 3.1 shows the maximal sampling periods with guaranteed stability for the systems

$$M_1(s) = \frac{s+1}{(s+2)(s^2+10s+41)} \quad \text{and} \quad M_2(s) = \frac{1}{5(s+10)^2},$$

obtained by applying the averaging method with the previously described approximation for $L \in \{1, 25, 100\}$ and the ones achieved by our approach. Since the bounds on h derived in [69] are improved in [33], we only compare our results with the latter.

Table 3.1: Maximal sampling times, $z_{\min} > 0$

	$L=1$	$L=25$	$L=100$	IQC γ_1	IQC γ_2
M_1	1.63	0.28	0.11	56.70	11.34
M_2	3.68	0.73	0.36	$5 \cdot 10^3$	$1 \cdot 10^3$

For M_1, the second row of Table 3.1 illustrates that even if we disregard the jump completely ($L = 1$), the averaging approach leads to significantly smaller sampling times if compared to the IQC approach for both γ_1, γ_2. For better approximations, i.e., larger L, the improvement achieved by the IQC approach becomes greater. This is even more pronounced for M_2.

Averaging approach

Let us also consider the case of $z_{\min} = 0$. Hou and Michel propose a method in [82] to translate the continuous-time interconnection of M realized by $(A, B, C, 0)$ and Δ into an autonomous discrete-time system with a dynamic matrix $e^{Ah}(I - hW_n)$. Here W_n depends on A, B, C and the time constant τ_n. Since A is Hurwitz stable, there exists $P > 0$ such that $(e^{Ah})^T P e^{Ah} - P = I$. The basic idea is to define the quadratic Lyapunov function $V(x) = x^T P x$ and derive bounds on the sampling time such that V is also a Lyapunov function for the system

$$x(t_{n+1}) = e^{Ah}(I - hW_n)x(t_n) \tag{3.43}$$

when $h > 0$. Then it is shown that stability of this system implies stability of the original interconnection. Since W_n depends on τ_n and the chosen realization of G, in order to prove stability we need to show that there exists some realization such that (3.43) is stable for all $\tau_n \in [0, h)$. This is implemented using 1000 randomly generated state coordinate transformations and a time grid on $[0, h)$. The maximal sampling period for which the system is stable is then approximated by optimizing over different realizations and grid refinements. To compare our results, we consider the family of LTI systems given by the transfer functions

$$M_a(s) := \frac{1}{(s + a)^2} \quad \text{with} \quad a \in \{0.1, 0.3, 0.4, 0.5\}.$$

Table 3.2 shows the maximal sampling period for which the respective approaches prove stability for a given a. Since we could not establish stability for $h > 10^{-9}$ and any a with the averaging approach in [69],

Table 3.2: Maximal sampling times, $z_{\min} = 0$

a	0.1	0.3	0.4	0.5
Hou, Michel	$< 10^{-6}$	$< 10^{-5}$	0.075	0.39
IQC approach	0.0012	0.032	0.075	0.15

this is not considered here. For $a = 0.1$ and $a = 0.3$ we were not able to prove stability with [82] even for $h = 10^{-6}$ and $h = 10^{-5}$, respectively. Here the IQC framework leads to much better results. For $a = 0.4$ we obtain similar results and for $a = 0.5$ the maximal sampling periods obtained by Hou and Michel are better than ours. However, as the approach in [82] relies on randomized realizations, it is computationally significantly more expensive than solving the LMI corresponding to (3.43) and there is no guarantee of finding an optimal solution. Moreover, the latter approach verifies stability only for the chosen grid points in $[0, h)$ and hence there is no certification for all $\tau_n \in [0, h)$.

3.5 Summary and possible extensions

In this chapter we present a first generalization of the IQC framework that allows us to significantly broaden its range of applications, now also including sampled-data type systems. We demonstrate how to apply lifting and frequency-domain techniques in order to render the resulting stability test computational without the need for approximations. The effectiveness is illustrated for the particular case of a PWM feedback interconnection that has so far been impossible to treat within IQC theory. This permits to draw significantly less conservative stability conclusions for such systems, if compared to state of the art techniques.

As is probably best seen from the realization of $\widetilde{SM}$ given in Lemma 3.7, the problem considered in this chapter only slightly differs from the question of deciding stability of an uncertain sampled-data interconnection [50, 65, 16]. The main difference is that the sampling S

at the output of the linear system results in a lifted system with zero direct feedthrough. For general uncertain sampled-data interconnections this is not the case and, thus, the transfer matrix will be a genuine infinite dimensional operator; a decomposition as in (3.12) will not result in an essentially finite dimension mapping. However, we believe that with a suitable approximation of the infinite dimensional operator, the framework established in this chapter may be extended to comprise uncertain sampled-data interconnections.

A further generalization worth investigating is the application of more general multipliers for the equivalent nonlinearity. As discussed in detail in Chapter 5, saturations allow for much tighter capturing if using frequency dependent (and unstructured) multipliers. Yet, the reduction to a finite dimensional LMI given herein relies on the use of static, diagonally structured multipliers. Thus we expect a further improvement of the obtained results for more sophisticated IQC descriptions of the uncertainty.

Chapter 4

A general framework for stability and performance analysis based on dissipativity constraints

4.1 Introduction

In Chapter 2, we describe the emergence of very general stability criteria from the early contributions of Zames and Yakubovich to the later, more sophisticated ones by Safonov and Teel. In this line of developments, the IQC framework as portrayed in Section 2.2 may be seen as a step backwards, at least in terms of generality. However, in conjunction with the celebrated KYP lemma (Lemma 2.12), it constitutes the most effective stability test of the ones discussed above if it comes to numerical verification.

This chapter is devoted to a further relaxation of the assumptions in our preliminary stability result, Theorem 3.4, towards general function spaces. Furthermore, we generalize the notion of truncation (Definition 2.1) towards a continuation based approach that allows for the incorporation of signal spaces with additional regularity requirements. We will highlight this advantage by considering the particular case of Sobolev spaces that are, as demonstrated in the subsequent chapters, of great practical importance. Following the line of thought of Megretski and Rantzer, we not only present an abstract stability result but a whole framework that also comprises novel performance measures and the means for verification of stability and a given performance criterion.

In contrast to what is typically seen in the literature, the proposed separating functionals, both for stability and performance, are derived from the underlying principle of dissipation [180, 181]. This direct connection to dissipation theory is a key ingredient as it enables the verification of stability and performance using LMIs by means of a generalization of the KYP lemma that is valid on Sobolev spaces. We illustrate the application of our framework for the example of an interconnection involving time-varying parametric uncertainties. For a clear and concise presentation of the framework we move some proofs to the appendix.

Finally, we emphasize that the present chapter has its roots in the publications [58, 56] where stability and performance are considered separately and from a different point of view. In the derivation at hand, this artificial partitioning is now removed in order to emphasize our unifying approach to global stability and performance analysis. This necessitated some adaption to the statement of the main stability theorem and its proof that now permit a natural transition from stability to performance analysis. Still, apart from these changes, large parts of the text overlap with [58, 56].

4.2 Function spaces, causality and boundedness

Let us begin by rigorously specifying the technical requirements on the underlying function spaces that are imposed in the sequel. The following construction is closely related to the one of resolution Hilbert spaces as discussed in the exceptionally well-written monograph [54]. As we will see, the introduced generalizations require slight modifications to the definitions of causality, boundedness and well-posedness if compared to the ones stated in Chapters 2 and 3.

Assumption 4.1.
Fix $k \in \mathbb{N}$ and let

a) $(\mathcal{X}_T, \|\cdot\|_{\mathcal{X}_T})_{T>0}$ be a family of normed function spaces with elements $\mathcal{X}_T \ni u : [0,T] \to \mathbb{R}^k$;

b) $\mathcal{X}_e := \left\{ u : [0,\infty) \to \mathbb{R}^k \;\middle|\; u_T := u|_{[0,T]} \in \mathcal{X}_T \text{ for all } T > 0 \right\}$ and define the family of semi-norms $\|\cdot\|_{\mathcal{X}_e,T}$ on $\mathcal{X}_e$ as $\|u\|_{\mathcal{X}_e,T} := \|u_T\|_{\mathcal{X}_T}$ for all $T > 0$ and $u \in \mathcal{X}_e$;

c) $\mathcal{X} := \{u \in \mathcal{X}_e \mid \|u\|_{\mathcal{X}} := \sup_{T>0} \|u\|_{\mathcal{X}_e,T} < \infty\}$;

d) there exist some $K_{\mathcal{X}} \geq 1$ such that for every $u \in \mathcal{X}_e$ and $T > 0$ there is a function $u^T \in \mathcal{X}$ with[1]

$$u_T = (u^T)_T \quad \text{and} \quad \|u^T\|_{\mathcal{X}} \leq K_{\mathcal{X}} \|u\|_{\mathcal{X}_e,T}. \tag{4.1}$$

⋆

We say that $\mathcal{X}_e$ **satisfies Assumption 4.1** if there exists a family of normed spaces as in 4.1 a), $\mathcal{X}_e$ and $\mathcal{X}$ are defined as in 4.1 b) and c), respectively, and d) holds.

Remark 4.2.
Note that an essential requirement in the previous chapter concerns $\mathscr{E}_e$, the domain of definition of Δ, which is assumed to satisfy the property $(\mathscr{E}_e)_T \subset \mathscr{E}$ (3.1). This assumption essentially dates back to the work of Zames [195] with applications mainly focussing on $\mathscr{L}_p$ as underlying function spaces. However, if $\mathscr{E}_e$ denotes a space consisting of smooth signals $u \in \mathscr{E}_e$, the instantaneous truncation $P_T u$ is typically not even continuous. This poses a major motivation for Assumption 4.1.d), which merely requires the existence of some continuation. By Assumption 4.1, every element $u \in \mathcal{X}_e$ admits a continuation $u^T \in \mathcal{X}$ that satisfies the

[1] Note that the definition of u_T now differs from Chapters 2 and 3. The classical (instantaneous) truncation at time T of some signal $u : [0,\infty) \to \mathbb{R}^n$ is denoted by $P_T u$ with the truncation operator P_T defined in Definition 2.1.

key property (4.1). Intuitively speaking, given some $T > 0$ and some signal u, we allow, e.g., for the steering of $u(T)$ to zero – as long as the resulting error if compared to the instantaneous truncation can be bounded as stated. $\star$

Based on Assumption 4.1, it is now natural to define causality and boundedness of operators as follows.

Definition 4.3.
Let $\mathscr{U}_e$, $\mathscr{Z}_e$ satisfy Assumption 4.1. An operator $S : \mathscr{U}_e \to \mathscr{Z}_e$ is said to be **causal** if

$$\widetilde{u}_T = u_T \quad\Longrightarrow\quad S(u)_T = S(\widetilde{u})_T \quad \text{for all} \quad T > 0 \quad \text{and} \quad u,\, \widetilde{u} \in \mathscr{U}_e.$$

The **gain** $\|S\|$ of $S : \mathscr{U}_e \to \mathscr{Z}_e$ is the infimal $\gamma \geq 0$ such that there exists some $\gamma_0 \in \mathbb{R}$ with

$$\|S(u)\|_{\mathscr{Z}} \leq \gamma\|u\|_{\mathscr{U}} + \gamma_0 \quad \text{for all} \quad u \in \mathscr{U}.$$

S is **bounded** if $\|S\|$ is finite. Recall that we may choose $\gamma_0 = 0$ if S is linear. $\star$

A sufficient condition for boundedness of S is given by

$$\|S(u)\|_{\mathscr{Z}_e,T} \leq \widetilde{\gamma}\|u\|_{\mathscr{U}_e,T} + \gamma_0 \quad \text{for all} \quad T > 0,\ u \in \mathscr{U}_e. \tag{4.2}$$

Indeed, $u \in \mathscr{U}$ implies $u \in \mathscr{U}_e$ and hence

$$\|S(u)\|_{\mathscr{Z}_e,T} \leq \widetilde{\gamma}\|u\|_{\mathscr{U}_e,T} + \gamma_0 \leq \widetilde{\gamma}\sup_{T>0}\|u\|_{\mathscr{U}_e,T} + \gamma_0 = \widetilde{\gamma}\|u\|_{\mathscr{U}} + \gamma_0;$$

the claim follows by taking the supremum over $T > 0$ on the left. The fact that the converse holds if S is also causal will play a key role in the derivation of our stability result.

Lemma 4.4
Let $S : \mathscr{U}_e \to \mathscr{Z}_e$ be a causal operator satisfying $\|S(u)\|_{\mathscr{Z}} \leq \gamma\|u\|_{\mathscr{U}} + \gamma_0$ for some pair $\gamma \geq 0$, $\gamma_0 \in \mathbb{R}$ and all $u \in \mathscr{U}$. Then (4.2) *holds with $\widetilde{\gamma} = \gamma K_{\mathscr{U}}$.*

Proof. Let $T > 0$ and $u \in \mathscr{U}_e$. Since $u^T \in \mathscr{U}$ and $u_T = (u^T)_T$, causality of S implies

$$\|S(u)\|_{\mathscr{Z}_e,T} = \|S(u)_T\|_{\mathscr{Z}_T} = \|S(u^T)_T\|_{\mathscr{Z}_T} = \|S(u^T)\|_{\mathscr{Z}_e,T} \leq \|S(u^T)\|_{\mathscr{Z}}.$$

With boundedness of S and (4.1) we further obtain

$$\begin{aligned}\|S(u)\|_{\mathscr{Z}_e,T} &\leq \|S(u^T)\|_{\mathscr{Z}} \leq \gamma\|u^T\|_{\mathscr{U}} + \gamma_0 \\ &\leq \gamma K_{\mathscr{U}} \|u\|_{\mathscr{U}_e,T} + \gamma_0 = \widetilde{\gamma}\|u\|_{\mathscr{U}_e,T} + \gamma_0.\end{aligned}$$

□

4.3 Fundamental stability result

Having established the basic properties of the function spaces and operators under consideration, let us now turn to the interconnection and our central stability result. With $M : \mathscr{U}_e \to \mathscr{Z}_e$, $\Delta : \mathscr{Z}_e \to \mathscr{W}_e$ and $N : \mathscr{V} \to \mathscr{Z}$ we consider the feedback interconnection

$$z = Mw + Mu + N(v), \qquad w = \Delta(z) \tag{4.3}$$

with external signals $(u, v) \in \mathscr{U}_e \times \mathscr{V}$ as in Figure 4.1 under the following standing hypotheses.

Assumption 4.5.

a) $\mathscr{U}_e, \mathscr{W}_e, \mathscr{Z}_e$ are function spaces satisfying Assumption 4.1 and the compatibility condition $\mathscr{W}_e \subset \mathscr{U}_e$ with the natural inclusion map $J : \mathscr{W}_e \to \mathscr{U}_e$, $Jw = w$ being bounded.

b) $M : \mathscr{U}_e \to \mathscr{Z}_e$ and $\Delta : \mathscr{Z}_e \to \mathscr{U}_e$ are causal and bounded while M is linear.

c) With a subset $\mathscr{V}$ of a normed space with norm $\|\cdot\|_{\mathscr{V}}$, the map $N : \mathscr{V} \to \mathscr{Z}$ satisfies $\|N(v)\|_{\mathscr{Z}} \leq \gamma_N \|v\|_{\mathscr{V}}$ for some $\gamma_N \geq 0$ and all $v \in \mathscr{V}$. The possible choice $\mathscr{V} = \{0\}$ implies $N(\mathscr{V}) = \{0\}$.

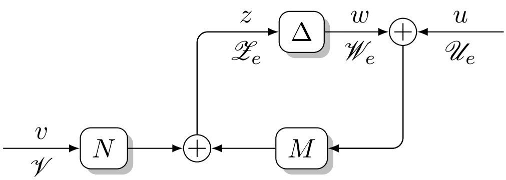

Figure 4.1: Interconnection for main stability result.

⋆

This general setup necessitates a slight modification of the standard definition of well-posedness that will be shown to pose no extra trouble in subsequent applications (see Chapter 5).

Definition 4.6.
The interconnection (4.3) is **well-posed** if for each $u \in \mathscr{U}_e$, $v \in \mathscr{V}$, and $\tau \in [0,1]$ there exists a unique $z \in \mathscr{Z}_e$ satisfying $z - M\tau\Delta(z) = Mu + N(v)$ and such that the correspondingly defined response map $(u,v) \to z = R_\tau(u,v)$ is causal in the first argument, i. e.,

$$R_\tau(u,v)_T = R_\tau(\widetilde{u},v)_T \quad \text{for all} \quad T > 0,\ v \in \mathscr{V}$$
$$\text{and} \quad u, \widetilde{u} \in \mathscr{U}_e \quad \text{with} \quad u_T = \widetilde{u}_T.$$

If $\mathscr{Z}_e = \mathscr{L}_{2e}^k$, well-posedness is clearly a consequence of $I - \tau M\Delta : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^k$ having a causal inverse for all $\tau \in [0,1]$, since the response map then equals $R_\tau(u,v) = (I - M\Delta)^{-1}(Mu + N(v))$. For $\mathscr{V} = \mathscr{L}_2$ and $N = \mathrm{id}$, this property is equivalent to the well-posedness defined in Definition 2.5, which is from now on referred to as **well-posedness in the classical sense** of (4.3). ⋆

In Theorem 3.4 we introduce a function $\delta_0 : \mathscr{V} \to [0,\infty)$ that allowed to incorporate IQCs depending on the initial conditions of the LTI system into the framework (see Theorem 3.4 b)). This was inspired by the treatment of the Popov stability criterion in (2.24). Having the dependence on the initial condition in mind, we further generalize this idea in the subsequent theorem and make use of a functional $l : \mathscr{Z} \to [0, \infty)$ of which we require the following property:

$$\forall z \in \mathscr{Z}_e,\ \exists c > 0 : \quad \sup_{T>0} l(z^T) < c \ \text{ for all continuations } z^T \text{ of } z. \tag{4.4}$$

If l is bounded and only depends on the initial value $z(0)$ (assuming that the evaluation of z at $t = 0$ is well-defined), (4.4) is obviously satisfied. As will become clear in the subsequent chapter, this change is instrumental for the generalization of the Popov criterion towards only proper LTI systems.

Theorem 4.7
In addition to Assumption 4.5 let

a) the feedback system (4.3) *be well-posed;*

b) $\Sigma : \mathscr{Z} \times \mathscr{W} \to \mathbb{R}$ *be a map such that, for some* $\sigma_{ij} \in \mathbb{R}$ *and all* $(u, v, w) \in \mathscr{U} \times \mathscr{V} \times \mathscr{W}$,

$$\Sigma \begin{pmatrix} Mw + Mu + N(v) \\ w \end{pmatrix} - \Sigma \begin{pmatrix} Mw \\ w \end{pmatrix} \leq$$
$$\leq \begin{pmatrix} \|w\|_{\mathscr{W}} \\ \|u\|_{\mathscr{U}} \\ \|v\|_{\mathscr{V}} \end{pmatrix}^T \begin{pmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \begin{pmatrix} \|w\|_{\mathscr{W}} \\ \|u\|_{\mathscr{U}} \\ \|v\|_{\mathscr{V}} \end{pmatrix}; \tag{4.5}$$

c) there exist some function $l : \mathscr{Z} \to [0, \infty)$ *satisfying* (4.4) *and such that*

$$\Sigma \begin{pmatrix} z \\ \tau\Delta(z) \end{pmatrix} \geq -l(z)^2 \tag{4.6}$$

for $\tau \in [0, 1]$, $z = Mu + N(v)$ *and* $(u, v) \in \mathscr{U} \times \mathscr{V}$;

d) there exists some $\varepsilon > 0$ *such that*

$$\Sigma \begin{pmatrix} Mw \\ w \end{pmatrix} \leq -\varepsilon \|w\|_{\mathscr{W}}^2 \quad \text{for all} \quad w \in \mathscr{W}. \tag{4.7}$$

Then the interconnection (4.3) *is stable, i. e., there exist some* $\gamma > 0$, $\gamma_0 \in \mathbb{R}$ *(only depending on* J, M, N *and* Σ*) such that* $z := R(u, v) \in \mathscr{Z}$ *and*

$$\|z\|_{\mathscr{Z}} \leq \gamma(\|u\|_{\mathscr{U}} + \|v\|_{\mathscr{V}}) + \gamma_0 l(z) \quad \textit{for all} \quad (u, v) \in \mathscr{U} \times \mathscr{V}. \tag{4.8}$$

Proof. A proof of Theorem 4.7 is found in Appendix C.3.1. □

The most noticeable difference with Theorems 3.4 is certainly the ability to work with general function spaces. Yet, as will become apparent shortly, the concept of continuation of functions in the extended space in conjunction with (4.1) is equally important.

Remark 4.8.
For $\mathcal{Y} := \mathscr{Z} \times \mathscr{W}$ with norm $|||(z, w)||| = \sqrt{\|z\|^2_{\mathscr{Z}} + \|w\|^2_{\mathscr{W}}}$, all maps Σ in the subsequent chapters are defined as $\Sigma(x) = \langle x, x\rangle$, where $\langle ., .\rangle : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ is additive in the first and second argument and continuous in the following sense: there exists a constant c with

$$|\langle x, y\rangle| \leq c|||x||| |||y||| \quad \text{for all} \quad x, y \in \mathcal{Y}.$$

It is easily seen that this together with the additivity in both arguments implies Theorem 4.7 b). Indeed, let $x = (Mw, w)$ and $y = (Mu + N(v), 0)$. Then $\langle x + y, x + y\rangle - \langle x, x\rangle = \langle y, y\rangle + \langle x, y\rangle + \langle y, x\rangle \leq c(|||y|||^2 + 2|||x||| |||y|||)$ and thus we infer (4.5) with boundedness of M, N and J. ⋆

4.4 From stability to performance analysis

Apart from stability, our primary concern is the verification of some performance criterion. Hence, consider the interconnection depicted in Figure 4.2 as governed by the equations

$$z = Mw + Nd, \qquad w = \Delta(z), \qquad d \in \mathscr{D} \tag{4.9}$$

$$e = N_{21}w + N_{22}d \tag{4.10}$$

with linear systems $N_{21} : \mathscr{U} \to \mathscr{E}$ and $N_{22} : \mathscr{D} \to \mathscr{E}$. Note that in the performance setup the operator $N : \mathscr{D} \to \mathscr{Z}$ is assumed to be LTI and we denote the (single) external disturbance by d. Yet as demonstrated next, the previously developed stability results immediately apply.

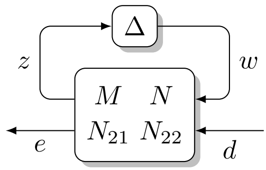

Figure 4.2: Performance interconnection

Suppose that the corresponding uncertainty loop (4.3) is well-posed and assume that we are given a map $\Sigma_p : \mathscr{E} \times \mathscr{D} \to \mathbb{R}$ satisfying

$$\Sigma_p \begin{pmatrix} e \\ 0 \end{pmatrix} \geq 0 \quad \text{for all} \quad e \in \mathscr{E}. \tag{4.11}$$

Our objective is to verify the performance criterion

$$\Sigma_p \begin{pmatrix} e \\ d \end{pmatrix} \leq -\varepsilon \|d\|_{\mathscr{D}}^2 + l_p(z)^2 \quad \text{for all} \quad d \in \mathscr{D}, \tag{4.12}$$

for some $\varepsilon > 0$, and some map $l_p : \mathscr{Z} \to [0, \infty)$. As is standard in dissipation theory, the function l_p quantifies the price to pay for non-zero initial conditions; as it is intertwined with the map l in (4.6), we will keep it as a degree of freedom that will be specified later. Let us now argue that stability and performance are guaranteed if there exists $\varepsilon > 0$ and some map Σ satisfying (4.5) and (4.6) such that for all $(w, d) \in \mathscr{W} \times \mathscr{D}$ we have

$$\Sigma \begin{pmatrix} Mw + Nd \\ w \end{pmatrix} + \Sigma_p \begin{pmatrix} N_{21}w + N_{22}d \\ d \end{pmatrix} \leq -\varepsilon\big(\|d\|_{\mathscr{D}}^2 + \|w\|_{\mathscr{W}}^2\big). \tag{4.13}$$

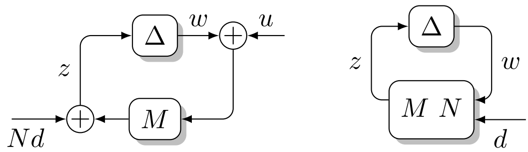

Figure 4.3: From stability to performance

Indeed, for $d = 0$ we recover (4.7) from (4.11), (4.13) and thus both interconnections depicted in Figure 4.3 are stable. Moreover, we may use the loop equations and the fact that $\mathscr{W} \subset \mathscr{U}$ (a consequence of Assumption 4.5 a)) in order to infer that (4.13) implies

$$\Sigma \begin{pmatrix} z \\ \Delta(z) \end{pmatrix} + \Sigma_p \begin{pmatrix} e \\ d \end{pmatrix} \leq -\varepsilon(\|d\|_{\mathscr{D}}^2 + \|\Delta(z)\|_{\mathscr{W}}^2)$$

for all $(w, d) \in \mathscr{W} \times \mathscr{D}$. This, in turn, implies with (4.6) that

$$\Sigma_p \begin{pmatrix} e \\ d \end{pmatrix} \leq -\varepsilon\|d\|_{\mathscr{D}}^2 + l(z)^2 \tag{4.14}$$

holds for all $(w, d) \in \mathscr{W} \times \mathscr{D}$. In conclusion, we have proven the following corollary.

Corollary 4.9
Let Σ_p satisfy (4.11) *and let the assumptions of Theorem 4.7 with* (4.7) *replaced by* (4.13) *be valid. Then there exist $\varepsilon > 0$ such that* (4.12) *holds with $l_p = l$.*

Once stability of (4.3) has been verified and as the above arguments reveal, performance is already guaranteed if the map Σ in (4.13) is exchanged by any map $\widetilde{\Sigma}$ satisfying $\widetilde{\Sigma}(z, \Delta(z)) \geq -l(z)^2$ for all $z = Mw + Nd$ with $(w, d) \in \mathscr{W} \times \mathscr{D}$. This observation opens the way to guarantee stability and performance with different maps Σ and $\widetilde{\Sigma}$, which is often not emphasized in the literature but might be of practical relevance in applications.

4.5 Application to Sobolev spaces

Let us now bring the above introduced generalizations to life by considering the specific example of LTI systems and quadratic forms defined on Sobolev spaces $\mathscr{H}^r$ for $r \in \mathbb{N}_0$ defined as follows.

Definition 4.10.
Let $r \in \mathbb{N}_0$. Then $\mathscr{H}^r$ and $\mathscr{H}_e^r$ denote the (extended) Sobolev spaces of functions $u : [0, \infty) \to \mathbb{R}^k$ with $\partial^j u \in \mathscr{L}_2$, or $\partial^j u \in \mathscr{L}_{2e}$ for $j \in \{0, \dots, r\}$, respectively. $\mathscr{H}^r$ is equipped with the norm $\|u\|_r^2 = \sum_{j=0}^r \|\partial^j u\|^2$. ⋆

4.5.1 Sobolev spaces

We first show that the spaces $\mathscr{H}^r$ satisfy all requirements on the underlying function spaces. To this end, let $\mathscr{H}^r[0,T]$ denote the space of functions $u : [0,T] \to \mathbb{R}^k$ with $\partial^j u \in \mathscr{L}_2[0,T]$ for $j \in \{0, \dots, r\}$ and equipped with the norm $\|u\|^2_{\mathscr{H}^r[0,T]} = \sum_{j=0}^r \|\partial^j u\|^2_{\mathscr{L}_2[0,T]}$. Then we obtain $\mathscr{H}_e^r$ and $\mathscr{H}^r$ by proceeding in accordance with Assumption 4.1. As already mentioned, $P_T u$ is typically not contained in $\mathscr{H}^r$ if $u \in \mathscr{H}^r$. Still we can prove the existence of a sufficiently smooth continuation in order to meet all requirements of Assumption 4.1. For clarity of display, we define the differential operator

$$\mathcal{D}^r z := \mathrm{col}(z, \partial z, \dots, \partial^r z) \quad \text{on} \quad \mathscr{H}^r.$$

Lemma 4.11
Let $r \in \mathbb{N}_0$. Then $\mathscr{H}_e^r$ and $\mathscr{H}^r$ satisfy Assumption 4.1 d).

Proof. For $r = 0$ and $u^T := P_T u$ we obtain (4.1) with $K_{\mathscr{H}^0} = 1$. Suppose $r > 0$. If $T > 0$ and $\bar{u} \in \mathscr{H}_e^r$ fix $\xi := (\mathcal{D}^{r-1}\bar{u})(T)$. We then construct $u := \bar{u}^T$ on $[T, \infty)$ with an optimal constant $K_{\mathscr{H}^r}$ by minimizing the functional

$$\int_T^\infty \sum_{j=0}^r \|\partial^j u(t)\|^2 \, dt$$

subject to $u \in \mathscr{H}^r$, $u_T = \bar{u}_T$ and $(\mathcal{D}^{r-1}u)(T) = \xi$. This is achieved by solving a linear quadratic optimal control problem with stability. Indeed, with $x := \mathcal{D}^{r-1}u$, $w := \partial^r u$, our problem amounts to minimizing the cost functional $\int_T^\infty x(t)^T x(t) + w(t)^T w(t)\, dt$ over all trajectories of

$$\dot{x} = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ I \end{pmatrix} w, \qquad x(T) = \xi \tag{4.15}$$

with $w \in \mathscr{L}_2$ and $x \in \mathscr{L}_2$. Since the system in (4.15) is stabilizable and with the special cost function, there exists a stabilizable solution P of the corresponding LQ Riccati equation such that the optimal cost equals $\xi^T P \xi$. Now recall that Proposition 3 in [19] (see also [20]) implies

$$\max_{t\in[0,T]} \|v(t)\| \le \|v\|_{\mathscr{H}^1[0,T]} \quad \text{for all} \quad T > 0,\ v \in \mathscr{H}_e^1. \tag{4.16}$$

Thus, with the maximal eigenvalue of P, denoted by $\lambda_{\max}(P)$, we get

$$\xi^T P \xi \le \lambda_{\max}(P) \sum_{l=0}^{r-1} \|\partial^l \bar{u}(T)\|^2 \le 2\lambda_{\max}(P) \|\bar{u}\|^2_{\mathscr{H}^r[0,T]}.$$

This proves the claim with $K_{\mathscr{H}^r} \le 1 + \sqrt{2\lambda_{\max}(P)}$. □

4.5.2 Systems and quadratic forms on $\mathscr{H}^r$

Let us now consider operators and quadratic forms on $\mathscr{H}^r$. In particular, we assume that the linear system $G = \left(\begin{smallmatrix} M & N \\ N_{21} & N_{22} \end{smallmatrix}\right)$ in (4.9), (4.10) is LTI and realized as

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 d, \qquad x(0) = 0, \\ z &= C_1 x + D_{11} w + D_{12} d, \\ e &= C_2 x + D_{21} w + D_{22} d \end{aligned} \tag{4.17}$$

with $A \in \mathbb{R}^{n\times n}$ being Hurwitz. Then G defines a bounded operator on $\mathscr{H}^r \times \mathscr{H}^r$ for any $r \in \mathbb{N}_0$. With a pair of symmetric matrices Q_1, Q_2, define the quadratic form

$$\Sigma_{(Q_1,Q_2)} \begin{pmatrix} z \\ w \end{pmatrix} := \sigma_{Q_1} \begin{pmatrix} z \\ w \end{pmatrix} + [(\mathcal{D}^{r-1}w)(0)]^T Q_2 [(\mathcal{D}^{r-1}w)(0)] \tag{4.18}$$

on $\mathscr{H}^r \times \mathscr{H}^r$, where

$$\sigma_{Q_1}\begin{pmatrix} z \\ w \end{pmatrix} := \int_0^\infty \begin{pmatrix} \mathcal{D}^r z(t) \\ \mathcal{D}^r w(t) \end{pmatrix}^T Q_1 \begin{pmatrix} \mathcal{D}^r z(t) \\ \mathcal{D}^r w(t) \end{pmatrix} dt. \tag{4.19}$$

For $r = 0$ the matrix Q_2 is empty; for $r > 0$ it will be instrumental to incorporate the initial values of w and its derivatives; still it is easily seen that $\Sigma_{(Q_1,Q_2)}$ satisfies (4.5).

Also performance criteria are modeled with the map Σ_p in (4.12) defined as

$$\sigma_P\begin{pmatrix} e \\ d \end{pmatrix} := \int_0^\infty \begin{pmatrix} \mathcal{D}^r e(t) \\ \mathcal{D}^r d(t) \end{pmatrix}^T P \begin{pmatrix} \mathcal{D}^r e(t) \\ \mathcal{D}^r d(t) \end{pmatrix} dt, \tag{4.20}$$

where P is some specified symmetric matrix. This allows, e.g., to put special emphasis on some derivatives or to consider weighted combinations thereof. Moreover, we obtain a generalization of $\mathscr{L}_2$-gain to $\mathscr{H}^r$-gain performance with bound γ by setting $P = \operatorname{diag}(I, -\gamma^2 I)$. However, note that $\mathscr{H}^r$-gain bounds do not, in general, translate right away into bounds on the $\mathscr{L}_2$-gain. The form σ_P as in (4.20) is sometimes called to be a quadratic differential form as considered, e. g., in [186, 185] in the context of behavioral systems. Yet, we are not aware of any references in which such performance specifications have been considered in the context of robustness analysis with (integral) quadratic constraints.

4.5.3 Verification of stability and performance

Let us now embed the stability and performance analysis for LTI systems on Sobolev spaces into classical dissipation theory and thus open the way for the verification of (4.13) using LMIs. Observe for $u \in \mathscr{H}^r$ with $r \geq 1$ that

$$\mathcal{D}^{r-1}u(t) = \mathcal{D}^{r-1}u(0) + \int_0^t \mathcal{D}^{r-1}\dot{u}(\tau)\,d\tau.$$

This implies that the initial values $\mathcal{D}^{r-1}u(0)$ and the highest derivative $\partial^r u \in \mathscr{H}^0 = \mathscr{L}_2$ generate the whole signal $\mathcal{D}^{r-1}u$. Now fix $r \in \mathbb{N}$ and apply the same reasoning to both inputs w, $d \in \mathscr{H}^r$ of the LTI system (4.17). Then $x_e := \mathrm{col}(x, \mathcal{D}^{r-1}w, \mathcal{D}^{r-1}d)$ is generated as the solution of the extended system $\dot{x}_e = A_e x_e + B_e u$ defined by

$$\dot{x}_e = \left(\begin{array}{c|cc|cc} A & B_1 & 0 & B_2 & 0 \\ \hline 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) x_e + \left(\begin{array}{cc} 0 & 0 \\ \hline 0 & 0 \\ I & 0 \\ \hline 0 & 0 \\ 0 & I \end{array}\right) \underbrace{\begin{pmatrix} \partial^r w \\ \partial^r d \end{pmatrix}}_{=u} \tag{4.21}$$

for $x_e(0) = \mathrm{col}(0, \mathcal{D}^{r-1}w(0), \mathcal{D}^{r-1}d(0))$. Moreover, with the abbreviations

$$K_i = \begin{pmatrix} C_i \\ C_i A \\ \vdots \\ C_i A^r \end{pmatrix}, \; L_{ij} = \begin{pmatrix} D_{ij} & 0 & \dots & 0 \\ C_i B_j & D_{ij} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_i A^{r-1} B_j & \dots & \dots & D_{ij} \end{pmatrix}, \tag{4.22}$$

the derivatives of the inputs and outputs of G are related as

$$\left(\begin{array}{c} \mathcal{D}^r z \\ \mathcal{D}^r w \\ \hline \mathcal{D}^r e \\ \mathcal{D}^r d \end{array}\right) = \left(\begin{array}{ccc} K_1 & L_{11} & L_{12} \\ 0 & I & 0 \\ \hline K_2 & L_{21} & L_{22} \\ 0 & 0 & I \end{array}\right) \begin{pmatrix} x \\ \mathcal{D}^r w \\ \mathcal{D}^r d \end{pmatrix} =: \left(\begin{array}{c} T_\Delta \\ \hline T_p \end{array}\right) \begin{pmatrix} x_e \\ \partial^r w \\ \partial^r d \end{pmatrix}. \tag{4.23}$$

For notational compactness, we further introduce the abbreviation

$$E^r(s) := \mathrm{col}\left(\frac{1}{s^r} I, \dots, \frac{1}{s} I, I\right), \quad F^r(s) := s^r E^r(s).$$

The main result of this section now gives a novel and precise extension of the classical triple of equivalent conditions to Sobolev spaces, relating the dissipativity constraint (4.13) to an FDI and an LMI.

Theorem 4.12
Fix $r \in \mathbb{N}$, $Q_1 = Q_1^T$ and $P = P^T$. Then the following statements are equivalent.

a) There exist $\varepsilon > 0$, $R = R^T$ *such that for all* $w, d \in \mathscr{H}^r$*:*

$$\sigma_{Q_1}\begin{pmatrix} Mw + Nd \\ w \end{pmatrix} + \sigma_P \begin{pmatrix} N_{21}w + N_{22}d \\ d \end{pmatrix} \leq$$
$$\leq -\varepsilon\big(\|w\|_r^2 + \|d\|_r^2\big) + (\star)^T R \begin{pmatrix} \mathcal{D}^{r-1}w(0) \\ \mathcal{D}^{r-1}d(0) \end{pmatrix}. \quad (4.24)$$

b) There exists $X = X^T$ *with*

$$(\star)^T \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A_e & B_e \end{pmatrix} + (\star)^T \begin{pmatrix} Q_1 & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} T_\Delta \\ T_p \end{pmatrix} \prec 0. \quad (4.25)$$

c) There exists $\varepsilon > 0$ *such that*

$$(\star)^* \begin{pmatrix} Q_1 & 0 \\ 0 & P \end{pmatrix} \left(\begin{array}{cc} E^r M & E^r N \\ E^r & 0 \\ \hdashline E^r N_{21} & E^r N_{22} \\ 0 & E^r \end{array}\right) \preccurlyeq -\varepsilon I \quad \text{on} \quad i\mathbb{R} \setminus \{0\} \quad (4.26)$$

and

$$(\star)^* \begin{pmatrix} Q_1 & 0 \\ 0 & P \end{pmatrix} \left(\begin{array}{cc} F^r M & F^r N \\ F^r & 0 \\ \hdashline F^r N_{21} & F^r N_{22} \\ 0 & F^r \end{array}\right) (0) \prec 0 \quad (4.27)$$

Proof. A proof is found in Appendix C.3.2 □

Remark 4.13.
Theorem 4.12 is a cornerstone of our analysis framework as it reduces the verification of stability and performance of an interconnection, Theorem 4.12 a), to the feasibility of an LMI, Theorem 4.12 b). Thus, in contrast to [195, 136, 159], we not only formulate an abstract stability result, but also provide the means for its efficient numerical verification. As illustrated in detail in Chapter 5, this opens the way to a comprehensive treatment of one of the most important classes of nonlinearities in control. ⋆

Finally, let us briefly comment on the structure of the quadratic form (4.18) and the use of the extended system (4.21) in the light of Theorem 4.12. In contrast to [134, 118], where stability of a similarly extended system interconnected with a nonlinearity is verified in order to conclude stability of the original interconnection, we only need the system (4.21) to establish the connection between the quadratic constraint in Theorem 4.12 a) and the LMI in Theorem 4.12 b). As visible from the proof of Theorem 4.12, both are related through a dissipation inequality that naturally involves the initial condition of the *extended* system. This, in turn, necessitates the inclusion of the term $[(\mathcal{D}^{r-1}w)(0)]^T Q_2[(\mathcal{D}^{r-1}w)(0)]$ in the definition of Σ in (4.18), and the analogue for Σ_p.

4.5.4 Application to parametric uncertainties

Finally, let us highlight the main characteristics of our novel approach by considering the concrete example of time-varying parametric uncertainties. We analyze the following problem.

Let $\delta(.)$ be any sufficiently smooth time-varying parameter such that some bounds $\partial^j\delta(t) \in [\alpha_j, \beta_j]$ for $j \in \{0, \ldots, r\}$ are known. For any such curve δ, the uncertainty $\Delta : \mathscr{L}_{2e} \to \mathscr{L}_{2e}$ in Figure 4.2 is defined as $\Delta_\delta(z) := \delta z$. If, in addition, we assume $z \in \mathscr{H}^r$ then $w = \Delta_\delta(z) \in \mathscr{H}^r$ and the following relations hold:

$$
\begin{aligned}
w(t) &= \delta(t)z(t),\\
\dot{w}(t) &= \dot{\delta}(t)z(t) + \delta(t)\dot{z}(t),\\
\ddot{w}(t) &= \ddot{\delta}(t)z(t) + 2\dot{\delta}(t)\dot{z}(t) + \delta(t)\ddot{z}(t),\\
&\vdots
\end{aligned}
$$

This can be expressed as $\mathcal{D}^r w(t) = \Delta(t)\mathcal{D}^r z(t)$ with $\Delta(t) \in \mathbf{\Delta}$ for all $t \in [0, \infty)$, where $\mathbf{\Delta}$ is some simple-to-specify compact polytope of structured matrices depending on the bounds α_j, β_j. If $\mathbf{\Delta}_{ext}$ denotes the set of extreme points of $\mathbf{\Delta}$ and with

$$\boldsymbol{Q} = \left\{ Q = Q^T \middle| (\star)^T Q \begin{pmatrix} I \\ \Delta \end{pmatrix} \succ 0 \text{ for all } \Delta \in \mathbf{\Delta}_{ext} \text{ where } Q_{22} \prec 0 \right\},$$

routine arguments[2] show, for all $Q_1 \in \boldsymbol{Q}$ and any parameter trajectory δ, that Δ_δ satisfies the following IQC:

$$\sigma_{Q_1} \begin{pmatrix} z \\ \Delta_\delta(z) \end{pmatrix} \geq 0 \quad \text{for all} \quad z \in \mathscr{H}^r. \tag{4.28}$$

By Theorem 4.12, feasibility of (4.25) for some $Q_1 \in \boldsymbol{Q}$ guarantees stability of (4.3) and the existence of $\varepsilon > 0$ with

$$\sigma_P \begin{pmatrix} e \\ d \end{pmatrix} \leq -\varepsilon \|d\|_r^2 \quad \text{for all} \quad d \in \mathscr{H}_0^r \tag{4.29}$$

along all trajectories of (4.9)-(4.10). This offers the possibility to exploit bounds on higher derivatives of time-varying parametric uncertainties in robustness analysis. Note that standard techniques relying on parameter-dependent Lyapunov functions as outlined, e.g., in [90], only allow to exploit bounds up to the first derivative. As yet another approach, the so-called swapping lemma (see [160] and also [117, 144]) allows to incorporate bounds on δ and $\dot{\delta}$ in the classical IQC setting [91, 102]. It would be interesting to extend this technique to higher derivatives and compare the obtained results to ours.

Finally, for a concrete numerical example we assume that G in (4.17) is defined by

$$A = \begin{pmatrix} -.9 & 1 & 0 & 0 \\ 0 & -2 & 0.5 & 0 \\ 0 & 0 & -0.4 & 4 \\ 0 & 0 & -3.9 & -0.4 \end{pmatrix}, \quad \begin{pmatrix} B_1 & B_2 \end{pmatrix} = \begin{pmatrix} 0.9 & 0 \\ -2 & 0 \\ 1 & -5 \\ 8 & 1 \end{pmatrix},$$

[2] See Section 6.3 for a detailed exposition.

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} -5\ 0\ 0\ 0 \\ -1\ 0\ 0\ 0 \end{pmatrix}, \qquad D = 0, \tag{4.30}$$

and in feedback with Δ_δ for $r = 2$ as in Figure 4.2. Since we are interested in bounding the standard $\mathscr{L}_2$-gain of $d \to e$, we take $\mathscr{W} = \mathscr{Z} = \mathscr{H}^2$, $\mathscr{D} = \mathscr{E} = \mathscr{L}_2$ and $P = \operatorname{diag}(1, -\gamma^2)$. Moreover, $\delta(t)$, $\dot{\delta}(t)$, $\ddot{\delta}(t)$ for $t \geq 0$ are assumed to be contained in

$$[\alpha_0, \beta_0] = [0, 1.5], \quad [\alpha_1, \beta_1] = [0, 6], \quad [\alpha_2, \beta_2] = [0, 6],$$

respectively. Then $\Delta_\delta : \mathscr{H}^2 \to \mathscr{H}^2$ is causal and bounded. Since $D = 0$, $C_1 B_2 = 0$ and $C_1 A B_2 = 0$, let us highlight at this point that it is not required to work with a full system extension with respect to the input d in (4.21) since d, $\dot{d}$ and $\ddot{d}$ are not feed through to the uncertainty output z, and $\dot{d}$, $\ddot{d}$ are not involved in the performance specification. We can hence work with the extended system

$$\begin{pmatrix} \dot{x} \\ \dot{w} \\ \ddot{w} \end{pmatrix} = \begin{pmatrix} A & B_1 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ w \\ \dot{w} \end{pmatrix} + \begin{pmatrix} 0 & B_2 \\ 0 & 0 \\ I & 0 \end{pmatrix} \begin{pmatrix} \ddot{w} \\ d \end{pmatrix}.$$

By applying Theorem 4.12 for this extension and the IQC (4.28) we obtain the following results.

If only exploiting the bound on $\delta(.)$, we are not able to verify stability of the interconnection. If, in addition, we utilize the bound on $\dot{\delta}(.)$ we can guarantee stability with a certified $\mathscr{L}_2$-gain bound of $\gamma_1 = 14.89$. By also including the constraints on $\ddot{\delta}(.)$, the guaranteed bound improves to $\gamma_2 = 1.72$. It is important to emphasize that we do not require X in (4.25) to be positive definite, as is usually required in standard Lyapunov arguments. If we artificially impose such an extra constraint, the performance bounds increase to the values of $16.38 > \gamma_1 = 14.89$ and $1.97 > \gamma_2 = 1.72$, respectively.

4.6 Summary

In this chapter, we derive a unified framework for stability and performance analysis on general function spaces. The modifications may be seen as an extension of the IQC framework towards the setting of Safonov and Zames as discussed in Chapter 2. An important strength of the present approach is that it naturally covers the special, yet highly relevant, case of Sobolev spaces. Here, we not only give sufficient conditions for stability and performance, but equivalently recast their numerical verification as an LMI feasibility problem. Finally, we illustrate how this general setup enfolds for the concrete example of time-varying parametric uncertainties.

Chapter 5

Full-block multipliers for repeated, slope-restricted scalar nonlinearities – continuous-time case

5.1 Introduction

In this chapter, we take advantage of the framework established in Chapter 4 by deriving a unified approach to the stability and performance analysis of the feedback interconnection (4.3), where Δ is a static repeated nonlinearity defined through a scalar slope-restricted function $\varphi : \mathbb{R} \to \mathbb{R}$. Even though this is a rather particular setting, it is nevertheless of great practical relevance, as it comprises some of the most important nonlinearities in control, such as multiple saturations and dead-zones. Examples of such interconnections in engineering applications typically stem from systems with actuator saturations (see, e.g., [83, 157, 79, 125, 40]), but slope-restricted nonlinearities also arise naturally in recurrent neural networks (see, e.g., [61, 95, 11] and [198] for an overview) and in more mathematical applications such as the analysis and design of optimization algorithms (see, e.g., [104, 53]).

As briefly touched upon in Chapter 2, the investigation of absolute stability of such interconnections has a long standing history in control, probably starting with the works of Lurye and Postnikov [106], and is hence also termed Lurye's problem (see [10] for an excellent historical overview). The main contributions to its solution date back to the 1960s with the works of Popov [126], Yakubovich [188], Zames [195] and Zames and Falb [197]. Even today, the stability criteria associated with these names remain the most effective analysis tools. As each of them focusses on a different aspect of the nonlinearity φ, it is typically desirable to apply several methods simultaneously in order to achieve the best results.

The circle [195] and Popov criteria [126] both rely on sector bounds for the nonlinearity. While the first criterion just exploits relations between the input and the output of the nonlinearity, the second one involves the derivative of the input. The method developed by Yakubovich [188] proceeds further along this line of thought and uses relations between the derivatives of both the input and the output if the nonlinearities have a bounded derivative. In contrast, the result by Zames and Falb [197] does not require the differentiability of signals and is applicable for non-smooth slope-restricted nonlinearities as well.

As a major advantage, our general analysis framework permits the direct application and combination of circle, Popov, Zames-Falb and Yakubovich multipliers, even if the nonlinearities are not everywhere differentiable and the underlying LTI system is not strictly proper. Besides the modularity and the possibility for combined application of different criteria, another aspect that separates the present approach from existing ones in the literature is its focus on repeated nonlinearities, as they are often emerging in practical applications. By treating each nonlinearity individually, the scalar versions of the above discussed multipliers can be easily combined to obtain stability tests for repeated nonlinearities that involve structured diagonal multipliers (see, e. g., [137, 139, 74, 76, 37]). It is known how to employ unstructured full-block multipliers for the circle [175] and the Zames-Falb criteria [42, 103, 166] in order to potentially reduce conservatism in the stability analysis. In a further contribution, we propose novel full-block multipliers for the Yakubovich stability criterion and suggest a parametrization of the complete class of full-block Zames-Falb multipliers for effective computations. Finally, our approach allows a seamless combination of all four multiplier stability tests in a modular fashion, which leads to computational stability tests in terms of LMIs.

Perhaps most closely related to our approach is the one proposed by Altshuller using delay integral quadratic constraints [10]. As a distinguishing feature, this framework allows for the inclusion of (diagonal) Yakubovich multipliers [9, 10], but at the expense of reduced flexibility and by requiring individual proofs for each new multiplier. Moreover, the emphasis of the present exposition lies on the derivation of full-block criteria and their (combined) verification using LMIs; these aspects are not touched upon or play a subordinate role in [10].

In the remainder of this chapter, we show how all the new ingredients of Theorem 4.7 come to flourish even if applied to the special case when M is an LTI system, Δ is a repeated nonlinearity defined through a sector-bounded and slope-restricted scalar function, and the underlying function spaces are either $\mathscr{L}_{2e}$ or $\mathscr{H}_e^1$ (see Definition 4.10).

After precisely defining the operators, the signal spaces and the considered quadratic constraints in Section 5.2, we carefully address the issue of well-posedness even if the LTI system is not strictly proper. Section 5.3 is devoted to a detailed presentation of stability analysis with full-block circle and Zames-Falb as well as standard Popov multipliers. For the circle criterion we generalize [175, 176] by using the so-called Pólya relaxation and reveal new insights into the relation with the classical circle criterion or other relaxation schemes; thus permitting to systematically exploit the full power of this test. After briefly discussing the incorporation of classical Popov multipliers into our framework, we turn to full-block Zames-Falb multipliers, as described in [42] and further generalized in [103], where we extend the parametrization in [37] to repeated scalar nonlinearities and prove it to be asymptotically exact. This new result enables us to tap the complete potential of the Zames-Falb stability test in computations.

Section 5.4 reveals that our framework allows for the inclusion of the Yakubovich criterion [188, 46] with new full-block multipliers, and it permits to drop the typically encountered restriction of Popov and Yakubovich tests to strictly proper plants. The translation into LMIs relies on Theorem 4.12. And finally, Section 5.5 serves to differentiate our results from related ones in the literature, as supported by further numerical examples in Section 5.6. We conclude by emphasizing that the results in this chapter have already appeared, in parts even literally, in [58, 56].

5.2 Application to slope-restricted nonlinearities

Let us now start by specializing the general framework of Chapter 4 to the particular setting featuring slope-restricted nonlinearities. As already mentioned in the introduction, it suffices to consider derivatives up to order one of the involved signals.

5.2.1 Slope-restricted nonlinearities

First, we specify the class of uncertainties under consideration.

Definition 5.1.
Let $\mu_1 \leq 0 \leq \mu_2$. Then $\varphi : \mathbb{R} \to \mathbb{R}$ is **slope-restricted**, in short $\varphi \in \mathrm{slope}(\mu_1, \mu_2)$, if

$$\varphi(0) = 0 \quad \text{and} \quad \mu_1 \leq \frac{\varphi(x) - \varphi(y)}{x - y} \leq \sup_{x \neq y} \frac{\varphi(x) - \varphi(y)}{x - y} < \mu_2 \qquad \text{for all} \quad x, y \in \mathbb{R},\, x \neq y. \tag{5.1}$$

If $\mu_1 = 0$ and the bound on the right is absent, φ is just **monotone** and we write $\varphi \in \mathrm{slope}(0, \infty)$. If there exist some $\alpha \leq 0 \leq \beta$ such that φ satisfies

$$(\varphi(x) - \alpha x)(\beta x - \varphi(x)) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}, \tag{5.2}$$

it is said to be **sector-bounded** which is expressed as $\varphi \in \sec[\alpha, \beta]$. ⋆

With such a nonlinearity φ let the map $\Phi : \mathbb{R}^k \to \mathbb{R}^k$ be given as $\Phi(x_1, \ldots, x_k) = \big(\varphi(x_1) \; \ldots \; \varphi(x_k)\big)^T$. In the sequel we restrict our attention to the static (and obviously causal) operators defined as

$$\Delta_\varphi(z)(t) := \Phi(z(t)) \text{ for almost all } t \in [0, \infty) \text{ and all } \; z \in \mathscr{L}_{2e}. \tag{5.3}$$

We say that $\Delta_\varphi \in \operatorname{slope}(\mu_1, \mu_2)^k$ and $\Delta_\varphi \in \sec[\alpha, \beta]^k$ if $\varphi \in \operatorname{slope}(\mu_1, \mu_2)$ or $\varphi \in \sec[\alpha, \beta]$, respectively. As an immediate consequence of Definition 5.1 we infer

$$\Delta_\varphi \in \operatorname{slope}(\mu_1, \mu_2)^k \implies \Delta_\varphi \in \sec[\mu_1, \mu_2]^k. \tag{5.4}$$

Thus finite slope restrictions always translate into finite sector bounds with the same constants. However, often tighter sector bounds are known, i. e., $\varphi \in \operatorname{slope}(\mu_1, \mu_2) \cap \sec[\alpha, \beta]$ with $\beta < \mu_2$ or $\mu_1 < \alpha$ implying $\Delta_\varphi \in \operatorname{slope}(\mu_1, \mu_2)^k \cap \sec[\alpha, \beta]^k$; in numerical examples we will demonstrate that this additional information can be beneficially exploited.

Remark 5.2.
Note that the assumptions $0 \in [\alpha, \beta]$ and $0 \in [\mu_1, \mu_2]$ are not restrictive, since we can always perform a loop transformation such that both are met. As a benefit from $0 \in [\alpha, \beta]$, we immediately infer that $\Delta_\varphi \in \sec[\alpha, \beta]^k$ implies $\tau\Delta_\varphi \in \sec[\alpha, \beta]^k$ for $\tau \in [0, 1]$ and the analog holds true for slope restrictions. If applying Theorem 4.7, it hence suffices to verify properties a) and c) for $\tau = 1$ and all uncertainties Δ_φ in the respective class. ⋆

5.2.2 Signal spaces and operators

For the purpose of this chapter it suffices to consider the spaces $\mathscr{L}_2$ and $\mathscr{H}^1$. As the number of repetitions of φ in Φ plays an important role, we include the dimension k of the signals in the signal space symbols, i.e., $\mathscr{L}_2^k$ and $\mathscr{H}^{1,k}$. In order to distinguish the norm on $\mathscr{L}_1$ from the norm on $\mathscr{H}^{1,k}$, we denote the latter by $\|\cdot\|_{\mathscr{H}}$. By Lemma 4.11, $\mathscr{L}_{2e}^k$

and $\mathscr{H}_e^{1,k}$ satisfy Assumption 4.1. We will only need the compatibility property in Assumption 4.5 a) for $\mathscr{H}_e^{1,k} \subset \mathscr{L}_{2e}^k$, which follows from the obvious inequality $\|w\|_{\mathscr{L}_{2e},T} \leq \|w\|_{\mathscr{H}_e,T}$ for all $T > 0$ and $w \in \mathscr{H}_e^{1,k}$. Following Section 4.5 and concerning the linear operator M, we restrict our attention to stable LTI systems represented as

$$\begin{aligned} \dot{x} &= Ax + Bw, \qquad\qquad x(0) = 0, \\ z &= Cx + Dw, \end{aligned} \tag{5.5}$$

with $A \in \mathbb{R}^{n \times n}$ being Hurwitz. Then both M and Δ_φ are causal and compatible with the considered spaces in the following sense. $M : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^k$ and $M : \mathscr{H}_e^{1,k} \to \mathscr{H}_e^{1,k}$ are bounded; in case of $D = 0$ also $M : \mathscr{L}_{2e}^k \to \mathscr{H}_e^{1,k}$ is well-defined and bounded. Moreover, for $\Delta_\varphi \in \mathrm{slope}(\mu_1, \mu_2)^k$, all the maps $\Delta_\varphi : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^k$, $\Delta_\varphi : \mathscr{H}_e^{1,k} \to \mathscr{L}_{2e}^k$ and $\Delta_\varphi : \mathscr{H}_e^{1,k} \to \mathscr{H}_e^{1,k}$ are bounded. The second property is a consequence of the first, while the third is stated next since it requires a proof.

Lemma 5.3
If $\varphi \in \mathrm{slope}(\mu_1, \mu_2)$ then $\Delta_\varphi : \mathscr{H}_e^{1,k} \to \mathscr{H}_e^{1,k}$ is well-defined and bounded.

Proof. A proof is found in Appendix C.4.3. □

Finally, $N : \mathscr{V} \to \mathscr{L}_2^k$ or $N : \mathscr{V} \to \mathscr{H}^{1,k}$ are assumed to obey the properties in Assumption 4.5 c).

5.2.3 Well-posedness

Let us briefly discuss the issue of well-posedness in this setting and based on the following result.

Lemma 5.4
Suppose $\varphi \in \mathrm{slope}(\mu_1, \mu_2)$ and let

$$\Theta(\{\mu_1, \mu_2\}, k) = \left\{ \mathrm{diag}(\delta_1, \ldots, \delta_k) \in \mathbb{R}^{k \times k} \;\middle|\; \delta_i \in \{\mu_1, \mu_2\} \right\}.$$

Then $I - D\Phi : \mathbb{R}^k \to \mathbb{R}^k$ *is invertible and* $(I - D\Phi)^{-1} : \mathbb{R}^k \to \mathbb{R}^k$ *is globally Lipschitz if and only if*

$$\det(I - D\Delta) > 0 \quad \textit{for all} \quad \Delta \in \Theta(\{\mu_1, \mu_2\}, k). \tag{5.6}$$

Proof. The proof of necessity is an adaption of [86, Proof of Claim 1] in the context of saturated systems. For sufficiency observe that $\det(I - D\Delta) > 0$ for all $\Delta \in \Theta(\{\mu_1, \mu_2\}, k)$ implies $\det(I - D\Delta) > 0$ for all $\Delta \in \Theta([\mu_1, \mu_2], k) = \left\{\operatorname{diag}(\delta_1, \ldots, \delta_k) \in \mathbb{R}^{k\times k} \;\middle|\; \delta_i \in [\mu_1, \mu_2]\right\}$ since $\det(I - D\Delta)$ is a multi-affine function in $(\delta_1, \ldots, \delta_k)$. Then the claim follows from Proposition 2 in [193]. $\square$

Lemma 5.5
Property (5.6) *implies that* $I - M\Delta_\varphi : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^k$ *has a causal inverse for all* $\Delta_\varphi \in \operatorname{slope}(\mu_1, \mu_2)^k$.

Proof. The map which takes $u \in \mathscr{L}_{2e}^k$ into $y = u - M\Delta_\varphi(u) \in \mathscr{L}_{2e}^k$ is described by $\dot{x} = Ax + B\Phi(u)$, $y = u - D\Phi(u) - Cx$ with $x(0) = 0$. Since $I - D\Phi$ is invertible, this is equivalent to

$$\begin{aligned} \dot{x} &= Ax + B\Phi(I - D\Phi)^{-1}(Cx + y), \quad x(0) = 0, \\ u &= (I - D\Phi)^{-1}(Cx + y). \end{aligned} \tag{5.7}$$

Since $\Phi(I - D\Phi)^{-1}$ is also globally Lipschitz, standard ODE theory implies that for each $y \in \mathscr{L}_{2e}^k$ there exists a unique response $u \in \mathscr{L}_{2e}^k$ with (5.7) that depends causally on y. $\square$

For repeated slope-restricted nonlinearities, we have thus identified the easily verifiable condition (5.6) that guarantees well-posedness of (4.3) in the classical sense. If $\varphi \in \sec[\alpha, \beta]$, other arguments are required to show well-posedness; for example, if $D = 0$ and φ is locally Lipschitz continuous, again standard ODE arguments guarantee this classical well-posedness property.

5.2.4 Quadratic forms and verification of constraints

In Section 5.3 we put Theorem 4.7 to use by employing quadratic forms Σ of the structure already discussed in Section 2.2:

$$\Sigma_\Pi \begin{pmatrix} z \\ w \end{pmatrix} := \int_{-\infty}^{\infty} (\star)^T \, \Pi(i\omega) \begin{pmatrix} \hat{z}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix} d\omega \ \text{ with } \ \Pi = \begin{pmatrix} Q & S \\ S^* & R \end{pmatrix}. \quad (5.8)$$

However, in contrast to (2.16), we only assume that the multiplier Π is a hermitian valued and measurable function defined on the imaginary axis. By omitting the (essential) boundedness assumption originally introduced by Megretski and Rantzer, we may seamlessly incorporate (unbounded) Popov multipliers (2.20). For reasons of compactness, we occasionally write $\Sigma_\Pi(z, w)$.

In case that Π is indeed essentially bounded on the imaginary axis, we note that Σ_Π is defined as described in Remark 4.8 and, hence, does satisfy the technical property Theorem 4.7 b). Moreover, with (5.8) and M as in (5.5), Theorem 4.7 d) reads for $\mathscr{W} = \mathscr{L}_2^k$ as follows:

$$\exists \varepsilon > 0, \ \forall w \in \mathscr{L}_2^k :$$
$$\int_{-\infty}^{\infty} \begin{pmatrix} M(i\omega)\hat{w}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix}^* \Pi(i\omega) \begin{pmatrix} M(i\omega)\hat{w}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix} d\omega \leq -\varepsilon \|w\|^2. \quad (5.9)$$

It is well-known that this is equivalent to FDI:

$$\exists \varepsilon > 0 : \quad (\star)^* \, \Pi(i\omega) \begin{pmatrix} M(i\omega) \\ I \end{pmatrix} \preccurlyeq -\varepsilon I \ \text{ for almost all } \ \omega \in \mathbb{R}. \quad (5.10)$$

If the left hand side of the inequality in (5.10) is also rational, the standard KYP lemma, Lemma 2.12, can be applied to computationally verify this property through the solution of an LMI feasibility problem. In Section 5.4 we encounter more general scenarios and derive the related LMI by means of Theorem 4.12.

5.2.5 Sketch of procedure

The general procedure for stability analysis now enfolds along the pattern outlined in Chapter 2. Well-posedness, Theorem 4.7 a), is verified separately. The key step is to capture the properties of the nonlinearity Δ_φ with a whole class of multipliers $\mathbf{\Pi}$ such that Theorem 4.7 b) and in particular 4.7 c) hold with Σ_Π and for all $\Pi \in \mathbf{\Pi}$. For guaranteeing stability, it then remains to verify the existence of one multiplier $\Pi \in \mathbf{\Pi}$ such that Σ_Π also satisfies Theorem 4.7 d); since all classes $\mathbf{\Pi}$ in the following Section 5.3 are convex cones, this latter search of $\Pi \in \mathbf{\Pi}$ boils down to solving a convex optimization problem as addressed in more detail in Section 5.3.4.

5.3 Derivation and application of multipliers

This and the next section are devoted to a comprehensive collection of stability tests for slope-restricted nonlinearities. A particular focus lies on the extension of standard criteria to facilitate the application of the more powerful full-block multipliers. Moreover, we present a framework that allows to combine all introduced multipliers in a modular fashion.

5.3.1 Full-block multipliers for the circle criterion

We start by considering the static (frequency independent) class of multipliers for the circle criterion and for nonlinearities that are merely sector-bounded. The property $\Delta_\varphi \in \sec[\alpha, \beta]^k$ is traditionally captured using multipliers that share the structure of the nonlinearity [139, 79, 21]. This set of diagonally repeated circle criterion multipliers is given by

$$\mathbf{\Pi}_{dr}[\alpha, \beta]^k = \left\{ \Pi \in \mathbb{S}^{2k} \,\middle|\, \Pi = \begin{pmatrix} -\alpha\beta \operatorname{diag}(\lambda) & \frac{\alpha+\beta}{2} \operatorname{diag}(\lambda) \\ \frac{\alpha+\beta}{2} \operatorname{diag}(\lambda) & -\operatorname{diag}(\lambda) \end{pmatrix}, \lambda \in \mathbb{R}^k_+ \right\}. \tag{5.11}$$

As will be seen, unstructured multipliers offer more freedom. Let us hence introduce the class of full-block circle criterion multipliers [176, Section 5.8.2.] adapted to the diagonally repeated case. With $\Omega \subset \mathbb{R}$,

$$\Theta(\Omega, k) := \{\Delta = \operatorname{diag}(\delta_1, \dots, \delta_k) \mid \delta_j \in \Omega\}$$

and

$$F_\Pi(\Delta) := \begin{pmatrix} I \\ \Delta \end{pmatrix}^T \Pi \begin{pmatrix} I \\ \Delta \end{pmatrix}$$

these are given by

$$\mathbf{\Pi}[\alpha, \beta]^k = \left\{\Pi \in \mathbb{S}^{2k} \,\middle|\, F_\Pi(\Delta) \succ 0 \;\; \forall \Delta \in \Theta([\alpha, \beta], k)\right\}. \tag{5.12}$$

With (5.12) we arrive at the following IQC.

Lemma 5.6 (Full-block circle IQC)
Let $\Pi \in \mathbf{\Pi}[\alpha, \beta]^k$ and Σ_Π be taken as in (5.8)*. Then*

$$\Sigma_\Pi \begin{pmatrix} z \\ \Delta_\varphi(z) \end{pmatrix} \geq 0 \quad \textit{for all} \quad \Delta_\varphi \in \sec[\alpha, \beta]^k \quad \textit{and} \quad z \in \mathscr{L}_2^k. \tag{5.13}$$

With Theorem 4.7 this IQC immediately translates into the following robust stability test.

Theorem 5.7 (Full-block circle criterion)
Let $N : \mathscr{V} \to \mathscr{L}_2^k$ and suppose that the interconnection (4.3) *is well-posed in the classical sense for all $\Delta_\varphi \in \sec[\alpha, \beta]^k$. If there exists some $\Pi \in \mathbf{\Pi}[\alpha, \beta]^k$ with* (5.10)*, then* (4.3) *is robustly stable: There exists $\gamma > 0$ such that*

$$\|z\| \leq \gamma(\|u\| + \|v\|_{\mathscr{V}}) \quad \textit{and all} \quad (u, v) \in \mathscr{L}_2 \times \mathscr{V}, \; \Delta_\varphi \in \sec[\alpha, \beta]^k.$$

Remark 5.8.
Note that $\mathscr{V} = \mathscr{L}_2^k$ and $N = I$ recovers the standard setting of [110] with two free inputs as in Figure 2.3. Robust stability then implies that $(I - M\Delta_\varphi)^{-1}$ maps $\mathscr{L}_2^k$ into $\mathscr{L}_2^k$. ⋆

Proof. As the IQC (5.13) is valid on $\mathscr{L}_2^k$, we take $\mathscr{W}_e = \mathscr{Z}_e = \mathscr{U}_e = \mathscr{L}_{2e}^k$. In view of Section 5.2, all requirements of Theorem 4.7 are fulfilled for Σ_Π with $\Pi \in \mathbf{\Pi}[\alpha,\beta]^k$ and $l = 0$. $\square$

Since $\mathbf{\Pi}[\alpha,\beta]^k$ is defined through infinitely many constraints, the application of Theorem 5.7 requires approximations in order to render this criterion computational. As it is also at the heart of the derivation of our novel full-block Yakubovich criterion, we summarize the most important relaxation schemes in some detail and give new insights regarding their interrelation. With the partition of Π as in (5.8), two nested inner approximations $\mathbf{\Pi}_c[\alpha,\beta]^k \subset \mathbf{\Pi}_{pc}[\alpha,\beta]^k \subset \mathbf{\Pi}[\alpha,\beta]^k$ are given by the so-called convex and partially convex relaxations

$$\mathbf{\Pi}_c[\alpha,\beta]^k = \left\{\Pi \in \mathbb{S}^{2k} \,\middle|\, R \prec 0 \text{ and } F_\Pi(\Delta) \succ 0 \;\; \forall\, \Delta \in \Theta(\{\alpha,\beta\},k)\right\} \tag{5.14}$$

and

$$\mathbf{\Pi}_{pc}[\alpha,\beta]^k = \left\{\Pi \in \mathbb{S}^{2k} \,\middle|\, R_{ii} < 0 \text{ and } F_\Pi(\Delta) \succ 0 \;\; \forall\, \Delta \in \Theta(\{\alpha,\beta\},k)\right\}, \tag{5.15}$$

respectively. Since $\mathbf{\Pi}_c$ or $\mathbf{\Pi}_{pc}$ are described by a finite number of LMIs they can be substituted for $\mathbf{\Pi}$ in Theorem 5.7 in order to arrive at a computationally tractable stability test. This substitution causes conservatism whose degree cannot be judged a priori. Based on a matrix version of a classical theorem of Pólya [127], this motivated the introduction of an asymptotically exact parameterization of $\mathbf{\Pi}$ in [148] as follows. Let Δ_j denote the $K := 2^k$ matrices in $\Theta(\{\alpha,\beta\},k)$ and define the Hermitian-valued polynomial matrix

$$\Lambda_d(\lambda,\Pi,[\alpha,\beta]^k) := (\lambda_1 + \ldots + \lambda_K)^d \left[\sum_{l=1}^K \lambda_l \begin{pmatrix} I \\ \Delta_l \end{pmatrix}\right]^T \Pi \left[\sum_{m=1}^K \lambda_m \begin{pmatrix} I \\ \Delta_m \end{pmatrix}\right]$$

in λ on the standard simplex $\mathcal{S} := \{(\lambda_1, \ldots, \lambda_K) \mid \lambda_j \geq 0,\ \sum \lambda_j = 1\} \subset \mathbb{R}^K$. This polynomial is homogenous of degree $d+2$ and can be expressed with the standard multi-index notation as

$$\Lambda_d(\lambda, \Pi, [\alpha, \beta]^k) = \sum_{\kappa \in \mathbb{N}_0^K,\ |\kappa| = d+2} C_{d,\kappa}(\Pi, [\alpha, \beta]^k)\lambda^\kappa. \tag{5.16}$$

Clearly $\Pi \in \mathbf{\Pi}[\alpha, \beta]^k$ is equivalent to $\Lambda_0(\lambda, \Pi, [\alpha, \beta]^k) \succ 0$ for all $\lambda \in \mathcal{S}$. If $d \in \mathbb{N}_0$ and because of $\sum \lambda_j = 1$ for $\lambda \in \mathcal{S}$, this is trivially equivalent to

$$\Lambda_d(\Pi, \lambda, [\alpha, \beta]^k) \succ 0 \quad \text{for all} \quad \lambda \in \mathcal{S}. \tag{5.17}$$

Since $\lambda \in \mathcal{S}$ implies $\lambda^\kappa \geq 0$ for all multi-indices with $|\kappa| = d + 2$, and since the inequality is strict for at least one of them, $C_{d,\kappa}(\Pi, [\alpha, \beta]^k) \succ 0$ for all $\kappa \in \mathbb{N}_0^K$ with $|\kappa| = d + 2$ does imply (5.17). This motivates to define the d-th order Pólya relaxation as

$$\begin{aligned} \mathbf{\Pi}_d^{\mathrm{Pol}}[\alpha, \beta]^k := \big\{\Pi \in \mathbb{S}^{2k} \big| C_{d,\kappa}(\Pi, [\alpha, \beta]^k) \succ 0 \\ \text{for all } \kappa \in \mathbb{N}_0^K \text{ with } |\kappa| = d + 2\big\}. \end{aligned} \tag{5.18}$$

We conclude $\mathbf{\Pi}_d^{\mathrm{Pol}}[\alpha, \beta]^k \subset \mathbf{\Pi}[\alpha, \beta]^k$ for all $d \in \mathbb{N}_0$. As a first insight, let us establish that the family $\mathbf{\Pi}_d^{\mathrm{Pol}}[\alpha, \beta]^k$ is nondecreasing with increasing d and recall the known fact that it contains any element of $\mathbf{\Pi}[\alpha, \beta]^k$ for $d \to \infty$, as formulated next. Note that we argue in terms of set inclusions and thus we recover the results in [120, Lemma 5] for a specific performance index as a special case.

Lemma 5.9

a) Suppose that $\Pi \in \mathbf{\Pi}[\alpha, \beta]^k$. *Then there exists an integer* $d \in \mathbb{N}_0$ *such that* $\Pi \in \mathbf{\Pi}_d^{\mathrm{Pol}}[\alpha, \beta]^k$.

b) If $d_1 < d_2$ *then* $\mathbf{\Pi}_{d_1}^{\mathrm{Pol}}[\alpha, \beta]^k \subset \mathbf{\Pi}_{d_2}^{\mathrm{Pol}}[\alpha, \beta]^k$.

Proof. Statement a) is a special case of Theorem 7.1 in [148]. For b) it suffices to show that $\mathbf{\Pi}_d^{\mathrm{Pol}}[\alpha, \beta]^k \subset \mathbf{\Pi}_{d+1}^{\mathrm{Pol}}[\alpha, \beta]^k$ holds for all

$d \in \mathbb{N}_0$. Assume that $C_{d,\kappa}(\Pi, [\alpha, \beta]^k) \succ 0$ for all multi-indices with $|\kappa| = 2 + d$. Since $\Lambda_{d+1}(\lambda, \Pi, [\alpha, \beta]^k) = (\lambda_1 + \ldots + \lambda_K)\Lambda_d(\Pi, \lambda, [\alpha, \beta]^k)$ and due to (5.16), every $C_{d+1,\tilde{\kappa}}(\Pi, [\alpha, \beta]^k)$ is a sum of suitable coefficients $C_{d,\kappa}(\Pi, [\alpha, \beta]^k)$ and thus positive definite. □

Together with $\mathbf{\Pi}_c[\alpha, \beta]^k \subset \mathbf{\Pi}_0^{\mathrm{Pol}}[\alpha, \beta]^k$ as shown in [102], Lemma 5.9 gives rise to the following chain of inclusions:

$$\mathbf{\Pi}_{dr}[\alpha, \beta]^k \subset \mathbf{\Pi}_c[\alpha, \beta]^k \subset \mathbf{\Pi}_0^{\mathrm{Pol}}[\alpha, \beta]^k \subset \mathbf{\Pi}_1^{\mathrm{Pol}}[\alpha, \beta]^k \subset \ldots \subset \mathbf{\Pi}[\alpha, \beta]^k. \tag{5.19}$$

It is possible to also consider the subset $\mathbf{\Pi}_{dc}[\alpha, \beta]^k$ of $\mathbf{\Pi}_c[\alpha, \beta]^k$ (or of $\mathbf{\Pi}_{pc}[\alpha, \beta]^k$) where we restrict the blocks Q, S and R in (5.8), (5.14) to be diagonal. Despite the fact that $\mathbf{\Pi}_{dc}[\alpha, \beta]^k$ is larger than $\mathbf{\Pi}_{dr}[\alpha, \beta]^k$, it is yet another new insight that its use does not provide any advantage over the classical circle criterion with diagonally repeated multipliers.

Lemma 5.10
With $\mathbf{\Pi}_{dc}[\alpha, \beta]^k$ as just defined, (5.10) *holds for some* $\Pi \in \mathbf{\Pi}_{dc}[\alpha, \beta]^k$ *if and only if there exists* $\Pi \in \mathbf{\Pi}_{dr}[\alpha, \beta]^k$ *with* (5.10).

Proof. A proof is found in Appendix C.4.1. □

Numerical examples (see Example 5.14) reveal that the extra freedom offered by the larger classes in (5.19) or by $\mathbf{\Pi}_{pc}[\alpha, \beta]^k$ over $\mathbf{\Pi}_{dr}[\alpha, \beta]^k$ can lead to a substantial reduction in conservatism. Due to their simple implementation and cheap computations (since involving only few constraints), all examples in the sequel tacitly employ the partially convex relaxation.

5.3.2 Classical Popov criterion

In general, stability criteria with frequency dependent multipliers are superior if compared to static ones. Historically, the most important dynamic stability test is the Popov criterion. Its incorporation into the IQC framework was proposed by Jönsson [93] and relies on a customized version of the IQC result in [110] (see Section 2.2.2). It is one of the major benefits of our formulation of the IQC stability Theorem 4.7 that the Popov criterion can be incorporated without any modifications. Even if there is no known full-block version of the Popov multipliers for slope-restricted nonlinearities in continuous time (see Chapter 6 for a discrete-time result), we state the result for diagonally structured ones for completeness. Translated into our setting, the Popov IQC on the Sobolev space $\mathscr{H}^{1,k}$ reads as follows.

Theorem 5.11 (Popov IQC)
Let Λ be a diagonal $k \times k$ matrix[1]*, i.e., $\Lambda \in \mathbb{D}^k$, and define*

$$\Pi_\Lambda(i\omega) := \begin{pmatrix} 0 & -i\omega\Lambda \\ i\omega\Lambda & 0 \end{pmatrix} \quad \textit{for all} \quad \omega \in \mathbb{R}. \tag{5.20}$$

Then there exists some $\delta \geq 0$ such that for all $\tau \in [0,1]$ the following IQC holds:

$$\Sigma_{\Pi_\Lambda}\begin{pmatrix} z \\ \tau\Delta_\varphi(z) \end{pmatrix} \geq -\delta\|z(0)\|^2$$
$$\textit{for all} \;\; z \in \mathscr{H}^{1,k} \;\; \textit{and all} \;\; \Delta_\varphi \in \sec[\alpha,\beta]^k. \tag{5.21}$$

Proof. Let $I_\varphi(x) := \int_0^x \varphi(s)\,ds$. With $\eta := \max\{|\alpha|, \beta\}$, elementary calculations show that

$$|I_\varphi(x)| \leq \frac{\eta}{2}x^2, \quad \frac{d}{dx}I_\varphi(x) = \varphi(x) \quad \text{and} \quad I_\varphi(x) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}. \tag{5.22}$$

Now let $\tau \in [0,1]$ and $z \in \mathscr{H}^{1,k}$. Using Plancherel's theorem, we arrive at

$$\int_0^\infty [\tau\Phi(z(t))]^T \Lambda\dot{z}(t)\,dt = \sum_{i=1}^k \tau\Lambda_{ii}\int_0^\infty \varphi(z_i(t))\dot{z}_i(t)\,dt. \tag{5.23}$$

[1] We denote with $\mathbb{D}^k$ the diagonal matrices in $\mathbb{R}^{k\times k}$. Note that this is a slight abuse of notation as $\mathbb{D}$ already denotes the unit disc in $\mathbb{C}$. However, the meaning will always be clear from the context.

With (5.22) and since $\lim_{t\to\infty} z_j(t) = 0$ for all $j \in \{1, \ldots, k\}$ we obtain

$$\begin{aligned}\sum_{i=1}^{k} \tau\Lambda_{ii} \int_0^\infty \varphi(z_i(t))\dot{z}_i(t)\,dt &= -\sum_{i=1}^{k} \tau\Lambda_{ii} I_\varphi(z_i(0)) \\ &\geq -\sum_{i=1}^{k} \frac{\eta}{2}\,|\Lambda_{ii}|\,\|z(0)\|_\infty^2,\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the maximum norm on $\mathbb{R}^k$. Finally, since all norms are equivalent on $\mathbb{R}^k$, we obtain (5.21). □

For the next theorem we require $D = 0$ in (5.5) in order to ensure $z = Mw \in \mathscr{H}^{1,k}$ for all $w \in \mathscr{L}_2^k$; together with the assumption $N : \mathscr{V} \to \mathscr{H}^{1,k}$ this permits the use of the Popov IQC in order to prove stability.

Theorem 5.12 (Popov criterion, strictly proper systems)
Assume that the interconnection (4.3), *with strictly proper M and $N : \mathscr{V} \to \mathscr{H}^{1,k}$, is well-posed in the classical sense. Moreover, let $\Lambda \in \mathbb{D}^k$. If* (5.10) *holds with $\Pi = \Pi_\Lambda$, then there exist constants $\gamma > 0$ and γ_0 such that,*

$$\begin{aligned}\|z\| \leq \gamma(\|u\| + \|v\|_{\mathscr{V}}) + \gamma_0\|z(0)\| \quad &\textit{for all} \quad (u, v) \in \mathscr{L}_2^k \times \mathscr{V}, \\ &\textit{and all} \quad \Delta_\varphi \in \sec[\alpha, \beta]^k. \quad (5.24)\end{aligned}$$

Proof. We apply Theorem 4.7 based on the uncertainty IQC in Theorem 5.11 which only holds on $\mathscr{H}^{1,k}$. This motivates the choice $\mathscr{Z}_e = \mathscr{H}_e^{1,k}$. Since M is strictly proper we know that $M : \mathscr{L}_{2e}^k \to \mathscr{H}_e^{1,k}$ and $\Delta_\varphi : \mathscr{H}_e^{1,k} \to \mathscr{L}_{2e}^k$ are bounded (see Section 5.2) and we can take $\mathscr{U}_e = \mathscr{W}_e = \mathscr{L}_{2e}^k$. From classical well-posedness and due to $M\mathscr{L}_{2e}^k \subset \mathscr{H}_e^{1,k}$ and $N(\mathscr{V}) \subset \mathscr{H}^{1,k}$, we can infer well-posedness of (4.3) in the sense of Definition 4.6. If observing that $l(z) = \sqrt{\delta}\|z(0)\|$ satisfies (4.4), Theorem 5.11 implies the validity of (4.6). Again since M is strictly proper, Theorem 4.7 d) follows from (5.10), where we factorize Π such that the FDI reads as

$$\begin{pmatrix} i\omega M(i\omega) \\ I \end{pmatrix}^* \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} i\omega M(i\omega) \\ I \end{pmatrix} \preccurlyeq -\varepsilon I \quad \text{on} \quad \mathbb{C}_0^\infty.$$

This obviously renders the resulting (passivity) multiplier bounded, while maintaining properness of the outer factors.

It remains to verify that Σ_{Π_Λ} satisfies (4.5) despite the fact that neither Σ_{Π_Λ} (as a quadratic form) nor Π_Λ (as a function on the imaginary axis) are bounded. Indeed, for $w, u \in \mathscr{L}_2^k$ and $v \in \mathscr{V}$ we have

$$\Sigma_{\Pi_\Lambda} \begin{pmatrix} M(w+u) + N(v) \\ w \end{pmatrix} - \Sigma_{\Pi_\Lambda} \begin{pmatrix} Mw \\ w \end{pmatrix} = \Sigma_{\Pi_\Lambda} \begin{pmatrix} Mu + N(v) \\ w \end{pmatrix}. \tag{5.25}$$

With $f = N(v)$, the response $z = Mu + N(v) = Mu + f$ satisfies

$$\begin{pmatrix} \dot{x} \\ z \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A & B \\ C & 0 \\ CA & CB \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ f \\ \dot{f} \end{pmatrix}, \quad x(0) = 0. \tag{5.26}$$

With $\gamma_1 := \|(sI - A)^{-1}B\|_\infty$ and $\|\dot{f}\| \leq \|N(v)\|_{\mathscr{H}} \leq \gamma_N \|v\|_{\mathscr{V}}$, we obtain

$$\begin{aligned} \|\dot{z}\| &\leq \|CA\|\|x\| + \|CB\|\|u\| + \|\dot{f}\| \\ &\leq (\gamma_1 \|CA\| + \|CB\|)\,\|u\| + \gamma_N \|v\|_{\mathscr{V}}. \end{aligned}$$

Hence, with z as above, (5.25) can be bounded as

$$\begin{aligned} \Sigma_{\Pi_\Lambda} \begin{pmatrix} z \\ w \end{pmatrix} &= 2 \int_0^\infty w(t)^T \Lambda \dot{z}(t)\, dt \leq \\ &\leq 2\|\Lambda\|\|w\| \big[\big(\gamma_1 \|CA\| + \|CB\|\big) \|u\| + \gamma_N \|v\|_{\mathscr{V}} \big] \end{aligned}$$

and the right-hand side is a quadratic form in $\|w\|$, $\|u\|$, $\|v\|_{\mathscr{V}}$ not depending on $\|w\|^2$. □

Remark 5.13.
In case of a nonzero initial condition x_0 as in (2.22), we set $\mathscr{V} := \{e^{A\bullet}x_0\} \subset \mathscr{H}^{1,k}$ and $N = C$. In analogy to Theorem 2.10, the conclusion of Theorem 5.12 can then be formulated as follows: There exist $\gamma > 0$ and γ_0 such that $\|z\| \leq \gamma\|u\| + \gamma_0\|x_0\|$ for all $u \in \mathscr{L}_2^k$ and all $\Delta_\varphi \in \sec[\alpha, \beta]^k$. ⋆

It is important to note that (5.10) with Π replaced by Π_Λ implies $(CB)^T\Lambda + \Lambda CB \prec 0$ for $\omega \to \infty$; hence the Popov criterion as formulated cannot be applied directly if, e. g, CB is singular. For this reason, Popov multipliers should always be used in combination with those for the circle criterion, as in the following example which compares different classes of circle with Popov multipliers.

Example 5.14.
In order to compare diagonally repeated circle multipliers to unstructured ones and those for the Popov criterion, we consider interconnection (4.3) with M as in (5.5) and

$$A = \begin{pmatrix} -4 & -3 & 0 \\ 2 & 0 & 0 \\ -1 & -1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 4 & 1 & 3 \\ 2 & 0 & 3 & 1 \\ 1 & 0 & 3 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -0.1 & -0.2 & 1 \\ -1 & -0.3 & 0.1 \\ -0.2 & 0.1 & 1 \\ 0.1 & -0.2 & 0.2 \end{pmatrix},$$

as well as $D = 0$. For $\Delta_\varphi \in \sec[0, \beta]^4$, the goal is to estimate the maximal value of $\beta \geq 0$ such that the interconnection (4.3) for $\mathscr{V} = \{0\}$ is stable. In anticipation of the computational procedure in Section 5.3.4, we actually determine for various fixed values of $\beta \geq 0$ the infimal value of $\gamma > 0$ for which the stability conditions of Theorems 5.7/5.12 can be assured. The dotted curve in Figure 5.1 plots the results for the diagonally structured circle criterion (5.11), whereas for obtaining the dashed and solid ones we employed multipliers for the full-block circle criterion (5.12) and their combination with Popov multipliers (5.20), respectively. Already the unstructured circle multipliers lead to considerable improvements, since the computed gains only diverge for much larger values of β if compared to the dotted curve. The combination with Popov multipliers even allows for further increased values of β until stability is lost.

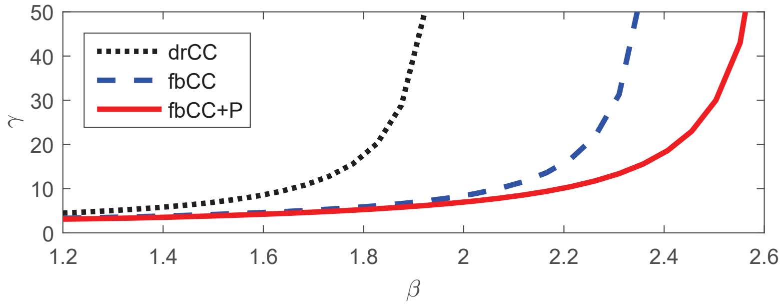

Figure 5.1: Comparison of $\mathscr{L}_2$-gain estimates for different approaches

⋆

5.3.3 Full-block Zames-Falb criterion

In the literature on slope-restricted nonlinearities, the Zames-Falb stability test is often labeled as the least conservative of all available criteria. However, from a computational point of view, it is also the most expensive one since it relies on the approximation of an infinite dimensional space of multipliers. After stating the Zames-Falb stability criterion, we give a new approximation family of full-block Zames-Falb multipliers and prove its asymptotic exactness.

Full-block Zames-Falb multipliers

The following Theorem is a combination of the main results in [42] and [103] that completely describes the class of full-block Zames-Falb multipliers.

Theorem 5.15 (Full-block Zames-Falb IQCs)
Let $H \in \mathscr{L}_1(-\infty,\infty)^{k\times k}$ and $G \in \mathbb{R}^{k\times k}$ satisfy

$$G_{ii} > \sum_{\substack{j=1 \\ j\neq i}}^{k} |G_{ij}| + \sum_{j=1}^{k} \|H_{ij}\|_1 \quad \text{for all} \quad i = 1,\ldots,k. \tag{5.27}$$

and

$$G_{ii} > \sum_{\substack{j=1 \\ j\neq i}}^{k} |G_{ji}| + \sum_{j=1}^{k} \|H_{ji}\|_1 \quad \text{for all} \quad i = 1,\ldots,k. \tag{5.28}$$

Assume either that φ is odd or that $G_{ij} \leq 0$ for $i \neq j$ and $H(t) \geq 0$ for almost all $t \in \mathbb{R}$. Then the following IQCs hold:

a) $\Sigma_{\Pi_{ZF,\infty}}(z, \Delta_\varphi(z)) \geq 0$ *for all* $z \in \mathscr{L}_2^k$ *and* $\Delta_\varphi \in \mathrm{slope}(0,\infty)^k \cap \sec[0,\beta]^k$ *with*

$$\Pi_{ZF,\infty}(i\omega) := \begin{pmatrix} 0 & G^T - \hat{H}(i\omega)^* \\ G - \hat{H}(i\omega) & 0 \end{pmatrix}. \tag{5.29}$$

b) $\Sigma_{\Pi_{ZF}}(z, \Delta_\varphi(z)) \geq 0$ *for all* $z \in \mathscr{L}_2^k$ *and* $\Delta_\varphi \in \mathrm{slope}(\mu_1,\mu_2)^k$ *if*

$$\Pi_{ZF}(i\omega) := (\star)^T \begin{pmatrix} 0 & G^T - \hat{H}(i\omega)^* \\ G - \hat{H}(i\omega) & 0 \end{pmatrix} \begin{pmatrix} \mu_2 I & -I \\ -\mu_1 I & I \end{pmatrix}. \tag{5.30}$$

Remark 5.16.
Note that the multipliers in Theorem 5.15 a) and b) are related via a loop transformation. It is one of the essential advantages of IQC theory over classical multiplier theory that such loop transformations can be incorporated into the multipliers and thus need not be carried out explicitly. ⋆

In case of $\Delta_\varphi \in \mathrm{slope}(0,\mu)^k$, both IQCs in Theorem 5.15 hold simultaneously with $\mu_1 = 0$ and $\mu_2 = \mu$. However, for stability analysis based on verifying the FDI (5.10), let us show next that (5.30) defines the stronger class in the following sense. For this reason we continue to work with (5.30) only.

Lemma 5.17
Let $\Delta_\varphi \in \mathrm{slope}(0,\mu)^k$ *for* $\mu > 0$. *Suppose* (5.10) *holds for* $\Pi = \Pi_{ZF,\infty} + \Pi_{ZF}$ *with* (5.29), (5.30) *defined through* (G,H), (G_1,H_1). *Then* Π *in* (5.30) *satisfies* (5.10) *for* $G_2 := \frac{1}{\mu}G + G_1$ *and* $H_2 := \frac{1}{\mu}H + H_1$.

Proof. With $Z := G - H$ and $Z_1 = G_1 - H_1$, the FDI (5.10) for $\Pi = \Pi_{ZF,\infty} + \Pi_{ZF}$ reads as

$$\mathrm{He}[(Z + \mu Z_1)M] - \mathrm{He}[Z_1] \preccurlyeq -\varepsilon I \quad \text{on} \quad i\mathbb{R}. \tag{5.31}$$

Now note that $|H_{ij}(i\omega)| \leq \|H_{ij}\|_1$ for all $i,j \in \{1,\ldots,k\}$ and all $\omega \in \mathbb{R}$. Using Geršgorin's theorem ([81, Theorem 6.1.1]), the constraints (5.27), (5.28) hence guarantee that the eigenvalues of $\mathrm{He}[Z(i\omega)]$ are contained in the open right half complex plane, i. e., $\mathrm{He}[Z(i\omega)] \succ 0$ for all $\omega \in \mathbb{R}$. We infer

$$\mathrm{He}[\mu(\frac{1}{\mu}Z + Z_1)M] - \mathrm{He}[\frac{1}{\mu}Z + Z_1] \preccurlyeq -\varepsilon I$$

and hence

$$\mathrm{He}[\mu Z_2 M] - \mathrm{He}[Z_2] \preccurlyeq -\varepsilon I \quad \text{on} \quad i\mathbb{R}$$

for $Z_2 := \frac{1}{\mu}Z + Z_1$, which is the FDI (5.10) for (5.30) with (G_2, H_2). It remains to observe that (G_2, H_2) also obeys (5.27), (5.28). □

Parametrization

In order to use Zames-Falb multipliers for computational stability analysis, we need to optimize over functions $H \in \mathscr{L}_1(-\infty,\infty)^{k\times k}$. In [42] this is done in a non-systematic way by fixing a small number of basis functions and optimizing in the resulting spanned subspace. Obviously, more freedom in the choice of H, i. e., an increase in the dimension of the considered subspaces, leads, in general, to improved results but also causes higher computational complexity. In the sequel, we present an approach that allows to balance both by extending the ideas for the scalar case [37, 176] which rely on a family of dense subspaces of $\mathscr{L}_1$.

Fix a real pole $\rho < 0$. With $\nu \in \mathbb{N}$ define

$$A_\nu := \begin{pmatrix} \rho & 0 & \dots & \dots & 0 \\ 1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & \rho \end{pmatrix} \in \mathbb{R}^{\nu\times\nu}, \qquad B_\nu := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{\nu\times 1},$$

and

$$Q_\nu(t) := e^{A_\nu t} B_\nu = e^{\rho t}\varphi_\nu(t),$$

where

$$\varphi_\nu(t) = \operatorname{diag}(0!, 1!, \dots, (\nu-1)!)\ \operatorname{col}(1, t, \dots, t^{\nu-1}).$$

This choice is motivated by the well-known fact that $e^{\rho t}p(t)$ with some polynomial p can approximate functions in $\mathscr{L}_1$ and $\mathscr{L}_2$ arbitrary closely (see, e. g., [155]). With coefficient matrices $C_1, C_2, C_3, C_4 \in \mathbb{R}^{k\times k\nu}$ and the identity matrix $I_k \in \mathbb{R}^{k\times k}$ we now select the function $H_\nu \in \mathscr{L}_1(-\infty,\infty)^{k\times k}$ as

$$\begin{aligned} H_\nu(t) &= H_{\nu,1}(t) - H_{\nu,2}(t) = (C_1 - C_2)(\varphi_\nu(-t)\otimes I_k)e^{-\rho t} \quad \text{for} \quad t<0, \\ H_\nu(t) &= H_{\nu,3}(t) - H_{\nu,4}(t) = (C_3 - C_4)(\varphi_\nu(t)\otimes I_k)e^{\rho t} \quad \text{for} \quad t\geq 0; \end{aligned} \tag{5.32}$$

if φ is odd, we impose the constraint $H_{\nu,l}(t) > 0$ for $l = 1, \dots, 4$ and all t in the respective domains, which still defines a function H_ν without sign-constraint; if φ is not odd, we take $C_2 = C_4 = 0$ and $H_{\nu,1}$, $H_{\nu,3}$ to be positive. Note that we use the same basis functions for all components of H_ν. With

$$\psi_\nu(i\omega) := \left(1 \ \frac{1}{i\omega-\rho} \ \cdots \ \frac{1}{(i\omega-\rho)^\nu}\right)^T \quad \text{realized as} \quad \psi_\nu = \left[\begin{array}{c|c} A_\nu & B_\nu \\ \hline 0 & 1 \\ I & 0 \end{array}\right],$$

the equations (5.32) give

$$\hat{H}_\nu(i\omega) = (\psi_\nu(i\omega) \otimes I_k)^* \begin{pmatrix} 0 & C_1 - C_2 \end{pmatrix}^T + \begin{pmatrix} 0 & C_3 - C_4 \end{pmatrix} (\psi_\nu(i\omega) \otimes I_k).$$

This leads to the multiplier

$$\begin{aligned} \Pi_\nu(i\omega) &= (\star)^T \begin{pmatrix} 0 & G^T - \hat{H}_\nu(i\omega)^* \\ G - \hat{H}_\nu(i\omega) & 0 \end{pmatrix} \begin{pmatrix} \mu_2 I & -I \\ -\mu_1 I & I \end{pmatrix} \\ &= (\star)^* P_{ZF} \begin{pmatrix} \mu_2 I & -I \\ -\mu_1 I & I \end{pmatrix} \begin{pmatrix} \psi_\nu(i\omega) \otimes I_k & 0 \\ 0 & \psi_\nu(i\omega) \otimes I_k \end{pmatrix}, \end{aligned} \tag{5.33}$$

where

$$P_{ZF} = \begin{pmatrix} 0 & P_{12}^T \\ P_{12} & 0 \end{pmatrix} \quad \text{with} \quad P_{12} = \begin{pmatrix} G & C_4 - C_3 \\ C_2^T - C_1^T & 0 \end{pmatrix}. \tag{5.34}$$

For $\nu = 0$ we choose $H_\nu(t) = 0$ for all $t \in \mathbb{R}$ and define Π_0 in the same fashion with $\psi_\nu = 1$ and empty coefficient matrices C_1, C_2, C_3, C_4. All this leads to the following computational stability test by combining Theorems 5.15 and 4.7.

Theorem 5.18 (Full-block Zames-Falb criterion)
Let $N : \mathscr{V} \to \mathscr{L}_2^k$ and suppose that the interconnection (4.3) *is well-posed in the classical sense for all $\Delta_\varphi \in \mathrm{slope}(\mu_1, \mu_2)^k$. With a pole location $\rho < 0$ and an expansion length $\nu \in \mathbb{N}_0$, the feedback interconnection* (4.3) *is robustly stable for $\Delta_\varphi \in \mathrm{slope}(\mu_1, \mu_2)^k$ if*

a) $C_l \in \mathbb{R}^{k \times k\nu}$ satisfy

$$C_l(\varphi_\nu(t) \otimes I) > 0 \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad l = 1, \ldots, 4; \tag{5.35}$$

b) for all $i = 1, \ldots, k$ the matrix $G \in \mathbb{R}^{k \times k}$ satisfies

$$\sum_{j=1}^{k} \|(H_\nu)_{ij}\|_1 + \sum_{\substack{j=1 \\ j \neq i}}^{k} |G_{ij}| < G_{ii} \quad \text{and}$$

$$\sum_{j=1}^{k} \|(H_\nu)_{ji}\|_1 + \sum_{\substack{j=1 \\ j\neq i}}^{k} |G_{ji}| < G_{ii}; \tag{5.36}$$

c) *either* φ *is odd or* $G_{ij} \leq 0$ *for* $i \neq j$, (5.35) *holds for* $l = 1, 3$ *and* $C_2 = C_4 = 0$.

d) *the FDI* (5.10) *is valid with* $\Pi = \Pi_\nu$ *given in* (5.33), (5.34).

Proof. Since Π_ν is bounded on the imaginary axis and no particular signal regularity requirements are needed, the result is an immediate consequence of Theorem 5.15 and Theorem 4.7 with all extended spaces taken as $\mathscr{L}_{2e}$ and $l = 0$. □

Note that both (5.35) and (5.36) can be easily turned into standard finite dimensional LMI constraints, along the same lines as for the scalar case in [37, 176]. This allows for a straightforward implementation of the Zames-Falb stability test, which is also applicable to nonlinearities φ that are not odd, in contrast to the one proposed in [163].

Remark 5.19.
For $\nu = 0$ the Zames-Falb multiplier is static (and independent from ρ). In [169, Lemma 3] a similar multiplier (with the unnecessary additional constraint $G = G^T$) is, slightly misleadingly, introduced as a less conservative substitute for a circle criterion multiplier in certain cases. As a consequence of our exposition (see also [42]), this multiplier neither requires a separate proof for its validity nor does it serve as a replacement for circle criterion multipliers. Instead, applying them both is computationally inexpensive and often beneficial, as demonstrated by Example 5.20. ⋆

Example 5.20.
In order to highlight the advantages of Zames-Falb multipliers over

circle and Popov ones, we consider interconnection (4.3) with M as in (5.5) defined by

$$A = \begin{pmatrix} -10 & -2.5 & -2.5 \\ 3 & -1 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1.5 & 0.5 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 0 \end{pmatrix},$$

and D=0. Let $\Delta_\varphi \in \text{slope}(0,1)$ with φ being odd and $\mathscr{U} = \mathscr{L}_2$ as well as $\mathscr{V} = \{0\}$. Table 5.1 shows bounds on the $\mathscr{L}_2$-gain from u to z computed with different criteria. As can be seen, neither the full-block circle criterion (fbCC) nor its combination with Popov (fbCC+P) can guarantee stability of the interconnection. However, a combination

Table 5.1: $\mathscr{L}_2$-gain estimates

fbCC	fbCC+P	$\nu = 0$	$\nu = 1$	$\nu = 3$	$\nu = 5$
∞	∞	25.0	11.25	7.98	7.55

of multipliers for the circle criterion even with only static (full-block) Zames-Falb multipliers ($\nu = 0$) allows to verify stability. If increasing the order of the Zames-Falb multipliers with pole $\rho = -1$ up to $\nu = 5$, we obtain improved bounds for the $\mathscr{L}_2$-gain that approach the open loop gain $\gamma_{ol} = 7.52$ quite fast. ⋆

Example 5.21.
As made precise in [24, 25], Popov multipliers may be thought of as Zames-Falb multipliers of order one with a pole at infinity (Note that this has been pointed out on numerous occasions; see, e.g., [140, 195, 181]). In order to illustrate this effect, we choose an example that is particularly

Table 5.2: Maximal values of β

ρ	-1	-10	-10^2	-10^3	-10^4	fbCC+P
$\beta_{\max}$	22	40	160	360	407	434

well suited for the analysis with Popov multipliers (see Example 3 in [123]). Here M is described with $A = -\operatorname{diag}(1,4,6,2,9,8,3,10,12)$,

$$B^T = -\begin{pmatrix} 1&0&0&0&1&0&0&1&0\\ 0&1&0&1&0&0&0&0&1\\ 0&0&1&0&0&1&1&0&0 \end{pmatrix}, \quad C = \begin{pmatrix} 1&1&1&0&0&0&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&0&0&0&0&0&1&1&1 \end{pmatrix}, \quad D = 0.$$

For $\Delta_\varphi \in \sec[0,\beta]^3 \cap \operatorname{slope}(0,2\beta)^3$ with φ being odd, the goal is to estimate the maximal value of $\beta \geq 0$ such that the interconnection (4.3) for $\mathscr{V} = \{0\}$ is stable. We analyze this interconnection by combining a full-block Zames-Falb multiplier for decreasing values of ρ and $\nu = 1$ with a full-block multiplier for the circle criterion which leads to the first five stability margins in Table 5.2. Indeed, these margins improve for larger negative values of ρ, i. e., for better approximations of the Popov multiplier. However, even for $\rho = -10000$ we are still more conservative than the margin obtained by combining full-block Circle and Popov criteria (last value in Table 5.2) which is all the more astonishing as both only exploit the sector constraint. Yet, it is important to note that this example works with tighter sector bounds than slope restrictions, which tips the scale towards the circle and Popov criteria. $\star$

The parametrization of Zames-Falb multipliers in [37], which is also at the heart of ours, is often criticized for being difficult to implement and computationally expensive (see, e. g., [123, 28, 29]). Far from that and in view of the possibility to combine them for different poles (see Section 5.3.4), this class allows for a flexible trade-off between computational load and accuracy, which is all the more important for larger values of k. Moreover, in contrast to what is often claimed, it does not require a line search over the pole location as we will clarify in the next section.

Asymptotic exactness of the parametrization

As one of the distinguishing features of the above proposed parametrization, if compared for example to the one in [163] (extended to the full-block case in [169]), even for a fixed choice of $\rho < 0$ it can be proven to be asymptotically exact.

Theorem 5.22
Fix $\rho < 0$ and a stable transfer matrix M of dimension $k \times k$. Then there exist $H \in \mathscr{L}_1(-\infty, \infty)^{k\times k}$ and $G \in \mathbb{R}^{k\times k}$ that satisfy the FDI (5.10) with Π_{ZF} as in (5.27), (5.30) if and only if there exists $\nu \in \mathbb{N}_0$ and $C_1, C_2, C_3, C_4 \in \mathbb{R}^{k\times k\nu}$ such that (5.35), (5.36) hold and (5.10) is satisfied with Π_ν in (5.33), (5.34).

This result holds (with a simpler proof) in the same fashion if including the extra constraints that are required in case the nonlinearity φ is not odd. The following technical fact (Lemma A.1 in [175], due to Jonathan R. Partington) provides the foundation for our proof of Theorem 5.22.

Lemma 5.23
Let $\rho < 0$ and $h \in \mathscr{L}_1[0, \infty)$ be nonnegative. Then for all $\varepsilon > 0$ there exists a real polynomial p such that $q(t) = e^{\rho t} p(t)$ satisfies $\|h - q\|_1 < \varepsilon$ and $p(t) > 0$ for all $t \geq 0$.

Proof of Theorem 5.22. Since Π_ν is a Zames-Falb multiplier, one implication is trivial. Now assume that (5.10) holds with Π_{ZF}. Since the inequalities (5.36) are strict, there exists some $\delta > 0$ such that (5.10) persists to hold for all $K \in \mathscr{L}_1(-\infty, \infty)^{k\times k}$ with

$$\|\hat{H}_{ij} - \hat{K}_{ij}\|_\infty \leq \delta \quad \text{for all} \quad i, j \in \{1, \ldots, k\}. \tag{5.37}$$

We split H_{ij} into two nonnegative functions in $\mathscr{L}_1(-\infty, \infty)$ as

$$H_{+,ij}(t) := \max\{H_{ij}(t), 0\} \quad \text{and} \quad H_{-,ij}(t) := -\min\{H_{ij}(t), 0\}.$$

Let us further choose $\varepsilon \in (0, \delta/4)$ with

$$G_{ii} > \sum_{\substack{j=1\\j\neq i}}^{k} |G_{ij}| + \sum_{j=1}^{k} \|H_{ij}\|_1 + 4k\varepsilon \quad \text{for all} \quad i = 1, \ldots, k;$$

$$G_{ii} > \sum_{\substack{j=1\\j\neq i}}^{k} |G_{ji}| + \sum_{j=1}^{k} \|H_{ji}\|_1 + 4k\varepsilon \quad \text{for all} \quad i = 1, \ldots, k. \tag{5.38}$$

Since $H_{+,ij}, H_{-,ij}$ are nonnegative, by Lemma 5.23, there exists some $\nu \in \mathbb{N}$ and coefficient vectors $c_{l,ij} \in \mathbb{R}^{1\times\nu}$ for $l = 1, \ldots, 4$ with (5.35) and such that the $\mathscr{L}_1$-norms of

$$H_{+,ij}(t) - c_{1,ij}Q_\nu(-t), \quad H_{-,ij}(t) - c_{2,ij}Q_\nu(-t) \quad \text{for} \quad t \in (-\infty, 0),$$

$$H_{+,ij}(t) - c_{3,ij}Q_\nu(t), \quad H_{-,ij}(t) - c_{4,ij}Q_\nu(t) \quad \text{for} \quad t \in [0, \infty)$$

are smaller than ε for all $i, j = 1, \ldots, k$. In view of (5.32) we now define the components of H_ν as

$$H_{ij,\nu}(t) := (c_{1,ij} - c_{2,ij})Q_\nu(-t) \;\; \text{for} \;\; t < 0$$

and

$$H_{ij,\nu}(t) := (c_{3,ij} - c_{4,ij})Q_\nu(t) \;\; \text{for} \;\; t \geq 0.$$

The triangle inequality implies $\|H_{ij,\nu} - H_{ij}\|_1 < 4\varepsilon$, and hence the $\mathscr{L}_\infty$-norm of $\hat{H}_{ij,\nu} - \hat{H}_{ij}$ is bounded by $4\varepsilon < \delta$ for all $i, j = 1, \ldots, k$. Therefore, by our choice of δ in (5.37), the FDI (5.10) still holds for $\hat{H}_\nu$. Finally, we also have $\|H_{ij,\nu}\|_1 \leq \|H_{ij,\nu} - H_{ij}\|_1 + \|H_{ij,\nu}\|_1 \leq 4\varepsilon + \|H_{ij,\nu}\|_1$ for all $i, j = 1, \ldots, k$ which implies by (5.38) that (5.36) is true as well. $\square$

5.3.4 Combination of multipliers in the frequency domain

Let us now address in detail how to implement combinations of the developed stability tests for $\Delta_\varphi \in \mathrm{slope}(\mu_1, \mu_2)^k \cap \sec[\alpha, \beta]^k$ if φ is odd and under the well-posedness assumption that

$$I - M\Delta_\varphi : \mathscr{L}_{2e}^k \to \mathscr{L}_{2e}^k \text{ has a causal inverse}$$
$$\text{for all } \Delta_\varphi \in \operatorname{slope}(\mu_1, \mu_2)^k \cap \sec[\alpha, \beta]^k. \quad (5.39)$$

As seen in Example 5.21 (see also [173]), for small lengths ν the pole location ρ influences the stability guarantees achieved with the Zames-Falb multiplier Π_ν in (5.33). This motivates to choose several pole locations $0 > \rho_1 > \ldots > \rho_L$ with lengths $\nu_1, \ldots, \nu_L \in \mathbb{N}_0$ and to merge the corresponding multipliers as follows. Each individual one reads as

$$\Pi_{\nu_l,\rho_l} = \Psi_{\nu_l,\rho_l}^* P_{ZF,l} \Psi_{\nu_l,\rho_l}$$

where

$$\Psi_{\nu_l,\rho_l} = \begin{pmatrix} \mu_2 I & -I \\ -\mu_1 I & I \end{pmatrix} \begin{pmatrix} \psi_{\nu_l,\rho_l} \otimes I_k & 0 \\ 0 & \psi_{\nu_l,\rho_l} \otimes I_k \end{pmatrix}$$

and

$$P_{ZF,l} \text{ as in } (5.34) - (5.36)$$

are defined for ν_l and ρ_l. With $P_{CC} \in \mathbf{\Pi}_{pc}[\alpha, \beta]^k$, the sum $P_{CC} + \sum_{l=1}^L \Pi_{\nu_l,\rho_l}$ is a valid IQC multiplier for Δ_φ due to Lemma 5.6 and Theorem 5.15. With $P := \operatorname{diag}(P_{CC}, P_{ZF,1}, \ldots, P_{ZF,L})$, and $\Psi := \operatorname{col}(I_{2k}, \Psi_{\nu_1,\rho_1}, \ldots, \Psi_{\nu_L,\rho_L})$, this sum is described as

$$\Pi(P) := P_{CC} + \sum_{l=1}^{L} \Pi_{\nu_l,\rho_l} = \Psi^* P\, \Psi \tag{5.40}$$

with a fixed stable dynamic part Ψ and a real symmetric matrix P varying in some set $\mathbf{P}$; in view of (5.15) and as emphasized for $P_{ZF,1}, \ldots, P_{ZF,L}$ satisfying the constraints (5.34)-(5.36) in Section 5.3.3, the set $\mathbf{P}$ is a convex cone with an LMI description. If $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is a minimal realization of $\Psi \operatorname{col}(M, I_k)$ and by Lemma 2.12, (5.10) holds for $\Pi(P)$ and some $P \in \mathbf{P}$ if and only if there exist $X = X^T$ and $P \in \mathbf{P}$ such that the following LMI is satisfied:

$$\begin{pmatrix} I & 0 \\ \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}^T \begin{pmatrix} 0 & X & 0 \\ X & 0 & 0 \\ 0 & 0 & P \end{pmatrix} \begin{pmatrix} I & 0 \\ \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \prec 0. \tag{5.41}$$

In this fashion, the robust stability test with multiplier class (5.40) for $P \in \mathbf{P}$ boils down to solving a standard LMI feasibility problem.

If $M(\infty) = 0$, we use the stable transfer matrix $H(s) = sM(s)$ to include the Popov multiplier Π_Λ from Theorem 5.11 as follows. We extend Ψ as $\Psi_{\rm Pop} := \operatorname{diag}(I_{2k}, \Psi)$ and now take a minimal realization $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ of $\Psi_{\rm Pop}\operatorname{col}(H, I_k, M, I_k)$. Testing robust stability then amounts to checking the feasibility of (5.41) with $P \in \mathbf{P}$ replaced by

$$\operatorname{diag}\left(\begin{pmatrix} 0 & \Lambda \\ \Lambda & 0 \end{pmatrix}, P\right) \quad \text{for} \quad \Lambda \in \mathbb{D}^k \ \text{ and } \ P \in \mathbf{P}.$$

As explained in more detail, for example, in [176], let us finally address how the performance setting illustrated in Section 4.4 specializes to the computation of guaranteed bounds on the $\mathscr{L}_2$-gain of $d \to e$ in the uncertain interconnection in Figure 5.2 for stable transfer matrices M, N, N_{21}, N_{22} (of compatible dimension) under the assumption (5.39).

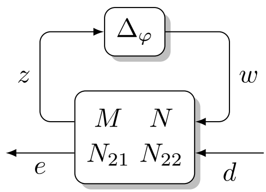

Figure 5.2: Performance setting

For $\gamma > 0$ suppose there exist some $P \in \mathbf{P}$ and $\varepsilon > 0$ such that the following FDI is valid:

$$(\star)^* \,\Pi(P) \begin{pmatrix} M & M_{12} \\ I & 0 \end{pmatrix} + (\star)^* \begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I \end{pmatrix} \begin{pmatrix} N_{21} & N_{22} \\ 0 & I \end{pmatrix} \preccurlyeq -\varepsilon I \ \text{ on } \ i\mathbb{R}. \tag{5.42}$$

Since the left upper block implies (5.10), our stability results imply that $(I - M\Delta_\varphi)^{-1}$ maps $\mathscr{L}_2^k$ into $\mathscr{L}_2^k$ for all considered uncertainties Δ_φ

(Remark 5.8). It is then elementary to show that (5.42) guarantees the following robust performance property:

$$\sup_{d \in \mathscr{L}_2 \setminus \{0\}} \frac{\|e\|}{\|d\|} < \gamma \quad \text{for all} \quad \Delta_\varphi \in \mathrm{slope}(\mu_1, \mu_2)^k \cap \sec[\alpha, \beta]^k.$$

With the KYP lemma, the FDI (5.42) is easily turned into an LMI, and it is even possible to find the smallest possible value of γ for which the FDI is satisfied with some $P \in \mathbf{P}$ and $\varepsilon > 0$. If M and N are strictly proper, the inclusion of Popov multipliers proceeds as for stability.

It remains to note that (4.3) for $\mathscr{V} = \{0\}$ (Figure 4.1) can be subsumed into the interconnection in Figure 5.2 with the choices $e = z$, $d = u$ and $N = N_{21} = N_{22} = M$. This is the way how all the optimal $\mathscr{L}_2$-gain bounds of $u \to z$ and for a chosen multiplier class have been computed in the present chapter.

Finally, let us highlight the necessity to combine stability multipliers for different criteria with the following example.

Example 5.24.
For single input single output (SISO) nonlinearities it has been shown in [24, 25] that the inclusion of Popov multipliers cannot improve the Zames-Falb stability test for a particular system class and under the assumption $[\alpha, \beta] = [\mu_1, \mu_2]$. In general, this is not true for a computationally tractable finite dimensional approximation of Zames-Falb multipliers or if $[\alpha, \beta] \neq [\mu_1, \mu_2]$, as [167] suggests. Let us illustrate this effect by continuing with Example 5.14, for which we now impose the slope restriction $\Delta_\varphi \in \mathrm{slope}(0, 4\beta)^4$ in addition to the sector constraint $\Delta_\varphi \in \sec[0, \beta]^4$. Naturally, this does not affect the stability margins obtained by using circle and Popov multipliers as recalled in Table 5.3 (CCP). If combining full-block circle and Zames-Falb multipliers for $\rho = -1000$, we can compute an increased margin for $\nu = 1$ that cannot be improved for larger values of ν (CCZF in Table 5.3). However, by further adding a Popov multiplier (CCZFP), this improvement gets much more pronounced even for small basis lengths. $\star$

Table 5.3: Maximal values of β for different multipliers

	CCP	CCZF ($\nu = 1$)	CCZF ($\nu = 5$)	CCZFP ($\nu = 1$)
β_{max}	2.66	3.30	3.30	9.48

5.4 General Popov and Yakubovich criteria

In the previous section we saw that the incorporation of a Popov multiplier may reduce conservativeness significantly. However, the given derivation is limited to strictly proper LTI systems M due to the required filtering property. Yakubovich introduced a stability test [188] that is based on a relation between the derivative of both the input z and the output w of the uncertainty Δ_φ. Here again we fail in applying classical IQC theory due to higher regularity properties needed for the involved signals, and not even strict properness of M serves as a remedy. In the present section we further exploit Theorem 4.7 in order to overcome these problems and derive stability results in both cases.

The general idea of both Popov and Yakubovich stability tests is to capture the operation of Δ_φ by exploiting relations between the signals $z, w, \dot{z}$ and $\dot{w}$. This can easily be realized using the quadratic forms in (4.18) that, specialized to the present setting, read as

$$\Sigma_{(P,P_0)}\begin{pmatrix} z \\ w \end{pmatrix} = \int_0^\infty (\star)^T P \begin{pmatrix} z(t) \\ w(t) \\ \hdashline \dot{z}(t) \\ \dot{w}(t) \end{pmatrix} dt + w(0)^T P_0 w(0) \quad (5.43)$$

for $(z, w) \in \mathscr{H}^{1,k} \times \mathscr{H}^{1,k}$ and with $P \in \mathcal{S}^{4k}$ and $P_0 \in \mathcal{S}^k$. Note that we rearranged the signals if compared to (4.18) which will permit a straightforward application of the already derived criteria. Theorem 4.7 d) then requires to certify

$$\exists \varepsilon > 0 : \quad \Sigma_{(P,P_0)}\begin{pmatrix} Mw \\ w \end{pmatrix} \leq -\varepsilon \|w\|_{\mathscr{H}}^2 \quad \text{for all} \quad w \in \mathscr{H}^{1,k}. \quad (5.44)$$

From Theorem 4.12, we immediately deduce the following corollary that relates (5.44) to an LMI and an FDI. We use the notations and definitions introduced in Section 4.5.3.

Corollary 5.25
Let $P \in \mathbb{S}^{4k}$. Then the following statements are equivalent:

a) There exists some $P_0 \in \mathbb{S}^k$ such that (5.44) *holds.*

b) There exist $X = X^T$ with

$$\begin{pmatrix} I & 0 \\ A_e & B_e \end{pmatrix}^T \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A_e & B_e \end{pmatrix} + T_\Delta^T P T_\Delta \prec 0. \tag{5.45}$$

c) There exists $\varepsilon > 0$ such that for all $\omega \in \mathbb{R} \setminus \{0\}$

$$(\star)^* P \begin{pmatrix} \frac{1}{i\omega} M(i\omega) \\ \frac{1}{i\omega} I \\ \hdashline M(i\omega) \\ I \end{pmatrix} \preccurlyeq -\varepsilon I \quad \text{and} \quad (\star)^T P \begin{pmatrix} M(0) \\ I \\ \hdashline 0 \\ 0 \end{pmatrix} \prec 0. \tag{5.46}$$

5.4.1 Full-block Yakubovich criterion

Let us first turn to the Yakubovich stability criterion [46, 188] (see also [123, 133, 134, 10]), which has originally been formulated for differentiable nonlinearities $\varphi : \mathbb{R} \to \mathbb{R}$ that satisfy the inequalities

$$0 \leq \varphi(x)/x \leq \kappa \quad \text{and} \quad -\kappa_1 \leq \varphi'(x) \leq \kappa_2 \quad \text{for all} \quad x \in \mathbb{R} \setminus \{0\}$$

with $\kappa_1 \geq 0$ and $\kappa_2 \geq \kappa$. Since some practically relevant nonlinearities, such as the saturation and the dead-zone function, are not differentiable, we need the following lemma for their rigorous treatment within our framework.

Lemma 5.26 (Yakubovich IQC)
Let $\varphi \in$ slope(μ_1, μ_2) and define $w(t) = \varphi(z(t))$ for $t \in [0, \infty)$ and

$z \in \mathscr{H}_e^{1,1}$. Then for almost every $t \in [0, \infty)$ the following inequality holds:

$$(\dot{w}(t) - \mu_1 \dot{z}(t))(\mu_2 \dot{z}(t) - \dot{w}(t)) \geq 0. \tag{5.47}$$

Proof. See Appendix C.4.2. □

Observe that (5.47) is just a sector constraint on the derivative of both the input and the output of the nonlinearity Δ_φ. Hence, for $\Delta_\varphi \in \operatorname{slope}(\mu_1, \mu_2)^k$ we can employ $P_Y \in \mathbf{\Pi}[\mu_1, \mu_2]^k$ (in complete analogy to the full-block circle criterion) in order to arrive at the following IQC:

$$\int_0^\infty \begin{pmatrix} \dot{z}(t) \\ \dot{w}(t) \end{pmatrix}^T P_Y \begin{pmatrix} \dot{z}(t) \\ \dot{w}(t) \end{pmatrix} dt \geq 0 \quad \text{for all} \quad z \in \mathscr{H}^{1,k}$$
$$\text{where} \quad w = \Delta_\varphi(z). \tag{5.48}$$

As a distinguishing feature of our stability theorem, this Yakubovich IQC may be seamlessly included into our framework. We differentiate between strictly proper or just proper LTI systems M. In the first case, a specialization of the general quadratic form (5.43) to the one in (5.48) immediately results in a novel full-block generalization of Yakubovich's stability theorem [188].

Theorem 5.27 (Full-block Yakubovich criterion, strictly proper systems)
Consider the interconnection (4.3) *with strictly proper M and $N : \mathscr{V} \to \mathscr{H}^{1,k}$. If there exist $P_Y \in \mathbf{\Pi}[\mu_1, \mu_2]^k$ and $P_0 \in \mathbb{S}^k$ such that* (5.44) *holds with $P = \operatorname{diag}(0, P_Y)$, then there exits $\gamma \geq 0$ and γ_0 with*

$$\|z\|_{\mathscr{H}} \leq \gamma(\|u\|_2 + \|v\|_{\mathscr{V}}) + \gamma_0 \|z(0)\| \quad \textit{for all} \quad (u, v) \in \mathscr{L}_2^k \times \mathscr{V}$$

and all $\Delta_\varphi \in \operatorname{slope}(\mu_1, \mu_2)^k$.

Like in the Popov criterion for $D = 0$, this is again a guarantee for disturbances u in the full space $\mathscr{L}_2^k$, in contrast to what is often seen in the literature [46, 134]. Example 5.32 reveals the great benefit of combining Yakubovich multipliers with those from Section 5.3.

Proof. Well-posedness follows as in Theorem 5.12. We apply Theorem 4.7 for $\mathscr{U}_e = \mathscr{L}_{2e}^k$ and $\mathscr{W}_e = \mathscr{Z}_e = \mathscr{H}_e^{1,k}$. Recall that $\Sigma_{(P,P_0)}$ satisfies Theorem 4.7 b). Due to (5.44), also d) holds. For $\delta_0 := \|P_0\| \max\{|\mu_1|, |\mu_2|\}^2$ we next note that

$$[\Delta_\varphi(z)(0)]^T P_0 [\Delta_\varphi(z)(0)] \geq -\|\Phi(z(0))\|^2 \|P_0\| \geq -\delta_0 \|z(0)\|^2$$

which means

$$\Sigma_{(0,P_0)}(z, \Delta_\varphi(z)) \geq -\delta_0 \|z(0)\|^2 \quad \text{for all} \quad z \in \mathscr{H}^{1,k} \tag{5.49}$$

and all $\Delta_\varphi \in \text{slope}(\mu_1, \mu_2)^k$. If $z \in M\mathscr{U} + N(\mathscr{V}) \subset \mathscr{H}^{1,k}$ and $\Delta_\varphi \in \text{slope}(\mu_1, \mu_2)^k$, (5.48) reads as

$$\Sigma_{(P,0)}(z, \Delta_\varphi(z)) \geq 0$$

and together with (5.49) we get

$$\Sigma_{(P,P_0)}(z, \Delta_\varphi(z)) \geq -\delta_0 \|z(0)\|^2 \quad \text{for all} \quad z \in M\mathscr{U} + N(\mathscr{V}); \tag{5.50}$$

which, with Remark 5.2, implies Theorem 4.7 c) with

$$l(z)^2 = \delta_0 \|z(0)\|^2.$$

$\square$

In case of $M(\infty) \neq 0$, we only need to confine the disturbance set $\mathscr{U}_e$ to $\mathscr{H}_e^{1,k}$. We emphasize that well-posedness is part of the conclusion in the next result.

Theorem 5.28 (Full-block Yakubovich criterion, general case)
Let $N : \mathscr{V} \to \mathscr{H}^{1,k}$. If there exist $P_Y \in \mathbf{\Pi}[\mu_1, \mu_2]^k$ and $P_0 \in \mathbb{S}^k$ such that (5.44) holds with $P = \text{diag}(0, P_Y)$, then there exist $\gamma \geq 0$ and γ_0 with

$$\|z\|_{\mathscr{H}} \leq \gamma(\|u\|_{\mathscr{H}} + \|v\|_{\mathscr{V}}) + \gamma_0 \|z(0)\| \quad \textit{for all} \quad (u, v) \in \mathscr{H}^{1,k} \times \mathscr{V}$$

and all $\Delta_\varphi \in \text{slope}(\mu_1, \mu_2)^k$.

Proof. We first show that (5.44) guarantees well-posedness in the classical sense. Indeed, due to (5.46) for $\omega \to \infty$, (5.44) implies $\left(\begin{smallmatrix} D \\ I \end{smallmatrix}\right)^T P_Y \left(\begin{smallmatrix} D \\ I \end{smallmatrix}\right) \prec 0$. Further, since $\left(\begin{smallmatrix} I \\ \Delta \end{smallmatrix}\right)^T P_Y \left(\begin{smallmatrix} I \\ \Delta \end{smallmatrix}\right) \succ 0$ for all $\Delta \in \Theta([\mu_1, \mu_2], k)$, we infer that $\det(I - D\Delta) > 0$ for all $\Delta \in \Theta([\mu_1, \mu_2], k)$; then the claim follows from Lemma 5.5. The remaining proof proceeds as for Theorem 5.27 but with $\mathscr{U}_e = \mathscr{W}_e = \mathscr{Z}_e = \mathscr{H}_e^{1,k}$. $\square$

5.4.2 Popov criterion for $D \neq 0$

Let us finally provide a stability result based on combining full-block circle and Yakubovich with Popov multipliers for the interconnection (4.3) with $\Delta_\varphi \in \mathrm{slope}(\mu_1, \mu_2)^k \cap \sec[\alpha, \beta]^k$, φ odd, and a general proper M. We use (5.43) with $P_0 \in \mathbb{S}^k$ and

$$P = \left(\begin{array}{cc|cc} Q_1 & S_1 & 0 & 0 \\ S_1^T & R_1 & \Lambda & 0 \\ \hline 0 & \Lambda & Q_2 & S_2 \\ 0 & 0 & S_2^T & R_2 \end{array}\right), \tag{5.51}$$

where $\Lambda \in \mathbb{D}^k$,

$$\begin{pmatrix} Q_1 & S_1 \\ S_1^T & R_1 \end{pmatrix} \in \mathbf{\Pi}[\alpha, \beta]^k, \quad \text{and} \quad \begin{pmatrix} Q_2 & S_2 \\ S_2^T & R_2 \end{pmatrix} \in \mathbf{\Pi}[\mu_1, \mu_2]^k. \tag{5.52}$$

Remark 5.29.
We further generalize the multiplier P in (5.51) to a completely unstructured version in the subsequent chapter. The results derived there, for the discrete-time case, immediately carry over to the present setting. $\star$

Theorem 5.30 (Popov criterion, general case)
Let $N : \mathscr{V} \to \mathscr{H}^{1,k}$. If there exist P with (5.51), (5.52) *and $P_0 \in \mathbb{S}^k$ such that* (5.44) *holds, then* (4.3) *is robustly stable in the sense of Theorem 5.28 for all $\Delta_\varphi \in \mathrm{slope}(\mu_1, \mu_2)^k \cap \sec[\alpha, \beta]^k$ with φ being odd.*

Proof. The proof proceeds as for Theorem 5.28. With the IQCs from Lemma 5.6, Theorem 5.11 and (5.48) we have $\Sigma_{(P,0)}(z, \Delta_\varphi(z)) \geq$

$-\delta\|z(0)\|^2$ for all considered uncertainties and $z \in \mathscr{H}^{1,k}$; together with (5.49) we now get $\Sigma_{(P,P_0)}(z, \Delta_\varphi(z)) \geq -(\delta+\delta_0)\|z(0)\|^2$ replacing (5.50); the remainder stays unchanged. $\square$

To conclude this section let us link, for scalar nonlinearities $\varphi \in \mathrm{slope}(\mu_1,\mu_2) \cap \sec[0,\beta]$ (with $\beta > 0$) and strictly proper LTI systems, Theorem 5.30 to the classical frequency-domain inequality as, e.g., stated in [46]; yet, with much more limiting regularity requirements. We need to restrict the class of multipliers by substituting $\mathbf{\Pi}[0,\beta]$, $\mathbf{\Pi}[\mu_1,\mu_2]$ through $\mathbf{\Pi}_{dr}[0,\beta]$, $\mathbf{\Pi}_{dr}[\mu_1,\mu_2]$ in (5.52), respectively. An analogous reasoning as in the proof of Theorem 4.12 then reveals that the existence of such a pair (P,P_0) in Theorem 5.30 is equivalent to the existence of positive scalars λ, κ and some real Λ with

$$(\star)^* \left[\lambda \begin{pmatrix} 0 & \frac{\beta}{2} \\ \frac{\beta}{2} & -1 \end{pmatrix} + \Lambda \begin{pmatrix} 0 & i\omega \\ -i\omega & 0 \end{pmatrix} + \kappa\omega^2 \begin{pmatrix} -\mu_1\mu_2 & \frac{\mu_1+\mu_2}{2} \\ \star & -1 \end{pmatrix}\right] \begin{pmatrix} M(i\omega) \\ 1 \end{pmatrix} \prec 0 \tag{5.53}$$

for all $\omega \geq 0$ and

$$\begin{pmatrix} M(\infty) \\ 1 \end{pmatrix}^T \left[\kappa \begin{pmatrix} -\mu_1\mu_2 & \frac{\mu_1+\mu_2}{2} \\ \star & -1 \end{pmatrix}\right] \begin{pmatrix} M(\infty) \\ 1 \end{pmatrix} \prec 0. \tag{5.54}$$

By homogeneity we can set $\lambda = 1/\beta$ in (5.53), while (5.54) trivially holds due to $M(\infty) = 0$; then (5.54) indeed just boils down to the inequality (6) in [46]. Note that this relation is also confirmed by the numerical results in Example 5.31.

5.4.3 Combination of multipliers in the state space

Let us show how to incorporate Zames-Falb multipliers in Theorem 5.30 and how to render the resulting - even stronger - stability test computational on the basis of Corollary 5.25. The inclusion of Zames-Falb multipliers is indeed possible since the corresponding IQCs persist to hold on the subspace $\mathscr{H}^{1,k}$ of $\mathscr{L}_2^k$ as well.

Suppose $\Pi(P_{\rm zf}) = \Psi^* P_{\rm zf} \Psi$ with $P_{\rm zf} \in \mathbf{P}$ is a family of Zames-Falb multipliers for various poles and lengths as described in Section 5.3.4 (without P_{CC}). To guarantee stability it is required to verify

$$\exists \varepsilon > 0: \quad \Sigma_{\Pi(P_{\rm zf})}\begin{pmatrix} Mw \\ w \end{pmatrix} + \Sigma_{(P,P_0)}\begin{pmatrix} Mw \\ w \end{pmatrix} \leq -\varepsilon \|w\|^2_{\mathscr{H}} \quad \text{for all} \quad w \in \mathscr{H}^{1,k} \tag{5.55}$$

with suitable $P_{\rm zf} \in \mathbf{P}$ and P, P_0 as in Theorem 5.30.

We proceed in the state-space by minimally realizing the stable transfer matrix Ψ as $(A_\Psi, B_\Psi, C_\Psi, D_\Psi)$. For $w \in \mathscr{H}^{1,k}$ and $z = Mw$ we infer that $z_\Psi = \Psi\,\mathrm{col}(z, w)$ is the output of

$$\begin{aligned} \dot{\xi} &= A_\Psi \xi + B_{\Psi,1} z + B_{\Psi,2} w, \quad \xi(0) = 0, \\ z_\Psi &= C_\Psi \xi + D_{\Psi,1} z + D_{\Psi,2} w. \end{aligned}$$

In view of (4.21), (4.23) all relevant trajectories in (5.55) hence satisfy the differential equation

$$\begin{pmatrix} \dot{\xi} \\ \dot{x} \\ \dot{w} \\ \hline z_\Psi \\ \hline z \\ w \\ \dot{z} \\ \dot{w} \end{pmatrix} = \left(\begin{array}{ccc|c} A_\psi & B_{\Psi,1}C & B_{\Psi,1}D + B_{\Psi,2} & 0 \\ 0 & A & B & 0 \\ 0 & 0 & 0 & I \\ \hline C_\Psi & D_{\Psi,1}C & D_{\Psi,1}D + D_{\Psi,2} & 0 \\ \hline 0 & C & D & 0 \\ 0 & 0 & I & 0 \\ 0 & CA & CB & D \\ 0 & 0 & 0 & I \end{array}\right) \begin{pmatrix} \xi \\ x \\ w \\ \hline \dot{w} \end{pmatrix} =: \left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C}_1 & \mathcal{D}_1 \\ \hline \mathcal{C}_2 & \mathcal{D}_2 \end{array}\right) \begin{pmatrix} \xi \\ x \\ w \\ \hline \dot{w} \end{pmatrix}$$

with initial conditions $\xi(0) = 0$, $x(0) = 0$ and $w(0)$ specified by $w \in \mathscr{H}^{1,k}$. This allows us to simply apply Corollary 5.25 in order to infer that (5.55) holds if and only if there exists $X = X^T$ such that

$$\begin{pmatrix} I & 0 \\ \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C}_1 & \mathcal{D}_1 \\ \hline \mathcal{C}_2 & \mathcal{D}_2 \end{pmatrix}^T \left(\begin{array}{cc|c|c} 0 & X & 0 & 0 \\ X & 0 & 0 & 0 \\ \hline 0 & 0 & P_{\rm zf} & 0 \\ \hline 0 & 0 & 0 & P \end{array}\right) \begin{pmatrix} I & 0 \\ \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C}_1 & \mathcal{D}_1 \\ \hline \mathcal{C}_2 & \mathcal{D}_2 \end{pmatrix} \prec 0. \tag{5.56}$$

In this fashion we end up with an LMI stability test that incorporates all multipliers of this chapter. The introduction of performance criteria proceeds in complete analogy to Section 5.3.4 and with the help of Theorem 4.12.

We illustrate the advantages with the next three examples. The first one is based on a combination of two systems in [46] and demonstrates the effectiveness of Yakubovich multipliers over Zames-Falb multipliers in terms of computational effort. The second example is a slightly modified version of one given in [46] that nicely exhibits the benefits from the combination of all stability criteria. Finally, in the third one we impose an $\mathscr{H}^{1,k}$-gain performance criterion and compare the result obtained by a combination of Zames-Falb and circle multipliers with our comprehensive approach.

Example 5.31.
Let the 2×2 system M be defined with the strictly proper elements $M_{12}(s) = M_{21}(s) = (s+1)^{-1}$,

$$M_{11}(s) = \frac{-s^2 - 1}{(s^2 + \delta s + 1)(s + 10)}$$

and

$$M_{22}(s) := \frac{-40}{(s+\delta)(s+1)(s^2 + 0.8s + 16)} \tag{5.57}$$

for $\delta = 0.0001$. We examine stability of the interconnection (4.3) with $\Delta_\varphi \in \text{slope}(0, \mu)^2$ and $\mathscr{U} = \mathscr{L}_2^2$. By combining circle, Popov and Yakubovich multipliers, we may ensure stability of (4.3) for up to $\mu = 0.73$. If replacing the Yakubovich by a Zames-Falb multiplier with $\rho = -1$, we need to choose a length $\nu \geq 7$ to also conclude stability for $\mu = 0.73$. The computation time is considerably smaller for the first approach since it only involves 89 decision variables, in contrast to at least 898 for the second. $\star$

Example 5.32.
Consider (4.3) for $\Delta_\varphi \in \text{slope}(0, \mu)$ and $M = M_{22}$ from (5.57). For

$\delta = 0$ it is shown in [46] that the combination of circle and Popov criteria guarantees stability up to $\mu = 0.65$, while the addition of a Yakubovich multiplier results in a maximal value of $\mu = 1.43$. Note that these conclusions were drawn under the assumption that φ is differentiable and that u, $\dot{u}$ and $\ddot{u}$ are all contained in $\mathscr{L}_2$. If using Theorem 5.27 in conjunction with circle and Popov multipliers for $\mathscr{V} = \{0\}$, we obtain the very same results for all nonlinearities $\Delta_\varphi \in \mathrm{slope}(0, \mu)$ and all $u \in \mathscr{L}_2$. If adding Zames-Falb multipliers with $(\nu_1, \nu_2) = (8, 1)$, $(\rho_1, \rho_2) = (-2, -0.5)$, the guaranteed margin increases to $\mu = 1.68$. $\star$

Example 5.33.
For the interconnection in Figure 5.2 with $\Delta_\varphi \in \mathrm{slope}(0, 1)^3$ let $N = N_{21} = N_{22} = M$ be realized as

$$A = \begin{pmatrix} -10 & -2.5 & -2.5 \\ 3 & -1 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1.5 & 0.5 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 0 \end{pmatrix},$$

and $D = 0.5I$. Then the inverse of $I - D\Phi$ exists and is Lipschitz, which guarantees well-posedness for all considered nonlinearities. For our application, we choose $\mathscr{W} = \mathscr{Z} = \mathscr{H}^{1,3}$ and, for simplicity of the presentation, $\mathscr{D} = \mathscr{H}_0^{1,3}$, that is, the subspace of $\mathscr{H}^{1,3}$ containing functions with zero initial condition. The goal is to bound the $\mathscr{H}_0^1$-gain of $d \to e$, i. e., to estimate the smallest $\gamma > 0$ such that $\|e\|_1^2 \leq \gamma^2 \|d\|_1^2$ for all $d \in \mathscr{D}$ by simply choosing $P = \mathrm{diag}(I, -\gamma^2 I)$ for $r = 1$ in (4.20).

Table 5.4 shows the obtained estimates for these gain bounds if only using a combination of Zames-Falb and circle criterion multipliers (Π_{CZF}) and if employing the combination (5.55). Clearly, the possibility of including Popov and Yakubovich multipliers allows for a significant improvement even for small values of ν, while the classical approach using the circle and Zames-Falb multipliers does not even guarantee stability. This is especially important since the number of decision variables increases substantially with ν, which makes it desirable to keep ν small; in our example, the computation of the bound for the multiplier

Table 5.4: $\mathscr{H}_0^1$-gain bounds

ν	0	1	2	3	4
Π_{CZF}	∞	∞	∞	22.98	22.77
Π in (5.40)	7.41	7.36	7.36	7.36	7.36

combination and $\nu = 0$ involves 109 decision variables, whereas the one for Π_{CZF} and $\nu = 4$ requires 667 variables and still achieves worse results. $\star$

5.5 Related stability tests

Let us discuss in this brief section some related stability tests in the literature and highlight the connection between these and our results.

5.5.1 Multiplier proposed by Park

The stability test in [123] relies on a Lurye-Postnikov type Lyapunov function and is an LMI representation of the circle, Popov and Yakubovich stability criteria as proposed much earlier in [188, 46]. However, both [188] and [46] require additional constraints for their Lyapunov based proof. In contrast to what seems to be implied by [123], this approach only captures first order Zames-Falb multipliers as shown in [24, 25]. Moreover, all proposed criteria are confined to diagonally structured multipliers. Since our approach allows a search in the full class of Zames-Falb multipliers in combination with Circle, Popov and Yakubovich tests, it is at least as strong as (and often much stronger than) Park's method, as illustrated in the subsequent examples.

5.5.2 Stability criterion by Hu et al.

The approach by Hu et al. (see, e. g., [51, 84] and [83] for an overview) is specifically designed for the local stability analysis of saturated systems. This means that it can incorporate bounds $d_{\max}$ on the disturbance energy $\|d\|$ (or its amplitude) in the performance setting as described in Section 5.3.4, which typically improves the obtained results especially for small values of $d_{\max}$. One can show that, in the limit $d_{\max} \to \infty$, the obtained results correspond to those achievable with the full-block circle criterion as discussed in Section 5.3.1. For global stability and performance analysis, this implies that our method typically outperforms the one proposed by Hu et al., sometimes even substantially as illustrated in Examples 5.34 and 5.35.

5.5.3 Zames-Falb implementation by Turner et al.

As mentioned before, there exist other ways of parameterizing Zames-Falb multipliers (see, e. g., [163, 167] and [29] for an overview). These methods for $k = 1$ offer the advantage that the multiplier poles do not have to be chosen a priori which allows to optimize over their location as well. However, this comes at the expense of various significant drawbacks. As probably the most important one, the occurring $\mathscr{L}_1$-norm constraint (5.27) is enforced by using LMIs that involve intrinsic conservatism and a line-search over some parameter. In addition, the algorithm is based on a common Lyapunov function for both the $\mathscr{L}_1$-constraint and the stability LMI. Furthermore, the original approach is limited to multipliers of the same degree as the system that are either causal or anti-causal and, more severely, only apply to odd nonlinearities. In a series of papers (see, e. g., the non-exhaustive list [26, 165, 163, 167, 27, 28, 164, 170]) various authors tried to reduce several of the non-intrinsic drawbacks of this method, for example, by combining the obtained Zames-Falb multipliers with those corresponding to the Popov criterion or by allowing for higher order approximations. Still, for k-fold repeated nonlinearities, numerical tractability suffers since a non-convex search must be performed for a k-dimensional parameter [168, 166, 169]; also the use of a common Lyapunov function for all $\mathscr{L}_1$-constraints and stability LMIs might cause additional conservatism for increasing values of k. Our approach avoids these troubles and is guaranteed to achieve no worse results, since it freely combines multiple pole Zames-Falb multipliers (each based on asymptotically exact parameterizations) with those from other criteria for repeated nonlinearities.

5.6 Numerical examples

In this section we compare the results achieved within our framework with some related stability test addressed in the previous section.

Example 5.34.
The canonical application for slope-restricted nonlinearities arises from systems with saturations or dead-zones. Although a great number of papers focus on so-called local stability and performance issues (see, e. g., [157, 83] and references therein for an overview), global tests are also discussed (see, e. g., [40], [85]). We adopt Example 1 from [85] and compute an $\mathscr{L}_2$-gain estimate for the channel $d \to e$ in Figure 5.2 with $\Delta_\varphi \in \mathrm{slope}(0,1)^2$ for the unit saturation function $\varphi = \mathrm{sat}$ and

$$\begin{pmatrix} M & N \\ N_{21} & N_{22} \end{pmatrix} = \left[\begin{array}{c|c:c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hdashline C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{ccc|cc:cc} 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 1 & -3 & 1 & -1 & 1 & 1 \\ \hline 1 & 0 & 1 & -3 & -1.3 & 1 & -1 \\ 0 & 1 & 0 & -2.3 & -4 & 0 & 1 \\ \hdashline 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \end{array}\right].$$

We compare the results achieved in [85] to those obtained by the full-block circle criterion (fbCC), its combination with full-block Zames-Falb (fbZF) as well as a diagonally repeated combination of Circle and Zames-Falb (drZF, [37, 176]) criteria. Note that the techniques in [166] and

[123] are not applicable because they require M to be strictly proper. Since the approach by Hu et al. [85] can take a bound $d_{\max}$ on the energy of d into account, we plot the computed $\mathscr{L}_2$-gain bounds γ over $d_{\max}$ in Figure 5.3 (with numerical values in the first column of Table 5.5). Figure 5.3 shows that [85] slightly outperforms all other

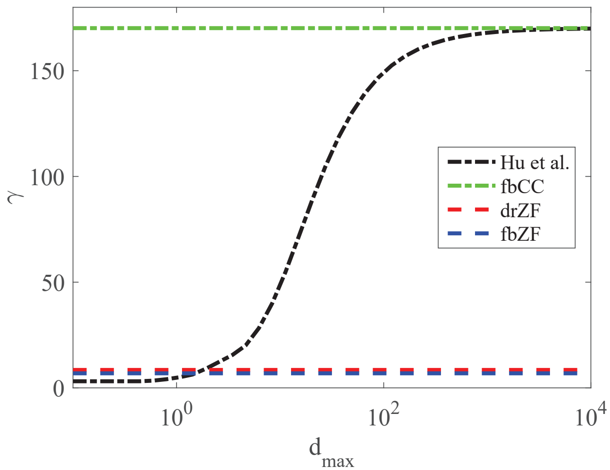

Figure 5.3: $\mathscr{L}_2$-gains for Example 7.15

techniques for small values of $d_{\max}$. However, the guaranteed $\mathscr{L}_2$-gains quickly increase substantially above those levels as globally guaranteed by the Zames-Falb based techniques and approach the ones for the full-block circle criterion if $d_{\max} \to \infty$ (see Section 5.5.2). $\star$

Example 5.35.
In [40] two variants of the above example are considered where $D_{11}^{\#1} := D_{11}$ is exchanged with

$$D_{11}^{\#2} = \begin{pmatrix} -3 & -1 \\ -2 & -4 \end{pmatrix} \quad \text{and} \quad D_{11}^{\#3} = \begin{pmatrix} -3 & -2 \\ -2 & -4 \end{pmatrix}$$

and multiple approaches for global $\mathscr{L}_2$-gain estimation using different Lyapunov function based techniques are compared. The paper ultimately proposes a non-convex search relying on piecewise quadratic Lyapunov functions that leads to the best results for all three examples, with the related values appearing in the first row of Table 5.5. Only recently, these results were further refined in [171] where a convex relaxation of the approach in [40] is proposed using sum of squares (SOS) methods (see second row of Table 5.5). All Lyapunov function based techniques

Table 5.5: Global $\mathscr{L}_2$-gain estimates with different techniques

case	#1	#2	#3
piecewise (pw.) quadratic [40]	17.19	15.13	25.86
pw. quadratic using SOS relaxation [171]	12.39	12.04	17.79
quadratic [85]	170.15	38.96	∞
drZF ($\nu = 4$, $\rho = -1$)	8.47	8.47	9.81
fbZF ($\nu = 2$, $\rho = -1$)	6.92	7.09	9.43
fbZF ($\nu = 4$, $\rho = -1$)	6.86	7.02	9.43

are, however, clearly outperformed by diagonally structured Zames-Falb multipliers even for small lengths ν, with further improvements for full-block versions. ⋆

Example 5.36.
The next example only features one nonlinearity and serves to disprove

the claims in [123] and [163] that Park's method [123] is less conservative than the one by Chen and Wen [37].[2] In addition, we also aim at demonstrating the advantage of being able to employ both causal and non-causal multipliers simultaneously. In order to do so, we adopt three examples from [28] defined by the transfer functions

$$\begin{aligned} M_1(s) &= \frac{s^2}{s^4 + 0.2s^3 + 6s^2 + 0.1s + 1} \\ M_2(s) &= -M_1(s) \\ M_3(s) &= \frac{s^2}{s^4 + 5.001s^3 + 7.005s^2 + 5.006s + 6}. \end{aligned}$$

As both [123] and [28] only consider stability and not performance, we apply the same stability criterion as in [28]. We compute the maximal value of μ for which (4.3) remains stable for all $\Delta_\varphi \in \text{slope}(0, \mu)$ where φ is odd. The values in the first three columns of Table 5.6 are copied from [28] and correspond to the approaches by Park and a combination of anticausal (causal) Zames-Falb and Popov multipliers, denoted as ACP (CP). The last column displays the maximal slope restrictions under which we can guarantee stability using a combination of Circle, Popov and two Zames-Falb multipliers (fbZFP) for lengths $(\nu_1, \nu_2) = (6; 1)$

Table 5.6: Maximal slope constraints obtained with different techniques

Transfer function	Park [123]	ACP [28]	CP [167]	fbZFP
M_1	0.79	1.45	0.78	1.75
M_2	0.71	0.72	1.08	1.21
M_3	26.01	91.09	13.78	3510

and pole locations $(\rho_1, \rho_2) = (-3; -10)$. Higher values of ν further

[2] In fact, as [37] proposes an asymptotically exact parameterization of SISO Zames-Falb multipliers (for nonlinearities that are not odd) and Park emploies circle, Popov and Yakubovich ones, a sensible comparison is impossible.

improve the results at the expense of higher computational effort. Park's approach is always outperformed by Zames-Falb methods, either for causal or anticausal multipliers. For the two systems M_1 and M_2, only slight improvements are gained from our approach, while for M_3 (designed to display the key features of [28]), the improvements are rather significant. $\star$

Example 5.37.
Finally, we compare our results to those achieved by [166] for repeated nonlinearities. To this end we revisit Example 5.31 and, in view of the limitation of [166], we restrict our attention to odd nonlinearities. Table 5.7 displays the maximal values of β achieved by the different approaches. In our implementation of the method in [166] we chose $\lambda_1 = \lambda_2 = \lambda$ and performed a line search over $\lambda \in [0, \infty)$ with 1000 points which leads to a maximal tolerable value of $\beta = 0.26$. As the approach by Turner et al. (cZFP) already includes a Popov multiplier, we first compare it to the same combination of multipliers (CCZFP) for $\nu = 8$ and $\rho = -1$ and later, as in Example 5.31, to one where the Zames-Falb multiplier is exchanged with a Yakubovich one. We conclude that the approach by Turner et al. (cZFP) is significantly more conservative, and the combination of all multipliers (in the last two columns of Table 5.7) even allows us to guarantee stability up to the Nyquist value of $\beta_N = 0.767$.

Table 5.7: Maximal values of β for different approaches

cZFP [166]	CCZFP($\nu = 8$)	CCYP	CCZFYP(2)	CCZFYP(8)
0.26	0.73	0.73	0.75	0.766

$\star$

5.7 Summary and recommendations

This chapter presents a novel and flexibly applicable framework, which is used to obtain the least conservative results available both inside and outside the IQC framework for the global analysis of feedback systems featuring repeated slope-restricted scalar nonlinearities. The extensions relative to previous works are manifold.

For the circle criterion, we give new relations between different computationally tractable subsets of the set of full-block multipliers, and prove that these genuinely extend the classical diagonally structured versions. In order to make the full potential of the Zames-Falb stability test accessible for computations, we propose a new tractable family of full-block Zames-Falb multipliers that is asymptotically exact, thus solving an open problem postulated in [29]. In addition, we rigorously include Yakubovich multipliers into IQC theory which makes it possible to propose a new full-block version thereof. We fully exploit the generality of our analysis framework which permits a tractable modular combination of the circle, Popov, Yakubovich and Zames-Falb tests even for non-proper systems. Note in particular that the assumptions for the Yakubovich criterion vary significantly in the literature [46, 188, 10], while the ones presented here are the least restrictive.

Yet many problems still need to be addressed. Of primary concern is the application of the developed tests for robust estimator or controller design [175]. A particularly interesting application would be anti-windup synthesis, which has received much attention in the literature, see e.g., [87, 66, 158, 100]. Yet, this topic is closely linked to that of the analysis of systems that are only locally stable, which is one of the major assets of Lyapunov theory. We will present an approach that allows for the effective merging of both IQC and Lyapunov theory in Chapter 7 and focus, in particular, on the class of Zames-Falb multipliers in Chapter 8.

Furthermore, as mentioned several times in this chapter, the application of full-block Zames-Falb multipliers can be very expensive in terms of computational effort. One remedy could be the choice of more effective basis functions that, for optimal performance, should be adapted to the problem at hand. This would pave the way for much more accurate stability test even for large scale (real world) applications.

Chapter 6

Absolute stability analysis of discrete-time feedback interconnections

6.1 Introduction

LET us revisit the feedback interconnection of an LTI system M and a nonlinear operator Δ defined via a slope-restricted or sector-bounded nonlinearity. Complementary to the previous chapter, we now focus on the discrete-time case. Our main goal is to highlight the fundamental principles behind the stability results and, in this way, establish the connection to Chapter 5.

In contrast to the continuous-time case, we do not exploit a distinction between the inputs u and v affection the interconnection (4.3). Thus it suffices to consider the simplified interconnection depicted in Figure 6.1, where the external disturbance d is square summable. In order to provide additional insights if compare to the continuous-time derivation, we also discuss non-repeated nonlinearities in the present chapter, that, as will be revealed, also admit full-block multiplier descriptions.

Stability analysis of such interconnections also has a long standing history, probably starting with the works of Tsypkin [162] and Jury and Lee [96]. Both approaches employed ideas developed by Popov for continuous-time systems [126]. However, in contrast to the Popov criterion that only requires the existence of some sector bound on the nonlinearity, it became apparent that the discrete-time counterpart also necessitated the assumption of monotonicity.

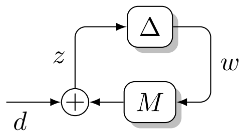

Figure 6.1: Feedback interconnection

Under the additional hypothesis that the derivative of the nonlinearity $\varphi : \mathbb{R} \to \mathbb{R}$ is bounded, O'Shea and Younis [121] proposed a discrete-time counterpart to the celebrated Zames-Falb stability criterion [197] that was later generalized by Willems and Brockett [184]. O'Shea and Younis already claim that their criterion is less restrictive than the one proposed in [96], which was the most effective test at that time.

Following these early results, many researchers have contributed to this field of study and, in particular, extended the above described stability tests to multi-variable nonlinearities (see, e.g, [122, 75, 103]). Recently, there seems to be renewed interest in the subject with several publications proposing seemingly different yet closely related criteria (see, e.g., [5, 7, 2, 3, 4, 177, 3, 17]).

The wealth of earlier and more recent results makes it rather difficult to judge which of the proposed stability tests is the most effective one, in the sense that it leads to the least conservative estimates for stability margins. In part, this is due to the fact that most proofs are based on Lurye-type Lyapunov function arguments by adding extra degrees of freedom whose generating principles often remain somewhat obscure. Moreover, the resulting tests are formulated in terms of linear matrix inequalities whose intricate structures prevent insightful comparisons.

In case of just one scalar nonlinearity, the result obtained by [177], which is based on Zames-Falb multipliers, is shown to subsume all earlier stability tests and thus leads to the least conservative estimates for stability margins (see also [184]). However, as will be revealed in the present chapter, for multiple nonlinearities it is beneficial to combine Zames-Falb multipliers with those corresponding to other stability criteria.

As one of the contributions of this chapter, we highlight the fundamental principles underlying all above mentioned stability tests and provide insights into their interrelation. On the one hand, this allows us to show how even the most recent versions can actually be derived from the ones proposed in [121], [184] and [103]. On the other hand, we may seamlessly extend the classical results in order to arrive at less conservative stability tests for multiple nonlinearities. Moreover, it is then easy to reveal that (at least for scalar nonlinearities) both discrete-time counterparts of the Popov and the Yakubovich [188, 46] stability criteria are already included in the one based on Zames-Falb multipliers. This should be contrasted with the situation in the previous chapter, where both tests may only be approximately handled by using Zames-Falb multipliers [140, 24, 25].

Apart from the work of [103], all the above discussed stability tests employ diagonally structured multipliers even if considering repeated multi-variable nonlinearities. As another contribution of this chapter we demonstrate how unstructured full-block multipliers may be combined with diagonal (full-block) Zames-Falb multipliers for non-repeated (repeated) nonlinearities in order to generate more powerful novel tests, as shown by numerical examples.

The chapter is structured as follows. After setting the stage in Section 6.2 we derive in Section 6.3 our full-block stability multipliers and give their relation to previous ones in Section 6.4. Subsequently, we discuss the implementation of these multipliers in Section 6.5 and close with numerical examples in Section 6.6. We emphasize that a condensed version of this chapter is accepted for presentation at the IFAC World Congress 2017 [57].

6.2 Preliminaries

Assume that we are given $k \in \mathbb{N}$ nonlinearities $\varphi_1, \ldots, \varphi_k$ that are sector-bounded or slope-restricted according to Definition 5.1. With such nonlinearities, let $\Phi : \mathbb{R}^k \to \mathbb{R}^k$ be given as $\Phi(x_1, \ldots, x_k) = \big(\varphi_1(x_1) \ \ldots \ \varphi_k(x_k)\big)^T$ and let the operator Δ_Φ be defined as

$$\Delta_\Phi(z)(t) := \Phi(z(t)) \quad \text{for all} \quad t \in \mathbb{N}_0, \quad z \in \ell_{2e}^k. \tag{6.1}$$

In contrast to (5.3) we now allow for different nonlinearities with varying sector and slope constraints. In order to take this into account, we write $\Delta_\Phi \in \operatorname{slope}(\mu, \nu)$ or $\Delta_\Phi \in \sec[\alpha, \beta]$ (as well as $\Phi \in \operatorname{slope}(\mu, \nu)$ or $\Phi \in \sec[\alpha, \beta]$) if $\varphi_j \in \operatorname{slope}(\mu_j, \nu_j)$ or $\varphi_j \in \sec[\alpha_j, \beta_j]$ for all $j \in \{1, \ldots, k\}$ and with $\mu = \operatorname{diag}(\mu_j)$, $\nu = \operatorname{diag}(\nu_j)$, $\alpha = \operatorname{diag}(\alpha_j)$, $\beta = \operatorname{diag}(\beta_j)$, respectively. In a natural extension of this notation, we write $[\alpha, \beta]$ for the following set of diagonal matrices:

$$[\alpha, \beta] := \big\{\Delta = \operatorname{diag}(\delta_1, \ldots, \delta_k) \in \mathbb{D}^k \big| \alpha_i \leq \delta_i \leq \beta_i \\ \text{for all} \quad i \in \{1, \ldots, n\}\big\}.$$

For the special case when all nonlinearities coincide, i.e., $\varphi_j = \varphi$ for all j, we say that Φ is a repeated nonlinearity and indicate this by $\Delta_\Phi \in \operatorname{slope}(\mu I, \nu I)$ or $\Delta_\Phi \in \sec[\alpha I, \beta I]$ for the operator.

Given such a nonlinearity Δ_Φ, we consider its feedback interconnection (see Figure 6.1) with a stable LTI system M described through a state-space realization as follows:

$$\begin{aligned} x(t+1) &= Ax(t) + Bw(t), \qquad x(0) = 0, \qquad w = \Delta_\Phi(z), \\ z(t) &= Cx(t) + Dw(t) + d(t), \end{aligned} \tag{6.2}$$

for $t \in \mathbb{N}_0$. Here we assume that $A \in \mathbb{R}^{n \times n}$ is Schur stable, i.e., $\operatorname{eig}(A) \subset \mathbb{D}$, and that the external disturbance d is square summable, i.e., $d \in \ell_2^k$. We denote the standard norm on ℓ_2^k by $\|\cdot\|$.

In complete analogy to the previous chapter, we define well-posedness and stability as follows.

Definition 6.1.
The interconnection (6.2) is said to be **well-posed** if for each $d \in \ell_2^k$ and each $\tau \in [0,1]$ there exists a unique response $z \in \ell_{2e}^k$ of (6.2) with Δ_Φ replaced by $\tau\Delta_\Phi$ which depends causally on d. Moreover, (6.2) is **stable** if there exists some $\gamma > 0$ such that

$$\|z\| \leq \gamma \|d\| \quad \text{for all} \quad d \in \ell_2^k. \tag{6.3}$$

$\star$

In discrete time, the quadratic form (5.8) translates into

$$\Sigma_\Pi \begin{pmatrix} z \\ w \end{pmatrix} = \int_0^{2\pi} \begin{pmatrix} \hat{z}(e^{i\omega}) \\ \hat{w}(e^{i\omega}) \end{pmatrix}^* \Pi(e^{i\omega}) \begin{pmatrix} \hat{z}(e^{i\omega}) \\ \hat{w}(e^{i\omega}) \end{pmatrix} d\omega,$$

where Π is measurable, bounded and Hermitian valued on $\mathbb{T}$ and $\hat{z}$ as well as $\hat{w}$ denote the z-transforms of the ℓ_2^k signals z and w, respectively. As before, a causal operator $\Delta : \ell_2^k \to \ell_2^k$ satisfies the IQC imposed by Π in case that

$$\Sigma_\Pi \begin{pmatrix} z \\ \Delta(z) \end{pmatrix} \geq 0 \quad \text{for all} \quad z \in \ell_2^k. \tag{6.4}$$

We end this section by stating a particular version of Theorem 3.4 adapted to our special configuration (see also [97]).

Theorem 6.2
Assume that the interconnection (6.2) *with* Δ_Φ *as in* (6.1) *is well-posed. Then* (6.2) *is stable if*

a) $\tau\Delta_\Phi$ *satisfies the IQC defined by* Π *for all* $\tau \in [0,1]$*;*

b) the following FDI holds:

$$\begin{pmatrix} M(e^{i\omega}) \\ I \end{pmatrix}^* \Pi(\mathrm{e}^{i\omega}) \begin{pmatrix} M(e^{i\omega}) \\ I \end{pmatrix} \prec 0 \quad \textit{for all} \quad \omega \in [0, 2\pi]. \tag{6.5}$$

The verification of well-posedness for $\Delta_\Phi \in \sec[\alpha,\beta]$ or $\Delta_\Phi \in \sec(\mu,\nu)$ in Section 5.2.3 literally carries over to discrete-time interconnections; in the sequel we tacitly assume that (6.2) is well-posed.

6.3 Principles of stability multipliers

Complementarily to Chapter 5, we divide this section according to the generating principles of the derived stability criteria, which allows for a straightforward categorization of all multiplier based results in the literature. As will become apparent, the stability multipliers employed in the papers cited in the introduction either rely on a subgradient argument or on polytopic bounding for the creation of inequalities. Another focus of this section is the formulation of stability test using full-block multipliers even if the nonlinearities are not repeated. Aiming at a self-contained exposition of stability criteria we briefly summarize the results from the previous chapter if necessary.

6.3.1 Methods based on polytopic bounding

Let $\Delta_\Phi \in \sec[\alpha,\beta]$. Conceptually, the circle criterion exploits the simple fact that $w(t) = \Delta_\Phi(z)(t) = \Phi(z(t))$ for $z \in \ell_2^k$ can be expressed, due to (5.2), as

$$w(t) = \Delta(t)z(t) \quad \text{for all} \quad t \in \mathbb{N}_0 \tag{6.6}$$

with $\Delta(t) \in [\alpha,\beta]$; indeed we can take $\Delta(t) = \mathrm{diag}(\delta_j(t))$ and $\delta_j(t) = \varphi_j(z_j(t))/z_j(t)$ if $z_j(t) \neq 0$ or $\delta_j(t) = 0$ if $z_j(t) = 0$ for all $j \in \{1,\ldots,k\}$. If we now choose any element Π in the following class of full-block multipliers

$$\mathbf{\Pi}[\alpha,\beta] = \left\{ \Pi \in \mathbb{S}^{2k} \,\middle|\, \begin{pmatrix} I \\ \Delta \end{pmatrix}^T \Pi \begin{pmatrix} I \\ \Delta \end{pmatrix} \succ 0 \quad \text{for all} \quad \Delta \in [\alpha,\beta] \right\} \tag{6.7}$$

(which seamlessly extends the definition in (5.12)), we obviously infer

$$\begin{pmatrix} z(t) \\ w(t) \end{pmatrix}^T \Pi \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} = z(t)^T \begin{pmatrix} I \\ \Delta(t) \end{pmatrix}^T \Pi \begin{pmatrix} I \\ \Delta(t) \end{pmatrix} z(t) \geq 0$$

for all $t \in \mathbb{N}_0$. By summation we conclude that Δ_Φ satisfies the IQC

$$\Sigma_\Pi \begin{pmatrix} z \\ \Delta_\Phi(z) \end{pmatrix} \geq 0 \quad \text{for all} \quad z \in \ell_2^k. \tag{6.8}$$

Since the class $\mathbf{\Pi}[\alpha, \beta]$ was originally defined to handle time-varying parametric uncertainties in polytopes [89, 147], these multipliers are said to be generated by polytopic bounding.

It is now straightforward to derive a full-block circle and a full-block Yakubovich stability criterion in our setting.

Circle criterion

Let $\Delta_\Phi \in \sec[\alpha, \beta]$. Since (6.8) holds for all $\Pi \in \mathbf{\Pi}[\alpha, \beta]$, Theorem 6.2 implies the following result.

Corollary 6.3 (Circle criterion)
The interconnection (6.2) *with* $\Delta_\Phi \in \sec[\alpha, \beta]$ *is stable if there exists some* $\Pi \in \mathbf{\Pi}[\alpha, \beta]$ *with*[1]

$$\begin{pmatrix} M(\mathrm{z}) \\ I \end{pmatrix}^* \Pi \begin{pmatrix} M(\mathrm{z}) \\ I \end{pmatrix} \prec 0 \quad \textit{for all} \quad \mathrm{z} \in \mathbb{T}. \tag{6.9}$$

Note that Corollary 6.3 is actually a generalization of Lemma 5.6 in the discrete-time setting, as it allows for non-repeated nonlinearities while still employing full-block multipliers. As emphasized above, all discrete-time circle criteria for stability in the literature restrict the search of Π to the subclass $\mathbf{\Pi}_{dr}[\alpha, \beta] \subset \mathbf{\Pi}[\alpha, \beta]$ of diagonally structured multipliers. Therefore, full-block multipliers will not be worse than

[1] As above, we use the symbol z to distinguish the frequency domain variable z form the signal z.

the conventional ones (see Lemma 5.10), and it can be concluded from numerical examples that they typically reduce conservatism significantly (see Example 5.14).

Yakubovich criterion

Let us now turn to the discrete-time analogue of the Yakubovich criterion for $\Delta_\Phi \in \mathrm{slope}(\mu, \nu)$. In order to demonstrate the underlying ideas, we first assume that the nonlinearities φ_j are continuously differentiable for $j \in \{1, \ldots, k\}$. The general case is merely more technical but proceeds in similar fashion.

Choose $z \in \ell_2^k$ and let $w := \Delta_\Phi(z)$. By the mean value theorem there exist ξ_j^t (depending on $t \in \mathbb{N}_0$) such that

$$\begin{aligned} w_j(t+1) - w_j(t) &= \varphi_j(z_j(t+1)) - \varphi_j(z_j(t)) \\ &= \varphi_j'(\xi_j^t)(z_j(t+1) - z_j(t)). \end{aligned}$$

Since the slope restriction $\Phi \in \mathrm{slope}(\mu, \nu)$ translates into

$$\mu_j \le \varphi_j'(\xi) \le \nu_j \quad \text{for all} \quad j \in \{1, \ldots, k\} \quad \text{and all} \quad \xi \in \mathbb{R},$$

we infer, in complete analogy to the circle criterion, that there exist $\Delta(t) \in [\mu, \nu]$ with

$$w(t+1) - w(t) = \Delta(t)(z(t+1) - z(t)) \tag{6.10}$$

for all $t \in \mathbb{N}_0$. Thus, for $\Pi_Y \in \mathbf{\Pi}[\mu, \nu]$ we obtain

$$\begin{pmatrix} z(t+1) - z(t) \\ w(t+1) - w(t) \end{pmatrix}^T \Pi_Y \begin{pmatrix} z(t+1) - z(t) \\ w(t+1) - w(t) \end{pmatrix} \ge 0 \tag{6.11}$$

for all $t \in \mathbb{N}_0$. As the time shift in the outer factors of (6.11) gives rise to a multiplication with $\mathrm{z}-1$ in the frequency domain, we obtain, by summation and with Parseval's theorem, the IQC $\Sigma_\Pi(z, w) \ge 0$ for the dynamic (z-dependent) multiplier

$$\Pi(\mathrm{z}) := \begin{pmatrix} (\mathrm{z}-1)I & 0 \\ 0 & (\mathrm{z}-1)I \end{pmatrix}^* \Pi_Y \begin{pmatrix} (\mathrm{z}-1)I & 0 \\ 0 & (\mathrm{z}-1)I \end{pmatrix} = |\,\mathrm{z}-1|^2 \Pi_Y. \tag{6.12}$$

We deal with the general case, where the functions φ_j are only differentiable almost everywhere, in the following Lemma.

Lemma 6.4
With the definitions above, let $\Phi \in \mathrm{slope}(\mu, \nu)$. *Then there exist* $\Delta(t) \in [\mu, \nu]$ *with* (6.10) *for all* $t \in \mathbb{N}_0$.

Proof. The proof employs Clarke's generalized derivatives and the mean value theorem of Lebourg (see [39]) and is found in Appendix C.5.1. □

In summary, with Theorem 6.2, we arrive at the following result.

Corollary 6.5 (Yakubovich criterion)
The interconnection (6.2) *with* $\Delta_\Phi \in \mathrm{slope}(\mu, \nu)$ *is stable if there exists some* $\Pi_Y \in \mathbf{\Pi}[\mu, \nu]$ *with*

$$\begin{pmatrix} (\mathrm{z}-1)M(\mathrm{z}) \\ (\mathrm{z}-1)I \end{pmatrix}^* \Pi_Y \begin{pmatrix} (\mathrm{z}-1)M(\mathrm{z}) \\ (\mathrm{z}-1)I \end{pmatrix} \prec 0 \quad \textit{for all} \quad \mathrm{z} \in \mathbb{T}.$$

Combined polytopic criterion

Let us now discuss how we may combine the circle and Yakubovich multipliers and embed the combination into a more general class of completely unstructured multipliers. The following discussion immediately translates to the continuous-time case, also resulting in a larger class of suitable multipliers.

Assume that $\Delta_\Phi \in \sec[\alpha, \beta] \cap \mathrm{slope}(\mu, \nu)$. Of course, the most simple way of exploiting both constraints simultaneously is to just add up the according multipliers. The corresponding FDI then reads as

$$\begin{pmatrix} M(\mathrm{z}) \\ I \\ (\mathrm{z}-1)M(\mathrm{z}) \\ (\mathrm{z}-1)I \end{pmatrix}^* \begin{pmatrix} \Pi_C & 0 \\ 0 & \Pi_Y \end{pmatrix} \begin{pmatrix} M(\mathrm{z}) \\ I \\ (\mathrm{z}-1)M(\mathrm{z}) \\ (\mathrm{z}-1)I \end{pmatrix} \prec 0 \quad \text{for all} \quad \mathrm{z} \in \mathbb{T} \tag{6.13}$$

with some $\Pi_C \in \Pi[\alpha, \beta]$ and $\Pi_Y \in \Pi[\mu, \nu]$. Yet, this obviously results in a potentially conservative block diagonal structure. Based on the above described generating principle, the generalization to unstructured multipliers is simple. Indeed, for $w = \Delta_\Phi(z)$ and $z \in \ell_2^k$ we have

$$\begin{pmatrix} w(t) \\ w(t+1) - w(t) \end{pmatrix} = \begin{pmatrix} \Delta_C(t) & 0 \\ 0 & \Delta_Y(t) \end{pmatrix} \begin{pmatrix} z(t) \\ z(t+1) - z(t) \end{pmatrix}$$

with suitable $\Delta_C(t) \in [\alpha, \beta]$, $\Delta_Y(t) \in [\mu, \nu]$ and for all $t \in \mathbb{N}_0$. Since

$$\mathrm{diag}(\Delta_C(t), \Delta_Y(t)) \in [\mathrm{diag}(\alpha, \mu), \mathrm{diag}(\beta, \nu)],$$

we infer for any $\Pi_{CY} \in \mathbf{\Pi}[\mathrm{diag}(\alpha, \mu), \mathrm{diag}(\beta, \nu)]$ that

$$\begin{pmatrix} z(t) \\ z(t+1) - z(t) \\ \hline w(t) \\ w(t+1) - w(t) \end{pmatrix}^T \Pi_{CY} \begin{pmatrix} z(t) \\ z(t+1) - z(t) \\ \hline w(t) \\ w(t+1) - w(t) \end{pmatrix} \geq 0 \quad \text{for all} \quad t \in \mathbb{N}_0.$$

In exactly the same fashion as described above this leads to a stability test that is formulated with the FDI

$$\begin{pmatrix} M(\mathrm{z}) \\ (\mathrm{z}-1)M(\mathrm{z}) \\ \hline I \\ (\mathrm{z}-1)I \end{pmatrix}^* \Pi_{CY} \begin{pmatrix} M(\mathrm{z}) \\ (\mathrm{z}-1)M(\mathrm{z}) \\ \hline I \\ (\mathrm{z}-1)I \end{pmatrix} \prec 0 \quad \text{for all} \quad \mathrm{z} \in \mathbb{T}. \tag{6.14}$$

We arrive at the following general full-block stability test.

Corollary 6.6
The interconnection (6.2) *with* $\Delta_\Phi \in \mathrm{sec}[\alpha, \beta] \cap \mathrm{slope}(\mu, \nu)$ *is stable if there exists some* $\Pi_{CY} \in \mathbf{\Pi}[\mathrm{diag}(\alpha, \mu), \mathrm{diag}(\beta, \nu)]$ *with* (6.14).

6.3.2 Subgradient based arguments

We start this subsection by giving a direct convexity proof for full-block finite impulse response (FIR) Zames-Falb IQCs as originally proposed by Willems and Brockett [184]. The derivation will serve as a foundation for the subsequent comparison of multipliers in the literature.

Full-block FIR Zames-Falb multipliers

Let us recall some definitions introduced in [184].

Definition 6.7.
Let $L = (L_{ij}) \in \mathbb{R}^{k\times k}$. Then L is a **Z-matrix** if $L_{ij} \leq 0$ for $i \neq j$. Moreover, L is **doubly hyperdominant** if it is a Z-matrix and if, in addition,

$$Le \geq 0 \quad \text{and} \quad e^T L \geq 0.$$

It is said to be **doubly dominant** if, for all $i \in \{1, \dots, n\}$,

$$L_{ii} \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |L_{ij}| \quad \text{and} \quad L_{ii} \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |L_{ji}|\,.$$

⋆

Remark 6.8.
Any doubly dominant matrix L can be decomposed as $L = L_d + L_{od}$ where L_d and L_{od} contain the diagonal and off-diagonal elements, respectively; then $L_d - |L_{od}|$ is doubly hyperdominant if we take the absolute value element-wise. ⋆

The following lemma provides the foundation for discrete-time Zames-Falb multipliers. We formulate it for repeated nonlinearities that comprise scalar ones as a special case.

Lemma 6.9
Let $\Phi \in \mathrm{slope}(0I, \infty I)$. If $L \in \mathbb{R}^{k\times k}$ is doubly hyperdominant then

$$\Phi(x)^T Lx \geq 0 \quad \textit{for all} \quad x \in \mathbb{R}^k.$$

In case that, in addition, φ is odd, this holds for any doubly dominant matrix L.

This is a matrix version of a result in [184]; our direct proof highlights the role of the underlying principles, namely convexity and permutation invariance.

Proof. Suppose that φ is not necessarily odd and choose the convex primitive I_φ satisfying $I_\varphi(0) = 0$. Define the convex function

$$\Psi(x) := I_\varphi(x_1) + \cdots + I_\varphi(x_k).$$

Since $\nabla\Psi(x) = \mathrm{col}(\varphi(x_1), \ldots, \varphi(x_k)) = \Phi(x)$ we infer by convexity that

$$\Phi(x)^T(x - y) \geq \Psi(x) - \Psi(y) \quad \text{for all} \quad x, y \in \mathbb{R}^k. \tag{6.15}$$

We now exploit that Φ is repeated by observing $\Psi(Px) = \Psi(x)$ for any permutation matrix P. Thus (6.15) implies

$$\Phi(x)^T(x - Px) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}^k.$$

By the Birkhoff-von Neumann theorem [81, Theorem 8.7.1] we infer for all doubly stochastic matrices S that

$$\Phi(x)^T(I - S)x \geq 0 \quad \text{for all} \quad x \in \mathbb{R}^k.$$

For the given Z-matrix L with $Le \geq 0$ and $e^TL \geq 0$ it is now clearly possible to choose $r > 0$ small enough such that $I - rL \geq 0$ and $1 - re^TLe \geq 0$. Thus

$$S := \begin{pmatrix} I - rL & rLe \\ re^TL & 1 - re^TLe \end{pmatrix} \geq 0$$

and S is obviously doubly stochastic. As just seen, we can conclude that

$$\Phi(x)^TLx = \frac{1}{r}\begin{pmatrix} \Phi(x) \\ 0 \end{pmatrix}^T (I - S)\begin{pmatrix} x \\ 0 \end{pmatrix} \geq 0 \quad \text{for all} \quad x \in \mathbb{R}^k.$$

If φ is also odd, the result follows from $|\Phi(x)| = \Phi(|x|)$ for all $x \in \mathbb{R}$ and with $L = L_d + L_{od}$ in Remark 6.8. Indeed, since $L_d - |L_{od}|$ is doubly hyperdominant, we get

$$\Phi(x)^TLx = \Phi(x)^TL_dx + \Phi(x)^TL_{od}x$$

$$
\begin{aligned}
&\geq |\Phi(x)|^T L_d |x| - |\Phi(x)|^T |L_{od}||x| \\
&= \Phi(|x|)^T (L_d - |L_{od}|)|x| \geq 0 \quad \text{for all} \quad x \in \mathbb{R}^k.
\end{aligned}
$$

□

It is now standard to extend Lemma 6.9 from monotone to slope-restricted nonlinearities (see, e.g., [197], [42]). For later reference, we state the result in terms of a quadratic form as follows.

Corollary 6.10
Let $\Phi \in \operatorname{slope}(\mu I, \nu I)$ with $\mu \leq 0 \leq \nu$ and assume that L is doubly hyperdominant or that φ is odd and L is doubly dominant. Then

$$
\begin{pmatrix} x \\ \Phi(x) \end{pmatrix}^T \begin{pmatrix} \nu I & -I \\ -\mu I & I \end{pmatrix}^T \begin{pmatrix} 0 & L^T \\ L & 0 \end{pmatrix} \begin{pmatrix} \nu I & -I \\ -\mu I & I \end{pmatrix} \begin{pmatrix} x \\ \Phi(x) \end{pmatrix} \geq 0
$$

for all $x \in \mathbb{R}^k$.

The extension to infinite block matrices defining operators on ℓ_2^k follows as in [184]. Let $L = (L_{ij})_{i,j\in\mathbb{Z}}$ be an infinite block matrix with $L_{ij} \in \mathbb{R}^{k\times k}$ such that there exists some $b \geq 0$ with

$$
\sum_{i\in\mathbb{Z}} \|L_{ji}\| \leq b \quad \text{and} \quad \sum_{i\in\mathbb{Z}} \|L_{ij}\| \leq b \quad \text{for all} \quad j \in \mathbb{Z}. \tag{6.16}
$$

It is then well-known that

$$
L : \ell_2^k \to \ell_2^k, \quad (Lx)_i := \sum_{j\in\mathbb{Z}} L_{ij} x_j
$$

defines a bounded linear operator. Now suppose that L is a Z-matrix. Due to (6.16) and if $e_\infty \in \ell_{2e}^k$ is the sequence of all-ones (column) vectors then Le_∞ and $e_\infty^T L$ are well-defined sequences in ℓ_{2e}^k. Let us assume that, in addition, $Le_\infty \geq 0$ and $e_\infty^T L \geq 0$ element-wise. Then we obtain the following result as a consequence of Corollary 6.10.

Corollary 6.11 (Zames-Falb IQC)
With $\mu \le 0 \le \nu$ *let* $\Phi \in \mathrm{slope}(\mu I, \nu I)$ *and assume that* L *with* (6.16) *is either an (infinite) doubly hyperdominant matrix or that* φ *is odd and* L *is doubly dominant. Then*

$$(\star)^T \begin{pmatrix} 0 & L^T \\ L & 0 \end{pmatrix} \begin{pmatrix} \nu\mathbb{1} & -\mathbb{1} \\ -\mu\mathbb{1} & \mathbb{1} \end{pmatrix} \begin{pmatrix} z \\ \Delta_\Phi(z) \end{pmatrix} \ge 0 \quad \textit{for all} \quad z \in \ell_2^k. \tag{6.17}$$

For the subsequent discussion it suffices to restrict the attention to block Toeplitz matrices with the structure

$$L = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ \ldots 0 & L_{l_+} & \ldots & L_0 & \ldots & L_{-l_-} & 0 \ldots & & & & \\ & \ldots 0 & L_{l_+} & \ldots & L_0 & \ldots & L_{-l_-} & 0 \ldots & & & \\ & & \ldots 0 & L_{l_+} & \ldots & L_0 & \ldots & L_{-l_-} & 0 \ldots & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \end{pmatrix} \tag{6.18}$$

for some chosen $l_\pm \in \mathbb{N}_0$; it is then required that L_0 is a Z-matrix, $L_{-j} \le 0$ for $j \in \{1, \ldots, l_-\}$, and $L_j \le 0$ for $j \in \{1, \ldots, l_+\}$ as well as

$$e^T \Big(\sum_{j=-l_-}^{l_+} L_j \Big) \ge 0 \quad \text{and} \quad \Big(\sum_{j=-l_-}^{l_+} L_j \Big) e \ge 0.$$

For $y \in \ell_2^k$ we then infer

$$\widehat{Ly}(\mathrm{z}) = \Big(\sum_{j=-l_-}^{l_+} L_j \, \mathrm{z}^j \Big) \hat{y}(\mathrm{z}) =: H_L(\mathrm{z}) \hat{y}(\mathrm{z})$$

and, due to the structure of L,

$$\widehat{L^T y}(\mathrm{z}) = \Big(\sum_{j=-l_-}^{l_+} L_j^T \frac{1}{\mathrm{z}^j} \Big) \hat{y}(\mathrm{z}) = H_L(1/\mathrm{z})^T \hat{y}(\mathrm{z}).$$

Based on (6.18) let us now define the class of FIR Zames-Falb multipliers as the set

$$\mathbf{\Pi}_L(\mu I, \nu I) = \left\{ \Pi \,\middle|\, \Pi = (\star)^T \begin{pmatrix} 0 & H_L^* \\ H_L & 0 \end{pmatrix} \begin{pmatrix} \nu I & -I \\ -\mu I & I \end{pmatrix} \right\} \tag{6.19}$$

where $H_L(\mathrm{z})^* = H_L(1/\mathrm{z})^T$ and

$$H_L(\mathrm{z}) = \sum_{j=-l_-}^{l_+} L_j\, \mathrm{z}^j\,. \tag{6.20}$$

If $\Delta_\Phi \in \mathrm{slope}(\mu I, \nu I)$ then (6.17) implies (via Parseval's theorem) for all $\Pi \in \mathbf{\Pi}_L(\mu I, \nu I)$ that

$$\Sigma_\Pi \begin{pmatrix} z \\ \Delta_\Phi(z) \end{pmatrix} \geq 0 \quad \text{for all} \quad z \in \ell_2^k. \tag{6.21}$$

In the light of the parametrization of continuous-time Zames-Falb multipliers in (5.33), the choices (6.19) and (6.20) correspond to basis functions with pole location zero. In discrete time, it remains an interesting research topic if nonzero pole locations also admit a parameterization as in (6.19), (6.20) and whether different poles would be computationally beneficial.

The multiplier classes corresponding to (6.19) for $\nu = \infty$ and $\mu = -\infty$ are derived analogously and take the form

$$\mathbf{\Pi}_L(\mu I, \infty I) = \left\{ \Pi \,\middle|\, \Pi = \begin{pmatrix} -\mu(H_L^* + H_L) & H_L^* \\ H_L & 0 \end{pmatrix} \right\} \tag{6.22}$$

and

$$\mathbf{\Pi}_L(\mu I, \infty I) = \left\{ \Pi \,\middle|\, \Pi = \begin{pmatrix} \nu(H_L^* + H_L) & -H_L \\ -H_L^* & 0 \end{pmatrix} \right\}, \tag{6.23}$$

respectively. This leads us to the following stability result, as a consequence of (6.21) and Theorem 6.2.

Corollary 6.12 (FIR Zames-Falb criterion)
Let $\mu \leq 0 \leq \nu$, $\Delta_\Phi \in \mathrm{slope}(\mu I, \nu I)$ and assume that L in (6.18) is either a doubly hyperdominant matrix or that φ is odd and L is doubly dominant. Then the interconnection (6.2) is stable if there exists $\Pi \in \mathbf{\Pi}_L(\mu I, \nu I)$ such that

$$\begin{pmatrix} M(\mathrm{z}) \\ I \end{pmatrix}^* \Pi(\mathrm{z}) \begin{pmatrix} M(\mathrm{z}) \\ I \end{pmatrix} \prec 0 \quad \textit{for all} \quad \mathrm{z} \in \mathbb{T}. \tag{6.24}$$

Finally, multipliers for some non-repeated uncertainty $\Delta_\Phi \in \text{slope}(\mu,\nu)$ may be obtained by choosing scalar functions $H_{L,j}$ for each φ_j and combining them diagonally, which just amounts to the restriction that all L_j are diagonal; we denote the respective multiplier class by $\mathbf{\Pi}_L(\mu,\nu)$.

6.4 Relation to multipliers in the literature

First note that the complete class of full-block Zames-Falb multipliers was already described in [103]. The multipliers in (6.19), (6.20) are the full-block versions of the FIR Zames-Falb multipliers as suggested for the scalar case in [177] and can be easily implemented numerically; this renders the results in [103] computational.

6.4.1 Zames-Falb multipliers of order one

In this section we prove that the criteria proposed in [162] and in [96] as well as all later derivatives thereof (see, e.g., [75, 5, 7, 2, 6, 4]) are special cases of (6.19), (6.20) for $l_- = l_+ = 1$. This reveals that all variants of the discrete-time counterpart to the Popov criterion are rendered obsolete by a Zames-Falb stability test using (6.19), (6.20) with $l_\pm \geq 1$.

Let us hence assume $\Phi \in \text{slope}(\mu,\nu)$, choose $l_- = l_+ = 1$, and select L with the zeroth block row

$$\begin{pmatrix} \ldots \; 0 \; L_1 \; L_0 \; L_{-1} \; 0 \; \ldots \end{pmatrix}$$

where $L_0 \geq 0$, $L_{-1} \leq 0$, $L_1 \leq 0$ are diagonal and satisfy

$$(L_{-1} + L_0 + L_1)e \geq 0, \qquad e^T(L_{-1} + L_0 + L_1) \geq 0.$$

These requirements are obviously fulfilled for the more special (and potentially restrictive; see Example 6.15) choices

$$L_{-1} = -\widetilde{\Lambda}, \quad L_0 = \Lambda + \widetilde{\Lambda}, \quad L_1 = -\Lambda.$$

with diagonal $\Lambda \geq 0$ and $\widetilde{\Lambda} \geq 0$. We denote the resulting multiplier classes corresponding to (6.19), (6.22) and (6.23) by

$$\mathbf{\Pi}_{(\Lambda,\widetilde{\Lambda})}(\mu,\nu), \quad \mathbf{\Pi}_{(\Lambda,\widetilde{\Lambda})}(\mu,\infty) \quad \text{and} \quad \mathbf{\Pi}_{(\Lambda,\widetilde{\Lambda})}(-\infty,\nu), \tag{6.25}$$

respectively.

Note that all multiplier classes in the present an previous chapter are convex cones and their combination is just obtained by summing them up. Therefore, both $\mathbf{\Pi}_{(\Lambda_1,\widetilde{\Lambda}_1)}(\mu,\infty)+\mathbf{\Pi}_{(\Lambda_2,\widetilde{\Lambda}_2)}(-\infty,\nu)$ and $\mathbf{\Pi}_{(\Lambda,\widetilde{\Lambda})}(\mu,\nu)$ (with varying Λ_i, $\widetilde{\Lambda}_i$ and Λ, $\widetilde{\Lambda}$ as described above) define valid multiplier classes for $\Delta_\Phi \in \text{slope}(\mu,\nu)$.

Table 6.1 illustrates how various stability tests proposed in the literature relate to Corollary 6.12 with (6.25). In the second column we state the uncertainty class under consideration in the respective paper as listed in the first column, while the third one gives the employed multiplier combination. Even if considering uncertainties in $\text{slope}(\mu,\nu)$, several papers just only exploit the fact that they are contained in either $\text{slope}(\mu,\infty)$ or $\text{slope}(-\infty,\nu)$. Yet some use, e.g., the information $\Delta_\Phi \in \text{slope}(0,\nu) = \text{slope}(-\infty,\nu) \cap \text{slope}(0,\infty)$ by additively combining multipliers for $\text{slope}(-\infty,\nu)$ and $\text{slope}(0,\infty)$, respectively. We devote the subsequent section to the proof that this is not beneficial if compared to using the dedicated multipliers for the class $\text{slope}(0,\nu)$ directly.

Several more aspects are worth pointing out in Table 6.1. All cited papers employ a combination of diagonally structured circle criterion multipliers ($\mathbf{\Pi}_{dr}[\alpha,\beta]$) and first order Zames-Falb multipliers. [6] also includes one of Yakubovich type ($\mathbf{\Pi}_{Y,dr}[\mu,\nu]$, see (6.27) below), but this is also covered by Zames-Falb multipliers as shown later. Further note that several approaches either take $\Lambda = 0$ or $\widetilde{\Lambda} = 0$ which, of course, increases conservativeness if computing stability margins. Thus, a combination of the multipliers proposed in the present paper is guaranteed to lead to the same or improved stability estimates.

We can further conclude that both the multipliers proposed by [162] and [96] (as well as the later proposed derivatives thereof) are special cases of Zames-Falb multipliers of order one. Hence, $\mathbf{\Pi}_L(\mu,\nu)$ with

Table 6.1: Overview of some multipliers employed in the literature

Reference	Uncertainty class	Multiplier class combination
[162]	$\sec[0,\beta] \cap \mathrm{slope}(0,\infty)$	$\mathbf{\Pi}_{dr}[0,\beta] + \mathbf{\Pi}_{(0,\widetilde{\Lambda})}(0,\infty)$
[96]	$\sec[0,\beta] \cap \mathrm{slope}(-\nu,\nu)$	$\mathbf{\Pi}_{dr}[0,\beta] + \mathbf{\Pi}_{(0,\widetilde{\Lambda})}(-\infty,\nu)$
[75]	$\sec[0,\beta] \cap \mathrm{slope}(0,\nu)$	$\mathbf{\Pi}_{dr}[0,\beta] + \mathbf{\Pi}_{(0,\widetilde{\Lambda})}(-\infty,\nu)$
[5]	$\sec[0,\beta] \cap \mathrm{slope}(0,\nu)$	$\mathbf{\Pi}_{dr}[0,\beta] + \mathbf{\Pi}_{(0,\widetilde{\Lambda})}(-\infty,\nu) + \mathbf{\Pi}_{(\Lambda,0)}(0,\infty)$
[7]	$\sec[0,\beta] \cap \mathrm{slope}(0,\nu)$	$\mathbf{\Pi}_{dr}[0,\beta] + \mathbf{\Pi}_{(0,\widetilde{\Lambda})}(-\infty,\nu) + \mathbf{\Pi}_{(\Lambda,0)}(0,\infty)$
[2]	$\sec[0,\nu] \cap \mathrm{slope}(0,\nu)$	$\mathbf{\Pi}_{dr}[0,\nu] + \mathbf{\Pi}_{(\Lambda,\widetilde{\Lambda})}(0,\nu)$
[6]	$\sec[0,\beta] \cap \mathrm{slope}(0,\nu)$	$\mathbf{\Pi}_{dr}[0,\beta] + \mathbf{\Pi}_{Y,dr}[\mu,\nu] + \mathbf{\Pi}_{(0,\widetilde{\Lambda}_1)}(-\infty,\nu) + \mathbf{\Pi}_{(\Lambda_2,\widetilde{\Lambda}_2)}(0,\infty)$
[4]	$\sec[0,\nu] \cap \mathrm{slope}(0,\nu)$	$\mathbf{\Pi}_{dr}[0,\nu] + \mathbf{\Pi}_{(\Lambda,\widetilde{\Lambda})}(0,\nu)$

L as in (6.20) and $l_\pm = 1$ could be seen as a full-block generalization of Tsypkin multipliers that, to the best of the authors knowledge, has not been described anywhere in the literature. As a side-remark, we emphasize that our approach does not require the LTI system in the loop to be strictly proper, as is typically encountered in the literature.

6.4.2 Redundant multiplier combinations

We have seen that a large number of papers handle $\varphi \in \mathrm{slope}(\mu,\nu)$ by combining Zames-Falb multipliers for $\mathrm{slope}(\mu,\infty)$ and $\mathrm{slope}(-\infty,\nu)$; let us now settle that it is more beneficial to work with the single class of dedicated multipliers $\mathbf{\Pi}_L(\mu,\nu)$.

Lemma 6.13
Let $\Pi \in \mathbf{\Pi}_L(\mu,\nu)$, $\Pi_1 \in \mathbf{\Pi}_L(\mu,\infty)$, $\Pi_2 \in \mathbf{\Pi}_L(-\infty,\nu)$ *be three given Zames-Falb multipliers. Then there exists another Zames-Falb multiplier* $\widetilde{\Pi} \in \mathbf{\Pi}_L(\mu,\nu)$ *such that* $\widetilde{\Pi} \preccurlyeq \Pi + \Pi_1 + \Pi_2$ *on* $\mathbb{T}$.

Proof. A proof is found in the Appendix C.5.2. □

In summary, we can just work with the tightest slope restriction in Corollary 6.12, i.e., $\varphi \in \mathrm{slope}(\mu,\nu)$, since the validity of (6.24) for a combination of multipliers as in Lemma 6.13 implies the existence of some multiplier in $\mathbf{\Pi}_L(\mu,\nu)$ also satisfying (6.24).

In case of a single nonlinearity, let us finally stress that Yakubovich multipliers for $\Delta_\Phi \in \mathrm{slope}(\mu,\nu)$ are also covered by first order Zames-Falb multipliers. Indeed if $k=1$, Lemma 5.10 shows that $\mathbf{\Pi}[\mu,\nu]$ can be parameterized as

$$\mathbf{\Pi}[\mu,\nu] = \left\{ \Pi \;\middle|\; \Pi = \lambda \begin{pmatrix} -2\mu\nu & \mu+\nu \\ \mu+\nu & -2 \end{pmatrix},\ \lambda > 0 \right\} \tag{6.26}$$

The claim then follows by the simple observation that, with (6.26), $|\mathrm{z}-1|^2\, \mathbf{\Pi}[\mu,\nu] = \mathbf{\Pi}_{(\lambda,\lambda)}(\mu,\nu)$ as in (6.25). Also for $k>1$ this shows that diagonally structured Yakubovich multipliers

$$\mathbf{\Pi}_{Y,dr}[\mu,\nu] := |\mathrm{z}-1|^2\, \mathbf{\Pi}_{dr}[\mu,\nu] \tag{6.27}$$

offer no benefit if combined with Zames-Falb multipliers. Yet, this no longer holds true for the full-block versions (see Example 6.16).

6.5 Implementation

In order to keep the derivation and comparison of multipliers as insightful as possible, we relied on the formulation of our tests in terms of inequalities in the frequency domain. Still, the translation to LMIs with an insightful structure is routine. Indeed, via multiplication with $1 = 1/(\mathrm{z}\,\bar{\mathrm{z}})$ on $\mathbb{T}$, observe that (6.14) holds on $\mathbb{T}$ if and only if we have

$$(\star)^* \Pi_{CY} \underbrace{\begin{pmatrix} \frac{1}{\mathrm{z}} & 0 \\ 1-\frac{1}{\mathrm{z}} & 0 \\ 0 & \frac{1}{\mathrm{z}} \\ 0 & 1-\frac{1}{\mathrm{z}} \end{pmatrix}}_{\Psi_{CY}(\mathrm{z})} \begin{pmatrix} M(\mathrm{z}) \\ I \end{pmatrix} \prec 0 \quad \text{for all} \quad \mathrm{z} \in \mathbb{T}; \tag{6.28}$$

clearly, Ψ_{CY} is a proper and stable transfer function.

Let us now sketch how to render Corollary 6.12 computational for some pair $l = (l_-, l_+)$. For brevity of display, consider the case $l_+ \geq l_-$ and define

$$\psi_{l_+} = \begin{pmatrix} I & \frac{1}{\mathrm{z}} I & \ldots & \frac{1}{\mathrm{z}^{l_+}} I \end{pmatrix}^T, \quad \Psi_{l_+} = \mathrm{diag}(\psi_{l_+}, \psi_{l_+})$$

as well as the square matrix $P_l \in \mathbb{R}^{(l_+ + 1)k \times (l_+ + 1)k}$ given by

$$P_l = \begin{pmatrix} L_0 & L_{-1} & \ldots & L_{-l_-} & 0 \ldots 0 \\ L_1 & 0 & \ldots & 0 & 0 \ldots 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \; \vdots \; \vdots \\ L_{l_+} & 0 & \ldots & 0 & 0 \ldots 0 \end{pmatrix}. \tag{6.29}$$

Then the multiplier (6.19), (6.20) may be expressed as

$$(\star)^* \Pi_l T \Psi_{l_+} = (\star)^* \begin{pmatrix} 0 & P_l^T \\ P_l & 0 \end{pmatrix} \begin{pmatrix} \nu I & -I \\ -\mu I & I \end{pmatrix} \Psi_{l_+}; \tag{6.30}$$

again $\Psi_{ZF} := T\Psi_{l_+}$ is proper and stable. The case of $l_- > l_+$ is treated analogously.

In this way we obtain the following combined stability test.

Corollary 6.14

Let $\Delta_\Phi \in \mathrm{sec}[\alpha I, \beta I] \cap \mathrm{slope}(\mu I, \nu I)$ and fix $l_-, l_+ \geq 0$. Suppose there exists some $\Pi_{CY} \in \mathbf{\Pi}[\mathrm{diag}(\alpha I, \mu I), \mathrm{diag}(\beta I, \nu I)]$ and some L as in

(6.18) *that is either doubly hyperdominant or doubly dominant (if* φ *is odd) with*

$$(\star)^* \begin{pmatrix} \Pi_{CY} & 0 \\ 0 & \Pi_l \end{pmatrix} \begin{pmatrix} \Psi_{CY} \\ \Psi_{ZF} \end{pmatrix} \begin{pmatrix} M \\ I \end{pmatrix} \prec 0 \quad on \quad \mathbb{T}. \tag{6.31}$$

Then the interconnection (6.2) *is stable*

This also holds for $\Delta_\Phi \in \sec[\alpha, \beta] \cap \operatorname{slope}(\mu, \nu)$ if just restricting all matrices L_j in (6.29) to be diagonal and assuming that $\Pi_{CY} \in \mathbf{\Pi}[\operatorname{diag}(\alpha, \mu), \operatorname{diag}(\beta, \nu)]$.

It is now routine to turn the verification of (6.31) (for some Π_{CY}, Π_l satisfying the respective constraints) into an LMI; just choose a realization $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ of

$$\begin{pmatrix} \Psi_{CY} \\ \Psi_{ZF} \end{pmatrix} \begin{pmatrix} M \\ I \end{pmatrix}$$

and apply the discrete-time KYP lemma ([128]); then (6.31) holds iff there exist some $X = X^T$ with

$$\left(\begin{array}{cc} I & 0 \\ \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right)^T \left(\begin{array}{cc|cc} X & & & \\ & -X & & \\ \hline & & \Pi_{CY} & \\ & & & \Pi_l \end{array}\right) \left(\begin{array}{cc} I & 0 \\ \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right) \prec 0. \tag{6.32}$$

This should be compared to the continuous-time analogue, (5.41), where the structure of the LMI is identical, apart from the fact that the left upper block of the middle matrix takes the form $\left(\begin{smallmatrix} 0 & X \\ X & 0 \end{smallmatrix}\right)$. In summary, the IQC framework allows us to state and combine stability criteria in an intuitive fashion, as demonstrated for (6.31), and to generate the corresponding LMI conditions (6.32) in a structured, insightful and routine way. Unfortunately, in the literature, this structure is usually not as apparent, which renders the task of comparing different approaches and extracting the underlying generating principles for the employed criteria unnecessarily tedious.

6.6 Examples

Let us finally provide some numerical illustrations.

Example 6.15.
First, we adopt an example from [2], where M is given by

$$M(\mathrm{z}) = \begin{pmatrix} \frac{0.2}{\mathrm{z}-0.98} & \frac{-0.2}{\mathrm{z}-0.92} \\ \frac{0.3}{\mathrm{z}-0.97} & \frac{0.1}{\mathrm{z}-0.91} \end{pmatrix}.$$

Our goal is to estimate the largest $r > 0$ such that the feedback interconnection (6.2) remains stable for all $\Delta_\Phi \in \mathrm{slope}(0, \nu)$ with $\nu = \mathrm{diag}(r, r)$. We first assume that the nonlinearities are non-identical. As can be inferred from Table 6.1, the stability test proposed in [2] is the least conservative of all discussed approaches. The maximal r estimated therein is $r = 3.626$. Using diagonally structured Zames-Falb multipliers in (6.32) (with $l_\pm = 1$), we can still improve on that and obtain $r = 3.808$ which is already very close to the Nyquist value $r_N = 3.85$. This supports the fact that diagonally structured first order Zames-Falb multipliers may already lead to improved estimates if compared to the Popov tests in the literature. If we further assume that the nonlinearities are repeated, stability can be guaranteed up to r_N by means of full-block multipliers. $\star$

Example 6.16.
Let the LTI system M in (7.2) be defined by

$$A = \begin{pmatrix} 0.74 & -0.3 & 0 & 0 & -0.1 \\ 0 & 0.98 & 0 & 0 & 0 \\ 0 & 0 & 0.97 & 0 & 0 \\ 0 & 0 & 0 & 0.72 & 0 \\ 0 & 0.1 & 0.31 & 0 & 0.9 \end{pmatrix}, \quad B = \begin{pmatrix} 2.2 & 0.2 \\ 0.3 & 0 \\ 0.5 & 0 \\ -0.5 & 0.5 \\ 0 & 0.05 \end{pmatrix},$$

$$C = \begin{pmatrix} -0.21 & -0.4 & -0.01 & 0.40 & 0 \\ 0.3 & 0.3 & -0.3 & 0 & -0.36 \end{pmatrix}, \quad D = 0,$$

and assume, for simplicity, that $\Delta_\Phi \in \mathrm{slope}(\mu,\nu)$ with $\mu = 0$ and $\nu = I$, yet with non-identical functions φ_j. Let us now compare the standard approach from the literature, namely a combination of diagonally structured Zames-Falb and circle multipliers, to Corollary 6.14. In order to contrast both approaches, we compute ℓ_2-gain estimates, i.e., the smallest $\gamma > 0$ such that (6.3) holds. This is easily achieved by following the procedure outlined, for the continuous-time case, in Section 5.3.4.

Table 6.2: ℓ_2-gain estimates for Example 6.16

multiplier	$l_\pm = 1$	$l_\pm = 2$	$l_\pm = 3$	$l_\pm = 4$
ZF+Π_{dr}	148.43	104.35	89.39	82.08
Corollary 6.14	95.23	76.47	69.99	66.58

The results depicted in Table 6.2 nicely illustrate that, unlike the diagonally structured circle and Yakubovich ones, the combined polytopic multipliers also provide additional benefit if employed together with those for the Zames-Falb criterion. ⋆

6.7 Summary

In this chapter, we present a comprehensive approach to the problem of absolute stability of feedback interconnections in the discrete-time setting. We reveal that all multiplier based stability criteria in the literature can be subsumed into one general framework that even allows for seamless extension and generalisation of the previously proposed stability tests. This leads to the derivation of novel and completely unstructured full-block multipliers that are, in numerical examples, shown to lead to less conservative stability estimates if compared to existing criteria. Moreover, the present chapter serves to illustrate that, if formulated correctly, the translation of stability results from continuous time to discrete time is rather straightforward.

Part II

From input-output properties to the analysis of internal behavior

Introduction to Part II

Let us briefly take one step back and focus on the conceptual development of the IQC stability results presented so far; from the classical ones discussed in Chapter 2, their first extension in Chapter 3, and, finally, the general framework derived in Chapter 4.

All the above mentioned results share the key common theme that stability is inferred solely from input-output properties of the systems composing the interconnection. These systems are given maps defined on the full underlying signal spaces and the constraints that form the basis of our stability proof are assumed to hold for all signals in the respective space. This global perspective is one of the reasons for the flexibility of the derived framework and, furthermore, allows us to entirely avoid the internal dynamics in stability considerations. However, we make fundamental use of the state-space description of the LTI system for the verification of stability based on LMIs.

On the downside, the exclusive focus on the input-output behavior also comes at the expense of limited applicability. So far, this is probably best visible by reviewing the generalizations made possible by the framework derived in Chapter 4. While in Theorem 3.4 we relax the assumption on the domain of definition of the uncertainty and its IQC description, we further proceed along this line of thought in Theorem 4.7 by carefully addressing the external inputs. Nevertheless, the restriction to arbitrary input *sets* or to IQC only valid for signals in a certain sub*set* of the underlying signal space could not be achieved. As most severe consequences, this prevents us from locally analyzing feedback interconnections, and here in particular the analysis of only locally stable systems or the verification of state and output constraints.

It will be the main contribution of the two subsequent chapters to establish a natural link between the general input-output theory based on IQCs and the local analysis of feedback interconnections as is standard in Lyapunov or dissipation theory. In Chapter 7 we present a comprehensive framework that, under rather mild conditions, allows to derive a Lyapunov function from the standard FDI condition in IQC theory. We furthermore illustrate that this puts us in the position to proceed with local analysis using identical arguments as in standard Lyapunov theory. Building on the framework established in Chapter 7, we propose in Chapter 8 a unified approach where the uncertainty is described with both hard and soft IQCs. On the one hand, this further widens the applicability of our framework, while, one the other hand, it effectively reduces the conservativeness in stability analysis. Beyond that, a particular focus in Chapter 8 lies on the class of Zames-Falb multipliers, where both the subclasses of causal and anti-causal Zames-Falb multipliers are shown to admit lossless incorporation into the framework developed in Chapter 7.

Finally, we emphasize that a slightly different version of Chapter 7 is provisionally accepted for publication in the IEEE Transactions on Automatic Control [60]. Furthermore, the results derived in Chapter 8 are, in a self contained form, accepted for publication in the IEEE Control Systems Letters [59].

Chapter 7

Invariance with dynamic multipliers

7.1 Introduction

As already indicated in Chapter 2, robust or absolute stability analysis of feedback interconnections composed of an LTI system and an uncertain component Δ, as shown in Figure 7.1, can essentially be divided into two still not fully connected fields. In fact, trajectory oriented time-domain techniques such as Lyapunov function methods (see, e.g., [106, 188, 118]) are considered in parallel to operator based functional analytic approaches. The latter may be further divided into multiplier theory [195, 196, 197, 44] and graph separation techniques [138, 139, 72, 159, 89], formalized, for example, in the framework of IQCs (see Section 2.2). In this thesis we have so far restricted our attention to the IQC approach that has been shown to offer the additional advantages over classical multiplier results that there is generally no need for loop transformations and that noncausal multipliers may be incorporated with ease (see [112, 63] for relations among [195, 196, 136, 138] and [110]).

Lyapunov techniques permit to exploit properties of signals as they are generated within the interconnection, which often allows to reduce conservativeness by taking the specifics of the interconnection structure

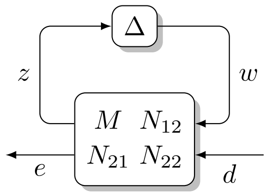

Figure 7.1: Performance setting

or the external stimuli into account. As a major asset, Lyapunov methods facilitate regional analysis[1], such as the verification of hard state and output constraints in the time domain, and, therefore, even allow for the analysis of interconnections that are not globally stable. This is often achieved by restricting the input signals to a certain (bounded) subset of the encompassing signal space and by imposing suitable additional constraints on the Lyapunov function.

In contrast, stability conditions obtained from the operator based approach are usually formulated in the frequency domain using multipliers. While Lyapunov arguments typically require an individual proof if additional aspects of the uncertain component are taken into account, refined uncertainty descriptions in terms of added multipliers may be easily incorporated into the operator framework. This allows for a simple reduction of conservativeness by capturing the uncertainty more accurately. However, in the configuration of Figure 7.1, interconnection stability boils down to guaranteeing global boundedness of the inverse $(I - M\Delta)^{-1}$ on the underlying function space, which prevents regional analysis with operator approaches.

Already this cursory exposition reveals that merging both rather complementary techniques may drastically improve stability and performance estimates and simultaneously widen the range of applications.

[1]In the literature, the term *regional* is sometimes used to distinguish the analysis on fixed, often large set form *local* results that merely hold in some neighborhood. In the sequel, we will use both terms synonymously.

However, even though both have their origins around the 1960s, their effective combination at a high level of generality is still largely open.

Following an earlier result by Safonov and Athans [138], which establishes the link between a classical theorem of Zames on conic sectors [195] and quadratic Lyapunov functions, there have been multiple attempts to bridge the gap between Lyapunov theory and IQCs (see, e.g., [74, 79, 125, 13, 124, 153, 32] and [174, 150]). A well-known tool in linking the FDIs arising in IQC analysis with the LMIs underlying the Lyapunov approach is the KYP lemma [128, 187]. This result equivalently characterizes the validity of an FDI by the feasibility of a related LMI; the corresponding LMI solution is a so-called certificate for the FDI. Since such certificates are, in general, indefinite, it is a pivotal question how they can be used to construct positive definite Lyapunov functions in order to enable the inclusion of extra constraints for regional analysis.

The aforementioned papers all address this issue from various angles. In fact, in [79, 125] the circle [195] and Popov [126] criteria are employed in the analysis of regionally stable saturated systems. For the circle criterion, these results were strengthened in [124] by also exploiting information regarding the derivative of the output of the LTI system. Not necessarily globally stable interconnections are also the subject of study in [153, 32], where hard IQCs (2.14) are employed in order to verify regional stability. Finally, in [52], an attempt was made to first prove contractive invariance of an ellipsoid and regional stability of the interconnection using general soft IQCs (2.16). Unfortunately, the authors incorrectly claim that the validity of an IQC on some invariant set suffices to conclude stability by applying the standard IQC theorem [110, Thm. 1]. The critical stumbling block in [52] is also at the very heart of the present chapter, namely the fact that the stability proof given in [110] fundamentally requires the IQCs to hold on the full space $\mathscr{L}_2$ of square integrable functions. As shown in Chapters 3 and 4 this can indeed be relaxed to subspaces of $\mathscr{L}_2$, but not to arbitrary subsets thereof.

The paper by Balakrishnan [13] stands out in this list, since it does connect dynamic IQC techniques with the Lyapunov approach and exhibits the possibility for regional analysis. However, the paper focusses on structured LTI uncertainties that are handled with so-called dynamic D-scalings, and the proposed method (including the proofs) heavily relies on these particularities.

In conclusion, the cited papers either focus on rather specific classes of uncertainties or employ a pretty limiting set of IQC multipliers. By contrast, in [174] and later in [150][2] first attempts were made to overcome these restrictions, by giving dissipation based proofs of the IQC stability theorem in [110] for a rich class of multipliers. It was hoped that these would permit the inclusion of time-domain constraints into general (frequency-domain) IQC theory, and vice versa. Yet, as a major obstacle, both derivations rely on the solution of an indefinite algebraic Riccati equation (ARE) that acts as a shift to the KYP certificate in order to obtain a positive definite Lyapunov function. Unfortunately, a simultaneous search over the class of multipliers and the corresponding shift matrix is impossible, since the indefinite ARE cannot be turned into an LMI constraint. In addition, even though the proofs in [174, 150] are both trajectory based in nature, the derived stability results require globally valid IQC descriptions of the uncertainty which prevents regional analysis.

As the main contributions of this chapter we present a framework that allows to overcome both limitations of the approaches in [174, 150], even under fewer assumptions. We propose a novel, multiplier dependent shift of the KYP certificate that is described by LMI constraints and enables the formulation of a regional stability result within IQC theory and based on solving LMIs. As an additional benefit, we can consider uncertainties, IQCs and external disturbances that are only defined on general *subsets* of $\mathscr{L}_2$. We demonstrate how our extension of the standard IQC framework permits the merging of multiplier descriptions of uncertainties with concepts from Lyapunov and dissipation

[2] A correction of the paper [151] that contains a technical glitch.

theory [182, 183]. In particular, we reveal how to guarantee state as well as output constraints, and how these can be systematically exploited in order to tighten IQCs for the reduction of conservatism. A selection of numerical examples illustrate that our approach can substantially improve on existing ones, some of which are even tailored to specific scenarios.

The chapter is structured as follows. After setting the stage by adapting the IQC framework to our regional scenario and stating the relation between the emerging FDIs and dissipation inequalities in Section 7.2, we formulate and prove our central local IQC stability result in Section 7.3. Furthermore, we discuss its novel features if compared to related approaches in the literature in Section 7.4, and highlight how to derive several concrete regional stability and performance criteria in Section 7.5. Section 7.6 concludes the chapter with a selection of numerical examples that demonstrate the benefit of our approach if compared to various state of the art methods in the literature for different scenarios. Again, we emphasise the fact that the results from this chapter are submitted for journal publication [60].

For convenience, we remind the reader of the following facts from Chapter 2 that are particularly relevant for the present chapter. If Ψ is a stable transfer matrix realized by (A, B, C, D) and $P = P^T$, we say that $X = X^T$ is a **certificate** for the FDI $\Psi^* P \Psi \prec 0$ on $\mathbb{C}_0^\infty$ in case that X solves the (KYP-) LMI

$$\begin{pmatrix} I & 0 \\ A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} 0 & X & 0 \\ X & 0 & 0 \\ 0 & 0 & P \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \\ C & D \end{pmatrix} \prec 0. \tag{7.1}$$

Moreover, for $H \succcurlyeq 0$ and $\eta \geq 0$ we use

$$\mathcal{E}(H, \eta) := \left\{ x \in \mathbb{R}^n \;\middle|\; x^T H x \leq \eta \right\}.$$

7.2 Preliminaries

This section serves to introduce the setup for this paper and to discuss the distinctions to the classical IQC framework (see Section 2.2.1) made necessary by our focus on regional analysis. Furthermore, we discuss the connection between IQCs and dissipation theory [182, 183] in some detail.

7.2.1 Setup

As depicted in Figure 7.1, we consider the interconnection

$$\begin{pmatrix} z \\ e \end{pmatrix} = \underbrace{\begin{pmatrix} M & N_{12} \\ N_{21} & N_{22} \end{pmatrix}}_{N} \begin{pmatrix} w \\ d \end{pmatrix}, \qquad w = \Delta(z), \tag{7.2}$$

involving the stable LTI system N, as realized by

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 d, \qquad\qquad x(0) = 0, \\ z &= C_1 x + D_{11} w + D_{12} d, \\ e &= C_2 x + D_{21} w + D_{22} d \end{aligned} \tag{7.3}$$

with $A \in \mathbb{R}^{n \times n}$ being Hurwitz, in feedback with some uncertainty Δ. As the first deviation from the assumptions imposed in Part I of this thesis, it is not required that Δ is defined globally. Instead, its domain of definition is some subset

$$\mathscr{Z}_e \subset \mathscr{L}_{2e} \quad \text{satisfying} \quad \{z_T \mid z \in \mathscr{Z}_e\} \subset \mathscr{Z} \quad \text{for all} \quad T > 0, \tag{7.4}$$

where $\mathscr{Z} := \mathscr{Z}_e \cap \mathscr{L}_2$ denotes the set of finite energy signals in $\mathscr{Z}_e$.

Remark 7.1.
Unfortunately, the assumptions on $\mathscr{Z}_e$ are closer to the ones in Chapter 3 than the more general ones in Chapter 4. As a main consequence, the requirement $(\mathscr{Z}_e)_T \subset \mathscr{Z}$ in (7.4) will prevent us from employing multipliers with additional regularity assumptions in regional analysis. ⋆

As in all previous chapters, uncertainties are causal maps

$$\Delta : \mathscr{Z}_e \to \mathscr{L}_{2e} \quad \text{but now satisfying only} \quad \Delta(\mathscr{Z}) \subset \mathscr{L}_2. \tag{7.5}$$

The latter inclusion is interpreted as a weak form of stability of Δ, while causality means, as usual, that $\Delta(z)_T = \Delta(z_T)_T$ holds[3] for all $z \in \mathscr{Z}_e$ and $T > 0$. Additionally, in (7.2) we also allow for the external disturbance d to be confined to some set $\mathscr{D} \subset \mathscr{L}_2$. The definitions of well-posedness and stability of (7.2) are adapted to the current setting as follows.

Definition 7.2.
The feedback interconnection (7.2) is **well-posed on** $\mathscr{D}$ if, for each $d \in \mathscr{D}$, there exists a unique response $z \in \mathscr{Z}_e$ such that the map $d \to z$ is causal. It is **stable on** $\mathscr{D}$ if

$$\text{there exists} \quad \gamma > 0 \quad \text{with} \quad \|z\| \leq \gamma \|d\| \quad \text{for all} \quad d \in \mathscr{D}. \tag{7.6}$$

$\star$

Remark 7.3.
For the choices $\mathscr{Z}_e = \mathscr{L}_{2e}$, $\mathscr{D} = \mathscr{L}_2$ and $N_{12} = I$, well-posedness of (7.2) translates into $I - M\Delta : \mathscr{L}_{2e} \to \mathscr{L}_{2e}$ having a causal inverse, while stability means that, in addition, $(I - M\Delta)^{-1}$ maps $\mathscr{L}_2$ into $\mathscr{L}_2$ and is bounded; therefore, this setting generalizes the classical notions of well-posedness from Definition 2.5 insofar as we do not require it for all Δ replaced by $\tau\Delta$ and $\tau \in [0, 1]$. $\star$

In deviating from our general framework developed in Chapter 4, we restrict our attention in the present chapter to classical IQCs (2.16). Two signals $z, w \in \mathscr{L}_2$ with Fourier transforms $\hat{z}, \hat{w}$ are said to satisfy the IQC defined by a Hermitian valued multiplier $\Pi \in RL_\infty$ if

$$\Sigma_\Pi \begin{pmatrix} z \\ w \end{pmatrix} = \int_{-\infty}^{\infty} \begin{pmatrix} \hat{z}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix}^* \Pi(i\omega) \begin{pmatrix} \hat{z}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix} d\omega \geq 0.$$

[3] $(\cdot)_T$ denotes the standard truncation on $\mathscr{L}_{2e}$, see Definition 2.1.

Definition 7.4.
The uncertainty (7.5) satisfies the (soft) local IQC imposed by $\Pi \in RL_\infty$, in short $\Delta \in \mathrm{IQC}_{\mathscr{Z}}(\Pi)$, if

$$\Sigma_\Pi \begin{pmatrix} z \\ \Delta(z) \end{pmatrix} \geq 0 \quad \text{for all} \quad z \in \mathscr{Z}. \tag{7.7}$$

$\star$

As a specifically relevant feature of our framework, we can and will beneficially exploit the ability to work with uncertainty IQCs that are only satisfied for signals in a genuine subset $\mathscr{Z}$ of $\mathscr{L}_2$. In this sense, (7.7) is a local IQC (despite the fact that $\mathscr{Z}$ might be "large") and, in the sequel, our stability and performance guarantees are addressed as local IQC results.

Locality not only allows us to deal with uncertainties that are unbounded or even undefined on the full space $\mathscr{L}_{2e}$ (see Example 7.16 in Section 7.6), but it also permits us to construct stronger IQCs for reducing the conservatism of global results, as illustrated with several applications of our main theorem in Section 7.5. Even if Δ is defined on $\mathscr{L}_{2e}$, we emphasize that it might still be beneficial to consider its restriction to some subset $\mathscr{Z}_e$ on which it exhibits a more desirable behavior.

Let us now specify the classes of multipliers considered in this chapter in more detail. As noted in Section 2.3, any $\Pi = \Pi^* \in RL_\infty$ can be factorized as

$$\Pi = \Psi^* P \Psi \quad \text{with a real} \quad P = P^T \quad \text{and} \quad \Psi \in RH_\infty; \tag{7.8}$$

this relation will be denoted by $\Pi \sim (P, \Psi)$. For extensive lists of multipliers that satisfy (7.8) and capture the behavior of practically relevant uncertainties, we refer the reader to [110, 176].

In the sequel we tacitly assume that $\Pi = (\Pi_{ij})$ and $\Psi = (\Psi_1 \ \Psi_2)$ are partitioned according to the dimensions of the signals z and w; moreover, Ψ is supposed to be realized as

$$\dot{\xi} = A_\Psi \xi + B_{\Psi,1} z + B_{\Psi,2} w, \qquad \xi(0) = 0,$$

$$z_\Psi = C_\Psi \xi + D_{\Psi,1} z + D_{\Psi,2} w, \tag{7.9}$$

where A_Ψ is Hurwitz. This allows us to emphasize the first relevance of multiplier factorizations, since they provide a means for translating IQCs in the frequency domain into their corresponding time-domain versions. Indeed, using (7.8), (7.9) and $w = \Delta(z)$, we infer from Parseval's theorem that the IQC (7.7) is equivalent to the (infinite horizon) time-domain constraint

$$\int_0^\infty z_\Psi(t)^T P z_\Psi(t)\, dt \geq 0 \quad \text{for all} \quad z \in \mathscr{Z}. \tag{7.10}$$

We indicate the fact that (7.10) depends on the factorization $\Pi \sim (P, \Psi)$ by abbreviating it as $\Delta \in \mathrm{IQC}_Z(P, \Psi)$.

All this motivates to work with classes of multipliers that are parameterized as in (7.8) with a fixed stable (usually tall) outer factor Ψ and with a variable symmetric matrix P that is constrained by (7.10). Using the notation introduced above, P thus varies in the set

$$\left\{ P = P^T \,\middle|\, \Delta \in \mathrm{IQC}_Z(\Psi^* P \Psi) \right\}.$$

The fact that we allow for arbitrary stable Ψ in (7.8) is an important strength of the present approach and allows to readily employ, e.g., Zames-Falb multipliers with tall outer factors exactly as in Chapter 5. This is in contrast to the works in [150, 174] that require J-spectral factorizations which may only be obtained from (7.8) in a non-convex additional step.

7.2.2 Performance specifications

Let us now discuss performance specification and their verification using LMIs as these provide a connection between IQCs and dissipation inequalities, thus enabling regional analysis. Suppose that (7.2) is well-posed and has been shown to be stable. It is then standard to characterize a certain desired behavior of the interconnection (7.2) by imposing quadratic performance criteria defined through a symmetric matrix P_p and expressed in the time domain as

$$\int_0^\infty \begin{pmatrix} e(t) \\ d(t) \end{pmatrix}^T P_p \begin{pmatrix} e(t) \\ d(t) \end{pmatrix} dt \leq 0 \tag{7.11}$$

for all trajectories of (7.2) in response to $d \in \mathscr{D}$. Let us recapitulate the well-known fact that one can guarantee this performance specification in terms of the FDI

$$(\star)^* \begin{pmatrix} P & 0 \\ 0 & P_p \end{pmatrix} \begin{pmatrix} \Psi & 0 \\ 0 & I \end{pmatrix} \left(\begin{array}{cc} M & N_{12} \\ I & 0 \\ \hline N_{21} & N_{22} \\ 0 & I \end{array}\right) \prec 0 \quad \text{on} \quad \mathbb{C}_0^\infty, \tag{7.12}$$

as made precise in the subsequent Theorem.

Remark 7.5.
Note that we could incorporate dynamic performance criteria defined by a multiplier $\Pi_p \sim (P_p, \Psi_p)$ in complete analogy to the stability multiplier Π. This only results in an additional filter Ψ_p instead of the identity matrix in (7.12). However, it remains to be explored if the enlargement of the certificate for (7.12), resulting from the additional dynamics, will introduce conservatism in the subsequent derivation. $\star$

Theorem 7.6
Assume that the interconnection (7.2) *is well-posed and stable on* $\mathscr{D}$. *Moreover, suppose there exists some* $P = P^T$ *with the following two properties*

a) $\Delta \in \mathrm{IQC}_{\mathscr{X}}(P, \Psi)$;

b) P *satisfies the performance FDI* (7.12).

Then (7.11) *holds for all trajectories of* (7.2) *with* $d \in \mathscr{D}$.

***Proof*.** Choose $d \in \mathscr{D}$. By well-posedness of (7.2) on $\mathscr{D}$, the response of (7.2) satisfies $z \in \mathscr{Z}_e$ and by stability we also get $z \in \mathscr{L}_2$. This implies $z \in \mathscr{Z}$ and, due to (7.5), also $w = \Delta(z) \in \mathscr{L}_2$; then stability of N guarantees $e \in \mathscr{L}_2$. Right- and left-multiplying the inequality in (7.12) with $\mathrm{col}(\hat{w}(i\omega), \hat{d}(i\omega))$ and its conjugate transpose, respectively, leads to

$$\begin{pmatrix} \hat{z}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix}^* \Psi(i\omega)^* P \Psi(i\omega) \begin{pmatrix} \hat{z}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix} + \begin{pmatrix} \hat{e}(i\omega) \\ \hat{d}(i\omega) \end{pmatrix}^* P_p \begin{pmatrix} \hat{e}(i\omega) \\ \hat{d}(i\omega) \end{pmatrix} \leq 0$$

for almost all $\omega \in \mathbb{R}$. After integration over frequency and exploiting $\hat{w} = \widehat{\Delta(z)}$, we get

$$\Sigma_{\Psi^* P \Psi} \begin{pmatrix} z \\ \Delta(z) \end{pmatrix} + \int_{-\infty}^{\infty} \begin{pmatrix} \hat{e}(i\omega) \\ \hat{d}(i\omega) \end{pmatrix}^* P_p \begin{pmatrix} \hat{e}(i\omega) \\ \hat{d}(i\omega) \end{pmatrix} d\omega \leq 0.$$

Using a), (7.7) and Parseval's theorem finishes the proof. $\square$

In addition to their role in the translation from frequency-domain to time-domain constraints, and as a second relevance of the factorization (7.8), we recall how to characterize the validity of (7.12) in terms of an LMI feasibility test. In fact we just need to take a realization of

$$\left(\begin{array}{cc|cc} \Psi_1 & \Psi_2 & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right) \left(\begin{array}{cc} M & N_{12} \\ I & 0 \\ \hline N_{21} & N_{22} \\ 0 & I \end{array}\right)$$

and apply the KYP lemma [128] to justify the following fact.

Lemma 7.7
For $\Pi \sim (P, \Psi)$ described as in (7.8), the FDI (7.12) is equivalent to the existence of a solution $X = X^T$ of the LMI

$$(\star)^T \begin{pmatrix} 0 & X & 0 & 0 \\ X & 0 & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P_p \end{pmatrix} \left(\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline A_\Psi & B_{\Psi,1}C_1 & B_{\Psi,1}D_{11} + B_{\Psi,2} & B_{\Psi,1}D_{12} \\ 0 & A & B_1 & B_2 \\ \hline C_\Psi & D_{\Psi,1}C_1 & D_{\Psi,1}D_{11} + D_{\Psi,2} & D_{\Psi,1}D_{12} \\ \hline 0 & C_2 & D_{21} & D_{22} \\ 0 & 0 & 0 & I \end{array}\right) \prec 0. \tag{7.13}$$

In order to provide a link between the frequency-domain constraints within the IQC framework and the time-domain descriptions used in Lyapunov arguments, we rely on dissipation theory. Let us hence recall how the relation of FDIs and LMIs in Lemma 7.7 allows us to extract the corresponding dissipation inequality. In the sequel we tacitly assume that $X = (X_{ij})$ in (7.13) is partitioned according to the dimensions of A_Ψ and A, respectively.

Lemma 7.8
For $\Pi \sim (P, \Psi)$ described as in (7.8) suppose that (7.13) holds. Then, for any $w, d \in \mathscr{L}_{2e}$ and a nonzero initial condition $x(0) = x_0$ of N, the trajectory of the nominal system (7.3) interconnected with the filter (7.9) satisfies

$$\begin{pmatrix} \xi(T) \\ x(T) \end{pmatrix}^T \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} \xi(T) \\ x(T) \end{pmatrix} + \int_0^T z_\Psi(t)^T P z_\Psi(t)\, dt + \\ + \int_0^T \begin{pmatrix} e(t) \\ d(t) \end{pmatrix}^T P_p \begin{pmatrix} e(t) \\ d(t) \end{pmatrix} dt - x_0^T X_{22} x_0 \leq 0 \tag{7.14}$$

for all $T > 0$.

Let us now establish the connection between Lemma 7.8 and classical Lyapunov and dissipation theory while simultaneously highlighting the key challenges if using soft IQCs.

7.2.3 Technical motivation of contributions

Suppose we take $P_p = \mathrm{diag}(I, -\gamma^2 I)$ and assume that the system N is structured as $(N_{21}\ N_{22}) = (M\ N_{12})$; then the gain inequality in (7.6) is equivalent to (7.11). Let us pinpoint why, even in this particular case, the hypotheses in Theorem 7.6 do not allow to *conclude* stability. Indeed, along the response of (7.2) to $d \in \mathscr{D}$, the dissipation inequality (7.14) reads as

$$\begin{pmatrix}\xi(T)\\ x(T)\end{pmatrix}^T X \begin{pmatrix}\xi(T)\\ x(T)\end{pmatrix} + \int_0^T z_\Psi(t)^T P z_\Psi(t)\,dt + \\ + \int_0^T \|z(t)\|^2\,dt \le \gamma^2 \int_0^T \|d(t)\|^2\,dt. \quad (7.15)$$

If trying to assure that $\int_0^T \|z(t)\|^2\,dt$ is bounded for $T \to \infty$ (in order to guarantee $z \in \mathscr{L}_2$), we need to argue that the sum of the first two terms in (7.15) is bounded from below for $T \to \infty$. However, in general, the solution X of the LMI (7.13) will not be positive (semi)definite. Moreover, since it is not clear whether z is contained in $\mathscr{Z}$, we cannot directly apply (7.10) in order to draw the conclusion that $\int_0^T z_\Psi(t)^T P z_\Psi(t)\,dt$ is bounded from below for $T \to \infty$. Yet, if assuming $X \succ 0$ and the validity of the so-called hard IQC

$$\int_0^T z_\Psi(t)^T P z_\Psi(t)\,dt \ge 0 \quad \text{for all} \quad T > 0, \quad (7.16)$$

it is straightforward to infer $z \in \mathscr{L}_2$ and $\|z\| \le \gamma\|d\|$ from (7.15); the latter two properties are standard hypothesis in robust stability proofs based on dissipation arguments that often appear in the literature, with [182, 138] being early references.

However, if working with the more general soft IQC (7.10) and without imposing a positivity constraint on X, this direct reasoning fails [174, 150]. In the following section, we will use (7.10) in order to derive a computationally tractable lower bound on the finite horizon integral in (7.16) which does allow to conclude robust stability.

7.3 Main result

Let us now present the key technical local IQC stability theorem of this chapter. Besides being of independent interest, it provides the foundation for all subsequent concrete novel regional stability and performance analysis results based on local IQCs.

Along the lines of standard IQC theory, stability of (7.2) will be characterized solely in terms of M and involves the FDI

$$\begin{pmatrix} M \\ I \end{pmatrix}^* \Psi^* P \Psi \begin{pmatrix} M \\ I \end{pmatrix} \prec 0 \quad \text{on} \quad \mathbb{C}_0^\infty \tag{7.17}$$

as certified by some symmetric matrix X^s satisfying

$$(\star)^T \begin{pmatrix} 0 & X^s & 0 \\ X^s & 0 & 0 \\ 0 & 0 & P \end{pmatrix} \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & I & 0 \\ \hline A_\Psi & B_{\Psi,1}C_1 & B_{\Psi,1}D_{11} + B_{\Psi,2} \\ 0 & A & B_1 \\ \hline C_\Psi & D_{\Psi,1}C_1 & D_{\Psi,1}D_{11} + D_{\Psi,2} \end{array}\right) \prec 0. \tag{7.18}$$

We need to restrict our attention to multipliers (7.8) that satisfy the additional property

$$\Pi_{22} = \Psi_2^* P \Psi_2 \prec 0 \quad \text{on} \quad \mathbb{C}_0^\infty \tag{7.19}$$

which is certified by some $Y_{22} = Y_{22}^T$ solving the LMI

$$\begin{pmatrix} A_\Psi^T Y_{22} + Y_{22} A_\Psi & Y_{22} B_{\Psi,2} \\ B_{\Psi,2}^T Y_{22} & 0 \end{pmatrix} + \begin{pmatrix} C_\Psi & D_{\Psi,2} \end{pmatrix}^T P \begin{pmatrix} C_\Psi & D_{\Psi,2} \end{pmatrix} \prec 0. \tag{7.20}$$

We emphasize that this property is automatically satisfied for most of the multiplier classes proposed in the literature (as can be extracted from the non-exhaustive survey in [176]). However, note that, e.g., the Pólya relaxation presented in Section 5.3.1 does not lead to multipliers satisfying (7.19). We will discuss the incorporation of these multipliers in the subsequent chapter.

Theorem 7.9
Suppose that the interconnection (7.2) *of the stable LTI system* N *and the causal uncertainty* Δ*, satisfying* (7.5)*, is well-posed on* $\mathscr{D}$*. Then* (7.2) *is stable on* $\mathscr{D}$ *if there exists some* $P = P^T$ *with the following two properties*

a) $\Delta \in \mathrm{IQC}_{\mathscr{X}}(P, \Psi)$*;*

b) there exists a certificate X^s *of* (7.17) *and a certificate* Y_{22} *of* (7.19) *which are coupled as*

$$\begin{pmatrix} X^s_{11} - Y_{22} & X^s_{12} \\ X^s_{21} & X^s_{22} \end{pmatrix} \succ 0. \tag{7.21}$$

This provides a generalization of [110, Thm. 1] towards stability analysis of the interconnection (7.2) on the basis of local (soft) integral quadratic constraints and general disturbance sets $\mathscr{D}$. In contrast to [110], we require well-posedness and the validity of the uncertainty IQC (7.7) to only hold for Δ and not for the whole set of all $\tau\Delta$ with $\tau \in [0, 1]$. If contrasted with the stability results in [174, 150, 13] (involving global IQCs), ours only requires the a priori restriction (7.19) on the multipliers and is, most importantly, not depending on any special features of the factorization of $\Pi \sim (P, \Psi)$; in particular, there is no need to work with a J-spectral factorization, which is the essential aspect that renders our results computational. As the price to pay, we require to enforce the coupling (7.21) between the certificates for the stability FDI (7.17) and (7.19). It comes to our benefit that all conditions impose LMI constraints on P.

Proof. *Step 1.* We start by proving the following fact: There exists some $\gamma > 0$ such that for all trajectories of (7.3) with $d \in \mathscr{D}$, $w \in \mathscr{L}_{2e}$ and for all $T > 0$ one has

$$\int_0^T \frac{1}{\gamma}\|z(t)\|^2 - \gamma\|d(t)\|^2\,dt \leq -\int_0^T z_\Psi(t)^T P z_\Psi(t)\,dt - (\star)^T X^s \begin{pmatrix}\xi(T)\\ x(T)\end{pmatrix}. \tag{7.22}$$

Let us consider (7.13) for the given P and the choices $(C_2\ D_{21}\ D_{22}) := (C_1\ D_{11}\ D_{12})$, $X := X^s$, and $P_p := \operatorname{diag}(\gamma^{-1}I, -\gamma I)$ with $\gamma > 0$; then, the performance measure (7.11) implies stability as in (7.6). If we denote the left-hand side of (7.18) by K_{11}, a simple computation shows that the left-hand side of (7.13) can then be expressed as

$$\begin{pmatrix} K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix} + \begin{pmatrix} \frac{1}{\gamma}L_{11} & \frac{1}{\gamma}L_{12}\\ \frac{1}{\gamma}L_{21} & \frac{1}{\gamma}L_{22} - \gamma I\end{pmatrix}, \tag{7.23}$$

where K_{ij}, L_{ij} are matrix blocks that do not depend on γ. Since $K_{11} \prec 0$, there does exist some (large) $\gamma > 0$ such that (7.23) is negative definite. Now apply Lemma 7.8 with $x_0 = 0$. Due to our choice of the matrices in (7.13), the dissipation inequality (7.14) is identical to (7.22), which proves the claim.

Step 2. In this step we derive a bound on the integral on the right-hand side of (7.22) in terms of the solution to a suitable LQ problem. We employ similar arguments as in [150] but extend these to only locally defined uncertainties and local IQCs. For inputs $z, w \in \mathscr{L}_{2e}$ and the response z_Ψ of (7.9) let us use the abbreviation

$$F_\Psi(z,w) := z_\Psi^T P z_\Psi = \left[\Psi\begin{pmatrix} z\\ w\end{pmatrix}\right]^T P \left[\Psi\begin{pmatrix} z\\ w\end{pmatrix}\right].$$

The assumption $\Delta \in \mathrm{IQC}_{\mathscr{Z}}(P,\Psi)$ then translates into

$$\int_0^\infty F_\Psi(z,\Delta(z))(t)\,dt \geq 0 \quad \text{for all} \quad z \in \mathscr{Z}, \tag{7.24}$$

and we immediately extract the trivial lower bound

$$\int_0^T F_\Psi(z, \Delta(z))(t)\, dt \geq -\int_T^\infty F_\Psi(z, \Delta(z))(t)\, dt \tag{7.25}$$

for the finite horizon integral in (7.22) and all $T > 0$, $z \in \mathscr{Z}$.

Note, however, that this inequality cannot be directly applied for the response of (7.2) to $d \in \mathscr{D}$, since well-posedness merely guarantees that the truncated signal z_T (and not $z \in \mathscr{Z}_e$ itself) is contained in $\mathscr{Z}$. As a remedy, we construct a finite energy signal $\widetilde{z} \in \mathscr{Z}$ that coincides with $z \in \mathscr{Z}_e$ on $[0, T]$, but not necessarily on (T, ∞). Then we use the resulting freedom in order to arrive at a lower bound in (7.25) that can be related to Y_{22}.

To do so, fix any $d \in \mathscr{D}$ and the resulting response z of (7.2) as well as any $T > 0$. We concatenate $z_T|_{[0,T]}$ with another signal $z_f \in \mathscr{L}_2(T, \infty)$ in such a way that the concatenation $\widetilde{z}$, defined as

$$\widetilde{z}(t) := (z_T \underset{T}{\wedge} z_f)(t) := \begin{cases} z_T(t), & t \in [0, T], \\ z_f(t), & t > T, \end{cases} \tag{7.26}$$

is contained in $\mathscr{Z}$. This is always possible since $z_f = 0|_{(T,\infty)}$ is a valid choice due to $(\mathscr{Z}_e)_T \subset \mathscr{Z}$. Let $\widetilde{w} := \Delta(\widetilde{z})$. Causality of Δ then implies

$$\widetilde{w}_T = \Delta(\widetilde{z})_T = \Delta(\widetilde{z}_T)_T = \Delta(z_T)_T = \Delta(z)_T = w_T.$$

Hence the signals $\widetilde{z}, \widetilde{w}$ coincide with the actual system trajectories $z, w = \Delta(z)$ on $[0, T]$ and we infer from (7.24) and (7.25) applied to $\widetilde{z} \in \mathscr{Z}$ that

$$\int_0^T F_\Psi(z, \Delta(z))(t)\, dt \geq -\int_T^\infty F_\Psi(\widetilde{z}, \Delta(\widetilde{z}))(t)\, dt. \tag{7.27}$$

We can tightly bound the left hand side from below as

$$\int_0^T F_\Psi(z, \Delta(z))(t)\, dt \geq J(\xi(T)) \tag{7.28}$$

if we define $J(\xi(T))$ as the value of the optimization problem

$$J(\xi(T)) := \sup_{\substack{z_f \in \mathscr{L}_2(T,\infty) \\ \widetilde{z} \in \mathscr{Z}}} -\int_T^\infty F_\Psi(\widetilde{z}, \widetilde{w})(t)\, dt$$

with $\widetilde{z}$ as in (7.26), $\widetilde{w} = \Delta(\widetilde{z})$ and subject to

$$\begin{aligned} \dot{\xi} &= A_\Psi \xi + B_{\Psi,1} \widetilde{z} + B_{\Psi,2} \widetilde{w}, \quad \xi(0) = 0, \\ \widetilde{z}_\Psi &= C_\Psi \xi + D_{\Psi,1} \widetilde{z} + D_{\Psi,2} \widetilde{w}. \end{aligned} \tag{7.29}$$

Since $\widetilde{z}, \widetilde{w}$ coincide with z, w on $[0, T]$, respectively, the state $\xi(T)$ to which $\xi(.)$ has evolved at time T is independent of $z_f \in \mathscr{L}_2(T,\infty)$. Furthermore, due to $\widetilde{z}|_{(T,\infty)} = z_f$, the response $\widetilde{z}_\Psi$ of (7.29) also coincides on (T, ∞) with the one of

$$\begin{aligned} \dot{\widetilde{\xi}} &= A_\Psi \widetilde{\xi} + B_{\Psi,1} z_f + B_{\Psi,2} w_f, \quad \widetilde{\xi}(T) = \xi(T), \\ \widetilde{z}_\Psi &= C_\Psi \widetilde{\xi} + D_{\Psi,1} z_f + D_{\Psi,2} w_f \end{aligned} \tag{7.30}$$

in case that $w_f = \tilde{w}|_{(T,\infty)}$. Due to $\widetilde{z} \in \mathscr{Z}$ and by (7.5) we infer $\widetilde{w} = \Delta(\widetilde{z}) \in \mathscr{L}_2$ and thus $\tilde{w}|_{(T,\infty)} \in \mathscr{L}_2(T,\infty)$. This implies

$$-\int_T^\infty F_\Psi(\widetilde{z}, \widetilde{w})(t)\, dt \geq \inf_{w_f \in \mathscr{L}_2(T,\infty)} -\int_T^\infty \tilde{z}_\Psi(t)^T P \tilde{z}_\Psi(t)\, dt$$

under the constraints (7.30). Always subject to (7.30) we conclude

$$\begin{aligned} &\sup_{\substack{z_f \in \mathscr{L}_2(T,\infty) \\ \widetilde{z} \in \mathscr{Z}}} -\int_T^\infty F_\Psi(\widetilde{z}, \widetilde{w})(t)\, dt \\ &\geq \sup_{\substack{z_f \in \mathscr{L}_2(T,\infty) \\ \widetilde{z} \in \mathscr{Z}}} \inf_{w_f \in \mathscr{L}_2(T,\infty)} -\int_T^\infty \tilde{z}_\Psi(t)^T P \tilde{z}_\Psi(t)\, dt && (7.31) \\ &\geq \sup_{z_f = 0} \inf_{w_f \in \mathscr{L}_2(T,\infty)} -\int_T^\infty \tilde{z}_\Psi(t)^T P \tilde{z}_\Psi(t)\, dt. && (7.32) \end{aligned}$$

Clearly (7.30), (7.32) describes a standard LQ problem on (T, ∞); it is worth noting that the final estimate completely eliminates the dependence of this problem on the set $\mathscr{X}$.

Step 3. If $\tilde{J}(\xi(T))$ denotes the value of

$$\inf_{w \in \mathscr{L}_2} \int_0^\infty y(t)^T(-P)y(t)\,dt \tag{7.33}$$

subject to the dynamics

$$\begin{aligned} \dot{x} &= A_\Psi x + B_{\Psi,2} w, \quad x(0) = \xi(T), \\ y &= C_\Psi x + D_{\Psi,2} w, \end{aligned} \tag{7.34}$$

we hence infer from Step 2. that $J(\xi(T)) \geq \tilde{J}(\xi(T))$. With

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} := \begin{pmatrix} C_\Psi & D_{\Psi,2} \end{pmatrix}^T (-P) \begin{pmatrix} C_\Psi & D_{\Psi,2} \end{pmatrix}$$

the ARE corresponding to the LQ problem (7.33), (7.34) reads according to [180, 116] as

$$A_\Psi^T Z_{22} + Z_{22} A_\Psi + Q - (Z_{22} B_{\Psi,2} + S) R^{-1} (\star)^T = 0. \tag{7.35}$$

We now exploit $-(7.20)$ in order to infer $R \succ 0$ and, by taking the Schur complement with respect to this block, that $-Y_{22}$ satisfies the corresponding strict algebraic Riccati inequality

$$-A_\Psi^T Y_{22} - Y_{22} A_\Psi + Q - (-Y_{22} B_{\Psi,2} + S) R^{-1} (\star)^T \succ 0.$$

Standard results from Riccati theory (as formulated, e.g., in [145, Theorem 2.23]) then show, on the one hand, that the stabilizing solution $Z_{22,-}$ of (7.35) does exist and, on the other hand, that it satisfies $Z_{22,-} \succ -Y_{22}$. Furthermore, we infer that the optimal value $\tilde{J}(\xi(T))$ equals $\xi(T)^T Z_- \xi(T)$ and can, hence, be estimated as $\tilde{J}(\xi(T)) \geq \xi(T)^T(-Y_{22})\xi(T)$. In summary, we have shown

$$J(\xi(T)) \geq \xi(T)^T(-Y_{22})\xi(T). \tag{7.36}$$

Step 4. For any $d \in \mathscr{D}$ and the responses z, w of (7.2) driving (7.9), we can combine (7.28) with (7.36) to conclude

$$\int_0^T z_\Psi(t)^T P z_\Psi(t)\, dt \geq \xi(T)^T(-Y_{22})\xi(T) \tag{7.37}$$

for all $T > 0$. In combination with (7.22) we end up with

$$\frac{1}{\gamma}\int_0^T \|z(t)\|^2\, dt \leq \gamma \int_0^T \|d(t)\|^2\, dt - (\star)^T \begin{pmatrix} X_{11}^s - Y_{22} & X_{12}^s \\ X_{21}^s & X_{22}^s \end{pmatrix} \begin{pmatrix} \xi(T) \\ x(T) \end{pmatrix}$$

for all $T > 0$. If finally using (7.21), we infer

$$\int_0^T \|z(t)\|^2\, dt \leq \gamma^2 \int_0^T \|d(t)\|^2\, dt \leq \gamma^2 \|d\|^2 < \infty$$

for all $T > 0$. This shows $z \in \mathscr{L}_2$ and $\|z\| \leq \gamma \|d\|$ by taking $T \to \infty$. Since $d \in \mathscr{D}$ was arbitrary, the proof is finished. □

7.4 Relation to existing local IQC results

As pointed out in the introduction, several articles have appeared in the literature that employ dynamic IQC descriptions of uncertainties for local analysis in particular settings. In order to emphasize the general applicability of our novel approach, we use this section to illustrate its connections to previous ones. Following up on the discussion in the introduction and with Theorem 7.9 at hand, we may now contrast our approach to the above cited works that also employ (dynamic) multipliers in regional analysis.

The first part of the proof of Theorem 7.9 was inspired by the derivation in [150] for $\mathscr{Z} = \mathscr{L}_2$ and multipliers satisfying

$$\Pi_{11} = \Psi_1^* P \Psi_1 \succ 0 \quad\text{and}\quad \Pi_{22} = \Psi_2^* P \Psi_2 \prec 0 \quad\text{on}\quad \mathbb{C}_0^\infty.$$

Seiler proves in [150] that the value of (7.31) subject to (7.30) equals $\xi(T)^T Y_c \xi(T)$ where Y_c is the stabilizing solution of the indefinite ARE

$$A_\Psi^T Y_c + Y_c A_\Psi + Q_c - (Y_c B_\Psi + S_c) R_c^{-1} (Y_c B_\Psi + S_c)^T = 0 \tag{7.38}$$

with

$$\begin{pmatrix} Q_c & S_c \\ S_c^T & R_c \end{pmatrix} = \left(C_\Psi \; D_\Psi\right)^T P \left(C_\Psi \; D_\Psi\right).$$

Moreover, he shows that all certificates of (7.17) satisfy

$$\begin{pmatrix} X_{11}^s - Y_c & X_{12}^s \\ X_{21}^s & X_{22}^s \end{pmatrix} \succ 0, \tag{7.39}$$

which permits to apply Lyapunov arguments to prove stability. Yet, as a major stumbling block and due to the non-convex constraint (7.38), there seems to be no possibility to incorporate Y_c in the convex search for multipliers to verify stability. On the one hand, this prevents the use of Y_c for merging IQC theory with Lyapunov techniques for computational regional stability and performance analysis as addressed in the remainder of this paper. And, on the other hand, this motivates us to work with the lower bound $-Y_c \succcurlyeq -Y_{22}$, which can be included in computations since Y_{22} is coupled to the multipliers by the convex LMI constraint (7.20). In the sequel, it will be a key additional benefit over [150] that the uncertainty IQCs are not required to be valid on the whole space $\mathscr{L}_2$.

In [153, 32] the authors work with hard IQCs and assume positivity of the certificates for the stability and performance FDIs. As discussed in Section 7.2.3, these restrictions simplify proofs considerably and provide the main motivation for the current work. Indeed, it is to be expected from global analysis (see Chapter 5) that the possibility to employ soft IQCs can substantially reduce conservativeness in local analysis as well; this will be illustrated in Examples 7.15 and 7.16 of Section 7.6.

Balakrishnan focusses in [13] on diagonally repeated LTI uncertainties Δ captured by dynamic D-scalings. Consider the special case of a single k-times repeated uncertainty, i.e., Δ defined by $\hat{w}(i\omega) = \delta(i\omega) I_k \hat{z}(i\omega)$

with $\delta \in RH_\infty$ satisfying $\|\delta\|_\infty \leq 1$. Then [13] uses the multipliers (7.8) with $\Psi := \mathrm{diag}(\psi, \psi)$ for a fixed $\psi \in RH_\infty^{\nu \times k}$ and $\nu \in \mathbb{N}$ as well as

$$P \in \mathbf{P} := \left\{ \begin{pmatrix} P_{11} & 0 \\ 0 & -P_{11} \end{pmatrix} \,\middle|\, \psi^* P_{11} \psi \succ 0 \ \text{ on } \ \mathbb{C}_0^\infty \right\}.$$

As main technical results in [13], it is shown that the feasibility of (7.18) with $P \in \mathbf{P}$ is equivalent to the feasibility of (7.18) where both P_{11} and X^s are positive definite; again this allows to infer robust stability with hard IQC arguments. However, the proofs heavily rely on the special structure of $\Psi^* \mathbf{P} \Psi$, including a classical commutation property with the uncertainties and a particular choice of ψ; this limits the applicability of [13] to a rather specific setting. We demonstrate the enhanced flexibility of our general approach for parametric uncertainties with D/G scalings in Example 7.14 of Section 7.6.

Finally, in [52] a generalized sector condition [157, 83] is employed in the verification of contractive invariance of an ellipsoid. Theorem 7.9 now justifies the conclusion in [52] that it suffices to guarantee the validity of a local (static) IQC defined through this ellipsoid in order to conclude interconnection stability.

7.5 Application to regional performance criteria

Let us now support our claim that local IQCs and the choice of bounded disturbance input sets $\mathscr{D}$ significantly widen the applicability of the IQC framework. This is done by demonstrating how to verify several regional invariance properties using dynamic IQCs with only minor modifications. The subsequent list is by no means complete, but the extension to other questions (as shown, e.g., in [13]) is pretty straightforward.

7.5.1 Invariance with general dynamic IQC multipliers

As a first application, we consider one of the most common examples for regional analysis, namely the computation of invariant sets of the state-space

$$d \in \mathscr{D}_\alpha := \{d \in \mathscr{L}_2 \mid \|d\| \leq \alpha\} \quad \text{for some fixed} \quad \alpha > 0. \tag{7.40}$$

Theorem 7.10
Suppose that the stable LTI system N and the causal $\Delta : \mathscr{Z}_e \to \mathscr{L}_{2e}$ with (7.5) are interconnected as in (7.2). Further assume that (7.2) is well-posed on $\mathscr{D}_\alpha$ and that there exists some $P = P^T$ such that

a) $\Delta \in \mathrm{IQC}_{\mathscr{Z}}(P, \Psi)$*;*

b) there exists a certificate X of the FDI (7.12) with $P_p = \mathrm{diag}(0, -I)$, a certificate Y_{22} of (7.19), and some $H = H^T$ with

$$\begin{pmatrix} H & 0 & I \\ 0 & X_{11} - Y_{22} & X_{12} \\ I & X_{12}^T & X_{22} \end{pmatrix} \succ 0. \tag{7.41}$$

Then (7.2) is stable on $\mathscr{D}_\alpha$ and, for any $d \in \mathscr{D}_\alpha$, the state trajectories $x(.)$ of (7.2) (starting at the origin) satisfy

$$x(t) \in \mathcal{E}(H^{-1}, \alpha^2) \quad \textit{for all} \quad t \geq 0. \tag{7.42}$$

For the formulation of this result it is relevant to emphasize that, due to Theorem 7.9, we are in the position to proceed exactly as is often done in Lyapunov theory. In order to verify some desired extra properties for the trajectories of the interconnection (7.2), we solely impose appropriate additional LMI constraints on the matrix in (7.21) that plays the role of defining a suitable Lyapunov function. This reasoning will also be at the heart of all subsequent results.

Proof. Stability of (7.2) on $\mathscr{D}_\alpha$ follows immediately from Theorem 7.9. To see this, we cancel the last block row/column of (7.13); due to the structure of P_p we infer that (7.18) holds for X^s replaced by X; since (7.41) for $X^s = X$ implies (7.21), all hypothesis in Theorem 7.9 are satisfied.

To show invariance we fix any $d \in \mathscr{D}_\alpha$ and consider the response of (7.2) driving (7.9). Lemma 7.8 with $P_p = \operatorname{diag}(0, -I)$ and $x_0 = 0$ leads to

$$\begin{pmatrix} \xi(T) \\ x(T) \end{pmatrix}^T X \begin{pmatrix} \xi(T) \\ x(T) \end{pmatrix} + \int_0^T z_\Psi(t)^T P z_\Psi(t)\, dt \le \int_0^T \|d(t)\|^2\, dt \quad (7.43)$$

for all $T > 0$. We then combine (7.37), as shown in the proof of Theorem 7.9, with (7.43) and $\|d\| \le \alpha$ to infer

$$\begin{pmatrix} \xi(T) \\ x(T) \end{pmatrix}^T \begin{pmatrix} X_{11} - Y_{22} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix} \begin{pmatrix} \xi(T) \\ x(T) \end{pmatrix} \le \alpha^2 \quad (7.44)$$

for all $T > 0$. By taking the Schur complement, (7.41) leads to

$$\begin{pmatrix} 0 \\ I \end{pmatrix} H^{-1} \begin{pmatrix} 0 & I \end{pmatrix} \prec \begin{pmatrix} X_{11} - Y_{22} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix}. \quad (7.45)$$

A combination of (7.44) and (7.45) then shows $x(T)^T H^{-1} x(T) \le \alpha^2$ for all $T > 0$, which proves (7.42). □

In practical applications one is often interested in bounds on the individual components z_j of the output signal z. For $j \in \{1, \ldots, k\}$ let $C_{1,j}$ denote the rows of C_1 in (7.3). Then we get the following result for the interconnection (7.2) under the further assumption that $\begin{pmatrix} M & N_{12} \end{pmatrix}$ is strictly proper.

Corollary 7.11
In addition to the assumptions of Theorem 7.10 let $D_{11} = D_{12} = 0$ and replace b) by the following hypothesis:

b') there exists a certificate X of the FDI (7.12) *with* $P_p = \operatorname{diag}(0, -I)$, *a certificate* Y_{22} *of* (7.19), *and some* $\gamma_j > 0$ *such that*

$$\begin{pmatrix} \gamma_j & 0 & C_{1,j} \\ 0 & X_{11} - Y_{22} & X_{12} \\ C_{1,j}^T & X_{12}^T & X_{22} \end{pmatrix} \succ 0 \quad \textit{for all} \quad j \in \{1, \ldots, k\}. \tag{7.46}$$

Then (7.2) *is stable on* $\mathscr{D}_\alpha$ *and, for all* $d \in \mathscr{D}_\alpha$*, the components of the response* z *of* (7.2) *are bounded as*

$$|z_j(t)| \leq \sqrt{\gamma_j}\alpha \quad \textit{for all} \quad t \geq 0,\ j \in \{1, \ldots, k\}. \tag{7.47}$$

Proof. The changes in our assumptions necessitate only minor alterations to the proof of Theorem 7.10. By the Schur complement formula and for any $j \in \{1, \ldots, k\}$, (7.46) implies

$$\begin{pmatrix} 0 \\ C_{1,j}^T \end{pmatrix} \begin{pmatrix} 0 & C_{1,j} \end{pmatrix} \prec \gamma_j \begin{pmatrix} X_{11} - Y_{22} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix}. \tag{7.48}$$

Hence, for $d \in \mathscr{D}_\alpha$ and $T > 0$ we conclude with (7.44) that

$$\begin{aligned} |z_j(T)|^2 = \begin{pmatrix} \xi(T) \\ x(T) \end{pmatrix}^T \begin{pmatrix} 0 \\ C_{1,j}^T \end{pmatrix} \begin{pmatrix} 0 & C_{1,j} \end{pmatrix} \begin{pmatrix} \xi(T) \\ x(T) \end{pmatrix} \leq \\ \leq \gamma_j \begin{pmatrix} \xi(T) \\ x(T) \end{pmatrix}^T \begin{pmatrix} X_{11} - Y_{22} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix} \begin{pmatrix} \xi(T) \\ x(T) \end{pmatrix} \leq \gamma_j \alpha^2. \end{aligned}$$

□

This result can be interpreted as providing guaranteed bounds on the energy to peak gain for the channel $d \to z$ of the uncertain interconnection (7.2) and based on soft local dynamic IQCs. A mere substitution of matrices leads to a similar result for the performance channel $d \to e$. Our focus on $d \to z$ is motivated by the next section, in which these bounds are exploited in order to improve stability tests with local IQCs.

7.5.2 Invariance using regionally valid IQCs

In the previous section we derived bounds on the states and outputs of the LTI system N in (7.2), depending on the maximal disturbance energy α. Let us now demonstrate how these bounds may be used in order to tighten the uncertainty description and thus reducing conservatism by adapting the domain of local IQCs accordingly. A similar idea also appears in the analysis of regionally stable saturated systems (see, e.g., [79, 125, 124] and the monographs [157], [83]). In order to do so, we distinguish between $\mathscr{Z}_e$, the domain of definition of the uncertainty, and subsets $V_e \subset \mathscr{Z}_e$ such that the IQC for Δ is satisfied on $V := V_e \cap \mathscr{L}_2$ only.

Specifically, we work with the amplitude bounded sets

$$V_{R,e} := \{z \in \mathscr{L}_{2e} \,|\, |z_j(t)| \leq R_j \quad \text{for all} \quad j \in \{1, \dots, k\},\ t \geq 0\} \tag{7.49}$$

and $V_R := V_{R,e} \cap \mathscr{L}_2$ parameterized by $R = (R_1, \dots, R_k) \in \mathbb{R}^k_+$.

Theorem 7.12
Suppose that the stable LTI system N with $D_{11} = D_{12} = 0$ and the causal $\Delta : \mathscr{Z}_e \to \mathscr{L}_{2e}$ with (7.5) are interconnected as in (7.2). With $R \in \mathbb{R}^k_+$ and (7.49) let $V_{R,e} \subset \mathscr{Z}_e$ and $V_R := V_{R,e} \cap \mathscr{L}_2$. Further assume that (7.2) is well-posed on $\mathscr{D}_\alpha$ and that there exists some $P = P^T$ such that

a) $\Delta \in \mathrm{IQC}_{V_R}(P, \Psi)$;

b) there exist certificates X of the FDI (7.12) with $P_p = \mathrm{diag}(0, -I)$ and Y_{22} of (7.19) such that (7.46) is valid for $\gamma_j := R_j^2/\alpha^2$ and all $j \in \{1, \dots, k\}$.

Then (7.2) is stable on $\mathscr{D}_\alpha$ and (7.47) holds for all $d \in \mathscr{D}_\alpha$.

Proof. Fix $d \in \mathscr{D}_\alpha$ and consider the response of (7.2) driving (7.9). Since $D_{11} = D_{12} = 0$, $z \in \mathscr{Z}_e$ satisfies $z(0) = 0$ and is continuous. Hence it makes sense to define

$$\overline{T} := \sup\{T > 0 \,|\, z_T \in V_e\} \in (0, \infty].$$

We then infer $|z_j(t)| \leq R_j$ for all $j \in \{1, \ldots, k\}$ and all $t \in [0, \overline{T})$, which shows $z_T \in V_R$ for all $T \in (0, \overline{T})$. For $T \in (0, \overline{T})$, this allows us to follow Steps 2. and 3. of the proof of Theorem 7.9 with $\widetilde{z} = z_T \underset{T}{\wedge} z_f$ where z_f is now chosen in such a way that $\widetilde{z} \in V_R$. We can hence exploit $\Delta \in \mathrm{IQC}_{V_R}(P, \Psi)$ and conclude as before that

$$\int_0^T z_\Psi(t)^T P z_\Psi(t)\, dt \geq \xi(T)^T(-Y_{22})\xi(T) \qquad \text{for all} \quad T \in (0, \overline{T}). \tag{7.50}$$

As in the proof of Theorem 7.10 we have (7.43) for all $T \in (0, \overline{T})$. This can be combined with (7.50) to infer (7.44) for all $T \in (0, \overline{T})$.

Let us now assume that $\overline{T} < \infty$. On the one hand, by continuity, (7.44) then also holds for $T = \overline{T}$ and with (7.48) as well as $\gamma_j = R_j^2/\alpha^2$ we infer

$$|z_j(\overline{T})|^2 < R_j^2 \quad \text{for all} \quad j \in \{1, \ldots, k\}. \tag{7.51}$$

On the other hand and again by continuity, we have $|z_j(\overline{T})| \leq R_j$ for all $j \in \{1, \ldots, k\}$ and, due to the definition of $\overline{T}$, there must exist some index $j_0 \in \{1, \ldots, k\}$ with

$$|z_{j_0}(\overline{T})| = R_{j_0}. \tag{7.52}$$

This contradiction to (7.51) allows to infer $\overline{T} = \infty$, and thus also $|z_j(t)| \leq \sqrt{\gamma_j}\alpha$ for all $t \geq 0$ and all $j \in \{1, \ldots, k\}$.

In summary, we have shown $z \in V_{R,e}$ for the response of (7.2) to any $d \in \mathscr{D}_\alpha$. We may hence restrict Δ to the smaller set $V_{R,e} \subset \mathscr{Z}_e$ while still maintaining well-posedness of the resulting interconnection (7.2) on $\mathscr{D}_\alpha$. This permits us to apply Corollary 7.11 for $\mathscr{Z}$ replaced by V_R; hence (7.2) is stable on $\mathscr{D}_\alpha$. $\square$

7.5.3 Excitation through nonzero initial conditions

As a final topic, let us consider the interconnection (7.2) without performance channel as in

$$z = Mw, \quad w = \Delta(z). \tag{7.53}$$

The only excitation is given by the nonzero initial condition of the LTI system M with the realization

$$\begin{aligned} \dot{x} &= Ax + B_1 w, \quad x(0) = x_0 \in \mathbb{R}^n, \\ z &= C_1 x. \end{aligned} \tag{7.54}$$

As is standard [93, 58], we say that (7.53) is well-posed for all initial conditions if (7.2) is well-posed on the subspace $\mathscr{D} := \left\{ C_1 e^{A\bullet} x_0 \,\middle|\, x_0 \in \mathbb{R}^n \right\}$ with $N_{12} = I$.

Theorem 7.13

Suppose that the stable LTI system M (7.54) and the causal $\Delta : \mathscr{Z}_e \to \mathscr{L}_{2e}$ with (7.5) are interconnected as in (7.53). Let R, $V_{R,e}$ and V_R be defined as in Theorem 7.12. Further, suppose that (7.53) is well-posed for all initial conditions and that there exists some $P = P^T$ such that

a) $\Delta \in \mathrm{IQC}_{V_R}(P, \Psi)$;

b) there exist a certificate X of (7.17), a certificate Y_{22} of (7.19), and some $H = H^T$ such that (7.41) holds;

c) for all $j \in \{1, \ldots, k\}$ the matrix H also satisfies

$$\begin{pmatrix} H & HC_{1,j}^T \\ C_{1,j}H & R_j^2 \end{pmatrix} \succ 0.$$

Then the state trajectories x of (7.53), (7.54) with initial conditions $x_0 \in \mathcal{E}(X_{22}, 1)$ satisfy

$$x(t) \in \mathcal{E}(H^{-1}, 1) \quad \text{for all} \quad t \geq 0. \tag{7.55}$$

Proof. We follow the proof of Theorem 7.12 and consider the response of (7.53) to some fixed $x_0 \in \mathcal{E}(X_{22}, 1)$. Using the Schur complement, assumptions b) and c) imply (7.45) and

$$C_{1,j}^T C_{1,j} \prec R_j^2 H^{-1} \quad \text{for all} \quad j \in \{1, \ldots, k\}, \tag{7.56}$$

respectively. Since (7.56) is strict, we can choose some $\varepsilon > 0$ such that (7.45) with (7.56) imply

$$|z_j(T)|^2 \leq (R_j - \varepsilon)^2 \begin{pmatrix} \xi(T) \\ x(T) \end{pmatrix}^T \begin{pmatrix} X_{11} - Y_{22} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix} \begin{pmatrix} \xi(T) \\ x(T) \end{pmatrix} \tag{7.57}$$

for all $j \in \{1, \ldots, k\}$ and $T \geq 0$.

For $T = 0$, we infer with $\xi(0) = 0$ and $x_0^T X_{22} x_0 \leq 1$ that

$$|z_j(0)|^2 < R_j^2 \quad \text{for all} \quad j \in \{1, \ldots, k\}.$$

Thus we can define $\overline{T} \in (0, \infty]$ as in the proof of Theorem 7.12.

By Lemma 7.7 and 7.8 applied to (7.17), the dissipation inequality

$$\begin{pmatrix} \xi(T) \\ x(T) \end{pmatrix}^T X \begin{pmatrix} \xi(T) \\ x(T) \end{pmatrix} + \int_0^T z_\Psi(t)^T P z_\Psi(t)\, dt \leq x_0^T X_{22} x_0$$

holds for all $T > 0$. In complete analogy to previous arguments, we first obtain (7.50) and, with $x_0^T X_{22} x_0 \leq 1$, also

$$\begin{pmatrix} \xi(T) \\ x(T) \end{pmatrix}^T \begin{pmatrix} X_{11} - Y_{22} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix} \begin{pmatrix} \xi(T) \\ x(T) \end{pmatrix} \leq 1 \quad \text{for all} \quad T \in (0, \overline{T}). \tag{7.58}$$

In combination with (7.57) we finally arrive at

$$|z_j(T)|^2 \leq (R_j - \varepsilon)^2 \quad \text{for all} \quad j \in \{1, \ldots, k\},\ T \in (0, \overline{T}). \tag{7.59}$$

Now suppose $\overline{T} < \infty$. Since z is continuous, we can argue again that there exists some $j_0 \in \{1, \ldots, k\}$ with (7.52), while (7.59) shows $|z_{j_0}(\overline{T})| < R_{j_0}$, a contradiction.

Consequently, $\overline{T} = \infty$ and $z \in V_{R,e}$. Moreover, (7.58) and (7.45) clearly show (7.55). Finally, $z \in \mathscr{L}_2$ is proven as for Theorem 7.12. $\square$

7.6 A selection of concrete applications

The great practical relevance of the stability and performance analysis problems discussed in the previous sections has sparked the development of many specialized approaches for in itself relevant subproblems. In this section we illustrate the benefits of our novel framework, in that it is not only generally applicable to a large variety of such specializations but even leads to often much less conservative results if compared to several recently developed alternative techniques.

7.6.1 Real parametric uncertainties

Let us start by considering real parametric uncertainties where Δ is defined as $\Delta(z) = \delta I_k z$ for $z \in \mathscr{L}_{2e}^k$ and with $\delta \in \mathbb{R}$ satisfying $|\delta| \leq \kappa$. IQCs for such uncertainties may be described by dynamic D or D/G scalings as discussed in detail, e.g., in [176, Sec. 5.3.1]. Dynamic D scalings were already employed in [13] in order to perform regional analysis. As one of the main motivation for the current paper, we emphasize that the techniques developed in [13] do not extend to other multiplier classes for parametric uncertainties or to general IQCs. Let us hence demonstrate that our approach opens the way for regional stability analysis with D/G scalings, thus allowing for significant reduction of conservatism.

Example 7.14.
Specifically, consider the interconnection (7.2) with $k = 1$ where N is realized by (A, B, C, D) given as

$$A = \begin{pmatrix} -2 & -1 & -1 \\ 1 & 0 & 0.1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ 0 & -0.1 \\ -0.1 & 0.2 \end{pmatrix},$$
$$C = \begin{pmatrix} -1.1 & 0.5 & 0.1 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

Our goal is to determine an ellipsoid of smallest size that bounds the state of N for any $d \in \mathscr{D}_\alpha$ using Theorem 7.10. As a measure for the size of the ellipsoid in (7.42), $\mathcal{E}(H^{-1}, \alpha^2)$, we choose the trace of H.

With a vector of stable basis functions $\psi_\nu \in RH_\infty^{\nu+1}$ of McMillan degree $\nu \geq 0$ and free matrix variables $P_{11} = P_{11}^T$, P_{12} consider the multiplier

$$\Pi_{DG} := \begin{pmatrix} \kappa\psi_\nu & 0 \\ 0 & \psi_\nu \end{pmatrix}^* \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^T & -P_{11} \end{pmatrix} \begin{pmatrix} \kappa\psi_\nu & 0 \\ 0 & \psi_\nu \end{pmatrix}. \tag{7.60}$$

If P_{11} satisfies $\psi_\nu^* P_{11} \psi_\nu \succ 0$ and P_{12} is skew symmetric, one easily checks $\Delta \in \mathrm{IQC}_{\mathscr{L}_2}(\Pi_{DG})$ [176]. We obtain the conditions for D-scalings, as derived in [13], by setting $P_{12} = 0$, assuming $P_{11} \succ 0$, and dropping Y_{22} in the LMI (7.41). In order to compare our results to [13], we also choose ψ_ν as

$$\psi_\nu(s) = \begin{pmatrix} 1 & \frac{1}{(s-p)} & \cdots & \frac{1}{(s-p)^\nu} \end{pmatrix}^T \quad \text{with} \quad p < 0, \tag{7.61}$$

and consider the two parameter bounds $\kappa = 0.4$ and $\kappa = 0.8$.

The computed values for the optimal trace of H as achieved with the pole location $p = -1$ and with different basis lengths ν are shown in Table 7.1 (for $\kappa = 0.4$) and Table 7.2 (for $\kappa = 0.8$).

If $\nu = 0$, the skew symmetric matrix $P_{12} \in \mathbb{R}$ vanishes and, as expected, both multiplier classes lead to the same sizes of the invariant ellipsoid. By contrast, for increasing McMillan degrees of the basis functions, the use of Π_{DG} instead of Π_D leads to much tighter ellipsoidal bounds as seen in Table 7.1.

We may further improve these results as follows. Using the optimal multiplier $\Pi_{\mathrm{opt}} \sim (P_{\mathrm{opt}}, \Psi)$ as obtained from the optimization conducted in the second row of Table 7.1, we compute the shift Y_c by solving the ARE (7.38). By fixing $P = P_{\mathrm{opt}}$ and $Y_{22} = Y_c$ and again optimizing trace(H) now subject to X and H, we achieve the slightly improved values of trace(H) as stated in the last row of Table 7.1. Note that this two step procedure will always lead to the same or improved results due to the relation $-Y_c \succcurlyeq -Y_{22}$.

If we increase the parameter bound to $\kappa = 0.8$, the LMIs from [13] involving Π_D are infeasible for values of ν up to 8, while the use of Π_{DG} leads to the ellipsoidal bounds as shown in the second row of Table 7.2.

Table 7.1: Bounds on trace(H) for $\kappa = 0.4$

ν	0	1	2	4	8
Π_D [13]	285	265.2	235.2	234.9	234.9
Π_{DG}	285	15.1	14.1	14.03	14.02
Π_{DG}, improved	285	13.77	13.35	13.39	13.38

Table 7.2: Bounds on trace(H) for $\kappa = 0.8$

ν	0	1	2	4	8
Π_{DG}	∞	32.7	27.1	26.1	25.9
Π_{DG}, improved	∞	25.6	24.0	23.9	24.0

Moreover, applying the two step procedure discussed above, we infer the improved estimates stated in the last row. These results nicely illustrate the benefit of extra dynamics in the D/G scalings (increased lengths ν) to improve the bounds; we emphasize again that no other multiplier-based technique in the literature is able to provide these guarantees.

$\star$

7.6.2 Locally stable saturated systems

As one of the major driving forces behind regional stability analysis, we now consider open loop unstable systems with a stabilizing saturated unity output feedback controller. Specifically, let the interconnection be given by

$$\begin{aligned} &\dot{x} = Ax + B_1 w + B_2 d, \quad x(0) = 0, \quad w = \operatorname{sat}(z), \\ &z = C_1 x \end{aligned} \tag{7.62}$$

with the unit saturation function $\mathrm{sat} : \mathbb{R} \to \mathbb{R}$. Here we assume that $A + B_1C_1$ is Hurwitz (but not necessarily so is A). With the standard dead-zone nonlinearity $\mathrm{dz} = \mathrm{id} - \mathrm{sat}$, (7.62) may be equivalently expressed as

$$\begin{aligned} &\dot{x} = (A + B_1C_1)x - B_1w + B_2d, \quad x(0) = 0, \qquad w = \mathrm{dz}(z), \\ &z = C_1x. \end{aligned} \tag{7.63}$$

Let us briefly recap from [79] how, adopted to our framework, one may capture local properties of the dead-zone nonlinearity in order to exploit the classical circle criterion for regional stability analysis. Fix some $R \geq 1$. Then $\mathrm{dz}(.)$ satisfies the local sector constraint

$$\mathrm{dz}(x)((1 - 1/R)x - \mathrm{dz}(x)) \geq 0 \quad \text{for all} \quad x \in [-R, R];$$

in short $\mathrm{dz} \in \mathrm{sec}_R[0, 1 - 1/R]$. With

$$\mathbf{P} := \left\{ \lambda \begin{pmatrix} 0 & 1 - \frac{1}{R} \\ 1 - \frac{1}{R} & -2 \end{pmatrix} : \lambda > 0 \right\}, \quad \Psi := I_2,$$

this implies for any $P \in \mathbf{P}$ that

$$\begin{pmatrix} x \\ \mathrm{dz}(x) \end{pmatrix}^T \Psi^* P \Psi \begin{pmatrix} x \\ \mathrm{dz}(x) \end{pmatrix} \geq 0 \quad \text{for} \quad x \in [-R, R].$$

With (7.49) ($k = 1$) and $V_R := V_{R,e} \cap \mathscr{L}_2$ we immediately infer

$$\mathrm{dz} \in \mathrm{IQC}_{V_R}(P, \Psi) \quad \text{for all} \quad P \in \mathbf{P}.$$

This puts us in the position to apply Theorem 7.12 to (7.63) for $\mathscr{Z}_e := \mathscr{L}_{2e}$ and draw the following conclusion: If there exists some $P \in \mathbf{P}$ such the assumption b) is valid (which amounts to solving an LMI feasibility problem) then (7.63) is stable on $\mathscr{D}_\alpha$; with an LMI optimization problem one can directly determine the supremal value of $\alpha > 0$ for which this is true. Note that α depends on the chosen $R \geq 1$, and either a plot of α over R or a line-search finally allows us to compute the largest value

$\alpha > 0$ such that interconnection stability on $\mathscr{D}_\alpha$ is guaranteed by the chosen class of multipliers.

All this has been proposed in the literature and extended to multivariable saturations with diagonal multipliers for the circle criterion. If only using static multipliers, we point out that Y_{22} is an empty matrix and, thus, our approach recovers these results as special cases.

In global stability analysis, the benefit of, e.g., Zames-Falb multipliers has been often emphasized in the literature. This is also visible from our detailed exposition of multiplier implementations based on (7.61) in Chapter 5.

Our new approach now opens the way to exploit this superiority for regional analysis with ease. We just employ *any* valid class of multipliers for the dead-zone nonlinearity and apply Theorem 7.12 in exactly the same fashion as for the circle criterion. It is as well possible to exploit the extra information that dz is odd and its slope is confined to $[0, 1]$ with soft IQCs, for which the introduction of Y_{22} is instrumental.

Let us now compare our results to the ones achieved by Lyapunov function techniques in [51] which are specifically designed to deal with the saturation or dead-zone. In contrast to our approach that extends [79] to dynamic IQCs, the method proposed by Fang et al. [51] relies on a reformulation of the dead-zone nonlinearity as a time dependent parametric uncertainty and does not involve any line search.

Example 7.15.
Let the system in (7.62) be defined by

$$A = \begin{pmatrix} 0.05 & 1 & 2 \\ 0 & -0.4 & -2 \\ 0 & 1 & -0.7 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 12 \\ -0.2 \\ -1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.2 \\ -0.1 \\ 0.5 \end{pmatrix},$$

and $C_1 = \begin{pmatrix} -.1 & -1.5 & -1 \end{pmatrix}$. Then A is obviously not Hurwitz but $A + B_1 C_1$ is. Hence we can locally analyze the interconnection in (7.63). If $R \in [0, 1]$ we note that the system operates in the so-called linear region where the output of the dead-zone is zero. This is the case for

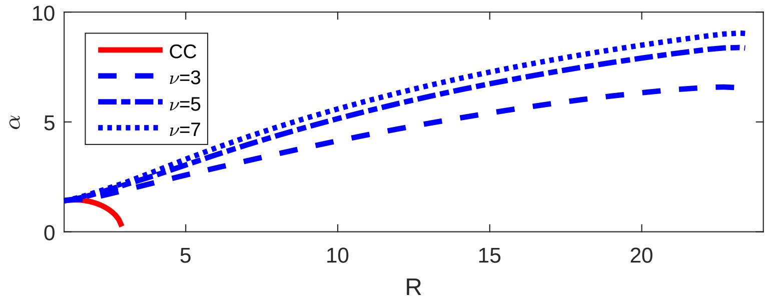

Figure 7.2: Maximal disturbance energy over R

disturbance energies smaller than $\alpha = 1.41$. The approaches in [79] and [51] permit an increase of the allowed energy level without endangering stability to $\alpha = 1.45$ and $\alpha = 2.07$, respectively.

As seen next, dynamic multipliers certify levels of $\alpha = 9.04$ which amounts to a reduction of conservativeness by more than a factor of four if compared to [51]. Figure 7.2 shows estimates of the maximal tolerable energy level α plotted versus R for the local circle criterion (CC) [79] and a combination of circle criterion and Zames-Falb multipliers for different basis lengths ν and $p = -1$ in (7.61) (as detailed in Chapter 5). We observe that the LMIs remain feasible for radii up to $R = 23.6$ and all three depicted basis lengths ν. Moreover, dynamic multipliers allow for a significant increase of the maximal tolerable disturbance energy, with improvements that get more pronounced for larger values of ν; indeed, we reach $\alpha = 9.04$ for $\nu = 7$.

Let us now augment the interconnection (7.63) with a performance output $e = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} x$ and estimate the local $\mathscr{L}_2$-gain γ from d to e for $\alpha \in [0, 9.04]$; since interconnection stability is guaranteed, we can use Theorem 7.6 with $P_p = \mathrm{diag}(I, -\gamma^2 I)$ for computations. As the approach by Fang et al. in [51] outperforms the local circle criterion for

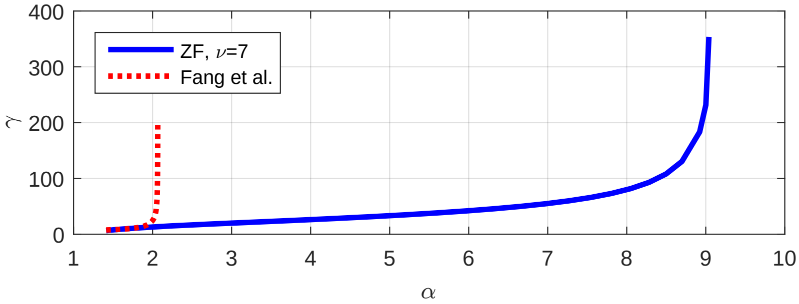

Figure 7.3: Local $\mathscr{L}_2$-gain γ plotted over disturbance energy bound α.

the present example, we only compare our method for $\nu = 7$, $p = -1$ to theirs. Figure 7.3 depicts the computed values for $\alpha \geq 1.41$. For energy bounds close to $\alpha = 1.41$, both methods return the $\mathscr{L}_2$-gain of the underlying linear system. If α approaches the maximal tolerable values for the the respective procedure, the estimated $\mathscr{L}_2$-gains tend to infinity; with the technique in [51] this happens for considerably lower values of α.

Thus we conclude that dynamic multipliers allow for a much more accurate description of the nonlinear effects of the dead-zone, which permits the system to enter further into the nonlinear regime as driven by higher disturbance energies and without loosing stability. In addition, this also translates into much less conservative $\mathscr{L}_2$-gain estimates. The price to pay, however, is higher computational complexity. This is illustrated in Table 7.3 which displays the number of decision variables and the computation times of our implementation (not optimized for computational performance) for a fixed value of R, together with the maximal disturbance energy α obtained for different basis lengths. Although Zames-Falb multipliers for $\nu = 3$ significantly outperform the circle criterion ones, the computation time is only slightly increased.

Table 7.3: Computational burden in Example 7.15

ν	0	3	5	7
Maximal disturbance energy α	1.45	6.59	8.39	9.04
Number of decision variables	10	87	195	351
Computation time [s]	0.25	0.35	0.83	3.67

For larger basis lengths ν, however, the increase in computational effort intensifies. $\star$

7.6.3 Unbounded nonlinearities

In our last example, let us consider unbounded nonlinearities as discussed in [153], but for hard IQCs only. It is well worth to illustrate how unbounded nonlinearities may be easily incorporated into our soft multiplier framework. Consider (7.2) with $\Delta_\varphi : \mathscr{L}_{2e} \to \mathscr{L}_{2e}$ defined by $\varphi(x) = x^3$ as

$$\Delta_\varphi(z)(t) = \varphi(z(t)) \quad \text{for almost all} \quad t \geq 0 \tag{7.64}$$

and $z \in \mathscr{L}_{2e}$. If the response of the loop is guaranteed to satisfy $z \in V_{R,e}$ with $V_{R,e}$ from (7.49), we may modify $\varphi(x) = x^3$ to

$$\widetilde{\varphi}(x) := \begin{cases} \varphi(x), & |x| \leq R, \\ \operatorname{sgn}(x) R^3, & |x| > R, \end{cases}$$

and in turn replace Δ_φ by $\Delta_{\widetilde{\varphi}}$ while still maintaining the same loop dynamics. Moreover, $\Delta_{\widetilde{\varphi}}$ is globally bounded on $\mathscr{L}_{2e}$ and $\widetilde{\varphi}$ satisfies the local sector constraint $\widetilde{\varphi} \in \sec_R[0, R^2]$, while its slope is locally restricted by $3R^2$. All this permits to perform a regional stability and performance analysis on the basis of soft local IQCs on V_R exactly as in the previous paragraph.

Before we consider a concrete example, let us briefly point out a special feature of nonlinearities of the form (7.64). If $\theta \in \mathbb{R}_+$ then $\theta^3\varphi(\theta^{-1}x) = \varphi(x)$ for all $x \in \mathbb{R}$. Thus, for $\theta = (\theta_1, \ldots, \theta_k)^T \in \mathbb{R}_+^k$ and $T_\theta : \mathscr{L}_{2e} \to \mathscr{L}_{2e}$ defined by $(T_\theta z)(t) = \mathrm{diag}(\theta_1, \ldots, \theta_k)z(t)$ for $t \in [0, \infty)$ we infer

$$T_{\theta^3}\Delta_\varphi T_{\theta^{-1}} = \Delta_\varphi \quad \text{on} \quad \mathscr{L}_{2e}. \tag{7.65}$$

Hence, for all $\theta \in \mathbb{R}_+^k$, the interconnection (7.2) with $\Delta = \Delta_\varphi$ and $d \in \mathscr{D}_\alpha$ remains invariant if we shift the transformation in (7.65) to the LTI system, i.e., consider $N \to N_\theta$ with

$$N_\theta = \begin{pmatrix} T_{\theta^{-1}} & 0 \\ 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} M & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} T_{\theta^3} & 0 \\ 0 & \mathrm{id} \end{pmatrix}. \tag{7.66}$$

In the sequel we apply our results to this family of interconnections by optimizing over $\theta \in \mathbb{R}_+^k$. Note that, due to the symmetry of φ and $\mathscr{D}_\alpha$, it suffices to consider positive θ_j's only.

Example 7.16.
Let us adopt an example from [153], where the interconnection depicted in Figure 7.4 is studied. Here $\Gamma = -1.05$ is a static gain and each subsystem S is given by the upper feedback interconnection of the nonlinearity Δ_φ in (7.64) with the LTI system $\widetilde{N} \sim (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$ for

$$\widetilde{A} = -1, \quad \widetilde{B} = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad \widetilde{C} = \widetilde{B}^T, \quad \widetilde{D} = 0.$$

It is straightforward to redescribe the interconnection of Figure 7.4 as in (7.2) where Δ is defined through φ or $\widetilde{\varphi}$ being 3-times repeated. The goal is to estimate the maximal disturbance energy bound $\alpha > 0$ such that the interconnection in Figure 7.4 remains stable for all $d \in \mathscr{D}_\alpha$ and, subsequently, to compute the $\mathscr{L}_2$-gain from d to e.

In [153] two different dissipation approaches are compared. First, novel hard IQCs are used to regionally capture the effect of Δ_φ in (7.64). Second, sum-of-squares (SOS) algorithms from [156] with polynomial

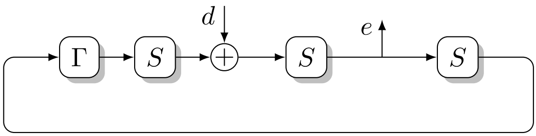

Figure 7.4: Feedback interconnection in [153]

storage functions of different degrees are applied. The first three graphs in Figure 7.5 are readily estimated from a plot in [153]. As can be seen, the maximal disturbance energies as well as the corresponding $\mathscr{L}_2$-gains obtained using hard IQCs are significantly more conservative than those obtained using polynomial storage functions of degree two (SOS (2)) or four (SOS (4)), respectively. Yet, as emphasized in [153], one should note that SOS techniques are computationally much more expensive than those with IQCs.

We apply Theorem 7.12 to the interconnection (7.2) with N replaced by N_θ, and for fixed $R_j = 0.5$ for all j using a gridding approach over θ. The search over θ rather than R is equivalent in terms of computational burden but offers the additional advantage that the three nonlinearities $\widetilde{\varphi}$ remain identical, as they are truncated at the same value of R. This enables the application of the larger class of full-block multipliers for the Zames-Falb criterion [58]. Our goal is to minimize the linear functional $\sum_{j=1}^{3} \gamma_j / R^2$ using full-block circle and Zames-Falb multipliers with pole $p = -3$ and $\nu = 1$; if the optimal value is v_{opt}, the estimated energy level then equals $1/\sqrt{v_{\mathrm{opt}}}$.

In the dashed plot in Figure 7.5 we only show the results obtained using the local (full-block) circle criterion, as Zames-Falb multipliers offer no additional improvement for the present case. Yet, already the application of the static multipliers outperforms the dynamic hard IQC multipliers proposed in [153] both in terms of maximal disturbance energy and local $\mathscr{L}_2$-gain. For a wide range of disturbance energies, our IQC approach also improves on the $\mathscr{L}_2$-gain estimates obtained using SOS techniques. However, the derived estimates of the maximal

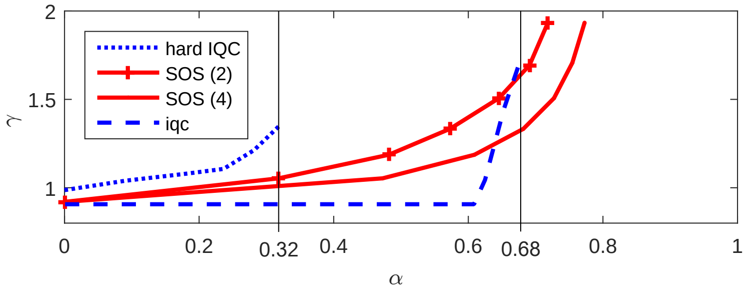

Figure 7.5: Local $\mathscr{L}_2$-gain γ plotted over disturbance energy bound α.

disturbance energy for which the interconnection remains stable are more conservative.

The lack of improvement if using dynamic multipliers can probably be attributed to the overly simple dynamics of $\widetilde{N}$. Let us hence modify $\widetilde{N}$ to

$$\widetilde{A} = \begin{pmatrix} -3.5 & -6 \\ 0.1 & -1 \end{pmatrix}, \; \widetilde{B} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}, \; \widetilde{C} = \begin{pmatrix} 0.4 & 0.6 \\ 0.7 & 0.1 \end{pmatrix}$$

and $\widetilde{D} = 0$. Even though this is a minor change, we now computed $\alpha = 22.3$ for the circle criterion and more than $\alpha = 10^5$ if including a full-block Zames-Falb multiplier (with $\nu = 1$, $p = -3$). If only using diagonally repeated Zames-Falb multipliers the obtained bounds on the maximal energy drop by about a factor of four. Moreover, note that for moderate basis lengths ν, the IQC analysis is computationally very cheap (requiring only 0.8 s for their solution with fixed R_j) while even for such academic examples with a small number of states, the SOS analysis quickly becomes intractable. $\star$

7.7 Summary and recommendations

We present a new local stability theorem that allows for uncertainty descriptions with general soft dynamic integral quadratic constraints. Based on this result, we develop a framework that extends classical IQC theory and allows to merge Lyapunov techniques with multiplier approaches. This enables the verification of several performance specifications, such as invariance of sets in the state and output space, just by imposing additional LMI constraints. As our novel approach allows for a very broad class of IQC multipliers, the theory developed is immediately applicable to a large variety of different problems. This fact is illustrated by means of various examples and comparisons with related techniques.

Yet, we only touch upon the wealth of possibilities for local analysis offered by our local IQC stability result. Many further results, similar to the ones listed in [13] can be derived in exactly the same fashion as outlined in the present chapter, namely by imposing additional LMI constraints on the Lyapunov matrix. In addition, and beyond the analysis of feedback interconnections with LTI systems in the forward path, we believe that our results allow for straightforward extensions to linear parameter varying (LPV) systems, which generalizes [178] to regional analysis with local IQCs. Furthermore, it is expected that our analysis approach provides the foundation for controller synthesis as in [149, 175].

However, there are also some obvious limitations of the present approach. First, we require an additional definiteness property from the multiplier ($\Pi_{22} \prec 0$ on $\mathbb{C}_0^\infty$) if compared to standard IQC theory that is often a priori satisfied but also violated in certain interesting cases. Furthermore, and most importantly, we have to artificially enforce positivity of the Lyapunov matrix $X - \operatorname{diag}(Y_{22}, 0)$. This will typically result in added conservatism. By contrast, the approach proposed in [13] allows to work with $Y_{22} = 0$ while even guaranteeing positivity of the certificate X. In the subsequent chapter, we illustrate how to avoid these limitations of the present approach for some concrete example resulting from multipliers corresponding to slope-restricted nonlinearities.

Chapter 8

Hard Zames-Falb factorizations for invariance

8.1 Introduction

In Chapter 7 we developed a rather general extension of classical IQC theory towards local analysis, i.e., the analysis of locally stable systems or the verification of state and output constraints under the assumption that the input signals are restricted to a certain (bounded) subset of $\mathscr{L}_2$. In the sequel, we recapitulate the setting and main stability theorem of Chapter 7 in order to highlight several limitations that sparked the development of the present, more general approach.

Given an uncertain operator $\Delta : \mathscr{Z}_e \to \mathscr{L}_{2e}$, defined on a set $\mathscr{Z}_e$ satisfying $(\mathscr{Z}_e)_T \subset \mathscr{Z}$, and a stable LTI system $N = \left(\begin{smallmatrix} M & N_{12} \\ N_{21} & N_{22} \end{smallmatrix}\right)$, realized as

$$
\begin{aligned}
\dot{x} &= Ax + B_1 w + B_2 d, \qquad\qquad x(0) = 0, \\
z &= C_1 x + D_{11} w + D_{12} d, \\
e &= C_2 x + D_{21} w + D_{22} d
\end{aligned}
\tag{8.1}
$$

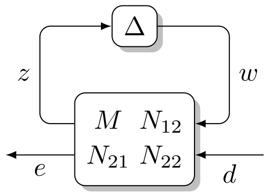

Figure 8.1: Performance setting

with $A \in \mathbb{R}^{n\times n}$ being Hurwitz, we consider the standard performance interconnection (see Figure 8.1)

$$z = Mw + N_{12}d \qquad w = \Delta(z), \tag{8.2a}$$

$$e = N_{21}w + N_{22}d. \tag{8.2b}$$

Here, the disturbance d is assumed to be contained in some subset $\mathscr{D}$ of $\mathscr{L}_2$ and the uncertainty is required to satisfy the weak boundedness condition $\Delta(\mathscr{Z}) \subset \mathscr{L}_2$. Well-posedness of (8.2a) on $\mathscr{D}$ is then simply defined by the existence of a unique response $z \in \mathscr{Z}_e$ for each $d \in \mathscr{D}$ such that the map $d \mapsto z$ is causal.

One of the key contributions of Chapter 7 was the transformation of general soft IQCs into hard ones in such a way that the residual convexly depends on the multiplier data. In particular, let $\Pi \in RL_\infty$ be factorized as $\Psi^* P \Psi$ with real symmetric P and $\Psi \in RH_\infty$ (see Section 2.3). Moreover, let Π_{ij} be structured according to the dimensions of the signals z, w and satisfy the FDI

$$\Pi_{22} \prec 0 \quad \text{on} \quad \mathbb{C}_0^\infty. \tag{8.3}$$

Then, $\Delta \in \mathrm{IQC}_{\mathscr{Z}}(P,\Psi)$, i.e., Δ satisfying the soft IQC

$$\int_0^\infty \left[\Psi\begin{pmatrix} z \\ \Delta(z)\end{pmatrix}(t)\right]^T P\Psi\begin{pmatrix} z \\ \Delta(z)\end{pmatrix}(t)\,dt \geq 0 \quad \text{for all} \quad z \in \mathscr{Z} \tag{8.4}$$

implies the shifted hard IQC

$$\int_0^T (\star)^T P\Psi \begin{pmatrix} z \\ \Delta(z) \end{pmatrix}(t)\, dt \geq \xi(T)^T(-Y_{22})\xi(T) \quad \text{for all} \quad z \in \mathscr{Z}, \tag{8.5}$$

where Y_{22} is a certificate for (8.3). We can now use this bound together with the FDI

$$\begin{pmatrix} M \\ I \end{pmatrix}^* \Psi^* P \Psi \begin{pmatrix} M \\ I \end{pmatrix} \prec 0 \quad \text{on} \quad \mathbb{C}_0^\infty \tag{8.6}$$

in order to arrive at a local IQC stability result using general dynamic multipliers. For completeness, we restate Theorem 7.9 below.

Theorem 8.1
Suppose that the interconnection (8.2) *of the stable LTI system N and the causal uncertainty Δ, satisfying* (7.5)*, is well-posed on $\mathscr{D}$. Then* (8.2) *is stable on $\mathscr{D}$ if there exists some $P = P^T$ with the following two properties*

a) $\Delta \in \mathrm{IQC}_{\mathscr{Z}}(P, \Psi)$;

b) there exists a certificate X^s of (8.6) *and a certificate Y_{22} of* (8.3) *which are coupled as*

$$\begin{pmatrix} X_{11}^s - Y_{22} & X_{12}^s \\ X_{21}^s & X_{22}^s \end{pmatrix} \succ 0. \tag{8.7}$$

If compared to the classical global result, Theorem 2.6, we immediately infer that the price to pay for locality (apart form $\Pi_{22} \prec 0$ on $\mathbb{C}_0^\infty$) is the additional constraint (8.7). As Y_{22} is empty for static multipliers Π, the shifting of X^s in (8.7) is only necessary for dynamic IQCs. However, for hard IQCs the right hand side in (8.5) is zero. Hence, one would expect that general hard IQCs can be incorporated without relying on Y_{22}.

This brings us to the main motivation for the present chapter. In the first part, we illustrate how hard and soft IQCs can be effectively combined in order to avoid unnecessary conservatism introduced by the lower bound in (8.5). In the second part, we focus on the particularly interesting example of Zames-Falb multipliers that, as illustrated in Chapter 5 allow for significant improvements in global stability analysis if compared to static multipliers. By considering causal and anti-causal Zames-Falb multipliers separately, another contribution of this chapter is to prove that both can be losslessly incorporated into the framework presented in Chapter 7. In particular, for causal and anti-causal Zames-Falb multipliers the additional constraint (8.7) is a priori satisfied. Consequently, these multiplier classes admit local analysis with identical hypothesis if compared to the global case (Theorem 2.6). As supported by a concrete numerical example, this leads to significantly improved stability margins.

Finally, we emphasize again that a self-contained version of the results developed in the present chapter is accepted for publication [59].

8.2 Local analysis with hard and soft IQCs

We lay the foundation for the combination of hard and soft IQCs by immediately turning our focus to the case where we have two available IQCs for the same uncertainty Δ that are both valid on $\mathscr{X}$ and imposed by $\Pi^{(k)} \sim (P^{(k)}, \Psi^{(k)})$ for $k = 1, 2$. Using the factorization in (7.8) for both $\Pi^{(k)}$, we obtain a structured factorization of the sum $\Pi := \Pi^{(1)} + \Pi^{(2)}$ as

$$\Pi \sim (P, \Psi) = \left(\begin{pmatrix} P^{(1)} & 0 \\ 0 & P^{(2)} \end{pmatrix}, \begin{pmatrix} \Psi^{(1)} \\ \Psi^{(2)} \end{pmatrix} \right), \tag{8.8}$$

and infer from (8.4) that $\Delta \in \mathrm{IQC}_{\mathscr{X}}(P, \Psi)$ holds. In the sequel we tacitly assume that $\Pi^{(k)} = (\Pi^{(k)}_{ij})$ and $\Psi^{(k)} = (\Psi^{(k)}_1 \ \ \Psi^{(k)}_2)$ are partitioned according to the dimensions of z and w in (8.4); moreover, $\Psi^{(k)}$ is supposed to be realized as

$$\dot{\xi}^{(k)} = A^{(k)}_{\Psi} \xi^{(k)} + B^{(k)}_{\Psi,1} z + B^{(k)}_{\Psi,2} w, \qquad \xi^{(k)}(0) = 0,$$

$$z_{\Psi}^{(k)} = C_{\Psi}^{(k)}\xi^{(k)} + D_{\Psi,1}^{(k)}z + D_{\Psi,2}^{(k)}w \tag{8.9}$$

with $A_{\Psi}^{(k)}$ being Hurwitz and $k = 1, 2$.

As will become clear in the sequel, it is indeed beneficial to distinguish in this way between multipliers defining hard (subsumed in $\Pi^{(1)}$) and soft IQCs (in $\Pi^{(2)}$). Thus, we further require the validity of the hard IQC constraint

$$\int_0^T z_{\Psi}^{(1)}(t)^T P^{(1)} z_{\Psi}^{(1)}(t)\, dt \geq 0 \quad \text{for all} \quad T > 0, \quad z \in \mathscr{X}, \tag{8.10}$$

imposed by the first multiplier $\Pi^{(1)} \sim (P^{(1)}, \Psi^{(1)})$. We abbreviate (8.10) as $\Delta \in \mathrm{HIQC}_{\mathscr{X}}(P^{(1)}, \Psi^{(1)})$ in the sequel.

In deviating from the setup in the previous chapter, we require the additional property

$$\Pi_{22}^{(2)} = (\Psi_2^{(2)})^* P^{(2)} \Psi_2^{(2)} \prec 0 \quad \text{on} \quad \mathbb{C}_0^{\infty}, \tag{8.11}$$

which is certified by the LMI

$$\begin{pmatrix} (A_{\Psi}^{(2)})^T Y_{22}^{(2)} + Y_{22}^{(2)} A_{\Psi}^{(2)} & Y_{22}^{(2)} B_{\Psi,2}^{(2)} \\ (B_{\Psi,2}^{(2)})^T Y_{22}^{(2)} & 0 \end{pmatrix} + (\star)^T P^{(2)} \begin{pmatrix} C_{\Psi}^{(2)} & D_{\Psi,2}^{(2)} \end{pmatrix} \prec 0, \tag{8.12}$$

only for the second multiplier. Apart from the fact that (8.11) is, e.g., not valid for certain multipliers capturing sector-bounded nonlinearities (see Chapter 5), it will be of crucial importance for our non-conservative incorporation of Zames-Falb multipliers into local analysis that we require (8.11) only for $\Pi^{(2)}$.

8.2.1 Two local IQC results

Naturally, is possible to embed all local stability and performance theorems derived in Chapter 7 into the present setting. We illustrate this exemplarily for two results.

Let us start with a reformulation of Theorem 8.1 that allows for the simultaneous use of hard and soft IQCs. The key ingredients are the lower bound (8.5) for the second (soft) multiplier

$$\int_0^T z_\Psi^{(2)}(t)^T P^{(2)} z_\Psi^{(2)}(t)\, dt \geq \xi^{(2)}(T)^T(-Y_{22}^{(2)})\xi^{(2)}(T) \quad \text{for all} \quad T > 0, \tag{8.13}$$

and the following stability FDI, adjusted to the split in hard and soft constraints:

$$(\star)^* \begin{pmatrix} P^{(1)} & 0 \\ 0 & P^{(2)} \end{pmatrix} \begin{pmatrix} \Psi_1^{(1)} & \Psi_2^{(1)} \\ \Psi_1^{(2)} & \Psi_2^{(2)} \end{pmatrix} \begin{pmatrix} M \\ I \end{pmatrix} \prec 0 \quad \text{on} \quad \mathbb{C}_0^\infty. \tag{8.14}$$

Theorem 8.2

Suppose that the interconnection (8.2a) *is well-posed on* $\mathscr{D}$. *Then* (8.2a) *is stable on* $\mathscr{D}$ *if the following conditions hold:*

a) $\Delta \in \mathrm{HIQC}_{\mathscr{X}}(P^{(1)}, \Psi^{(1)})$*;*

b) $\Delta \in \mathrm{IQC}_{\mathscr{X}}(P^{(2)}, \Psi^{(2)})$ *and* (8.11) *is satisfied;*

c) there exists a certificate X^s *of* (8.14) *and a certificate* $Y_{22}^{(2)}$ *of* (8.11) *which are coupled as*

$$\begin{pmatrix} X_{11}^s & X_{12}^s & X_{13}^s \\ X_{21}^s & X_{22}^s - Y_{22}^{(2)} & X_{23}^s \\ X_{31}^s & X_{32}^s & X_{33}^s \end{pmatrix} \succ 0. \tag{8.15}$$

If either $\Pi^{(1)}$ or $\Pi^{(2)}$ is empty, Theorem 8.2 remains valid if we merely drop the respective assumption a) or b) and cancel the corresponding block rows and columns in (8.15).

Proof. Since the proof proceeds in exactly the same fashion as the one of Theorem 7.9, we only state the necessary alterations here.

The sole use of the FDI (8.14) is to guarantee, for the trajectories of the LTI system in (8.2) with $(N_{21}\ N_{22}) = (M\ N_{12})$ driving (8.9) and with $\xi = \mathrm{col}(\xi^1, \xi^2)$, the existence of some $\gamma > 0$ with

$$\int_0^T \frac{1}{\gamma}\|z(t)\|^2 - \gamma\|d(t)\|^2\, dt \le -\begin{pmatrix}\xi(T)\\ x(T)\end{pmatrix}^T X^s \begin{pmatrix}\xi(T)\\ x(T)\end{pmatrix} - \\ - \int_0^T \begin{pmatrix} z_\Psi^{(1)}(t)\\ z_\Psi^{(2)}(t)\end{pmatrix}^T \begin{pmatrix} P^{(1)} & 0\\ 0 & P^{(2)}\end{pmatrix} \begin{pmatrix} z_\Psi^{(1)}(t)\\ z_\Psi^{(2)}(t)\end{pmatrix} dt \quad (8.16)$$

for all $d \in \mathscr{D}$, $w \in \mathscr{L}_{2e}$, and $T > 0$. By exploiting

$$\Delta \in \mathrm{HIQC}_{\mathscr{Z}}(P^{(1)}, \Psi^{(1)}) \cap \mathrm{IQC}_{\mathscr{Z}}(P^{(2)}, \Psi^{(2)}),$$

$P^{(2)}$ satisfying (8.11), and (8.13), we deduce (8.5) with (ξ, Y) replaced by $(\xi^{(2)}, Y_{22}^{(2)})$; here we make crucial use of $\int_0^T z_\Psi^{(1)}(t)^T P^{(1)} z_\Psi^{(1)}(t)\, dt \ge 0$ for all $T > 0$ and all $z \in \mathscr{Z}$. If combining with (8.15), we directly infer stability of (8.2a). □

Theorem 8.2 offers two main advantages over Theorem 7.9, as it allows for the incorporation of general hard IQCs into $\Pi^{(1)}$, irrespective of the way by which they are generated and even if they violate (8.11).

As already mentioned in Chapter 7, Balakrishnan proves in [13] that dynamic D-scaling multipliers (7.60) admit hard factorizations. His arguments only rely on a special choice of Ψ in (7.8) and a classical commutation property. Even though derived in a fundamentally different way, we may now incorporate the resulting hard factorized multipliers into our framework and thus merge them with others for a more accurate local analysis without any modifications.

A second advantage of our novel formulation of Theorem 8.2 arises from the fact that it allows to consider the most effective full-block multipliers corresponding to the celebrated circle criterion (see Section 5.3.1), namely those based on Pólya relaxations. Although these do not satisfy (8.11), they can still be employed in the realm of Theorem 8.2 since defining hard IQCs.

As a concrete application of Theorem 8.2, let us consider the scenario introduced in Section 7.5.2 that is mainly motivated by the analysis of only locally stable saturated systems. The underlying idea is to capture the action of the uncertainty Δ on some set of input signals while simultaneously guaranteeing invariance of a related set in the state-space (see [79] for an early reference).

In Section 7.5.2 we specifically work with amplitude bounded signals in

$$V_{R,e} := \{z \in \mathscr{L}_{2e} \mid \|z\|_\infty \leq R \quad \text{for all} \quad t \geq 0\} \tag{8.17}$$

and $V_R := V_{R,e} \cap \mathscr{L}_2$ where $R > 0$. Moreover, we restrict the inputs d to the set $\mathscr{D}_\alpha := \{d \in \mathscr{L}_2 \mid \|d\|_2 \leq \alpha\}$. The following corollary of Theorem 8.2 can be established along the lines of the proof of Theorem 7.12.

Corollary 8.3
Suppose that N in (8.1) *is realized with $D_{11} = D_{12} = 0$. With $R > 0$ and* (8.17) *let $V_{R,e} \subset \mathscr{Z}_e$. Further assume that* (8.2a) *is well-posed on $\mathscr{D}_\alpha$ and that*

a) $\Delta \in \mathrm{HIQC}_{V_R}(P^{(1)}, \Psi^{(1)})$*;*

b) $\Delta \in \mathrm{IQC}_{V_R}(P^{(2)}, \Psi^{(2)})$ *and* (8.11) *is satisfied;*

c) there exist certificates X and $Y_{22}^{(2)}$ of the FDIs (7.12) *with $P_p =$* $\mathrm{diag}(0, -I)$ *and* (8.11)*, respectively, such that*

$$\left(\begin{array}{c|ccc} R^2/\alpha^2 & 0 & 0 & C_1 \\ \hline 0 & X_{11} & X_{12} & X_{13} \\ 0 & X_{21} & X_{22} - Y_{22}^{(2)} & X_{23} \\ C_1^T & X_{31} & X_{32} & X_{33} \end{array}\right) \succ 0. \tag{8.18}$$

Then (8.2a) *is stable on $\mathscr{D}_\alpha$ and the response z of* (8.2a) *is bounded as $\|z(t)\| \leq R$ for all $t \geq 0$ and all $d \in \mathscr{D}_\alpha$.*

Remark 8.4.
For fixed $R > 0$, we emphasize that (8.18) is an LMI in X, $Y_{22}^{(2)}$ and $1/\alpha^2$. By a line search over R we can thus determine the maximal admissible disturbance energy α such that (8.2a) remains stable. $\star$

After having established two local analysis results, we focus in the subsequent section on a rather specific yet very relevant class of multipliers in order to illustrate how the above derived setup enfolds in a concrete scenario.

8.3 Zames-Falb multipliers

The claim that Zames-Falb multipliers define one of the most effective classes of soft IQCs for global analysis was supported by many comparisons and examples in Part I of the present thesis. Furthermore, in Chapter 7, we demonstrated that Zames-Falb multipliers also play an important role in the verification of local criteria. The present section is devoted to the proof that both the subclasses of causal and anticausal Zames-Falb multipliers can, individually, be incorporated into our local analysis framework without any conservatism. This is based on two contributions. First we reveal that simple factorizations of both subclasses, as used for parameterization and subsequent numerical computations, lead to hard IQC constraints. Based on these factorizations, we prove in a second step that all certificates X^s of (8.14) with empty (P^2, Ψ^2) are positive definite. Thus we may drop the assumption b) in Theorem 8.2 and infer that (8.15) does not cause extra limitations. In summary, we may proceed with local analysis as in Corollary 8.3 without any added conservatism.

For simplicity of presentation, we restrict the class of uncertainties under consideration if compared to Chapter 5 as follows. Let $\varphi : \mathbb{R} \to \mathbb{R}$ and assume that $\varphi(0) = 0$ and

$$0 \leq \frac{\varphi(x) - \varphi(y)}{x - y} \leq \sup_{x \neq y} \frac{\varphi(x) - \varphi(y)}{x - y} < \mu \tag{8.19}$$

for all $x, y \in \mathbb{R}$ with $x \neq y$. If the bound on the right is absent but φ remains locally Lipschitz continuous and bounded as in $|\varphi(x)| \leq k\,|x|$, it is said to be **bounded and monotone**. As before, we define the nonlinear operator Δ_Φ as

$$\Delta_\Phi(z)(t) = \varphi(z(t)) \quad \text{for all} \quad z \in \mathscr{Z} \tag{8.20}$$

and almost all $t \geq 0$. The following fundamental theorem from [197] provides the basis for Zames-Falb multipliers.

Theorem 8.5
Let Δ_Φ be as in (8.20) with a bounded and monotone function φ. Moreover, let $h \in \mathscr{L}_1(-\infty, \infty)$ be nonnegative and satisfy $\|h\|_1 < g$ for some $g > 0$. Then (8.4) holds on $\mathscr{X} = \mathscr{L}_2$ with

$$\Pi = \begin{pmatrix} 0 & g - \hat{h}^* \\ g - \hat{h} & 0 \end{pmatrix}. \tag{8.21}$$

If φ further satisfies (8.19), then (8.4) is valid on $\mathscr{L}_2$ with

$$\Pi_{[0,\mu]} := \begin{pmatrix} \mu & -1 \\ 0 & 1 \end{pmatrix}^T \Pi \begin{pmatrix} \mu & -1 \\ 0 & 1 \end{pmatrix}. \tag{8.22}$$

Remark 8.6.
Note that Theorem 8.5 is merely a specialization of Theorem 5.15 for a single nonlinearity and under the assumptions on φ stated above. $\star$

Now observe that any $h \in \mathscr{L}_1(-\infty, \infty)$ can be split up as $h = h_- + h_+$ with h_- and h_+ supported on $(-\infty, 0]$ and $[0, \infty)$. By setting either h_- or h_+ to zero we obtain causal or anticausal Zames-Falb multipliers that may be factorized according to (7.8) as

$$\Pi_+ \sim (P_+, \Psi_+) := \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} g - \hat{h}_+ & 0 \\ 0 & 1 \end{pmatrix} \right), \tag{8.23}$$

or

$$\Pi_- \sim (P_-, \Psi_-) := \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & g - \hat{h}_-^* \end{pmatrix} \right), \tag{8.24}$$

respectively. Let us show next that both define hard IQCs.

Theorem 8.7
Let φ be bounded and monotone. Then $\Delta_\Phi \in \mathrm{HIQC}(P_+, \Psi_+)$ as well as $\Delta_\Phi \in \mathrm{HIQC}(P_-, \Psi_-)$.

The following proofs of both claims in Theorem 8.7 will be fundamentally different. We show $\Delta_\Phi \in \mathrm{HIQC}(P_+, \Psi_+)$ from scratch by tracing the multiplier back to its generating principle, namely convexity. This allows to justify the claim without resorting to Theorem 8.5. By contrast, we derive the hard IQC $\Delta_\Phi \in \mathrm{HIQC}(P_-, \Psi_-)$ form the corresponding soft one in Theorem 8.5. This is achieved by truncating the input to the uncertainty at time T, in complete analogy to the case of general IQCs discussed in the proof of Theorem 7.9.

Proof. Let us first give an elementary proof for the validity of a hard IQC with causal Zames-Falb multipliers. The assumed bound on φ implies

$$|\varphi(x)|^2 \leq k^2|x|^2 \quad \text{for all} \quad x \in \mathbb{R}. \tag{8.25}$$

Since $s\varphi(s) \geq 0$ for $s \in \mathbb{R}$, the C^1-function $\chi(x) = \int_0^x \varphi(s)\,ds$ satisfies $\chi(x) \geq 0$ and $\chi'(x) = \varphi(x)$ for all $x \geq 0$. By monotonicity of the derivative, χ is convex and hence, by the subgradient inequality,

$$\chi(x) - \chi(y) \leq \varphi(x)(x-y) \quad \text{for all} \quad x, y \in \mathbb{R}. \tag{8.26}$$

For $y = 0$ and with $\chi(0) = 0$ we infer $\chi(x) \leq \varphi(x)x \leq |\varphi(x)||x|$ and thus

$$0 \leq \chi(x) \leq k|x|^2 \quad \text{for all} \quad x \in \mathbb{R}. \tag{8.27}$$

Now fix $z \in \mathscr{L}_2$ and $\tau \geq 0$. From (8.25) we have $\varphi(z(.)) \in \mathscr{L}_2$; moreover, as $z(.-\tau) \in \mathscr{L}_2 = \mathscr{L}_2[0,\infty)$ and by (8.27) also $\chi(z(.-\tau)) \in \mathscr{L}_1[0,\infty)$. Using (8.26) we arrive at

$$\chi(z(t)) - \chi(z(t-\tau)) \leq \varphi(z(t))(z(t) - z(t-\tau)) \tag{8.28}$$

for almost all $t \geq 0$. Since $\chi(z(.-\tau)) = 0$ on $[0,\tau]$ we note for $0 \leq T \leq \tau$ that

$$\int_0^T \chi(z(t)) - \chi(z(t-\tau))\,dt = \int_0^T \chi(z(t))\,dt \geq 0 \tag{8.29}$$

and, similarly for $T > \tau$, that

$$\int_0^T \chi(z(t)) - \chi(z(t-\tau))\, dt = \int_{T-\tau}^T \chi(z(t))\, dt \geq 0. \tag{8.30}$$

For all $\tau \geq 0$, (8.29), (8.30) in combination with (8.28) imply the validity of the hard IQC $\int_0^T \varphi(z(t))[z(t) - z(t-\tau)]\, dt \geq 0$ for all $T \geq 0$. If $h \in \mathscr{L}_1[0,\infty)$ is non-negative, multiplication with $h(\tau)$ and integration over $\tau \in [0,\infty)$ shows nonnegativity of

$$\int_0^T \varphi(z(t)) \left[z(t) \int_0^\infty h(\tau)\, d\tau - \int_0^\infty z(t-\tau) h(\tau)\, d\tau \right] dt$$

for all $T \geq 0$. It remains to note for $t \geq 0$ and $z \in \mathscr{L}_2$ that

$$\int_0^\infty z(t-\tau) h(\tau)\, d\tau = \int_0^t h(t-\sigma) z(\sigma)\, d\sigma$$

in order to arrive at

$$\int_0^T \varphi(z(t)) \left[\|h\|_1 z(t) - \int_0^t h(t-\tau) z(\tau)\, d\tau \right] dt \geq 0$$

for all $T \geq 0$. The claim follows by noting $g > \|h\|_1$ and simple application of the Fourier transform.

Let us now turn to the anticausal case. Consequently, let $h \in \mathscr{L}_1(-\infty, 0]$ be non-negative and define $f(t) = h(-t)$ for almost all $t \in [0,\infty)$. Then $f \in \mathscr{L}_1[0,\infty)$, $\hat{h}^* = \hat{f}$ and the soft IQC defined by (P_-, Ψ_-) reads as

$$\int_0^\infty z(t) \left[g\varphi(z(t)) - \int_0^t f(t-\tau) \varphi(z(\tau))\, d\tau \right] dt \geq 0$$

for all $z \in \mathscr{L}_2$. In particular, for $z_T \in \mathscr{L}_2$ we infer

$$\int_0^T z_T(t) \left[g\varphi(z_T(t)) - \int_0^t f(t-\tau) \varphi(z_T(\tau))\, d\tau \right] dt \geq 0.$$

As z and z_T coincide on $[0,T]$ and φ is static, this implies

$$\int_0^T z(t)\left[g\varphi(z(t)) - \int_0^t f(t-\tau)\varphi(z(\tau))\,d\tau\right] dt \geq 0$$

for all $z \in \mathscr{L}_2$. $\square$

With arguments given in [197, Ch. 7], one can likewise show that the transformed multipliers corresponding to (8.22), namely $\Pi_{+,[0,\mu]} \sim (P_+, \Psi_+ \left(\begin{smallmatrix}\mu & -1\\ 0 & 1\end{smallmatrix}\right))$ and $\Pi_{-,[0,\mu]} \sim (P_-, \Psi_- \left(\begin{smallmatrix}\mu & -1\\ 0 & 1\end{smallmatrix}\right))$, define hard IQCs for Δ_Φ's with (8.19).

We proceed by introducing the following parameterization of $\Pi_{[0,\mu]}$ that comprises the one proposed in Section 5.3.3 for the case of a single nonlinearity. Given a strictly proper and stable column vector $\phi \in RH_\infty^l$, let h in (8.21)-(8.22) be defined through

$$\hat{h} = \hat{h}_- + \hat{h}_+ = \phi^*\lambda_- + \lambda_+^T\phi \;\text{ with }\; \lambda_-,\ \lambda_+ \in \mathbb{R}^l \tag{8.31}$$

to obtain

$$\Pi_{[0,\mu]} = \begin{pmatrix}\mu & -1\\ 0 & 1\end{pmatrix}^T \begin{pmatrix} 0 & g - \lambda_-^T\phi - \phi^*\lambda_+ \\ g - \phi^*\lambda_- - \lambda_+^T\phi & 0\end{pmatrix} \begin{pmatrix}\mu & -1\\ 0 & 1\end{pmatrix}. \tag{8.32}$$

Let us now prove that both the factorizations (8.23) and (8.24), if employed individually, imply that all certificates X^s of (8.14) are a priori positive definite. We first consider causal multipliers and take $(P^{(1)}, \Psi^{(1)}) := (P_+, \Psi_+ \left(\begin{smallmatrix}\mu & -1\\ 0 & 1\end{smallmatrix}\right))$ while leaving $(P^{(2)}, \Psi^{(2)})$ empty in Theorem 8.2. Then (8.14) is

$$(\star)^* \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \begin{pmatrix} g - \lambda_+^T\phi & 0\\ 0 & 1\end{pmatrix} \begin{pmatrix}\mu M - 1\\ 1\end{pmatrix} \prec 0 \quad \text{on} \quad \mathbb{C}_0^\infty. \tag{8.33}$$

Due to the passivity structure of P_+ and the stability of $\mu M - 1$, it is a matter of direct verification that all solutions X^s of the LMI related to (8.33) are positive definite. Indeed, let $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$ be an minimal realization of the stable LTI system $(g-\lambda_+^T\phi)(\mu M-1)$. Then, (8.33) is equivalent to the existence of some symmetric X^s satisfying

$$\begin{pmatrix} I & 0\\ \widetilde{A} & \widetilde{B}\end{pmatrix}^T \begin{pmatrix} 0 & X^s\\ X^s & 0\end{pmatrix}\begin{pmatrix} I & 0\\ \widetilde{A} & \widetilde{B}\end{pmatrix} + \begin{pmatrix} \widetilde{C} & \widetilde{D}\\ 0 & 1\end{pmatrix}^T \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} \widetilde{C} & \widetilde{D}\\ 0 & 1\end{pmatrix} \prec 0.$$

By inspection of the left upper block, we infer that X^s satisfies $\widetilde{A}^T X^s + X^s\widetilde{A} \prec 0$. As $\widetilde{A}$ was Hurwitz, the claim follows.

For anticausal multipliers we choose $(P^{(1)}, \Psi^{(1)}) := (P_-, \Psi_-\left(\begin{smallmatrix}\mu & -1\\ 0 & 1\end{smallmatrix}\right))$. With (8.31) and a minimal realization $(A_\phi, B_\phi, C_\phi, 0)$ of ϕ, (8.14) is then certified by X^s satisfying

$$(\star)^T \begin{pmatrix} 0 & X^s\\ X^s & 0\end{pmatrix}\left(\begin{array}{cc|c} I & 0 & 0\\ 0 & I & 0\\ \hline A_\phi & 0 & B_\phi\\ 0 & A & B\end{array}\right) + (\star)^T\begin{pmatrix} 0 & I\\ I & 0\end{pmatrix}\left(\begin{array}{cc|c} 0 & \mu C & -1\\ -\lambda_-^T C_\phi & 0 & g\end{array}\right) \prec 0.$$

After a congruence transformation eliminating $-\lambda_-^T C_\phi$, the left upper block reads as

$$(\star)^T X^s + X^s\left(\begin{smallmatrix}\widetilde{A}_\phi & 0\\ \star & A\end{smallmatrix}\right) \prec 0 \quad \text{with} \quad \widetilde{A}_\phi := A_\phi + g^{-1}\lambda_-^T C_\phi B_\phi.$$

It remains to show that $\widetilde{A}_\phi$ is Hurwitz in order to conclude that X^s is positive definite. Indeed, using the $\mathscr{L}_1$-norm constraint in Theorem 8.5, a close inspection of the proof of Lemma 5.17 reveals that $-(g-\lambda_-^T\phi)$ is strictly negative real; this translates into the existence of some $K=K^T$ with

$$(\star)^T\begin{pmatrix} 0 & K\\ K & 0\end{pmatrix}\begin{pmatrix} I & 0\\ A_\phi & B_\phi\end{pmatrix} + (\star)^T\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} -\lambda_-^T C_\phi & g\\ 0 & -1\end{pmatrix} \prec 0, \tag{8.34}$$

whose left upper block is $A_\phi^T K + K A_\phi \prec 0$; since A_ϕ is Hurwitz, we obtain $K \succ 0$; if applying the same congruence transformation as before to (8.34), we infer

$$(\star)^T\begin{pmatrix} 0 & K\\ K & 0\end{pmatrix}\begin{pmatrix} I & 0\\ \widetilde{A}_\phi & B_\phi\end{pmatrix} + (\star)^T\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & g\\ \star & -1\end{pmatrix} \prec 0 \tag{8.35}$$

with left upper block $\widetilde{A}_\phi^T K + K\widetilde{A}_\phi \prec 0$; hence $\widetilde{A}_\phi$ is Hurwitz due to $K \succ 0$. Consequently, $X^s \succ 0$ holds in both cases.

Since $\Pi_{-,[0,\mu]}$ satisfies $\Pi_{22} \prec 0$ on $\mathbb{C}_0^\infty$, we could alternatively select $(P^{(2)}, \Psi^{(2)}) := (P_-, \Psi_- \left(\begin{smallmatrix} \mu & -1 \\ 0 & 1 \end{smallmatrix}\right))$, while leaving $(P^{(1)}, \Psi^{(1)})$ empty in Theorem 8.2. The set of certificates X^s of (8.14) remains identical to that for the previous choice. However, we now need to introduce the shift $Y_{22}^{(2)}$ in (8.15) which satisfies (8.12). Since (8.12) is in fact identical to (8.34) with $Y_{22}^{(2)}$ replacing K, we infer $Y_{22}^{(2)} \succ 0$ and conclude that (8.15) now imposes more stringent constraints on the certificates if compared to those in the previous paragraph, which is a severe disadvantage. The same conclusion can be drawn for causal multipliers $\Pi_{+,[0,\mu]}$.

To sum up, both parameterizations of causal and anticausal Zames-Falb multipliers, if employed individually, can be and should be treated as hard IQCs; since the related KYP certificates X^s are all positive definite, we conclude that (8.15) then imposes no extra limitation and may be applied in local analysis without introducing any further conservatism.

If working with the full $\Pi_{[0,\mu]}$ in (8.32) the picture is slightly more complicated. As $\Pi_{[0,\mu]}$ defines a genuine soft IQC, we are led to incorporate it into $\Pi^{(2)}$ and cannot expect all certificates X^s to be positive definite. Still, also in this case we have $Y_{22}^{(2)} \succ 0$; in fact, if factorizing $\Pi_{[0,\mu]}$ in (8.32) as

$$\left(P^{(2)}, \left(\Psi_1^{(2)} \;\; \Psi_2^{(2)}\right)\right) = \left(\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \left(\begin{array}{cc} \mu & -1 \\ \mu\phi & -\phi \\ \hline 0 & g - \lambda_-^T\phi \\ 0 & -\lambda_+ \end{array}\right) \right),$$

the resulting realization of $\Psi_2^{(2)}$ is given by $(A_\Psi, B_{\Psi_2}, C_\Psi, D_{\Psi_2})$ equal to

$$\left(\begin{pmatrix} A_\phi & 0 \\ 0 & A_\phi \end{pmatrix}, \begin{pmatrix} -B_\phi \\ B_\phi \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ C_\phi & 0 \\ 0 & -\lambda_-^T C_\phi \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ g \\ -\lambda_+ \end{pmatrix} \right);$$

we extract $C_\Psi^T P^{(2)} C_\Psi = 0$, and (8.12) shows again $Y_{22}^{(2)} \succ 0$.

Most importantly, also in this case we can avoid the need to work with $Y_{22}^{(2)}$ and thus reduce conservatism. We only need to decompose $\Pi_{[0,\mu]} = \Pi_{-,[0,\mu]} + \Pi_{+,[0,\mu]}$ into its anticausal and causal parts as above and consider $\Pi_{-,[0,\mu]}$, $\Pi_{+,[0,\mu]}$ as defining hard IQC constraints by incorporating both of them into $\Pi^{(1)}$. The significant advantages of this approach over the former is illustrated in the subsequent section by means of a numerical example.

8.4 Concrete numerical example

Let us now demonstrate the benefit gained by our refined approach for the very specific application of saturated systems that are of great practical importance and, thus, have been extensively researched (see, e.g., [157, 83] and references therein). One of the associated analysis problems may be stated as follows: Given an exponentially unstable LTI system that is locally (but not globally) stabilized by saturated state feedback, what is the maximal admissible input energy such that the feedback interconnection remains stable?

As addressed in detail in Section 7.6.2 (see also [157, 83]), a standard loop transformation reduces the described problem to the stability analysis of the interconnection (8.2), with an uncertainty Δ_Φ and $\varphi = \mathrm{dz}$, the unit dead-zone function. In addition to the Zames-Falb multipliers discussed in Section 8.3, we may also use multipliers corresponding to the celebrated circle criterion. Following [79], we capture the restriction of Δ_{dz} to the amplitude bounded set (8.17) by means of local sectors. Furthermore, we combine the resulting local circle multipliers with those for the Zames-Falb stability criterion, which allows us to also take the slope restriction into account.

Specifically, let us revisit Example 7.15, where the linear system is described after the loop transformation by

$$A = \begin{pmatrix} -1.15 & -17 & -10 \\ 0.02 & -0.1 & -1.8 \\ 0.1 & 2.5 & 0.3 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -12 \\ 0.2 \\ 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.2 \\ -0.1 \\ 0.5 \end{pmatrix}$$

and $C_1 = \begin{pmatrix} -0.1 & -1.5 & -1 \end{pmatrix}$ with $\mathscr{D} = \mathscr{D}_\alpha$. We leave the output e void, since we are only interested in stability. As the dead-zone is globally Lipschitz continuous, the interconnection is well-posed for $\mathscr{Z}_e = \mathscr{L}_{2e}$. We apply Corollary 8.3 by performing a line search over $R > 0$. For each R we adapt our combination of multipliers corresponding to the circle and Zames-Falb criteria [58] such that (8.4) holds for $\mathscr{Z} = V_R$ defined in (8.17).

Let us briefly recap the results derived in Example 7.15. For $R \in [0, 1]$, the interconnection operates in the so-called linear region such that the output of the dead-zone nonlinearity is zero; thus stability is guaranteed. As stated in Table 8.1, this is the case for $\alpha \leq 1.41$. Using a local version of the circle criterion, stability is proven in [79] for values of α up to 1.45. This was further increased by Fang et al. in [51] guaranteeing stability up until $\alpha = 2.07$. The derivation in the previous section builds on the one in [79] by adding soft factorized Zames-Falb multipliers to the local circle criterion. This leads to a significant push of the threshold to 9.04. Finally, our novel approach based on a combination of hard factorized causal and anticausal Zames-Falb multipliers as described in Section 8.3 allows for a further increase of the admissible energies up to the bound $\alpha = 12.03$. This amounts to nearly six times the energy level obtained with [51].

Table 8.1: Maximal disturbance energy levels α

lin. region	[79]	[51]	[60]	novel approach
≤ 1.41	1.45	2.07	9.04	12.03

8.5 Summary

The contribution in this chapter is twofold. First, we provide a comprehensive framework that allows to optimally combine hard and soft IQCs for local analysis of feedback interconnections. This allows for the nonconservative incorporation of hard IQC regardless of generating principles. It is expected that this lays the foundation for the merging of dissipation based results in the literature with the framework outlined in Chapter 7. Second, we prove that both the subclasses of causal and anticausal Zames-Falb multipliers may easily be factorized such that they, individually, impose hard IQC constraints and may be losslessly incorporated into our local framework. In combination both contributions allow, on the one hand, to significantly reduce the conservativeness in the approach presented in Chapter 7. On the other hand, the possibility for adding general hard IQCs considerably widens the are of applications.

Chapter 9

Concluding remarks

THIS thesis provides several crucial steps towards its main goal of a comprehensive robustness analysis theory that provides the framework for the computational verification of global as well as local stability and performance criteria.

In case of global robust stability and performance analysis, we present a general theorem that merges ideas from the abstract graph separation results proposed by Safonov and Teel with those from classical IQC theory established by Megretski and Rantzer. As one of the key features of the first part of the present thesis, this allows us to develop a unified framework for global robustness analysis on Sobolev spaces that is shown to subsume and extend all multiplier based results for the classical problem of absolute stability. Consequently, and as supported by numerous examples, the results presented in this thesis define the least conservative robust stability and performance estimates available in the literature for this essential problem.

The second part of this thesis is devoted to local analysis of feedback structures. For interconnections consisting of an LTI system and an uncertain component, we demonstrate that standard soft IQCs can be incorporated into classical dissipation theory in order to guarantee state and output constraints, or even to handle only locally bounded nonlinearities within the IQC framework. The full power of our novel approach is illustrated for the particular case of Zames-Falb multipliers. Here it is shown that the subclasses of causal and anticausal Zames-Falb multipliers can be employed in local analysis without any conservatism.

However, in line with the title of the present thesis which does not claim completeness of our comprehensive framework, there remain several issues that we believe should be addressed in the future. As we discussed individual recommendations already at the end of each chapter, we only highlight some major general questions in the sequel.

The formulation of our main global stability result on Sobolev spaces allows to incorporate constraints and also performance specification involving higher order derivatives with ease. This is hoped to provide the basis for more accurate descriptions of nonlinearities and also for more sophisticated performance criteria. However, apart from our treatment of time-varying parametric uncertainties, the generation of novel stability criteria based on higher order derivatives of the involved signals remains an open problem.

Moreover, as indicated in Chapter 7, our approach to local robustness analysis of feedback interconnections based on IQCs is still subject to certain limitations. First and foremost, we are as of yet unable to include multipliers that require additional signal regularity into the framework. Although one would expect that all Lyapunov based criteria are also applicable in our setting, Popov multipliers remain, so far, out of reach. In addition, the ability to exploit general soft IQCs comes at the expense of added conservatism. It remains unclear whether the overall approach can be refined in order to improve the obtained stability estimates or, as illustrated in Chapter 8, whether it is more beneficial to enhance the present framework by exploiting individual properties of given multiplier classes.

Part III

Appendices

Appendix A

Explanation of symbols

A.1 Sets and matrices

$\mathbb{N}$, $\mathbb{N}_0$	positive, nonnegative integers
$\mathbb{R}$, $\mathbb{C}$	real and complex numbers
$\mathbb{R}_+$	set of positive real numbers, i.e., $(0, \infty)$.
$\mathbb{C}_0$, $\mathbb{C}_0^\infty = \mathbb{C}_0 \cup \{\infty\}$	imaginary axis and extension thereof
$\mathbb{D}$, $\mathbb{T}$	open unit disc in the complex plane and its boundary
$\mathrm{sgn}(a)$	sign of a real scalar, i.e., $\mathrm{sgn}(a) = a\,\lvert a \rvert$
$\mathbb{R}^{n\times m}$	real valued matrices of dimension $n \times m$
$\mathbb{S}^n, \mathbb{D}^n$	subspace of symmetric, diagonal matrices of $\mathbb{R}^{n\times n}$
A^*, A^T	conjugate transpose and transpose of matrix A
$A > (\geq)\ 0$	used for $A \in \mathbb{R}^{n\times m}$ if $A_{ij} > (\geq)\ 0$ for all i, j
$A \prec (\preccurlyeq)\ B$	used for A, $B \in \mathbb{S}^n$ if $B - A$ is positive (semi-) definite
$A \otimes B$	Kronecker product of matrices A and B

$\det(A)$	determinant of A
$\mathrm{trace}(A)$	trace of A
$\mathrm{Ran}(A)$, $\mathrm{Ker}(A)$	range space and kernel of A (also if A is a bounded operator between Hilbert spaces)
$\mathrm{diag}(A_1, \ldots, A_n)$	block diagonal matrix with matrices A_j on diagonal
$\mathrm{col}(A_1, \ldots, A_n)$	for matrices A_j with appropriate dimensions
$[\alpha, \beta]$	for $\alpha = \mathrm{diag}(\alpha_i)$, $\beta = \mathrm{diag}(\beta_i) \in \mathbb{D}^n$, the set of diagonal matrices $\{\mathrm{diag}(\delta_1, \ldots, \delta_k) : \alpha_i \leq \delta_i \leq \beta_i \text{ for all } i \in \{1, \ldots, n\}\}$.
$\mathrm{eig}(A)$	set of eigenvalues of A
e	the all ones vector in $\mathbb{R}^n$

A.2 Function spaces and signals

In the following table, Ω denotes a measurable subset of $\mathbb{R}$. Note that for function spaces, we typically omit the superscript indicating the dimension k of the contained signals.

$\mathscr{L}_2^k(\Omega)$, $\mathscr{L}_{2e}^k(\Omega)$	space of square integrable, locally square integrable functions mapping $\Omega \subset \mathbb{R}$ into $\mathbb{R}^k$ with $k \in \mathbb{N}$
$\mathscr{L}_1^k(\Omega)$, $\mathscr{L}_\infty^k(\Omega)$	absolute integrable, essentially bounded functions mapping $\Omega \subset \mathbb{R}$ into $\mathbb{R}^k$
$\|\cdot\|_p$	norm on $\mathscr{L}_p^k(\Omega)$ for $p = 1, 2, \infty$
ℓ_2^k, (ℓ_{2e}^k)	the space of (locally) square summable functions mapping $\mathbb{N}_0$ into $\mathbb{R}^k$; ℓ_2^k is equipped with the norm $\|u\|^2 = \sum_{j=0}^\infty \|u(j)\|^2$

$\mathscr{H}^{r,k}$	Sobolev space of functions $u : [0,\infty) \to \mathbb{R}^k$ such that u and its distributional derivatives $\partial^j u$ for $j \in \{1,\ldots,r\}$ are all contained in $\mathscr{L}_2$; it is equipped with the norm $\|u\|_r^2 := \sum_{j=0}^r \|\partial^j u\|_0^2$
$C[0,\infty)$ $(PC[0,\infty))$	set of (piecewise) continuous functions mapping $[0,\infty)$ into $\mathbb{R}^k$
RL_∞	space of real-rational and proper matrix functions without poles on $\mathbb{C}_0$
$RH_\infty \subset RL_\infty$	subspace of RL_∞ containing proper and stable transfer matrices
$\star$	convolution operator and objects that can be inferred by symmetry
$\hat{u}$	Fourier transform of a signal u in $\mathscr{L}_2^k$ or $\mathscr{L}_1^k$
$G(s)$, (A,B,C,D)	transfer matrix and its realization, i.e., $G(s) = C(sI-A)^{-1}B + D$
u_T	denotes either the restriction of a signal $u : [0,\infty) \to \mathbb{R}^k$ to $[0,T]$, i.e., $u_T = u\|_{[0,T]}$, or its truncation, i.e., $u(t) = u$ on $[0,T]$ and zero otherwise
u^T	extension of $u : [0,\infty) \to \mathbb{R}^k$ that coincides with u on $[0,T]$, i.e., $u^T\|_{[0,T]} = u\|_{[0,T]}$
$D_\pm$, $D^\pm$	lower and upper Dini derivatives

Appendix B

List of terms

ARE algebraic Riccati equation.

FDI frequency-domain inequality.

IQC integral quadratic constraint.

KYP Kalman Yakubovich Popov.

LMI linear matrix inequality.

LTI linear time-invariant.

PWM pulse-width modulator.

SISO single input single output.

SOS sum of squares.

Appendix C

Some additional proofs

C.1 For Chapter 2

C.1.1 Proof of Theorem 2.2

Step 1. Since M is bounded, there exist some $\tilde{\gamma}, \tilde{\gamma}_0 \geq 0$ with $\|M(w)_T\|^2 \leq \tilde{\gamma}^2 \|w_T\|^2 + \tilde{\gamma}_0^2$ and thus

$$\left\| \begin{pmatrix} M(w)_T \\ w_T \end{pmatrix} \right\| \leq \sqrt{(1+\tilde{\gamma}^2)\|w_T\|^2 + \tilde{\gamma}_0^2} \leq \sqrt{(1+\tilde{\gamma}^2)}\|w_T\| + \tilde{\gamma}_0$$

for all $T > 0$ and $w \in \mathscr{L}_{2e}^k$. Applying (2.10) to $u = \mathrm{col}(M(w)_T, w_T)$ and $v = \mathrm{col}(d_T, 0)$ hence leads to

$$\begin{aligned} \Sigma \begin{pmatrix} M(w)_T + d_T \\ w_T \end{pmatrix} - \Sigma \begin{pmatrix} M(w)_T \\ w_T \end{pmatrix} & \\ \leq 2c\left(\sqrt{(1+\tilde{\gamma}^2)}\|w_T\| + \tilde{\gamma}_0\right)\|d_T\| + c\|d_T\|^2 & \\ = \begin{pmatrix} 1 \\ \|w_T\| \\ \|d_T\| \end{pmatrix}^T \begin{pmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \begin{pmatrix} 1 \\ \|w_T\| \\ \|d_T\| \end{pmatrix} & \quad \text{(C.1)} \end{aligned}$$

for all $T>0$ and $(w,d)\in\mathscr{L}_{2e}^k\times\mathscr{L}_{2e}^l$, with σ_{ij} only depending on M and Σ. If M is linear one can choose $\widetilde{\gamma}_0=0$ which implies $\sigma_{13}=0$.

Step 2. In this crucial step we show that there exist $\gamma>0$ and $\hat{\gamma}_0$ such that

$$\Sigma\begin{pmatrix} M(w)_T+d_T\\ w_T\end{pmatrix}+\frac{1}{\gamma}\|M(w_T)_T+d_T\|^2-\gamma\|d_T\|^2\leq\hat{\gamma}_0 \tag{C.2}$$

for all $T>0$ and $(w,d)\in\mathscr{L}_{2e}^k\times\mathscr{L}_{2e}^l$ as follows. Add (C.1) and (2.11) to get

$$\Sigma\begin{pmatrix} M(w)_T+d_T\\ w_T\end{pmatrix}\leq\begin{pmatrix}1\\ \|w_T\|\\ \|d_T\|\end{pmatrix}^T\begin{pmatrix} m_0 & 0 & \sigma_{13}\\ 0 & -\varepsilon & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{pmatrix}\begin{pmatrix}1\\ \|w_T\|\\ \|d_T\|\end{pmatrix} \tag{C.3}$$

for all $T>0$ and $(w,d)\in\mathscr{L}_{2e}^k\times\mathscr{L}_{2e}^l$. With $\|M(w)_T\|\leq\widetilde{\gamma}\|w_T\|+\widetilde{\gamma}_0$ and the triangle inequality we infer for all $\gamma>0$ that

$$\frac{1}{\gamma}\|M(w)_T+d_T\|^2-\gamma\|d_T\|^2\leq\frac{1}{\gamma}\left(\widetilde{\gamma}\|w_T\|+\widetilde{\gamma}_0+\|d_T\|\right)^2-\gamma\|d_T\|^2=$$

$$=\begin{pmatrix}1\\ \|w_T\|\\ \|d_T\|\end{pmatrix}^T\begin{pmatrix} m_{11}/\gamma & m_{12}/\gamma & m_{13}/\gamma\\ m_{12}/\gamma & m_{22}/\gamma & m_{23}/\gamma\\ m_{13}/\gamma & m_{23}/\gamma & m_{33}/\gamma-\gamma\end{pmatrix}\begin{pmatrix}1\\ \|w_T\|\\ \|d_T\|\end{pmatrix} \tag{C.4}$$

for all $T>0$ and $(w,d)\in\mathscr{L}_{2e}^k\times\mathscr{L}_{2e}^l$, where $m_{ij}\in\mathbb{R}$ do not depend upon γ. For any $\hat{\gamma}_0>m_0$ observe that there exists some (sufficiently large) $\gamma>0$ for which

$$\begin{pmatrix} m_0 & 0 & \sigma_{13}\\ 0 & -\varepsilon & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{pmatrix}+\begin{pmatrix} m_{11}/\gamma & m_{12}/\gamma & m_{13}/\gamma\\ m_{12}/\gamma & m_{22}/\gamma & m_{23}/\gamma\\ m_{13}/\gamma & m_{23}/\gamma & m_{33}/\gamma-\gamma\end{pmatrix}\preccurlyeq\begin{pmatrix}\hat{\gamma}_0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{C.5}$$

If we add (C.3) and (C.4), we can exploit (C.5) to arrive at (C.2). If M is linear we can choose $\widetilde{\gamma}_0=0$ which implies $m_{11}=m_{12}=m_{13}=0$. We can then take $\hat{\gamma}_0=m_0$.

Step 3. To finish the proof choose $d \in \mathscr{D}$ and a corresponding response $z \in \mathscr{L}_{2e}^l$ of (2.9). Then $w_T = \Delta(z)_T$ and $z_T = M(w)_T + d_T$. Now observe that the inequality in (2.12) holds for $v := M(w_T) + d \in M(\mathscr{L}_2^k) + \mathscr{D}$. On the other hand, by causality we have $v_T = M(w_T)_T + d_T = M(w)_T + d_T = z_T$ and $\Delta(v)_T = \Delta(v_T)_T = \Delta(z_T)_T = \Delta(z)_T = w_T$. This allows to combine (2.12) with (C.2) in order to infer (2.13) for $\gamma_0 = \hat{\gamma}_0 + \delta_0$ (which equals $m_0 + \delta_0$ if M is linear).

C.2 For Chapter 3

C.2.1 Proof of Theorem 3.4

We use the abbreviations $\mathscr{D} := \ell_2(\mathscr{H}^k) \times \mathscr{V}$ and $\mathscr{E} := \mathscr{E}_e \cap \ell_2(\mathscr{H}^l)$.

Step 1. Only for proving the following key fact we make use of the properties of Σ and the constraints b), c): There exists a (τ-independent) $\gamma > 0$ such that

$$\begin{aligned} &\tau \in [0,1] \;\text{ and }\; R_\tau(\mathscr{D}) \subset \mathscr{E} \;\text{ imply} \\ &\|R_\tau(u,v)\|^2 \leq \gamma^2 \left(\|u\|^2 + \|v\|^2\right) + \gamma\delta_0(v) \;\text{ for all }\; (u,v) \in \mathscr{D}. \end{aligned} \tag{C.6}$$

Observe for all $\gamma > 0$ and all $w \in \ell_2(\mathscr{H}^k)$, $(u,v) \in \mathscr{D}$ that

$$\begin{aligned} &\frac{1}{\gamma}\|Mw + Mu + Nv\|^2 - \gamma\left(\|u\|^2 + \|v\|^2\right) \\ &\qquad \leq \frac{1}{\gamma}\left(\|M\|\|w\| + \|M\|\|u\| + \|N\|\|v\|\right)^2 - \gamma\left(\|u\|^2 + \|v\|^2\right) \\ &\qquad = \begin{pmatrix} \|w\| \\ \|u\| \\ \|v\| \end{pmatrix}^T \begin{pmatrix} m_{11}/\gamma & m_{12}/\gamma & m_{13}/\gamma \\ m_{12}/\gamma & m_{22}/\gamma - \gamma & m_{23}/\gamma \\ m_{13}/\gamma & m_{23}/\gamma & m_{33}/\gamma - \gamma \end{pmatrix} \begin{pmatrix} \|w\| \\ \|u\| \\ \|v\| \end{pmatrix}, \end{aligned} \tag{C.7}$$

where the m_{ij} only depend on $\|M\|$ and $\|N\|$. Now add (3.2), b), and (C.7) to get

$$\Sigma\begin{pmatrix} Mw + Mu + Nv \\ w \end{pmatrix} + \frac{1}{\gamma}\|Mw + Mu + Nv\|^2 - \gamma(\|u\|^2 + \|v\|^2)$$

$$\leq \Big(\star\Big)^T \begin{pmatrix} -\varepsilon + m_{11}/\gamma & m_{12}/\gamma + \sigma_{12} & m_{13}/\gamma + \sigma_{13} \\ m_{12}/\gamma + \sigma_{12} & m_{22}/\gamma - \gamma + \sigma_{22} & m_{23}/\gamma + \sigma_{23} \\ m_{13}/\gamma + \sigma_{13} & m_{23}/\gamma + \sigma_{23} & m_{33}/\gamma - \gamma + \sigma_{33} \end{pmatrix} \begin{pmatrix} \|w\| \\ \|u\| \\ \|v\| \end{pmatrix}$$

for $(u, v) \in \mathscr{D}$ and $w \in \ell_2(\mathscr{H}^k)$. Since $\varepsilon > 0$ there exists some large $\gamma > 0$ such that

$$\Sigma \begin{pmatrix} Mw + Mu + Nv \\ w \end{pmatrix} + \frac{1}{\gamma}\|Mw + Mu + Nv\|^2 - \gamma\left(\|u\|^2 + \|v\|^2\right) \leq 0 \tag{C.8}$$

for all $w \in \ell_2(\mathscr{H}^k)$, $(u, v) \in \mathscr{D}$. Now fix any $(u, v) \in \mathscr{D}$ and $\tau \in [0, 1]$. Due to the hypothesis in (C.6), we infer that $z := R_\tau(u, v) \in \mathscr{E}$ and thus, since Δ was assumed to be bounded, also $w := \tau\Delta(z) \in \ell_2(\mathscr{H}^k)$. Moreover, with the loop equation $z = Mw + Mu + Nv$ by (3.3), we can exploit (C.8) to get

$$\Sigma \begin{pmatrix} z \\ \tau\Delta(z) \end{pmatrix} + \frac{1}{\gamma}\|R_\tau(u, v)\|^2 - \gamma\left(\|u\|^2 + \|v\|^2\right) \leq 0. \tag{C.9}$$

Since $w + u \in \ell_2(\mathscr{H}^k)$ and $v \in \mathscr{V}$ it remains to use c) in order to obtain from (C.9) that $\frac{1}{\gamma}\|R_\tau(u, v)\|^2 \leq \gamma\left(\|u\|^2 + \|v\|^2\right) + \delta_0(v)$ as was to be shown.

Step 2. Since Δ is bounded, there exist $\hat{\delta} > 0$, $\hat{\delta}_0 \geq 0$ such that $\|\Delta(z)_T\| \leq \hat{\delta}\|z_T\| + \hat{\delta}_0$ for all $T \in \mathbb{N}_0$, $z \in \mathscr{E}_e$. With $\gamma > 0$ from Step 1 we now fix any $\rho_0 > 0$ with $\gamma\rho_0\hat{\delta} < 1$. In this step we show that

$$\tau \in [0, 1],\ \tau + \rho \in [0, 1],\ |\rho| \leq \rho_0,\ R_\tau(\mathscr{D}) \subset \mathscr{E} \ \text{ imply } \ R_{\tau+\rho}(\mathscr{D}) \subset \mathscr{E}. \tag{C.10}$$

Choose ρ and τ as in (C.10) and any $(u, v) \in \mathscr{D}$. We have to show that $z = R_{\tau+\rho}(u, v) \in \ell_2(\mathscr{H}^l)$. Observe that $z - \tau M\Delta(z) - \rho M\Delta(z) = Mu + Nv$ can be written as

$$z - \tau M\Delta(z) = M(\rho\Delta(z) + u) + Nv \iff z = R_\tau(\rho\Delta(z) + u, v).$$

As in [110, 93], the key idea is to just employ a small-gain argument based on $1-\gamma\rho_0\hat{\delta}>0$ as follows. We know that $z\in\mathscr{E}_e$. The hypothesis in (C.10) allows us to exploit Step 1; since R_τ is causal in the first argument, we infer with $\gamma_0=\sqrt{\gamma\delta_0(v)}$ for $T\in\mathbb{N}_0$ that

$$\begin{aligned}\|z_T\| &= \|R_\tau(\rho\Delta(z)_T+u_T,v)_T\| \leq \|R_\tau(\rho\Delta(z)_T+u_T,v)\|\\ &\leq \gamma\|\rho\Delta(z)_T+u_T\|+\gamma\|v\|+\gamma_0\\ &\leq (\gamma\rho_0\hat{\delta})\|z_T\|+\gamma\|u\|+\gamma\|v\|+\gamma\rho_0\hat{\delta}_0+\gamma_0.\end{aligned}$$

Hence

$$(1-\gamma\rho_0\hat{\delta})\|z_T\| \leq \gamma\|u\|+\gamma\|v\|+\gamma\rho_0\hat{\delta}_0+\gamma_0 \quad \text{for all} \quad T\in\mathbb{N}_0,$$

which implies $z\in\mathscr{E}_e\cap\ell_2(\mathscr{H}^l)=\mathscr{E}$.

Step 3. Due to boundedness of M, N and assumption a) we have $R_0(\mathscr{D})\subset\mathscr{E}$. Since ρ_0 in Step 2 does not depend on τ, we can inductively apply (C.10) in order to infer $R_\tau(\mathscr{D})\subset\mathscr{E}$ for $\tau\in[0,\nu\rho_0]\cap[0,1]$ and all $\nu=1,2,\ldots$ and thus in particular also for $\tau=1$. Then (C.6) implies (3.4).

C.3 For Chapter 4

C.3.1 Proof of Theorem 4.7

In view of item a) and for $\tau\in[0,1]$, we can introduce the notation $z=R_\tau(u,v)$ for the response of the interconnection (4.3) if $(u,v)\in\mathscr{U}_e\times\mathscr{V}$ and if replacing Δ with $\tau\Delta$.

Step 1. Only for proving the following fact we make use of the properties of Σ and and the quadratic constraints in c) and d): There exist (τ-independent) $\gamma>0$, γ_0 such that

$$\begin{aligned}&\tau\in[0,1] \ \text{ and } \ R_\tau(\mathscr{U}\times\mathscr{V})\subset\mathscr{Z} \ \text{ imply}\\ &\qquad \|R_\tau(u,v)\|_{\mathscr{Z}} \leq \gamma(\|u\|_{\mathscr{U}}+\|v\|_{\mathscr{V}})+\gamma_0 l(R_\tau(u,v))\\ &\qquad\qquad\qquad \text{for all} \quad (u,v)\in\mathscr{U}\times\mathscr{V}. \qquad \text{(C.11)}\end{aligned}$$

Indeed, by exploiting boundedness of $J : \mathscr{W} \to \mathscr{U}$, $M : \mathscr{U} \to \mathscr{Z}$ and $N : \mathscr{V} \to \mathscr{Z}$, observe for all $\bar{\gamma} > 0$ and all $(u, v, w) \in \mathscr{U} \times \mathscr{V} \times \mathscr{W}$ that

$$\begin{aligned}
&\frac{1}{\bar{\gamma}}\|Mw + Mu + N(v)\|_{\mathscr{Z}}^2 - \bar{\gamma}\left(\|u\|_{\mathscr{U}}^2 + \|v\|_{\mathscr{V}}^2\right) \\
&\leq \frac{1}{\bar{\gamma}}\left(\|M\|\|J\|\|w\|_{\mathscr{W}} + \|M\|\|u\|_{\mathscr{U}} + \gamma_N\|v\|_{\mathscr{V}}\right)^2 - \bar{\gamma}\left(\|u\|_{\mathscr{U}}^2 + \|v\|_{\mathscr{V}}^2\right) \\
&= \begin{pmatrix} \|w\|_{\mathscr{W}} \\ \|u\|_{\mathscr{U}} \\ \|v\|_{\mathscr{V}} \end{pmatrix}^T \begin{pmatrix} m_{11}/\bar{\gamma} & m_{12}/\bar{\gamma} & m_{13}/\bar{\gamma} \\ m_{12}/\bar{\gamma} & m_{22}/\bar{\gamma} - \bar{\gamma} & m_{23}/\bar{\gamma} \\ m_{13}/\bar{\gamma} & m_{23}/\bar{\gamma} & m_{33}/\bar{\gamma} - \bar{\gamma} \end{pmatrix} \begin{pmatrix} \|w\|_{\mathscr{W}} \\ \|u\|_{\mathscr{U}} \\ \|v\|_{\mathscr{V}} \end{pmatrix},
\end{aligned} \tag{C.12}$$

where the constants m_{ij} only depend on J, M and N. If we add (4.5) and (4.7) we infer

$$\Sigma\begin{pmatrix} Mw + Mu + N(v) \\ w \end{pmatrix} \leq \left(\star\right)^T \begin{pmatrix} -\varepsilon & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \begin{pmatrix} \|w\|_{\mathscr{W}} \\ \|u\|_{\mathscr{U}} \\ \|v\|_{\mathscr{V}} \end{pmatrix} \tag{C.13}$$

for all $(u, v, w) \in \mathscr{U} \times \mathscr{V} \times \mathscr{W}$. We can clearly fix some sufficiently large $\bar{\gamma} > 0$ with

$$\begin{pmatrix} m_{11}/\bar{\gamma} & m_{12}/\bar{\gamma} & m_{13}/\bar{\gamma} \\ m_{12}/\bar{\gamma} & m_{22}/\bar{\gamma} - \bar{\gamma} & m_{23}/\bar{\gamma} \\ m_{13}/\bar{\gamma} & m_{23}/\bar{\gamma} & m_{33}/\bar{\gamma} - \bar{\gamma} \end{pmatrix} + \begin{pmatrix} -\varepsilon & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \prec 0.$$

Thus, by adding (C.12) and (C.13) we get

$$\begin{aligned}
&\Sigma\begin{pmatrix} Mw + Mu + N(v) \\ w \end{pmatrix} + \frac{1}{\bar{\gamma}}\|Mw + Mu + N(v)\|_{\mathscr{Z}}^2 - \\
&\quad - \bar{\gamma}\left(\|u\|_{\mathscr{U}}^2 + \|v\|_{\mathscr{V}}^2\right) \leq 0 \quad \text{for all} \quad (u, v, w) \in \mathscr{U} \times \mathscr{V} \times \mathscr{W}.
\end{aligned} \tag{C.14}$$

Now fix any $(u, v) \in \mathscr{U} \times \mathscr{V}$. Due to the hypothesis in (C.11), we infer that $z := R_\tau(u, v)$ satisfies $z \in \mathscr{Z}$ and thus, since the bounded uncertainty Δ maps $\mathscr{Z}$ into $\mathscr{W}$, also $w := \tau\Delta(z) \in \mathscr{W}$. From the loop equation $z = Mw + Mu + N(v)$ we conclude with (C.14) that

$$\|R_\tau(u,v)\|_{\mathscr{Z}}^2 \leq \bar{\gamma}^2 \left(\|u\|_{\mathscr{U}}^2 + \|v\|_{\mathscr{V}}^2\right) - \bar{\gamma}\Sigma \begin{pmatrix} z \\ \tau\Delta(z) \end{pmatrix}. \tag{C.15}$$

Since Assumption 4.5.a) implies $\mathscr{W} \subset \mathscr{U}$, we get $w + u \in \mathscr{U}$ and with $v \in \mathscr{V}$ we conclude $z \in M\mathscr{U} + N(\mathscr{V})$. Hence, we can exploit (4.6) in order to obtain from (C.15) the inequality

$$\|R_\tau(u,v)\|_{\mathscr{Z}}^2 \leq \bar{\gamma}^2 \left(\|u\|_{\mathscr{U}}^2 + \|v\|_{\mathscr{V}}^2\right) + \bar{\gamma} l(R_\tau(u,v))^2,$$

and thus (C.11).

Step 2. By Lemma 4.4 there exist $\delta > 0$, $\delta_0 \geq 0$ with $\|\Delta(z)\|_{\mathscr{U}_e,T} \leq \delta\|z\|_{\mathscr{Z}_e,T} + \delta_0$ for all $T > 0$, $z \in \mathscr{Z}_e$. With the constant $K_{\mathscr{U}}$ for the space $\mathscr{U}_e$ as in (4.1) and with γ from the first step, let us fix any $\rho_0 > 0$ satisfying $\rho_0\gamma\delta K_{\mathscr{U}} < 1$. In this second step we show that

$$\tau \in [0,1],\ \tau + \rho \in [0,1],\ |\rho| \leq \rho_0,\ R_\tau(\mathscr{U} \times \mathscr{V}) \subset \mathscr{Z} \quad \text{imply} \quad R_{\tau+\rho}(\mathscr{U} \times \mathscr{V}) \subset \mathscr{Z}. \tag{C.16}$$

Choose ρ and τ as in (C.16) and take any $(u,v) \in \mathscr{U} \times \mathscr{V}$. Then $z = R_{\tau+\rho}(u,v)$ is known to be contained in $\mathscr{Z}_e$; due to loop equation, we have $z - \tau M\Delta(z) - \rho M\Delta(z) = Mu + N(v)$ which is equivalent to $z - \tau M\Delta(z) = M(\rho\Delta(z) + u) + N(v)$ and hence also to $z = R_\tau(\rho\Delta(z) + u, v)$. Fix any $T > 0$. Since $\rho\Delta(z) \in \mathscr{W}_e \subset \mathscr{U}_e$ and thus $\rho\Delta(z) + u \in \mathscr{U}_e$ we get $((\rho\Delta(z) + u)^T, v) \in \mathscr{U} \times \mathscr{V}$, which implies by the assumption in (C.16) that $R_\tau((\rho\Delta(z) + u)^T, v) \in \mathscr{Z}$. Hence, due to (C.11),

$$\|R_\tau((\rho\Delta(z) + u)^T, v)\|_{\mathscr{Z}} \leq \gamma\|(\rho\Delta(z) + u)^T\|_{\mathscr{U}} + \gamma\|v\|_{\mathscr{V}} + \\ + \gamma_0 l(R_\tau((\rho\Delta(z) + u)^T, v)). \tag{C.17}$$

Since R_τ is causal in the first argument, we get $z_T = R_\tau(\rho\Delta(z) + u, v)_T = R_\tau((\rho\Delta(z) + u)^T, v)_T$ and thus $\|z\|_{\mathscr{Z}_e,T} = \|z_T\|_{\mathscr{Z}_T} = \|R_\tau((\rho\Delta(z) + u)^T, v)_T\|_{\mathscr{Z}_T} \leq \|R_\tau((\rho\Delta(z) + u)^T, v)\|_{\mathscr{Z}}$. Now note that $z^T = R_\tau((\rho\Delta(z)+u)^T, v)$ is actually a valid extension of z at time T. If we combine with (C.17) and exploit (4.1), we obtain

$$\begin{aligned}\|z\|_{\mathscr{Z}_e,T} &\leq \gamma\|(\rho\Delta(z)+u)^T\|_{\mathscr{U}} + \gamma\|v\|_{\mathscr{V}} + \gamma_0 l(z^T)\\ &\leq \gamma K_{\mathscr{U}}\|\rho\Delta(z)+u\|_{\mathscr{U}_e,T} + \gamma\|v\|_{\mathscr{V}} + \gamma_0 l(z^T)\\ &\leq \gamma K_{\mathscr{U}}(|\rho|\delta\|z\|_{\mathscr{Z}_e,T} + |\rho|\delta_0 + \|u\|_{\mathscr{U}_e,T}) + \gamma\|v\|_{\mathscr{V}} + \gamma_0 l(z^T)\\ &\leq (\rho_0\gamma\delta K_{\mathscr{U}})\|z\|_{\mathscr{Z}_e,T} + \gamma K_{\mathscr{U}}(\rho_0\delta_0 + \|u\|_{\mathscr{U}}) + \gamma\|v\|_{\mathscr{V}} + \gamma_0 l(z^T).\end{aligned}$$

This implies with (4.4) that $(1 - \rho_0\gamma\delta K_{\mathscr{U}})\|z\|_{\mathscr{Z}_e,T} \leq \gamma K_{\mathscr{U}}(\rho_0\delta_0 + \|u\|_{\mathscr{U}}) + \gamma\|v\|_{\mathscr{V}} + \gamma_0 c$. Since $T > 0$ was arbitrary and the right-hand side does not depend on T, we can exploit $1 - \rho_0\gamma\delta K_{\mathscr{U}} > 0$ to infer $\sup_{T>0}\|z\|_{\mathscr{Z}_e,T} < \infty$ and thus $z \in \mathscr{Z}$ as was to be shown.

Step 3. Clearly $R_0(\mathscr{U}\times\mathscr{V}) \subset \mathscr{Z}$. Since ρ_0 in Step 2 does not depend on τ, we can inductively apply (C.16) in order to infer $R_\tau(\mathscr{U}\times\mathscr{V}) \subset \mathscr{Z}$ for $\tau \in [0, \nu\rho_0] \cap [0,1]$ and all $\nu = 1, 2, \ldots$, and thus this inclusion holds in particular also for $\tau = 1$. Then (C.11) for $\tau = 1$ implies (4.8).

C.3.2 Proof of Theorem 4.12

b) $\Rightarrow$ *a)*: Choose $w, d \in \mathscr{H}^r$ and let $u := \mathrm{col}(\partial^r w, \partial^r d)$. By right- and left-multiplying a perturbed version of (4.25) with a trajectory $\mathrm{col}(x_e(t), u(t))$ of (4.21) and its transpose, we infer for some $\varepsilon > 0$ and almost all $t \geq 0$ that

$$\begin{pmatrix}\star\end{pmatrix}^T \begin{pmatrix}0 & X\\ X & 0\end{pmatrix}\begin{pmatrix}x_e(t)\\ \dot{x}_e(t)\end{pmatrix} + \begin{pmatrix}\star\end{pmatrix}^T\begin{pmatrix}Q_1 & 0\\ 0 & P\end{pmatrix}\begin{pmatrix}T_\Delta\\ T_p\end{pmatrix}\begin{pmatrix}x_e(t)\\ \partial^r w(t)\\ \partial^r d(t)\end{pmatrix}$$
$$\leq -\varepsilon\sum_{l=0}^{r}\left(\|\partial^l w(t)\|^2 + \|\partial^l d(t)\|^2\right).$$

Since x_e vanishes at infinity and $x(0) = 0$, integration over $[0, T]$ and taking the limit $T \to \infty$ results in

$$\sigma_{Q_1}\begin{pmatrix} Mw+Nd \\ w \end{pmatrix} - \left(\star\right)^T \left(\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array}\right) \left(\begin{array}{c} 0 \\ \hline \mathcal{D}^{r-1}w(0) \\ \mathcal{D}^{r-1}d(0) \end{array}\right) +$$
$$+\,\sigma_P\begin{pmatrix} N_{21}w+N_{22}d \\ d \end{pmatrix} \leq -\varepsilon\big(\|w\|_r^2+\|d\|_r^2\big).$$

The statement follows by setting $R=-X_{22}$.

$a) \Rightarrow b)$: With the real matrix

$$H := \begin{pmatrix} T_\Delta \\ T_p \end{pmatrix}^T \begin{pmatrix} Q_1 & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} T_\Delta \\ T_p \end{pmatrix}$$

define

$$q_\eta(x,\widetilde{w},\widetilde{d},u) := \left(\star\right)^T [H+\eta I]\,\mathrm{col}(x,\widetilde{w},\widetilde{d},u) \quad \text{for} \quad \eta\in\mathbb{R}.$$

By (4.24) there exists some $\varepsilon>0$ such that the trajectories of (4.17) with $w,d\in\mathscr{H}^r$ satisfy

$$\int_0^\infty q_0\big(x(t),\mathcal{D}^{r-1}w(t),\mathcal{D}^{r-1}d(t),u(t)\big)\,dt \leq -\varepsilon\big(\|w\|_r^2+\|d\|_r^2\big)+r_0(w,d)$$

with

$$r_0(w,d) := \begin{pmatrix} \mathcal{D}^{r-1}w(0) \\ \mathcal{D}^{r-1}d(0) \end{pmatrix}^T R \begin{pmatrix} \mathcal{D}^{r-1}w(0) \\ \mathcal{D}^{r-1}d(0) \end{pmatrix}.$$

Since A is Hurwitz, we have (with $\|\cdot\|$ denoting the norm on $\mathscr{L}_2$)

$$\|x\| \leq \gamma\big(\|w\|+\|d\|\big) \leq \gamma\big(\|w\|_r+\|d\|_r\big)$$

with $\gamma := \|(sI-A)^{-1}\begin{pmatrix} B_1 & B_2 \end{pmatrix}\|_\infty$ along all the above trajectories. For some sufficiently small $\widetilde{\varepsilon}>0$ this implies

$$\int_0^\infty q_{2\widetilde{\varepsilon}}\big(x(t),\mathcal{D}^{r-1}w(t),\mathcal{D}^{r-1}d(t),u(t)\big)\,dt$$

$$\leq -\frac{\varepsilon}{2}\big(\|w\|_r^2 + \|d\|_r^2\big) + r_0(w,d).$$

Now consider trajectories of the system in (4.17) for $w, d \in \mathscr{H}^r$ and $x(0) = \xi$; the state response is $x_\xi := x + v$ if x is the response with $x(0) = 0$ and $v = e^{A\bullet}\xi$. Note that there exists some $Q_v = Q_v^T$ with $\|v\|^2 = \xi^T Q_v \xi$. Moreover, there also exists some (large) $\widetilde{\gamma} > 0$ such that

$$T^T[H + \widetilde{\varepsilon} I]T \preccurlyeq \begin{pmatrix} \widetilde{\gamma} I & 0 \\ 0 & H + 2\widetilde{\varepsilon} I \end{pmatrix} \quad \text{for} \quad T := \left(\begin{array}{c|ccc} I & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right).$$

This implies

$$q_{\widetilde{\varepsilon}}\big(x(t), \mathcal{D}^{r-1}w(t), \mathcal{D}^{r-1}d(t), u(t)\big) \leq \\ \leq \widetilde{\gamma} v(t)^T v(t) + q_{2\widetilde{\varepsilon}}\big(x(t), \mathcal{D}^{r-1}w(t), \mathcal{D}^{r-1}d(t), u(t)\big)$$

and thus, by combining with what we have derived so far,

$$\int_0^\infty q_{\widetilde{\varepsilon}}\big(x(t), \mathcal{D}^{r-1}w(t), \mathcal{D}^{r-1}d(t), u(t)\big)\, dt \leq \widetilde{\gamma}\xi^T Q_v \xi - \\ - \frac{\varepsilon}{2}\big(\|w\|_r^2 + \|d\|_r^2\big) + r_0(w,d). \quad \text{(C.18)}$$

Now we use the fact that the above considered trajectories are also trajectories of the system in (4.21) with $x_e(0) = \mathrm{col}(\xi, \mathcal{D}^{r-1}w(0), \mathcal{D}^{r-1}d(0))$.

Let $U(\xi_e)$ denote the set of all control functions $u \in \mathscr{L}_2 \times \mathscr{L}_2$ such that the response of $\dot{x}_e = A_e x_e + B_e u$ with $x_e(0) = \xi_e$ satisfies $x_e \in \mathscr{L}_2$. With $\widetilde{\varepsilon}$ from above introduce

$$V(\xi_e) := \inf_{u \in U(\xi_e)} \left\{ \int_0^\infty -q_{\widetilde{\varepsilon}}\big(x_\xi, \mathcal{D}^{r-1}w, \mathcal{D}^{r-1}d, u\big)\, dt \;\middle|\; \dot{x}_e(t) = A_e x_e(t) + B_e u(t),\ x_e(0) = \xi_e \right\}.$$

Then (C.18) implies that $V(\xi_e) > -\infty$ for all $\xi_e \in \mathbb{R}^{n+rk}$. The remaining part of this proof is now an application of classical dissipation arguments [180, 116]. We infer by Theorem 1 in [116] and controllability of (A_e, B_e) that V satisfies the dissipation inequality

$$V(x_e(t_1)) \leq \int_{t_1}^{t_2} q_{\widetilde{\varepsilon}}\big(x_e(t), \dot{w}(t)\big)\, dt + V(x_e(t_2)) \tag{C.19}$$

for any trajectory of (4.21) and any $0 \leq t_1 \leq t_2$. Finally, $\widetilde{\varepsilon} > 0$ and Theorems 2 and 3 in [180] imply the existence of a symmetric solution of the strict inequality (4.25).

$b) \Leftrightarrow c)$: By the generalized KYP lemma, Lemma 2.11 (see also [14, Lemma 1]), the LMI (4.25) is feasible if and only if (by setting $1/\infty = 0$)

$$\big(\star\big)^* \begin{pmatrix} Q_1 & 0 \\ 0 & P \end{pmatrix} \left(\begin{array}{cc} E^r M & E^r N \\ E^r & 0 \\ \hdashline E^r N_{21} & E^r N_{22} \\ 0 & E^r \end{array}\right) (\infty) \prec 0 \tag{C.20}$$

and

$$\underbrace{\begin{pmatrix} x_e \\ u \end{pmatrix}}_{\neq 0} \in \operatorname{Ker} \begin{pmatrix} A_e - i\omega I & B_e \end{pmatrix}$$

$$\Longrightarrow \big(\star\big)^T P T_M \begin{pmatrix} x_e \\ u \end{pmatrix} \prec 0 \text{ for all } \omega \in \mathbb{R}. \tag{C.21}$$

With a tedious yet elementary calculation we can equivalently reformulate (C.21) as

$$\big(\star\big)^* \begin{pmatrix} Q_1 & 0 \\ 0 & P \end{pmatrix} \left(\begin{array}{cc} E^r M & E^r N \\ E^r & 0 \\ \hdashline E^r N_{21} & E^r N_{22} \\ 0 & E^r \end{array}\right) \prec 0 \text{ on } i\mathbb{R} \setminus \{0\} \tag{C.22}$$

and (4.27).

If (4.26) holds, then obviously both (C.20) and (C.22) are satisfied. The proof of the converse implication relies on a continuity argument and proceeds in complete analogy to the one given in [58].

C.4 For Chapter 5

C.4.1 Proof of Lemma 5.10

It suffices to prove the statement for $k = 1$ and $\alpha \neq \beta$ (since otherwise we have $\alpha = 0 = \beta$ and $\varphi = 0$). As $\mathbf{\Pi}_{dr}[\alpha, \beta]^k \subset \mathbf{\Pi}_{dc}[\alpha, \beta]^k$, the "if" statement is trivial. To prove "only if" along the lines of an argument in [45], assume that we found a multiplier $\Pi \in \mathbf{\Pi}_{dc}[\alpha, \beta]$ with (5.10). The constraints $R \prec 0$ and $F_\Pi(\Delta) \succ 0$ for all $\Delta \in \Theta(\{\alpha, \beta\}, k)$ translate into $r < 0$, $m_1 > 0$ and $m_2 > 0$ for

$$\begin{pmatrix} m_1 & n \\ n & m_2 \end{pmatrix} := \begin{pmatrix} 1 & 1 \\ \alpha & \beta \end{pmatrix}^T \begin{pmatrix} q & s \\ s & r \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \alpha & \beta \end{pmatrix}^T \Pi \begin{pmatrix} 1 & 1 \\ \alpha & \beta \end{pmatrix}.$$

Since $\alpha \neq \beta$, the equation

$$\begin{pmatrix} 1 & 1 \\ \alpha & \beta \end{pmatrix}^T \Pi_\tau \begin{pmatrix} 1 & 1 \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \alpha & \beta \end{pmatrix}^T \begin{pmatrix} q_\tau & s_\tau \\ s_\tau & r_\tau \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} \tau m_1 & n \\ n & \tau m_2 \end{pmatrix}$$

has a unique solution Π_τ for every $\tau \in [0, 1]$. Moreover, because the derivative of the right hand side with respect to τ is positive definite, we infer $\dot{\Pi}_\tau \succ 0$ for all $\tau \in [0, 1]$ and thus $\Pi_0 \prec \Pi_1 = \Pi$. Hence (5.10) also holds with Π replaced by Π_0. Moreover, we obtain

$$r_0 < r < 0 \quad \text{and} \quad \begin{pmatrix} 1 \\ \delta \end{pmatrix}^T \begin{pmatrix} q_0 & s_0 \\ s_0 & r_0 \end{pmatrix} \begin{pmatrix} 1 \\ \delta \end{pmatrix} = 0 \quad \text{for} \quad \delta \in \{\alpha, \beta\}. \qquad \text{(C.23)}$$

With $\widetilde{q} = q_0 / |r_0|$, $\widetilde{s} = s_0 / |r_0|$ we get from (C.23) that $\widetilde{q} + 2\delta\widetilde{s} - \delta^2 = 0$ for $\delta \in \{\alpha, \beta\}$, which implies $\widetilde{q} = -\alpha\beta$ and $\widetilde{s} = \frac{\alpha+\beta}{2}$. Hence $\Pi_0 = |r_0| \begin{pmatrix} -\alpha\beta & \frac{\alpha+\beta}{2} \\ \frac{\alpha+\beta}{2} & -1 \end{pmatrix} \in \mathbf{\Pi}_{dr}[\alpha, \beta]$ which completes the proof.

C.4.2 Proof of Lemma 5.26

The following proof uses Dini derivatives and some properties of absolutely continuous functions. A comprehensive treatment of the concepts relevant for our purpose is for example given in [131].

Set $\mu := \max\{|\mu_1|, \mu_2\}$ and suppose that z is differentiable at t (which is the case for almost every $t \in [0, \infty)$). If $\dot{z}(t) = 0$ then $w(.) := \varphi(z(.))$ is differentiable with derivative zero at t: Indeed, with (5.1) we infer

$$\frac{|\varphi(z(t+h)) - \varphi(z(t))|}{|h|} \leq \frac{\mu|z(t+h) - z(t)|}{|h|} \to 0 \quad \text{for} \quad h \to 0.$$

Now suppose $\dot{z}(t) \neq 0$. We can then choose $\delta > 0$ such that for $0 < |h| < \delta$ we have $\left|\frac{z(t+h)-z(t)}{h} - \dot{z}(t)\right| \leq \frac{1}{2}|\dot{z}(t)|$. This implies $z(t+h) - z(t) \neq 0$ and hence the right-hand side in

$$\frac{\varphi(z(t+h)) - \varphi(z(t))}{h} = \frac{\varphi(z(t+h)) - \varphi(z(t))}{z(t+h) - z(t)} \frac{z(t+h) - z(t)}{h}$$

is well-defined for $0 < |h| < \delta$.

Let us now consider the limit $h \searrow 0$. First, if $\dot{z}(t) > 0$ then $z(t+h) - z(t) > 0$ for $h \in (0, \delta)$. Suppose $D^+\varphi$, D^+w and $D_+\varphi$, D_+w are the right upper and right lower Dini derivatives of φ, w respectively. We can then choose $h_\nu \in (0, \delta)$ with $\frac{w(t+h_\nu)-w(t)}{h_\nu} \to D^+w(t)$. With $z_\nu := z(t+h_\nu) - z(t) > 0$ we infer that

$$\limsup_{\nu\to\infty} \frac{\varphi(z(t) + z_\nu) - \varphi(z(t))}{z_\nu} \leq$$
$$\leq \limsup_{h\searrow 0} \frac{\varphi(z(t) + h) - \varphi(z(t))}{h} = D^+\varphi(z(t)).$$

Note that $z_\nu \searrow 0$ for $\nu \to \infty$. This finally implies

$$D^+w(t) = \limsup_{\nu\to\infty} \left[\frac{\varphi(z(t) + z_\nu) - \varphi(z(t))}{z_\nu} \frac{z(t+h_\nu) - z(t)}{h_\nu}\right]$$

$$\leq D^+\varphi(z(t))\dot{z}(t).$$

We argue for $D_+w(t)$ in a similar fashion to get

$$D_+\varphi(z(t))\dot{z}(t) \leq D_+w(t) \leq D^+w(t) \leq D^+\varphi(z(t))\dot{z}(t).$$

Now consider the case $\dot{z}(t) < 0$. Then $z_\nu = z(t+h_\nu) - z(t) < 0$ and thus

$$\liminf_{\nu\to\infty} \frac{\varphi(z(t)+z_\nu) - \varphi(z(t))}{z_\nu} \geq$$
$$\geq \liminf_{h\nearrow 0} \frac{\varphi(z(t)+h) - \varphi(z(t))}{h} = D_-\varphi(z(t))$$

with D_-, D^- denoting left Dini derivatives. Then

$$D^+w(t) = \limsup_{\nu\to\infty} \left[\frac{\varphi(z(t)+z_\nu) - \varphi(z(t))}{z_\nu} \frac{z(t+h_\nu) - z(t)}{h_\nu}\right] =$$
$$= \liminf_{\nu\to\infty} \left[\frac{\varphi(z(t)+z_\nu) - \varphi(z(t))}{z_\nu}\right] \dot{z}(t) \leq D_-\varphi(z(t))\dot{z}(t).$$

By a similar argument for $D_+w(t)$ we infer

$$D^-\varphi(z(t))\dot{z}(t) \leq D_+w(t) \leq D^+w(t) \leq D_-\varphi(z(t))\dot{z}(t).$$

If both z and w are differentiable at t we conclude

$$D_+\varphi(z(t))\dot{z}(t) \leq \dot{w}(t) \leq D^+\varphi(z(t))\dot{z}(t) \quad \text{if} \quad \dot{z}(t) \geq 0,$$
$$D^-\varphi(z(t))\dot{z}(t) \leq \dot{w}(t) \leq D_-\varphi(z(t))\dot{z}(t) \quad \text{if} \quad \dot{z}(t) < 0.$$

Now note that $\varphi \in \mathrm{slope}(\mu_1, \mu_2)$ implies

$$\mu_1 \leq D_-\varphi(z) \leq D^-\varphi(z) \leq \mu_2 \quad \text{and}$$
$$\mu_1 \leq D_+\varphi(z) \leq D^+\varphi(z) \leq \mu_2 \quad \text{for all} \quad z \in \mathbb{R}.$$

For $z \in \mathscr{H}_e^{1,1}$, $w = \varphi(z)$ and $t \geq 0$ such that both z and w are differentiable at t (which is true for almost all such points), we infer

$$\mu_1\dot{z}(t) \leq \dot{w}(t) \leq \mu_2\dot{z}(t) \quad \text{and} \quad \mu_2\dot{z}(t) \leq \dot{w}(t) \leq \mu_1\dot{z}(t) \tag{C.24}$$

for $\dot{z}(t) \geq 0$ and $\dot{z}(t) < 0$, respectively. Consequently, irrespective of the sign of $\dot{z}(t)$, we get (5.47).

C.4.3 Proof of Lemma 5.3

Let $z \in \mathscr{H}^{1,k}$ and set $\mu := \max\{|\mu_1|, \mu_2\}$. Then z is locally absolutely continuous. As $\Delta_\Phi : \mathscr{L}_2^k \to \mathscr{L}_2^k$ is bounded with $\|\Delta_\Phi\| \leq \mu$ and φ is Lipschitz continuous, $w = \Delta(z)$ is square integrable with $\|w\|^2 \leq \mu^2 \|z\|^2$ and locally absolutely continuous. Hence $\dot{w}(t)$ exists for almost every $t \in [0, \infty)$ and, by (C.24), $\|\dot{w}(t)\| \leq \mu \|\dot{z}(t)\|$ for almost every $t \in [0, \infty)$. Thus $\dot{z} \in \mathscr{L}_2^k$ implies $\dot{w} \in \mathscr{L}_2^k$ and $\|w\|_{\mathscr{H}}^2 = \|w\|^2 + \|\dot{w}\|^2 \leq \mu^2(\|z\|^2 + \|\dot{z}\|^2) = \mu^2 \|z\|_{\mathscr{H}}^2$. This proves the claim.

C.5 For Chapter 6

C.5.1 Proof of Lemma 6.4

It suffices to prove the claim for a scalar nonlinearity $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying $\varphi \in \operatorname{slope}(\mu, \nu)$ for some $\mu \leq 0 \leq \nu$. We use the notation from [39].

Let $x \in \mathbb{R}$ and $v > 0$. Then we have for the generalized directional derivative

$$\varphi^\circ(x; v) = \limsup_{\substack{y \to x \\ t \searrow 0}} \frac{\varphi(y + tv) - \varphi(y)}{t} = v \limsup_{\substack{y \to x \\ t \searrow 0}} \frac{\varphi(y + tv) - \varphi(y)}{tv} \geq \mu v$$

With the same reasoning, we obtain for $v < 0$ that $\varphi^\circ(x; v) \geq \nu v$. In conclusion, we obtain

$$\mu \leq \frac{\varphi^\circ(x; v)}{v} \leq \nu$$

Finally, with $\varphi^\circ(x; 0) \geq 0$ and the definition of the generalized gradient $\partial\varphi$, we arrive at

$$\partial\varphi(x) \subset [\mu, \nu] \quad \text{for all} \quad x \in \mathbb{R}.$$

Now let $w(t) = \varphi(z(t))$ for $z \in \ell_2^k$ and $t \in \mathbb{N}_0$. Then the statement follows form Lebourg's mean value theorem ([39, Theorem 2.3.7]).

C.5.2 Proof of Lemma 6.13

The proof relies on an argument similarly to one made in [45], which is also used in the proof of Lemma 5.10. Let Π, Π_1 and Π_2 be parameterized by L, L_1 and L_2, respectively. A short computation reveals that the combined multiplier

$$\Pi := \Pi + \Pi_1 + \Pi_2$$

satisfies

$$\begin{pmatrix} 1 & 1 \\ \mu & \nu \end{pmatrix}^T \Pi \begin{pmatrix} 1 & 1 \\ \mu & \nu \end{pmatrix} - \begin{pmatrix} 0 & (\nu-\mu)H_L^* \\ (\nu-\mu)H_L & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} 0 & (\nu-\mu)H_{L_1}^* \\ (\nu-\mu)H_{L_1} & (\nu-\mu)(H_{L_1}^* + H_{L_1}) \end{pmatrix} +$$
$$+ \begin{pmatrix} (\nu-\mu)(H_{L_2}^* + H_{L_2}) & (\nu-\mu)H_{L_2}^* \\ (\nu-\mu)H_{L_2} & 0 \end{pmatrix}.$$

Define

$$\begin{pmatrix} Q & S^* \\ S & R \end{pmatrix} := \begin{pmatrix} 1 & 1 \\ \mu & \nu \end{pmatrix}^T \Pi \begin{pmatrix} 1 & 1 \\ \mu & \nu \end{pmatrix}$$

to infer $Q = (\nu-\mu)(H_{L_2}^* + H_{L_2})$ and $R = (\nu-\mu)(H_{L_1}^* + H_{L_1})$ which are both positive semi-definite on $\mathbb{T}$. Since $\mu < \nu$, the equation

$$\begin{pmatrix} \tau Q & S^* \\ S & \tau R \end{pmatrix} := \begin{pmatrix} 1 & 1 \\ \mu & \nu \end{pmatrix}^T \Pi_\tau \begin{pmatrix} 1 & 1 \\ \mu & \nu \end{pmatrix}$$

has a unique solution Π_τ for all $\tau \in [0,1]$. Moreover, the derivative of the left hand side with respect to τ is positive semi-definite. Hence, we infer $\dot{\Pi}_\tau \succ 0$ and thus $\Pi_0 \preccurlyeq \Pi_1 = \Pi$ on $\mathbb{T}$. Now note that

$$S = \begin{pmatrix} I & \nu I \end{pmatrix} \Pi_0 \begin{pmatrix} I \\ \mu I \end{pmatrix} = \begin{pmatrix} I & \nu I \end{pmatrix} \Pi \begin{pmatrix} I \\ \mu I \end{pmatrix}$$

$$= (\nu - \mu)(H_L + H_{L_1} + H_{L_2}) =: (\nu - \mu) H_{L_3}$$

where H_{L_3} satisfies all requirements of a Zames-Falb multiplier (since the constraints define convex cones). We finally get

$$\begin{aligned}
\Pi_0 &= (\nu - \mu) \begin{pmatrix} 1 & 1 \\ \mu & \nu \end{pmatrix}^{-T} \begin{pmatrix} 0 & H_{L_3}^* \\ H_{L_3} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \mu & \nu \end{pmatrix}^{-1} \\
&= \frac{1}{\nu - \mu} \begin{pmatrix} \nu I & -I \\ -\mu I & I \end{pmatrix}^{T} \begin{pmatrix} 0 & H_3^* \\ H_3 & 0 \end{pmatrix} \begin{pmatrix} \nu I & -I \\ -\mu I & I \end{pmatrix}.
\end{aligned}$$

Thus the lemma follows with $H_{\widetilde{L}} = (\nu - \mu)^{-1} H_{L_3}$, which defines again a Zames-Falb multiplier.

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Declaration

I hereby certify that this thesis has been composed by myself, and describes my own work, unless otherwise acknowledged in the text. All references and verbatim extracts have been quoted, and all sources of information have been specifically acknowledged.

Stuttgart, June 2017

Matthias Fetzer